THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: SUMMER 2021

Norms, Metrics and Topologies

Time Allowed: **2 hours**

Read all instructions carefully. Please note also the guidance you have received in advance on the departmental 'Warwick Mathematics Exams 2021' webpage.

Calculators, wikipedia and interactive internet resources are not needed and are not permitted in this examination. You are not allowed to confer with other people. You may use module materials and resources from the module webpage.

ANSWER COMPULSORY QUESTION 1 AND TWO FURTHER QUESTIONS out of the three optional questions 2, 3 and 4.

On completion of the assessment, you must upload your answer to Moodle as a single PDF document if possible, although multiple files (2 or 3) are permitted. You have an additional 45 minutes to make the upload, and instructions are available on the departmental 'Warwick Mathematics Exams 2021' webpage.

You must not upload answers to more than 3 questions, including Question 1. If you do, you will only be given credit for your Question 1 and the first two other answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question. The compulsory question is worth 40 marks, while each optional question is worth 30 marks.

COMPULSORY QUESTION

1. a) State whether each of the following is a norm on the given vector space. You **do not** have to justify your answers.

(i) $\|(x,y,z)\| = |x| + 2|y| + |z|$ on the vector space $\mathbb{R}^3$. **[2]**

(ii) $\|(x,y,z)\| = |x| - |y| + |z|$ on the vector space $\mathbb{R}^3$. **[2]**

(iii)

$$\|(x_k)_{k=1}^{\infty}\| = \sum_{k=1}^{\infty} |x_k|$$

on the vector space ℓ^2. **[2]**

(iv)

$$\|f\| = \int_0^1 |f(x)|\, x^2 \, dx$$

on the vector space $C[0,1]$. **[2]**

b) Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^2$ and let d be the standard metric, i.e. for $x, y \in \mathbb{R}^2$, $d(x,y) = \|x - y\|$. The *sunflower metric* d_{sf} on $\mathbb{R}^2$ is defined by

$$d_{\text{sf}}(x,y) = \begin{cases} \|x-y\| & \text{if } x \text{ and } y \text{ lie on same line through the origin,} \\ \|x\| + \|y\| & \text{otherwise.} \end{cases}$$

(i) Sketch the open unit ball centred at $(0, 1/2)$ in the sunflower metric. (You **do not** have to justify your answer.) **[2]**

Let $I : \mathbb{R}^2 \to \mathbb{R}^2$ denote the identity map.

(ii) Show that $I : (\mathbb{R}^2, d_{\text{sf}}) \to (\mathbb{R}^2, d)$ is continuous. **[3]**

(iii) Show that $I : (\mathbb{R}^2, d) \to (\mathbb{R}^2, d_{\text{sf}})$ is not continuous. **[3]**

You may use without proof any convenient criterion for continuity.

c) (i) Write down the definition of a sequence $(x_n)_{n=1}^{\infty}$ in a topological space $(T, \mathcal{T})$ converging to a point $x \in T$. **[1]**

(ii) Consider the sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}^2$ given by

$$x_n = \left(1 + \frac{1}{n^2}, \frac{1}{n}\right).$$

Use the definition to show that this sequence converges to $(1,0) \in \mathbb{R}^2$ with respect to the standard topology on $\mathbb{R}^2$. **[3]**

(iii) Now suppose that $\mathbb{R}^2$ has the co-finite topology. Show that the sequence $(x_n)_{n=1}^{\infty}$ in (ii) converges to every point in $\mathbb{R}^2$ with respect to this topology. **[4]**

Question 1 continued overleaf

d) Show that the set

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{7\sin(x^2 + y) - 3}{x - y^3 + z^4} < 2 \right\}$$

is open in $\mathbb{R}^3$ (with respect to the standard topology). **[7]**

e) Let $(T, \mathcal{T})$ be a topological space.

(i) Define the closure $\overline{A}$ of a set $A \subset T$. **[2]**

(ii) Let $A, B \subset T$. Is it always true that

$$\overline{A \cap B} = \overline{A} \cap \overline{B} \ ?$$

Give a proof or a counterexample. **[3]**

(iii) Suppose that $\mathcal{T}$ is the topology induced by a metric d. Show that if $x \in \overline{A}$ then

$$\inf_{a \in A} d(x, a) = 0.$$

You may use results proved in the course provided you state them clearly. **[4]**

OPTIONAL QUESTIONS

2. a) For each of the following spaces, decide whether or not it is **compact**, and give a brief reason (one or two sentences will suffice).

(All subsets of $\mathbb{R}$ and $\mathbb{R}^2$ below are considered with the standard topology.)

