THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: APRIL 2020

GEOMETRY

Time Allowed: **3 hours**

Read all instructions carefully. Please note also the guidance you have received in advance on the departmental 'Warwick Mathematics Exams 2020' webpage.

Calculators, wikipedia and interactive internet resources are not needed and are not permitted in this examination. You are not allowed to confer with other people. You may use module materials and resources from the module webpage.

ANSWER COMPULSORY QUESTION 1 AND TWO FURTHER QUESTIONS out of the four optional questions 2, 3, 4 and 5.

On completion of the assessment, you must upload your answer to the AEP as a single PDF document. You have an additional 45 minutes to make the upload, and instructions are available on the departmental 'Warwick Mathematics Exams 2020' webpage.
You must not upload answers to more than 3 questions, including Question 1. If you do, you will only be given credit for your Question 1 and the first two other answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question. The compulsory question is worth twice the number of marks of each optional question. Note that the marks do not sum to 100.

COMPULSORY QUESTION

1. a) Give the definition of a metric d on a set X, and of an isometry. **[3]**

b) Let X be a set and $f\colon X \to [0, \infty)$ a function such that $f(x_0) = 0$ for a unique point $x_0 \in X$. Show that the function

$$d(x,y) = \begin{cases} f(x) + f(y) & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is a metric on X. **[3]**

c) Let (X, d) and (X', d') be two metric spaces, and $f\colon X \to X'$ a map such that $d'(f(x), f(y)) = d(x, y)$. Is f injective? Is f surjective? Justify your answers with either a proof or a counterexample. **[4]**

 Question 1 continued overleaf

d) Define a great circle of the 2-dimensional sphere S^2 of radius 1, and show that two distinct great circles intersect in two antipodal points. **[5]**

e) Give the definition of an orthogonal $n \times n$-matrix A and show that multiplication by A defines an isometry of $\mathbb{R}^n$, equipped with the Euclidian metric. **[5]**

f) Define the Lorentz inner product on $\mathbb{R}^{n+1}$, the Lorentz norm, and the hyperbolic space $\mathcal{H}^n$, for every $n \geq 1$. **[5]**

g) Let $x = (x_1, x_2)$ be a time-like vector and $y = (y_1, y_2)$ be a non-zero vector Lorentz-orthogonal to x. Show that y is space-like. **[5]**

h) Give one of the equivalent definitions of the n-dimensional projective space $\mathbb{P}^n$. **[5]**

i) Show that a bijective linear map $L: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ induces a well-defined bijective map $\overline{L}: \mathbb{P}^n \to \mathbb{P}^n$, for every $n \geq 0$. **[5]**

OPTIONAL QUESTIONS

2. Let us denote by $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ the Euclidian inner product on $\mathbb{R}^n$, and $\|x\| = \sqrt{\langle x, x\rangle}$ the Euclidian norm.

a) State without proof the Cauchy-Schwarz inequality for $\langle -, -\rangle$. **[5]**

b) Define the Euclidian metric on $\mathbb{R}^n$ and show that it satisfies the triangle inequality. **[5]**

c) Find the subset of $\mathbb{R}^3$ consisting of the points (a, b, c) such that multiplication by the matrix

$$\begin{pmatrix} 1-2a^2 & -2ab & -2ac \\ -2ba & 1-2b^2 & -2bc \\ -2ca & -2cb & 1-2c^2 \end{pmatrix}$$

is an isometry of $\mathbb{R}^3$ with respect to the Euclidian metric. **[10]**

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3. a) Define the n-dimensional sphere S^n of radius 1 and the spherical metric, and justify why it is a well-defined function (you may use the properties of the Euclidian inner product seen in the course). **[4]**

b) Show that a 2×2-matrix A is orthogonal if and only if it is of the form

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \qquad or \qquad A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix},$$

for some $\theta \in \mathbb{R}$. **[6]**

c) Determine all the values $a \in \mathbb{R}$ such that the map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x-(a+1)y+az+1 \\ (a+1)x+ay-z+a+1 \\ ax+y+(a+1)z+a \end{pmatrix}$$

is an isometry with respect to the Euclidian metric. **[6]**

d) Let T be a spherical triangle of S^2 with angles α, β, γ, with $\alpha = \beta = \pi/2$, and of area $\pi/4$. What is the value of γ? **[4]**

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4. a) Define the hyperbolic sine and cosine functions sinh and cosh. Define the hyperbolic metric on $\mathcal{H}^n$ and justify why it is a well-defined function (you may use the properties of cosh and sinh and of the Lorentz inner product seen in the course). **[4]**

Question 4 continued overleaf

b) Show that for every $a \in \mathbb{R}$ multiplication by the matrix

$$A = \begin{pmatrix} 1 + \frac{a^2}{2} & -\frac{a^2}{2} & a \\ \frac{a^2}{2} & 1 - \frac{a^2}{2} & a \\ a & -a & 1 \end{pmatrix}$$

is a Lorentz transformation $\mathbb{R}^3 \to \mathbb{R}^3$. **[6]**

c) Give the definition of a hyperbolic line in $\mathcal{H}^2$, and show that two distinct hyperbolic lines of $\mathcal{H}^2$ intersect in at most one point. **[5]**

d) Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ be positive and time-like. Show that $x+y$ is time-like. Is the sum of positive space-like vectors also space-like? (Justify your answer). **[5]**

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5. a) A hyperbolic circle centered at $P \in \mathcal{H}^2$ of radius $r > 0$ is the set of points of $\mathcal{H}^2$ whose hyperbolic distance from P is r. Show that the hyperbolic circle centered at $(1,0,0) \in \mathcal{H}^2$ of radius $r > 0$ is a Euclidian circle in $\mathbb{R}^3$. What is its Euclidian radius? **[6]**

b) Give the definition of a projective line of $\mathbb{P}^2$, and show that two distinct projective lines of $\mathbb{P}^2$ intersect in exactly one point. **[4]**

c) Let L be the projective line of $\mathbb{P}^2$ defined by the equation $x + y + z = 0$, and L' the projective line defined by the equation $ax + by + z = 0$, for some fixed real numbers $a \neq b$. Calculate the intersection point of L and L'. **[5]**

d) Construct an axiomatic projective plane which has seven points and seven lines. **[5]**
