THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: APRIL 2017

ANALYSIS III

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

COMPULSORY QUESTION

1. When answering the questions below, you may cite any statement proved during the lectures.

a) A. Give an example of a step function $[0, 1] \to \mathbb{R}$ that takes four different values and is discontinuous at three points. **[5]**

b) Let $\varphi : [a, b] \to \mathbb{R}$ be a step function, and fix $K > 0$ in $\mathbb{R}$. Define $\psi : [Ka, Kb] \to \mathbb{R}$ by $\psi(x) := \varphi(x/K)$. Prove that ψ is a step function, and that $\int_{Ka}^{Kb} \psi = K \int_a^b \varphi$. **[5]**

c) Let $\varphi, \psi : [a, b] \to \mathbb{R}$ be step functions. Prove that $\int_a^b |\varphi + \psi| \leq \int_a^b |\varphi| + \int_a^b |\psi|$. **[5]**

d) If $\varphi : [0, 1] \to \mathbb{R}$ is a step function and $f : [0, 1] \to \mathbb{R}$ is given by $f(0) = 1, f(x) = \cos(1/x^4), 0 < x \leq 1$ show that $\|\varphi - f\|_\infty \geq 1$. Deduce that f is not regulated. **[5]**

e) Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Show that there exists $c \in (a, b)$ such that $\int_a^b f(x)dx = (b - a) \cdot f(c)$. **[5]**

f) Find the derivatives of the following functions:

 Question 1 continued overleaf

(i) $G(x) = \int_x^0 \sqrt{1+t^8}dt,\ x \in \mathbb{R};$ **[2]**

(ii) $H(x) = \int_x^{x^2} e^{-t^3}dt,\ x \in \mathbb{R};$ **[3]**

g) Consider the series $\sum_{n=1}^{\infty} \frac{x}{n^{4/5}(1+nx^2)}$. Prove that this series converges uniformly on $\mathbb{R}$. **[5]**

h) Determine the sign of the following integral:

$$\int_0^{2\pi} f(x)dx,$$

where $f(0) = 1,\ f(x) = \frac{sin(x)}{x}$ for $x > 0$. **[5]**

OPTIONAL QUESTIONS

2. a) Consider $f : [0,1] \to \mathbb{R},\ f(x) := 0$ if $x \notin \mathbb{Q}$, and $f(p/q) := 1/q$, where $p, q \in \mathbb{N}$ are coprime. Show that f is continuous at $1/\sqrt{5}$ but not at $2/5$. **[5]**

b) Show that f is *not* a step function. **[5]**

c) For any $\varepsilon > 0$ construct a step function $\varphi : [0,1] \to \mathbb{R}$ such that $\|f - \varphi\|_\infty := \sup\{|f(t) - \varphi(t)| : t \in [0,1]\} < \varepsilon$. [Suggestion: restrict to $q < 1/\varepsilon$.]
Show that f is regulated. What is $\int_0^1 f$? **[10]**

3.

B. Evaluate the improper integrals (or establish their divergence):

a) $\int_0^1 \log(t)dt;$ **[5]**

b) $\int_0^1 \frac{1}{t^p}dt,\ p > 0;$ **[5]**

c) $\int_1^{\infty} \frac{1}{t^p}dt,\ p > 0;$ **[5]**

d) $\int_0^{\infty} \cos(x)dx.$ **[5]**

(Note: different values of p in parts (b) and (c) may lead to different types of behaviour.)

4.

Applying term-wise differentiation and integration, find the sums of the series in the indicated intervals:

a) $\sum_{k=1}^{\infty} \frac{x^k}{k},\ x \in [a,b],\ -1 < a < 0 < b < 1$ **[6]**

b) $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$, $x \in [a,b]$, $-1 < a < 0 < b < 1$ **[7]**

c) $\sum_{k=1}^{\infty}\frac{x^{2k-1}}{2k-1}$, $x \in [a,b]$, $-1 < a < 0 < b < 1$. **[7]**

Justify all steps of your calculations.

5.

Consider S_F, the space of real sequences $\mathbf{a} = (a_n)_{n=1}^{\infty}$, such that **all but finitely many** of the a_n's are zero. (In other words, each sequence $\mathbf{a} \in S_F$ is eventually zero.)

a) Show that the following definitions all give norms on S_F, **[10]**

$$||\mathbf{a}||_\infty = \max_{n\geq 1} |a_n|, \tag{1}$$

$$||\mathbf{a}||_w = \max_{n\geq 1} |na_n|, \tag{2}$$

$$||\mathbf{a}||_1 = \sum_{n=1}^{\infty} |a_n|. \tag{3}$$

b) Show that norms (2), (3) are NOT equivalent to norm (1). **[10]**

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As advertised to the students, all exam questions are based on the questions of the assignments (only a minor part of which was submitted for marking and even less marked). The compulsory question is based on the A-questions and the optional questions—on the B-questions of the assignments.

