Find reflexions on a 3d surface from a point and a unit vector

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Contents

1 Notations

- A is the point of observation
- $\underline{\mathrm{e}}_{\mathrm{A}}$ is the associated vector
- C is the local coordinate center of the 3d surface
- D is the intersection of $(A, \underline{\mathrm{e}}_{\mathrm{A}})$ with the 3d surface
- $\underline{\mathrm{e}}_{\mathrm{B}}$ is the associated reflected vector
- $\underline{\mathrm{n}}(D)$ is the normal vector of the 3d surface at point D

2 General equations

2.1 intersection: finding D

Point D is such that:

$$\underline{\mathrm{AD}} = k\,\underline{\mathrm{e}}_{\mathrm{A}} = \underline{\mathrm{AC}} + \underline{\mathrm{CD}}$$

So:

$$k\,\underline{\mathrm{e}}_{\mathrm{A}} \cdot \underline{\mathrm{n}} = (\underline{\mathrm{AC}} + \underline{\mathrm{CD}}) \cdot \underline{\mathrm{n}} \tag{1}$$

2.2 reflection: deriving angle of incidence

The angle of incidence (with respect to the dioptre) is:

$$\alpha = \arcsin(-\,\underline{\mathrm{e}}_{\mathrm{A}} \cdot \underline{\mathrm{n}})$$

2.3 reflection: deriving $\underline{\mathrm{e}}_{\mathrm{B}}$

The point D on the 3d sufarce is such that $\underline{\mathrm{n}}(D)$ is in the same plane as (A, D, B), which is written:

Using vector components:

$$\begin{aligned} \underline{\mathrm{e}}_{\mathrm{B}} &= (-\,\underline{\mathrm{e}}_{\mathrm{A}} \cdot \underline{\mathrm{n}})\,\underline{\mathrm{n}} + (\underline{\mathrm{e}}_{\mathrm{A}} - (\underline{\mathrm{e}}_{\mathrm{A}} \cdot \underline{\mathrm{n}})\,\underline{\mathrm{n}}) \\ &= \underline{\mathrm{e}}_{\mathrm{A}} - 2(\underline{\mathrm{e}}_{\mathrm{A}} \cdot \underline{\mathrm{n}})\,\underline{\mathrm{n}} \end{aligned}$$

We can check that in both cases:

$$\begin{cases} \underline{\mathrm{e}}_{\mathrm{B}} \cdot (\underline{\mathrm{e}}_{\mathrm{A}} \wedge \underline{\mathrm{n}}) = 0 & \text{(coplanar)} \\ \underline{\mathrm{e}}_{\mathrm{B}} \cdot \underline{\mathrm{n}} = -\,\underline{\mathrm{e}}_{\mathrm{A}} \cdot \underline{\mathrm{n}} & \text{(same angle)} \end{cases}$$

3 Plane

If the 3d surface is a plane, then $\underline{\mathrm{n}}$ is constant and does not depend on k.

The equations become:

$$\begin{aligned}(1) \quad \Leftrightarrow \quad & k\,\underline{\mathrm{e}}_{\mathrm{A}} \cdot \underline{\mathrm{n}} && = \underline{\mathrm{AC}} \cdot \underline{\mathrm{n}} \\ \Leftrightarrow \quad & k = \frac{\underline{\mathrm{AC}} \cdot \underline{\mathrm{n}}}{\underline{\mathrm{e}}_{\mathrm{A}} \cdot \underline{\mathrm{n}}} &&\end{aligned}$$

4 Cylinder

Consider a cylinder of axes $(O, \underline{\mathrm{z}})$, with $\underline{\mathrm{z}}$ unit vector and radius r. The equations become:

