Find reflexion points on a a 3d surface

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1 Notations

- $\underline{\mathrm{A}}$ is the point of observation
- $\underline{\mathrm{B}}$ is the point observed
- $\underline{\mathrm{C}}$ is the local coordinate center of the 3d surface
- $\underline{\mathrm{D}}$ is the projection of $\underline{\mathrm{B}}$ on the 3d surface as seen from $\underline{\mathrm{A}}$
- $\underline{\mathrm{e}}$ is the unit vector such that $\underline{\mathrm{AB}} = \| \underline{\mathrm{AB}} \| \, \underline{\mathrm{e}} = l \, \underline{\mathrm{e}}$
- $\underline{\mathrm{E}}$ is a point on (AB) parameterized by $\underline{AE} = kl \, \underline{\mathrm{e}}$
- $\underline{\mathrm{n}}(D)$ is the normal vector of the 3d surface at point $\underline{\mathrm{D}}$

Point $\underline{\mathrm{D}}$ on the 3d sufarce is the reflexion point connecting A and B. It means that $\underline{\mathrm{n}}(D)$ is in the same plane as (A, D, B).

Point $\underline{\mathrm{E}}$ on is the projection, on line (A, B) of point $\underline{\mathrm{D}}$ along $\underline{\mathrm{n}}$.

The idea is to look for $\underline{\mathrm{E}}$, which is parameterized by k, and then derive $\underline{\mathrm{D}}$ from $\underline{\mathrm{E}}$.

Hence both $\underline{\mathrm{n}}(\underline{\mathrm{D}})$ and $d_E = \| \underline{\mathrm{ED}} \|$ are parametrized by k: $\underline{\mathrm{n}}(k)$ and $d_E(k)$.

2 General equations

2.1 co-planarity

The point $\underline{\mathrm{D}}$ on the 3d sufarce is such that $\underline{\mathrm{n}}(D)$ is in the same plane as (A, D, B), which is written:

$$(\underline{\mathrm{DA}} \wedge \underline{\mathrm{n}}) \cdot (\underline{\mathrm{DB}} \wedge \underline{\mathrm{n}}) = 0 \tag{1}$$

$$\begin{aligned} & (\underline{\mathrm{DA}} \wedge \underline{\mathrm{n}}) \wedge (\underline{\mathrm{DB}} \wedge \underline{\mathrm{n}}) = 0 \\ \Leftrightarrow \quad & (\underline{\mathrm{EA}} \wedge \underline{\mathrm{n}}) \wedge (\underline{\mathrm{EB}} \wedge \underline{\mathrm{n}}) = 0 \\ & ((-kl)\, \underline{\mathrm{e}} \wedge \underline{\mathrm{n}}) \wedge ((1-k)le \wedge \underline{\mathrm{n}}) = 0 \\ & k(1-k)l^2 (\underline{\mathrm{e}} \wedge \underline{\mathrm{n}}) \wedge (\underline{\mathrm{e}} \wedge \underline{\mathrm{n}}) = 0 \end{aligned}$$

Which is true by construction of E.

2.2 equal angles

Since it is a specular reflexion, angles (A, D, E) and (B, D, E) are equal, which means $\underline{\mathrm{E}}$ is necessarily standing on the bisector of angle (A, D, B).

As such, the distance between $\underline{\mathrm{E}}$ and line (A, D) is equal to the distance between $\underline{\mathrm{E}}$ and line (B, D), which is written:

$$d_{E,(A,D)} = \frac{\| \underline{\mathrm{AD}} \wedge \underline{AE} \|}{\| \underline{\mathrm{AD}} \|} = \frac{\| \underline{\mathrm{BD}} \wedge \underline{BE} \|}{\| \underline{\mathrm{BD}} \|} = d_{E,(B,D)}$$

Knowing that:

$$\begin{aligned} \| \underline{\mathrm{AD}} \|^2 &= \| \underline{AE} + \underline{\mathrm{ED}} \|^2 \\ &= k^2 l^2 + d_E^2 + 2\underline{AE} \cdot \underline{\mathrm{ED}} \\ &= k^2 l^2 + d_E^2 + 2kl\, \underline{\mathrm{e}} \cdot (-d_E)\, \underline{\mathrm{n}} \\ &= k^2 l^2 + d_E(k)^2 - 2kl d_E(k)\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}(k) \end{aligned}$$