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Model Solution No: 1

a) There is, for example, a step function $[0,1] \to \mathbb{R}$ taking the values $1,2,0,3,2$ on the subsets $[0,\frac{1}{3}),[\frac{1}{3},\frac{2}{3}),\{\frac{2}{3}\},(\frac{2}{3},1),\{1\}$, which is continuous except at the three points $\frac{1}{3},\frac{2}{3},1$.

b) Take a partition $a = p_0 < p_1 < \ldots < p_{l-1} < p_l = b$ with $\varphi(x) = c_j$ for $p_{j-1} < x < p_j$. Define $q_j = Kp_j$ for $0 \le j \le l$ to get a partition $Ka = q_0 < q_1 < \ldots < q_{l-1} < q_l = Kb$ for which $\psi(x) = c_j$ for $q_{j-1} < x < q_j$, which shows that ψ is a step function. Then $\int_{Ka}^{Kb} \psi = \sum_1^l c_j(q_j - q_{j-1}) = \sum_1^l c_j(Kp_j - Kp_{j-1}) = K\int_a^b \varphi$.

c) Choose $k \ge 1$ and a partition $a = p_0 < \ldots < p_{k-1} < p_k = b$ such that $\forall j \in \{1,\ldots,k\}$, both φ and ψ (and so $\varphi + \psi$) are constant on (p_{j-1}, p_j) (say, c_j and c_j' respectively). Then $\int_a^b |\varphi+\psi| = \sum_1^k |c_j + c_j'|(p_j - p_{j-1}) \le \sum_1^k (|c_j| + |c_j'|)(p_j - p_{j-1}) = \int_a^b |\varphi| + \int_a^b |\psi|$.

d) The step function φ takes some value c_1 on $(0,p_1)$, and f takes the values ± 1 here (and other values between these), so $\|f-\varphi\|_\infty \ge \max\{|c_1 - 1|, |c_1 + 1|\} \ge 1$. Since $\|f-\varphi\|_\infty$ cannot be small, f is not regulated.

e) Due to the continuity of f, $F(x) = \int_a^x f(t)\,dt$ is differentiable on (a,b) with $F'(x) = f(x)$ (FTC1). By the mean value theorem, there is $c \in (a,b)$: $F(b) - F(a) = F'(c)(b-a) = f(c)(b-a)$. As $F(a) = 0$, the statement follows.

f) The chain rule and FTC lead to the following formula:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = b'(x)f(b(x)) - a'(x)f(a(x)).$$

Its application gives:

(i) $G'(x) = -\sqrt{1+x^8}$.

(ii) $H'(x) = 2xe^{-x^6} - e^{-x^3}$

g) $f_n(x) := \frac{x}{n^{4/5}(1+nx^2)}$ is an odd function with maximum/minimum at $\pm\sqrt{\frac{1}{n}}$ (where $f_n' = 0$) and $f_n(x) \to 0$ as $x \to \pm\infty$. Hence, $\forall x \in \mathbb{R}$ $|f_n(x)| \le f(\sqrt{\frac{1}{n}}) = \frac{1}{2}n^{-1.3}$. Since $\sum \frac{1}{2}n^{-1.3}$ converges, $\sum \frac{x}{n^{4/5}(1+nx^2)}$ converges uniformly by the Wierstrass M-test.

h) Function f is continuous on $[0, 2\pi]$, so the integral exists in the usual sense.

$$\int_0^{2\pi} f = \int_0^{\pi} f + \int_{\pi}^{2\pi} f = \int_0^{\pi} (f(x) + f(x+\pi))dx,$$

$f(x) + f(x+\pi) = \sin(x)/x + \sin(x+\pi)/(x+\pi) = \sin(x)(1/x - 1/(x+\pi)) = \sin(x)\frac{\pi}{x(x+\pi)} > 0$ for $x \in (0, \pi)$. Therefore, $\int f > 0$.

