THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: APRIL 2019

GEOMETRY

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

COMPULSORY QUESTION

1. a) Give the definition of a metric space (X, d) and an isometry. [5]

b) Let n be a positive integer, and consider the set $X_n = \{0,1\}^n$. For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in X_n define the number $d(x,y)$ as the number of indices i such that $x_i \neq y_i$.

(i) Show that d is a metric on X_n. [3]

(ii) For an integer r with $1 \leq r \leq n$ and $x \in X_n$ determine the number of elements in the "ball" of radius r around x:

$$B(x,r) := \{y \in X_n \mid d(x,y) \leq r\}.$$

[2]

c) Let P_0, P_1, P_2 and P'_0, P'_1, P'_2 in $\mathbb{E}^2$ be two ordered triples of distinct noncollinear points in the Euclidean plane $\mathbb{E}^2$ such that $d(P_i, P_j) = d(P'_i, P'_j)$ for all i, j. Prove that there exists a unique motion $T\colon \mathbb{E}^2 \to \mathbb{E}^2$ taking P_i to P'_i for all i.

[You can assume known that there are exactly two motions T_1, T_2 that take P_0 to P_0' and P_1 to P_1'.] [4]

d) Define the metric d_{S^2} of spherical geometry on the sphere $S^2 \subset \mathbb{R}^3$ of radius one. [2]

e) Give the definition of the hyperbolic plane $(\mathcal{H}^2, d_{\mathcal{H}^2})$ together with its metric $d_{\mathcal{H}^2}$ and also define the notion of a hyperbolic line. [4]

f) Show that in the hyperbolic plane, the circle centred at $(1,0,0)$ of radius r has circumference

$$2\pi \sinh r.$$

[4]

g) Give the definition of projective n-space $\mathbb{P}^n$ and say what a projectivity and projective linear subspace are. [6]

h) Write down the affine transformation $\mathbb{A}^2 \to \mathbb{A}^2$ taking

$$(0,0), (1,0), (0,1) \mapsto (2,1), (0,1), (3,8).$$

Repeat the exercise with target vectors $(2,1), (5,-1), (1,2)$.

Why would this not have worked if the target vectors had been $(2,1), (5,-2), (-1,4)$?

[5]

i) For unknowns $a, b, c \in \mathbb{R}^*$, let

$$A = \begin{pmatrix} a & 0 & c \\ a & b & 2c \\ 0 & b & -c \end{pmatrix}$$

and consider the projectivity $T = T_A \colon \mathbb{P}^2 \to \mathbb{P}^2$ determined by the matrix A.

(i) Show that for any $a, b, c \in \mathbb{R}^*$, T takes the first three points of the standard frame of reference to

$$(1:1:0), \quad (0:1:1), \quad (1:2:-1).$$

[1]

(ii) Find values of $a, b, c \in \mathbb{R}^*$ so that $T(1:1:1) = (1:2:3)$. [2]

(iii) Show that there are no values of $a, b, c \in \mathbb{R}^*$ for which $T(1:1:1) = (1:2:1)$. [2]

[Recall: The standard frame of reference in $\mathbb{P}^2$ is the set of the 4 points

$$(1:0:0), \quad (0:1:0), \quad (0:0:1), \quad (1:1:1).\ ]$$

OPTIONAL QUESTIONS

2. a) Consider standard Euclidean n-space $\mathbb{R}^n$, and write $||x|| = \sqrt{x_1^2 + \cdots + x_n^2}$ for the Euclidean norm of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$ for the Euclidean inner product of $x, y \in \mathbb{R}^n$. Considering $\mathbb{R}^n$ with its natural vector space structure, we call a bijective linear map $f \colon \mathbb{R}^n \to \mathbb{R}^n$ *conformal* if it preserves Euclidean angles. A bijective linear map $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is called a *similarity* if there is a positive constant $\varrho = \varrho(f) > 0$ with $||f(v)|| = \varrho ||v||$ for all $v \in V$.

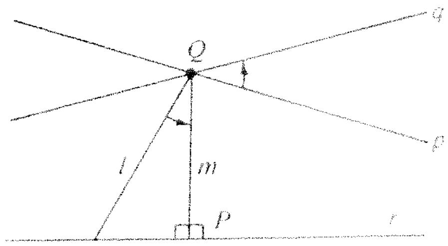
(i) If f is conformal, prove the following [3]

$$(*) \quad \forall v, w \in V : \ (\langle v, w \rangle = 0 \implies \langle f(v), f(w) \rangle = 0).$$

(ii) Show that property $(*)$ implies that f is a similarity. [Hint: First prove that $||v|| = ||w||$ implies $||f(v)|| = ||f(w)||$ by considering $\langle v + w, v - w \rangle$.] [4]

(iii) Show that a similarity f is conformal. [3]

b) If l is a line in the Euclidean plane, we denote by σ_l the reflection in l; if P is a point, we denote by τ_P the half turn (rotation through 180°) around P (also called the reflection in P). Suppose p, q, r are 3 lines, with p, q intersecting at Q as in the picture. Let $m = PQ$ be the perpendicular dropped from Q to line r.



