THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: April 2020

ANALYSIS III

Time Allowed: **3 hours**

Read all instructions carefully. Please note also the guidance you have received in advance on the departmental 'Warwick Mathematics Exams 2020' webpage.

Calculators, wikipedia and interactive internet resources are not needed and are not permitted in this examination. You are not allowed to confer with other people. You may use module materials and resources from the module webpage.

ANSWER COMPULSORY QUESTION 1 AND TWO FURTHER QUESTIONS out of the four optional questions 2, 3, 4 and 5.

On completion of the assessment, you must upload your answer to the AEP as a single PDF document. You have an additional 45 minutes to make the upload, and instructions are available on the departmental 'Warwick Mathematics Exams 2020' webpage.

You must not upload answers to more than 3 questions, including Question 1. If you do, you will only be given credit for your Question 1 and the first two other answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question. The compulsory question is worth twice the number of marks of each optional question. Note that the marks do not sum to 100.

COMPULSORY QUESTION

1. a) Let $f : [a, b] \to \mathbb{R}$. Define what it means for f to be absolutely continuous. Define what it means for f to be uniformly continuous. **[2]**

b) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that it is uniformly continuous. **[4]**

c) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove that f is Riemann integrable if and only if for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that **[5]**

$$|U(f, P) - L(f, P)| < \varepsilon.$$

Here $U(f, P)$ and $L(f, P)$ correspond to the Upper and Lower Riemann sums associated to the partition P.

d) Show that $f(z) = \dfrac{1}{z}$ is analytic in $|z - 2| < 2$ and compute its Taylor Series expansion centred at 2. **[4]**

e) Evaluate the following integrals, assuming all contours are oriented counter-clockwise **[4]**

$$\int_{|z-1|=1} \bar{z}|\mathrm{d}z| \qquad\qquad \int_{|z|=2} |z+1|^2 \mathrm{d}z.$$

f) State Cauchy's theorem for analytic functions. **[5]**

g) Let $f : \Omega \to \mathbb{C}$ be an analytic function on an open set Ω. State the Cauchy-Riemann equations satisfied by the real and imaginary part of f and prove that both are harmonic functions. **[4]**

h) Find the harmonic conjugate for $u(x, y) = \sin x \cosh y$. That is, find $v(x, y)$ so that the function $f(z) = u(x, y) + iv(x, y)$ for $z = x + \mathrm{i}y$ is analytic. **[4]**

i) Prove Liouville's Theorem, that is, show that if $f : \mathbb{C} \to \mathbb{C}$ is analytic and bounded in $\mathbb{C}$, then it is constant. **[6]**

j) Is Liouville's Theorem still true if we only require a bound away from zero? That is if $f : \mathbb{C} \to \mathbb{C}$ is analytic and $|f(z)| > c > 0$, then is f constant? Justify your answer. **[2]**

OPTIONAL QUESTIONS

2. a) State and prove the Weierstrass M-test for series of functions. **[6]**

b) This part uses contour integration to calculate

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2} \mathrm{d}x.$$

Consider the integral

$$\frac{1}{\mathrm{i}} \int_{\gamma_R} \frac{z \mathrm{e}^{\mathrm{i}z}}{1+z^2} \mathrm{d}z,$$

where γ_R is the contour formed by the straight line joining $(-R, 0)$ and $(R, 0)$, together with the half circumference $|z| = R$, Im $(z) \geq 0$, oriented counter-clockwise.

(i) Show that the integral over the half circumference in γ_R goes to zero as R goes to infinity. **[4]**

(ii) Use the deformation of contours Theorem to replace γ_R by a suitable circumference. **[4]**

(iii) Use Cauchy's Theorem to calculate the integral over the new contour (circumference). (You can also work it out directly if you prefer.) **[4]**

