THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: APRIL 2022

GEOMETRY

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER ALL 4 QUESTIONS.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. a) State without proof the Cauchy-Schwarz inequality for the Euclidean inner product. **[5]**

b) Prove the triangle inequality for the Euclidean metric (you may use without proof the Cauchy-Schwarz inequality). **[8]**

c) Give the definition of an isometry between metric spaces. **[5]**

d) Prove that the set of isometries from a metric space to itself forms a group under the composition of maps. **[8]**

e) Give the definition of the n-dimensional projective space $\mathbb{P}^n$, and of a projective line of $\mathbb{P}^n$. **[6]**

f) Show that two distinct projective lines of $\mathbb{P}^2$ intersect in exactly one point. **[8]**

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2. a) Show that the map $f\colon S^2 \to S^2$ defined by

$$f(x,y,z) = \left(\frac{2x-2y-z}{3}, \frac{-2x-y-2z}{3}, \frac{-x-2y+2z}{3}\right)$$

is a spherical isometry. **[5]**

b) Show that f is a reflection, and determine the equation of the corresponding plane. **[5]**

c) Let $(a_1, a_2) \in \mathbb{R}^2$ be a unit vector. Write the matrix of the linear reflection of $\mathbb{R}^2$ which fixes $(0,0)$ and (a_1, a_2), as a function of a_1, a_2. **[10]**

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3. a) Let (X, d_1) and (Y, d_2) be two metric spaces. Show that the function

$$D\colon (X \times Y) \times (X \times Y) \to [0, \infty)$$

defined by

$$D((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$$

is a metric on $X \times Y$. **[5]**

b) Give the definition of the Lorentz inner product, of the n-dimensional hyperbolic space $\mathcal{H}^n$, and of a hyperbolic line of $\mathcal{H}^n$ (The definition of the hyperbolic metric on $\mathcal{H}^n$ is not required). **[5]**

c) Let L, L' and L'' be the hyperbolic lines of $\mathcal{H}^2$ determined respectively by the planes

$$\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 + x_3 = 0\}$$
$$\Pi' = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 - 3x_3 = 0\}$$
$$\Pi'' = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0\}$$

Determine which pairs of hyperbolic lines among L, L' and L'' intersect, and calculate the intersection points. **[10]**

 Question 4 continued overleaf

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4. Recall that the hyperbolic distance on $\mathcal{H}^2$ is defined by

$$d_{\mathcal{H}^2}(x, y) = \text{arccosh}(-\langle x, y\rangle_L),$$

for all $x, y \in \mathcal{H}^2$.

a) A hyperbolic circle centered at $P \in \mathcal{H}^2$ of radius $r > 0$ is the set of points of $\mathcal{H}^2$ whose hyperbolic distance from P is r. Show that the hyperbolic circle centered at $(1, 0, 0) \in \mathcal{H}^2$ of radius $r > 0$ is a Euclidean circle in $\mathbb{R}^3$, and determine its radius. **[5]**

b) For $\theta \in \mathbb{R}$, let $\rho^h_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the "hyperbolic reflection" , defined by the matrix

$$\rho^h_\theta = \begin{pmatrix} \cosh(\theta) & -\sinh(\theta) \\ \sinh(\theta) & -\cosh(\theta) \end{pmatrix}.$$

Calculate the eigenvalues of ρ^h_θ, and show that the distinct eigenspaces are Lorentz orthogonal. **[5]**

c) Show that every positive Lorentz transformation $L \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the composite of at most two hyperbolic reflections, as defined in 4b). **[10]**
