THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: APRIL 2021

MA243 GEOMETRY

Time Allowed: **2 hours**

Read all instructions carefully. Please note also the guidance you have received in advance on the departmental 'Warwick Mathematics Exams 2021' webpage.

Calculators, wikipedia and interactive internet resources are not needed and are not permitted in this examination. You are not allowed to confer with other people. You may use module materials and resources from the module webpage.

ANSWER ALL THREE QUESTIONS.

On completion of the assessment, you must upload your answer to Moodle as a single PDF document if possible, although multiple files (2 or 3) are permitted. You have an additional 45 minutes to make the upload, and instructions are available on the departmental 'Warwick Mathematics Exams 2021' webpage.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

You can freely use any of the results covered in the module without giving a proof, but please carefully cite the result's number from the lecture notes or the assignments.

1. Decide for each of the following assertions whether it is true or false, and justify your answer with either a proof or a counterexample.

We recall that $\langle -, - \rangle$ and $\| - \|$ denote respectively the Euclidian inner product and the Euclidian norm, and that $\langle -, - \rangle_L$ and $\| - \|_L$ denote respectively the Lorentz inner product and the Lorentz norm.

a) Let X be a set and $d \colon X \times X \to X$ be the function defined by $d(x, y) = 1/2$ if $x \neq y$ and $d(x, x) = 0$, for every $x, y \in X$. Then d is a metric on X. **[4]**

b) Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a Euclidian isometry. Then the image $T(C)$ of a circle C of centre $x \in \mathbb{R}^2$ and radius $r > 0$ is a circle of centre $T(x)$ and radius r. **[4]**

c) Any Euclidian isometry $T \colon \mathbb{R}^n \to \mathbb{R}^n$ is linear, for every $n \geq 1$. **[4]**

d) Let $n \geq 1$ and $f \colon \mathbb{R}^n \to \mathbb{R}^n$ be a map which satisfies $\|f(x)\| = \|x\|$ for all $x \in \mathbb{R}^n$. Then $\langle f(x), f(y) \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{R}^n$. **[4]**

e) Let $n \geq 1$ and $L\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear Euclidian isometry with a real eigenvalue λ. Then $\lambda = \pm 1$. **[4]**

f) For every $n \geq 2$, every (Euclidian) line L in $\mathbb{R}^n$ and point $P \in \mathbb{R}^n$ not belonging to L, there is a line L' passing through P such that $L \cap L' = \emptyset$. **[4]**

g) For every $n \geq 2$, every great circle C in S^n and point $P \in S^n$ not belonging to C, there is a great circle C' passing through P such that $C \cap C' = \emptyset$. **[4]**

h) For every $n \geq 2$, every hyperbolic line L in $\mathcal{H}^n$ and point $P \in \mathcal{H}^n$ not belonging to L, there is a hyperbolic line L' passing through P such that $L \cap L' = \emptyset$. **[4]**

i) Let $x, y \in \mathbb{R}^2$ be space-like. Then $|\langle x, y\rangle_L| \geq \|x\|_L \|y\|_L$. **[4]**

j) Let λ and μ be distinct eigenvectors of a Lorentz transformation, and v and w eigenvectors respectively of λ and μ. Then v and w are Lorentz-orthogonal. **[4]**

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2. a) Show that the map $f\colon S^2 \to S^2$ defined by

$$f(x, y, z) = \left(\frac{x - 2(y+z)}{3}, \frac{y - 2(x+z)}{3}, \frac{z - 2(x+y)}{3}\right)$$

is a spherical isometry. **[6]**

b) Show that f is a reflection, and determine the equation of the corresponding plane. **[6]**

c) Let $R\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the Euclidian isometry defined by

$$R(x, y, z) = (z, x, y).$$

What is the minimum number of reflections needed for decomposing R as a composition of reflections? Find such a decomposition. **[6]**

d) Give an example of an isometry of $\mathbb{R}^3$ which is the composition of 3 reflections, but no fewer than 3 (and justify why). **[6]**

e) Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be an isometry which is the composition 4 reflections, and no fewer. Show that there is a point $x \in S^2$ such that $f(x)$ does not belong to S^2. **[6]**

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3. Let V be a vector subspace of $\mathbb{R}^{n+1}$. The Lorentz-orthogonal complement $V^{\perp_L}$ of V is defined as the set

$$V^{\perp_L} = \{w \in \mathbb{R}^{n+1} \mid \langle w, v\rangle_L = 0 \text{ for all } v \in V\}.$$

a) Show that $V^{\perp_L}$ is a vector subspace of $\mathbb{R}^{n+1}$ of dimension $n + 1 - dim(V)$, and that $(V^{\perp_L})^{\perp_L} = V$. **[6]**

Question 3 continued overleaf

b) Show that if V contains a time-like vector, then $V \cap V^{\perp_L} = \{0\}$. **[6]**

c) Let $V \subset \mathbb{R}^{n+1}$ be a vector subspace of dimension n containing a time-like vector. Show that there is a Lorentz transformation $L: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $L(v) = v$ for all $v \in V$ and $L(w) = -w$ for all $w \in V^{\perp_L}$. **[6]**

d) Let $n = 1$ now. Write the matrix of L from $c)$ with respect to the standard basis of $\mathbb{R}^2$, where $V \subset \mathbb{R}^2$ is the line spanned by a time-like vector $v = (a_1, a_2)$ with $\|v\|_L = i$. **[6]**

e) Sketch the image under L from $d)$ of the (Euclidian) circle of centre 0 and radius one, for $v = (\cosh(\theta), \sinh(\theta))$ and $\theta = 0, 1/4, 1/2$. **[6]**
