Integrals of bsplines of degree 0,1,2 and 3

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Another tool for ToFu

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Contents

1. INTRODUCTION

1.1. GENERAL PROBLEM

Quadrature[1] is numerical integration by summing the weighted values of the integrand assessed at well-chosen fixed points $\int_a^b f(x)dx = \sum_{i=1}^n w_i f(x_i)$. The points are found by computing the roots of a set of special orthogonal polynomials.

Gaussian quadrature is a method of choosing n points and weights such that the result is exact for a polynomial of degree $d \leq 2n-1$. Hence, for a bspline of degree d, the gaussian quadrature is exact if we have $n \geq (d+1)/2$ points.

More generally, if $f(x) = h(x)g(x)$ where at least g is a polynom with proper degree and h is know, then modified points x_i^l and weights w_i^l can be used. When the weight function is $h(x) = 1$ (i.e. when f is a polynomial of appropriate regularity), the best weights and points are the Gauss-Legendre or Gauss-Lobatto ones (we will focus on Gauss-Legendre in the following). In other cases, some special points and weights can be obtained for specific weighting funtions which are not polynomials:

Table 1.1: Quadrature formulas vs weighting function

$f(x) = h(x)g(x)$ where g is a polynomial and h is the weighting function

interval	h	orthogonal polynomials
$[-1;1]$	1	Gauss-Legendre
$[-1;1]$	$(1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$	Jacobi
$[-1;1]$	$1/\sqrt{1-x^2}$	1st kind Chebyshev
$[-1;1]$	$\sqrt{1-x^2}$	2nd kind Chebyshev
$[0;\infty]$	e^{-x}	Laguerre
$[0;\infty]$	$x^\alpha e^{-x}$, $\alpha > -1$	Generalized Laguerre
$[-\infty;\infty]$	e^{-x^2}	Hermite

1.2. GAUSS-LEGENDRE

In this section, the domain of integration is $[-1;1]$.

Table 1.2: Gauss-Legendre quadrature formulas on $[-1;1]$

The interval of integration is $[1;1]$

Degree d	Nb. of points n	Points x_i	Weights w_i
0	1	0	2
1	1	0	2
2	2	$\pm 1/\sqrt{3}$	1
3	2	$\pm 1/\sqrt{3}$	1
4	3	$0, \pm\sqrt{\frac{3}{5}}$	$\frac{8}{9}, \frac{5}{9}$
5	3	$0, \pm\sqrt{\frac{3}{5}}$	$\frac{8}{9}, \frac{5}{9}$
6	4	$\pm\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}, \pm\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18+\sqrt{30}}{36}, \frac{18-\sqrt{30}}{36}$

1.3. RESCALING

Since $\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right)dx$, we can derive;

$$\int_a^b f(x)dx = \frac{b-a}{2}\sum_{i=1}^n w_i f\left(\frac{b-a}{2}x_i + \frac{a+b}{2}\right)$$

[1] see https://en.wikipedia.org/wiki/Gaussian_quadrature

2. 1D B-SPLINES

2.1. LINEAR FUNCTIONALS

In the following, we tr to use quadrature formulas to derive, when possible, operators for matrix computation of integrals, noting $\underline{C} = \begin{pmatrix} c_0 \\ \vdots \\ c_j \\ \vdots \\ c_{N-1} \end{pmatrix}$ the vector of N coefficients associated to each b-spline. In the case of linear functionals (i.e.: D0, D0N1, D1, D1N2, D2, D2N2, D3, D3N2), we used Gauss-Legendre quadrature because all integrands are themselves polynomials. By noting $\partial_m b_{d,0}$ the m-th derivative of b-spline $b_{d,0}$, where $m \leq d$, if $g(x) = \sum_{j=0}^{N-1} c_j b_{d,j}$ is a sum of b-splines, then, since a sum of polynomials of any degree is also a polynomial of the same degree:

2.1.1. D0, D1, D2, D3

$$\begin{aligned}
\int_a^b g(x)dx &= \tfrac{b-a}{2}\textstyle\sum_{i=1}^n w_i g\left(\tfrac{b-a}{2}x_i + \tfrac{a+b}{2}\right) && \text{(faster evaluation with known coefs)} \\
&= \tfrac{b-a}{2}\textstyle\sum_{i=1}^n w_i \sum_{j=0}^{N-1} c_j b_{d,j}\left(\tfrac{b-a}{2}x_i + \tfrac{a+b}{2}\right) \\
&= \textstyle\sum_{j=0}^{N-1} c_j \times \left[\tfrac{b-a}{2}\sum_{i=1}^n w_i b_{d,j}\left(\tfrac{b-a}{2}x_i + \tfrac{a+b}{2}\right)\right] \\
&= \textstyle\sum_{j=0}^{N-1} c_j \times A_j && \text{(coefs factorized for pre-computing)}
\end{aligned}$$

So here

$$\int_a^b g(x)dx = \underline{AC} = \left(A_0 \dots A_j \dots A_{N-1}\right)\underline{C}$$

2.1.2. D0N2

Similarly, a polynomial of degree d squared is a polynomial of degree $2d$, but since:

$$\begin{aligned}
\int_a^b g^2(x)dx &= \tfrac{b-a}{2}\textstyle\sum_{i=1}^n w_i g^2\left(\tfrac{b-a}{2}x_i + \tfrac{a+b}{2}\right) && \text{(faster evaluation with known coefs)} \\
&= \tfrac{b-a}{2}\textstyle\sum_{i=1}^n w_i \left(\sum_{j=0}^{N-1} c_j b_{d,j}\left(\tfrac{b-a}{2}x_i + \tfrac{a+b}{2}\right)\right)^2
\end{aligned}$$

Then the factorisation depends on the degree of the b-splines (which determines the overlap, i.e.: the number of terms in the squared brackets).

