THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: April 2020

ANALYSIS III

Time Allowed: **2 hours**

Read all instructions carefully. Please note also the guidance you have received in advance on the departmental 'Warwick Mathematics Exams 2021' webpage.

Calculators, wikipedia and interactive internet resources are not needed and are not permitted in this examination. You are not allowed to confer with other people. You may use module materials and resources from the module webpage.

ANSWER ALL THREE QUESTIONS.

On completion of the assessment, you must upload your answer to Moodle as a single PDF document if possible, although multiple files (2 or 3) are permitted. You have an additional 45 minutes to make the upload, and instructions are available on the departmental 'Warwick Mathematics Exams 2021' webpage.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. a) Let $f : [2, \infty) \to \mathbb{R}$ be a nonnegative, decreasing function. Prove that $\sum_{n=2}^{\infty} f(n)$ converges if and only if $\int_2^{\infty} f(x)\mathrm{d}x < \infty$. **[5]**

Provide upper and lower bounds for $\sum_{n=2}^{\infty} f(n)$ in terms of the integral of f.

b) Consider the function $f : [0, 1] \to \mathbb{R}$ given by **[5]**

$$f(x) = \begin{cases} 0 & x \in [0,1]\backslash\mathbb{Q} \\ \dfrac{1}{q} & \text{for } x = \dfrac{p}{q}, \quad \text{with } p, q \in \mathbb{N} \text{ coprime.} \end{cases}$$

Prove that f is Riemann integrable in $[0, 1]$.

c) Calculate the derivative of the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \int_x^{x^2} \mathrm{e}^{-t^2}\mathrm{d}t.$$

Indicate which results you use. (You may refer to theorems used in the notes without the need to restate them.) **[5]**

d) The oscillation of a bounded function f on a set A is defined by

$$\operatorname*{osc}_A f = \sup_A f - \inf_A f.$$

Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions, with g Riemann integrable on $[a, b]$. Show that if there exists $C > 0$ such that **[5]**

$$\operatorname*{osc}_I f \le C \operatorname*{osc}_I g$$

on every interval $I \subset [a, b]$, then f is Riemann integrable.

e) Let $u, v : [a, b] \to \mathbb{R}$ be two continuous functions. Assume that $v \ge 0$ on $[a, b]$. Show that there exists $y \in [a, b]$ such that **[5]**

$$\int_a^b u(x)v(x)\mathrm{d}x = u(y) \int_a^b v(x)\mathrm{d}x.$$

f) Let $f : [0, 1] \to \mathbb{R}$ be a non-integrable function. Is it possible that $|f|$ is Riemann integrable? Either find an example, or prove that it is impossible. **[5]**

g) Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Prove that given $\varepsilon > 0$, there exists a partition of $[a, b]$, say $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$, for some $n \in \mathbb{N}$, such that for any choice of points $c_k \in [x_k, x_{k+1}]$ we have **[5]**

$$\int_a^b f(x)\mathrm{d}x - \sum_{k=0}^{n-1} f(c_k)[x_{k+1} - x_k] < \varepsilon.$$

h) Let $f : [a, b] \to \mathbb{R}$ be a monotone increasing function with a jump discontinuity at some point $c \in (a, b)$. Consider the function

$$F(x) = \int_a^x f(t)\mathrm{d}t.$$

[5]

(i) Explain why F is a well defined function on $[a, b]$.

(ii) Determine whether F is continuous at c. Justify your answer.

(iii) Determine whether F is differentiable at c. Justify your answer.

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2. a) Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Define the sequence

$$f_n(x) = f(x + \frac{1}{n}).$$

Show that f_n converges uniformly to f. Give an example of a continuous function (that is not uniformly continuous) for which the result is false. **[6]**

b) Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of C^1 functions that converges pointwise to a function f. **[6]**

(i) Is f necessarily differentiable? Either prove that f is differentiable or exhibit a counterexample.

(ii) What about if we assume that f_n converges uniformly to f. Prove that f is differentiable or exhibit a counterexample.

c) Determine whether or not the function $f : \mathbb{R} \to \mathbb{R}$ **[6]**

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos(n^2 x) + e^x}{n^4}$$

is a C^1 function, that is, whether it has one continuous derivative or not. Indicate all results that you use.

d) Consider the sequence of functions

$$f_n(x) = \frac{nx^n}{1 + nx^n}.$$

[6]

(i) Determine the pointwise limit of the sequence.

