Find reflexion points on a a 3d surface

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1 Notations

- $\underline{\mathrm{A}}$ is the point of observation
- $\underline{\mathrm{B}}$ is the point observed
- $\underline{\mathrm{C}}$ is the local coordinate center of the 3d surface
- $\underline{\mathrm{D}}$ is the projection of $\underline{\mathrm{B}}$ on the 3d surface as seen from $\underline{\mathrm{A}}$
- $\underline{\mathrm{e}}$ is the unit vector such that $\underline{\mathrm{AB}} = \|\,\underline{\mathrm{AB}}\,\|\,\underline{\mathrm{e}} = l\,\underline{\mathrm{e}}$
- $\underline{\mathrm{E}}$ is a point on (AB) parameterized by $\underline{AE} = kl\,\underline{\mathrm{e}}$
- $\underline{\mathrm{n}}(D)$ is the normal vector of the 3d surface at point $\underline{\mathrm{D}}$

Point $\underline{\mathrm{D}}$ on the 3d sufarce is the reflexion point connecting A and B. It means that $\underline{\mathrm{n}}(D)$ is in the same plane as (A, D, B).

Point $\underline{\mathrm{E}}$ on is the projection, on line (A, B) of point $\underline{\mathrm{D}}$ along $\underline{\mathrm{n}}$.

The idea is to look for $\underline{\mathrm{E}}$, which is parameterized by k, and then derive $\underline{\mathrm{D}}$ from $\underline{\mathrm{E}}$.

Hence both $\underline{\mathrm{n}}(\underline{\mathrm{D}})$ and $d_E = \|\,\underline{\mathrm{ED}}\,\|$ are parametrized by k: $\underline{\mathrm{n}}(k)$ and $d_E(k)$.

2 General equations

2.1 co-planarity

The point $\underline{\mathrm{D}}$ on the 3d sufarce is such that $\underline{\mathrm{n}}(D)$ is in the same plane as (A, D, B), which is written:

$$(\underline{\mathrm{DA}} \wedge \underline{\mathrm{n}}) \cdot (\underline{\mathrm{DB}} \wedge \underline{\mathrm{n}}) = 0 \tag{1}$$

$$\Leftrightarrow \begin{array}{l} (\underline{\mathrm{DA}} \wedge \underline{\mathrm{n}}) \wedge (\underline{\mathrm{DB}} \wedge \underline{\mathrm{n}}) = 0 \\ (\underline{\mathrm{EA}} \wedge \underline{\mathrm{n}}) \wedge (\underline{\mathrm{EB}} \wedge \underline{\mathrm{n}}) = 0 \\ ((-kl)\,\underline{\mathrm{e}} \wedge \underline{\mathrm{n}}) \wedge ((1-k)le \wedge \underline{\mathrm{n}}) = 0 \\ k(1-k)l^2(\underline{\mathrm{e}} \wedge \underline{\mathrm{n}}) \wedge (\underline{\mathrm{e}} \wedge \underline{\mathrm{n}}) = 0 \end{array}$$

Which is true by construction of E.

2.2 equal angles

Since it is a specular reflexion, angles (A, D, E) and (B, D, E) are equal, which means $\underline{\mathrm{E}}$ is necessarily standing on the bisector of angle (A, D, B).

As such, the distance between $\underline{\mathrm{E}}$ and line (A, D) is equal to the distance between $\underline{\mathrm{E}}$ and line (B, D), which is written:

$$d_{E,(A,D)} = \frac{\|\,\underline{\mathrm{AD}} \wedge \underline{AE}\|}{\|\,\underline{\mathrm{AD}}\,\|} = \frac{\|\,\underline{\mathrm{BD}} \wedge \underline{BE}\|}{\|\,\underline{\mathrm{BD}}\,\|} = d_{E,(B,D)}$$

Knowing that:

$$\begin{aligned} \|\,\underline{\mathrm{AD}}\,\|^2 &= \|\underline{AE} + \underline{\mathrm{ED}}\,\|^2 \\ &= k^2l^2 + d_E^2 + 2\underline{AE} \cdot \underline{\mathrm{ED}} \\ &= k^2l^2 + d_E^2 + 2kl\,\underline{\mathrm{e}} \cdot (-d_E)\,\underline{\mathrm{n}} \\ &= k^2l^2 + d_E(k)^2 - 2kld_E(k)\,\underline{\mathrm{e}} \cdot \underline{\mathrm{n}}(k) \end{aligned}$$

