THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: April 2019

ANALYSIS III

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

COMPULSORY QUESTION

1. a) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Prove that f is Riemann integrable if and only if for every $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that **[5]**

$$|U(f,P) - L(f,P)| < \varepsilon.$$

Here $U(f,P)$ and $L(f,P)$ correspond to the Upper and Lower Riemann sums associated to the partition P.

b) Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Prove that there exists $c \in [a,b]$ such that **[6]**

$$\int_a^b f(t)\mathrm{d}t = (b-a)f(c).$$

[Note that f is not assumed to be differentiable, so the standard Mean Value Theorem cannot be applied to f.]

c) Let $G(x) := \sum_{n=1}^{\infty} g_n(x)$, for $g_n : [0,1] \to \mathbb{R}$. State sufficient conditions that make differentiability term by term possible, that is, ensuring that **[5]**

$$G'(x) = \sum_{n=1}^{\infty} g_n'(x).$$

d) Let $f(z) = 2|z|^2 - \bar{z}^2$ for $z \in \mathbb{C}$. At which points is f complex differentiable? At which points is f analytic? Justify your answers. **[5]**

e) Evaluate the following integrals, assuming all contours are oriented counter-clockwise **[4]**

$$\int_{|z-1|=1} \bar{z}\mathrm{d}z \qquad\qquad \int_{|z|=2} (z+1)^2 |\mathrm{d}z|.$$

f) For $z \neq 0$ define the multi-valued argument function $\arg(z)$. Explain the number of values of z^α for $z \neq 0$ depending on α. Consider only the cases: $\alpha \in \mathbb{N}$ and $\alpha = p/q$ for p, q coprime. **[6]**

g) State Cauchy's theorem for analytic functions. **[4]**

h) Let $f : \Omega \to \mathbb{C}$ be an analytic function on an open set Ω. State the Cauchy-Riemann equations satisfied by the real and imaginary part and prove that both are harmonic functions. **[5]**

OPTIONAL QUESTIONS

2. a) Prove Liouville's Theorem, that is, show that if $f : \mathbb{C} \to \mathbb{C}$ is analytic and bounded in $\mathbb{C}$, then it is constant. **[6]**

b) In this part you are asked to use contour integration to calculate

$$\int_{-\infty}^{\infty} \frac{\cos(5x)}{1+x^2} \mathrm{d}x.$$

Consider the integral

$$\int_{\gamma_R} \frac{\mathrm{e}^{\mathrm{i}5z}}{1+z^2} \mathrm{d}z,$$

where γ_R is the contour formed by the straight line joining $(-R, 0)$ and $(R, 0)$, together with the half circumference $|z| = R$, Im $(z) \geq 0$, oriented counter-clockwise.

(i) Show that the integral over the half circumference in γ_R goes to zero as R goes to infinity. **[4]**

(ii) Use the deformation of contours Theorem to replace γ_R by a suitable circumference. **[4]**

(iii) Use Cauchy's Theorem to calculate the integral over the new contour (circumference). (You can also work it directly if you prefer.) **[4]**

(iv) Use (i), (ii) and (iii) to evaluate **[2]**

$$\int_{-\infty}^{\infty} \frac{\cos(5x)}{1+x^2} \mathrm{d}x.$$

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3. a) Let $(f_n)_{n=1}^{\infty}$, with $f_n : [a,b] \to \mathbb{R}$ be a sequence of Riemann integrable functions such that **[5]**

$$\int_a^b |f_n(x) - f(x)| \mathrm{d}x \to 0, \qquad \text{as } n \to \infty,$$

for some Riemann integrable function $f : [a,b] \to \mathbb{R}$. Does f_n necessarily converge to f pointwise? Justify your answer.

b) Let $(f_n)_{n=1}^{\infty}$, with $f_n : [a,b] \to \mathbb{R}$ be a sequence of Riemann integrable functions that converges pointwise to $f : [a,b] \to \mathbb{R}$. Do we necessarily have **[5]**

$$\int_a^b f_n(x)\mathrm{d}x \to \int_a^b f(x)\mathrm{d}x?$$

Justify your answer. If the result is false state sufficient conditions that will make this convergence hold, and prove the result.