(ii) Does f_n converge uniformly in $\mathbb{R}$? Justify your answer.

(iii) Does the sequence converge uniformly in $[1, 2]$? Justify your answer.

e) Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Show that there exist $a, b > 0$ such that **[6]**

$$|f(x)| \leq a|x| + b \qquad \text{for every } x \in \mathbb{R}.$$

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3. a) Let $f(x + \mathrm{i}y) = \phi(x) + \mathrm{i}\psi(y)$ to be an analytic function in $\mathbb{C}$, where $\phi(x)$ and $\psi(y)$ are real valued, and x, y are real. Determine the most general form for $\phi(x)$ and $\psi(y)$. **[6]**

b) Assume $\zeta \in \mathbb{C}$ and $a \in \mathbb{R}$. Show that $\log \zeta = \mathrm{Log}\, a$ if and only if $\zeta = a$. Use this result to find all complex roots z of $\log \cosh z = \mathrm{Log} \cosh \pi$. (Hint: you may find it convenient to use trigonometric functions instead of hyperbolic ones.) **[6]**

c) Let C be the positively oriented curve formed by a square of size 2 centred at $z_0 = \mathrm{i}$. For $f(z)$ defined as

$$f(z) = 2z + \frac{1}{z - \mathrm{i}}$$

determine

$$\int_C f(z)\ \mathrm{d}z.$$

Indicate all the results you use, either by referencing the theorem number from the notes, or stating the theorem. **[4]**

d) Let $f \colon \mathbb{C} \to \mathbb{C}$ be an analytic function not identically zero that satisfies $|f(z)| \leq K|z|^k$ for a fixed real K and positive integer k. Prove that $f(z)$ has at most k zeros in the complex plane. **[6]**

e) Let $\lambda \in \mathbb{R}$ be positive. By considering a suitable contour integral, show that

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\lambda x}}{x - \mathrm{i}}\ \mathrm{d}x = 2\pi\mathrm{i}\mathrm{e}^{-\lambda}.$$

You may use the inequality $\sin\theta \geq \dfrac{2\theta}{\pi}$ for $0 \leq \theta \leq \dfrac{\pi}{2}$ without proof. **[8]**

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Fulls solutions are provided below. Some additional comments about the exam.

- This was the first April exam taken by this cohort (with exams cancelled last year).
- The averages for Q2 and Q3 were effectively the same (Q3 2% higher). The average for Q1 was 15% higher.

For Q1, parts b) and and d) had appeared in the module and solutions had been provided. Many answers for Q1b) proved that the function was continuous everywhere, when the function is discontinuous at the rationals.
Q1a) was mostly answered correctly but in many cases with rather intricate approaches rather than attempting to compare the integral of f with the area of rectangles with base $[n, n+1]$.
Common mistakes in Q1c) included forgetting to differentiate x^2 as part of the chain rule, integrating e^{-t^2} first (incorrectly), and in some claiming (incorrectly) that $\int f(x)g(x) = \int f(x) \int g(x)$ and using that to find an answer.
Q1e) was meant to be the most challenging, requiring the IVT, but was similar to questions seen in the course.
Q1f) most correct answers found the example in the solutions
Q1g) was meant to be straight forward from the main criteria used in the course for integrability but in many cases was left blank or some very convoluted route was attempted.
Q1h) was mostly answered correctly though in many cases the use of the fundamental theorem of calculus to prove that F is not differentiable were wrong. Loosely speaking, FTC says that if f is continuous, then F is differentiable. It does NOT say that if f is discontinuous, then F is not differentiable.

For Q2, it's hard to find general comments that apply to a large number of students. The main issues were with the reasoning provided for justifying a specific answer or with the counterexamples used, which were not suitable for that part.
An incorrect claim that was repeated a few times for example: "since f it is uniformly continuous it is differentiable".
Q2e) was not attempted by many people but the correct approach was used by most people who did.

