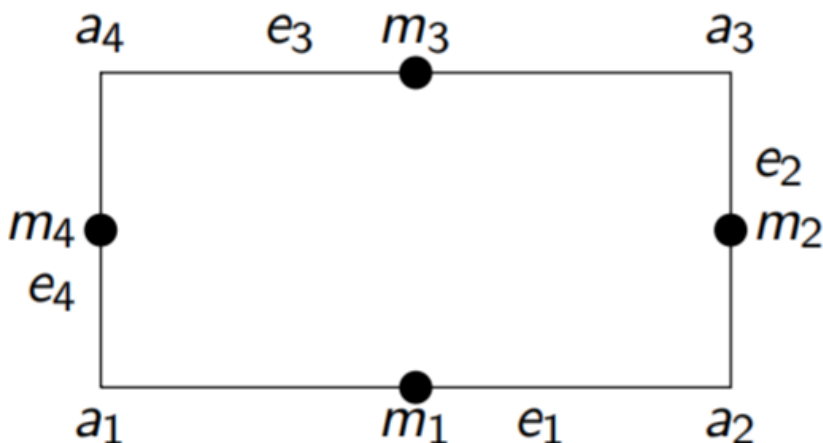


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1. 如果 K 是矩形, $P_K = Q_1(K)$, 证明如图所示的边的中点值所确定的节点参数对 P_K 不是唯一可解的 (Hint 试着构造 $v \in Q_1(K)$ 使得 $v(m_i) = 0, 1 \leq i \leq 4$)



Proof. 令 $\xi = \frac{x-x_0}{\ell_1}$, $\eta = \frac{y-y_0}{\ell_2}$.

考虑 $v = \xi\eta$. 显然 $v \in Q_1(K)$ 且 $v(m_i) = 0$, 且 $v \not\equiv 0$

□

2. 如果 K 是矩形, $P_K = \{1, x, y, \xi^2 - \eta^2\}$, 证明 $\mathcal{N}_K = \{N_i, 1 \leq i \leq 4\}$, 其中 $N_i(v) = \frac{1}{|e_i|} \int_{e_i} v ds, 1 \leq i \leq 4$ 对 P_K 是唯一可解的.

Proof. 易得 $P_K = \{1, \xi, \eta, \xi^2 - \eta^2\}$, 设 $v = a_1(\xi^2 - \eta^2) + a_2\xi + a_3\eta + a_4$.

由 $N_i(v) = 0, i = 1, 2, 3, 4$, 可得

$$\int_{-1}^1 -a_1(1 - \eta^2) + a_2 + a_3\eta + a_4 d\eta = 0$$

$$\int_{-1}^1 -a_1(1 - \xi^2) + a_2 + a_3\xi + a_4 d\xi = 0$$

$$\int_{-1}^1 -a_1(1 - \eta^2) - a_2 + a_3\eta + a_4 d\eta = 0$$

$$\int_{-1}^1 -a_1(1 - \xi^2) - a_2 + a_3\xi + a_4 d\xi = 0$$

即

$$-\frac{2}{3}a_1 + 2(a_1 + a_2 + a_4) = 0$$

$$-\frac{2}{3}a_1 + 2(a_1 + a_2 + a_3) = 0$$

$$-\frac{2}{3}a_1 + 2(a_1 - a_2 + a_4) = 0$$

$$-\frac{2}{3}a_1 + 2(a_1 - a_2 + a_3) = 0$$

解得

$$a_1 = a_2 = a_3 = a_4 = 0$$

从而 $v \equiv 0$. □

3. 证明 Morley 有限元空间 V_h 满足 $V_h \not\subset H^2(\Omega)$ 且 $V_h \not\subset H^1(\Omega)$

Morley 元的有限元空间:

$V_h = \{v \in L^2(\Omega) \mid v|_K \in P_2(K), \forall K \in T_h, v \text{ 在 } T_h \text{ 的所有顶点连续, } \frac{\partial v}{\partial \nu_e} \text{ 在 } T_h \text{ 所有内边 } e \text{ 的中点连续} \}$

Proof. 不妨设某条内边 $e = (-1, 0) \rightarrow (1, 0)$, 相邻单元为 K_1, K_2 .