$d = 0 \Rightarrow$ NO OVERLAPPING AND $n = 1$ $(2 \times 0 = 0)$

$$\begin{aligned}
\int_a^b g^2(x)dx &= \tfrac{b-a}{2}\textstyle\sum_{i=1}^n w_i \sum_{j=0}^{N-1} c_j^2 b_{0,j}^2\left(\tfrac{b-a}{2}x_i + \tfrac{a+b}{2}\right) \\
&= \textstyle\sum_{j=0}^{N-1} c_j^2 \times \left[\tfrac{b-a}{2}\sum_{i=1}^n w_i b_{0,j}^2\left(\tfrac{b-a}{2}x_i + \tfrac{a+b}{2}\right)\right] \\
&= \textstyle\sum_{j=0}^{N-1} c_j^2 \times A_j && \text{(coefs factorized for pre-computing)}
\end{aligned}$$

So, here an operator can be derived in a matrix form:

$$\begin{aligned}
\int_a^b g^2(x)dx &= {}^t\,\underline{C}\,\underline{\underline{A}}\,\underline{C} \\
&= {}^t\,\underline{C}\begin{pmatrix} A_0 & & & & \\ & \ddots & & & \\ & & A_j & & \\ & & & \ddots & \\ & & & & A_{N-1} \end{pmatrix}\underline{C}
\end{aligned}$$

$d=1 \Rightarrow$ **2 B-SPLINES ON EACH INTERVAL** $n=2$ $(2\times 1=2)$ Here, we decompose the total interval $[a;b]$ into elementary intervals $I_k=[a_k;b_k]$ matching the knots of the bsplines $\int_a^b g^2(x)dx = \sum_{k=0}^{K-1}\int_{x_k}^{x_{k+1}} g^2(x)dx$, where $N=K-1-d$ To simplify the equations, we note $X_{i,k}=\frac{x_{k+1}-x_k}{2}x_i+\frac{x_k+x_{k+1}}{2}$.
With this notation, each b-spline $b_{d,j}$ lives on an interval $I_j=[x_j;x_{j+1+d}]$. Hence, if we consider just one mesh element $[x_k;x_{k+1}]$, two halves of two bsplines of $d=1$ live on it:

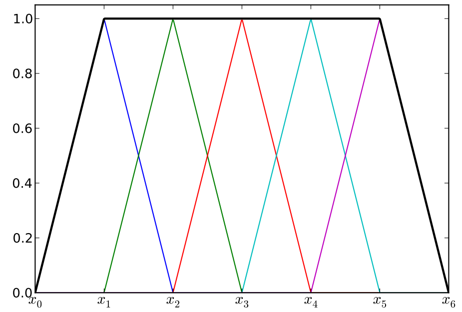

Figure 2.1: ToFu-created $d=1$ bsplines

$$
\begin{aligned}
&\int_{x_k}^{x_{k+1}} g^2(x)dx\\
=\ &\frac{x_{k+1}-x_k}{2}\sum_{i=1}^n w_i\left(c_{k-1}b_{1,k-1}\left(X_{i,k}\right)+c_k b_{1,k}\left(X_{i,k}\right)\right)^2\\
=\ &\frac{x_{k+1}-x_k}{2}\sum_{i=1}^n w_i\left(c_{k-1}^2 b_{1,k-1}^2\left(X_{i,k}\right)+2c_{k-1}c_k b_{1,k-1}\left(X_{i,k}\right)b_{1,k}\left(X_{i,k}\right)+c_k^2 b_{1,k}^2\left(X_{i,k}\right)+\right)\\
=\ &c_{k-1}^2\frac{x_{k+1}-x_k}{2}\sum_{i=1}^n w_i b_{1,k-1}^2\left(X_{i,k}\right)+2c_{k-1}c_k\frac{x_{k+1}-x_k}{2}\sum_{i=1}^n w_i b_{1,k-1}\left(X_{i,k}\right)b_{1,k}\left(X_{i,k}\right)+c_k^2\frac{x_{k+1}-x_k}{2}\sum_{i=1}^n w_i b_{1,k}^2\left(X_{i,k}\right)\\
=\ &c_{k-1}^2 A_{k-1,k-1,k}+2c_{k-1}c_k A_{k-1,k,k}+c_k^2 A_{k,k,k}
\end{aligned}
$$

Hence, by summing over all intervals:

$$
\begin{aligned}
\int_a^b g^2(x)dx &= c_0^2\int_{x_0}^{x_1} b_{2,0}^2 && (k=0)\\
&\quad +c_0^2\int_{x_1}^{x_2} b_{2,0}^2+c_1^2\int_{x_1}^{x_2} b_{2,1}^2+2c_0c_1\int_{x_1}^{x_2} b_{2,0}b_{2,1} && (k=1)\\
&\quad +c_1^2\int_{x_2}^{x_3} b_{2,1}^2+c_2^2\int_{x_2}^{x_3} b_{2,2}^2+2c_1c_2\int_{x_2}^{x_3} b_{2,1}b_{2,2} && (k=2)\\
&\quad +c_2^2\int_{x_3}^{x_4} b_{2,2}^2+c_3^2\int_{x_3}^{x_4} b_{2,3}^2+2c_2c_3\int_{x_3}^{x_4} b_{2,2}b_{2,3} && (k=3)\\
&\quad +\dots\\
&= c_0^2\int_{I_0} b_{2,0}^2+2c_0c_1\int_{I_0\bigcap I_1} b_{2,0}b_{2,1}+c_1^2\int_{I_1} b_{2,1}^2+2c_1c_2\int_{I_1\bigcap I_2} b_{2,1}b_{2,2}+\dots
\end{aligned}
$$

Where we have introduce I_j the interval on which $b_{1,j}$ lives. Thus noting $A_{i,j}=\int_{I_i\bigcap I_j} b_{2,i}b_{2,j}=A_{j,i}$.

$$
\int_a^b g^2(x)dx = {}^t\underline{C}\underline{\underline{A}}\underline{C} = {}^t\underline{C}
\begin{pmatrix}
\int_{I_0} b_{1,0}^2 & A_{0,1} & & & & & \\
A_{0,1} & \ddots & \ddots & & & & \\
 & \ddots & \ddots & A_{j,j-1} & & & \\
 & & A_{j,j-1} & \int_{I_j} b_{1,j}^2 & A_{j,j+1} & & \\
 & & & A_{j,j+1} & \ddots & \ddots & \\
 & & & & \ddots & \ddots & A_{N-2,N-1}\\
 & & & & & A_{N-2,N-1} & \int_{I_{N-1}} b_{1,N-1}^2
\end{pmatrix}
\underline{C}
$$

$d = 2 \Rightarrow$ 3 B-SPLINES ON EACH INTERVAL $n = 3$ $(2\times 2 = 4)$ Following the same logic, we have here for each interval:

