MA251 Algebra 1 - Week 5

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1 Week 5

Question 1.

Find the JCF J of

$$A = \begin{pmatrix} -3 & -1 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 5 & 2 & -1 & -5 \\ -2 & -1 & 0 & 1 \end{pmatrix}$$

and find a matrix P such that $P^{-1}AP = J$. For your convenience: the characteristic polynomial of A is $(x+1)^4$.

Solution.

For the JCF of A, we use nullity method.

Since we know that $c_A(x) = (x+1)^4$, hence $\lambda = -1$, also

$$A + I_4 = \begin{pmatrix} -2 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 5 & 2 & 0 & -5 \\ -2 & -1 & 0 & 2 \end{pmatrix},$$

the nullity of this matrix is 2, and hence by Theorem 2.9.1, the number of Jordan blocks with eigenvalue -1 is 2. From $c_A(x)$, we know the sum of the degrees is 4, so the JCF can be $J_{-1,1} \oplus J_{-1,3}$ or $J_{-1,2} \oplus J_{-1,2}$.

We need to check the minimal polynomial in this case: we see that

$$(A + I_4)^2 = \mathbf{0},$$

and hence the JCF must be in the form of $J_{-1,2} \oplus J_{-1,2}$.

In order to find P, we need to find the Jordan basis. Since $\dim(A + I_4) = 4$, we need to find 4 vectors.

Choose $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $\mathbf{v}_1 = (A + I_4)\mathbf{v}_2 = \begin{pmatrix} -2 \\ 0 \\ 5 \\ -2 \end{pmatrix}$.

Choose $\mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, (since $\mathbf{v}_2$ and $\mathbf{v}_4$ need to be linearly independent) then $\mathbf{v}_3 = (A + I_4)\mathbf{v}_4 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$.

Hence,

$$P = \begin{pmatrix} -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & 0 & 2 & 0 \\ -2 & 0 & -1 & 0 \end{pmatrix}.$$

□

Question 2.

Use your answer to Question 1 to find A^{2022}. How about e^A?

Solution.

By telescoping product,

$$A^{2002} = PJ^{2002}P^{-1}.$$

By formula of powers of Jordan block,

$$J_{\lambda,k}^{n} = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & ... & \binom{n}{k-1}\lambda^{n-k+1} \\ 0 & \lambda^n & ... & \vdots \\ \vdots & \vdots & ... & n\lambda^{n-1} \\ 0 & 0 & ... & \lambda^n \end{pmatrix}.$$

Hence,

$$J_{-1,2}^{2022} = \begin{pmatrix} (-1)^{2022} & 2022(-1)^{2021} \\ 0 & (-1)^{2022} \end{pmatrix} = \begin{pmatrix} 1 & -2022 \\ 0 & 1 \end{pmatrix}.$$

Therefore,

$$A^{2022} = \begin{pmatrix} -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & 0 & 2 & 0 \\ -2 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2022 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2022 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0-1 & \\ 0 & 0 & -2 & -5 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4045 & 2022 & 0 & -4044 \\ 0 & 1 & 0 & 0 \\ -10110 & -4044 & 1 & 10110 \\ 4044 & 2022 & 0 & -4043 \end{pmatrix}.$$

Similar for e^A,

$$e^A = Pe^JP^{-1}.$$

By definition of $f(J)$,

$$f(J_{\lambda,k}) = \begin{pmatrix} f(\lambda) & f'(\lambda) & ... & f^{[k-1]}(\lambda) \\ 0 & f(\lambda) & ... & f^{[k-2]}(\lambda) \\ \vdots & \vdots & ... & \vdots \\ 0 & 0 & ... & f(\lambda) \end{pmatrix},$$

where

$$f^{[k]}(\lambda) = \frac{1}{k!}f^k(\lambda).$$

Hence,

$$e^J = \begin{pmatrix} e^{-1} & e^{-1} & 0 & 0 \\ 0 & e^{-1} & 0 & 0 \\ 0 & 0 & e^{-1} & e^{-1} \\ 0 & 0 & 0 & e^{-1} \end{pmatrix}.$$

Therefore,

$$
\begin{aligned}
e^A &= \begin{pmatrix} -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & 0 & 2 & 0 \\ -2 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-1} & e^{-1} & 0 & 0 \\ 0 & e^{-1} & 0 & 0 \\ 0 & 0 & e^{-1} & e^{-1} \\ 0 & 0 & 0 & e^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & -2 & -5 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
&= e^{-1} \begin{pmatrix} -1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 5 & 2 & 1 & -5 \\ -2 & -1 & 0 & 3 \end{pmatrix}.
\end{aligned}
$$

□

Question 3.

What is the minimal polynomial of A? Find the Lagrange interpolation polynomial of μ_A and use it to calculate A^{2022} and e^A again.

Solution.

From question 1, we see that the minimal polynomial of A is $(x+1)^2$. Given the degree of $\mu_A(x)$ is 2, our Lagrange interpolation polynomial is linear, so $h(x) = \alpha x + \beta$. To determine α and β, we need to solve

$$
\begin{cases} (-1)^n = (-1)^{2022} = h(-1) = -\alpha + \beta \\ n(-1)^{n-1} = 2022(-1)^{2021} = h'(-1) = \alpha \end{cases}.
$$

We have

$$
\begin{cases} \alpha & = -2022 \\ \beta & = -2021 \end{cases}.
$$

Hence,

$$
\begin{aligned}
A^{2022} &= -2022A - 2021I_4 \\
&= -2022 \begin{pmatrix} -3 & -1 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 5 & 2 & -1 & -5 \\ -2 & -1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2021 & 0 & 0 & 0 \\ 0 & 2021 & 0 & 0 \\ 0 & 0 & 2021 & 0 \\ 0 & 0 & 0 & 2021 \end{pmatrix} \\
&= \begin{pmatrix} 4045 & 2022 & 0 & -4044 \\ 0 & 1 & 0 & 0 \\ -10110 & -4044 & 1 & 10110 \\ 4044 & 2022 & 0 & -4043 \end{pmatrix}.
\end{aligned}
$$

Similarly, our Lagrange interpolation polynomial is linear, so $h(x) = ax + b$. To determine a and b, we need to solve

$$
\begin{cases} e^\lambda = e^{-1} = h(-1) = -a + b \\ e^\lambda = e^{-1} = h'(-1) = a. \end{cases}
$$

Solving it we get

$$
\begin{cases} a = e^{-1} \\ b = 2e^{-1} \end{cases}.
$$

Hence,

$$
\begin{aligned}
e^A &= e^{-1}A + 2e^{-1}I_4 \\
&= e^{-1}\begin{pmatrix} -3 & -1 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 5 & 2 & -1 & -5 \\ -2 & -1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2e^{-1} & 0 & 0 & 0 \\ 0 & 2e^{-1} & 0 & 0 \\ 0 & 0 & 2e^{-1} & 0 \\ 0 & 0 & 0 & 2e^{-1} \end{pmatrix} \\
&= e^{-1}\begin{pmatrix} -1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 5 & 2 & 1 & -5 \\ -2 & -1 & 0 & 3 \end{pmatrix}.
\end{aligned}
$$

□