四维单位球的体积

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习题 (12.28(超体积)). 求四维单位球 $x^2+y^2+z^2+w^2\leq 1$ 的体积.

解. 易得积分为

$$\int_{-1}^{1}\int_{-\sqrt{1-w^2}}^{\sqrt{1-w^2}}\int_{-\sqrt{1-w^2-x^2}}^{\sqrt{1-w^2-x^2}}\int_{-\sqrt{1-w^2-x^2-y^2}}^{\sqrt{1-w^2-x^2-y^2}}dzdydxdw.$$

我们进行球坐标变量替换.令

$$w=\rho\cos\xi, z=\rho\sin\xi\cos\phi, y=\rho\sin\xi\sin\phi\sin\theta, x=\rho\sin\xi\sin\phi\cos\theta.$$

其中 $\xi\in[0,\pi],\phi\in[0,\pi],\theta\in[0,2\pi)$.则我们来看Jacobi 行列式

$$\begin{vmatrix}\frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\xi} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta}\\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\xi} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta}\\ \frac{\partial w}{\partial\rho} & \frac{\partial w}{\partial\xi} & \frac{\partial w}{\partial\phi} & \frac{\partial w}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\sin\xi\sin\phi\cos\theta & \rho\cos\xi\sin\phi\cos\theta & \rho\sin\xi\cos\phi\cos\theta & -\rho\sin\xi\sin\phi\sin\theta\\ \sin\xi\sin\phi\sin\theta & \rho\cos\xi\sin\phi\sin\theta & \rho\sin\xi\cos\phi\sin\theta & \rho\sin\xi\sin\phi\cos\theta\\ \sin\xi\cos\phi & \rho\cos\xi\cos\phi & -\rho\sin\xi\sin\phi & 0\\ \cos\xi & -\rho\sin\xi & 0 & 0\end{vmatrix}$$

下面我们来算.先把一些该提取的给提取掉,得到

$$\rho^3\sin^2\xi\sin\phi\begin{vmatrix}\sin\xi\sin\phi\cos\theta & \cos\xi\sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta\\ \sin\xi\sin\phi\sin\theta & \cos\xi\sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta\\ \sin\xi\cos\phi & \cos\xi\cos\phi & -\sin\phi & 0\\ \cos\xi & -\sin\xi & 0 & 0\end{vmatrix}$$

我们考虑行列式的拉普拉斯展开,得到

$$
\begin{aligned}
&-\cos\xi\begin{vmatrix}\cos\xi\sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta\\ \cos\xi\sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta\\ \cos\xi\cos\phi & -\sin\phi & 0\end{vmatrix}-\sin\xi\begin{vmatrix}\sin\xi\sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta\\ \sin\xi\sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta\\ \sin\xi\cos\phi & -\sin\phi & 0\end{vmatrix}\\
&=-\cos^2\xi\begin{vmatrix}\sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta\\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta\\ \cos\phi & -\sin\phi & 0\end{vmatrix}-\sin^2\xi\begin{vmatrix}\sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta\\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta\\ \cos\phi & -\sin\phi & 0\end{vmatrix}
\end{aligned}
$$

我们只用来计算三阶行列式

$$
T=\begin{vmatrix}\sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta\\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta\\ \cos\phi & -\sin\phi & 0\end{vmatrix}
$$

即可.在正式计算之前,请允许我节外生枝.我们来看在我们计算三维单位球体积 时出现的三阶 Jacobi 行列式

$$
\begin{vmatrix}\frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta}\\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\\ \cos\phi & -\rho\sin\phi & 0\end{vmatrix}=T\rho^2\sin\phi.
$$

可见其关系.易得 $T=1$.因此,我们最终算得4阶 Jacobi 行列式为

$$
-\rho^3\sin^2\xi\sin\phi.
$$

因此,我们可以把上述积分化为

$$
\begin{aligned}
\int_0^{2\pi}\int_0^1\int_0^{\pi}\int_0^{\pi}\rho^3\sin^2\xi\sin\phi d\phi d\xi d\rho d\theta &= \int_0^{2\pi}\int_0^1\int_0^{\pi}2\rho^3\sin^2\xi d\xi d\rho d\theta\\
&=\int_0^{2\pi}\int_0^1\pi\rho^3 d\rho d\theta\\
&=\int_0^{2\pi}\frac{1}{4}\pi d\theta\\
&=\frac{1}{2}\pi^2.
\end{aligned}
$$

□