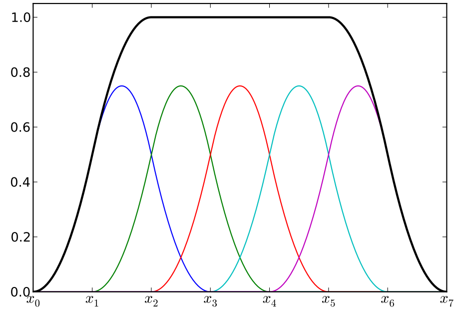

Figure 2.2: ToFu-created $d = 2$ bsplines

$$
\begin{aligned}
& \int_{x_k}^{x_{k+1}} g^2(x)dx \\
= & \int_{x_k}^{x_{k+1}} \left(c_{k-2}b_{2,k-2} + c_{k-1}b_{2,k-1} + c_k b_{2,k}\right)^2(x)dx \\
= & \int_{x_k}^{x_{k+1}} c_{k-2}^2 b_{2,k-2}^2 + c_{k-1}^2 b_{2,k-1}^2 + 2c_{k-2}b_{2,k-2}c_{k-1}b_{2,k-1} + c_k^2 b_{2,k}^2 + 2c_k b_{2,k}c_{k-2}b_{2,k-2} + 2c_k b_{2,k}c_{k-1}b_{2,k-1} \\
= & c_{k-2}^2 \int_{x_k}^{x_{k+1}} b_{2,k-2}^2 + c_{k-1}^2 \int_{x_k}^{x_{k+1}} b_{2,k-1}^2 + c_k^2 \int_{x_k}^{x_{k+1}} b_{2,k}^2 \dots \\
& + 2c_{k-2}c_{k-1}\int_{x_k}^{x_{k+1}} b_{2,k-2}b_{2,k-1} + 2c_k c_{k-2}\int_{x_k}^{x_{k+1}} b_{2,k}b_{2,k-2} + 2c_k c_{k-1}\int_{x_k}^{x_{k+1}} b_{2,k}b_{2,k-1}
\end{aligned}
$$

Hence, by summing over all intervals:

$$
\begin{aligned}
\int_a^b g^2(x)dx = & \; c_0^2\int_{x_0}^{x_1} b_{2,0}^2 && (k=0) \\
& + c_0^2\int_{x_1}^{x_2} b_{2,0}^2 + c_1^2\int_{x_1}^{x_2} b_{2,1}^2 + 2c_0c_1\int_{x_1}^{x_2} b_{2,0}b_{2,1} && (k=1) \\
& + c_0^2\int_{x_2}^{x_3} b_{2,0}^2 + c_1^2\int_{x_2}^{x_3} b_{2,1}^2 + c_2^2\int_{x_2}^{x_3} b_{2,2}^2 + 2c_0c_1\int_{x_2}^{x_3} b_{2,0}b_{2,1} + 2c_0c_2\int_{x_2}^{x_3} b_{2,0}b_{2,2} + 2c_1c_2\int_{x_2}^{x_3} b_{2,1}b_{2,2} && (k=2) \\
& + c_1^2\int_{x_3}^{x_4} b_{2,1}^2 + c_2^2\int_{x_3}^{x_4} b_{2,2}^2 + c_3^2\int_{x_3}^{x_4} b_{2,3}^2 + 2c_1c_2\int_{x_3}^{x_4} b_{2,1}b_{2,2} + 2c_1c_3\int_{x_3}^{x_4} b_{2,1}b_{2,3} + 2c_2c_3\int_{x_3}^{x_4} b_{2,2}b_{2,3} && (k=3) \\
& + \dots \\
= & \; c_0^2\int_{I_0} b_{2,0}^2 + 2c_0c_1\int_{I_0\cap I_1} b_{2,0}b_{2,1} + 2c_0c_2\int_{I_0\cap I_2} b_{2,0}b_{2,2} + c_1^2\int_{I_1} b_{2,1}^2 + 2c_1c_2\int_{I_1\cap I_2} b_{2,1}b_{2,2} + \dots
\end{aligned}
$$

So in matrix form, still noting $A_{i,j} = \int_{I_i\cap I_j} b_{2,i}b_{2,j} = A_{j,i}$:

$$
\int_a^b g^2(x)dx = {}^t\underline{C}\,\underline{\underline{A}}\,\underline{C} = {}^t\underline{C}
\begin{pmatrix}
\int_{I_0} b_{2,0}^2 & A_{0,1} & A_{0,2} & & & & & & & & \\
A_{0,1} & \ddots & \ddots & \ddots & & & & & & & \\
A_{0,2} & \ddots & \ddots & \ddots & \ddots & & & & & & \\
 & \ddots & \ddots & \ddots & \ddots & A_{j,j-2} & & & & & \\
 & & \ddots & \ddots & \ddots & A_{j,j-1} & \ddots & & & & \\
 & & & A_{j,j-2} & A_{j,j-1} & \int_{I_j} b_{2,j}^2 & A_{j,j+1} & A_{j,j+2} & & & \\
 & & & & \ddots & A_{j,j+1} & \ddots & \ddots & \ddots & & \\
 & & & & & A_{j,j+2} & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & & \ddots & \ddots & \ddots & \ddots & A_{N-1,N-3} \\
 & & & & & & & \ddots & \ddots & \ddots & A_{N-1,N-2} \\
 & & & & & & & & A_{N-1,N-3} & A_{N-1,N-2} & \int_{I_{N-1}} b_{2,N-1}^2
\end{pmatrix}
\underline{C}
$$

$d=3 \Rightarrow$ **4 B-SPLINES ON EACH INTERVAL** $n=4$ $(2\times 3=6)$ The degree determines the amount of overlapping (i.e.: the number of non-zero diagonals in the matrix). Apart from that, the rest remains similar since we are still deriving the squared total function.

$$
\begin{aligned}
& \int_{x_k}^{x_{k+1}} g^2(x)\,dx \\
= \; & \int_{x_k}^{x_{k+1}} \left(c_{k-3}b_{3,k-3}+c_{k-2}b_{3,k-2}+c_{k-1}b_{3,k-1}+c_k b_{3,k}\right)^2(x)\,dx \\
= \; & c_{k-3}^2\int_{x_k}^{x_{k+1}} b_{3,k-3}^2 + c_{k-2}^2\int_{x_k}^{x_{k+1}} b_{3,k-2}^2 + c_{k-1}^2\int_{x_k}^{x_{k+1}} b_{3,k-1}^2 + c_k^2\int_{x_k}^{x_{k+1}} b_{3,k}^2 \dots \\
& +2c_{k-3}c_{k-2}\int_{x_k}^{x_{k+1}} b_{3,k-3}b_{3,k-2} + 2c_{k-3}c_{k-1}\int_{x_k}^{x_{k+1}} b_{3,k-3}b_{3,k-1} + 2c_{k-3}c_k\int_{x_k}^{x_{k+1}} b_{3,k-3}b_{3,k} \dots \\
& +2c_{k-2}c_{k-1}\int_{x_k}^{x_{k+1}} b_{3,k-2}b_{3,k-1} + 2c_k c_{k-2}\int_{x_k}^{x_{k+1}} b_{3,k}b_{3,k-2} + 2c_k c_{k-1}\int_{x_k}^{x_{k+1}} b_{3,k}b_{3,k-1}
\end{aligned}
$$

