MA 249

THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: June 2019

ALGEBRA II: GROUPS AND RINGS

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. Let G be a group, K a subgroup of G and N a normal subgroup of G. In what happens you may use the fact that the kernel of a group homomorphism is a normal subgroup.

a) Explain what it means for N to be a normal subgroup of G. [2]

b) Consider a map $\pi : G \to G/N$ defined by $\pi(g) = Ng$.

(i) Show that π is a group homomorphism. [1]

(ii) Find the kernel of π. [1]

c) State the First Isomorphism Theorem. [3]

d) Show that NK is a subgroup of G. [4]

e) Show that $K \cap N \lhd K$ and $NK/N \cong K/K \cap N$. [6]

f) Let $G = S_4$, $N = \{1, (12)(34), (13)(24), (14)(23)\}$ and $K = \{1, (123), (321), (12), (13), (23)\}$.

(i) Show that $N \lhd G$. [4]

(ii) Explain why $K \leq G$. [1]

(iii) Prove that $G = NK$. [3]

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2. Let G be a finite group, $g \in G$, X a set and $x \in X$.

a) Explain what it means for G to act on X. [3]

b) Prove that if G acts on X non-trivially, then there exists a non-trivial homomorphism
$\phi : G \to Sym(X)$ with $\phi(g)(x) = g \cdot x$. [6]

c) Let G be a finite group, $|G| = p^n \cdot m$ where p is a prime, $n \geq 1$ and $hcf(p, m) = 1$.

(i) Explain what it means for $P \subseteq G$ to be a *Sylow* p-subgroup of G. [2]

(ii) State all three parts of Sylow's Theorem. [3]

d) Explain what it means for a group to be *simple*. [1]

e) Show that there are no simple groups of order 36. [10]

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3. Let R be a domain and $r \in R \setminus \{0\}$.

a) Explain what it means for r to be *irreducible*. [2]

b) Explain what it means for r to be *prime*. [2]

c) Show that in a domain any prime is irreducible. [5]

d) Explain what it means for R to be a PID (principal ideal domain). [2]

e) Prove that in a PID any irreducible element is prime (you may use the existence of gcd in a PID and the results about it that you need as long as you state them clearly). [7]

f) Consider a domain $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.
Show that R is not a PID.
(Hint: you may use the fact that $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$.) [7]

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4. Let R be a domain, $a, b \in R$, x an independent variable.

a) Explain what it means for $u \in R$ to be a *unit*. [1]

b) Explain what it means for a and b be *associate* in R. [1]

c) Explain what it means for R to be a UFD, *unique factorization domain.* [3]

d) Let R be a UFD.

(i) Explain what it means for $f \in R[x]$ to be *primitive.* [1]

(ii) Show that the product of two primitive polynomials is primitive. [6]

e) State Gauss' Lemma. [2]

f) Let $f = a_0 + a_1x + ... + a_nx^n$ be a primitive polynomial in $\mathbb{Z}[x]$. Suppose there is a prime $p \in \mathbb{Z}$ such that $p \not| a_n$, $p \mid a_i$ for $0 \leq i < n$ and $p^2 \not| a_0$. Show that f is irreducible in $\mathbb{Z}[x]$. [7]

g) Decide whether polynomial $3 + 12x - 27x^4 + 5x^5$ is irreducible in $\mathbb{Q}[x]$. [4]

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5. Determine whether the following statements are true or false. In each case give a brief justification or counterexample. (You will get no marks unless you give either a justification or a counterexample.)

a) If H is a subgroup of G, then $gH = Hg$ for all $g \in G$. [3]

b) Let G and H be two groups. If $\phi : G \to H$ is a homomorphism, then $Im(\phi) \triangleleft H$. [4]

c) Group S_7 has exactly 7 conjugacy classes. [3]

d) Let G be a group and H a non-empty subset of G. If $ab^{-1} \in H$ for all $a, b \in H$, then H is a subgroup of G. [4]

e) $\mathbb{Z}[x]$ is a PID. [3]

f) $\sin(\frac{3\pi}{4})$ is an algebraic number. [3]

g) If R and S are two non-zero rings, then the map $f : R \to S$ such that $f(r) = 0_S$ for $r \in R$, is a ring homomorphism. [2]

h) If R is a domain, then u is a unit in R if and only if u is a unit in $R[x]$. [3]

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Comments for the external can be placed here.