

Find reflexions on a 3d surface from a point and a unit vector

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1 Notations

- A is the point of observation
- \underline{e}_A is the associated vector
- C is the local coordinate center of the 3d surface
- D is the intersection of (A, \underline{e}_A) with the 3d surface
- \underline{e}_B is the associated reflected vector
- $\underline{n}(D)$ is the normal vector of the 3d surface at point D

2 General equations

2.1 intersection: finding D

Point D is such that:

$$\underline{AD} = k \underline{e}_A = \underline{AC} + \underline{CD}$$

So:

$$k \underline{e}_A \cdot \underline{n} = (\underline{AC} + \underline{CD}) \cdot \underline{n} \quad (1)$$

2.2 reflection: deriving angle of incidence

The angle of incidence (with respect to the dioptré) is:

$$\alpha = \arcsin(-\underline{e}_A \cdot \underline{n})$$

2.3 reflection: deriving \underline{e}_B

The point D on the 3d surface is such that $\underline{n}(D)$ is in the same plane as (A, D, B) , which is written:

Using vector components:

$$\underline{e}_B = (-\underline{e}_A \cdot \underline{n}) \underline{n} + (\underline{e}_A - (\underline{e}_A \cdot \underline{n}) \underline{n})$$

We can check that in both cases:

$$\begin{cases} \underline{e}_B \cdot (\underline{e}_A \wedge \underline{n}) = 0 & (\text{coplanar}) \\ \underline{e}_B \cdot \underline{n} = -\underline{e}_A \cdot \underline{n} & (\text{same angle}) \end{cases}$$

3 Plane

If the 3d surface is a plane, then \underline{n} is constant and does not depend on k .

The equations become:

$$\begin{aligned} (1) \quad &\Leftrightarrow k \frac{\underline{e}_A \cdot \underline{n}}{\underline{e}_A \cdot \underline{n}} = \underline{AC} \cdot \underline{n} \\ &\Leftrightarrow k = \frac{\underline{AC} \cdot \underline{n}}{\underline{e}_A \cdot \underline{n}} \end{aligned}$$

4 Cylinder

Consider a cylinder of axes (O, \underline{z}) , with \underline{z} unit vector and radius r . The equations become:

$$\begin{aligned}
 (1) \quad &\Leftrightarrow k \underline{e}_A \cdot \underline{n} = \underline{AO} \cdot \underline{n} + \underline{OD} \cdot \underline{n} \\
 &\Leftrightarrow k \underline{e}_A \cdot \underline{n} = \underline{AO} \cdot \underline{n} + \underline{OE} \cdot \underline{n} + \underline{ED} \cdot \underline{n} \\
 &\Leftrightarrow k \underline{e}_A \cdot \underline{n} = \underline{AO} \cdot \underline{n} + \underline{ED} \cdot \underline{n} \\
 &\Leftrightarrow k \underline{e}_A \cdot \underline{n} = \underline{AO} \cdot \underline{n} + (-r \underline{n}) \cdot \underline{n} \\
 &\Leftrightarrow k \underline{e}_A \cdot \underline{n} = \underline{AO} \cdot \underline{n} - r
 \end{aligned}$$

Now, for any point on line (A, \underline{e}_A) , the unit vector \underline{n} is:

$$\underline{n}(D) = \frac{(\underline{OD} \wedge \underline{z}) \wedge \underline{z}}{\|(\underline{OD} \wedge \underline{z}) \wedge \underline{z}\|} = \frac{(\underline{OD} \wedge \underline{z}) \wedge \underline{z}}{\|\underline{OD} \wedge \underline{z}\|}$$

And:

$$\begin{aligned}
 \underline{OD} \wedge \underline{z} &= \underline{OA} \wedge \underline{z} + k \underline{e}_A \wedge \underline{z} \\
 \Rightarrow (\underline{OD} \wedge \underline{z}) \wedge \underline{z} &= (\underline{OA} \wedge \underline{z}) \wedge \underline{z} + k(\underline{e}_A \wedge \underline{z}) \wedge \underline{z}
 \end{aligned}$$

And:

$$\|\underline{OD} \wedge \underline{z}\|^2 = \|\underline{OA} \wedge \underline{z}\|^2 + k^2 \|\underline{e}_A \wedge \underline{z}\|^2 + 2k(\underline{OA} \wedge \underline{z}) \cdot (\underline{e}_A \wedge \underline{z})$$