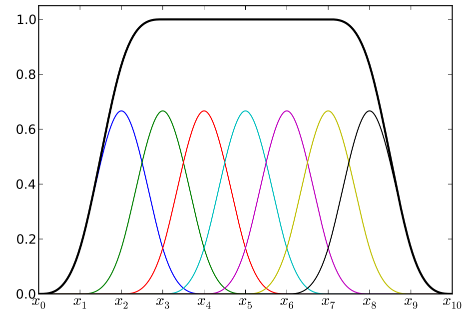

Figure 2.3: ToFu-created $d=3$ bsplines

hence, following the same logic:

$$
\int_a^b g^2(x)\,dx = {}^t\,\underline{C}\,\underline{\underline{A}}\,\underline{C} = {}^t\,\underline{C}
\begin{pmatrix}
\int_{I_0} b_{3,0}^2 & A_{0,1} & A_{0,2} & A_{0,3} & & & & & & & & & & & \\
A_{0,1} & \ddots & \ddots & \ddots & \ddots & & & & & & & & & & \\
A_{0,2} & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & & & & \\
A_{0,3} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & A_{j,j-3} & & & & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & A_{j,j-2} & \ddots & & & & & & \\
 & & & \ddots & \ddots & \ddots & \ddots & A_{j,j-1} & \ddots & \ddots & & & & & \\
 & & & & A_{j,j-3} & A_{j,j-2} & A_{j,j-1} & \int_{I_j} b_{3,j}^2 & A_{j,j+1} & A_{j,j+2} & A_{j,j+3} & & & & \\
 & & & & & \ddots & \ddots & A_{j,j+1} & \ddots & \ddots & \ddots & & & & \\
 & & & & & & \ddots & A_{j,j+2} & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & & & & & A_{j,j+3} & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & A_{N-1,N-4} \\
 & & & & & & & & & & \ddots & \ddots & \ddots & \ddots & A_{N-1,N-3} \\
 & & & & & & & & & & & \ddots & \ddots & \ddots & A_{N-1,N-2} \\
 & & & & & & & & & & & A_{N-1,N-4} & A_{N-1,N-3} & A_{N-1,N-2} & \int_{I_{N-1}} b_{3,N-1}^2
\end{pmatrix}
\underline{C}
$$

2.1.3. D1N2

With squared derivatives of any order, the structure of A, determined by the number of overlaps, is identical to $D0N2$.
However, the integrands are the chosen derivatives, and computing each integral requires less quadrature points per mesh element (smaller degree).
Here $A_{i,j} = \int_{I_i \bigcap I_j} \partial_x b_{d,i} \partial_x b_{d,j} = A_{j,i}$

$d = 1 \Rightarrow$ 2 B-SPLINES ON EACH INTERVAL $n = 1$ $(2 \times 0 = 0)$

$$\int_a^b \left(\partial_x g\right)^2 = {}^t\underline{C} \begin{pmatrix} \int_{I_0}\left(\partial_x b_{1,0}\right)^2 & A_{0,1} & & & & & \\ A_{0,1} & \ddots & \ddots & & & & \\ & \ddots & \ddots & A_{j,j-1} & & & \\ & & A_{j,j-1} & \int_{I_j}\left(\partial_x b_{1,j}\right)^2 & A_{j,j+1} & & \\ & & & A_{j,j+1} & \ddots & \ddots & \\ & & & & \ddots & \ddots & A_{N-2,N-1} \\ & & & & & A_{N-2,N-1} & \int_{I_{N-1}}\left(\partial_x b_{1,N-1}\right)^2 \end{pmatrix} \underline{C}$$

$d = 2 \Rightarrow$ 3 B-SPLINES ON EACH INTERVAL $n = 2$ $(2 \times 1 = 2)$

$$\int_a^b \left(\partial_x g\right)^2 = {}^t\underline{C} \begin{pmatrix} \int_{I_0}\left(\partial_x b_{2,0}\right)^2 & A_{0,1} & A_{0,2} & & & & & & & & \\ A_{0,1} & \ddots & \ddots & \ddots & & & & & & & \\ A_{0,2} & \ddots & \ddots & \ddots & \ddots & & & & & & \\ & \ddots & \ddots & \ddots & \ddots & A_{j,j-2} & & & & & \\ & & \ddots & \ddots & \ddots & A_{j,j-1} & \ddots & & & & \\ & & & A_{j,j-2} & A_{j,j-1} & \int_{I_j}\left(\partial_x b_{2,j}\right)^2 & A_{j,j+1} & A_{j,j+2} & & & \\ & & & & \ddots & A_{j,j+1} & \ddots & \ddots & \ddots & & \\ & & & & & A_{j,j+2} & \ddots & \ddots & \ddots & \ddots & \\ & & & & & & \ddots & \ddots & \ddots & \ddots & A_{N-1,N-3} \\ & & & & & & & \ddots & \ddots & \ddots & A_{N-1,N-2} \\ & & & & & & & & A_{N-1,N-3} & A_{N-1,N-2} & \int_{I_{N-1}}\left(\partial_x b_{2,N-1}\right)^2 \end{pmatrix} \underline{C}$$

