THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: SUMMER 2022

ALGEBRA II: GROUPS AND RINGS

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

Notation, conventions and terminology

- D_n is the *dihedral group* of the regular n–gon (of order $2n$).
- C_n is the *cyclic group* of order n.
- $\mathbb{Z}_n$ is the finite ring $\{0, 1, 2, \ldots, n-1\}$ with addition and multiplication modulo n.
- All rings have a multiplicative identity element 1, and all ring homomorphisms map 1 to 1.
- If R is a commutative ring and $a \in R$, then (a) denotes the principal ideal of R generated by a.
- A PID is a *principal ideal domain* and a UFD is a *unique factorisation domain.*

COMPULSORY QUESTION

1. a) Decide whether the following statements are true or false, and in each case give a proof or a counterexample, stating any standard results you use.

(i) The conjugacy class of $(1, 2, 3, 4, 5)(6, 7)$ in the symmetric group S_7 contains exactly 504 elements. **[4]**

(ii) There is a ring homomorphism $\phi\colon \mathbb{Z}[i] \to \mathbb{Z}$. **[4]**

(iii) Let R be a nonzero ring. If $f, g \in R[x]$ with $\deg(f) = 2$ and $\deg(g) = 3$, then $\deg(fg)$ must equal 5. **[2]**

(iv) If G is a group and H is a subgroup of G with $|G : H| = 2$, then $H \triangleleft G$. **[4]**

(v) All groups of order 6 are isomorphic. **[4]**

b) Answer the following questions. Justify your answers.

(i) It is known that the minimal polynomial of $\alpha = \sqrt{3 + \sqrt{5}}$ over $\mathbb{Q}$ has degree 4. Find the minimal polynomial of α over $\mathbb{Q}$. **[2]**

(ii) Find all the generators of the cyclic group C_{20}. **[4]**

(iii) Let C be the multiplicative group of nonzero complex numbers. Let $U = \{z \in \mathbb{C} \mid |z| = 1\}$ be a subgroup of C. Let R be the multiplicative group of positive real numbers. Show that $C/U \cong R$. **[4]**

c) Let R be a commutative ring, I a subset of R.

(i) Explain what it means for I to be a *maximal ideal* of R. **[4]**

(ii) If I and J are ideals of R, show that $I{+}J = \{i{+}j : i \in I, j \in J\}$ is an ideal of R. **[4]**

(iii) Show that if I is a maximal ideal of R, then for any $x \in R \setminus I$, the coset $I{+}x$ has a multiplicative inverse in R/I, and hence R/I is a field. **[4]**

OPTIONAL QUESTIONS

2. a) Let G be a finite group of order $n = p^a \cdot m$ where p is a prime, $m, a \in \mathbb{N}$ and $gcd(p, m) = 1$.

(i) Explain what a Sylow p-subgroup of G is. **[2]**

(ii) State all three Sylow's Theorems. **[4]**

(iii) Let P be a Sylow p-subgroup of G, and n_p be the number of distinct Sylow p-subgroups of G.

(1) For any $g \in G$, show that gPg^{-1} is a Sylow p-subgroup of G. **[3]**

(2) Show that $n_p = 1$ if and only if P is normal in G. **[3]**

b) Let G be a group of order $2022 = 2 \cdot 3 \cdot 337$.

(i) Find the possible numbers of subgroups of order 2 that G might have. **[3]**

(ii) Find the possible numbers of subgroups of order 3 that G might have. **[2]**

(iii) Prove that G contains a normal subgroup of order 337. **[2]**

(iv) Is it possible for G to be simple? Explain your answer. **[1]**

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3. a) Explain what a cyclic group is. **[1]**

b) Let G be the *quaternion group* Q_8 of order 8 generated by two elements a and b subject to the defining relations $a^4 = 1$, $b^2 = a^2$ and $ba = a^{-1}b$.

(i) Show that Q_8 is not cyclic. **[2]**

(ii) Let $H = \{1, a^2, b, a^2b\}$. Show that H is a subgroup of G. **[5]**

(iii) Show that $Q_8 \ncong D_4$. **[2]**

(iv) Is H a normal subgroup of G? Justify your answer. **[2]**

c) Let $\phi \colon G \to H$ be a group homomorphism.

(i) Show that $\ker(\phi)$ is a normal subgroup of G. **[6]**

(ii) Is it true that $\mathrm{im}(\phi)$ is a normal subgroup of H? Explain your answer. You may assume that $\mathrm{im}(\phi)$ is a subgroup of H. **[2]**

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4. a) Let R be a domain.

(i) Explain what it means for R to be a PID. **[1]**

(ii) Explain what an irreducible element of R is. **[2]**

(iii) Explain what a prime element of R is. **[2]**

(iv) If R is a PID, show that every irreducible element of R is prime. You may use results about PIDs proven in the course, but you need to state them carefully. **[8]**

b) Let $R = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$. Is R a PID? Explain your answer. (Hint: consider $(1 + \sqrt{-3})(1 - \sqrt{-3}) = 4 = 2 \cdot 2$.) **[7]**

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5. a) (i) State Eisenstein's Criterion. **[4]**

(ii) Let $f(x, y) = y^3 + 2xy^2 - 2y^2 - x^2y + 2xy - y + x - 1 \in \mathbb{C}[x, y]$. By rewriting f as an element of $(\mathbb{C}[x])[y]$, use Eisenstein's Criterion to prove that f is irreducible in $\mathbb{C}[x, y]$. **[6]**

b) Let $F = \mathbb{Z}_2[x]/(x^2+x+1)$. Show that F is a field of order 4. **[5]**

c) Let $K = \mathbb{Z}_3[x]/(x^2+x+1)$. Show that K is a ring of order 9, but not a field. **[5]**