Q3a) Most students used Cauchy–Riemann, in most cases getting the correct answer
Q3b) was meant to test multi-valued functions, the Log and trig functions, but proved to be the wrong question, with many students producing very long answers in ways that were not intended.
Q3b) was meant to be a direct application of Cauchy's theorem, but many people tried to parametrise the 4 curves and compute the integrals.
Q3d) was essentially done in a tutorial (proving that if is a polynomial of degree k). It was not attempted by more than half of the class.
Q3e) was answered correctly by most students.

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Course Title: ANALYSIS III

Model Solution No: 1

a) Notice that for $k \in \mathbb{N}$ we have

$$f(k+1) \leq \int_k^{k+1} f(x)\mathrm{d}x \leq f(k).$$

This follows from $f(k+1) \leq f(x) \leq f(k)$ for $x \in [k, k+1]$, since f is decreasing. Therefore

$$\sum_{k=2}^{N} f(k+1) \leq \int_2^{N+1} f(x)\mathrm{d}x \leq \sum_{k=2}^{N} f(k).$$

If we assume that $\sum_{k=2}^{\infty} f(k) < \infty$ then we obtain that $\int_2^{\infty} f(x)\mathrm{d}x < \infty$. Similarly if $\int_2^{\infty} f(x)\mathrm{d}x < \infty$ we have $\sum_{k=2}^{\infty} f(k+1) = \sum_{k=3}^{\infty} f(k) < \infty$, from where the convergence of the full series follows.

We have the bounds

$$\int_2^{\infty} f(x)\mathrm{d}x \leq \sum_{k=2}^{\infty} f(k)$$

and

$$\sum_{k=2}^{\infty} f(k) = f(2) + \sum_{k=2}^{\infty} f(k+1) \leq f(2) + \int_2^{\infty} f(x)\mathrm{d}x.$$

b) In order to prove integrability we use Theorem 1.9 from the Notes, namely

Theorem.
Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon$$

In order to prove that f is integrable, given $\varepsilon > 0$, we notice that there are only finitely many rationals in $[0, 1]$ of the form p/q (with p, q coprime) such that $\frac{1}{q} > \frac{\varepsilon}{2}$. Say N is the total number of rationals with those property.

We construct a partition by describing the corresponding endpoints.. For each rational p/q above we choose the endpoints $p - \frac{\varepsilon}{4N}$ and $p + \frac{\varepsilon}{4N}$. (We take the left endpoint as 0 if $p - \frac{\varepsilon}{4N}$ is negative.)

First notice that for every partition $L(f, P)$ will always be 0, as there is always a point where there function equals 0.

Therefore all we have to do is show that for the partition we have constructed we have $U(f,P) < \varepsilon$. We have to types of intervals in the partition. Those that contain a rational where $f(p/q) > \varepsilon/2$ and those who do not. For the first kind, we have chosen the endpoints so that the interval containing the rational is of length is $\frac{\varepsilon}{2N}$, and since there are N such intervals (and $f(x) \leq 1$) the total contribution to $U(f,P)$ is less than $\varepsilon/2$. For the rest of the intervals, we use that $f(x) \leq \varepsilon/2$. Since the total length of those intervals is less that 1, the contribution is again less than or equal to $\varepsilon/2$.

Therefore $U(f,P) < \varepsilon$, as required.

c) We use the Fundamental Theorem of Calculus (Theorem 1.30).

Given any x we can consider an interval $[a,b]$ large enough to include the domain of integration for f. Since the integrand is a continuous function we obtain

$$f'(x) = 2xe^{-x^4} - 1e^{-x^2}.$$

Students need to justify that the derivative corresponding to the lower limit carries a negative sign as when applying the result above we consider

$$\int_x^{x^2} e^{-t^2}\mathrm{d}t = \int_\alpha^{x^2} e^{-t^2}\mathrm{d}t + \int_x^{\alpha} e^{-t^2}\mathrm{d}t = \int_\alpha^{x^2} e^{-t^2}\mathrm{d}t - \int_\alpha^{x} e^{-t^2}\mathrm{d}t$$

for some α in the domain of integration. Applying Theorem 1.30 yields the result above.

d) Let $P = \{I_1, \cdots, I_n\}$ be a partition of $[a,b]$ then

$$U(f,P) - L(f,P) = \sum [\sup_{I_k} - \inf_{I_k} f]|I_k| = \sum \operatorname*{osc}_{I_k} f|I_k| \leq C \sum \operatorname*{osc}_{I_k} g|I_k| = C[U(g,P) - U(f,P)].$$