Similarly:

$$\begin{aligned} \| \underline{\mathrm{BD}} \|^2 &= \| \underline{BE} + \underline{\mathrm{ED}} \|^2 \\ &= (1-k)^2 l^2 + d_E^2 + 2\underline{BE} \cdot \underline{\mathrm{ED}} \\ &= (1-k)^2 l^2 + d_E^2 + 2(1-k)l(-\underline{\mathrm{e}}) \cdot (-d_E)\, \underline{\mathrm{n}} \\ &= (1-k)^2 l^2 + d_E(k)^2 + 2(1-k) l d_E(k)\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}(k) \end{aligned}$$

And the cross-products:

$$\begin{aligned} \| \underline{\mathrm{AD}} \wedge \underline{AE} \|^2 &= \| \underline{\mathrm{ED}} \wedge \underline{AE} \|^2 \\ &= \| (-d_E)\, \underline{\mathrm{n}} \wedge kl\, \underline{\mathrm{e}} \|^2 \\ &= k^2 d_E(k)^2 l^2 \| \underline{\mathrm{n}}(k) \wedge \underline{\mathrm{e}} \|^2 \end{aligned}$$

And:

$$\begin{aligned} \| \underline{\mathrm{BD}} \wedge \underline{BE} \|^2 &= \| \underline{\mathrm{ED}} \wedge \underline{BE} \|^2 \\ &= \| (-d_E)\, \underline{\mathrm{n}} \wedge (1-k)l(-\underline{\mathrm{e}}) \|^2 \\ &= (1-k)^2 d_E(k)^2 l^2 \| \underline{\mathrm{n}}(k) \wedge \underline{\mathrm{e}} \|^2 \end{aligned}$$

So in the end:

$$\begin{aligned} & d_{E,(A,D)}^2 = d_{E,(B,D)}^2 \\ \Leftrightarrow \quad & \| \underline{\mathrm{AD}} \wedge \underline{AE} \|^2 \| \underline{\mathrm{BD}} \|^2 = \| \underline{\mathrm{BD}} \wedge \underline{BE} \|^2 \| \underline{\mathrm{AD}} \|^2 \\ \Leftrightarrow \quad & k^2 d_E^2 l^2 \| \underline{\mathrm{n}} \wedge \underline{\mathrm{e}} \|^2 \left[(1-k)^2 l^2 + d_E^2 + 2(1-k) l d_E\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \right] \\ & = (1-k)^2 d_E^2 l^2 \| \underline{\mathrm{n}} \wedge \underline{\mathrm{e}} \|^2 \left[k^2 l^2 + d_E^2 - 2kl d_E\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \right] \end{aligned}$$

So assuming that:

$$\begin{cases} \| \underline{\mathrm{n}}(k) \wedge \underline{\mathrm{e}} \| \neq 0 \\ l \neq 0 \\ d_E(k) \neq 0 \end{cases}$$

So in the end:

$$\begin{aligned} & d_{E,(A,D)}^2 = d_{E,(B,D)}^2 \\ \Leftrightarrow \quad & k^2 \left[(1-k)^2 l^2 + d_E^2 + 2(1-k) l d_E\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \right] = (1-k)^2 \left[k^2 l^2 + d_E^2 - 2kl d_E\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \right] \\ \Leftrightarrow \quad & k^2 \left[d_E^2 + 2(1-k) l d_E\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \right] = (1-k)^2 \left[d_E^2 - 2kl d_E\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \right] \\ \Leftrightarrow \quad & k^2 d_E^2 + 2k^2(1-k) l d_E\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} = (1-k)^2 d_E^2 - 2(1-k)^2 kl d_E\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \\ \Leftrightarrow \quad & 2k d_E^2 - d_E^2 + 2k(1-k) l d_E\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}(k + 1 - k) = 0 \\ \Leftrightarrow \quad & (2k-1) d_E + 2k(1-k) l\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} = 0 \end{aligned}$$

Hence:

$$(2) \Leftrightarrow (2k-1) d_E(k) + 2k(1-k) l\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}(k) = 0$$

2.3 equal angles 2

This time we use the scalar product:

$$(\underline{\mathrm{DA}} \cdot \underline{\mathrm{n}})^2 \|DB\|^2 = (\underline{\mathrm{DB}} \cdot \underline{\mathrm{n}})^2 \|DA\|^2$$

