THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: APRIL 2020

ALGEBRA I: Advanced Linear Algebra

Time Allowed: **3 hours**

Read all instructions carefully. Please note also the guidance you have received in advance on the departmental 'Warwick Mathematics Exams 2020' webpage.

Calculators, wikipedia and interactive internet resources are not needed and are not permitted in this examination. You may use module materials and resources from the module webpage.

ANSWER 3 QUESTIONS.

On completion of the assessment, you must upload your answer to the AEP as a single PDF document. You have an additional 45 minutes to make the upload, and instructions are available on the departmental 'Warwick Mathematics Exams 2020' webpage.

You must not upload answers to more than 3 questions. If you do, you will only be given credit for your first 3 answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question. Note that the marks do not sum to 100.

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1. Let V be a vector space of finite dimension n over a field K and let $T : V \to V$ be a linear map.

Define the *eigenvalues*, the *eigenspaces*, and the *generalised eigenspaces* of T. **[2,2,2]**

Let λ be an eigenvalue of T, and suppose that the associated eigenspace W has dimension r. Prove that the characteristic polynomial $c_A(x)$ of an $n \times n$-matrix A representing T is divisible by $(\lambda - x)^r$, and deduce that r is less than or equal to the multiplicity of λ as an eigenvalue of A. **[8,1]**

Give two examples of linear maps T with an eigenvalue λ such that:

- in the first example, $r = 2$, $n = 3$, and r is equal to the multiplicity of λ as an eigenvalue of A;

- and in the second example, $r = 2$, $n = 3$, and r is less than the multiplicity of λ as an eigenvalue of A.

[3,3]

In both of your examples calculate the dimensions of all generalised eigenspaces of T associated to λ. **[2,2]**

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2. Once again, let V be a vector space of finite dimension n over a field K and let $T : V \to V$ be a linear map.

Define the *minimal polynomial* of T. **[2]**

Suppose that $T^k = 0_V$ (the zero map on V) for some $k > 0$. What can you say about the minimal polynomial of T? **[2]**

Prove that 0 is an eigenvalue of T, and that T has no other eigenvalues. **[3,4]**

Suppose now that $K = \mathbb{C}$ and that T is defined by the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \end{pmatrix}.$$

Find the least k such that $A^k = 0$, and write down the minimal polynomial μ_A and the characteristic polynomial c_A of A. **[3,1,2]**

What are the possible Jordan Canonical Forms of a matrix B with with $\mu_B = \mu_A$ and $c_B = c_A$? **[4]**

Find the Jordan Canonical Form of A. **[4]**

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3. Let V and W be vector spaces over a field K. Define a bilinear map $\tau : W \times V \to K$. **[4]**

Let τ be a bilinear map. Assuming that V and W are finite dimensional spaces, describe how τ can be represented by a matrix over K. **[3]**

If U is a subspace of V, then define

$$U^\perp = \{\mathbf{w} \in W \mid \tau(\mathbf{w}, \mathbf{u}) = \mathbf{0} \ \ \forall \mathbf{u} \in U\}.$$

Prove that $U^\perp$ is a subspace of W. **[3]**

Similarly, if X is a subspace of W, then we define

$$X^\perp = \{\mathbf{v} \in V \mid \tau(\mathbf{x}, \mathbf{v}) = \mathbf{0} \ \ \forall \mathbf{x} \in X\}.$$

Prove that, for all subspaces U, U_1, U_2 of V, we have:

Question 3 continued overleaf

(i) $U \subseteq (U^{\perp})^{\perp}$, **[3]**

(ii) $U_1 \subseteq U_2 \Rightarrow U_2^{\perp} \subseteq U_1^{\perp}$, **[3]**

(iii) $U^{\perp} = ((U^{\perp})^{\perp})^{\perp}$. **[5]**

Find a basis for $U^{\perp}$ when $K = \mathbb{R}$, $V = W = \mathbb{R}^2$, τ is represented (using the standard basis of $\mathbb{R}^2$) by the matrix $\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$, and U is 1-dimensional with basis $(2, -1)$. **[4]**

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4. (a) What does it mean to say that an $n \times n$ matrix over $\mathbb{R}$ is *orthogonal*? **[2]**

(b) Let A be $n \times n$ symmetric matrix over $\mathbb{R}$. Prove that, if λ_1 and λ_2 are distinct eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. **[6]**

(c) Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix},$$

and find an orthogonal matrix P such that $P^{-1}AP$ is diagonal. **[12]**

(d) Classify the following curve and surface, respectively, in geometric terms (such as ellipse, parabola, etc.):
(i) $x^2 + y^2 - 2xy + y - 1 = 0$ (in 2-dimensional x, y space);
(ii) $x^2 + y^2 + z^2 - 2xz + 2yz - 4x + 4z + 5 = 0$ (in 3-dimensional x, y, z space). **[5]**

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5. (a) Calculate the Smith Normal Form of the following matrices.

$$\text{(i)} \quad \begin{pmatrix} 6 & 6 & 0 \\ 9 & -12 & 9 \end{pmatrix} \qquad \text{(ii)} \quad \begin{pmatrix} 45 & 5 & -50 & -5 \\ 10 & -20 & 0 & 10 \\ -50 & -10 & 60 & 10 \\ -10 & 10 & 0 & -10 \end{pmatrix}.$$

[6]

(b) State (but do not prove) the fundamental theorem of finitely generated abelian groups. **[5]**

(c) Let G be a group and $g \in G$ an element of order 2. If $\varphi \colon G \to H$ is an isomorphism, prove that $\varphi(g) \in H$ has order 2. **[4]**

(d) Prove that no two of $\mathbb{Z}_{16}$, $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$ are isomorphic. **[6]**

(e) Write down three finitely generated abelian groups of order 24, no two of which are isomorphic. (You do not need to prove that your examples are pairwise non-isomorphic.) **[4]**
