MA251 Algebra 1 - Week 3

Louis Li

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1 Week 3

Question 1.

Calculate the characteristic and minimal polynomials of the following matrices

$$A = \begin{pmatrix} 13 & 0 \\ 0 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 4 & 0 \\ 1 & 0 & 5 \end{pmatrix}$$

Solution.

(a) The characteristic polynomial for A is just

$$c_A(x) = (x-13)(x-7).$$

So does the minimal polynomial $\mu_A(x)$ as A is a diagonal matrix.

(b) The characteristic polynomial for B is also

$$c_B(x) = (x+i)^2.$$

Since $\mu_B(x)|c_B(x)$, $\mu_B(x) = (x+i)^2 \text{or} (x+i)$. By definition of minimal polynomial, it should be the least degree, so $\mu_B(x) = x + i$.

(c) The characteristic polynomial for C is

$$\begin{aligned} c_C(x) &= \det(C - xI_3) \\ c_C(x) &= \det\left(\begin{pmatrix} 1-x & 0 & -3 \\ 0 & 4-x & 0 \\ 1 & 0 & 5-x \end{pmatrix}\right) \\ c_C(x) &= (4-x)(x^2-6x+8) \\ c_C(x) &= -(x-4)^2(x-2) \\ c_C(x) &= -(x-4)^2(x-2). \end{aligned}$$

Since $\mu_C(x)|c_C(x)$, then $\mu_C(x) = (x-4)(x-2) \text{or} (x-4)^2(x-2)$. Now we can only check $\mu_C(C)$ to see which one equals to 0. We shall $(x-4)(x-2)$ first since if it is not 0, then $\mu_C(C) = (x-4)^2(x-2)$

and if it is 0, then $\mu_C(C) = (x-4)(x-2)$ since it has less degree:

$$\begin{aligned}\mu_C(C) &= (C-4I_3)(C-4I_2)\\ &= \begin{pmatrix} -3 & 0 & -3 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & -3 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

Hence, $\mu_C(C) = (x-4)(x-2)$.

□

Question 2.

The characteristic polynomial of

$$A = \begin{pmatrix} 3 & -1 & -1 & -2 \\ 1 & 1 & -5 & -10 \\ -10 & -14 & -8 & -28 \\ 5 & 7 & 7 & 20 \end{pmatrix}$$

is $(x-2)^2(x-6)^2$.

(a) Find the generalised eigenspaces of A of index $i > 0$ for the eigenvalue 2. (You may assume that the full generalised eigenspace for eigenvalue 2 is 2-dimensional, this does follow from Theorem 2.7.4)

(b) Find the generalised eigenspaces of A of index $i > 0$ for the eigenvalue 6. (You may assume that the full generalised eigenspace for eigenvalue 6 is 2-dimensional).

(c) Is A diagonalisable?

Solution.

(a) Since we know that the full generalised eigenspace for eigenvalue 2 is 2-dimensional, then it is sufficient to check $(A-2I_4)$ and $(A-2I_4)^2$. Therefore we are looking for

$$N_1 = \left\{\mathbf{v} \in \mathbb{R}^4 : (A-2I_4)\mathbf{v} = 0\right\} \quad \text{and} \quad N_2 = \left\{\mathbf{v} \in \mathbb{R}^4 : (A-2I_4)^2\mathbf{v} = 0\right\}.$$

All the $\mathbf{v}$s in N_1 are in the form of $\begin{pmatrix} 1 \\ 1 \\ 6 \\ -3 \end{pmatrix}$ and all the $\mathbf{v}$s in N_2 are in the form of $\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}$.

(b) Similarly, the full generalised eigenspace for eigenvalue 6 is 2-dimensional, then it is sufficient to check $(A-6I_4)$ and $(A-6I_4)^2$. Therefore we are looking for

$$N_1 = \left\{\mathbf{v} \in \mathbb{R}^4 : (A-6I_4)\mathbf{v} = 0\right\} \quad \text{and} \quad N_2 = \left\{\mathbf{v} \in \mathbb{R}^4 : (A-6I_4)^2\mathbf{v} = 0\right\}.$$

All the $\mathbf{v}$s in N_1 are in the form of $\begin{pmatrix} 0 \\ 2 \\ 0 \\ -1 \end{pmatrix}$ and all the $\mathbf{v}$s in N_2 are in the form of $\begin{pmatrix} 0 \\ 0 \\ 2 \\ -1 \end{pmatrix}$.

(c) No, it is not diagonlisable. If it is diagonalisable, it must have a basis of eigenvectors. For the 4 generalised eigenvectors, we see that the one in N_2 is not actually an eigenvector. Check that

$$A \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} \neq 2 \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, there are only 3 eigenvectors which cannot form a basis of eigenvectors for this matrix, and hence it is not diagonlisable.