$d = 3 \Rightarrow$ **4 B-SPLINES ON EACH INTERVAL** $n = 3$ $(2 \times 2 = 4)$

$$\int_a^b (\partial_x g)^2 = {}^t\underline{C} \begin{pmatrix}
\int_{I_0} (\partial_x b_{3,0})^2 & A_{0,1} & A_{0,2} & A_{0,3} & & & & & & & & & & & \\
A_{0,1} & \ddots & \ddots & \ddots & \ddots & & & & & & & & & & \\
A_{0,2} & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & & & & \\
A_{0,3} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & A_{j,j-3} & & & & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & A_{j,j-2} & \ddots & & & & & & \\
 & & & \ddots & \ddots & \ddots & \ddots & A_{j,j-1} & \ddots & \ddots & & & & & \\
 & & & & A_{j,j-3} & A_{j,j-2} & A_{j,j-1} & \int_{I_j} (\partial_x b_{3,j})^2 & A_{j,j+1} & A_{j,j+2} & A_{j,j+3} & & & & \\
 & & & & & \ddots & \ddots & A_{j,j+1} & \ddots & \ddots & \ddots & & & & \\
 & & & & & & \ddots & A_{j,j+2} & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & & & & & A_{j,j+3} & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & A_{N-1,N-4} \\
 & & & & & & & & & & \ddots & \ddots & \ddots & \ddots & A_{N-1,N-3} \\
 & & & & & & & & & & & \ddots & \ddots & \ddots & A_{N-1,N-2} \\
 & & & & & & & & & & & A_{N-1,N-4} & A_{N-1,N-3} & A_{N-1,N-2} & \int_{I_{N-1}} (\partial_x b_{3,N-1})^2
\end{pmatrix} \underline{C}$$

2.1.4. D2N2

Here again, A has the same structure, but the integrals can be evaluated with fewer points and $A_{i,j} = \int_{I_i \cap I_j} \partial_x^2 b_{d,i} \partial_x^2 b_{d,j} = A_{j,i}$

$d = 2 \Rightarrow$ **3 B-SPLINES ON EACH INTERVAL** $n = 1$ $(2 \times 0 = 0)$

$d = 3 \Rightarrow$ **4 B-SPLINES ON EACH INTERVAL** $n = 2$ $(2 \times 1 = 2)$

2.1.5. D3N2

Here again, A has the same structure, but the integrals can be evaluated with fewer points and $A_{i,j} = \int_{I_i \cap I_j} \partial_x^3 b_{3,i} \partial_x^3 b_{3,j} = A_{j,i}$

$d = 3 \Rightarrow$ **4 B-SPLINES ON EACH INTERVAL** $n = 1$ $(2 \times 0 = 0)$

2.2. Non-linear functionals

In this section, we consider two non-linear functionals: the entropy (D0ME) and the Fisher information (D1FI).
Obviously, the non-linearity prevents from building a matrix operator. Instead, we simply want to assess the value of the integral as fast as possible for one set of coefficients.
Since there is no quadrature rule dedicated to these expressions (here g is a sum of b-splines of degree 0,1,2 or 3), we resort to the Gauss-Legendre quadrature, with possibly more points that would be required based on the degree of g.

2.2.1. D0ME

Applicable to all degrees, $D0ME\left(g\right) = -\int g \ln(g)$.
However, the entropy is in principle computed with a distribution, so alternatively: $D0ME\left(g\right) = -\int \frac{g}{\int g} \ln\left(\frac{g}{\int g}\right)$.

Since in most cases the considered function g will decrease to 0 at the edge of the domain, it may be necessary to introduce a threshold value ϵ below which the term in the natural logarithm is replaced by ϵ:

$$D0ME\left(g\right) = -\left[\int_{g\geq\epsilon} \frac{g}{\int g} \ln\left(\frac{g}{\int g}\right) + \int_{g<\epsilon} \frac{g}{\int g} \ln\left(\frac{\epsilon}{\int g}\right)\right]$$

The value of ϵ is an input, provided as an absolute value, as a fraction of $\max(g)$ or of $\int g$. Finally, we also use teh absolute value of g to be able to apply the same method to negative profiles, with ϵ defined from $\max(\|g\|)$ or of $\int \|g\|$:

$$D0ME\left(g\right) = -\left[\int_{\|g\|\geq\epsilon} \frac{\|g\|}{\int \|g\|} \ln\left(\frac{\|g\|}{\int \|g\|}\right) + \int_{\|g\|<\epsilon} \frac{\|g\|}{\int \|g\|} \ln\left(\frac{\epsilon}{\int \|g\|}\right)\right]$$

2.2.2. D1FI

Applicable to degrees $d \geq 1$, $D1FI\left(g\right) = \int \frac{(\partial_x g)^2}{g}$ or optionally (experimental): $\int \frac{(\partial_x g)^2}{\|g\|}$
However, the same numerical problem arises: if g goes to 0 the integral diverges or is dominated by its weakest values, hence we also introduce a threshold ϵ from $\max(\|g\|)$ or of $\int \|g\|$:

$$D1FI\left(g\right) = \int_{\|g\|\geq\epsilon} \frac{\|\partial_x g\|^2}{\|g\|} + \int_{\|g\|<\epsilon} \frac{\|\partial_x g\|^2}{\epsilon}$$

3. 2D B-SPLINES

A. D0,D1,D2,D3 - Surf - Exact formulations

A.1. Deriv = 0 - Deg = 0

$$\int_{x_0}^{x_1} b_{0,0}(x)dx = \int_{x_0}^{x_1} dx = x_1 - x_0$$

A.2. Deriv = 0 - Deg = 1

$$\begin{aligned}
\int_{x_0}^{x_2} b_{1,0}(x)dx &= \int_{x_0}^{x_1} \frac{x-x_0}{x_1-x_0}dx + \int_{x_1}^{x_2} \frac{x_2-x}{x_2-x_1}dx \\
&= \frac{\left[(x-x_0)^2\right]_{x_0}^{x_1}}{2(x_1-x_0)} + \frac{\left[-(x_2-x)^2\right]_{x_1}^{x_2}}{2(x_2-x_1)} \\
&= \frac{(x_1-x_0)^2}{2(x_1-x_0)} + \frac{(x_2-x_1)^2}{2(x_2-x_1)} \qquad = \frac{x_2-x_0}{2}
\end{aligned}$$

