THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: APRIL 2022

MULTIVARIABLE CALCULUS

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER ALL 4 QUESTIONS.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

Throughout this examination:

- $L(\mathbb{R}^n, \mathbb{R}^k) := \{A : \mathbb{R}^n \to \mathbb{R}^k \mid A \text{ is linear}\}$; $L(\mathbb{R}^n) := L(\mathbb{R}^n, \mathbb{R}^n)$.
- $\mathbb{R}^{k\times n}$ will denote the space of $k \times n$ matrices with real entries in k rows and n columns.
- For $r > 0$ and $p \in \mathbb{R}^n$, $\mathbb{B}(p, r) := \{x \in \mathbb{R}^n : |x - p| < r\}$. Furthermore, $\mathbb{B}_r := \mathbb{B}(0, r)$.

1. a) Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) := \frac{\sin(x-y)}{x-y}, \text{ if } y \neq x, \qquad f(t,t) = 1 \;\forall\, t \in \mathbb{R}.$$

Prove that f is continuous at all points of $\mathbb{R}^2$. **[4]**

b) Given a continuous function $g\colon \mathbb{R}^n \to \mathbb{R}^k$ and a bounded function $h\colon \mathbb{R}^n \to \mathbb{R}$, define a function $F\colon \mathbb{R}^n \to \mathbb{R}^k$ as the product of g with the scalar h. Thus $F(x) := \big(h(x)\big)\big(g(x)\big)$. Prove that if $g(a) = 0$ then F is continuous at $a \in \mathbb{R}^n$. **[4]**

c) (i) Fix $A \in \mathbb{R}^{k\times n}$ and define $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(x) := \frac{1}{2}|Ax|^2$. Calculate the directional derivative $\partial_v f(x)$, $v \in \mathbb{R}^n$ and deduce that $\nabla f(x) = A^T Ax$, where A^T is the transpose of A. **[4]**

(ii) Fix $x \in \mathbb{R}^n$ and define $F\colon L(\mathbb{R}^n, \mathbb{R}^k) \to \mathbb{R}$ by $F(A) := \frac{1}{2}|Ax|^2$. Calculate the directional derivative $\partial_H F(A)$, $H \in L(\mathbb{R}^n, \mathbb{R}^k)$. **[3]**

d) Suppose that $f \in C^1(\mathbb{B}_1, \mathbb{R}^k)$ satisfies $\|Df(0)\| = 1$. Prove that $\exists\, \delta > 0$ such that

$$|f(x) - f(y)| \leqslant 2|x-y| \quad \forall\, x, y \in \mathbb{B}_\delta.$$ **[5]**

e) For each of the following statements, state whether it is true or false. Either provide a proof or a counterexample to justify your answers.

(i) For $A \in L(\mathbb{R}^n, \mathbb{R}^k)$, $\|A\| \neq 0 \Rightarrow Ax \neq 0 \;\forall\, x \in \mathbb{R}^n \setminus \{0\}$. **[2]**

(ii) The graph $\mathcal{G}_f$ of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is a closed subset of $\mathbb{R}^2$. (Recall that $\mathcal{G}_f := \{(x, f(x)) : x \in \mathbb{R}\}$.) **[2]**

(iii) If $f \in C^1(\mathbb{R}^n)$ and $C \in \mathbb{R}^n$ is a closed curve then $\oint_c \nabla f \cdot dr = 0$. **[2]**

(iv) Let $\underline{v}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable planar vector field. If $\nabla\cdot\underline{v} = 0$ and $\operatorname{curl}\underline{v} = 0$ then $\underline{v}$ is constant. **[2]**

f) Suppose $u \in C^2(\mathbb{R}^3)$ satisfies $\Delta u = 1$. Prove that for any bounded region $\Omega \subset \mathbb{R}^3$ we have

$$\mathrm{Vol}(\Omega) = \iint_{\partial\Omega} \nabla u \cdot n_+ \, dA\,,$$

where n_+ is the outward unit normal to Ω. **[3]**

g) (i) State Green's Theorem in the plane, taking care to define the *curl* of a planar vector field $\underline{v}$ and the positively oriented parameterisation of the boundary of a planar region. **[4]**

(ii) Let R denote the rectangle $[\alpha, \beta] \times [\xi, \eta] \subset \mathbb{R}^2$, let U be an open subset of $\mathbb{R}^2$ such that $R \subset U$ and let $\underline{v}\colon U \to \mathbb{R}^2$ be a C^1 vector field. Prove Green's Theorem for $\underline{v}$ over R. **[5]**

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2. a) Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) := \frac{x^2 y}{x^4 + y^2}, \text{ if } (x,y) \neq (0,0), \qquad f(0,0) = 0.$$

(i) Show that $\partial_{(a,b)} f(0,0)$ exists for all $(a,b) \in \mathbb{R}^2$. (You need to be careful when $b = 0$.) **[5]**

(ii) Show that, for $v = (a,b)$ such that $ab \neq 0$, $\partial_v f(0,0) \neq v \cdot \nabla f(0,0)$. **[2]**

