THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: APRIL 2021

ALGEBRA 1: ADVANCED LINEAR ALGEBRA

Time Allowed: **2 hours**

Read all instructions carefully. Please note also the guidance you have received in advance on the departmental 'Warwick Mathematics Exams 2021' webpage.

Calculators, wikipedia and interactive internet resources are not needed and are not permitted in this examination. You are not allowed to confer with other people. You may use module materials and resources from the module webpage.

ANSWER ALL THREE QUESTIONS.

On completion of the assessment, you must upload your answer to Moodle as a single PDF document if possible, although multiple files (2 or 3) are permitted. You have an additional 45 minutes to make the upload, and instructions are available on the departmental 'Warwick Mathematics Exams 2021' webpage.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. a) Let e_1, e_2, e_3, e_4 be the standard basis of $W = \mathbb{R}^4$. Consider the linear map $S : W \to W$ defined by

$$S(e_1) = e_1 + e_3, \quad S(e_2) = e_3 + e_4,$$

$$S(e_3) = -e_1 - e_3, \quad S(e_4) = e_1 + e_3.$$

Write the matrix B of S with respect to the standard basis of W. **[2]**

b) The characteristic polynomial of B is z^4. Find the minimal polynomial $m_B(z)$ of B. **[4]**

c) Find the Jordan normal form of B and calculate a Jordan basis of W (that is, a basis of W such that the matrix of S with respect to this basis is in Jordan normal form). **[6]**

d) Compute e^{tB}, explaining your method. **[6]**

 Question 1 continued overleaf

e) Using d) or otherwise, solve the system of differential equations

$$\frac{d}{dt}F(t) = S(F(t))$$

with the initial condition $F(0)^T = (1,1,1,1)$. **[3]**

f) Let e_1, e_2 be the standard basis of $V = \mathbb{R}^2$. Consider the linear map $T_a : V \to V$ defined by

$$T_a(e_1) = ae_1 + e_2, \quad T_a(e_2) = e_1 + ae_2,$$

where a is a real number. Write down the matrix A_a of T_a with respect to the standard basis of V. **[2]**

g) Find the minimal polynomial and the characteristic polynomial of A_a. **[4]**

h) Using either of the two methods from the course (Lagrangian interpolation or the JCF method), calculate A_2^{2021}. **[6]**

i) Let D be a 5×5 complex matrix such that $D^5 = 0$. Suppose there exist 6 vectors $v_1, \ldots, v_6 \in \mathbb{C}^5$ such that they span $\mathbb{C}^5$ and

$$Dv_1 = v_2,\ Dv_2 = v_3 \neq 0,\ Dv_4 = v_5 \ \text{ and } \ Dv_5 = v_6 \neq 0\ .$$

Determine all of the possible Jordan normal forms D could have. Justify your answer. **[7]**

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2. a) Let $V = \mathbb{R}[x]_{\leq n}$ be the vector space of real polynomials of degree at most n. Define a function

$$\tau : V \times V \to \mathbb{R} \ \text{ by } \ \tau(f(x), g(x)) = \int_0^1 x f(x) g(x) dx\ .$$

Prove that (V, τ) is a euclidean space. **[8]**

b) Consider the euclidean space (W, π) where $W = \mathbb{R}^3$ and the euclidean form is

$$\pi((x_1, y_1, z_1)^T, (x_2, y_2, z_2)^T) = x_1x_2 + 4y_1y_2 + 16z_1z_2\ .$$

Find an orthonormal basis f_1, f_2, f_3 of W. **[3]**

c) Consider the quadratic form $\beta : W \to \mathbb{R}$ defined by

$$\beta(x, y, z) = 11x^2 + 8y^2 + 8xy + 80z^2 - 64xz + 160yz\ .$$

Write the matrix A of β in the basis f_1, f_2, f_3. **[3]**

In the rest of the question, you may use the fact that A has characteristic polynomial

$$-t^3 + 18t^2 + 81t - 1458 = (18 - t)(9 - t)(-9 - t).$$

d) Find an orthonormal basis g_1, g_2, g_3 of W in which the matrix of the quadratic form β is diagonal. **[8]**

e) Write the matrix of β in the basis g_1, g_2, g_3. **[3]**

f) Suppose h_1, h_2, h_3 is another orthonormal basis in which the matrix of β is diagonal. What is the relation between this basis and the basis g_1, g_2, g_3? Justify your answer. **[5]**

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3. a) Calculate the Smith normal form of the integer-valued matrix A, briefly explaining your steps.

$$A = \begin{pmatrix} 2 & 12 & 24 \\ -3 & 2 & 2 \\ 2 & 6 & 12 \\ -1 & 6 & 12 \end{pmatrix}.$$

[6]

b) How many abelian groups are there, up to isomorphism, of order $2021 = 43 \times 47$? **[2]**

c) Calculate the order of the following three groups, justifying your answers:

$$G_1 = \langle x_1, x_2 \mid 52x_1 + 24x_2, -8x_1 - 4x_2 \rangle,$$

$$G_2 = \langle x_1, x_2, x_3 \mid 6x_1, -2x_2 + 14x_3, 18x_1 + 2x_2 - 14x_3 \rangle,$$

$$G_3 = \langle x_1, x_2, x_3 \mid 2x_1 + 6x_2 + 12x_3, -x_1 + 6x_2 + 12x_3, 2x_1 + 12x_2 + 24x_3, -3x_1 + 2x_2 + 2x_3 \rangle.$$

[6]

d) For each of the following statements decide if they are true or false. If true, provide a short proof. If false, provide a couterexample with justification that it contradicts the statement.

(i) Let G be an abelian group. For all $a, b \in G$ the order of $a + b$ is the lowest common multiple of the orders of a and b. **[5]**

(ii) Let G be a finite abelian group with presentation $\langle x_1, \ldots, x_n | R_1, \ldots, R_m \rangle$. Then $m \geq n$. **[5]**

(iii) Let G be an abelian group which is not finitely generated. Then there exists a proper subgroup of G which is not finitely generated. **[1]**

(iv) Let G be an abelian group of order 2^n for some $n \geq 1$. Suppose we know G has m involutions (elements of order 2) for some fixed m. Then the isomorphism class of G is uniquely determined. **[5]**