$$\begin{array}{llll}
(1) & \Leftrightarrow & k\,\underline{\mathrm{e}}_{\mathrm{A}}\cdot\underline{\mathrm{n}} & = \underline{\mathrm{AO}}\cdot\underline{\mathrm{n}} + \underline{\mathrm{OD}}\cdot\underline{\mathrm{n}} \\
 & \Leftrightarrow & k\,\underline{\mathrm{e}}_{\mathrm{A}}\cdot\underline{\mathrm{n}} & = \underline{\mathrm{AO}}\cdot\underline{\mathrm{n}} + \underline{OE}\cdot\underline{\mathrm{n}} + \underline{\mathrm{ED}}\cdot\underline{\mathrm{n}} \\
 & \Leftrightarrow & k\,\underline{\mathrm{e}}_{\mathrm{A}}\cdot\underline{\mathrm{n}} & = \underline{\mathrm{AO}}\cdot\underline{\mathrm{n}} + \underline{\mathrm{ED}}\cdot\underline{\mathrm{n}} \\
 & \Leftrightarrow & k\,\underline{\mathrm{e}}_{\mathrm{A}}\cdot\underline{\mathrm{n}} & = \underline{\mathrm{AO}}\cdot\underline{\mathrm{n}} + (-r\,\underline{\mathrm{n}})\cdot\underline{\mathrm{n}} \\
 & \Leftrightarrow & k\,\underline{\mathrm{e}}_{\mathrm{A}}\cdot\underline{\mathrm{n}} & = \underline{\mathrm{AO}}\cdot\underline{\mathrm{n}} - r
\end{array}$$

Now, for any point on line $(A, \underline{\mathrm{e}}_{\mathrm{A}})$, the unit vector $\underline{\mathrm{n}}$ is:

$$\underline{\mathrm{n}}(D) = \frac{(\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}})\wedge\underline{\mathrm{z}}}{\|(\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}})\wedge\underline{\mathrm{z}}\|} = \frac{(\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}})\wedge\underline{\mathrm{z}}}{\|\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}}\|}$$

And:

$$\begin{array}{lll}
 & \underline{\mathrm{OD}}\wedge\underline{\mathrm{z}} & = \underline{\mathrm{OA}}\wedge\underline{\mathrm{z}} + k\,\underline{\mathrm{e}}_{\mathrm{A}}\wedge\underline{\mathrm{z}} \\
\Rightarrow & (\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}})\wedge\underline{\mathrm{z}} & = (\underline{\mathrm{OA}}\wedge\underline{\mathrm{z}})\wedge\underline{\mathrm{z}} + k(\underline{\mathrm{e}}_{\mathrm{A}}\wedge\underline{\mathrm{z}})\wedge\underline{\mathrm{z}}
\end{array}$$

And:

$$\|\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}}\|^2 = \|\underline{\mathrm{OA}}\wedge\underline{\mathrm{z}}\|^2 + k^2\|\underline{\mathrm{e}}_{\mathrm{A}}\wedge\underline{\mathrm{z}}\|^2 + 2k(\underline{\mathrm{OA}}\wedge\underline{\mathrm{z}})\cdot(\underline{\mathrm{e}}_{\mathrm{A}}\wedge\underline{\mathrm{z}})$$

So

$$\begin{array}{llll}
(1) & \Leftrightarrow & k\,\underline{\mathrm{e}}_{\mathrm{A}}\cdot\underline{\mathrm{n}} & = \underline{\mathrm{AO}}\cdot\underline{\mathrm{n}} - r \\
 & \Leftrightarrow & k\,\underline{\mathrm{e}}_{\mathrm{A}}\cdot((\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}})\wedge\underline{\mathrm{z}}) & = \underline{\mathrm{AO}}\cdot((\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}})\wedge\underline{\mathrm{z}}) - r\,\|\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}}\| \\
 & \Leftrightarrow & -k(\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}})\cdot(\underline{\mathrm{e}}_{\mathrm{A}}\wedge\underline{\mathrm{z}}) & = (\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}})\cdot(\underline{\mathrm{OA}}\wedge\underline{\mathrm{z}}) - r\,\|\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}}\| \\
 & \Leftrightarrow & r\,\|\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}}\| & = k(\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}})\cdot(\underline{\mathrm{e}}_{\mathrm{A}}\wedge\underline{\mathrm{z}}) + (\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}})\cdot(\underline{\mathrm{OA}}\wedge\underline{\mathrm{z}}) \\
 & \Leftrightarrow & r\,\|\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}}\| & = (\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}})\cdot(k\,\underline{\mathrm{e}}_{\mathrm{A}}\wedge\underline{\mathrm{z}} + \underline{\mathrm{OA}}\wedge\underline{\mathrm{z}}) \\
 & \Leftrightarrow & r\,\|\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}}\| & = \|\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}}\|^2
\end{array}$$

