THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: APRIL 2020

MULTIVARIABLE CALCULUS

Time Allowed: **3 hours**

Read all instructions carefully. Please note also the guidance you have received in advance on the departmental 'Warwick Mathematics Exams 2020' webpage.

Calculators, wikipedia and interactive internet resources are not needed and are not permitted in this examination. You are not allowed to confer with other people. You may use module materials and resources from the module webpage.

ANSWER COMPULSORY QUESTION 1 AND TWO FURTHER QUESTIONS out of the four optional questions 2, 3, 4 and 5.

On completion of the assessment, you must upload your answer to the AEP as a single PDF document. You have an additional 45 minutes to make the upload, and instructions are available on the departmental 'Warwick Mathematics Exams 2020' webpage.
You must not upload answers to more than 3 questions, including Question 1. If you do, you will only be given credit for your Question 1 and the first two other answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question. The compulsory question is worth twice the number of marks of each optional question. Note that the marks do not sum to 100.

Throughout this examination,

- $L(\mathbb{R}^n, \mathbb{R}^k) := \{A : \mathbb{R}^n \to \mathbb{R}^k \mid A \text{ is linear}\}; \quad L(\mathbb{R}^n) := L(\mathbb{R}^n, \mathbb{R}^n)$.
- $GL(n, \mathbb{R}) := \{A \in L(\mathbb{R}^n) : A \text{ is invertible}\}$.
- $M(k \times n, \mathbb{R})$ will denote the space of $k \times n$ matrices with real entries in k rows and n columns; $\quad M(n, \mathbb{R}) := M(n \times n, \mathbb{R})$.
- For $r > 0$ and $p \in \mathbb{R}^n$, $\mathbb{B}(p, r) := \{x \in \mathbb{R}^n : |x - p| < r\}$.

COMPULSORY QUESTION

1. a) Show that if $A \in L(\mathbb{R}^n, \mathbb{R}^k)$ satisfies $\|A\| = 0$, then $Ax = 0 \; \forall\, x \in \mathbb{R}^n$. **[2]**

b) Show that if $A \in GL(n, \mathbb{R})$ and $\|A^{-1}\| = \mu$, then $|Ax| \geqslant \frac{1}{\mu}|x| \; \forall\, x \in \mathbb{R}^n$. **[3]**

c) Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \frac{\sin(xy)}{x^2+y^2}, \quad (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \qquad f(0,0) = 0.$$

Prove that f is separately continuous, but not continuous, at $(0,0)$. **[4]**

d) Prove that, if $f\colon \mathbb{R}^n \to \mathbb{R}^k$ is differentiable at x then, $\forall\, h \in \mathbb{R}^n$, the directional derivative $\partial_h f(x)$ exists and $\partial_h f(x) = Df(x)h$. **[3]**

e) Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} x - \frac{y^2}{x} & \text{if } 0 < |y| \leqslant |x|, \\ 0 & \text{if } 0 \leqslant |x| \leqslant |y|. \end{cases}$$

(i) Calculate the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ and the directional derivative $\partial_{(1,1)} f(0,0)$. **[3]**

(ii) Is f differentiable at $(0,0)$? Explain your answer briefly. **[2]**

f) (i) Define $S\colon M(n,\mathbb{R}) \to M(n,\mathbb{R})$ by $S(A) := A^2$. For $H \in M(n,\mathbb{R})$, calculate $DS(A)H$ directly from the definition of the derivative. **[3]**

(ii) Define $F\colon M(n,\mathbb{R}) \to \mathbb{R}$ and $f\colon M(n,\mathbb{R}) \to \mathbb{R}$ by

$$F(A) := \operatorname{trace} A^2, \qquad f(A) := (\operatorname{trace} A)^2.$$

For $H \in M(n,\mathbb{R})$, calculate $DF(A)H$ and $Df(A)H$. **[5]**

g) Define $\Psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Psi(x,y) := (\sin(x+y^2), 2x + e^y \cos x) =: (u(x,y), v(x,y)).$$

(i) Verify that $\partial\Psi(0,0)$ is invertible. **[3]**

(ii) Denote the local inverse in part (i) by $\Psi^{-1}(u,v) =: (x(u,v), y(u,v))$. Thus, since $\Psi(0,0) = (0,1)$ we have that $\Psi^{-1}(0,1) = (0,0)$. Calculate $\partial\Psi^{-1}(0,1)$ and read off from this matrix the values of $\frac{\partial x}{\partial u}(0,1)$ and $\frac{\partial y}{\partial u}(0,1)$. **[3]**

h) (i) State Green's Theorem in the plane, taking care to define the *curl* of a planar vector field $\underline{v}$ and the positively oriented parameterisation of the boundary of a planar region. **[4]**

(ii) Let R denote the rectangle $[\alpha, \beta] \times [\xi, \eta] \subset \mathbb{R}^2$, let U be an open subset of $\mathbb{R}^2$ such that $R \subset U$ and let $\underline{v}\colon U \to \mathbb{R}^2$ be a C^1 vector field. Prove Green's Theorem for $\underline{v}$ over R. **[5]**

OPTIONAL QUESTIONS

2. a) Given that the vector field $\underline{v}(x, y, z) := (x + yz, y + zx, z + xy)$ is conservative, find a scalar potential of $\underline{v}$. **[5]**

b) Let $\underline{w}(x, y) = (x^2, xy),\ (x, y) \in \mathbb{R}^2$ and let C_ρ be the circle of radius ρ centred at the origin in $\mathbb{R}^2$.

