THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: MOCK 1

ASYMPTOTICS AND INTEGRAL TRANSFORMS

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

The steepest descent saddle point contribution at z_s, where $g'(z_s) = 0$, is given by

$$\int f(z)\mathrm{e}^{\lambda g(z)}\,\mathrm{d}z \;\sim\; \sqrt{\frac{2\pi}{-\lambda g''(z_s)}}\,f(z_s)\mathrm{e}^{\lambda g(z_s)} \qquad \text{as } \lambda \to \infty.$$

COMPULSORY QUESTION

1. a) A rocket is powered by compressed gas, and accelerates vertically against gravity. The dimensional governing equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}(mv) = -(V - v)\frac{\mathrm{d}m}{\mathrm{d}t} - mg, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(m - m_0) = -k(m - m_0), \qquad (1)$$

where $m(t)$ is the mass of the rocket, m_0 is the empty (unfuelled) mass of the rocket, $v(t)$ is the vertical velocity of the rocket, V is the speed at which gas is ejected from the rocket, g is the acceleration due to gravity, and k is a (dimensional) constant. At time $t = 0$, the rocket is stationary, $v(0) = 0$, and has mass $m(0) = M$.

(i) Find a nondimensionalization of the governing equations and the initial conditions. **[8]**

(ii) What dimensionless constants affect the behaviour of the rocket? **[2]**

b) The equation $x^2(x^2 + x + 1)^2 - \epsilon = 0$ has two real roots as $\epsilon \to 0$. For each real root, find the first two terms of its asymptotic series. **[10]**

c) By using the Fourier transform, solve the differential equation **[10]**

$$2\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + t\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0 \qquad \text{with} \qquad y(0) = 1 \qquad \text{and} \qquad \lim_{t \to -\infty} y(t) = 0.$$

d) Find the function $f(t)$ which has the Laplace transform **[10]**

$$\hat{f}(s) = \frac{1 - (s + 1)\mathrm{e}^{-s}}{s^2}.$$

OPTIONAL QUESTIONS

2. Calculate the first term in the asymptotic series for $f(t)$ as $t \to \infty$, where **[20]**

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega^2 t}}{\omega^2 + 1} \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}\omega.$$

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3. Consider the differential equation below in the limit $\epsilon \to 0$.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + \epsilon y = 1, \qquad \text{with} \quad y(0) = \frac{\mathrm{d}y}{\mathrm{d}t}(0) = 0.$$

a) Without rescaling, find the first term in the asymptotic series for $y(t)$ for fixed t as $\epsilon \to 0$ that satisfies both boundary conditions. This is the inner solution. **[4]**

b) Find a rescaled independent variable T and a rescaled dependent variable $Y(T)$ such that a different combination of terms of the differential equation are dominant. This is the outer solution scaling. Using this scaling, find the first term in the asymptotic series for $Y(T)$ for fixed T as $\epsilon \to 0$. **[7]**

c) Match your two solutions to determine the constant in the outer solution. **[6]**

d) Write down a composite asymptotic solution that is valid for both fixed T and fixed t as $\epsilon \to 0$. **[3]**

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4. By using a Fourier transform, or otherwise, find the solution $f(x)$ to the integral equation **[20]**

$$f(x) \; - \; 4 \int_{-\infty}^{0} \mathrm{e}^{2t} f(x-t) \, \mathrm{d}t \; = \; \mathrm{e}^{-2|x|}.$$

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5. By using a Laplace transform in time, solve the differential equation

$$\frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} = 0 \qquad \text{for} \quad x > 0, \quad y > 0,$$

subject to boundary conditions $y(x, 0) = x\mathrm{e}^{-x}$ and $y(0, t) = t\mathrm{e}^{-t}$. **[20]**