(iv) Use (i), (ii) and (iii) to evaluate **[2]**

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2} \mathrm{d}x.$$

3. a) State and prove Fundamental Theorem of Calculus (for differentiating integrals). **[7]**

b) Let $F : (1, \infty) \to \mathbb{R}$ be given by

$$F(x) = \int_{1/\sqrt{x}}^{\sqrt{x}} \cos(t^2)\, \mathrm{d}t.$$

Show that F is differentiable, and compute F'. **[3]**

c) Connect the integral $\int_\gamma f \mathrm{d}z$ with the circulation and flux integrals for a suitable vector-field related to f. **[2]**

d) Define complex differentiability and state a theorem with equivalent conditions when considering the function as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. **[3]**

e) Let $f : [0, 2] \to \mathbb{R}$ be a continuous bounded function. Define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} f(x + \frac{k}{n}).$$

Show that f_n converges uniformly. (Hint: Try to guess the limit function first.) **[5]**

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4. a) Prove that the exist sequences of functions f_n and g_n that converge uniformly to f and g respectively but for which the product $f_n g_n$ coverges pointwise, but NOT uniformly. **[4]**

b) Assume that the sequences in part a) satisfy $|f_n| < M < \infty$ and $|g_n| < M < \infty$. Is it still possible that $f_n g_n$ does not converge uniformly? (Either exhibit a counterexample or prove the uniform convergence.) **[3]**

c) Let (f_n), $f_n : [a, b] \to \mathbb{R}$ be a sequence in Riemann integrable functions that converges uniformly to $f : [a, b] \to \mathbb{R}$. Prove that f is Riemann integrable and $\int f_n \to \int f$. **[5]**

d) Let $(f_n)_{n=1}^{\infty}$, with $f_n : [a, b] \to \mathbb{R}$ be a sequence of Riemann integrable functions such that **[4]**

$$\int_a^b |f_n(x) - f(x)| \mathrm{d}x \to 0, \qquad \text{as } n \to \infty,$$

for some Riemann integrable function $f : [a, b] \to \mathbb{R}$. Does f_n necessarily converge to f pointwise? Justify your answer.

e) Let $(f_n)_{n=1}^{\infty}$, with $f_n : [a, b] \to \mathbb{R}$ be a sequence of Riemann integrable functions that converges pointwise to $f : [a, b] \to \mathbb{R}$. Do we necessarily have **[4]**

$$\int_a^b f_n(x) \mathrm{d}x \to \int_a^b f(x) \mathrm{d}x?$$

Justify your answer. If the result is false state sufficient conditions that will make this convergence hold, and prove the result.

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5. a) Let (f_k), with $f_k : [a,b] \to \mathbb{R}$, be a sequence of integrable functions. Assume that $S_n = \sum_{k=1}^{n} f_k$ converges uniformly. Show that $\sum_{k=1}^{\infty} f_k$ is Riemann integrable and that **[4]**

$$\int \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int f_k.$$

State any results from the lectures that you use.

b) Given a sequence $(a_n)_{n=0}^{\infty}$, define what it means for the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

to have radius of convergence R. Indicate the range of values of R and a formula to calculate R in terms of the coefficients a_n. **[4]**

c) Calculate the radius of convergence of the following power series. For $a \neq 0$, **[4]**

$$\sum_{n=0}^{\infty} a^k z^k, \qquad \sum_{n=0}^{\infty} z^{k!}, \qquad \sum_{n=0}^{\infty} k! z^k.$$

State any convergence tests (either for series or power series) that you use.

d) Let $(f_n)_{n=1}^{\infty}$, with $f_n : [a,b] \to \mathbb{R}$ be a sequence of continuous functions such that $f_n \to f$ pointwise to $f : [a,b] \to \mathbb{R}$. Is f necessarily a continuous function. Justify your answer. **[4]**

e) Let $(f_n)_{n=1}^{\infty}$, with $f_n : [a,b] \to \mathbb{R}$ be a sequence of C^1 functions that converges uniformly to a function f. Is f necessarily differentiable? Suppose that f is actually infinitely differentiable. Does f_n' necessarily converge to f' pointwise? Justify your answers. **[4]**

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Model Solution No: 1

a) A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon$$

for every n and every disjoint collection of intervals $(a_1, b_1), \ldots, (a_n, b_n)$ with

$$\sum_{i=1}^{n} b_i - a_i < \delta.$$

Given $f : [a, b] \to \mathbb{R}$, we say that it is uniformly continuous if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$x, y \in [a, b] \text{ and } |x - y| < \delta \Longrightarrow |f(y) - f(x)| < \varepsilon.$$

b) We will argue by contradiction. That would mean the there exists $\varepsilon > 0$ and x_n, y_n such that $|x_n - y_n| \leq \frac{1}{n}$ but $|f(x_n) - f(y_n)| > \varepsilon$.