With:

$$\begin{cases} \underline{\mathrm{DA}} \cdot \underline{\mathrm{n}} = d - kl\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \\ \underline{\mathrm{DB}} \cdot \underline{\mathrm{n}} = d + (1-k)l\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \\ \| \underline{\mathrm{DA}} \|^2 = d^2 + k^2 l^2 - 2dkl\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \\ \| \underline{\mathrm{DB}} \|^2 = d^2 + (1-k)^2 l^2 + 2d(1-k)l\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \end{cases}$$

So:

$$\begin{aligned} & (\underline{\mathrm{DA}} \cdot \underline{\mathrm{n}})^2 \|DB\|^2 = (\underline{\mathrm{DB}} \cdot \underline{\mathrm{n}})^2 \|DA\|^2 \\ \Leftrightarrow\ & (d - kl\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}})^2 (d^2 + (1-k)^2 l^2 + 2d(1-k)l\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}) = (d - kl\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} + l\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}})^2 (d^2 + k^2 l^2 - 2dkl\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}) \\ \Leftrightarrow\ & (d - kl\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}})^2 ((1-k)^2 l^2 + 2d(1-k)l\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} - k^2 l^2 + 2dkl\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}) = (l^2\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}^2 + 2l\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}(d - kl\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}))(d^2 + k^2 l^2 - 2dkl\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}) \\ \Leftrightarrow\ & (d - kl\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}})^2 (l^2 - 2kl^2 + 2dl\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}) = l\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}(l\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}} + 2d - dkl\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}})(d^2 + k^2 l^2 - 2dkl\, \underline{\mathrm{e}} \cdot \underline{\mathrm{n}}) \\ \Leftrightarrow\ & \end{aligned}$$

3 Planar

If the 3d surface is a plane, then $\underline{\mathrm{n}}$ is constant and does not depend on k.

The equations become:

$$(2k-1)d_E(k) + 2k(1-k)l\,\underline{\mathrm{e}} \cdot \underline{\mathrm{n}} = 0 \tag{2}$$

In that case:

$$\begin{aligned} d_E(k) &= \underline{\mathrm{DE}} \cdot \underline{\mathrm{n}} \\ &= (\underline{\mathrm{DC}} + \underline{\mathrm{CA}} + \underline{AE}) \cdot \underline{\mathrm{n}} \\ &= \underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} + kl\,\underline{\mathrm{e}} \cdot \underline{\mathrm{n}} \end{aligned}$$

Which means:

$$\begin{aligned} (2) \quad &\Leftrightarrow 2k(\underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} + kl\,\underline{\mathrm{e}} \cdot \underline{\mathrm{n}}) - \underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} - kl\,\underline{\mathrm{e}} \cdot \underline{\mathrm{n}} + 2k(1-k)l\,\underline{\mathrm{e}} \cdot \underline{\mathrm{n}} = 0 \\ &\Leftrightarrow k^2(2l\,\underline{\mathrm{e}} \cdot \underline{\mathrm{n}} - 2l\,\underline{\mathrm{e}} \cdot \underline{\mathrm{n}}) + k(2\,\underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} - l\,\underline{\mathrm{e}} \cdot \underline{\mathrm{n}} + 2l\,\underline{\mathrm{e}} \cdot \underline{\mathrm{n}}) - \underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} = 0 \\ &\Leftrightarrow k(2\,\underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} + l\,\underline{\mathrm{e}} \cdot \underline{\mathrm{n}}) - \underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} = 0 \end{aligned}$$

So finally:

$$(2) \Leftrightarrow k = \frac{\underline{\mathrm{CA}} \cdot \underline{\mathrm{n}}}{2\,\underline{\mathrm{CA}} \cdot \underline{\mathrm{n}} + l\,\underline{\mathrm{e}} \cdot \underline{\mathrm{n}}}$$

4 Cylinder

Consider a cylinder of axes $(O, \underline{z})$, with $\underline{z}$ unit vector and radius r. The normal vector associated to any point E (not on the axis) is:

$$\underline{n}(E) = -\frac{(\underline{OE} \wedge \underline{z}) \wedge \underline{z}}{\|(\underline{OE} \wedge \underline{z}) \wedge \underline{z}\|} = -\frac{(\underline{OE} \wedge \underline{z}) \wedge \underline{z}}{\|\underline{\mathrm{OE}} \wedge \underline{z}\|}$$

Given that E in on the (A, B) line:

$$\underline{OE} = \underline{OA} + kl\,\underline{e}$$

So:

$$(\underline{OE} \wedge \underline{z}) \wedge \underline{z} = (\underline{OA} \wedge \underline{z}) \wedge \underline{z} + kl(\underline{e} \wedge \underline{z}) \wedge \underline{z}$$