c) Let $(f_n)_{n=1}^{\infty}$, with $f_n : [a,b] \to \mathbb{R}$ be a sequence of continuous functions such that $f_n \to f$ pointwise to $f : [a,b] \to \mathbb{R}$. Is f necessarily a continuous function. Justify your answer. **[5]**

d) Let $(f_n)_{n=1}^{\infty}$, with $f_n : [a,b] \to \mathbb{R}$ be a sequence of C^1 functions that converges uniformly to a function f. Is f necessarily differentiable? Suppose that f is actually infinitely differentiable. Does f_n' necessarily converge to f' pointwise? **[5]** Justify your answers.

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4. a) State and prove the Weierstrass M-test for series of functions. **[6]**

b) Calculate the Taylor series expansion of **[6]**

$$g(x) = \frac{1}{(x-2)^2 + 4}$$

around $x = 2$. Calculate the radius of convergence of the series using one of the tests covered in the lectures.

c) Let **[8]**

$$f(x) = \frac{1}{2}\text{atan}\left(\frac{x-2}{2}\right).$$

Here atan stands for the arc tangent. Use the fact that $f'(x) = g(x)$, with g given in part b) to calculate a Taylor series expansion for f. State any Theorems you use that allow for term-by-term integration, and verify that all hypothesis are satisfied. [Do not forget to deal with the constant of integration.]

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5. a) Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Prove that for every $\varepsilon > 0$ there exists a continuous function $g : [a, b] \to \mathbb{R}$ such that **[6]**

$$\int_a^b |f(x) - g(x)|\mathrm{d}x < \varepsilon.$$

b) Does the result in part a) extends to indefinite integrals involving unbounded functions? More specifically, assume $f : [0, 1] \to \mathbb{R}$ is unbounded at the origin, but Riemann integrable (as an improper integral). Is it true that for every $\varepsilon > 0$ we can find a continuous function g such that **[6]**

$$\int_0^1 |f(x) - g(x)|\mathrm{d}x < \varepsilon,$$

with the integral understood as an improper integral.

c) State and prove the Fundamental Theorem of Calculus (for differentiating integrals). **[5]**

d) Let $F : (1, \infty) \to \mathbb{R}$ be given by **[3]**

$$F(x) = \int_{1/\sqrt{x}}^{\sqrt{x}} \sin(t^2)\,\mathrm{d}t.$$

Show that F is differentiable, and compute F'.

MATHEMATICS DEPARTMENT
SECOND YEAR UNDERGRADUATE EXAMS – April 2019

Course Title: ANALYSIS III

Model Solution No: 1

a) By the properties of sup and inf in the respective definitions $U(f)$ and $L(f)$ we know that there exists partitions P_1 and P_2 such that

$$U(f,P_1) < U(f) + \frac{\varepsilon}{2} \qquad\qquad L(f,P_2) > L(f) - \frac{\varepsilon}{2}.$$

Therefore if we consider the partition P which is a refinement of both P_1 and P_2, (for example by considering the union of all the endpoints of both partitions) we have

$$U(f,P) \le U(f,P_1) < U(f) + \frac{\varepsilon}{2} \qquad\qquad L(f,P) \ge L(f,P_2) > L(f) - \frac{\varepsilon}{2}.$$

Notice that if f is integrable we have $U(f) = L(f)$, which implies, using the inequality above that

$$U(f,P) - L(f,P) < \varepsilon$$

For the other implication notice that since

$$U(f) - L(f) \le U(f,P) - L(f,P)$$

for every partition P the left-hand side, which is always greater or equal to zero, must be zero, as we can find partitions P that make it arbitrarily small.

b) Define the function $F(x) = \int_a^x f(t)\mathrm{d}t$. Notice that since f is continuous the Fundamental Theorem of Calculus that F is C^1, and the MVT implies that there exists $c \in [a,b]$ such that

$$F(b) - F(a) = F'(c)(b-a).$$

Since $F(a) = 0$ and $F'(c) = f(c)$ we obtained the desired result.

