THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: SUMMER 2021

ALGEBRA II: GROUPS AND RINGS

Time Allowed: **2 hours**

Read all instructions carefully. Please note also the guidance you have received in advance on the departmental 'Warwick Mathematics Exams 2021' webpage.

Calculators, wikipedia and interactive internet resources are not needed and are not permitted in this examination. You are not allowed to confer with other people. You may use module materials and resources from the module webpage.

ANSWER COMPULSORY QUESTION 1 AND TWO FURTHER QUESTIONS out of the three optional questions 2, 3 and 4.

On completion of the assessment, you must upload your answer to Moodle as a single PDF document if possible, although multiple files (2 or 3) are permitted. You have an additional 45 minutes to make the upload, and instructions are available on the departmental 'Warwick Mathematics Exams 2021' webpage.

You must not upload answers to more than 3 questions, including Question 1. If you do, you will only be given credit for your Question 1 and the first two other answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question. The compulsory question is worth 40 marks, while each optional question is worth 30 marks.

You may use any results from the lecture notes, assignments, tutorial sheets or supplementary example sheets without proving them, provided you make clear which results you are using and how you are using them.

Notation. D_n denotes the dihedral group of the regular n-gon (with order $2n$).
All rings have a multiplicative identity element.
If R is a commutative ring and $a \in R$, then (a) denotes the principal ideal of R generated by a.

COMPULSORY QUESTION

1. a) Decide whether the following statements are true or false, and in each case give a brief justification or counterexample. You will get no marks unless you give either a justification or counterexample.

(i) The dihedral group D_9 contains an element of order 6. **[4]**

(ii) The conjugacy class of $(1,2)(3,4,5,6,7,8)$ in the symmetric group S_8 contains exactly $28 \times 120 = 3360$ elements. **[4]**

(iii) There exists a ring homomorphism $\phi\colon \mathbb{C}[x] \to \mathbb{C}[x]$ such that the kernel of ϕ is equal to the set of polynomials of degree 0 in $\mathbb{C}[x]$. **[4]**

(iv) If R and S are non-zero rings and $\phi\colon R \to S$ is a ring homomorphism, then for every unit $u \in R$, $\phi(u)$ is a unit in S. **[4]**

(v) $x^3 + x + 3$ is irreducible in $\mathbb{Z}_5[x]$. **[4]**

(vi) $x^4 + 30x^3 - 50x^2 + 40x - 60$ is irreducible in $\mathbb{Z}[x]$. **[4]**

b) Give an example of each of the following. You do not need to explain why your example is correct or give any proofs. You will get full marks for a correct example.

(i) An element of order 4 in the alternating group A_8. **[3]**

(ii) A generator of the principal ideal $\{f \in \mathbb{Q}[x] : f(5) = 0\}$ in the ring $\mathbb{Q}[x]$. **[3]**

c) Describe all the zero divisors and units in the ring $\mathbb{Z} \times \mathbb{Q}$. No justification is required. **[4]**

d) Let R be an integral domain. Prove that $R/(p)$ is an integral domain if and only if p is a prime element of R. **[6]**

OPTIONAL QUESTIONS

2. Let G be a group of order 44.

a) Using Sylow's theorem, or otherwise, prove that G contains a normal subgroup N which is isomorphic to the cyclic group C_{11}. **[4]**

b) Let a be a generator of the normal subgroup N defined in part (a).
Suppose that G contains an element b of order 4 and that $ba = a^{-1}b$.

(i) Prove that the right cosets N, Nb, Nb^2 and Nb^3 are pairwise distinct and that they are the only right cosets of N in G. Deduce that every element of G has the form a^ib^j for some integers i and j with $0 \leq i \leq 10$, $0 \leq j \leq 3$. **[5]**

(ii) Show that the order of ab^2 is 22. **[2]**

Question 2 continued overleaf

(iii) Calculate the centraliser and conjugacy class of the element b in G. **[4]**

(iv) Does G contain a subgroup isomorphic to D_{11}? Justify your answer. **[5]**

c) Suppose now that G does not contain any element of order 4.

(i) Let H be a Sylow 2-subgroup of G. Which standard group is H isomorphic to? No justification is required. **[2]**

(ii) Label the elements of H as $1, x, y, z$.
Prove that $xax^{-1}, yay^{-1}, zaz^{-1}$ are all in the set $\{a, a^{10}\}$, where a is again a generator of the normal subgroup N defined in (a).
Also prove that at least one of $xax^{-1}, yay^{-1}, zaz^{-1}$ is equal to a. **[8]**

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3. a) Let $D = D_6$, the dihedral group of order 12. Let $a \in D$ be a rotation of order 6 and let $b \in D$ be a reflection. Let $S = \{1, a^4, a^8, b, a^4b, a^8b\}$.

(i) Prove that S is a subgroup of D. **[5]**

(ii) Is S a normal subgroup of D? Justify your answer. **[3]**

b) Let $\phi\colon G \to H$ be a group homomorphism, let N be a normal subgroup of H and let

$$K = \{k \in G : \phi(k) \in N\}.$$

(i) Prove that K is a normal subgroup of G. **[4]**

(ii) Prove that G/K is isomorphic to a subgroup of H/N. **[4]**

c) Again let $D = D_6$. The functions $\lambda_a\colon D \to D$ defined by $\lambda_a(g) = ag$ and $\lambda_b(g) = bg$ are permutations of D. Write the permutations λ_a and λ_b in cycle notation. (No justification is required.) **[6]**

d) Let G be a group of order $2n$, where n is an odd number. Such a group G contains an element x of order 2. (You do not need to prove this.)

(i) The left regular action of G gives rise to a homomorphism $\phi\colon G \to \mathrm{Sym}(G)$. Show that $\phi(x)$ is a product of n transpositions. **[4]**

(ii) By applying (b) to $N = \mathrm{Alt}(G)$, deduce that G contains a subgroup of index 2. **[4]**

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4. Let S denote the subset of $\mathbb{Z}[x]$ consisting of polynomials $a_d x^d + \cdots + a_1 x + a_0$ where a_0 is any integer and $a_1, a_2, \ldots, a_d$ are multiples of 2:

$$S = \{a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x] : a_0 \in \mathbb{Z} \text{ and } a_1, a_2, \ldots, a_d \in 2\mathbb{Z}\}.$$

The set S forms a ring under the usual operations of addition and multiplication of polynomials. (You may use this fact without proving it.)

a) Prove that S is an integral domain. **[2]**

b) What are the units in S? Justify your answer. **[3]**

c) Prove that an integer $n \in \mathbb{Z}$ is irreducible as an element of S if and only if it is irreducible as an element of $\mathbb{Z}$. **[3]**

d) For every integer $i \geq 0$, prove that $2x^i$ is an irreducible element of S. **[6]**

e) By considering factorisations of $4x^4$, or otherwise, prove that S is not a unique factorisation domain. **[3]**

f) Prove that the ideal $(2x) + (2x^2)$ of S is not principal. **[6]**

g) Do $2x$ and $2x^2$ have a least common multiple in S? Justify your answer. **[7]**