The sequences $\{x_n\}$ and $\{y_n\}$ are bounded, as they are in $[a, b]$, and therefore we can apply Bolzano–Weierstrass to obtain convergent subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ to x and $\{y_{n_k}\}_{k=1}^{\infty}$ to y. Notice that

$$|x - y_{n_k}| \leq |x - x_{n_k}| + |x_{n_k} - y_{n_k}| \leq |x - x_{n_k}| + \frac{1}{n_k} \xrightarrow[k\to\infty]{} 0,$$

which implies that $x = y$. However we know that $|f(x_{n_k}) - f(y_{n_k})| > \varepsilon$ for all k. Since f is continuous, taking limits as k goes to infinity we obtain $0 = |f(x) - f(x)| > \varepsilon$, which is a contradiction.

c) By the properties of sup and inf in the respective definitions $U(f)$ and $L(f)$ we know that there exists partitions P_1 and P_2 such that

$$U(f, P_1) < U(f) + \frac{\varepsilon}{2} \qquad\qquad L(f, P_2) > L(f) - \frac{\varepsilon}{2}.$$

Therefore if we consider the partition P which is a refinement of both P_1 and P_2, (for example by considering the union of all the endpoints of both partitions) we have

$$U(f, P) \leq U(f, P_1) < U(f) + \frac{\varepsilon}{2} \qquad\qquad L(f, P) \geq L(f, P_2) > L(f) - \frac{\varepsilon}{2}.$$

Notice that if f is integrable we have $U(f) = L(f)$, which implies, using the inequality above that

$$U(f,P) - L(f,P) < \varepsilon$$

For the other implication notice that since

$$U(f) - L(f) \leq U(f,P) - L(f,P)$$

for every partition P the left-hand side, which is always greater or equal to zero, must be zero, as we can find partitions P that make it arbitrarily small.

d) There are multiple ways to prove it is analytic (all, provided correct receive full credit). Since $g(z) = z$ is analytic, the function $f(x)$ is analytic in the disc $|z-2| < 2$, as g does not vanish in that set. Alternatively, the students might claim it is differentiable as map from $\mathbb{R}^2$ to $\mathbb{R}^2$, and that check that it satisfies the Cauchy Riemann equation.

To calculate a power series expansion, notice that

$$\frac{1}{z} = \frac{1}{2-(2-z)} = \frac{1}{2}\frac{1}{1-\frac{2-z}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{2-z}{2}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}(z-2)^n$$

e) We use repeatedly that $\int_0^{2\pi} \mathrm{e}^{ik\theta}\mathrm{d}\theta = 0$ for $k \neq 0$.

$$\int_{|z-1|=1} \bar{z}|\mathrm{d}z| = \int_0^{2\pi}(1+\mathrm{e}^{-\mathrm{i}\theta})\mathrm{d}\theta = 2\pi$$

$$\int_{|z=2|} |z+1|^2\mathrm{d}z = \int_0^{2\pi} |2\mathrm{e}^{\mathrm{i}\theta}+1|^2 2\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta = \int_0^{2\pi}(2\mathrm{e}^{\mathrm{i}\theta}+1)(2\mathrm{e}^{-\mathrm{i}\theta}+1)2\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta$$

$$= \int_0^{2\pi}(4+2\mathrm{e}^{\mathrm{i}\theta}+2\mathrm{e}^{-\mathrm{i}\theta}+1)2\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta = \int_0^{2\pi} 4\mathrm{i}\mathrm{d}\theta = 8\pi\mathrm{i}$$

f) Let $f : \Omega \to \mathbb{C}$ be an analytic function, with Ω an open, simply connected domain. Let γ be a C^1 curve closed curve in Ω. Then