(iii) Show that f is discontinuous at $(0,0)$. **[2]**

(iv) Give two substantially different reasons that demonstrate that f is not differentiable at $(0,0)$. **[2]**

b) Consider the equations

$$xe^{-y} = 2e^z, \qquad x^2 + y = 2z + 1 \tag{1}$$

and observe that $x = 2,\ y = -1,\ z = 1$ is a solution of (1). Define $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ by

$$F(x,y,z) = (xe^{-y} - 2e^z\,,\, x^2 + y - 2z).$$

(i) Explain how F satisfies the conditions for applying the Implicit Function Theorem (whose general formulation you need not state), and deduce the existence of $\delta > 0$ and continuously differentiable functions $x = g(z),\ y = h(z),\ z \in (1-\delta, 1+\delta)$ such that

$$g(1) = 2,\ h(1) = -1 \quad \text{and} \quad F(g(z), h(z), z) = (0,1)\ \forall\, z \in (1-\delta, 1+\delta).$$

[4]

(ii) Calculate $g'(z)$ and $h'(z)$ in terms of z, $g(z)$ and $h(z)$. **[5]**

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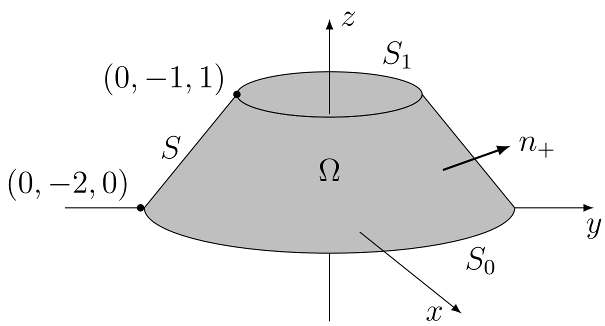
3. a) The vector field $\underline{v}(x,y,z) := (x+y-1, x-\cos z, y\sin z)$ is known to be conservative. Find a scalar potential of $\underline{v}$. **[6]**

b) Let $\underline{v}$ be the vector field

$$\underline{v}(x,y,z) := \left((2z-1)\cos x\,,\ \frac{y}{(z-2)^2}\,,\ (z^2-z)\sin x \right),\quad z<2$$

and let Ω be the solid truncated cone in the diagram below, in which

- S_0 is a disk of radius 2 in the x-y plane with centre at the origin,
- S_1 is a disk of radius 1 in the plane $z=1$ with centre at $(0,0,1)$ and
- the surface S is a truncated right circular cone.
- $\partial\Omega = S_0 \cup S \cup S_1$.



(i) Calculate the flux of $\underline{v}$ across S_0 and S_1. **[4]**

(ii) Calculate $\nabla\cdot\underline{v}$. **[2]**

(iii) Parameterise Ω and calculate $\displaystyle\iiint_\Omega \nabla\cdot\underline{v}\,dV$. **[6]**

(iv) Use the Divergence Theorem to calculate the flux of $\underline{v}$ across S. **[2]**

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4. a) Let $f\colon \mathbb{R}^n \to \mathbb{R}^k$ be differentiable at x, that is, $Df(x)$ exists. Prove that $\partial_v f(x)$ exists for all $v \in \mathbb{R}^n$ and that $\partial_v f(x) = Df(x)v$. **[4]**

b) Define $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $f(A) := A^2$. You may assume, and you need not prove, that f is continuously differentiable.

(i) Calculate $\partial_H f(A)$. **[3]**

For parts (ii) and (iii) below, let

$$S := \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \text{and} \qquad T := \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = f(S).$$

(ii) Calculate $Df(S)H = \partial_H f(S)$ and deduce that $\ker Df(S)$ is just the zero linear transformation of $\mathbb{R}^2$. **[2]**

(iii) Because of (ii), the Inverse Function Theorem may be invoked to assert the existence of a neighbourhood $\mathcal{N}_S$ of S and a corresponding neighbourhood $\mathcal{N}_T$ of T such that $f\colon \mathcal{N}_S \to \mathcal{N}_T$ is a local diffeomorphism with inverse (local square root) given by $g\colon \mathcal{N}_T \to \mathcal{N}_S$. For $K \in \mathbb{R}^{2\times 2}$, calculate $Dg(T)K$. **[3]**

`Hint:` Use the Chain Rule to differentiate with respect to A the relations $g(f(A)) = A \ \forall A \in \mathcal{N}_S$.

For parts (iv) and (v) below, let

$$J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = f(J), \qquad \text{and set} \quad H := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(iv) Calculate $Df(J)H = \partial_H f(J)$ as a 2×2 matrix whose entries depend linearly on a, b, c and d and deduce that $\ker Df(J)$ is equal to $\left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : b, c \in \mathbb{R} \right\}$. **[4]**

(v) Show that f is not injective near J by finding all matrices of the form $\begin{pmatrix} 1 & b \\ c & -1 \end{pmatrix}$ whose square is equal to I. **[4]**