Therefore by the Cauchy criteria we are done, i.e. f is Riemann integrable iff for every $\varepsilon > 0$ there exists P such that $U(f,P) - L(f,P) < \varepsilon$. Since g is integrable, the partition associated to ε/C suffices.

e) Since u is continuous on a compact set, it attains its maximum M and its minimum m. Therefore since $v \geq 0$ we have

$$mv(x) \leq u(x)v(x) \leq Mv(x),$$

which implies

$$\int_a^b mv(x)\mathrm{d}x \leq \int_a^b u(x)v(x)\mathrm{d}x \leq \int_a^b Mv(x)\mathrm{d}x.$$

If v is identically zero, then the equality required is trivial, as both sides equal 0. Otherwise since $v \geq 0$ we have that $\int_a^b v(x)\mathrm{d}x > 0$. Rearranging the inequality above we find

$$m \leq \frac{\int_a^b u(x)v(x)\mathrm{d}x}{\int_a^b v(x)\mathrm{d}x} \leq M.$$

Since u is continuous it attains all values between the minimum m and the maximum M, and therefore there exists y such that

$$u(y) = \frac{\int_a^b u(x)v(x)\mathrm{d}x}{\int_a^b v(x)\mathrm{d}x},$$

as required.

f) It is actually possible for $|f|$ to be Riemann integrable. One such example is $f : [0,1] \to \mathbb{R}$ given by

$$f(x) := \chi_{\mathbb{Q}\cap[0,1]}(x) - \chi_{[0,1]\setminus\mathbb{Q}}(x),$$

i.e. the function that equals 1 on the rationals and -1 on the irrationals. Of course $|f| = 1$ on $[0,1]$ which is trivially integrable (continuous, or monotone, or directly from the definition)

g) Since we know that f is Riemann integrable we know that for every $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that

$$\sum_{k=0}^{n-1} m_k \chi_{I_k}(x) \leq f(x) \leq \sum_{k=0}^{n-1} M_k \chi_{I_k}(x),$$

where m_k and M_k are given by the infimum and supremum of f on the interval $I_k = [x_k, x_{k+1}]$, and such that

$$U(f,P) - L(f,P) < \varepsilon.$$

This implies

$$\sum_{k=0}^{n-1} m_k[x_{k+1} - x_k] \leq \int_a^b f(x)\mathrm{d}x \leq \sum_{k=0}^{n-1} M_k[x_{k+1} - x_k]. \tag{1}$$

That partition defines x_k for $k = 0, \ldots, n$. For any $c_k \in [x_k, x_{k+1}]$ we have $m_k \leq c_k \leq M_k$ and therefore

$$\sum_{k=0}^{n} m_k \chi_{I_k}(x) \leq \sum_{k=0}^{n} f(c_k)\chi_{I_k}(x) \leq \sum_{k=0}^{n} M_k \chi_{I_k}(x)$$

which yields

$$\sum_{k=0}^{n-1} m_k[x_{k+1} - x_k] \leq \sum_{k=0}^{n-1} f(c_k)[x_{k+1} - x_k] \leq \sum_{k=0}^{n-1} M_k[x_{k+1} - x_k].$$

Equation (1) together with the above inequality prove the result as both expressions belong to the same interval of length ε.

h) (i) We have seen in the course (Theorem 1.1.5) that all monotonic functions are integrable

(ii) The fundamental theorem of Calculus (Theorem 1.30) ensures that F is continuous

(iii) Let
$$\alpha := \lim_{x\to c-} f(x) \qquad\qquad \beta := \lim_{x\to c+} f(x).$$
We can safely ignore the value of $f(c)$ (we know $\alpha \le f(c) \le \beta$), as changing the value of a function at one point does not change the value of the integral. We consider the increment quotients of F. For $h > 0$
$$\frac{F(c+h)-F(c)}{h} = \frac{1}{h}\int_c^{c+h} f(t)\mathrm{d}t \ge \frac{1}{h}\int_c^{c+h} \beta\mathrm{d}t = \beta$$

$$\frac{F(c-h)-F(c)}{-h} = \frac{1}{-h}\int_c^{c-h} f(t)\mathrm{d}t = \frac{1}{h}\int_{c-h}^{c} f(t)\mathrm{d}t \le \frac{1}{h}\int_{c-h}^{c} \alpha\mathrm{d}t \le \alpha$$
We have used used the monotonicity above on each interval to obtain the corresponding inequality in terms of α and β. Since there is a jump, $\alpha < \beta$,
$$\frac{F(c-h)-F(c)}{-h} \le \alpha < \beta \le \frac{F(c+h)-F(c)}{h}$$
shows that F is not differentiable.