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

g) Assume that $f(z)$ is given by $u(x,y) + \mathrm{i}v(x,y)$ for $z = x + \mathrm{i}y$. Then the Cauchy–Riemann equations are given by

$$u_x = v_y \qquad\qquad u_y = -v_x.$$

If we calculate the Laplacian of u for example,

$$\triangle u = u_{xx} + u_{yy} = (v_y)_x + (-v_x)_y = 0.$$

As for v, similarly

$$\triangle v = v_{xx} + v_{yy} = (-u_y)_x + (u_x)_y = 0.$$

h) The Cauchy–Riemann equations tell us that $u_x = v_y$. Therefore

$$v_y = \cos x \cosh y,$$

which implies that $v(x, y) = \cos x \sinh y + f(x)$. To find f we use the other C-R equation, namely $v_x = -u_y$. We have

$$v_x = -\sin x \sinh y + f'(x) = -u_y - -\sin x \sinh y.$$

This implies that $f'(x) = 0$, and up to constant, we have $v(x, y) = \cos x \sinh y$.

i) Assume that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let $a \neq b$ be two points in $\mathbb{C}$. Choose R large enough so that $2\max\{|a|, |b|\} < R$. That means that if we consider $w \in \partial B_R(0)$, that is $|w| = R$ then

$$|w - a| > \frac{R}{2} \qquad\qquad |w - b| > \frac{R}{2}.$$

Since f is analytic in $\mathbb{C}$ we can use Cauchy's formula to compute $f(a)$ and $f(b)$ using $\partial B_R(0)$ as the curve γ (of course positively oriented!). We have

$$f(a) - f(b) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_R(0)} \frac{f(w)}{w - a} \mathrm{d}w - \frac{1}{2\pi \mathrm{i}} \int_{\partial B_R(0)} \frac{f(w)}{w - b} \mathrm{d}w$$

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial B_R(0)} f(w) \left(\frac{1}{w - a} - \frac{1}{w - b} \right) \mathrm{d}w = \frac{a - b}{2\pi \mathrm{i}} \int_{\partial B_R(0)} \frac{f(w)}{(w - a)(w - b)} \mathrm{d}w.$$

Therefore

$$|f(a) - f(b)| \leq \frac{|a - b|}{2\pi} \frac{M}{R^2/4} \int_{\partial B_r(0)} 1 \mathrm{d}w = \frac{|a - b| 4M}{R},$$

as $\int_{\partial B_R(0} 1 \mathrm{d}w$ is just the length of the curve, which equals $2\pi R$. Notice that since R is arbitrary (provided that it is big enough, as indicated above) we can send R to infinity, showing that $|f(a) - f(b)| = 0$ for any a and b in $\mathbb{C}$, therefore proving that the function is constant.

j) Notice that since f does not vanish we can consider $\frac{1}{f}$, for which the original Liouville's Theorem applies, and therefore f is constant.

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Model Solution No: 2

a) The Weierstrass M-test: Let (f_k) be a sequence of functions $f_n : \Omega \to \mathbb{R}$, and assume that for every k there exists $M_k > 0$ such that $|f_k(x)| \leq M_k$ for every $x \in \Omega$ and $\sum_{k=1}^{\infty} M_k < \infty$. Then

$$\sum_{k=1}^{\infty} f_k$$

converges uniformly on Ω.

Notice that it suffices to show that $S_n := \sum_{k=1}^{n} f_k(x)$ is uniformly Cauchy. Now since $\sum_{k=1}^{\infty} M_k < \infty$, given $\varepsilon > 0$ there exists N such that

$$\sum_{k=m+1}^{n} M_k < \varepsilon \qquad \text{for all } m, n > N.$$

Now

$$|S_n(x) - S_m(x)| = \left|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x)\right| = \left|\sum_{k=m+1}^{n} f_k(x)\right| \leq \sum_{k=m+1}^{n} |f_k| \leq \sum_{k=m+1}^{n} M_k \leq \varepsilon,$$

for every x. Therefore S_n is uniformly Cauchy and the proof is complete.

b) Notice that

$$\int_{\infty}^{\infty} \frac{x \sin x}{1+x^2} \mathrm{d}x = \frac{1}{\mathrm{i}} \int_{-\infty}^{\infty} \frac{z\mathrm{e}^{\mathrm{i}z}}{1+z^2} \mathrm{d}z,$$

since the real part of the integral on the RHS vanishes as the integrand is odd (hence dividing the i).

