THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: SUMMER 2022

METRIC SPACES

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

COMPULSORY QUESTION

1. a) (i) For each of the following, just answer yes or no. (You do not need to justify your answers.)

(1) Does
$$\|(x, y, z)\| := |x| + 2|y| + 7|z|$$
define a norm on $\mathbb{R}^3$?

(2) Does
$$\|(x, y, z)\| := 2|x| - |y| + 5|z|$$
define a norm on $\mathbb{R}^3$?

(3) Does
$$\|f\| := \left| \int_0^1 f(x)\, dx \right|$$
define a norm on $C[0, 1]$?

(4) Does
$$\|f\| := \left(\int_0^1 \frac{|f(x)|^2}{2x + 1}\, dx \right)^{1/2}$$
define a norm on $C[0, 1]$? **[4]**

(ii) What does in mean for two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the same vector space to be *equivalent*? **[2]**

(iii) Show that
$$\|f\|_\infty := \sup_{x \in [0,1]} |f(x)| \quad \text{and} \quad \|f\|_{L^1} := \int_0^1 |f(x)|\, dx$$
are *not* equivalent norms on $C[0, 1]$. (Hint: construct a sequence of functions to show that the definition in (ii) does not hold.) **[4]**

b) Let $\|x\| = \|x\|_{\ell^2}$ denote the Euclidean norm on $\mathbb{R}^2$. Recall that the sunflower metric d_{sf} on $\mathbb{R}^2$ is defined by
$$d_{\mathrm{sf}}(x, y) = \begin{cases} \|x - y\| & \text{if } x \text{ and } y \text{ lie on the same line through the origin} \\ \|x\| + \|y\| & \text{otherwise.} \end{cases}$$

(i) What is the distance in the sunflower metric between the points $(3, 0)$ and $(-4, 3)$? **[2]**

(ii) Describe the open unit ball centred at $(1/2, 0)$ in $(\mathbb{R}^2, d_{\mathrm{sf}})$. (It is sufficient to draw a clearly marked diagram.) **[2]**

Question 1 continued overleaf

(iii) Does the sequence $((\cos(1/n), \sin(1/n)))_{n=1}^{\infty}$ converge to $(1, 0)$ in $(\mathbb{R}^2, d_{\mathrm{sf}})$? Justify your answer. **[3]**

(iv) Show that the set $\{x \in \mathbb{R}^2 : \|x\| = 1\}$ is *not* compact with respect to the topology given by the sunflower metric. (You may assume that a set in a metric space is compact if and only if it is sequentially compact.) **[4]**

c) (i) Let $(T, \mathcal{T})$ be a topological space and let $A \subset T$. What is the definition of the *closure*, the *interior* and the *boundary* of A? **[3]**

(ii) Consider $\mathbb{R}$ with the standard topology. Determine the *closure*, *interior* and *boundary* of each of the following sets. (You do not need to justify your answers.)

(1) $\mathbb{R} \setminus \mathbb{Q}$

(2) $[0, 1] \setminus C$, where C is the middle third Cantor set. **[6]**

d) Show that the set

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2 + e^y}{x + y^2 e^z} > 3 \right\}$$

is open in $\mathbb{R}^3$ with respect to the standard topology. **[10]**

OPTIONAL QUESTIONS

2. Let $(T, \mathcal{T})$ be a topological space.

a) What does it mean for $(T, \mathcal{T})$ to be a *Hausdorff* space? **[2]**

b) Show that if $(T, \mathcal{T})$ is Hausdorff then the limit of a convergent sequence in $(T, \mathcal{T})$ is unique. **[6]**

c) (i) Give the definition of the *co-finite topology* on $\mathbb{R}$. **[2]**

(ii) Show directly from the definition that $\mathbb{R}$ with the co-finite topology is not Hausdorff. **[4]**

(iii) What are the limits of the sequence $(2^{-n})_{n=1}^{\infty}$ in $\mathbb{R}$ with respect to the co-finite topology? Justify your answer. **[6]**

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3. a) Let $(T, \mathcal{T})$ be a topological space. What does it mean for this space to be *compact*? **[2]**

b) For each of the following spaces, give a brief reason why it is compact (one or two sentences will suffice).

(i) $E = \{(x,\, y) \in [-1, 1]^2 \;:\; x^3 + xy = \sin y\}$

(ii) $F = \{(y \sin x,\; y \cos x,\; y^2) \in \mathbb{R}^3 \;:\; 0 \le x \le 100,\; 0 \le y \le 100\}$ **[5]**

(The subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$ are considered with the standard topology.)

c) Let x be a point in a metric space X. Assume that for each positive integer n a compact set $K_n \subset X$ is given such that

$$x \in K_n \subset B(x, 1/n).$$

Show that

$$\bigcup_{n=1}^{\infty} K_n$$

is compact. **[8]**

d) Let $\mathcal{U}$ be an open cover of a metric space (X, d). Define what we mean by a Lebesgue number of $\mathcal{U}$. **[2]**

e) Give an open cover of the open interval $(0, 1)$ that does not have a Lebesgue number. (You do not have to justify your answer.) **[3]**

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4. a) Show that the continuous image of a connected set is connected. **[5]**

b) Consider two open squares in the plane,

$$S = \{(x, y) \in \mathbb{R}^2 \,:\, 0 < x < 1,\ 0 < y < 1\},$$

$$R = \{(x, y) \in \mathbb{R}^2 \,:\, -1 < x < 0,\ 0 < y < 1\}.$$

Find all those points $(x, y) \in \mathbb{R}^2$ for which

$$S \cup R \cup \{(x, y)\}$$

is a connected set. **[1]**

Justify (briefly)

- why these points work, **[3]**
- and why all others don't. **[3]**

c) Assume that A and B are non-empty topological spaces such that there is a homeomorphism $f : [0, 1] \to A \times B$. Show that either A or B is a singleton (that is, consists of one point only). **[8]**

(You may use without proof that the sets of the form $A \times \{b\}$, for any $b \in B$, are homeomorphic to A.)

5. a) Let A be a closed subset of a complete metric space (X, d). Show that the metric space A (as a subspace of X) is complete. **[5]**

b) We have seen that the Contraction Mapping Theorem can be used to prove the Picard–Lindelöf Theorem about uniqueness and existence of solutions to certain ordinary differential equations.

- Carefully state a version of the Picard–Lindelöf Theorem (as seen at lectures or in your preferred form).
- Describe the metric space and define the map for which the Contraction Mapping Theorem is applied in the proof of this theorem.

(You do not have to give any details of the proof; in particular, you do not have to prove that your map is a contraction.) **[7]**

c) (i) In the space $C([0, 1])$ of continuous functions $f : [0, 1] \to \mathbb{R}$ with the supremum norm, give a *uniformly bounded* sequence of functions (f_n) that has no convergent subsequence (that is, no subsequence converging in the supremum norm). **[4]**

(ii) In the space $C_b(\mathbb{R})$ of bounded continuous functions $g : \mathbb{R} \to \mathbb{R}$ with the supremum norm, give a *uniformly bounded uniformly equicontinuous* sequence of functions (g_n) that has no convergent subsequence (that is, no subsequence converging in the supremum norm). **[4]**

Give a brief justification why your sequences work (two or three sentences for each will suffice).
