MA 2590

THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: APRIL 2019

MULTIVARIABLE CALCULUS

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

Throughout this examination,

- $L(\mathbb{R}^n, \mathbb{R}^k) := \{A : \mathbb{R}^n \to \mathbb{R}^k \mid A \text{ is linear}\}$; $L(\mathbb{R}^n) := L(\mathbb{R}^n, \mathbb{R}^n)$.
- $M(k \times n, \mathbb{R})$ will denote the space of $k \times n$ matrices with real entries in k rows and n columns.
- You may use, without proof, the inequality

$$|xy| \leqslant \tfrac{1}{2}(x^2 + y^2) \quad \forall\, x, y \in \mathbb{R}.$$

COMPULSORY QUESTION

1. a) Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) := \frac{x^3}{x^2+y^2}\,, \quad \text{if } (x,y) \neq (0,0), \qquad f(0,0) := 0.$$

(i) Show that the directional derivative $D_{(\alpha,\beta)}f(0,0)$ exists $\forall\ (\alpha,\beta) \in \mathbb{R}^2 \setminus \{0\}$. [3]

(ii) Explain why the result in part (i) shows that f is not differentiable at $(0,0)$. [2]

b) Define the *operator norm* $\|A\|$ of $A \in L(\mathbb{R}^n, \mathbb{R}^k)$. [1]

c) Suppose that $A \in L(\mathbb{R}^n)$ and

$$\text{for some } \alpha > 0, \quad |Ax| \geqslant \alpha|x| \quad \forall\, x \in \mathbb{R}^n.$$

(i) Prove that A is invertible and that $\|A^{-1}\| \leqslant \alpha^{-1}$. [4]

(ii) Prove that if $H \in L(\mathbb{R}^n)$ and $\|H\| \leqslant \alpha - \delta$ for some $\delta \in [0,\alpha)$ then $A+H$ is also invertible. [2]

d) Let $U \subset \mathbb{R}^n$ be open and path connected. Use the Fundamental Theorem of Calculus and the Chain Rule to prove that if $f\colon U \to \mathbb{R}$ satisfies $\nabla f(x) = 0\ \forall\, x \in U$ then f is constant on U. [4]

e) Given that the vector field $\underline{v}(x,y,z) := (x+y-1, x-\cos z, y\sin z)$ is conservative, find a scalar potential of $\underline{v}$. [5]

f) Define $\Psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Psi(x,y) := (2x+y^2, e^x y).$$

(i) Calculate $D\Psi(x,y)$. [2]

(ii) At which points $(x,y) \in \mathbb{R}^2$ is $D\Psi(x,y)$ not invertible? [2]

(iii) Verify that the Inverse Function Theorem is applicable at $(x,y) = (-1,2)$ and state what it asserts about the existence of an inverse defined on an open set containing $\Psi(-1,2)$. [4]

(iv) Given that $\Psi(-1,2) = (2, 2/e)$, calculate $D\Psi^{-1}(2,2/e)$. [2]

g) Let U be an open subset of $\mathbb{R}^3$ and suppose that $f \in C^2(U)$ is harmonic. Let Ω be a bounded region in $\mathbb{R}^3$ such that $\overline{\Omega} \subset U$. Prove that $\iint_{\partial\Omega} \nabla f \cdot n_+\, dA = 0$. [2]

h) Let $f\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ be radial, i.e., $\exists\, \varphi\colon \mathbb{R}_{>0} \to \mathbb{R}$ such that $f(x) = \varphi(|x|)\ \forall\, x \in \mathbb{R}^3 \setminus \{0\}$.

(i) Assume that $\varphi \in C^1(\mathbb{R}_{>0})$ and calculate $\nabla f(x)$ in terms of φ' and x. [3]

(ii) Assume further that $\varphi \in C^2(\mathbb{R}_{>0})$ and that f is harmonic. Use the result in part g) with an appropriate choice of Ω and the result in (i) of part h) to derive a first order differential equation that has to be satisfied by φ. [4]

OPTIONAL QUESTIONS

2. a) (i) Let

$$f(x,y) := \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Use the ε-δ definition of continuity to prove that f is continuous at $(0,0)$. [3]

(ii) For $(x,y) \neq (0,0)$, let

$$g(x,y) := \frac{xy}{x^2+y^2}.$$

Prove that $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist. [2]

b) Let $h\colon \mathbb{R} \to \mathbb{R}$ be differentiable and define $u\colon \mathbb{R}^2 \to \mathbb{R}$ by $u(x,y) := e^{-x}h(2x-y^4)$. Show that u satisfies

$$2y^3\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + 2y^3 u = 0\,.$$ [5]

c) Let Ω be the closed solid upper half-ball of radius 2, i.e.,

$$\Omega := \{(x,y,z) : x^2+y^2+z^2 \leqslant 4,\ z \geqslant 0\}.$$

The boundary of Ω then consists of the upper hemisphere S,

$$S := \{(x,y,z) : x^2+y^2+z^2 = 4,\ z \geqslant 0\},$$

and the closed flat disk D of radius 2 centred at the origin in the x-y plane,

$$D := \{(x,y,0) : x^2+y^2 \leqslant 4\}.$$

Consider the vector field

$$\underline{v}(x,y,z) := (e^z - 1, 3y + \sin z, 1 - x)$$

and let n_+ be the unit normal on D and S which points out of Ω.
Calculate the flux of $\underline{v}$ across D in the direction of n_+ and, using the divergence theorem or otherwise, calculate the flux of $\underline{v}$ across S in the direction of n_+. [10]