We consider the contours (notice they are both oriented counter-clockwise) (see picture).

(i) Consider the integral over the arc $\gamma_1 = \gamma_R$.

$$\left|\int_{\text{arc}} \frac{z\mathrm{e}^{\mathrm{i}z}}{1+z^2} \mathrm{d}z\right| = \left|\int_0^{\pi} \frac{R\mathrm{e}^{\mathrm{i}t}\mathrm{e}^{-R\sin t + R\mathrm{i}\cos t}}{1R^2\mathrm{e}^{2\mathrm{i}t}} R\mathrm{i}\mathrm{e}^{\mathrm{i}t}\mathrm{d}t\right| \leq \int_0^{\pi} \frac{R^2}{R^2-1}\mathrm{e}^{-R\sin t}\mathrm{d}t$$

$$\leq 2\int_0^{\pi} \mathrm{e}^{-R\sin t}\mathrm{d}t = 4\int_0^{\pi/2} \mathrm{e}^{-R\sin t}\mathrm{d}t \leq 4\int_0^{\pi/2} \mathrm{e}^{-R2t/\pi}\mathrm{d}t = 4\frac{\pi}{2R}\mathrm{e}^{-R2t/\pi}\Big|_0^{\pi/2}$$

$$= \frac{2\pi}{R}\left[\mathrm{e}^{-R} - 1\right] \xrightarrow[R\to\infty]{} 0.$$

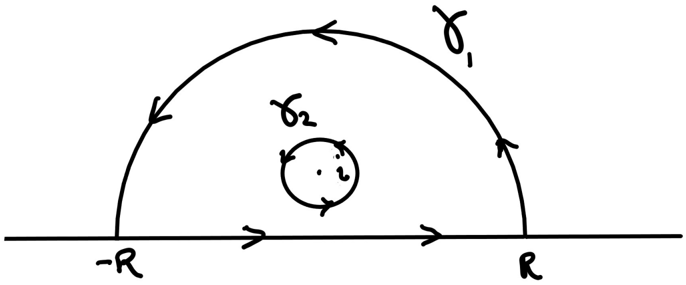

Figure 1: Contours

(ii) Given the zeroes of the denominator within γ_R we denote by γ_2 the circle centred at i of radius $1/2$, oriented counter-clockwise. By Cauchy's Theorem since the integrand is analytic in the region between the curves we have

$$\int_{\gamma_R} \frac{z\mathrm{e}^{\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z = \int_{\gamma} \frac{z\mathrm{e}^{\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z.$$

Also

$$\int_{\gamma} \frac{z\mathrm{e}^{\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z = \int_{-R}^{R} \frac{z\mathrm{e}^{\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z + \int_{\mathrm{arc}} \frac{z\mathrm{e}^{\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z.$$

(iii) Now for γ_2

$$\int_{\gamma_2} \frac{z\mathrm{e}^{\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z = \int_{\gamma_2} \frac{h(z)}{(z-\mathrm{i})}\mathrm{d}z = 2\pi\mathrm{i}h(\mathrm{i}),$$

for

$$h(z) = \frac{z\mathrm{e}^{\mathrm{i}z}}{z+\mathrm{i}},$$

and so

$$\int_{\gamma_2} \frac{z\mathrm{e}^{\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z = 2\pi\mathrm{i}\frac{\mathrm{i}\mathrm{e}^{-1}}{2\mathrm{i}} = \frac{\pi}{\mathrm{e}}\mathrm{i}.$$

(iv) Therefore

$$\int_{\infty}^{\infty} \frac{x\sin x}{1+x^2}\mathrm{d}x = \frac{1}{\mathrm{i}}\int_{\gamma_1} \frac{z\mathrm{e}^{\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z = \frac{1}{\mathrm{i}}\int_{\gamma_2} \frac{z\mathrm{e}^{\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z = \frac{\pi}{\mathrm{e}}.$$

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Model Solution No: 3

a) Let $f : [a, b] \to \mathbb{R}$ be an integrable function and define the function $F : [a, b] \to \mathbb{R}$ by

$$F(x) := \int_a^x f(t)\mathrm{d}t,$$

then F is continuous of $[a, b]$. Additionally if f is continuous at $c \in [a, b]$ then $F'(c) = f(c)$, with the derivatives at a and b understood as one-sided derivatives.

