MA 244

THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: APRIL 2018

ANALYSIS III

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

COMPULSORY QUESTION

1. When answering the questions below, you may cite any statement proved during the lectures.

a) Let $\varphi, \psi : [a,b] \to \mathbb{R}$ be step functions. Prove that:

(i) Prove that if $\varphi \geq \psi$, then $\int_a^b \varphi \geq \int_a^b \psi$; **[4]**

(ii) $|\varphi|$ is a step function and $|\int_a^b \varphi| \leq \int_a^b |\varphi|$; **[4]**

(iii) $\varphi\psi : [a,b] \to \mathbb{R}$ is a step function and $(\int_a^b \varphi\psi)^2 \leq (\int_a^b \varphi^2)(\int_a^b \psi^2)$. **[6]**

b) If the regulated function $f : [-\pi, \pi] \to \mathbb{R}$ is even (meaning that $\forall x \in [-\pi,\pi]\ f(x) = f(-x)$) show that $\int_{-\pi}^{\pi} f = 2\int_0^{\pi} f$. **[4]**

c) Let $\log : (0,\infty) \to \mathbb{R}$ be defined by $\log(x) := \int_1^x f$, where $f : (0,\infty) \to \mathbb{R} : x \mapsto \frac{1}{x}$. Show that $\log(xy) = \log(x) + \log(y)$. **[5]**

d) Give an example of a regulated function $f : [a,b] \to \mathbb{R}$ with the properties that $\forall x \in [a,b]\ f(x) \geq 0$, $\int_a^b f = 0$ and there is $c \in [a,b]$ with $f(c) > 0$. **[4]**

e) Show that a continuous function $f : [a, b] \to \mathbb{R}$ with the properties that $\forall x \in [a, b]\ f(x) \geq 0$ **and** $\int_a^b f = 0$ must be identically zero. [5]

f) Find the points of the extremum of the function

$$f(x) = \int_1^x \frac{\sin(t)}{t} dt$$

in the region $x > 1$. [4]

g) Calculate $\lim_{R\to\infty} \int_{-R}^{R} x\, dx$. Does your result imply that $\int_{-\infty}^{\infty} x\, dx$ exists? Explain your answer. [4]

OPTIONAL QUESTIONS

2. The goal of this question is to calculate the improper integral $I = \int_{-\infty}^{\infty} e^{-x^2} dx$.

a) Calculate the two-dimensional integral $g(R) = \int_D e^{-x^2-y^2} dx$ over the disk defined as $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$ (Hint: use the polar coordinates). [7]

b) Consider the two-dimensional integral $f(R) = \int_S e^{-x^2-y^2} dx$ over a square defined as $S := \{(x, y) \in \mathbb{R}^2 : |x| \leq R, |y| \leq R\}$. Show that $\lim_{R\to\infty} f(R) = \lim_{R\to\infty} g(R)$. [7]

c) Use the results of the previous two parts and calculate $I = \lim_{R\to\infty} \int_{-R}^{R} e^{-x^2} dx$. [6]

---

3. a) On the vector space $R[0, 1]$ of regulated functions define $||f||_1 := \int_0^1 |f|$. Show that $||f||_1$ satisfies all except one of the requirements to be a norm. [7]

b) Show that $||f||_1 := \int_0^1 |f|$ does define a norm on the subspace $C[0, 1]$ of continuous functions. [7]

c) Show that $||f|| := \int_0^1 t|f(t)|\, dt$ is a norm on $C[0, 1]$. [6]

---

4. Fix $a < b$ in $\mathbb{R}$ and consider the two norms $||f||_2 := \sqrt{\int_a^b |f|^2}$ and $||f||_\infty := \sup\{|f(x)| : a \leq x \leq b\}$ on the vector space $C[a, b]$. This question shows that these two norms are *not* equivalent.

a) Show that there is $K \in \mathbb{R}$ such that $\forall f \in C[a, b]\ \ ||f||_2 \leq K||f||_\infty$. [10]

b) Show that there is no $K \in \mathbb{R}$ such that $\forall f \in C[a, b]\ \ ||f||_\infty \leq K||f||_2$. [10]

Question 5 continued overleaf

---

5. Let $(V, ||\cdot||)$ be a normed vector space over $\mathbb{R}$.

a) Let $E \subset V$ be a non-empty subset of V. For each $x \in V$ define

$$d(x, E) = \inf_{y \in E} ||x - y||.$$

Show that the map $d(\cdot, E) : V \to \mathbb{R}$ is continuous. [8]

b) Let $\mathcal{E}$ be the set of non-empty closed bounded sets in V. If $K, L \in \mathcal{E}$, define

$$\rho(K, L) = \max(\sup_{k \in K} d(k, L), \sup_{l \in L} d(l, K))$$

Prove that the function $\rho : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ is well defined and satisfies the following properties:

(i) $L = K$ implies $\rho(L, K) = 0$; [8]

(ii) For any $L, K \in \mathcal{E}$, $\rho(L, K) = \rho(K, L)$. [4]

END