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3. a) Let $f(x, y) := (e^x + \sin(y))^2$. Calculate

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y^2}$$

and write down the Taylor expansion of f about $(0, 0)$ up to, and including, quadratic terms. [6]

b) Let $g(x, y) := x^3 - 3x + y^3 - 3y$.

(i) Find the coordinates of the four critical points of g. [4]

(ii) Classify these four critical points as saddles and local maxima and minima. [4]

c) (i) State Green's Theorem in the plane, taking care to define the *curl* of a planar vector field $\underline{v}$ and the positively oriented parameterisation of the boundary of a planar region. [4]

(ii) Show that if U is an open subset of $\mathbb{R}^2$ and $h \in C^2(U)$ then $\operatorname{curl}(\nabla h) = 0$ at all points of U. [2]

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4. a) Define $f\colon M(k \times n, \mathbb{R}) \to M(n \times n, \mathbb{R})$ by $f(A) := A^T A$ where the superscript T denotes transpose. Calculate $Df(A)$ directly from the definition of derivative. [4]

b) Let $U \subset \mathbb{R}^2$ be open and suppose that the two partial derivatives $D_1 f$ and $D_2 f$ of $f\colon U \to \mathbb{R}$ exist at all points of U and are continuous at $(\alpha, \beta) \in U$. Prove that f is differentiable at (α, β). [9]

c) Consider the vector field $\underline{v}(x, y, z) := (2xz + y^2, xy, 1)$ and let $p := (0, 0, 0)$ and $q := (1, 1, 1)$.

(i) Evaluate $\int_{C_{pq}} \underline{v} \cdot dr$ where C_{pq} is the straight line from p to q. [3]

(ii) Let Γ_{pq} be the curve from p to q parameterised by $\rho(t) := (t,\, t^2,\, t^3)$, $0 \leqslant t \leqslant 1$. Evaluate $\int_{\Gamma_{pq}} \underline{v} \cdot d\rho$. [3]

(iii) Does $\underline{v}$ admit a scalar potential? Justify your answer. [1]

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5. a) Let $K \subset \mathbb{R}^n$ be sequentially compact and let $f\colon K \to \mathbb{R}^k$ be continuous.

(i) Prove that $f(K)$ is sequentially compact. [3]

(ii) Assume further that $k = 1$. Prove that $\exists\, x_*,\, x^* \in K$ such that

$$f(x_*) \leqslant f(x) \leqslant f(x^*)\ \forall\, x \in K.$$

You may assume, without proof, that a subset of $\mathbb{R}^n$ is sequentially compact if, and only if, it is closed and bounded. [2]

b) Let U be an open subset of $\mathbb{R}^n$ and suppose that $f\colon U \to \mathbb{R}^k$ is differentiable at $p \in U$ and that,

$$\text{for some } \alpha > 0, \quad |(Df(p))h| \geqslant \alpha |h|\ \forall\, h \in \mathbb{R}^n.$$

Prove that $\exists\, \delta > 0$ such that

$$|h| < \delta \Rightarrow |f(p+h) - f(p)| \geqslant \tfrac{1}{2}\alpha |h|.$$ [5]

c) Define $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ by

$$F(x, y_1, y_2) := (x^4 + y_1^4 + y_2^4\,,\, y_1 + y_2).$$

Let $p := (0,1,0)$ and observe that $F(p) = (1,1)$.

(i) Calculate $DF(0,1,0)$. [2]

(ii) Verify, by appealing to the Implicit Function Theorem, that it is possible to solve the equations $F(x, y_1, y_2) = (1,1)$ for y_1 and y_2 in terms of x, $x \in (-\delta, \delta)$ for some $\delta > 0$. Furthermore, calculate $y_1'(x)$ and $y_2'(x)$ in terms of x, $y_1(x)$ and $y_2(x)$. [6]

(iii) Explain why there does not exist a C^1-map $h\colon (-\eta, \eta) \to \mathbb{R}^2$ for some $\eta > 0$ such that

$$h(0) = (0,1) \quad \text{and} \quad F(h(y_2), y_2) = (1,1)\ \forall\, y_2 \in (-\eta, \eta).$$ [2]

Hint: Differentiate the relation $F(h(y_2), y_2) = (1,1)$ with respect to y_2 and consider the result at $y_2 = 0$.

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