A.3. Deriv = 0 - Deg = 2

$$\begin{aligned}
\int_{x_0}^{x_3} b_{2,0}(x)dx &= \int_{x_0}^{x_1} \frac{(x-x_0)^2}{(x_2-x_0)(x_1-x0)}dx + \int_{x_1}^{x_2} \frac{(x-x_0)(x_2-x)}{(x_2-x_0)(x_2-x_1)} + \frac{(x-x_1)(x_3-x)}{(x_2-x_1)(x_3-x_1)}dx + \int_{x_2}^{x_3} \frac{(x_3-x)^2}{(x_3-x_2)(x_3-x_1)}dx \\
&= \frac{\left[\frac{(x-x_0)^3}{3}\right]_{x_0}^{x_1}}{(x_2-x_0)(x_1-x0)} + \frac{\left[-\frac{x^3}{3}+(x_0+x_2)\frac{x^2}{2}-xx_0x_2\right]_{x_1}^{x_2}}{(x_2-x_0)(x_2-x_1)} + \frac{\left[-\frac{x^3}{3}+(x_1+x_3)\frac{x^2}{2}-xx_1x_3\right]_{x_1}^{x_2}}{(x_2-x_1)(x_3-x_1)} + \frac{-\left[\frac{(x_3-x)^3}{3}\right]_{x_2}^{x_3}}{(x_3-x_2)(x_3-x_1)} \\
&= \frac{(x_1-x_0)^2}{3(x_2-x_0)} + \frac{-\frac{x_2^2+x_2x_1+x_1^2}{3}+(x_0+x_2)\frac{x_2+x_1}{2}-x_0x_2}{(x_2-x_0)} + \frac{-\frac{x_2^2+x_2x_1+x_1^2}{3}+(x_1+x_3)\frac{x_2+x_1}{2}-x_1x_3}{(x_3-x_1)} + \frac{(x_3-x_2)^2}{3(x_3-x_1)} \\
&= \frac{(x_1-x_0)^2}{3(x_2-x_0)} + \frac{-2(x_2^2+x_2x_1+x_1^2)+3(x_0+x_2)(x_2+x_1)-6x_0x_2}{6(x_2-x_0)} + \frac{-2(x_2^2+x_2x_1+x_1^2)+3(x_1+x_3)(x_2+x_1)-6x_1x_3}{6(x_3-x_1)} + \frac{(x_3-x_2)^2}{3(x_3-x_1)} \\
&= \frac{(x_1-x_0)^2}{3(x_2-x_0)} + \frac{x_2^2+x_2x_1-2x_1^2+3x_0(x_1-x_2)}{6(x_2-x_0)} + \frac{-2x_2^2+x_2x_1+x_1^2+3x_3(x_2-x_1)}{6(x_3-x_1)} + \frac{(x_3-x_2)^2}{3(x_3-x_1)}
\end{aligned}$$

A.4. Deriv = 0 - Deg = 3

$$b_{3,0} = \begin{cases}
\frac{x^3-3x^2x_0+3xx_0^2-x_0^3}{(x_3-x_0)(x_2-x_0)(x_1-x_0)} & \text{, if} \quad x \in [x_0, x_1[\\
\frac{x^3A+x^2B+xC+D}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)} & \text{, if} \quad x \in [x_1, x_2[\\
\frac{x^3A'+x^2B'+xC'+D'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} & \text{, if} \quad x \in [x_2, x_3[\\
\frac{-x^3+3x^2x_4-3xx_4^2+x_4^3}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} & \text{, if} \quad x \in [x_3, x_4[
\end{cases}$$

Hence

$$\begin{aligned}
\int_{x_0}^{x_4} b &= \int_{x_0}^{x_1} b + \int_{x_1}^{x_2} b + \int_{x_2}^{x_3} b + \int_{x_3}^{x_4} b \\
&= \int_{x_0}^{x_1} \frac{x^3-3x^2x_0+3xx_0^2-x_0^3}{(x_3-x_0)(x_2-x_0)(x_1-x_0)} + \int_{x_1}^{x_2} \frac{x^3A+x^2B+xC+D}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)} + \int_{x_2}^{x_3} \frac{x^3A'+x^2B'+xC'+D'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} + \int_{x_3}^{x_4} \frac{-x^3+3x^2x_4-3xx_4^2+x_4^3}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} \\
&= \frac{\frac{x_1^4-x_0^4}{4}-3x_0\frac{x_1^3-x_0^3}{3}+3x_0^2\frac{x_1^2-x_0^2}{2}-x_0^3(x_1-x_0)}{(x_3-x_0)(x_2-x_0)(x_1-x_0)} + \frac{A\frac{x_2^4-x_1^4}{4}+B\frac{x_2^3-x_1^3}{3}+C\frac{x_2^2-x_1^2}{2}+D(x_2-x_1)}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)} + \frac{A'\frac{x_3^4-x_2^4}{4}+x^2B'\frac{x_3^3-x_2^3}{3}+C'\frac{x_3^2-x_2^2}{2}+D'(x_3-x_2)}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} + \frac{-\frac{x_4^4-x_3^4}{4}+3x_4\frac{x_4^3-x_3^3}{3}-3x_4^2\frac{x_4^2-x_3^2}{2}+x_4^3(x_4-x_3)}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}
\end{aligned}$$

A.5. DERIV = 1 - DEG = 2

$$
\begin{aligned}
\int_{x_0}^{x_3} \partial_x b_{2,0} &= \begin{cases} \frac{2}{2(x_2-x_0)(x_1-x_0)} \int_{x_0}^{x_1} (x-x_0)^2 dx & \text{, on } [x_0, x_1[\\ \frac{\int_{x_1}^{x_2} -2x(x_3+x_2-x_1-x_0)+2(x_3x_2-x_1x_0)dx}{(x_2-x_1)(x_2-x_0)(x_3-x_1)} = \frac{\left[-x^2(x_3+x_2-x_1-x_0)+2x(x_3x_2-x_1x_0)\right]_{x_1}^{x_2}}{(x_2-x_1)(x_2-x_0)(x_3-x_1)} & \text{, on } [x_1, x_2[\\ \frac{-2}{-2(x_3-x_2)(x_3-x_1)} \int_{x_2}^{x_3} (x_3-x)^2 dx & \text{, on } [x_2, x_3[\end{cases} \\
&= \begin{cases} \frac{x_1-x_0}{x_2-x_0} & \text{, on } [x_0, x_1[\\ \frac{-x_3x_2^2-x_2^3+x_2^2x_1+x_2^2x_0+2x_3x_2^2-2x_2x_1x_0+x_3x_1^2+x_2x_1^2-x_1^3-x_1^2x_0-2x_3x_2x_1-2x_1^2x_0}{(x_2-x_1)(x_2-x_0)(x_3-x_1)} & \text{, on } [x_1, x_2[\\ -\frac{x_3-x_2}{x_3-x_1} & \text{, on } [x_2, x_3[\end{cases} \\
&= \begin{cases} \frac{x_1-x_0}{x_2-x_0} & \text{, on } [x_0, x_1[\\ \frac{x_3x_2^2-x_2^3+x_2^2x_1+x_2^2x_0-2x_2x_1x_0+x_3x_1^2+x_2x_1^2-x_1^3-2x_3x_2x_1+x_1^2x_0}{(x_2-x_1)(x_2-x_0)(x_3-x_1)} & \text{, on } [x_1, x_2[\\ -\frac{x_3-x_2}{x_3-x_1} & \text{, on } [x_2, x_3[\end{cases}
\end{aligned}
$$