Which entails:

$$\begin{aligned} \underline{e} \cdot \underline{n}(k) &= -\tfrac{1}{\|\underline{\mathrm{OE}} \wedge \underline{z}\|} \left[\underline{e} \cdot ((\underline{OE} \wedge \underline{z}) \wedge \underline{z})\right] \\ &= -\tfrac{1}{\|\underline{\mathrm{OE}} \wedge \underline{z}\|} \left[\underline{z} \cdot (\underline{e} \wedge (\underline{OE} \wedge \underline{z}))\right] \end{aligned}$$

And:

$$d_E = r - \|\underline{\mathrm{OE}} \wedge \underline{z}\|$$

So the equation becomes:

$$\begin{aligned} (2) \quad &\Leftrightarrow (2k-1)(r - \|\underline{\mathrm{OE}} \wedge \underline{z}\|) + 2k(1-k)l\,\underline{e} \cdot \underline{n} = 0 \\ &\Leftrightarrow (2k-1)(r - \|\underline{\mathrm{OE}} \wedge \underline{z}\|)\,\|\underline{\mathrm{OE}} \wedge \underline{z}\| - 2k(1-k)l\left[\underline{z} \cdot (\underline{e} \wedge (\underline{OE} \wedge \underline{z}))\right] = 0 \\ &\Leftrightarrow (2k-1)r\,\|\underline{\mathrm{OE}} \wedge \underline{z}\| - (2k-1)\,\|\underline{\mathrm{OE}} \wedge \underline{z}\|^2 - 2k(1-k)l\left[(\underline{OE} \wedge \underline{z}) \cdot (\underline{z} \wedge \underline{e})\right] = 0 \end{aligned}$$

Observing that:

$$\underline{OE} \wedge \underline{z} = \underline{\mathrm{OA}} \wedge \underline{z} + kl\,\underline{e} \wedge \underline{z}$$

And:

$$\|\underline{\mathrm{OE}} \wedge \underline{z}\|^2 = \|\,\underline{\mathrm{OA}} \wedge \underline{z} + kl\,\underline{e} \wedge \underline{z}\,\|^2 = (\underline{\mathrm{OA}} \wedge \underline{z})^2 + 2kl(\underline{\mathrm{OA}} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z}) + k^2l^2(\underline{e} \wedge \underline{z})^2$$

So

$$\begin{aligned} (2) \quad \Leftrightarrow \quad & (2k-1)r\,\|\underline{\mathrm{OE}} \wedge \underline{z}\| - (2k-1)(\underline{\mathrm{OA}} \wedge \underline{z})^2 - (2k-1)2kl(\underline{\mathrm{OA}} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z}) - (2k-1)k^2l^2(\underline{e} \wedge \underline{z})^2 \\ & -2k(1-k)l(\underline{OA} \wedge \underline{z}) \cdot (\underline{z} \wedge \underline{e}) - 2k^2(1-k)l^2(\underline{e} \wedge \underline{z}) \cdot (\underline{z} \wedge \underline{e}) && = 0 \\ \Leftrightarrow \quad & (2k-1)r\,\|\underline{\mathrm{OE}} \wedge \underline{z}\| - (2k-1)(\underline{\mathrm{OA}} \wedge \underline{z})^2 + 2kl(\underline{\mathrm{OA}} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z})(1-k-2k+1) + k^2l^2(\underline{e} \wedge \underline{z})^2(2-2k-2k+1) && = 0 \\ \Leftrightarrow \quad & (2k-1)r\,\|\underline{\mathrm{OE}} \wedge \underline{z}\| - (2k-1)(\underline{\mathrm{OA}} \wedge \underline{z})^2 + 2kl(\underline{\mathrm{OA}} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z})(2-3k) + k^2l^2(\underline{e} \wedge \underline{z})^2(3-4k) && = 0 \end{aligned}$$

5 Sphere

Consider a sphere of center O and radius r. The normal vector associated to any point E (not on O) is:

$$\underline{\mathrm{n}}(E) = -\frac{\underline{OE}}{\|\underline{\mathrm{OE}}\|}$$

Given that E in on the (A, B) line:

$$\begin{cases} \underline{OE} = \underline{\mathrm{OA}} + kl\,\underline{\mathrm{e}} \\ \|\underline{OE}\|^2 = \|\,\underline{\mathrm{OA}}\,\|^2 + k^2l^2 + 2kl\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}} \end{cases}$$