c) Let (f_k), with $f_k : [a,b] \to \mathbb{R}$, be a sequence of C^1 functions such that $S_n = \sum_{k=1}^n f_k$ converges pointwise. Assume that $\sum_{k=1}^n f_k'$ converges uniformly. Then

$$\left(\sum_{k=1}^{\infty} f_k(x)\right)' = \sum_{k=1}^{\infty} f_k'(x),$$

that is, the series is differentiable and can be differentiated term-by-term.

d) Notice that

$$f(z) = 2(x^2+y^2) - (x - \mathrm{i}y)^2 = 2(x^2+y^2) - x^2 + y^2 + 2xy\mathrm{i} = x^2 + 3y^2 + 2xy\mathrm{i}.$$

Therefore $f = u + \mathrm{i}v$ for $u = x^2 + 3y^2$ and $v = 2xy$. We have

$$u_x = 2x \qquad u_y = 6y \qquad v_x = 2y \qquad v_y = 2x,$$

and there for the C-R equations are satisfied at all points of the form $(x, 0)$. This is the case since $u_x = v_y$ but $u_y = -v_x$ is only satisfied for $y = 0$. Since the expressions for u and v are infinitely differentiable we have the f if complex differentiable at all points of the form $(x, 0)$. However it is not analytic at any point, since by definition to be analytic at z the functions must be complex differentiable at an open neighborhood of z.

e) We use repeatedly that $\int_0^{2\pi} \mathrm{e}^{\mathrm{i}k\theta}\mathrm{d}\theta = 0$ for $k \neq 0$.

$$\int_{|z-1|=1} \bar{z}\mathrm{d}z = \int_0^{2\pi} (1 + \mathrm{e}^{-\mathrm{i}\theta})\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta = \int_0^{2\pi} (\mathrm{i}\mathrm{e}^{\mathrm{i}\theta} + \mathrm{i})\mathrm{d}\theta = 2\pi\mathrm{i}.$$

$$\int_{|z|=2} (z+1)^2|\mathrm{d}z| = \int_0^{2\pi} (2\mathrm{e}^{\mathrm{i}\theta} + 1)^2 2\mathrm{d}\theta = \int_0^{2\pi} (4\mathrm{e}^{2\mathrm{i}\theta} + 4\mathrm{e}^{\mathrm{i}\theta} + 1)2\mathrm{d}\theta = 4\pi$$

f) For $z \neq 0$,

$$\arg(z) = \{\theta \in \mathbb{R} : z = |z|\mathrm{e}^{\mathrm{i}\theta}\}.$$

The multi-valuedness of arg means that the same is true for z^α. If we write

$$z^\alpha = \mathrm{e}^{\alpha\mathrm{Log}|z| + \alpha\mathrm{i}\arg(z)} = \mathrm{e}^{\alpha\mathrm{Log}|z| + \alpha\mathrm{i}\mathrm{Arg}(z) + 2\pi\alpha k\mathrm{i}} = \mathrm{e}^{\alpha\mathrm{Log}(z)}\mathrm{e}^{2\pi\alpha k\mathrm{i}}$$

for $k \in \mathbb{Z}$ the multi-valuedness becomes more evident.

If α is an integer then $\mathrm{e}^{2\pi\alpha k\mathrm{i}} = 1$, which means that in fact there is only one value for z^α. If α is rational, say $\alpha = p/q$, with p, q coprime, then z^α will have finitely many powers. For $\alpha = p/q$

$$\mathrm{e}^{2\pi\alpha k\mathrm{i}} = \mathrm{e}^{2\pi\alpha(k+q)\mathrm{i}}$$

and therefore z^α will take q different values

$$\mathrm{e}^{\alpha\mathrm{Log}(z)}\mathrm{e}^{2\pi\alpha k\mathrm{i}}, \qquad k = 0, 1, \ldots, q-1.$$

g) Let $f : \Omega \to \mathbb{C}$ be an analytic function, with Ω an open, simply connected domain. Let γ be a C^1 curve closed curve in Ω. Then

