MA 2510

THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: APRIL 2022

ALGEBRA 1: ADVANCED LINEAR ALGEBRA

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER ALL 4 QUESTIONS.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. a) State the Cayley-Hamilton Theorem. **[2]**

b) Let $M \in K^{n,n}$ and $\mu_M \in K[x]$ be its minimal polynomial. Prove that if $p \in K[x]$ with $p(M) = 0$ then μ_M divides p. **[4]**

c) Let $A \in \mathbb{C}^{3,3}$ be defined as follows

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

(i) Find the characteristic polynomial of A. **[2]**

(ii) Find the minimal polynomial of A, without calculating the Jordan canonical form of A. Provide justification for your calculations. **[4]**

(iii) Using the previous part, write down the Jordan canonical form of A. **[2]**

d) Let u, v, w be functions of t and suppose that

$$\begin{aligned} \frac{du}{dt} &= 3u + v \\ \frac{dv}{dt} &= 3v + w, \\ \frac{dw}{dt} &= 3w. \end{aligned}$$

By finding e^{tB} for an appropriate matrix B, solve the set of differential equations with initial conditions $u(0) = 1, v(0) = 0, w(0) = 2$. **[5]**

e) Find a QR-decomposition of the following invertible matrix

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

[8]

f) Let $V = \mathbb{C}^3$, $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V$ and let q be a quadratic form on V given by

$$q(v) = 2xy + 2xz + 2y^2 + z^2.$$

(i) Find a basis of V such that the corresponding matrix of q is diagonal. **[6]**

(ii) What is the rank of q? **[1]**

g) (i) Prove that if $v_1, \ldots, v_m$ are linearly independent in $\mathbb{Z}^n$ then $v_1, \ldots, v_m$ are linearly independent when considered as a sequence of vectors in $\mathbb{Q}^n$. **[4]**

(ii) Is $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ a free basis of $\mathbb{Z}^2$? **[1]**

(iii) Is $w_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ a free basis of $\mathbb{Z}^2$? **[1]**

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2. Recall that $A \in \mathbb{C}^{n,n}$ is called nilpotent if there exists an integer $r > 0$ such that $A^r = 0_{n,n}$ (the $n \times n$ matrix with all entries 0).

a) (i) Prove that A is nilpotent if and only if there exists an integer $0 < s \leq n$ such that $A^s = 0_{n,n}$. **[3]**

(ii) Prove that A is nilpotent if and only if 0 is the only eigenvalue of A. **[4]**

Let $B, C \in \mathbb{C}^{n,n}$ be invertible.

b) (i) Prove that if B is similar to C then B^{-1} is similar to C^{-1}. **[1]**

(ii) Suppose we are given the Jordan canonical form, J, of B. Find the Jordan canonical form of B^{-1}, with justification. **[6]**

Suppose that $D \in \mathbb{C}^{13,13}$ with a single eigenvalue λ (repeated 13 times) and that

$$\text{Nullity}(D - \lambda I_{13}) = 3,$$
$$\text{Nullity}((D - \lambda I_{13})^3) = 9.$$

Let J be the Jordan canonical form of D.

c) (i) How many Jordan blocks does J have? **[1]**

(ii) How many Jordan blocks of size 2 does J have? **[2]**

(iii) Write down all of the possible sequences of Jordan block sizes J could have (up to reordering the blocks). **[3]**

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3. Let $V = \mathbb{R}[x]_{\leq 2}$, the vector space of real polynomials of degree at most 2 with standard basis $e_1 = 1, e_2 = x, e_3 = x^2$. Let $r_1, r_2 \in \mathbb{R}$ and define $\tau : V \times V \to \mathbb{R}$ by

$$\tau(p, q) = p(r_1)q(r_2), \text{ for } p, q \in V.$$

a) Prove that τ is a bilinear form and write down the corresponding matrix A of τ with respect to the standard basis. **[5]**

b) For what values of r_1, r_2 is τ symmetric? **[1]**

c) Let $f_1 = x^2 + x$, $f_2 = x + 2$, $f_3 = x^2 + 3$. Write down the change of basis matrix P from the basis e_1, e_2, e_3 to the basis f_1, f_2, f_3. **[2]**

d) Write down an expression in terms of P and A for the matrix of τ in the basis f_1, f_2, f_3? (You do not need to do any matrix inversions/multiplications) **[1]**

Let $W = \mathbb{R}^2$ be a euclidean space with the standard dot product. Suppose $a \in \mathbb{R}$ with $a \geq 0$ and $w = \begin{pmatrix} x \\ y \end{pmatrix} \in W$. Let $q : W \to \mathbb{R}$ be the quadratic form defined by

$$q(w) = ax^2 + 2xy + ay^2.$$

e) Determine the signature of q (your answer will depend on a). **[8]**

f) Describe the geometry of the curve defined by $q(w) = 1$ (your answer will depend on a). **[3]**

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4. a) Let $\phi : G \to H$ be a homomorphism between abelian groups G and H. Prove that the order of $\phi(g)$ divides the order of g for all $g \in G$. **[4]**

b) Write down all abelian groups, up to isomorphism, of order 36. Prove that all the groups in your list are non-isomorphic. **[6]**

Let $G = \mathbb{Z}^3$ with standard free basis $x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Define H to be the subgroup of G generated by

$$h_1 = 3x_1 + 2x_2 - 5x_3, \quad h_2 = -3x_1 + 4x_2 + 7x_3, \quad h_3 = -7x_1 - 2x_2 + 13x_3.$$

c) Find a free basis y_1, y_2, y_3 of G such that H is generated by d_1y_1, d_2y_2, d_3y_3 for positive integers d_1, d_2, d_3 with $d_1 \mid d_2$ and $d_2 \mid d_3$. **[7]**

d) What is the order of H? **[1]**

e) What is the order of the group

$$K = \langle x_1, x_2, x_3 \mid 3x_1 - 3x_2 - 7x_3, 2x_1 + 4x_2 - 2x_3, -5x_1 + 7x_2 + 13x_3 \rangle?$$

[2]
