MA 2590

THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: APRIL 2021

MULTIVARIABLE CALCULUS

Time Allowed: **2 hours**

Read all instructions carefully. Please note also the guidance you have received in advance on the departmental 'Warwick Mathematics Exams 2021' webpage.

Calculators, wikipedia and interactive internet resources are not needed and are not permitted in this examination. You are not allowed to confer with other people. You may use module materials and resources from the module webpage.

ANSWER ALL THREE QUESTIONS.

On completion of the assessment, you must upload your answer to Moodle as a single PDF document if possible, although multiple files (2 or 3) are permitted. You have an additional 45 minutes to make the upload, and instructions are available on the departmental 'Warwick Mathematics Exams 2021' webpage.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

Throughout this examination,

- $L(\mathbb{R}^n, \mathbb{R}^k) := \{A : \mathbb{R}^n \to \mathbb{R}^k \mid A \text{ is linear}\}; \quad L(\mathbb{R}^n) := L(\mathbb{R}^n, \mathbb{R}^n)$.
- $GL(n, \mathbb{R}) := \{A \in L(\mathbb{R}^n) : A \text{ is invertible}\}$.
- $\mathbb{R}^{k\times n}$ will denote the space of $k\times n$ matrices with real entries in k rows and n columns.
- For $r > 0$ and $p \in \mathbb{R}^n$, $\mathbb{B}(p, r) := \{x \in \mathbb{R}^n : |x - p| < r\}$.
- You may use, without proof, the inequality

$$|xy| \leqslant \tfrac{1}{2}(x^2 + y^2) \quad \forall\, x, y \in \mathbb{R}.$$

1. a) Consider $A \in L(\mathbb{R}^n, \mathbb{R}^k)$ and suppose that $\exists\, \mu > 0$ such that

$$|Ah| \geqslant \mu |h| \;\forall\, h \in \mathbb{R}^n.$$

Prove that if $B \in L(\mathbb{R}^n, \mathbb{R}^k)$ satisfies $\|A - B\| < \lambda$ where $0 < \lambda < \mu$ and $\|\cdot\|$ denotes the operator norm then

$$|Bh| \geqslant (\mu - \lambda)|h| \;\forall\, h \in \mathbb{R}^n.$$ **[3]**

b) Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^4+y^4}}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(i) Prove that f is separately continuous at $(0,0)$. **[2]**

(ii) Prove that f is not continuous at $(0,0)$. **[2]**

(iii) Prove that f is bounded. **[2]**

Remark. A function which is separately continuous may still be unbounded.

c) Fix $A \in \mathbb{R}^{k \times n}$ and define $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(x) := \frac{1}{2}|Ax|^2$. Calculate the directional derivative $\partial_v f(x)$, $v \in \mathbb{R}^n$ and deduce that $\nabla f(x) = A^T A x$ where A^T is the transpose of A. **[5]**

d) Let $f(x,y) := xe^y$ and let C be the spiral parameterised by

$$r(t) := t\big(\cos(\pi/t)\,,\, \sin(\pi/t)\big),\; 0.1 \leqslant t \leqslant 1.$$

Evaluate $\int_C \nabla f \cdot dr$. **[4]**

e) Given that the vector field $\underline{v}(x,y,z) = (yz{+}2xz,\; xz{+}1,\; x^2{+}xy{+}2)$ is conservative, find a scalar potential of $\underline{v}$. **[5]**

f) Suppose $u \in C^2(\mathbb{R}^3)$ satisfies $\Delta u = 1$. Prove that for any bounded region $\Omega \subset \mathbb{R}^3$ we have

$$\mathrm{Vol}(\Omega) = \iint_{\partial\Omega} \nabla u \cdot n_+ \, dA$$

where n_+ is the outward unit normal to Ω. **[3]**

g) Consider a function $f\colon \mathbb{R} \to \mathbb{R}$ whose graph $\mathcal{G}_f$ is a closed subset of $\mathbb{R}^2$. Use the sequential definition of continuity to prove that if f is also bounded then it is continuous. **[4]**

h) Find all the critical points of $f(x,y) := x^4 + 4x^2y^2 - 2x^2 + 2y^2 - 1$ and classify them into local maxima, minima and saddles. **[10]**

2. a) Let $h\colon \mathbb{R} \to \mathbb{R}$ be differentiable and define define $u\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x, y) = \cos(x)\, h(3x - y).$$

Verify that u satisfies the partial differential equation

$$\cos(x) \left(\frac{\partial u}{\partial x} + 3 \frac{\partial u}{\partial y} \right) + \sin(x) u = 0.$$ **[4]**

b) Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} x^2 \sin(1/x) + y^2 \sin(1/y), & \text{if } xy \neq 0, \\ 0, & \text{if } xy = 0. \end{cases}$$

(i) Prove that f is differentiable at $(0, 0)$. **[3]**

(ii) Calculate the partial derivatives of f at points (x, y) for which $xy \neq 0$ and prove that f is not continuously differentiable at $(0, 0)$. **[3]**

c) Define $\Psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Psi(x, y) := (x^2 - 3e^y\, ,\, xe^{-y}) =: \big(u(x, y), v(x, y)\big).$$