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Model Solution No: 2

a) Take a sequence of irrationals $x_n \to 2/5$; then $f(x_n) = 0 \to 0 \neq 1/5 = f(2/5) = f(\lim x_n)$ so f is not continuous at $2/5$. For $c = 1/\sqrt{5}$, $f(c) = 0$. Then given $\varepsilon > 0$, only finitely many p/q have $q > 1/\varepsilon$, so let δ be the distance from c to the nearest of these to show that f is continuous at c.

b) Every open interval in $(0, 1)$ contains irrational points x where $f(x) = 0$ and rational x where $f(x) \neq 0$, so it is not constant there and f cannot be a step function.

c) Given $\varepsilon > 0$, define $\varphi := [0, 1] \to \mathbb{R}$ by $\varphi(p/q) := 1/q$ if $0 \leq p \leq q \in \mathbb{N}$ and p, q are coprime and $q \leq 1/\varepsilon$, and $\varphi(x) = 0$ otherwise. Then $f(x) = \varphi(x)$ unless $x = p/q, q > 1/\varepsilon$. Hence $\|f - \varphi\|_\infty = 1/\inf\{q \in \mathbb{N} : q > 1/\varepsilon\} < \varepsilon$ so f is regulated. $\int_0^1 f = \lim_{n\to\infty} \int_0^1 \varphi_n = \lim_{n\to\infty} 0 = 0$.

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Model Solution No: 3

In all cases, the relevant integrals are elementary, in (a) use integration by parts to find the anti-derivative of $\log(t)$.

a) $\int_0^1 \log(t)dt = \lim_{\epsilon\downarrow 0}\int_\epsilon^1 \log(t)dt = \lim_{\epsilon\downarrow 0}(t\log(t)-t)\mid_\epsilon^1 = -1$. Here we used $\lim_{\epsilon\to 0}\epsilon\log(\epsilon) = 0$, which follows from l'Hôpital's rule.

b) There are three cases to consider: $0 < p < 1$, $p = 1$ and $p > 1$. If $0 < p < 1$, $\int_0^1 \frac{1}{t^p}dt = \lim_{\epsilon\downarrow 0}\int_\epsilon^1 \frac{1}{t^p}dt = \lim_{\epsilon\downarrow 0}\frac{t^{1-p}}{1-p}|_\epsilon^1 = \frac{1}{1-p}$. The improper integral exists and equals to $\frac{1}{1-p}$. If $p = 1$, $\int_0^1 \frac{1}{t}dt = \lim_{\epsilon\downarrow 0}\int_\epsilon^1 \frac{1}{t}dt = \lim_{\epsilon\downarrow 0}\log(t)|_\epsilon^1$. The limit does not exist, so the improper integral diverges. If $p > 1$, $\int_0^1 \frac{1}{t^p}dt = \lim_{\epsilon\downarrow 0}\int_\epsilon^1 \frac{1}{t^p}dt = \lim_{\epsilon\downarrow 0}\frac{t^{1-p}}{1-p}|_\epsilon^1$. The limit does not exist, so the improper integral diverges.

c) There are three cases to consider: $p > 1$, $p = 1$ and $0 < p < 1$. If $p > 1$, $\int_1^\infty \frac{1}{t^p}dt = \lim_{R\to\infty}\int_1^R \frac{1}{t^p}dt = \lim_{\mathbb{R}\to\infty}\frac{t^{1-p}}{1-p}|_1^R = \frac{1}{p-1}$. The improper integral exists and equals to $\frac{1}{p-1}$. If $p = 1$, $\int_0^\infty \frac{1}{t}dt = \lim_{R\to\infty}\int_1^R \frac{1}{t}dt = \lim_{R\to\infty}\log(t)|_1^\infty$. The limit does not exist, so the improper integral diverges. If $0 < p < 1$, $\int_1^\infty \frac{1}{t^p}dt = \lim_{R\to\infty}\int_1^R \frac{1}{t^p}dt = \lim_{R\to\infty}\frac{t^{1-p}}{1-p}|_1^\infty$. The limit does not exist, so the improper integral diverges.

d) $\int_0^\infty \cos(x)dx = \lim_{R\to\infty}\int_0^R \cos(x)dx = \lim_{R\to\infty}\sin(R)$. The limit does not exist, the improper integral is divergent.

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Model Solution No: 4

In each of the three cases below, both the power series $S(x)$ and the series for $S'(x)$ converge uniformly on $[a, b]$. Therefore, the term-wise differentiation is justified. Integrating S' to obtain S is done using $FTC2$.

a) Let $S(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$. Then $S'(x) = \sum_{k=1}^{\infty} x^{k-1} = \frac{1}{1-x}$. Therefore, $S(x) = -\log(1-x) + C$, where C is an integration constant. As $S(0) = 0$, $C = 0$. Therefore, $\sum_{k=1}^{\infty} \frac{x^k}{k} = -\log(1-x)$.