(i) $[1, \infty)$

(ii) $\{1/n \in \mathbb{R} \,:\, n \text{ is a positive integer}\}$

(iii) $\{(x, y) \in \mathbb{R}^2 \,:\, x^4 + 5y^2 = 1\}$

(iv) $\{(\cos x, \, \sin x) \in \mathbb{R}^2 \,:\, x \text{ is an element of the Cantor set}\}$

(v) $C[0, 1]$, that is, the space of continuous functions $f : [0, 1] \to \mathbb{R}$ with the supremum norm $\| \cdot \|_\infty$. **[7]**

b) Consider $\mathbb{R}$ with the standard metric and its open cover

$$\{(n, \, n + 2) \,:\, n \in \mathbb{Z}\}.$$

Give the **largest** Lebesgue number of this open cover. You **do not** have to justify your answer. **[3]**

c) Let (X, d) be a metric space for which every open cover has a Lebesgue number. Show that every continuous function $f : X \to \mathbb{R}$ is uniformly continuous. **[6]**

d) Let T be a compact Hausdorff space. Suppose that $F \subset T$ is closed and $x \in T \setminus F$. Show that there are disjoint open sets U and V such that $x \in U$ and $F \subset V$. **[8]**

e) Show that there is no compact metric space (X, d) for which the set of all distances

$$\{d(x, y) \in \mathbb{R} \,:\, x, y \in X\}$$

is equal to

$$\{0\} \cup [1, 2].$$

[6]

3. a) Without using any results from lectures about connectedness, give a direct proof that the following two statements are equivalent for every topological space T.

S1. T has a subset that is both open and closed and is neither $\varnothing$ nor T.

S2. There is a continuous surjection from T onto the two-point set $\{0, 1\}$ with the discrete topology. **[6]**

b) For the following four pairs of topological spaces argue *briefly* why they are not homeomorphic (one or two sentences will suffice).

(i) $\mathbb{Q}$ and the Cantor set;

(ii) The Cantor set and $[0, 1]$;

(iii) $(0, 1) \cup (1, 2)$ and $\mathbb{R} \setminus \{0, 1\}$;

(iv) $[0, 1) \cup [2, 3)$ and $(0, 1) \cup (2, 3)$.

(All of these spaces are considered as subspaces of $\mathbb{R}$ with the standard topology.) **[8]**

c) Let $f : \mathbb{R} \to [-1, 1]$, and let G denote its graph, that is,

$$G = \{(x,\ f(x)) \in \mathbb{R}^2 \ : \ x \in \mathbb{R}\}.$$

(i) Show that if f is continuous, then $\mathbb{R}^2 \setminus G$ is disconnected. **[6]**

(ii) Show that if f is not continuous, then $\mathbb{R}^2 \setminus G$ is connected. **[10]**
[Hint: If this function f is not continuous, then there is $x_0 \in \mathbb{R}$ and a sequence $x_n \to x_0$ such that $f(x_n) \to y \neq f(x_0)$. If you want to use this, briefly argue why it is true.]

4. a) For each of the following spaces, decide whether or not it is **complete**, and give a brief reason (one or two sentences will suffice).

(i) The set of rational numbers $\mathbb{Q}$ (with the standard metric).

(ii) $\mathbb{R}$ with the metric $d(x,y) = \sqrt{|x-y|}$.

(iii) The space of continuous functions $f : \mathbb{R} \to [-1,1]$ with the supremum norm $\|\cdot\|_\infty$.

(iv) The space of continuous functions $f : \mathbb{R} \to [0,1)$ with the supremum norm $\|\cdot\|_\infty$. **[8]**

b) For each of the following families of functions decide whether it is

- uniformly equicontinuous, or
- equicontinuous but not uniformly equicontinuous, or
- not even equicontinuous.

You **do not** have to justify your answer.

(i) $\{f_n : \mathbb{R} \to \mathbb{R} \,:\, n = 1,2,3,\ldots\}$ where $f_n(x) = \sin(nx)$.

(ii) $\{f_{a,b} : \mathbb{R} \to \mathbb{R} \,:\, a,b \in [0,1]\}$ where $f_{a,b}(x) = ax + b$.

(iii) $\{g_{a,b} : \mathbb{R} \to \mathbb{R} \,:\, a,b \in [0,1]\}$ where $g_{a,b}(x) = ax^2 + b$.

(iv) $\{h_{a,b} : \mathbb{R} \to \mathbb{R} \,:\, a,b \in \mathbb{R}\}$ where $h_{a,b}(x) = ax + b$. **[4]**

c) Assume that the functions $f_n : [0,1] \to \mathbb{R}$ ($n = 1,2,\ldots$) satisfy $f_n(0) = 0$ and that

$$|f_n(x) - f_n(y)| \le \sqrt{|x-y|} \qquad \text{for every } x,y \in [0,1].$$

Show that f_n has a subsequence that converges with respect to the supremum norm $\|\cdot\|_\infty$. (You may use theorems from the lectures without proof, provided you state them clearly.) **[7]**

d) Let $C_b(\mathbb{R})$ denote the space of bounded continuous functions $\mathbb{R} \to \mathbb{R}$ with the supremum norm $\|\cdot\|_\infty$. Let $F : C_b(\mathbb{R}) \to C_b(\mathbb{R})$ be the map

$$F(f)(x) = \sin(x) + \left(\frac{f(2x)}{10}\right)^2.$$

(i) Show that F is not a contraction. **[2]**

(ii) Use the Contraction Mapping Theorem (on an appropriate space) to prove that there is a function $f \in C_b(\mathbb{R})$ satisfying

$$f(x) = \sin(x) + \left(\frac{f(2x)}{10}\right)^2 \qquad \text{for every } x \in \mathbb{R}.$$

[9]

[Hint: Perhaps show that there is such f with $\|f\|_\infty \le 2$.]