MA251 Algebra 1 - Week 4

Louis Li

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1 Week 4

Question 1.

Find the JCF of the following matrices:

$$A = \begin{pmatrix} 2 & 5 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3+i & 2+9i \\ -i & 1-i \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

Solution.

(a) Note that $c_A(x) = (x-2)^2$. Hence, we can determine its minimal polynomial by Cayley Hamilton Theorem, $mu_A(x) = (x-2)\text{or}(x-2)^2$. Check $(A - 2I_2)$, since the upper top right corner is 5, $A - 2I_2 \neq 0$. Therefore, the minimal polynomial is $\mu_A(x) = (x-2)^2$. Hence, the JCF is $J_{2,2}$.

(b) Note that $c_B(x) = x^2 - 4x - 5 = (x-5)(x+1)$. There are two distinct eigenvalues $x = 5$ and $x = -1$. Therefore, the JCF is $J_{5,2} \oplus J_{-1,2}$.

(c) Note that in this case it is already in its JCF, with $J_{3,2} \oplus J_{i,1}$.

□

Question 2.

Let $T : \mathbb{C}[x]_{\leq 3} \to \mathbb{C}[x]_{\leq 3}$ be the linear map defined by $T(p) = p'$. Let A be a matrix representing T. Find a Jordan basis for A and write down its JCF.

Solution.

The basis for $\mathbb{C}[x]_{\leq 3}$ is $\left\{1, x, x^2, x^3\right\}$ and apply T to it, we have

$$\begin{aligned} T(1) &= 0 \\ T(x) &= 1 \\ T(x^2) &= 2x \\ T(x^3) &= 3x^2. \end{aligned}$$

Therefore the matrix A representing T is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We immediately see that $c_A(\lambda) = \lambda^4$ with the eigenvalue $= 0$. The minimal polynomial can be $\mu_A(\lambda) = \lambda, \lambda^2, \lambda^3 \text{or} \lambda^4$. Check $A, A^2, A^3 \text{and} A^4$, we see that $A^4 = \mathbf{0}$ and hence $\mu_A(\lambda) = \lambda^4$. Therefore, the JCF of A is $J_{0,4}$:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To find the jordan basis, we are required to find

$$\{0\} \not\subseteq \{\ker(A)\} \not\subseteq \left\{\ker(A^2)\right\} \not\subseteq \left\{\ker(A^3)\right\},$$

since $\lambda = 0$ and $A^4 = \mathbf{0}$. Therefore first, take any vector in $\ker(A^4) \setminus \ker(A^3)$, i.e. Take $\mathbf{v}$ such that $A^4\mathbf{v} = 0$ while $A^3\mathbf{v} \neq 0$ and the obvious pick is $\mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

Therefore, we have

$$\mathbf{v}_3 = A\mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}.$$

Similarly,

$$\mathbf{v}_2 = A\mathbf{v}_3 = \begin{pmatrix} 0 \\ 6 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_1 = A\mathbf{v}_2 = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This means our Jordan basis is

$$\left\{6, 6x, 3x^2, x^3\right\}.$$

□

Question 3.

Let A be a matrix with characteristic polynomial $-(x-2)^5$. What could the possible JCFs of A be? What if we don't know the characteristic polynomial of A but its minimal polynomial is $(x-2)^5$?

Solution.

For the characteristic polynomial, we know the eigenvalue is 2. By Theorem 2.7.4, we know 5 is the sum of all degrees of Jordan blocks. Therefore, the possible JCFs of A could be

$$J_{2,5}; J_{2,4} \oplus J_{2,1}; J_{2,3} \oplus J_{2,2}; J_{2,2} \oplus J_{2,2} \oplus J_{2,1}; J_{2,3} \oplus J_{2,1} \oplus J_{2,1}$$

and

$$J_{2,2} \oplus J_{2,1} \oplus J_{2,1} \oplus J_{2,1}; J_{2,1} \oplus J_{2,1} \oplus J_{2,1} \oplus J_{2,1} \oplus J_{2,1}.$$

For the minimal polynomial, if $\mu_A(x) = (x-2)^5$, then also by Theorem 2.7.4, the maximum size of the Jordan block is 5, and there are infinitely man of these. We can write them as

$$J_{2,5}^a \oplus J_{2,4}^b \oplus J_{2,3}^c \oplus J_{2,2}^d \oplus J_{2,1}^e$$

with $a \geq 1$. □

Question 4.

Find the JCF J of the following matrix A.

$$A = \begin{pmatrix} 2 & 3 & 0 & 0 & 1 \\ 0 & -10 & 0 & 0 & -3 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 36 & 0 & 0 & 11 \end{pmatrix}.$$

Solution.

There are two ways of doing this.

Method 1:

Find the characteristic polynomial of A:

$$c_A(x) = -(x+1)^3(x-2)^2.$$

Hence, by Cayley Hamilton theorem,

$$\mu_A(x) = (x+1)^a(x-2)^b$$

where $a, b \in \mathbb{Z}^+ \cup \{0\}$ and $a \leq 3$, $b \leq 2$.

By calculation, we see that

$$\mu_A(x) = (x+1)^2(x-2)^2,$$

meaning the JCF of A must be $J_{-1,2} \oplus J_{-1,1} \oplus J_{2,2}$.

Method 2:

By Theorem 2.9.1, the number of Jordan blocks of J with eigenvalue λ and degree at least i is equal to nullity$(A - \lambda I_n)^i$−nullity$(A - \lambda I_n)^{i-1}$.

Since we know the characteristic polynomial of A is

$$c_A(x) = -(x+1)^3(x-2)^2,$$

meaning the eigenvalues of A are -1 and 2. Also

$$A - 2I_5 = \begin{pmatrix} 0 & 3 & 0 & 0 & 1 \\ 0 & -12 & 0 & 0 & -3 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 36 & 0 & 0 & 9 \end{pmatrix},$$

and the rank of it is 4, hence nullity$(A - 2I_5) = 1$. Therefore by Theorem 2.9.1, the number of Jordan blocks of A with eigenvalue $\lambda = 2$ is 1. From the characteristic polynomial, we see the degree of Jordan blocks with eigenvalue 2 is 2, so there must be one Jordan block of degree 2 with eigenvalue 2.

Similarly,

$$A + I_5 = \begin{pmatrix} 3 & 3 & 0 & 0 & 1 \\ 0 & -9 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 0 & 0 & 12 \end{pmatrix},$$

and this matrix has rank 3, thus nullity$(A + I_5) = 2$. By Theorem 2.9.1, the number of Jordan blocks of A with eigenvalue $\lambda = -1$ is 2. From the characteristic polynomial, we see that the degree of Jordan blocks with eigenvalue -1 is 3, hence, we must have one block with degree 2 and one with degree 1. That is to say, JCF of A is $J_{2,2} \oplus J_{-1,2} \oplus J_{-1,1}$.

□