MA 2510/2517

THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: APRIL 2018

ALGEBRA 1: ADVANCED LINEAR ALGEBRA

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. In this question the base field is $\mathbb{C}$. Put

$$A = \begin{pmatrix} 2 & 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & -2 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ -1 & -1 & -3 & -1 & 0 \end{pmatrix}.$$

a) Define Jordan block. **[5]**

b) Define Jordan Canonical Form of a matrix B. **[3]**

c) Find a Jordan Canonical Form J for A. Find invertible P such that $A = PJP^{-1}$. Before you find any vector, state a necessary and sufficient condition on the vector. **[11]**

d) Compute the entry of $(2I_5 + A)^5$ in position $(3,3)$. **[6]**

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2. Let K be a base field and $A \in K^{n,n}$.

a) For a polynomial $p \in K[x]$ define $p(A)$. [3]

b) Define the minimal polynomial μ_A of A. [4]

c) Prove that the minimal polynomial of A exists and is unique. [8]

d) Let A^T denote the transpose of A. Prove that A and A^T have the same minimal polynomial. State any results from the lectures that you use. [6]

e) Prove that every Jordan block is similar to its transpose. [4]

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3. In this question the base field is $\mathbb{R}$. Let q be a quadratic form on a finite-dimensional vector space V.

a) Define *bilinear form* on V. [3]

b) For $x, y \in V$ define $B(x,y) = q(x+y) - q(x) - q(y)$. Prove that B is bilinear. [4]

c) Prove Sylvester's inertia theorem, which states the following. Let $(x_i)_i$ and $(y_i)_i$ be two cobases (= bases of V^*) and suppose [8]

$$q = \sum_{i=1}^{t} x_i^2 - \sum_{i=t+1}^{u} x_i^2 = \sum_{i=1}^{t'} y_i^2 - \sum_{i=t'+1}^{u'} y_i^2.$$

Then $(t,u) = (t',u')$.

Hint: Assume $t > t'$ and prove $V_1 \cap V_2 \neq 0$ where we write

$$V_1 = \{v \in V \mid x_{t+1}(v) = \cdots = x_n(v) = 0\},$$
$$V_2 = \{v \in V \mid y_1(v) = \cdots = y_{t'}(v) = 0\}.$$

d) Define rank and signature of q. [2]

e) Find rank and signature of the symmetric real matrix [8]

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 10 & -6 \\ 0 & -6 & 2 \end{pmatrix}.$$

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4. In this question the base field is $\mathbb{C}$. Let V be a vector space with basis $E = (e_1, \ldots, e_n)$. Let $B: V \times V \to \mathbb{C}$ be a Hermitian form. Let A be the matrix of B with respect to E.

   a) Define *Hermitian form* $B: V \times V \to \mathbb{C}$. Don't use the concept *sesquilinear* unless you define it. **[5]**

   b) Define *Hermitian matrix*. **[2]**

   c) Define A (the matrix of B with respect to the basis E). **[1]**

   d) Prove that A is Hermitian. **[1]**

   e) Prove that the complex eigenvalues of A are real. **[10]**

   f) Let $P, Q \in \mathbb{C}^{n,n}$. Carefully prove $\overline{PQ} = \overline{P}\,\overline{Q}$. Clearly state what properties of complex numbers you use and where you use them. **[6]**

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5. a) Without proof give the abelian groups of order 16, up to isomorphism. **[4]**

   b) Let G be an abelian group. What does it mean to say that G is generated by $x_1, \ldots, x_n$? **[2]**

   c) Let G be an abelian group generated by $x_1, \ldots, x_n$. Let H be the subgroup of G generated by $x_1, \ldots, x_{n-1}$. Let K be any subgroup of G. Prove that K is generated by $K \cap H$ and (at most) one more element. **[8]**

   Hint: One of the cases to consider is where there are $h \in H$ and $t \in \mathbb{Z}_{>0}$ such that $h + tx_n \in K$ with t minimal.

   d) Define *Integral Smith Normal Matrix* ($\mathbb{Z}$SNM), that is, a matrix that is its own Integral Smith Normal Form ($\mathbb{Z}$SNF). **[4]**

   e) Put **[7]**

$$A = \begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}.$$

   Find an explicit $\mathbb{Z}$SNM $B \in \mathbb{Z}^{2,2}$ and unimodular $P, Q \in \mathbb{Z}^{2,2}$ such that $B = PAQ$.
