THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: SUMMER 2018

Algebra - II: Groups and Rings

Time Allowed: 2 hours

Read carefully the instructions on the answer book and make sure that the particulars required are entered.

Calculators are not needed and are not permitted in this exam.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers. Each question is worth 25 marks.

The numbers in the margin indicate **approximately** how many marks are available for each part of a question.

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1. a) Let R be a ring. Explain what it means for a pair of ideals of R to be comaximal. **[2]**

b) Are the ideals (27) and (10) of $\mathbb{Z}$ comaximal? Justify your answer. **[3]**

c) State and prove the Chinese Remainder Theorem in the abstract form: for a ring R and a finite collection of ideals $I_1, I_2, \ldots I_n$. **[9]**

d) Let A be a commutative ring, I, J comaximal ideals in A.

(i) Define the power I^n of the ideal I and the product ideal IJ. **[2]**

(ii) Prove that I^n and J^m are comaximal for all positive integers n and m. **[5]**

(iii) Prove that $I \cap J = IJ$. **[4]**

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2. There exists a group G of order 20, generated by two elements x and y such that $x^4 = y^5 = 1$ and $xyx^{-1} = y^2$. In this problem you will need to answer questions about this group G.

a) Prove that every element of G can be uniquely written as $y^b x^a$ with $a \in \mathbb{Z}_4, b \in \mathbb{Z}_5$. **[4]**

b) Compute the multiplication table of G. (*Hint: dont fill a* 20×20 *table, do a* 4×4 *table instead with* $y^b, y^b x, y^b x^2, y^b x^3$.) **[5]**

c) Compute the conjugation table for G: keep the same columns as for the multiplication table and use two rows for conjugations by x and y. **[5]**

d) Compute the conjugacy classes, the centraliser of an element in each conjugacy class, and the centre of G. Justify your answer. **[6]**

e) Is G isomorphic to D_{20}? Justify your answer. **[5]**

3. a) Define the order of an element in a group. [3]

b) We consider the additive group of the rational numbers $\mathbb{Q}^+$. The integers $\mathbb{Z}$ form a normal subgroup. Prove that the quotient group $G = \mathbb{Q}^+/\mathbb{Z}$ is an infinite group. [4]

c) Consider $x = \frac{17}{6} + \mathbb{Z} \in G$. Compute the order of x, explaining what you are doing. [3]

d) Consider $y = \frac{a}{b} + \mathbb{Z} \in G$ where a, b are some integers, $b \neq 0$. Compute the order of y (in terms of a an b), explaining what you are doing. [5]

From what you have already proved it follows that G is an infinite group, every element of which has a finite order.

e) Prove that any finite number of elements of G are contained in a cyclic subgroup. [5]

f) Prove that any subgroup of G generated by finitely many elements is cyclic. [5]

4. a) Explain what is meant by a domain, a principal ideal domain (PID) and a unique factorisation domain (UFD). [4]

b) Let $A = \mathbb{C}[z^6, z^9, z^{20}]$ be the McNuggets ring.

(i) Is A a domain? Justify your answer. [2]

(ii) Is A a unique factorisation domain (UFD)? Justify your answer. [2]

(iii) Is A a principal ideal domain (PID)? Justify your answer. [2]

c) State and prove Eisenstein's Criterion for the irreducibility of a polynomial. [8]

d) Is the polynomial $f(x,y) = x^2y^4 + 6x^3y^3 - 12$ irreducible in $\mathbb{Z}[x,y]$? Justify your answer. [3]

e) Is the polynomial $h(x,y) = xy^4 + x^3y^3 - 12$ irreducible in $\mathbb{Q}[x,y]$? Justify your answer. [4]

5. Compute the following. You will receive the full mark for a correct answer without any explanation. However, if your answer is incorrect, a partial credit will be given only if a satisfactory explanation is given so that I can see that it is just a computational mistake.

a) The smallest natural number that is 6 modulo 7, 8 modulo 11 and 5 modulo 13. [5]

b) The maximal possible order of an element in the symmetric group S_7. [5]

c) The number of distinct group homomorphisms from the cyclic group C_2 to the dihedral group D_{2018} of symmetries of a regular 1009-gon. [5]

d) The number of elements in the ring $M_2\Big(\mathbb{Z}[i]/(1+3i)\Big)$. [5]

e) The order of the group $SL_2(\mathbb{F}_9)$ where $\mathbb{F}_9$ is the field of 9 elements. [5]