(i) At which points (x, y) is $\partial\Psi(x, y)$ not invertible? **[4]**

(ii) Find the two points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ such that $\Psi(p_1) = \Psi(p_2) = (-2, 1)$. **[3]**

(iii) For $i \in \{1, 2\}$, let Ψ_i^{-1} be the branch of Ψ^{-1} which maps $(-2, 1)$ to p_i. Calculate $\partial\Psi_i^{-1}(-2, 1)$ and, writing $\Psi_1^{-1}(u, v)$ as $\big(x(u, v), y(u, v)\big)$, read off $\frac{\partial x}{\partial v}(-2, 1)$ and $\frac{\partial y}{\partial u}(-2, 1)$ from $\partial\Psi_1^{-1}(-2, 1)$. **[4]**

You are free to choose whichever branch you want to call Ψ_1^{-1}.

d) Let $U \subset \mathbb{R}^n$ be open and let $\Psi \in C^1(U, \mathbb{R}^n)$ be given. Prove that if $D\Psi(x)$ is invertible for all $x \in U$ then $\Psi(U)$ is an open subset of $\mathbb{R}^n$. **[4]**

e) Let $I^2 := \{(x, y) \in \mathbb{R}^2 : \max\{|x|, |y|\} < R\}$, i.e., I_2 is the open square in $\mathbb{R}^2$ of side length $2R$ centred at the origin. Suppose that both partial derivatives of $f\colon I^2 \to \mathbb{R}$ are bounded, i.e., $\exists\, M > 0$ such that $|\partial_1 f(x, y)| \leqslant M$ and $|\partial_2 f(x, y)| \leqslant M\ \forall\, (x, y) \in I^2$. Prove that f is Lispchitz continuous on I^2. **[5]**

Hint. If (x, y) and $(x + h, y + k)$ both lie in I^2 apply the 1-variable mean value theorem to f restricted to the horizontal line segment joining (x, y) to $(x + h, y)$ and to the vertical line segment joining $(x + h, y)$ to $(x + h, y + k)$.

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3. a) Define $F\colon \mathbb{R}^4 \to \mathbb{R}^2$ by

$$F(x, y, u, v) := (x^2 + y^2 + u^2 - v^2\,,\; xy + uv)$$

and let $S := \{(x, y, u, v) \in \mathbb{R}^4 : F(x, y, u, v) = (-1, 0)\}$.

(i) Demonstrate, using the Implicit Function Theorem, that it is possible to solve the equations

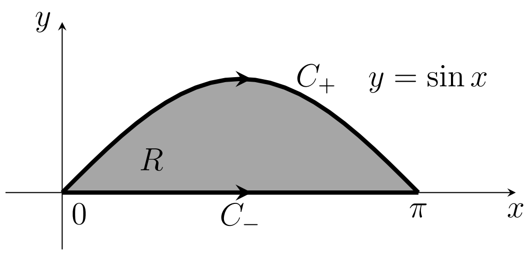
$$x^2 + y^2 + u^2 - v^2 = -1, \qquad xy + uv = 0$$

for u and v in terms of continuously differentiable functions of x and y near any point $p \in S$. **[5]**

(ii) Compute $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$ in terms of $x, y, u(x, y), v(x, y)$ where $u(x, y)$ and $v(x, y)$ are the functions whose existence is asserted in part a) (i). **[5]**

(iii) By part a) (i), S is a regular 2-dimensional submanifold of $\mathbb{R}^4$. Write down the two equations that define the tangent space T_pS as a subset of $\mathbb{R}^4$, $p = (0, 0, 0, 1)$. **[2]**

b) Let C_- be the line segment $[0, \pi]$ along the x-axis in the x-y plane, and let C_+ be the graph of $y = \sin x$, $x \in [0, \pi]$. Let t denote the unit tangent of C_+ and C_- oriented from left to right as shown in the diagram below.



Let $\underline{v}$ be the vector field defined by $\underline{v}(x, y) := (y + e^x, 1)$.

(i) Evaluate $\int_{C_-} \underline{v} \cdot t\, ds$ and $\int_{C_+} \underline{v} \cdot t\, ds$. **[5]**

(ii) Calculate $curl\,\underline{v}$. **[1]**

(iii) Let R be the region enclosed by C_- and C_+. Evaluate $\iint_R curl\,\underline{v}\, dx\, dy$ without applying Green's theorem. **[2]**

(iv) Explain how your calculation in parts b) (i) and b) (iii) verify Green's theorem. Be sure to account for the orientations of C_- and C_+ correctly. **[2]**

c) Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^3$ which is centred at the origin and let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be twice continuously differentiable and satisfy the equation

$$\nabla \cdot \left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = H, \quad H \in \mathbb{R}.$$

(i) Show that

$$\iint_{|x|=1} \frac{(\nabla f(x)) \cdot x}{\sqrt{1+|\nabla f|^2}}\, dA = \frac{4\pi}{3} H. \qquad [\mathbf{4}]$$

(ii) Deduce that $|H| \leqslant 3$. [**4**]