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

h) Assume that $f(z)$ is given by $u(x, y) + \mathrm{i}v(x, y)$ for $z = x + \mathrm{i}y$. Then the Cauchy–Riemann equations are given by

$$u_x = v_y \qquad\qquad u_y = -v_x.$$

If we calculate the Laplacian of u for example,

$$\triangle u = u_{xx} + u_{yy} = (v_y)_x + (-v_x)_y = 0.$$

As for v, similarly

$$\triangle v = v_{xx} + v_{yy} = (-u_y)_x + (u_x)_y = 0.$$

MATHEMATICS DEPARTMENT
SECOND YEAR UNDERGRADUATE EXAMS – April 2019

Course Title: ANALYSIS III

Model Solution No: 2

a) Assume that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let $a \neq b$ be two points in $\mathbb{C}$. Choose R large enough so that $2\max\{|a|,|b|\} < R$. That means that if we consider $w \in \partial B_R(0)$, that is $|w| = R$ then

$$|w-a| > \frac{R}{2} \qquad\qquad |w-b| > \frac{R}{2}.$$

Since f is analytic in $\mathbb{C}$ we can use Cauchy's formula to compute $f(a)$ and $f(b)$ using $\partial B_R(0)$ as the curve γ (of course positively oriented!). We have

$$f(a) - f(b) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_R(0)} \frac{f(w)}{w-a} \mathrm{d}w - \frac{1}{2\pi \mathrm{i}} \int_{\partial B_R(0)} \frac{f(w)}{w-b} \mathrm{d}w$$

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial B_R(0)} f(w) \left(\frac{1}{w-a} - \frac{1}{w-b} \right) \mathrm{d}w = \frac{a-b}{2\pi \mathrm{i}} \int_{\partial B_R(0)} \frac{f(w)}{(w-a)(w-b)} \mathrm{d}w.$$

Therefore

$$|f(a) - f(b)| \leq \frac{|a-b|}{2\pi} \frac{M}{R^2/4} \int_{\partial B_r(0)} 1 \mathrm{d}w = \frac{|a-b| 4M}{R},$$

as $\int_{\partial B_R(0} 1 \mathrm{d}w$ is just the length of the curve, which equals $2\pi R$. Notice that since R is arbitrary (provided that it is big enough, as indicated above) we can send R to infinity, showing that $|f(a) - f(b)| = 0$ for any a and b in $\mathbb{C}$, therefore proving that the function is constant.

b) We can rewrite this integral as

$$\mathbf{Re} \int_{\infty}^{\infty} \frac{\mathrm{e}^{5\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})} \mathrm{d}z,$$

and we can actually drop the **Re** part as the imaginary part will be an odd integrand and it will vanish.

(i) We consider first the integral over the arc (half circle of radius R). We have, for $R >> 1$

$$\left| \int_{\text{arc}} \frac{\mathrm{e}^{5\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})} \mathrm{d}z \right| \leq \int_{\text{arc}} \frac{\left|\mathrm{e}^{5\mathrm{i}z}\right|}{|z^2+1|} \mathrm{d}z \leq \int_{\text{arc}} \frac{\mathrm{e}^{-5\,\mathbf{Im}\,z}}{R^2-1} |\mathrm{d}z| \leq \frac{\pi R}{R^2-1} \xrightarrow[R\to\infty]{} 0,$$

where we have used that along the arc, $\mathbf{Im}\, z \geq 0$ and so $\mathrm{e}^{-5\,\mathbf{Im}\,z} \leq 1$.