A.6. DERIV = 1 - DEG = 3

$$
\begin{aligned}
\int_{x_0}^{x_4} \partial_x b_{3,0} &= \begin{cases} \int_{x_0}^{x_1} \frac{3x^2-6xx_0+3x_0^2}{(x_3-x_0)(x_2-x_0)(x_1-x_0)} & \text{, on } [x_0, x_1[\\ \int_{x_1}^{x_2} \frac{3x^2A+2xB+C}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)} & \text{, on } [x_1, x_2[\\ \int_{x_2}^{x_3} \frac{3x^2A'+2xB'+C'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} & \text{, on } [x_2, x_3[\\ \int_{x_3}^{x_4} \frac{-3x^2+6xx_4-3x_4^2}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} & \text{, on } [x_3, x_4[\end{cases} \\
&= \frac{\left[x^3-3x^2x_0+3x_0^2x\right]_{x_0}^{x_1}}{(x_3-x_0)(x_2-x_0)(x_1-x_0)} + \frac{\left[x^3A+x^2B+Cx\right]_{x_1}^{x_2}}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)} + \frac{\left[x^3A'+x^2B'+C'x\right]_{x_2}^{x_3}}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} + \frac{\left[-x^3+3x^2x_4-3x_4^2x\right]_{x_3}^{x_4}}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} \\
&= \frac{(x_1^3-x_0^3)-3x_0(x_1^2-x_0^2)+3x_0^2(x_1-x_0)}{(x_3-x_0)(x_2-x_0)(x_1-x_0)} + \frac{A(x_2^3-x_1^3)+B(x_2^2-x_1^2)+C(x_2-x_1)}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)} + \frac{A'(x_3^3-x_1^3)+B'(x_3^2-x_2^2)+C'(x_3-x_2)}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} + \frac{-(x_4^3-x_3^3)+3x_4(x_4^2-x_3^2)-3x_4^2(x_4-x_3)}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}
\end{aligned}
$$

A.7. DERIV = 2 - DEG = 3

$$
\begin{aligned}
\int_{x_0}^{x_4} \partial_x^2 b_{3,0} &= \begin{cases} \int_{x_0}^{x_1} \frac{6x-6x_0}{(x_3-x_0)(x_2-x_0)(x_1-x_0)} & \text{, on } [x_0, x_1[\\ \int_{x_1}^{x_2} \frac{6xA+2B}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)} & \text{, on } [x_1, x_2[\\ \int_{x_2}^{x_3} \frac{6xA'+2B'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} & \text{, on } [x_2, x_3[\\ \int_{x_3}^{x_4} \frac{-6x+6x_4}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} & \text{, on } [x_3, x_4[\end{cases} \\
&= \frac{3(x_1^2-x_0^2)-6x_0(x_1-x_0)}{(x_3-x_0)(x_2-x_0)(x_1-x_0)} + \frac{3A(x_2^2-x_1^2)+2B(x_2-x_1)}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)} + \frac{3A'(x_3^2-x_1^2)+2B'(x_3-x_2)}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} + \frac{-3(x_4^2-x_3^2)+6x_4(x_4-x_3)}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} \\
&= \frac{3(x_1-x_0)}{(x_3-x_0)(x_2-x_0)} + \frac{3A(x_2^2-x_1^2)+2B(x_2-x_1)}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)} + \frac{3A'(x_3^2-x_1^2)+2B'(x_3-x_2)}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} + \frac{3(x_4-x_3)}{(x_4-x_2)(x_4-x_1)}
\end{aligned}
$$

$$\int_{x_0}^{x_4} \partial_x^3 b_{3,0} = \begin{cases} \int_{x_0}^{x_1} \frac{6}{(x_3-x_0)(x_2-x_0)(x_1-x_0)} & \text{, on} \quad [x_0, x_1[\\ \int_{x_1}^{x_2} \frac{6A}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)} & \text{, on} \quad [x_1, x_2[\\ \int_{x_2}^{x_3} \frac{6A'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} & \text{, on} \quad [x_2, x_3[\\ \int_{x_3}^{x_4} \frac{-6}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} & \text{, on} \quad [x_3, x_4[\end{cases}$$

$$= \frac{6}{(x_3-x_0)(x_2-x_0)} + \frac{6A(x_2-x_1)}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)} + \frac{6A'(x_3-x_2)}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} + \frac{-6}{(x_4-x_2)(x_4-x_1)}$$

B. D0,D1,D2,D3 - VOL - EXACT FORMULATIONS

B.1. DERIV = 0 - DEG = 0

$$\int_{x_0}^{x_1} x b_{0,0}(x)dx = \int_{x_0}^{x_1} x dx = \frac{x_1^2 - x_0^2}{2}$$

B.2. DERIV = 0 - DEG = 1

$$\begin{aligned}
\int_{x_0}^{x_2} x b_{1,0}(x)dx &= \int_{x_0}^{x_1} x \frac{x-x_0}{x_1-x_0} dx + \int_{x_1}^{x_2} x \frac{x_2-x}{x_2-x_1} dx \\
&= \frac{1}{x_1-x_0}\left[\frac{x^3}{3} - x_0 \frac{x^2}{2}\right]_{x_0}^{x_1} + \frac{1}{x_2-x_1}\left[x_2 \frac{x^2}{2} - \frac{x^3}{3}\right]_{x_1}^{x_2} \\
&= \frac{1}{x_1-x_0}\left(\frac{x_1^3-x_0^3}{3} - x_0 \frac{x_1^2-x_0^2}{2}\right) + \frac{1}{x_2-x_1}\left(x_2 \frac{x_2^2-x_1^2}{2} - \frac{x_2^3-x_1^3}{3}\right) \\
&= \left(\frac{x_1^2+x_1x_0+x_0^2}{3} - x_0 \frac{x_1+x_0}{2}\right) + \left(x_2 \frac{x_2+x_1}{2} - \frac{x_2^2+x_2x_1+x_1^2}{3}\right) \\
&= \frac{2x_1^2-x_1x_0-x_0^2}{6} + \frac{x_2^2+x_2x_1-2x_1^2}{6} \qquad = \frac{x_2^2+x_1(x_2-x_0)-x_0^2}{6}
\end{aligned}$$