(i) Parameterise C_ρ (any orientation is acceptable) and evaluate the circulation $\oint_{C_\rho} \underline{w} \cdot dr$. **[3]**

(ii) Is $\underline{w}$ conservative? Justify your answer fully. **[4]**

c) Define $\underline{v} \colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\underline{v}(x) = x, \qquad x = (x_1, x_2, x_3)$$

and let $\Omega \subset \mathbb{R}^3$ be a region for which $|x| \leqslant R\ \forall\, x \in \overline{\Omega}$.

(i) Apply the divergence theorem to $\underline{v}$ on Ω to prove that

$$\mathrm{Vol}(\Omega) \leqslant \frac{R}{3} \mathrm{Area}(\partial\Omega). \tag{1}$$

`Hint:` The inequality $|a \cdot b| \leqslant |a|\,|b|\ \forall\, a, b \in \mathbb{R}^3$ may be helpful. **[6]**

(ii) What must the shape of Ω be for equality in (1) in part (i) to hold? **[2]**

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3. a) Define $f \colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) := (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right) \text{ if } (x, y) \neq (0, 0), \quad f(0, 0) = 1.$$

Prove that f is differentiable everywhere and that ∂f is not continuous at $(0, 0)$. **[7]**

b) Suppose that $f \in C^1(\mathbb{B}(p, r), \mathbb{R}^k)$ satisfies $\|\partial f(x)\| \leqslant M\ \forall\, x \in \mathbb{B}(p, r)$ for some $M \geqslant 0$. Prove that

$$|f(x) - f(y)| \leqslant M\,|x - y|\ \forall\, x, y \in \mathbb{B}(p, r).$$ **[4]**

c) Given $f \in C^1(\mathbb{B}(p, r), \mathbb{R}^k)$, suppose that $\exists\ \alpha > 0$ such that

$$|Df(p)\,h| \geqslant \alpha|h|\ \forall\, h \in \mathbb{R}^n.$$

Prove that $\exists\ \delta \in (0, r)$ such that f is injective on $\mathbb{B}(p, \delta)$, i.e., $x, y \in \mathbb{B}(p, \delta),\ x \neq y \Rightarrow f(x) \neq f(y)$. **[9]**

`Hint:` Let $A := Df(p)$ and define $F \colon \mathbb{B}(p, r) \to \mathbb{R}^k$ by $F(x) := f(x) - Ax$. Use the continuity of Df and apply the mean value inequality in part b) to F to assert the existence of $\delta \in (0, r)$ such that $|F(x) - F(y)| \leqslant \frac{1}{2}\alpha|x - y|\ \forall\, x, y \in \mathbb{B}(p, \delta)$.

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4. a) In parts (ii) and (iii) below, C_r will denote the circle of radius r centred at the origin in $\mathbb{R}^2$ and n will denote the unit normal to C_r that points away from the origin, i.e., n points out of the disc bounded by C_r.

Let $\underline{v}(x,y) := \frac{1}{x^2+y^2}(x,y),\ (x,y) \neq (0,0)$. Calculate

(i) $\nabla \cdot \underline{v}$, **[2]**

(ii) the flux of $\underline{v}$ across C_1 in the direction of n, **[2]**

(iii) the flux of $\underline{v}$ across C_2 in the direction of n. **[2]**

(iv) Explain why the answer in (ii) does not contradict the divergence theorem and the answer in (i) and explain, using the divergence theorem, why the answers in (ii) and (iii) are equal. **[4]**

b) Let $P(x,y) := \alpha x^3 + \beta x^2 y + \gamma x y^2 + \delta y^3$. Find the two linear conditions that have to be satisfied by $\alpha,\ \beta,\ \gamma,\ \delta$ for P to be harmonic. **[4]**

c) Let $a \in \mathbb{R}^n \setminus \{0\}$ be a fixed vector. Find all continuously differentiable functions $\varphi\colon \mathbb{R}_{>0} \to \mathbb{R}$ such that the divergence of the vector field $\big(\varphi(a \cdot x)\big)x,\ x \in \mathbb{R}^n$, is equal to 1 on $\{x \in \mathbb{R}^n : a \cdot x > 0\}$.

`Hint:` It may be helpful to express $a \cdot x$ as $a_1 x_1 + \cdots + a_n x_n$. **[6]**

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5. a) Let

$$E := \{(x, y) : x^2 - 4y^2 = 4\}, \qquad F := \{(x, y) : x^2 + 4y^2 = 4\}.$$

Prove that

(i) E is closed, but not sequentially compact, **[4]**

(ii) F is sequentially compact. **[3]**

b) Define $F\colon \mathbb{R}^4 \to \mathbb{R}^2$ by

$$F(x_1, x_2, y_1, y_2) := (y_1 + e^{y_1+x_1} + x_2^2 e^{y_1} - y_2 - e^{y_2}\ ,\ y_1 - e^{-y_1} + y_2 + e^{y_2+x_1} + x_2^2 e^{y_2}).$$

(i) Calculate $\partial F(x_1, x_2, y_1, y_2)$. **[4]**

(ii) Fix $(x_1^*, x_2^*, y_1^*, y_2^*) \in \mathbb{R}^4$ and set $(c_1, c_2) := F(x_1^*, x_2^*, y_1^*, y_2^*)$. Prove that it is possible to solve $F(x_1, x_2, y_1, y_2) = (c_1, c_2)$ for (y_1, y_2) in terms of continuously differentiable functions of (x_1, x_2) on a neighbourhood of (x_1^*, x_2^*). **[3]**
Furthermore, calculate the Jacobian matrix

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{pmatrix}$$

in terms of x_1, x_2, y_1 and y_2. You need not perform any matrix calculations; your answer should be expressed as the product of two matrices, one of which is an inverse matrix which should not be calculated explicitly. **[3]**

(iii) Prove that it is never possible to solve $F(x_1, x_2, y_1, y_2) = (c_1, c_2)$ for (x_1, x_2) in terms of differentiable functions of (y_1, y_2). **[3]**