(ii) Given the zeroes of the denominator within γ_R we denote by γ the circle centred at i of radius $1/2$, oriented counter-clockwise we have

$$\int_{\gamma_R} \frac{\mathrm{e}^{5\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z = \int_{\gamma} \frac{\mathrm{e}^{5\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z.$$

Also

$$\int_{\gamma} \frac{\mathrm{e}^{5\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z = \int_{-R}^{R} \frac{\mathrm{e}^{5\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z + \int_{\text{arc}} \frac{\mathrm{e}^{5\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}|\mathrm{d}z|.$$

(iii) Now for γ (oriented counter-clockwise)

$$\int_{\gamma} \frac{\mathrm{e}^{5\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z = \int_{\gamma} \frac{g(z)}{z-\mathrm{i}}\mathrm{d}z = 2\pi\mathrm{i}g(\mathrm{i}),$$

where

$$g(z) = \frac{\mathrm{e}^{5\mathrm{i}z}}{z+\mathrm{i}}$$

and we have used Cauchy's formula since g is analytic inside γ_2. We have

$$g(\mathrm{i}) = \frac{\mathrm{e}^{-5}}{2\mathrm{i}}.$$

(iv) Therefore

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{5\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z = \int_{\gamma_R} \frac{\mathrm{e}^{5\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})} = \int_{\gamma} \frac{\mathrm{e}^{5\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})} = 2\pi\mathrm{i}g(\mathrm{i}) = \pi\mathrm{e}^{-5}.$$

MA244

MATHEMATICS DEPARTMENT
SECOND YEAR UNDERGRADUATE EXAMS – April 2019

Course Title: ANALYSIS III

Model Solution No: 3

a) No. A counterexample is given by a sequence of moving intervals of decreasing length that are supported on intervals of decreasing length that move along the an interval. Consider functions defined on $[0, 1]$. Let

$$f_0(x) = \chi_{[0,1]}(x),$$

$$f_1(x) = \chi_{[0,1/2]}, \qquad f_2(x) = \chi_{[1/2,1]},$$

$$f_3(x) = \chi_{[0,1/4]}, \qquad f_4(x) = \chi_{[1/4,1/2]}, \qquad f_5(x) = \chi_{[1/2,3/4]}, \qquad f_6(x) = \chi_{[3/4,1]}.$$

Each function is an indicator of an interval, and that in each row above the intervals sweep $[0, 1]$. When we move to the next row the length of the corresponding intervals gets divided by 2 and therefore we consider twice as many functions for each group. A student that explains this much will get full credit.

A formula for the sequence is given by

$$n \in [\sum_{l=0}^{k-1} 2^l, \sum_{l=0}^{k} 2^l], \qquad k = 1, 2, \ldots$$

we set f_n as the indicator of the interval

$$\left[\frac{(n - \sum_{l=0}^{k-1} 2^l)}{2^k}, \frac{n - \sum_{l=0}^{k-1} 2^l + 1}{2^k}\right].$$

b) No. A possible counter example is given by $f_n(x) = n\chi_{I_n}(x)$, where $I_n = (0, 1/n)$. The sequence converges to $f(x) = 0$, but $\int f_n = 1$ for every n.

In order to ensure convergence of the integrals in the statement we require uniform convergence for the sequence f_n. In that case

$$\left|\int_a^b f_n - \int f\right| \leq \int_a^b |f_n - f| \leq (b - a)\|f_n - f\|_\infty.$$

The right-hand side goes to zero as n goes to infinity by the uniform convergence and we obtain the result.

c) No, pointwise convergence does not necessarily preserve continuity. A possible example is given by

$$f_n(x) = \begin{cases} 0 & |x| > 1/n \\ -n(x - 1/n) & x \in [0, 1/n] \\ n(x + 1/n) & x \in [-1/n, 0) \end{cases}$$

The functions f_n are supported on $[-1/n, 1/n]$ and formed by two straight lines such that $f_n(0) = 1$ for every n. As such the limit is

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

which is not continuous.

d) No. Consider

$$f_n(x) = \left(x^2 + 1/n\right)^{1/2}.$$

They are clearly C^1 as the $x^2 + 1/n$ never vanishes for fixed n. (f_n) converges uniformly to $f(x) = |x|$ which is not smooth at the origin. To see this notice that if

$$A := \left(x^2 + 1/n\right)^{1/2} - |x|$$

then

$$A \leq \left((x + 1/\sqrt{n})^2\right)^{1/2} - |x| \leq \frac{1}{\sqrt{n}},$$

and the uniform convergence follows.