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Model Solution No: 2

a) Recall that f is uniformly continuous if and only if for every $\varepsilon > 0$ there exists δ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

In order to show uniform convergence we need to prove that given $\varepsilon > 0$ there exists N such that

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| < \varepsilon.$$

Notice that

$$|f_n(x) - f(x)| = |f(x + \frac{1}{n}) - f(x)|$$

and therefore choosing N large enough so that $1/n < \delta$, with δ arising from the uniform continuity condition we obtain the result.

There are many counterexample, but $f(x) = x^2$ is one where the calculations become elementary. We have

$$|f_n(x) - f(x)| = |(x + 1/n)^2 - x^2| = |\frac{2x}{n} + \frac{1}{n^2}|.$$

Notice that we cannot make $2|x|/n$ arbitrarily small uniformly in x, as for example, for $x = n$ the difference equals $2 + 1/n^2$.

b) The answer is in fact negative in both cases. We can take as an example

$$f_n(x) = (x^2 + \frac{1}{n})^{1/2},$$

defined on $[-1, 1]$ for example. The sequence converges uniform to $f(x) = |x|$, which is not differentiable. To see this

$$\left|(x^2 + \frac{1}{n})^{1/2} - |x|\right| \leq \left|((|x| + \frac{1}{\sqrt{n}})^2)^{1/2} - |x|\right| = \frac{1}{\sqrt{n}}$$

c) We define

$$f_N(x) = \sum_{n=1}^{N} \frac{\cos(n^2 x) + e^x}{n^4}.$$

Notice that given x it suffices to consider the series restricted to a finite interval $[a, b]$, so that we can apply the following result from the notes

Theorem 2.27 *Let* (f_k)*, with* $f_k : [a, b] \to \mathbb{R}$*, be a sequence of* C^1 *functions such that* $S_n = \sum_{k=1}^n f_k$ *converges pointwise. Assume that* $\sum_{k=1}^n f_k'$ *converges uniformly. Then*

$$\left(\sum_{k=1}^{\infty} f_k(x)\right)' = \sum_{k=1}^{\infty} f_k'(x),$$

that is, the series is differentiable and can be differentiated term-by-term.

Notice that in our case the functions f_k, are clearly C^1. We just need to consider whether

$$\sum_{n=1}^{N} \frac{-n^2 \sin(n^2 x) + e^x}{n^4}$$

converges uniformly. Here it is crucial that we have restricted $x \in [a, b]$. The result follows from the Weierstrass M-test.

Theorem 2.28 *Let* (f_k) *be a sequence of functions* $f_k : \Omega \to \mathbb{R}$*, and assume that for every* k *there exists* $M_k > 0$ *such that* $|f_k(x)| \leq M_k$ *for every* $x \in \Omega$ *and* $\sum_{k=1}^{\infty} M_k < \infty$*. Then*

$$\sum_{k=1}^{\infty} f_k$$

converges uniformly on Ω*.*

In our case we can take $M_n = \dfrac{1 + e^b}{n^2}$, which is a convergent. Therefore the original function is differentiable and its derivative can be obtained by differentiating term by term. Since each term in the series defining the derivative if continuous, and we have proven uniform convergence, the derivative is continuous.

d) Notice the we can rewrite the sequence as

$$1 - \frac{1}{1 + nx^n}.$$

(i) For $|x| < 1$ the sequence nx^n converges to 0, while for $|x| \geq 1$, $|nx^n|$ goes to infinity. Therefore f_n converges to

$$f(x) = \begin{cases} 0 & |x| < 1, \\ 1 & |x| \geq 1. \end{cases}$$

(ii) No, the sequence does not converge uniformly. Since f_n are all continuous on $[1/2, 2]$if the convergence was uniform, the limit should be continuous, but it is not. Notice that $f_n(x)$ are not continuous in $\mathbb{R}$ for n odd. The function blows up at $(-1/n)^{1/n}$, which is another possible way of proving that the convergence is not uniform.