Similarly:

$$\begin{aligned} \|\,\underline{\mathrm{BD}}\,\|^2 &= \|\underline{BE} + \underline{\mathrm{ED}}\,\|^2 \\ &= (1-k)^2l^2 + d_E^2 + 2\underline{BE} \cdot \underline{\mathrm{ED}} \\ &= (1-k)^2l^2 + d_E^2 + 2(1-k)l(-\,\underline{\mathrm{e}}) \cdot (-d_E)\,\underline{\mathrm{n}} \\ &= (1-k)^2l^2 + d_E(k)^2 + 2(1-k)ld_E(k)\,\underline{\mathrm{e}} \cdot \underline{\mathrm{n}}(k) \end{aligned}$$

And the cross-products:

$$\begin{aligned} \| \underline{\mathrm{AD}} \wedge \underline{AE} \|^2 &= \| \underline{\mathrm{ED}} \wedge \underline{AE} \|^2 \\ &= \| (-d_E) \, \underline{\mathrm{n}} \wedge kl \, \underline{\mathrm{e}} \, \|^2 \\ &= k^2 d_E(k)^2 l^2 \| \, \underline{\mathrm{n}}(k) \wedge \underline{\mathrm{e}} \, \|^2 \end{aligned}$$

And:

$$\begin{aligned} \| \underline{\mathrm{BD}} \wedge \underline{BE} \|^2 &= \| \underline{\mathrm{ED}} \wedge \underline{BE} \|^2 \\ &= \| (-d_E) \, \underline{\mathrm{n}} \wedge (1-k) l (- \underline{\mathrm{e}}) \|^2 \\ &= (1-k)^2 d_E(k)^2 l^2 \| \, \underline{\mathrm{n}}(k) \wedge \underline{\mathrm{e}} \, \|^2 \end{aligned}$$

So in the end:

$$\begin{aligned} & d^2_{E,(A,D)} = d^2_{E,(B,D)} \\ \Leftrightarrow \quad & \| \underline{\mathrm{AD}} \wedge \underline{AE} \|^2 \| \, \underline{\mathrm{BD}} \, \|^2 = \| \underline{\mathrm{BD}} \wedge \underline{BE} \|^2 \| \, \underline{\mathrm{AD}} \, \|^2 \\ \Leftrightarrow \quad & k^2 d_E^2 l^2 \| \, \underline{\mathrm{n}} \wedge \underline{\mathrm{e}} \, \|^2 \left[(1-k)^2 l^2 + d_E^2 + 2(1-k) l d_E \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \right] \\ & = (1-k)^2 d_E^2 l^2 \| \, \underline{\mathrm{n}} \wedge \underline{\mathrm{e}} \, \|^2 \left[k^2 l^2 + d_E^2 - 2kld_E \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \right] \end{aligned}$$

So assuming that:

$$\begin{cases} \| \, \underline{\mathrm{n}}(k) \wedge \underline{\mathrm{e}} \, \| \neq 0 \\ l \neq 0 \\ d_E(k) \neq 0 \end{cases}$$

So in the end:

$$\begin{aligned} & d^2_{E,(A,D)} = d^2_{E,(B,D)} \\ \Leftrightarrow \quad & k^2 \left[(1-k)^2 l^2 + d_E^2 + 2(1-k) l d_E \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \right] = (1-k)^2 \left[k^2 l^2 + d_E^2 - 2kld_E \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \right] \\ \Leftrightarrow \quad & k^2 \left[d_E^2 + 2(1-k) l d_E \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \right] = (1-k)^2 \left[d_E^2 - 2kld_E \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \right] \\ \Leftrightarrow \quad & k^2 d_E^2 + 2k^2(1-k) l d_E \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} = (1-k)^2 d_E^2 - 2(1-k)^2 kld_E \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \\ \Leftrightarrow \quad & 2kd_E^2 - d_E^2 + 2k(1-k) l d_E \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} (k + 1 - k) = 0 \\ \Leftrightarrow \quad & 2kd_E - d_E + 2k(1-k) l \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} = 0 \end{aligned}$$

Hence:

$$(2) \Leftrightarrow (2k-1) d_E(k) + 2k(1-k) l \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}(k) = 0$$

3 Planar

If the 3d surface is a plane, then $\underline{\mathrm{n}}$ is constant and does not depend on k.