First, by the additivity Theorem we know that f is integrable on the interval $[0, x]$ and therefore F is well defined. Also since f is integrable we know that it is bounded, say $|f| \leq M$, and therefore

$$F(x+h)-F(x) = \int_a^{x+h} f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t = \int_a^x f(t)\mathrm{d}t + \int_x^{x+h} f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t$$
$$= \int_x^{x+h} f(t)\mathrm{d}t,$$

from which $|F(x + h) - F(x)| \leq M|h|$, proving that F is (Lipschitz) continuous. Also, from that equality we deduce

$$\frac{F(x + h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t.$$

The result will follow if we show that whenever f is continuous at x (in addition to integrable) we have

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t = f(x).$$

Notice that since we are integrating with respect to t, $f(x)$ is a constant, and therefore

$$\frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t - f(x) = \frac{1}{h}\int_x^{x+h} (f(t) - f(x))\mathrm{d}t.$$

It suffices to show that with the hypotheses above on f

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} (f(t) - f(x))\mathrm{d}t = 0.$$

Now, since f is continuous at x, given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Therefore, for $|h| < \delta$ we have

$$\lim_{h\to 0} \left|\frac{1}{h}\int_x^{x+h} (f(t) - f(x))\mathrm{d}t\right| \leq \lim_{h\to 0} \left|\frac{1}{h}\int_x^{x+h} |(f(t) - f(x))|\mathrm{d}t\right| \leq \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} \varepsilon\mathrm{d}t = \varepsilon,$$

which implies, taking limits as ε tends to zero, the desired result.

b) F is differentiable since it is the composition of differentiable functions, as the limits of integrations are smooth in the domain of definition of F, and $\sin(t^2)$ is smooth. By the chain rule, and the FTC from the previous section

$$F'(x) = \cos(x)\frac{1}{2\sqrt{x}} - \cos(1/x)\frac{-1}{2x^{3/2}}.$$

c) If we define the vector field $\underline{f} = (u, -v)$, we have just shown that

$$\int_\gamma f \mathrm{d}z = \mathrm{circulation}(\underline{f}) + \mathrm{i}\,\mathrm{flux}(\underline{f})$$

d) Let $\Omega \subset \mathbb{C}$ be an open set and $z_0 \in \Omega$. We say that f is complex differentiable at z if and only if the limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists. We denote the limit by $f'(z)$.

We have the following Theorem:

Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ with Ω open. f is complex differentiable of $z = a + \mathrm{i}b \in \Omega$ if and only if f, when considered as map from $\Omega \subset \mathbb{R}^2$ to $\mathbb{R}^2$ has a differential at the point (a, b) that satisfies the Cauchy–Riemann equation.

e) Notice that f_n looks like an approximation to the integral of f between x and $x+1$. We will show that $f_n \rightrightarrows h$, where

$$h(x) = \int_x^{x+1} f(y)\mathrm{d}y$$

We have

$$f_n(x) - h(x) = \frac{1}{n}\sum_{k=0}^{n-1} f(x + \frac{k}{n}) - \int_x^{x+1} f(y)\mathrm{d}y =$$

$$= \sum_{k=0}^{n-1} \int_{x+k/n}^{x+(k+1)/n} f(x + \frac{k}{n})\mathrm{d}y - \sum_{k=0}^{n-1} \int_{x+k/n}^{x+(k+1)/n} f(y)\mathrm{d}y$$

Therefore

$$|f_n(x) - h(x)| \leq \sum_{k=0}^{n-1} \int_{x+k/n}^{x+(k+1)/n} \left| f(x + \frac{k}{n}) - f(y) \right| \mathrm{d}y.$$

Since f is continuous on a compact set, it is uniformly continuous, and therefore the integrand can be made less than any arbitraty $\varepsilon > 0$ by taking n large enough (hence reducing the length of the interval of the domain of integration). Hence we obtain

$$|f_n(x) - h(x)| \leq \varepsilon,$$

as the total length of all intervals in the sum above is 1.