(iii) The answer in this case is yes. We have

$$\left| \frac{nx^n}{1 + nx^n} - 1 \right| = \left| \frac{-1}{1 + nx^n} \right| \leq \frac{1}{n},$$

which converges uniformly to 0.

e) Since f is uniformly continuous we know that given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$.

Lets denote by δ (from now on) the δ corresponding to $\varepsilon = 1$ above, and take $k \in \mathbb{N}$ large enough so that $1/k < \delta$. we have $|x - y| < 1/k$ implies $|f(x) - f(y)| < 1$.

Therefore in an interval of length $1/k$ the function increases by less than 1. That means that in an interval of length 1 it will increase at most by k, since the estimates are uniform and can be repeated in each subinterval of length $1/k$.

Given any x, we know that it belongs to an interval of the form $j/k < |x| \leq (j+1)/k$ for some $j \in \mathbb{N}$. For $x > 0$ we can write (for $x < 0$ we need to consider $-j/k, \ldots, -1/k$ instead of $j/k, \ldots, 1/k$)

$$|f(x)| = |f(x) - f(\frac{j}{k}) + f(\frac{j}{k}) - f(\frac{j-1}{k}) + f(\frac{j-1}{k}) \cdots - f(\frac{1}{k}) + f(\frac{1}{k}) - f(0) + f(0)|$$

$$\leq j + 1 + |f(0)| \leq k|x| + |f(0)| + 1.$$

since $j \leq k|x|$.

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Model Solution No: 3

a) The Cauchy-Riemann conditions state:

$$\frac{\partial\phi(x)}{\partial x}=\frac{\partial\psi(y)}{\partial y}\qquad\qquad \frac{\partial\phi(x)}{\partial y}=-\frac{\partial\psi(y)}{\partial x}.$$

This leads to $\phi'(x)=\psi'(y)$ (and the trivial 0=0). Since the left-hand side is a function of x alone, and the right-hand side is a function of y alone, both must be the same constant, i.e. $\phi'(x)=\psi'(y)=\lambda$.

Integrating yields

$$\phi(x)=\lambda x+c_1,\qquad \psi(y)=\lambda y+c_2,$$

where c_1 and c_2 are constants of integration. Since ϕ and ψ are real, the constants λ, c_1 and c_2 must be real. This is the most general form of $\phi(x)$ and $\psi(y)$. The corresponding $f(z)=\lambda z+(c_1+\mathrm{i}c_2)$.

b) $\log z$ is the multivalued function given by

$$\log(z)=\mathrm{Log}|z|+\mathrm{i}\arg(z)$$

with Log here meaning the real logarithm. The principal branch of the logarithm (also denoted by Log) is given by

$$\mathrm{Log}(z)=\mathrm{Log}|z|+\mathrm{i}\mathrm{Arg}(z),$$

where $\mathrm{Arg}(z)\in(-\pi,\pi]$. If $a\in\mathbb{R}$, then $\mathrm{Arg}(a)$ is either 0 if $a>0$ or π if $a<0$.

$\mathrm{Log}\zeta=\mathrm{Log}a$ means that $\mathrm{Arg}(\zeta)$ needs to be 0 or π, making ζ a real number. The fact that $\mathrm{Log}|\zeta|=\mathrm{Log}|a|$ with $\mathrm{Arg}(\zeta)=\mathrm{Arg}(a)$ forces $\zeta=a$.

To find the roots of $\log\cosh z=\mathrm{Log}\cosh a$, we first use the previous result and simplify the equation to $\cosh z=\cosh\pi$.

Notice that

$$\cosh(x+\mathrm{i}y)=\cosh(x)\cos y+\mathrm{i}\sinh(x)\sin y$$

which has to equal $\cosh\pi$. This forces that $\sinh(x)=0$ or that $\sin y=0$. In the first case $x=0$, and looking at the real part that would require $\cos y=\cosh\pi$, which has no solutions for $y\in\mathbb{R}$.