考虑 $v|_{K_1} = x^2 - 1, v|_{K_2} = 1 - x^2$. 满足 $v \in V_h$ 但 $v \notin H^1(\Omega)$ □

4. 验证课件中给出的函数是 Morley 元的节点基函数

$$\begin{aligned} p_i &= 1 - (\lambda_{i-1} + \lambda_{i+1}) + 2\lambda_{i-1}\lambda_{i+1} \\ &\quad - (\nabla\lambda_{i-1})^T \nabla\lambda_{i+1} \sum_{k=i-1, i+1} \frac{\lambda_k(\lambda_k - 1)}{\|\nabla\lambda_k\|^2}, \quad 1 \leq i \leq 3 \\ p_{i+3} &= \frac{\lambda_i(\lambda_i - 1)}{\|\nabla\lambda_i\|^2}, \quad 1 \leq i \leq 3. \end{aligned}$$

$$N_i(v) = v(a_i), \quad 1 \leq i \leq 3$$

$$N_{i+3}(v) = \frac{\partial v}{\partial \nu}(m_i), \quad 1 \leq i \leq 3$$

$$a_1 = (1, 0, 0), a_2 = (0, 1, 0), a_3 = (0, 0, 1)$$

$$m_1 = (0, 0.5, 0.5), m_2 = (0.5, 0, 0.5), m_3 = (0.5, 0.5, 0)$$

Proof. 注意到 $\nabla\lambda_i = \frac{1}{2|S_K|}(y_j - y_k, x_k - x_j), e_i = (x_k - x_j, y_k - y_j)$. 从而 $-\nabla\lambda_i^\top$ 为 e_i 的外法向量, $-\frac{\nabla\lambda_i}{\|\nabla\lambda_i\|}^\top$ 为 e_i 的单位外法向量 ν_{e_i} .

$$N_1(p_1) = 1 - (0 + 0) + 2 * 0 * 0 - 0 = 0$$

$$N_1(p_2) = 1 - (1 + 0) - 2 * 1 * 0 - 0 = 0$$

$$N_1(p_4) = \frac{0}{\|\nabla\lambda_1\|} = 0$$

计算得

$$\nabla p_k = \left(-\nabla\lambda_i - \nabla\lambda_j + 2(\lambda_i \nabla\lambda_j + \lambda_j \nabla\lambda_i) - \nabla\lambda_i^\top \nabla\lambda_j \sum_{k=i,j} \frac{(2\lambda_k - 1)\nabla\lambda_k}{\|\nabla\lambda_k\|^2} \right) \quad k = 1, 2, 3$$

$$N_4(p_1) = \nabla p_1(m_1) \cdot \nu_{e_1} = 0 \cdot \nu_{e_1} = 0$$

$$\begin{aligned} N_4(p_2) &= \nabla p_2(m_1) \cdot \nu_{e_1} \\ &= \left(-\nabla \lambda_2 + \nabla \lambda_1^\top \nabla \lambda_2 \frac{\nabla \lambda_1}{\|\nabla \lambda_1\|^2} \right) \cdot \left(-\frac{\nabla \lambda_1}{\|\nabla \lambda_1\|} \right)^\top \\ &= 0. (\text{利用单位外法向量}) \end{aligned}$$

计算得

$$\begin{aligned} \nabla p_{i+3} &= \frac{1}{\|\nabla \lambda_i\|} (2\lambda_i - 1) \nabla \lambda_i \quad i = 1, 2, 3 \\ N_4(p_4) &= \nabla p_4(m_1) \cdot \nu_{e_1} = \left(-\frac{\nabla \lambda_i}{\|\nabla \lambda_i\|} \right)^\top \cdot \left(-\frac{\nabla \lambda_i}{\|\nabla \lambda_i\|} \right) = 1 \\ N_4(p_5) &= \nabla p_5(m_1) \cdot \nu_{e_1} = 0 \cdot \nu_{e_1} = 0 \end{aligned}$$

结合对称性，验证完毕。

□