But $\|\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}}\| \neq 0$ because D cannot be of the axis of the cylinder.
So

$$\begin{array}{lll}
(1) & \Rightarrow & r^2 = \|\underline{\mathrm{OD}}\wedge\underline{\mathrm{z}}\|^2 \\
 & \Leftrightarrow & r^2 = \|\underline{\mathrm{OA}}\wedge\underline{\mathrm{z}}\|^2 + k^2\|\underline{\mathrm{e}}_{\mathrm{A}}\wedge\underline{\mathrm{z}}\|^2 + 2k(\underline{\mathrm{OA}}\wedge\underline{\mathrm{z}})\cdot(\underline{\mathrm{e}}_{\mathrm{A}}\wedge\underline{\mathrm{z}})
\end{array}$$

So ultimately:

$$(1) \quad \Rightarrow \quad k^2\|\underline{\mathrm{e}}_{\mathrm{A}}\wedge\underline{\mathrm{z}}\|^2 + 2k(\underline{\mathrm{OA}}\wedge\underline{\mathrm{z}})\cdot(\underline{\mathrm{e}}_{\mathrm{A}}\wedge\underline{\mathrm{z}}) + \|\underline{\mathrm{OA}}\wedge\underline{\mathrm{z}}\|^2 - r^2 = 0$$

5 Sphere

Consider a sphere of center O and radius r. The normal vector associated to any point E (not on O) is:

$$\underline{\mathrm{n}}(D) = -\frac{\underline{\mathrm{OD}}}{\|\underline{\mathrm{OD}}\|}$$

Given that E in on the (A, B) line:
Which is the same equation as for the cylinder, but with cross products replaced by norms.

$$\begin{array}{rlll}
(1) & \Leftrightarrow & k\,\underline{\mathrm{e}}_{\mathrm{A}} \cdot \underline{\mathrm{n}} & = \underline{\mathrm{AO}} \cdot \underline{\mathrm{n}} + \underline{\mathrm{OD}} \cdot \underline{\mathrm{n}} \\
& \Leftrightarrow & k\,\underline{\mathrm{e}}_{\mathrm{A}} \cdot \underline{\mathrm{n}} & = \underline{\mathrm{AO}} \cdot \underline{\mathrm{n}} - r \\
& \Leftrightarrow & -k\,\underline{\mathrm{e}}_{\mathrm{A}} \cdot \underline{\mathrm{OD}} & = -\,\underline{\mathrm{AO}} \cdot \underline{\mathrm{OD}} - r\,\|\underline{\mathrm{OD}}\| \\
& \Leftrightarrow & r\,\|\underline{\mathrm{OD}}\| & = \underline{\mathrm{OA}} \cdot \underline{\mathrm{OD}} + k\,\underline{\mathrm{e}}_{\mathrm{A}} \cdot \underline{\mathrm{OD}} \\
& \Leftrightarrow & r\,\|\underline{\mathrm{OD}}\| & = \|\underline{\mathrm{OD}}\|^2 \\
& \Rightarrow & r^2 & = \|\underline{\mathrm{OD}}\|^2 \\
& \Leftrightarrow & r^2 & = \|\,\underline{\mathrm{OA}}\,\|^2 + k^2 + 2k\,\underline{\mathrm{OA}} \cdot \underline{\mathrm{e}}_{\mathrm{A}}
\end{array}$$

So ultimately:

$$(1) \quad \Rightarrow \quad k^2 + 2k\,\underline{\mathrm{OA}} \cdot \underline{\mathrm{e}}_{\mathrm{A}} + \|\,\underline{\mathrm{OA}}\,\|^2 - r^2 = 0$$

6 Torus

Consider a torus of axes $(O, \underline{\mathrm{z}})$, with $\underline{\mathrm{z}}$ unit vector. It has major radius $\mathrm{r_a}$ and minor radius $\mathrm{r_b}$.

For any point E in space, we have:

$$\underline{\mathrm{e}}_{\mathrm{a}}(E) = -\frac{(\underline{OE} \wedge \underline{\mathrm{z}}) \wedge \underline{\mathrm{z}}}{\|(\underline{OE} \wedge \underline{\mathrm{z}}) \wedge \underline{\mathrm{z}}\|}$$