□

Question 3.

Check that the characteristic polynomial of $J_{\lambda,n}$ is $(\lambda - x)^n$ and the minimal polynomial of $J_{\lambda,n}$ is $(x - \lambda)^n$.

Proof.

The characteristic polynomial of $J_{\lambda,n}$ comes from the upper triangular matrix from the definition of Jordan blocks as all the entries on the diagonal are eigenvalues. Therefore,

$$c_{J_{\lambda,n}}(x) = (\lambda - x)^n.$$

We know that minimal polynomial is $(x - \lambda)^r$ for some $1 \leq r \leq n$. Calculate $(J_{\lambda,n} - \lambda I_n)^r$.

$$(J_{\lambda,n} - \lambda I_n)^r = \left(\begin{pmatrix} \lambda & 1 & 0 & 0 & ... \\ 0 & \lambda & 1 & 0 & ... \\ 0 & 0 & \lambda & 1 & ... \\ 0 & 0 & 0 & \lambda & ... \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 & ... \\ 0 & \lambda & 0 & 0 & ... \\ 0 & 0 & \lambda & 0 & ... \\ 0 & 0 & 0 & \lambda & ... \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \right)^r$$
$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & ... \\ 0 & 0 & 1 & 0 & 0 & ... \\ 0 & 0 & 0 & 1 & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}^r.$$

After performing some powers, we notice that as we take powers of this matrix, we move the diagonal line of 1s upward by one place. In particular, $(J_{\lambda,n} - \lambda I_n)^{n-1}$ is the matrix with all zeros apart from the $1, n$ entry which is a 1. Therefore, the minimal polynomial must be $(x - \lambda)^n$.

□

Question 4.

Prove Lemma 2.7.2: Suppose that $M = A \oplus B$ for matrices A and B with entries in K (i.e. M has block-diagonal form with blocks A and B along the diagonal.) Then the characteristic polynomial $c_M(x)$ is the product of $c_A(x)$ and $c_B(x)$, and the minimal polynomial $\mu_M(x)$ is the lowest common multiple of $\mu_A(x)$ and $\mu_B(x)$.

Proof.

The first property follows from how we calculate determinants. We need to prove that $\det(A \oplus B) = \det(A)\det(B)$. Note that

$$A \oplus B = (A \oplus I_B)(I_A \oplus B)$$

where I_x is an identity matrix of the same dimension as X.

Also by expanding via the last row, then the next row up to get $1 \times 1 \times 1 \times ... \times \det(A)$.

Therefore we have

$$\det(A \oplus I_B) = \det(A).$$

Hence we have $\det(A \oplus B) = \det(A)\det(B)$ and note that $A \oplus B - xI_{AB} = (A - xI_A) \oplus (B - xI_B)$, hence

$$c_M(x) = c_A(x)c_B(x).$$

For the minimal polynomial, we have to annihilate both A and B in $A \oplus B$, mathematically speaking,

$$\mu_{A\oplus B}(A \oplus B) = \mu_{A\oplus B}(A) \oplus \mu_{A\oplus B}(B).$$

Therefore,

$$\mu_A(x)|\mu_M(x) \quad \mu_B(x)|\mu_M(x),$$

where $M = A \oplus B$.

By division property,

$$\mu_A(x)\mu_B(x)|\mu_M(x)$$

and by definition of minimal polynomial,

$$\mu_M(x) = \text{lcm}(\mu_A(x), \mu_B(x)).$$

□

Question 5.

Let A be an $n \times n$ complex matrix with $\mu_A(x) = (-1)^n c_A(x)$. Determine the number of blocks in the JCF of A.

Solution.

Recall Theorem 2.7.4, we have

$$c_A(x) = (-1)^n \prod_{i=1}^{r}(x - \lambda_i)^{a_i}$$

where a_i is the sum of degrees of the Jordan blocks of A of eigenvalue λ_i and

$$\mu_A(x) = \prod_{i=1}^{r}(x - \lambda_i)^{b_i},$$

where b_i is the largest among the degrees of the Jordan blocks of A of eigenvalue λ_i.

Since $\mu_A(x) = (-1)^n c_A(x)$, we have

$$\prod_{i=1}^{r}(x - \lambda_i)^{a_i} = \prod_{i=1}^{r}(x - \lambda_i)^{b_i}.$$

Hence, $a_i = b_i$, meaning that the sum of all the sizes of Jordan blocks for λ_i is equal to the size of the largest one, thus there must be only one block for each eigenvalue of A. Hence, the number of blocks in the JCF of A is the number of eigenvalues of A.

□