THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: SUMMER 2022

ASYMPTOTICS AND INTEGRAL TRANSFORMS

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

The steepest descent saddle point contribution at z_s, where $g'(z_s) = 0$, is given by

$$\int f(z)\mathrm{e}^{\lambda g(z)}\,\mathrm{d}z \;\sim\; \sqrt{\frac{2\pi}{-\lambda g''(z_s)}}\,f(z_s)\mathrm{e}^{\lambda g(z_s)} \qquad \text{as } \lambda \to \infty.$$

COMPULSORY QUESTION

1. a) For $\epsilon > 0$, order the following elements to construct an asymptotic sequence $(\alpha_n(\epsilon))$ as $\epsilon \to 0$: **[10]**

$$\exp\{-1/\epsilon\}, \qquad \cos\epsilon, \qquad 1-\cos\epsilon, \qquad \sin\epsilon,$$
$$\log(1/\epsilon), \qquad \log\log(1/\epsilon), \qquad \frac{1}{\epsilon}\log(1/\epsilon), \qquad \big(\log(1/\epsilon)\big)^2.$$

b) The equation $\epsilon x^6 + x - 1 = 0$ has two real roots as $\epsilon \to 0$. For each real root, find the first two terms of its asymptotic series. **[15]**

c) The Fourier transform of $f(x)$ is $\tilde{f}(k)$. Express the Fourier transform of $g(x) = xf'(x-a)$ in terms of $\tilde{f}(k)$, where a is a constant and a prime denotes $\mathrm{d}/\mathrm{d}x$. **[6]**

d) Show how the Laplace transform may be used to solve the differential equation **[9]**

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} + y = t\mathrm{e}^t, \qquad \text{with} \quad y(0) = 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t}(0) = 1.$$

OPTIONAL QUESTIONS

2. Find the first term in the asymptotic series as $\lambda \to \infty$ of the integral

$$I(\lambda, a) = \int_0^1 x\exp\big\{-\lambda\sin(ax)\big\}\,\mathrm{d}x$$

a) for $\pi < a < 3\pi/2$; **[7]**

b) for $3\pi/2 < a < 7\pi/2$. **[9]**

c) Without performing further calculations, state and briefly justify what method you would use for each of $0 < a < \pi$ and $7\pi/2 < a < 11\pi/2$. **[4]**

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3. Consider the differential equation below in the limit $\epsilon \to 0$.

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - y(1-y) = 0, \qquad \text{with} \quad y(0) = 0, \quad \frac{\mathrm{d}y}{\mathrm{d}x}(0) = \frac{1}{2\epsilon}.$$

a) Without applying any boundary conditions, find the first term in the asymptotic series for $y(x)$ for fixed x as $\epsilon \to 0$. This is the outer solution. **[4]**

[Hint: You may find it useful that

$$\int \frac{\mathrm{d}y/\mathrm{d}x}{y(1-y)}\,\mathrm{d}x = \log y - \log(1-y) + C,$$

where C is a constant of integration.]

b) Determine a rescaling for the independent variable x for small x. Using this rescaling, and both boundary conditions, find the first term in the asymptotic series for $y(\xi)$ for fixed ξ as $\epsilon \to 0$, where ξ is your rescaled independent variable. **[8]**

c) Match your two solutions to determine the constant in the outer solution. **[6]**

d) Write down a composite asymptotic solution that is valid for both fixed ξ and fixed x as $\epsilon \to 0$. **[2]**

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4. a) Show that the Fourier transform of the function $g(x) = \exp\{-a|x|\}$, where $a > 0$ is a constant, is given by **[6]**

$$\tilde{g}(k) = \frac{2a}{k^2 + a^2}.$$

b) Find the solution $f(x)$ to the integral equation **[14]**

$$f(x) + \int_{-\infty}^{\infty} \frac{3}{4} \mathrm{e}^{-4|x-t|} f'(t)\,\mathrm{d}t \;=\; \mathrm{e}^{-4|x|}.$$

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5. a) Give a definition of both the Laplace transform and the inverse Laplace transform, being careful to specify the contour of integration for each. **[4]**

b) By using a Laplace transform in time, solve the differential equation

$$\frac{\partial y}{\partial t} + x\frac{\partial y}{\partial x} = x \qquad \text{for} \quad x > 0, \quad y > 0,$$

subject to boundary conditions $y(x, 0) = 0$ and $y(0, t) = 0$. **[16]**
