THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: MOCK 2

ASYMPTOTICS AND INTEGRAL TRANSFORMS

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

The steepest descent saddle point contribution at z_s, where $g'(z_s) = 0$, is given by

$$\int f(z)\mathrm{e}^{\lambda g(z)}\,\mathrm{d}z \;\sim\; \sqrt{\frac{2\pi}{-\lambda g''(z_s)}}\,f(z_s)\mathrm{e}^{\lambda g(z_s)} \qquad \text{as } \lambda \to \infty.$$

COMPULSORY QUESTION

1. a) A function $f(x,\epsilon)$ has the asymptotic series

$$f(x,\epsilon) \sim \sum_{n=0}^{N} f_n(x,\epsilon) \qquad \text{as} \qquad \epsilon \to 0.$$

(i) Define what it means for this to be an asymptotic series for $f(x,\epsilon)$. **[2]**

(ii) Define what it means for this to be a Poincare asymptotic series. **[1]**

(iii) Show that, if this is a Poincare asymptotic series, then the coefficients in the series are unique. **[4]**

b) Find the first two nonzero terms in the asymptotic series for the solution of $\tanh x = 1 - \epsilon$ as $\epsilon \to 0$. **[9]**

c) Find the function $f(t)$ which has the Fourier transform **[16]**

$$\tilde{f}(\omega) = \frac{-5\mathrm{i}}{(\omega + \mathrm{i})(\omega^2 - 2\mathrm{i}\omega - 2)}.$$

d) Express the Laplace transform of $g(t) = t\dot{f}(t-a)$ in terms of the Laplace transform $\hat{f}(s)$, where a is a constant and a dot denotes $\mathrm{d}/\mathrm{d}t$. **[8]**

OPTIONAL QUESTIONS

2. a) Show that, as $x \to \infty$:

(i) $\tanh x = 1 + O\big(\mathrm{e}^{-2x}\big)$; **[2]**

(ii) $\log\big(\cosh x\big) = x - \log 2 + O\big(\mathrm{e}^{-2x}\big)$. **[2]**

b) Find the first two terms of the asymptotic series for $I(\epsilon)$ as $\epsilon \to 0$, where **[16]**

$$I(\epsilon) = \int_0^\infty \frac{\tanh x}{1 + \epsilon x^2}\,\mathrm{d}x.$$

You may use that $\tan^{-1} x = x + O\big(x^3\big)$ *as* $x \to 0$ *(note this is* $\tan^{-1}$*, not* $\tanh^{-1}$*).*

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3. Consider the differential equation below in the limit $\epsilon \to 0$.

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - y\cos x = 0, \qquad \text{with} \quad y(0) = 1 \quad \text{and} \quad y(\pi) = -1.$$

a) Without rescaling, find the first term in the asymptotic series for $y(x)$ for fixed x as $\epsilon \to 0$ that satisfies the boundary condition at $x = \pi$. This is the outer solution. **[4]**

b) Determine a rescaling for the independent variable x for small x. Using this rescaling, and the boundary condition at $x = 0$, find the first term in the asymptotic series for $y(\xi)$ for fixed ξ as $\epsilon \to 0$, where ξ is your rescaled independent variable. Your solution will involve an arbitrary constant. **[7]**

c) Match your two solutions to determine the arbitrary constant. **[6]**

d) Write down a composite asymptotic solution that is valid for both fixed ξ and fixed x as $\epsilon \to 0$. **[3]**

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4. A flexible beam has displacement $u(x,t)$. Initially at rest, at time $t = 0$ a distributed load $F(x)$ is placed on the beam, causing it to bend. The displacement is governed by the differential equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = F(x), \qquad \text{with} \quad u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0.$$

a) By using a Fourier transform in x, show that the solution $u(x,t)$ may be written **[14]**

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-\cos\left(k^2 t\right)}{k^4}\widetilde{F}(k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k.$$

b) Comment on and briefly justify: (i) the nature of the singularity of the integrand at $k = 0$; and (ii) any difficulties with closing the contour of integration in the upper and lower half planes. **[6]**

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5. a) Find the solution $f(t)$ to the integral equation **[16]**

$$f(t) = t^2 + \int_0^t f'(t-\tau)\mathrm{e}^{-a\tau}\,\mathrm{d}\tau \qquad \text{with} \quad f(0) = 0.$$

b) Comment on how the solution would be different if instead $f(0) = b \neq 0$. **[4]**