(i) Show that there exists a line l through Q such that $\sigma_q \sigma_p = \sigma_m \sigma_l$. [3]

(ii) Deduce that $\sigma_r \sigma_q \sigma_p = \tau_P \sigma_l$, and prove that this is a glide. Describe its axis and translation vector in terms of l and P. [5]

(iii) Show that if lines a, b, c form a triangle with vertices A, B, C opposite a, b, c and the angle at B is not a right angle, then the axis of the glide reflection $\sigma_c \sigma_b \sigma_a$ can be constructed as the line joining the feet of the perpendiculars to a and c from A and C, respectively. [2]

3. a) In the Euclidean plane, let ABC be a right-angled isosceles triangle with $\angle BCA$ a right angle and AC, BC both of length a. Write D for the midpoint of AC and E for the midpoint of BC. Calculate the lengths of DE and AB as functions of a. [4]

b) In the unit sphere S^2, consider the spherical triangle ABC that is right-angled and isosceles, with $\angle BCA$ a right angle and AC, BC of equal length a. As before, write D and E for the midpoints of AC and BC. Calculate the length c' of the spherical line segment DE, and the length c of AB as functions of a. Prove that $c' > (1/2) \cdot c$. [You may use without proof the formula $\cos(2\alpha) = 2(\cos\alpha)^2 - 1$.] [8]

c) In the hyperbolic plane $\mathcal{H}^2$, consider the hyperbolic triangle ABC that is right-angled and isosceles, with $\angle BCA$ a right angle and AC, BC of equal length a. As before, write D and E for the midpoints of AC and BC. Calculate the length c' of the hyperbolic line segment DE, and the length c of AB as functions of a. Prove that $c' < (1/2) \cdot c$. [You may use without proof the formula $\cosh(2\alpha) = 2(\cosh(\alpha))^2 - 1$.] [8]

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4. a) For $t, \theta, s \in \mathbb{R}$ show that

$$L_t = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

$$N_s = \begin{pmatrix} 1 + \frac{s^2}{2} & -\frac{s^2}{2} & s \\ \frac{s^2}{2} & 1 - \frac{s^2}{2} & s \\ s & -s & 1 \end{pmatrix}$$

induce isometries of the hyperbolic plane $\mathcal{H}^2 \subset \mathbb{R}^3 = \mathbb{R}^{1,2}$. [6]

b) Denote coordinates in $\mathbb{R}^3 = \mathbb{R}^{1,2}$ by x_1, x_2, x_3. Show that each of L_t, R_θ, N_s preserves a family of affine planes in $\mathbb{R}^3$, as follows:

(i) Show that L_t preserves the planes $\{x_3 = c\}$, $c \in \mathbb{R}$. [1]

(ii) Show that R_θ preserves the planes $\{x_1 = c\}$, $c \in \mathbb{R}$. [1]

(iii) Show that N_s preserves the planes $\{x \in \mathbb{R}^3 = \mathbb{R}^{1,2} \mid v \cdot_L x = c\}$, $c \in \mathbb{R}$, where $\cdot_L$ is the Lorentzian inner product and $v = (1,1,0) \in \mathcal{H}^2$. [4]

c) For each of the three cases in b), determine the intersections of the planes with $\{x \in \mathbb{R}^3 \mid x \cdot_L x = -1\}$ provided this is nonempty. Say what type of conic section this is in each case: a parabola, ellipse or hyperbola. [6]

d) Show that there is a unique hyperbolic line that is carried into itself by all transformations L_t, and find the equation of this line. [2]

 Question 5 continued overleaf

5. a) Consider three pairwise disjoint lines l, m and n in four-dimensional projective space $\mathbb{P}^4$ (over any field k, but you can think of $k = \mathbb{R}$ if you wish). Suppose that l, m, n are not contained in any projective linear subspace of dimension 3. Show that there exists a unique projective line that intersects l, m and n. [7]

b) Formulate the statement dual to a) under projective duality. [6]

c) It is a theorem of Sylvester and Gallai that, given any finite set of points $\{P_i\}$ in $\mathbb{P}^2_{\mathbb{R}}$, not all collinear, there exists at least one line l of $\mathbb{P}^2_{\mathbb{R}}$ that contains exactly two of the P_i. [This is only intended to provide the context for the following problem, and the proof is not required.]

Write $\omega = \exp(2\pi i/3)$ for the usual primitive cube root of 1. Consider the following set of 9 points in $\mathbb{P}^2_{\mathbb{C}}$:

$$\begin{aligned} p_1 &= (0:1:-1), & p_2 &= (0:1:-\omega), & p_3 &= (0:1:-\omega^2), \\ p_4 &= (1:0:-1), & p_5 &= (1:0:-\omega^2), & p_6 &= (1:0:-\omega), \\ p_7 &= (1:-1:0), & p_8 &= (1:-\omega:0), & p_9 &= (1:-\omega^2:0) \end{aligned}$$

Show that the line through any two passes through a third. (Hence the theorem of Sylvester and Gallai does not hold for $\mathbb{P}^2_{\mathbb{C}}$).

[Hint: The 9 points lie 3 on each of 12 lines.] [7]