No, even if f is infinitely smooth we do not necessarily have convergence of f_n' to f. In fact the limit f_n' might not exist. For example

$$f_n(x) = \frac{1}{n}\sin(n^2 x).$$

The sequence converges uniformly to zero, which is a smooth function. However

$$f_n'(x) = n\cos(n^2 x)$$

does not converge to zero.

MA244

MATHEMATICS DEPARTMENT
SECOND YEAR UNDERGRADUATE EXAMS – April 2019

Course Title: ANALYSIS III

Model Solution No: 4

a) The Weierstrass M-test: Let (f_k) be a sequence of functions $f_n : \Omega \to \mathbb{R}$, and assume that for every k there exists $M_k > 0$ such that $|f_k(x)| \leq M_k$ for every $x \in \Omega$ and $\sum_{k=1}^{\infty} M_k < \infty$. Then

$$\sum_{k=1}^{\infty} f_k$$

converges uniformly on Ω.

Notice that it suffices to show that $S_n := \sum_{k=1}^{n} f_k(x)$ is uniformly Cauchy. Now since $\sum_{k=1}^{\infty} M_k < \infty$, given $\varepsilon > 0$ there exists N such that

$$\sum_{k=m+1}^{n} M_k < \varepsilon \qquad \text{for all } m, n > N.$$

Now

$$|S_n(x) - S_m(x)| = \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x) \right| = \left| \sum_{k=m+1}^{n} f_k(x) \right| \leq \sum_{k=m+1}^{n} |f_k| \leq \sum_{k=m+1}^{n} M_k \leq \varepsilon,$$

for every x. Therefore S_n is uniformly Cauchy and the proof is complete.

b) Notice that we can rewrite

$$\frac{1}{4 + (x-2)^2} = \frac{1}{4} \frac{1}{1 + \left(\frac{x-2}{2}\right)^2}$$

and since

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$$

we obtain

$$\frac{1}{4 + (x-2)^2} = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x-2}{2}\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} (x-2)^{2n}$$

To compute the radius of convergence using a test, we can use the ratio test, which will yield

$$\frac{\frac{|x-2|^{2(n+1)}}{4^{n+2}}}{\frac{|x-2|^{2n}}{4^{n+1}}} = \frac{|x-2|^2}{4}$$

and so the series will be convergence for

$$|x-2|^2 < 4$$

which implies the radius of convergence is 2.

c) We will use the following Theorems

Theorem: Let (f_k), with $f_k : [a,b] \to \mathbb{R}$, be a sequence of integrable functions. Assume that $S_n = \sum_{k=1}^{n} f_k$ converges uniformly. Then $\sum_{k=1}^{\infty} f_k$ is Riemann integrable and

$$\int \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int f_k.$$

and

Theorem: Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Then for every $r < R$ the sequence of functions

$$f_k := \sum_{n=0}^{k} a_n z^n$$

converges uniformly in $|z| \le r$. The M-Weierstrass Theorem ensures that for every x such that $|x-2| < r < 2$ the series converges uniformly. Therefore for x such that $|x-2| < r < 2$

$$f(x) = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} (x-2)^{2n} \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} \frac{1}{2n+1} (x-2)^{2n+1} + C.$$

Since r is arbitrary we obtain an expansion that works for $|x-2| < 2$. To find C, notice that $f(2) = C$. So $C = \frac{1}{2}\mathrm{atan}(0) = 0$.