b) Let $S(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$. Then $S'(x) = \sum_{k=1}^{\infty} (-x)^{k-1} = \frac{1}{1+x}$. Therefore, $S(x) = \log(1+x) + C$, where C is an integration constant. As $S(0) = 0$, $C = 0$. Therefore, $\sum_{k=1}^{\infty} \frac{x^k}{k} = \log(1+x)$.

c) Let $S(x) = \sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k-1}$. Then $S'(x) = \sum_{k=1}^{\infty} x^{2k-2} = \frac{1}{1-x^2}$. Therefore, $S(x) = \frac{1}{2} \log \frac{1+x}{1-x} + C$, where C is an integration constant. As $S(0) = 0$, $C = 0$. Therefore, $\sum_{k=1}^{\infty} \frac{x^k}{k} = -\frac{1}{2} \log \frac{1+x}{1-x}$.

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Model Solution No: 5

a) All three norms on S_F are well defined as the max and the sum are always taken over a finite set.

(i) Non-negativity: $||\mathbf{a}|| \geq \mathbf{0}$ for all $\mathbf{a} \in S_F$. Indeed,

$$||\mathbf{a}||_\infty = \max_{n\geq 1} |a_n| \geq 0, \tag{4}$$
$$||\mathbf{a}||_w = \max_{n\geq 1} |na_n| \geq 0, \tag{5}$$
$$||\mathbf{a}||_1 = \sum_{n=1}^{\infty} |a_n| \geq 0. \tag{6}$$

(ii) Separation of points: if $||\mathbf{a}|| = 0$ then $\mathbf{a} = \mathbf{0}$. Indeed, it is clear that the equal sign is realised in (4), (5) and (6) iff $\forall n \geq 1, a_n = 0$.

(iii) Absolute homogeneity: if $\lambda \in \mathbb{R}$ and $\mathbf{a} \in S_F$ then $||\lambda\mathbf{a}|| = |\lambda| \cdot ||\mathbf{a}||$. Indeed,

$$||\lambda\mathbf{a}||_\infty = |\lambda| \max_{n\geq 1} |a_n| = |\lambda|||\mathbf{a}||_\infty, \tag{7}$$
$$||\lambda\mathbf{a}||_w = |\lambda| \max_{n\geq 1} |na_n| = |\lambda|||\mathbf{a}||_w, \tag{8}$$
$$||\lambda\mathbf{a}||_1 = |\lambda| \sum_{n=1}^{\infty} |a_n| = |\lambda|||\mathbf{a}||_1. \tag{9}$$

(iv) Triangle inequality: if $\mathbf{a}, \mathbf{b} \in S_F$, then $||\mathbf{a}+\mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}||$. Indeed,

$$||\mathbf{a}+\mathbf{b}||_\infty = \max_{n\geq 1} |a_n + b_n| \leq \max_{n\geq 1}(|a_n| + |b_n|) \leq \max_{n\geq 1} |a_n| + \max_{\geq 1} |b_n| = ||\mathbf{a}||_\infty + ||\mathbf{b}||_\infty,$$
$$||\mathbf{a}+\mathbf{b}||_w = \max_{n\geq 1} n|a_n + b_n| \leq \max_{n\geq 1} n(|a_n| + |b_n|) \leq \max_{n\geq 1} n|a_n| + \max_{n\geq 1} n|b_n| = ||\mathbf{a}||_w + ||\mathbf{b}||_w,$$
$$||\mathbf{a}+\mathbf{b}||_1 = \sum_{n=1}^{\infty} |a_n + b_n| \leq \sum_{n=1}^{\infty}(|a_n| + |b_n|) = ||\mathbf{a}||_1 + ||\mathbf{b}||_1.$$

b) (i) (2)$\not\sim$(1). Let $(\mathbf{a}_n)_{n\geq 1} \subset S_F$ be a sequence:

$$\mathbf{a}_n = (\underbrace{1, 1, \ldots, 1}_{n\text{-times}}, 0, 0 \ldots).$$

For any $n \in \mathbb{N}$, $||\mathbf{a}_n||_\infty = 1$, $||\mathbf{a}_n||_w = n \to \infty$ for $n \to \infty$. Therefore $\nexists\ K_1 > 0$: for any $n \in \mathbb{N}$, $K_1||\mathbf{a}_n||_w < ||\mathbf{a}_n||_\infty$, meaning that $||\cdot||_w \not\sim ||\cdot||_\infty$.

(ii) (3)$\not\sim$(1). Using the sequence $(\mathbf{a}_n)_{n>1}$ from the previous step we see that $||\mathbf{a}_n||_1 = n \to \infty$ as $n \to \infty$. Therefore, (3)$\not\sim$(1) by the argument given in the previous step.