The equations become:

$$2kd_E(k) - d_E(k) + 2k(1-k) l \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} = 0 \tag{2}$$

In that case:

$$\begin{aligned} d_E(k) &= \underline{\mathrm{DE}} \cdot \underline{\mathrm{n}} \\ &= (\underline{\mathrm{DC}} + \underline{\mathrm{CA}} + \underline{AE}) \cdot \underline{\mathrm{n}} \\ &= \underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} + kl \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \end{aligned}$$

Which means:

$$\begin{aligned} (2) \quad & \Leftrightarrow 2k(\underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} + kl \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}) - \underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} - kl \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} + 2k(1-k) l \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} = 0 \\ & \Leftrightarrow k^2 (2l \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} - 2l \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}) + k(2 \, \underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} - l \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} + 2l \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}) - \underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} = 0 \\ & \Leftrightarrow k(2 \, \underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} + l \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}) - \underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} = 0 \end{aligned}$$

So finally:

$$(2) \Leftrightarrow k = \frac{\underline{\mathrm{CA}} \cdot \underline{\mathrm{n}}}{2 \, \underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} + l \, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}}$$

4 Cylinder

Consider a cylinder of axes $(O, \underline{\mathrm{z}})$, with $\underline{\mathrm{z}}$ unit vector and radius r. The normal vector associated to any point E (not on the axis) is:

$$\underline{\mathrm{n}}(E) = - \frac{(\underline{OE} \wedge \underline{\mathrm{z}}) \wedge \underline{\mathrm{z}}}{\| (\underline{OE} \wedge \underline{\mathrm{z}}) \wedge \underline{\mathrm{z}} \, \|} = - \frac{(\underline{OE} \wedge \underline{\mathrm{z}}) \wedge \underline{\mathrm{z}}}{\| \underline{\mathrm{OE}} \wedge \underline{\mathrm{z}} \, \|}$$

Given that E in on the (A, B) line:

$$\underline{OE} = \underline{OA} + kl \, \underline{\mathrm{e}}$$

So:

$$(\underline{OE} \wedge \underline{\mathrm{z}}) \wedge \underline{\mathrm{z}} = (\underline{OA} \wedge \underline{\mathrm{z}}) \wedge \underline{\mathrm{z}} + kl (\underline{\mathrm{e}} \wedge \underline{\mathrm{z}}) \wedge \underline{\mathrm{z}}$$

Which entails:

$$\begin{aligned}
\underline{\mathrm{e}} \cdot \underline{\mathrm{n}}(k) &= -\tfrac{1}{\|\underline{\mathrm{OE}} \wedge \underline{\mathrm{z}}\|} \left[\underline{\mathrm{e}} \cdot ((\underline{OE} \wedge \underline{\mathrm{z}}) \wedge \underline{\mathrm{z}})\right] \\
&= -\tfrac{1}{\|\underline{\mathrm{OE}} \wedge \underline{\mathrm{z}}\|} \left[\underline{\mathrm{z}} \cdot (\underline{\mathrm{e}} \wedge (\underline{OE} \wedge \underline{\mathrm{z}}))\right]
\end{aligned}$$

And:

$$d_E = r - \|\underline{\mathrm{OE}} \wedge \underline{\mathrm{z}}\|$$

So the equation becomes:

$$\begin{aligned}
(2) \quad &\Leftrightarrow (2k-1)(r - \|\underline{\mathrm{OE}} \wedge \underline{\mathrm{z}}\|) + 2k(1-k)l\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} = 0 \\
&\Leftrightarrow (2k-1)(r - \|\underline{\mathrm{OE}} \wedge \underline{\mathrm{z}}\|)\, \|\underline{\mathrm{OE}} \wedge \underline{\mathrm{z}}\| - 2k(1-k)l \left[\underline{\mathrm{z}} \cdot (\underline{\mathrm{e}} \wedge (\underline{OE} \wedge \underline{\mathrm{z}}))\right] = 0 \\
&\Leftrightarrow (2k-1)r\, \|\underline{\mathrm{OE}} \wedge \underline{\mathrm{z}}\| - (2k-1)\, \|\underline{\mathrm{OE}} \wedge \underline{\mathrm{z}}\|^2 - 2k(1-k)l \left[(\underline{OE} \wedge \underline{\mathrm{z}}) \cdot (\underline{\mathrm{z}} \wedge \underline{\mathrm{e}})\right] = 0
\end{aligned}$$

Observing that:

$$\underline{OE} \wedge \underline{\mathrm{z}} = \underline{\mathrm{OA}} \wedge \underline{\mathrm{z}} + kl\, \underline{\mathrm{e}} \wedge \underline{\mathrm{z}}$$

And:

$$\|\underline{\mathrm{OE}} \wedge \underline{\mathrm{z}}\|^2 = \|\, \underline{\mathrm{OA}} \wedge \underline{\mathrm{z}} + kl\, \underline{\mathrm{e}} \wedge \underline{\mathrm{z}}\, \|^2 = (\underline{\mathrm{OA}} \wedge \underline{\mathrm{z}})^2 + 2kl(\underline{\mathrm{OA}} \wedge \underline{\mathrm{z}}) \cdot (\underline{\mathrm{e}} \wedge \underline{\mathrm{z}}) + k^2 l^2 (\underline{\mathrm{e}} \wedge \underline{\mathrm{z}})^2$$