So

$$\begin{aligned}
 (1) \quad &\Leftrightarrow k \underline{e}_A \cdot \underline{n} = \underline{AO} \cdot \underline{n} - r \\
 &\Leftrightarrow k \underline{e}_A \cdot ((\underline{OD} \wedge \underline{z}) \wedge \underline{z}) = \underline{AO} \cdot ((\underline{OD} \wedge \underline{z}) \wedge \underline{z}) - r \|\underline{OD} \wedge \underline{z}\| \\
 &\Leftrightarrow -k(\underline{OD} \wedge \underline{z}) \cdot (\underline{e}_A \wedge \underline{z}) = (\underline{OD} \wedge \underline{z}) \cdot (\underline{OA} \wedge \underline{z}) - r \|\underline{OD} \wedge \underline{z}\| \\
 &\Leftrightarrow r \|\underline{OD} \wedge \underline{z}\| = k(\underline{OD} \wedge \underline{z}) \cdot (\underline{e}_A \wedge \underline{z}) + (\underline{OD} \wedge \underline{z}) \cdot (\underline{OA} \wedge \underline{z}) \\
 &\Leftrightarrow r \|\underline{OD} \wedge \underline{z}\| = (\underline{OD} \wedge \underline{z}) \cdot (k \underline{e}_A \wedge \underline{z} + \underline{OA} \wedge \underline{z}) \\
 &\Leftrightarrow r \|\underline{OD} \wedge \underline{z}\| = \|\underline{OD} \wedge \underline{z}\|^2
 \end{aligned}$$

But $\|\underline{OD} \wedge \underline{z}\| \neq 0$ because D cannot be of the axis of the cylinder.

So

$$\begin{aligned}
 (1) \quad &\Rightarrow r^2 = \|\underline{OD} \wedge \underline{z}\|^2 \\
 &\Leftrightarrow r^2 = \|\underline{OA} \wedge \underline{z}\|^2 + k^2 \|\underline{e}_A \wedge \underline{z}\|^2 + 2k(\underline{OA} \wedge \underline{z}) \cdot (\underline{e}_A \wedge \underline{z})
 \end{aligned}$$

So ultimately:

$$(1) \quad \Rightarrow k^2 \|\underline{e}_A \wedge \underline{z}\|^2 + 2k(\underline{OA} \wedge \underline{z}) \cdot (\underline{e}_A \wedge \underline{z}) + \|\underline{OA} \wedge \underline{z}\|^2 - r^2 = 0$$

5 Sphere

Consider a sphere of center O and radius r . The normal vector associated to any point E (not on O) is:

$$\underline{n}(D) = -\frac{\underline{OD}}{\|\underline{OD}\|}$$

Given that E is on the (A, B) line:

Which is the same equation as for the cylinder, but with cross products replaced by norms.

$$\begin{aligned}
 (1) \quad &\Leftrightarrow k \underline{e}_A \cdot \underline{n} = \underline{AO} \cdot \underline{n} + \underline{OD} \cdot \underline{n} \\
 &\Leftrightarrow k \underline{e}_A \cdot \underline{n} = \underline{AO} \cdot \underline{n} - r \\
 &\Leftrightarrow -k \underline{e}_A \cdot \underline{OD} = -\underline{AO} \cdot \underline{OD} - r \|\underline{OD}\| \\
 &\Leftrightarrow r \|\underline{OD}\| = \underline{OA} \cdot \underline{OD} + k \underline{e}_A \cdot \underline{OD} \\
 &\Leftrightarrow r \|\underline{OD}\| = \|\underline{OD}\|^2 \\
 &\Rightarrow r^2 = \|\underline{OD}\|^2 \\
 &\Leftrightarrow r^2 = \|\underline{OA}\|^2 + k^2 + 2k \underline{OA} \cdot \underline{e}_A
 \end{aligned}$$

So ultimately:

$$(1) \Rightarrow k^2 + 2k \underline{OA} \cdot \underline{e}_A + \|\underline{OA}\|^2 - r^2 = 0$$

6 Torus

Consider a torus of axes (O, \underline{z}) , with \underline{z} unit vector. It has major radius r_a and minor radius r_b .

For any point E in space, we have:

$$e_a(E) = -\frac{(\underline{OE} \wedge \underline{z}) \wedge \underline{z}}{\|(\underline{OE} \wedge \underline{z}) \wedge \underline{z}\|}$$