In the second case $y=\pi n$, for $n\in\mathbb{Z}$. Looking at the real part we need

$$\cosh(x)\cos(\pi n)=\cosh(\pi)$$

that is

$$(-1)^n \cosh(x) = \cosh(\pi),$$

forcing n to be even and $x = \pm\pi$

Thus, all possible roots are $\pm\pi + 2k\pi\mathrm{i}$, with $k \in \mathbb{Z}$.

c) We break up the integral into two. The first $\int_C 2z \, \mathrm{d}z$ vanishes by Cauchy's theorem 3.29 because $2z$ is analytic everywhere in C. The second part,

$$\int_C \frac{\mathrm{d}z}{z - i},$$

can be evaluated using Theorem 3.33, which states

$$g(a) = \frac{1}{2\pi\mathrm{i}} \int_C \frac{g(w)}{w - a} \, \mathrm{d}w.$$

In our case, $g(w) = 1$ and $a = \mathrm{i}$. Thus,

$$1 = \frac{1}{2\pi\mathrm{i}} \int_C \frac{\mathrm{d}w}{w - i}$$

and, therefore,

$$\int_C \frac{\mathrm{d}z}{z - i} = 2\pi\mathrm{i}.$$

Adding the two parts together yields the answer $2\pi\mathrm{i}$.

d) Since $f(z)$ is analytic, then by theorem 3.36, it has the following Taylor series expansion around $z = 0$,

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Moreover, the coefficients c_n can be determined using and positively oriented simple closed curve C that goes around the origin as

$$c_n = \frac{1}{2\pi\mathrm{i}} \int_C \frac{f(w)}{w^{n+1}} \, \mathrm{d}w.$$

We shall choose C to be a circle of radius R centred at the origin, so its parameterization may be written as

$$\gamma(t) = Re^{\mathrm{i}t}$$

with $0 \leq t < 2\pi$.

Now note that

$$2\pi|c_n| = \left| \int_C \frac{f(w)}{w^{n+1}} \, \mathrm{d}w \right| \leq \int_C \frac{K|w|^k}{|w^{n+1}|} \, |\mathrm{d}w| = \int_0^{2\pi} K \frac{R^k}{R^{n+1}} R \, \mathrm{d}t = 2\pi K R^{k-n}.$$

Since R is arbitrary in this inequality, we take the limit as $R \to \infty$ to conclude that $|c_n| = 0$ for $n > k$. This implies that the power-series terminates at most c_k and, therefore, $f(z)$ is a polynomial at most of degree k. Using the fundamental theorem of algebra, the polynomial has at most k roots.

e) Consider the contour C formed by C_1 given as $z = t$ where $t \in [-R, R]$, and C_2 given as $z = Re^{it}$ where $t \in [0, \pi]$.

Using theorem 3.33,

$$\int_C f(z)\mathrm{d}z = 2\pi i e^{-\lambda} \text{ where } f(z) = \frac{e^{i\lambda z}}{z - i}$$

The integral we want is $I = \lim_{R\to\infty} \int_{C_1} f(z)\ \mathrm{d}z$, hence the objective is to show that $\lim_{R\to\infty} I_2 = \int_{C_2} f(z)\ \mathrm{d}z = 0$.

We can bound I_2 as

$$\begin{aligned}
|I_2| &\leq \int_{C_2} \left| \frac{e^{\mathrm{i}\lambda z}}{z - i}\ \mathrm{d}z \right| \\
&\leq \int_0^{\pi} \frac{e^{-\lambda R \sin t}}{|R| - 1} R\mathrm{d}t \\
&\leq 2 \int_0^{\pi/2} \frac{e^{-2\lambda R t/\pi}}{|R| - 1} R\mathrm{d}t \\
&= \left(\frac{2R}{|R| - 1}\right) \left(\frac{\pi}{2\lambda R}\right) \left[e^{-2\lambda R t/\pi}\right]_0^{\pi/2}
\end{aligned}$$

Parameterizing and taking absolute values with inequalities.

Correctly using $\sin t > 2t/\pi$, including closing the loop on the top half-plane.

In the limit $R \to \infty$, $I_2 \to 0$, thus concluding the proof. Evaluating integrals and limits.