MATHEMATICS DEPARTMENT
SECOND YEAR UNDERGRADUATE EXAMS – April 2019

Course Title: ANALYSIS III

Model Solution No: 5

a) Since we know that f is Riemann integrable we know that for every $\varepsilon > 0$ there exists a partition of $[a, b]$ such that

$$\sum_{k=0}^{n} m_k \chi_{I_k}(x) \leq f(x) \leq \sum_{k=0}^{n} M_k \chi_{I_k}(x)$$

where m_k and M_k are given by the infimum and supremum of f on the interval I_k, and such that

$$U(f, P) - L(f, P) < \varepsilon.$$

If we did not require to have g continuous, the function $g(x) = \sum_{k=0}^{n} m_k \chi_{I_k}(x)$, or similarly for the RHS, would satisfy

$$\int |f - g| < \varepsilon.$$

So the only thing left to do is find a continuous function g whose integral is very close to the integral of $\sum_{k=0}^{n} m_k \chi_{I_k}(x)$. Notice that we can use linear interpolation near the endpoints of the intervals I_k this can be achieved. Indeed since f is bounded and there are finitely many points in the partition by using linear interpolation on an interval of length smaller than $\varepsilon/(Mn)$ this can be easily achieved. As a consequence we find g such that

$$\int |f - g| \leq \int |f - \sum_{k=0}^{n} m_k \chi_{I_k}(x)| + \int |\sum_{k=0}^{n} m_k \chi_{I_k}(x) - g| \leq 2\varepsilon,$$

but since ε is arbitrary we obtain the result.

b) The result is still true. Notice that we need to understand

$$\int_0^1 f(x)\mathrm{d}x = \lim_{\delta \to 0} \int_\delta^1 f(x)\mathrm{d}x.$$

That means that given $\varepsilon > 0$ there exists δ such that

$$|\int_0^1 f(x)\mathrm{d}x - \int_\delta^1 f(x)\mathrm{d}x| < \varepsilon/2$$

We can now apply the result from the previous part to $f\chi_{[\delta,1]}$, finding g continuous on $[0, 1]$ such that

$$\int_0^1 |f\chi_{[\delta,1]} - g| \leq \varepsilon/2.$$

In fact we can arrange for g to vanish on $[0, \delta]$ for example. That g satisfies the required conditions.

c) Let $f : [a,b] \to \mathbb{R}$ be an integrable function and define the function $F : [a,b] \to \mathbb{R}$ by

$$F(x) := \int_a^x f(t)\mathrm{d}t,$$

then F is continuous of $[a,b]$. Additionally if f is continuous at $c \in [a,b]$ then $F'(c) = f(c)$, with the derivatives at a and b understood as one-sided derivatives.

First, by the additivity Theorem we know that f is integrable on the interval $[0,x]$ and therefore F is well defined. Also since f is integrable we know that it is bounded, say $|f| \leq M$, and therefore

$$F(x+h)-F(x) = \int_a^{x+h} f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t = \int_a^x f(t)\mathrm{d}t + \int_x^{x+h} f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t$$

$$= \int_x^{x+h} f(t)\mathrm{d}t,$$

from which $|F(x+h) - F(x)| \leq M|h|$, proving that F is (Lipschitz) continuous. Also, from that equality we deduce

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t.$$

The result will follow if we show that whenever f is continuous at x (in addition to integrable) we have

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t = f(x).$$

Notice that since we are integrating with respect to t, $f(x)$ is a constant, and therefore

$$\frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t - f(x) = \frac{1}{h}\int_x^{x+h} (f(t) - f(x))\mathrm{d}t.$$

It suffices to show that with the hypotheses above on f

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} (f(t) - f(x))\mathrm{d}t = 0.$$

Now, since f is continuous at x, given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Therefore, for $|h| < \delta$ we have

$$\lim_{h\to 0} \left|\frac{1}{h}\int_x^{x+h} (f(t) - f(x))\mathrm{d}t\right| \leq \lim_{h\to 0} \left|\frac{1}{h}\int_x^{x+h} |(f(t) - f(x))|\mathrm{d}t\right| \leq \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} \varepsilon \mathrm{d}t = \varepsilon,$$

which implies, taking limits as ε tends to zero, the desired result.

d) F is differentiable since it is the composition of differentiable functions, as the limits of integrations are smooth in the domain of definition of F, and $\sin(t^2)$ is smooth. By the chain rule, and the FTC from the previous section

$$F'(x) = \sin(x)\frac{1}{2\sqrt{x}} - \sin(1/x)\frac{-1}{2x^{3/2}}.$$