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Model Solution No: 4

a) Consider $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = x$ and $g_n(x) = 1/n$. Trivially f_n converges uniformly to $f(x) = x$ with g_n converging uniformly go $g = 0$. Notice that $f_n g_n = x/n$ which converges pointwise to 0. However it does not converge uniformly, as $f_n(n)g_n(n) = 1$.

b) Notice that $|f_n|, |g_n| < M$ we have

$$|f_n g_n - fg| \leq |f_n g_n - f_n g||f_n g - fg| \leq |f_n||g_n - g| + |g||f_n - f| \leq M[|f_n - f| + |g_n - g|],$$

which does go to zero as n goes to infinity, proving the uniform convergence.

c) First we need to show that f is Riemann integrable, that is show that for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Now, since $f_n \rightrightarrows f$ we know that for any $\varepsilon > 0$ there exists N such that $\|f_n - f\|_\infty < \varepsilon/(4(b-a))$ for $n > N$. For a fixed $n > N$ since f_n is integrable we know that given $\varepsilon > 0$ there exists a partition P such that

$$U(f_n, P) - L(f_n, P) < \frac{\varepsilon}{2}.$$

Now, for that P

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum [\sup_{I_k} f - \inf_{I_k} f]|I_k| = \sum [\sup_{I_k}(f - f_n + f_n) - \inf_{I_k}(f - f_n + f_n)]|I_k| \\
&\leq \sum \left[\|f - f_n\|_\infty + \sup_{I_k} f_n + \|f - f_n\|_\infty - \inf_{I_k} f_n \right] |I_k| \\
&= 2 \sum \|f - f_n\|_\infty |I_k| + \sum [\sup_{I_k} f_n - \inf_{I_k} f_n]|I_k| \\
&\leq 2\|f - f_n\|_\infty (b-a) + U(f_n, P) - L(f_n, P) \\
&\leq 2 \frac{\varepsilon}{4(b-a)}(b-a) + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

To see that $\int f_n \to \int f$, notice that

$$\left| \int_a^b f_n - \int_a^b f \right| \leq \left| \int_a^b f_n - f \right| \leq \int_a^b |f_n - f| \leq \int_a^b \|f - f_n\|_\infty = \|f_n - f\|_\infty (b-a).$$

Clearly the right hand side goes to zero as n goes to infinity by the uniform convergence of (f_n) to f.

d) No. A counterexample is given by a sequence of moving intervals of decreasing length that are supported on intervals of decreasing length that move along the an interval. Consider functions defined on $[0,1]$. Let

$$f_0(x) = \chi_{[0,1]}(x),$$

$$f_1(x) = \chi_{[0,1/2]}, \qquad f_2(x) = \chi_{[1/2,1]},$$

$$f_3(x) = \chi_{[0,1/4]}, \qquad f_4(x) = \chi_{[1/4,1/2]}, \qquad f_5(x) = \chi_{[1/2,3/4]}, \qquad f_6(x) = \chi_{[3/4,1]}.$$

Each function is an indicator of an interval, and that in each row above the intervals sweep $[0,1]$. When we move to the next row the length of the corresponding intervals gets divided by 2 and therefore we consider twice as many functions for each group. A student that explains this much will get full credit.

A formula for the sequence is given by

$$n \in [\sum_{l=0}^{k-1} 2^l, \sum_{l=0}^{k} 2^l], \qquad k = 1, 2, \ldots$$

we set f_n as the indicator of the interval

$$\left[\frac{(n - \sum_{l=0}^{k-1} 2^l)}{2^k}, \frac{n - \sum_{l=0}^{k-1} 2^l + 1}{2^k}\right].$$

e) No. A possible counter example is given by $f_n(x) = n\chi_{I_n}(x)$, where $I_n = (0, 1/n)$. The sequence converges to $f(x) = 0$, but $\int f_n = 1$ for every n.