Which entails:

$$\underline{\mathrm{e}}\cdot\underline{\mathrm{n}}(k) \quad = -\tfrac{1}{\|\underline{\mathrm{OE}}\|}\left[\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}} + kl\right]$$

And:

$$d_E = r - \|\underline{OE}\|$$

So the equation becomes:

$$\begin{aligned}
(2) \quad \Leftrightarrow \quad & (2k-1)(r - \|\underline{\mathrm{OE}}\|) + 2k(1-k)l\,\underline{\mathrm{e}}\cdot\underline{\mathrm{n}} = 0 \\
\Leftrightarrow \quad & (2k-1)(r - \|\underline{\mathrm{OE}}\|)\,\|\underline{\mathrm{OE}}\| - 2k(1-k)l\left[\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}} + kl\right] = 0 \\
\Leftrightarrow \quad & (2k-1)r\,\|\underline{\mathrm{OE}}\| - (2k-1)\,\|\underline{\mathrm{OE}}\|^2 - 2k(1-k)l\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}} - 2k^2l^2(1-k) = 0 \\
\Leftrightarrow \quad & (2k-1)r\,\|\underline{\mathrm{OE}}\| - (2k-1)\|\,\underline{\mathrm{OA}}\,\|^2 - (2k-1)k^2l^2 - 2(2k-1)kl\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}} - 2k(1-k)l\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}} - 2k^2l^2(1-k) = 0 \\
\Leftrightarrow \quad & (2k-1)r\,\|\underline{\mathrm{OE}}\| - (2k-1)\|\,\underline{\mathrm{OA}}\,\|^2 - k^2l^2(2k-1+2-2k) - 2kl\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}}(2k-1+1-k) = 0 \\
\Leftrightarrow \quad & (2k-1)r\,\|\underline{\mathrm{OE}}\| - (2k-1)\|\,\underline{\mathrm{OA}}\,\|^2 - k^2l^2 - 2k^2l\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}} = 0 \\
\Leftrightarrow \quad & (2k-1)^2r^2\,\|\underline{\mathrm{OE}}\|^2 = \left[(2k-1)\|\,\underline{\mathrm{OA}}\,\|^2 + k^2\left(\mathrm{l}^2 + 2\mathrm{l}\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}}\right)\right]^2 \\
\Leftrightarrow \quad & (2k-1)^2r^2\|\,\underline{\mathrm{OA}}\,\|^2 + (2k-1)^2r^2k^2l^2 + (2k-1)^2r^22kl\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}} \\
& = (2k-1)^2\|\,\underline{\mathrm{OA}}\,\|^4 + 2(2k-1)\|\,\underline{\mathrm{OA}}\,\|^2k^2\left(\mathrm{l}^2 + 2\mathrm{l}\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}}\right) + k^4\left(\mathrm{l}^2 + 2\mathrm{l}\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}}\right)^2 \\
\Leftrightarrow \quad & (2k-1)^2(r^2 - \|\,\underline{\mathrm{OA}}\,\|^2)\|\,\underline{\mathrm{OA}}\,\|^2 + (2k-1)^2r^2k^2l^2 + (2k-1)^2r^22kl\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}} \\
& = 2(2k-1)l\|\,\underline{\mathrm{OA}}\,\|^2k^2(l + 2\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}}) + k^4l^2(l + 2\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}})^2 \\
\Leftrightarrow \quad & (4k^2-2k+1)C_0 + (4k^2-2k+1)k^2C_1 + (4k^2-2k+1)kC_2 = (2k-1)k^2C_3 + k^4C_4 \\
\Leftrightarrow \quad & k^4\left[4C_1 - C_4\right] + k^3\left[-2C_1 + 4C_2 - 2C_3\right] + k^2\left[4C_0 + C_1 - 2C_2 + C_3\right] + k\left[-2C_0 + C_2\right] + C_0 = 0
\end{aligned}$$

Where:

$$\begin{cases} C_0 = (r^2 - \|\,\underline{\mathrm{OA}}\,\|^2)\|\,\underline{\mathrm{OA}}\,\|^2 \\ C_1 = r^2l^2 \\ C_2 = 2lr^2\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}} \\ C_3 = 2l\|\,\underline{\mathrm{OA}}\,\|^2(l + 2\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}}) \\ C_4 = l^2(l + 2\,\underline{\mathrm{OA}}\cdot\underline{\mathrm{e}})^2 \end{cases}$$

6 Toroidal