THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: APRIL 2019

ALGEBRA I: Advanced Linear Algebra

Time Allowed: 2 hours

Read carefully the instructions on the answer book and make sure that the particulars required are entered on it.

Calculators are not needed and are not permitted in this exam.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of the question.

Each question is worth 25 marks.

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1. Let A be an $n \times n$ matrix over a field K.

Prove, without using the Cayley-Hamilton Theorem, that there is a non-zero polynomial $p \in K[x]$ such that $p(A)$ is the zero matrix. **[5]**

Define the *minimal polynomial* μ_A of A, and prove that μ_A divides any polynomial $f \in K[x]$ for which $f(A)$ is the zero matrix. **[8]**

Find the minimal polynomial of the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ with $K = \mathbb{R}$. **[4]**

Write down the possible minimal polynomials of a matrix over $\mathbb{R}$ with characteristic polynomial $(1 + x)(1 + x^3)$, and for each possibility write down a specific example of a matrix having that minimal polynomial (no proofs required). **[8]**

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2. What does it mean to say that an $n \times n$ matrix over the complex numbers is in *Jordan Canonical Form*? **[5]**

State (but do not prove) the main theorem concerning $n \times n$ matrices over the complex numbers and the Jordan Canonical Form. **[3]**

Let J be the Jordan Canonical Form of the matrix A and let λ be an eigenvalue of A. Prove that the number of Jordan blocks of J with eigenvalue λ is equal to the nullity of $A - \lambda I_n$. **[8]**

Let

$$A = \begin{pmatrix} 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & -2 & -1 \end{pmatrix}$$

The characteristic polynomial of A is $x^2(x+1)^2$ (you need not prove that). Calculate the Jordan Canonical Form of A. **[9]**

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3. (a) Let V be a vector space over a field K. Define a *symmetric bilinear form* $\tau : V \times V \to K$ and its associated quadratic form $q : V \to K$. **[3]**

Show how τ gives rise to a symmetric matrix A over K with respect to a given basis of V. **[3]**

Explain briefly why, if the basis of V is changed using a matrix P, then the matrix A is replaced by $P^{\mathrm{T}}AP$. **[4]**

(b) Two $n \times n$ symmetric matrices A and B over K are said to be *congruent* if there is an $n \times n$ invertible matrix P with $B = P^{\mathrm{T}}AP$. Prove that congruence is an equivalence relation. **[3]**

How many equivalence classes are there when

(i) $K = \mathbb{C}$ and $n = 5$; (ii) $K = \mathbb{R}$ and $n = 2$. **[3,4]**

(c) The following two matrices have characteristic polynomials $-x^3 + 3x^2 + 6x$ and $-x^3 + 4x^2 - 3x$. Determine whether they are congruent, firstly when $K = \mathbb{C}$, and secondly when $K = \mathbb{R}$.

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & -1 \\ -2 & -1 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

[5]

Continued

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4. What does it mean to say that an $n \times n$ matrix over $\mathbb{R}$ is *orthogonal*? **[2]**

Prove that the determinant of an orthogonal matrix is equal to 1 or -1. **[5]**

Let A be $n \times n$ symmetric matrix over $\mathbb{R}$. Prove that, if λ_1 and λ_2 are distinct eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. **[8]**

Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & -2 \\ -2 & -2 & 0 \end{pmatrix},$$

and find an orthogonal matrix P such that $P^{-1}AP$ is diagonal. **[2,8]**

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5. (a) State (but do not prove) the fundamental theorem of finitely generated abelian groups. **[5]**

(b) Let G be the abelian group defined by the presentation

$$\langle\, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \mid \begin{array}{rrr} 4\mathbf{x}_1 & +4\mathbf{x}_2 & +8\mathbf{x}_3, \\ 7\mathbf{x}_1 & +8\mathbf{x}_2 & +3\mathbf{x}_3, \\ 10\mathbf{x}_1 & +12\mathbf{x}_2 & -14\mathbf{x}_3 \end{array} \,\rangle.$$

Explain carefully what this notation means. **[4]**

Find integers $\alpha_i > 1$ such that

$$G \cong \mathbb{Z}_{\alpha_1} \oplus \mathbb{Z}_{\alpha_2} \oplus \cdots \oplus \mathbb{Z}_{\alpha_r},$$

where $\alpha_i | \alpha_{i+1}$ for $1 \leq i < r$. **[10]**

Write down a presentation of an abelian group H that has the same order as G but is not isomorphic to G, and prove that H and G are not isomorphic. **[3,3]**

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