In order to ensure convergence of the integrals in the statement we require uniform convergence for the sequence f_n. In that case

$$\left|\int_a^b f_n - \int f\right| \leq \int_a^b |f_n - f| \leq (b-a)\|f_n - f\|_\infty.$$

The right-hand side goes to zero as n goes to infinity by the uniform convergence and we obtain the result.

MA2440

MATHEMATICS DEPARTMENT
SECOND YEAR UNDERGRADUATE EXAMS – April 2020

Course Title: ANALYSIS III

Model Solution No: 5

a) S_n is a finite sum of integrable functions and therefore integrable (by additivity). Since S_n converges uniformly and therefore using Theorem X below, we have that S is integrable and moreover

$$\lim_{n\to\infty} \int S_n = \int \lim_{n\to\infty} S_n.$$

Since $\int S_n = \sum_{k=1}^{n} \int f_k$ and $\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} f_k$ we obtain the result.

Theorem X. Lef (g_n), $g_n : [a,b] \to \mathbb{R}$ be a sequence in Riemann integrable functions that converges uniformly to $g : [a,b] \to \mathbb{R}$. Then f is Riemann integrable and $\int g_n \to \int g$.

b) Given $(a_n)_{n=0}^{\infty}$ there exists $R \in [0,\infty]$, called the radius of convergence such that

$$\sum_{n=0}^{\infty} a_n z^n$$

converges for all $|z| < R$ and diverges for $|z| > R$. (As we will see in the proof $R = \frac{1}{\limsup |a_n|^{1/n}}$.)

c) We will use the following consequence ratio test, which states: Let $a_n \neq 0$ for all $n \geq N$, and assume that $\lim \frac{|a_{n+1}|}{|a_n|}$ exists. Then $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R = \lim \frac{|a_n|}{|a_{n+1}|}$.

In particular if $\lim \frac{|a_{n+1}|}{|a_n|}$ exists, and equals L we have convergence for $L < 1$ and divergence for $L > 1$. (The test is inconclusive if $L = 1$.)

For the first power series we have

$$\frac{|a|^{k+1}|z|^{k+1}}{|a|^k|z|^k} = |a||z|$$

which shows that the radius of convergence is $R = 1/|a|$

For the second we have

$$\frac{|z|^{(k+1)!}}{|z|^{k!}} = |z|^{k(k!)}$$

which shows the radius of convergence is 1. Finally for the last one

$$\frac{(k+1)!|z|^{k+1}}{k!|z|^k} = (k+1)|z|$$

which shows that the raidus of convergence is zero.

d) No, pointwise convergence does not necessarily preserve continuity. A possible example is given by

$$f_n(x) = \begin{cases} 0 & |x| > 1/n \\ -n(x - 1/n) & x \in [0, 1/n] \\ n(x + 1/n) & x \in [-1/n, 0) \end{cases}$$

The functions f_n are supported on $[-1/n, 1/n]$ and formed by two straight lines such that $f_n(0) = 1$ for every n. As such the limit is

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

which is not continuous.

e) No. Consider

$$f_n(x) = \left(x^2 + 1/n\right)^{1/2}.$$

They are clearly C^1 as the $x^2 + 1/n$ never vanishes for fixed n. (f_n) converges uniformly to $f(x) = |x|$ which is not smooth at the origin. To see this notice that if

$$A := \left(x^2 + 1/n\right)^{1/2} - |x|$$

then

$$A \leq \left((x + 1/\sqrt{n})^2\right)^{1/2} - |x| \leq \frac{1}{\sqrt{n}},$$

and the uniform convergence follows.

No, even if f is infinitely smooth we do not necessarily have convergence of f_n' to f. In fact the limit f_n' might not exist. For example

$$f_n(x) = \frac{1}{n}\sin(n^2 x).$$

The sequence converges uniformly to zero, which is a smooth function. However

$$f_n'(x) = n\cos(n^2 x)$$

does not converge to zero.