MA260

THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: JUNE 2019

NORMS, METRICS, & TOPOLOGIES

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

COMPULSORY QUESTION

1. a) What is meant by a norm on a vector space X? [2]

   b) For $1 \leq p < \infty$ define the sequence space ℓ^p, along with its norm. Show, without using the triangle inequality in ℓ^p, that if $\mathbf{x}, \mathbf{y} \in \ell^p$ then $\mathbf{x} + \mathbf{y} \in \ell^p$. [4]

   c) What is meant by a metric d on a set X? [2]

   d) Show that $d(x, y) := \max_{j=1,\ldots,n} |x_j - y_j|$ is a metric on $\mathbb{R}^n$ (where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$). [3]

   e) What is meant by an open set in a metric space (X, d)? Show directly from the definition that any open ball $B(x, r)$ is open. [4]

   f) Given a map $f : X \to Y$, define the preimage $f^{-1}(B)$ for $B \subset Y$. Show that $f(f^{-1}(B)) \subset B$ for any $B \subset Y$ and $f^{-1}(f(A)) \supset A$ for any $A \subset X$, and that both of these inclusions can be strict (i.e. not equalities). [4]

   g) Give the definition of a topology $\mathcal{T}$ on a set T. [3]

   h) Let $(T_1, \mathcal{T}_1)$ and $(T_2, \mathcal{T}_2)$ be topological spaces. What does it mean for a map $f : T_1 \to T_2$ to be continuous? [2]

   i) Let $(T_i, \mathcal{T}_i)$, $i = 1, 2, 3$, be topological spaces. Show that if $f : T_1 \to T_2$ and $g : T_2 \to T_3$ are continuous then $g \circ f : T_1 \to T_3$ is continuous. [3]

   j) If $(T, \mathcal{T})$ is a topological space and $S \subset T$ define the subspace topology on S, and show that it really is a topology. [4]

   k) What does it mean for a subset A of a topological space $(T, \mathcal{T})$ to be path connected? Show that any open ball in $\mathbb{R}^n$ (with its standard topology) is path connected. [4]

   l) What does it mean for a subset A of a topological space to be nowhere dense? State the Baire Category Theorem ('nowhere dense version'), and use this to show that the closed interval $[0, 1]$ is uncountable. [5]

OPTIONAL QUESTIONS

2. a) Given any uncountable set T, the co-countable topology $\mathcal{T}_{cc}$ on T consists of T, $\varnothing$, and all subsets of T whose complement is at most countable [i.e. either finite or countably infinite]. Show that this is indeed a topology on T. [4]

b) What is meant by saying that a topological space is Hausdorff? [2]

c) Show that any metric space is Hausdorff. [2]

d) Show that the topological space $(T, \mathcal{T}_{cc})$ from part (a) is not Hausdorff. [3]

e) What does it mean for a sequence to converge in a topological space? [2]

f) Show that in a Hausdorff space a sequence can have at most one limit. [3]

g) Show that a convergent sequence in $(T, \mathcal{T}_{cc})$ [the space from part (a)] is eventually constant and so has a unique limit. [4]

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3. Let $(T, \mathcal{T})$ be a topological space.

a) The closure $\overline{A}$ of a subset A of T is the intersection of all closed subsets of T that contain A. Show that $x \in \overline{A}$ if and only if every open set that contains x intersects A (i.e. if $x \in U$ and U is open then $U \cap A \neq \varnothing$). [5]

b) Using part (a) show that if A and B are open and $A \cap B = \varnothing$ then $\overline{A} \cap B = \varnothing$. [4]

c) What does it mean for a topological space to be connected? [2]

d) Show that a topological space is not connected if and only if there exists a continuous surjective map $f : T \mapsto \{0, 1\}$ (with the discrete topology on $\{0, 1\}$). [3]

e) Show that if A and B are connected subsets of T such that $\overline{A} \cap B \neq \varnothing$ then $A \cup B$ is connected. [6]

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4. a) What does it mean for a topological space to be compact? Explain the terms you use. [2]

b) Show that any compact subset of a Hausdorff topological space is closed. [5]

c) Suppose that X is a Hausdorff topological space and that (A_n) is a decreasing sequence of non-empty compact subsets of X ($A_{n+1} \subset A_n$ for every $n \in \mathbb{N}$). Show that $\cap_{n=1}^{\infty} A_n$ is not empty. [5]

d) What does it mean for a topological space to be sequentially compact? [2]

e) Suppose that X is a sequentially compact topological space and that (A_n) is a decreasing sequence of non-empty closed subsets of X. Show that $\cap_{n=1}^{\infty} A_n$ is not empty. [Hint: for each $n \in \mathbb{N}$ take some $x_n \in A_n$ and show that (x_n) has a subsequence that converges to a point x such that $x \in A_k$ for every $k \in \mathbb{N}$.] [6]

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5. a) What is meant by a Cauchy sequence in a metric space (X, d)? [2]

b) Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence in a metric space (X, d) such that

$$\sum_{j=1}^{\infty} d(x_{j+1}, x_j) < \infty$$

(i.e. the sum converges). Show that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. [4]

c) What does it mean for a metric space to be complete? [2]

d) State the Contraction Mapping Theorem. [2]

e) Use the Contraction Mapping Theorem to show that for every $M \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ with $|\alpha| < 1$ there is a unique real solution E of the equation [5]

$$M + \alpha \sin E = E.$$

f) What does it mean for a metric space to be totally bounded? [2]

g) Let X be an infinite set with the discrete metric. Show that X is bounded but not totally bounded. [3]

END