B.3. DERIV = 0 - DEG = 2

$$\begin{aligned}
\int_{x_0}^{x_3} x b_{2,0}(x)dx \quad &= \int_{x_0}^{x_1} x \frac{(x-x_0)^2}{(x_2-x_0)(x_1-x0)} dx + \int_{x_1}^{x_2} x \frac{(x-x_0)(x_2-x)}{(x_2-x_0)(x_2-x_1)} + x \frac{(x-x_1)(x_3-x)}{(x_2-x_1)(x_3-x_1)} dx + \int_{x_2}^{x_3} x \frac{(x_3-x)^2}{(x_3-x_2)(x_3-x_1)} dx \\
&= \frac{\left[\frac{x^4}{4} - 2x_0 \frac{x^3}{3} + x_0^2 \frac{x^2}{2}\right]_{x_0}^{x_1}}{(x_2-x_0)(x_1-x0)} + \frac{\left[-\frac{x^4}{4} + (x_0+x_2)\frac{x^3}{3} - x_0x_2\frac{x^2}{2}\right]_{x_1}^{x_2}}{(x_2-x_0)(x_2-x_1)} + \frac{\left[-\frac{x^4}{4} + (x_1+x_3)\frac{x^3}{3} - x_1x_3\frac{x^2}{2}\right]_{x_1}^{x_2}}{(x_2-x_1)(x_3-x_1)} + \frac{\left[\frac{x^4}{4} - 2x_3\frac{x^3}{3} + x_3^2\frac{x^2}{2}\right]_{x_2}^{x_3}}{(x_3-x_2)(x_3-x_1)} \\
&= \frac{3(x_1^4-x_0^4)-8x_0(x_1^3-x_0^3)+6x_0^2(x_1^2-x_0^2)}{12(x_2-x_0)(x_1-x0)} + \frac{-3(x_2^4-x_1^4)+4(x_0+x_2)(x_2^3-x_1^3)-6x_0x_2(x_2^2-x_1^2)}{12(x_2-x_0)(x_2-x_1)} + \frac{-3(x_2^4-x_1^4)+4(x_1+x_3)(x_2^3-x_1^3)-6x_1x_3(x_2^2-x_1^2)}{12(x_2-x_1)(x_3-x_1)} + \frac{3(x_3^4-x_2^2)-8x_3(x_3^3-x_2^3)+6x_3^2(x_3^2-x_2^2)}{12(x_3-x_2)(x_3-x_1)} \\
&= \frac{3(x_1^2+x_0^2)(x_1+x_0)-8x_0(x_1^2+x_1x_0+x_0^2)+6x_0^2(x_1+x_0)}{12(x_2-x_0)} \\
&\quad + \frac{-3(x_2^2+x_1^2)(x_2+x_1)+4(x_0+x_2)(x_2^2+x_2x_1+x_1^2)-6x_0x_2(x_2+x_1)}{12(x_2-x_0)} + \frac{-3(x_2^2+x_1^2)(x_2+x_1)+4(x_1+x_3)(x_2^2+x_2x_1+x_1^2)-6x_1x_3(x_2+x_1)}{12(x_3-x_1)} \\
&\quad + \frac{3(x_3^2+x_2^2)(x_3+x_2)-8x_3(x_3^2+x_3x_2+x_2^2)+6x_3^2(x_3+x_2)}{12(x_3-x_1)} \\
&= \frac{3x_1^3+3x_0^2x_1+3x_0x_1^2+3x_0^3-8x_0x_1^2-8x_0^2x_1-8x_0^3+6x_0^2x_1+6x_0^3}{12(x_2-x_0)} \\
&\quad + \frac{-3x_2^3-3x_1x_2^2-3x_1^2x_2-3x_1^3+4x_0x_2^2+4x_0x_1x_2+4x_0x_1^2+4x_2^3+4x_2^2x_1+4x_1^2x_2-6x_0x_2^2-6x_0x_1x_2}{12(x_2-x_0)} + \frac{-3x_2^3-3x_1x_2^2-3x_1^2x_2-3x_1^3+4x_1x_2^2+4x_1^2x_2+4x_1^3+4x_2^2x_3+4x_1x_2x_3+4x_1^2x_3-6x_1x_2x_3-6x_1^2x_3}{12(x_3-x_1)} \\
&\quad + \frac{3x_3^3+3x_2x_3^2+3x_2^2x_3+3x_2^3-8x_3^3-8x_2x_3^2-8x_2^2x_3+6x_3^3+6x_2x_3^2}{12(x_3-x_1)} \\
&= \frac{3x_1^3+x_0^2x_1-5x_0x_1^2+x_0^3}{12(x_2-x_0)} \\
&\quad + \frac{x_2^3+x_1x_2^2+x_1^2x_2-3x_1^3-2x_0x_2^2-2x_0x_1x_2+4x_0x_1^2}{12(x_2-x_0)} + \frac{-3x_2^3+x_1x_2^2+x_1^2x_2+x_1^3+4x_2^2x_3-2x_1x_2x_3-2x_1^2x_3}{12(x_3-x_1)} \\
&\quad + \frac{x_3^3+x_2x_3^2-5x_2^2x_3+3x_2^3}{12(x_3-x_1)} \\
&= \frac{(3x_1+x_0)(x_1-x_0)^2}{12(x_2-x_0)} + \frac{(x_2-x_1)\left(x_2^2+2x_1x_2+3x_1^2-2x_0(x_2+2x_1)\right)}{12(x_2-x_0)} - \frac{(x_2-x_1)\left(x_1^2+2x_1x_2+3x_2^2-2x_3(2x_2+x_1)\right)}{12(x_3-x_1)} + \frac{(3x_2+x_3)(x_3-x_2)^2}{12(x_3-x_1)}
\end{aligned}$$

B.4. Deriv = 0 - Deg = 3

$$\begin{aligned}
\int_{x_0}^{x_4} x b_{3,0}(x)dx &= \int_{x_0}^{x_1} x\frac{x^3-3x_0x^2+3x_0^2x-x_0^3}{(x_3-x_0)(x_2-x_0)(x_1-x_0)}dx+\int_{x_1}^{x_2} x\frac{Ax^3+Bx^2+Cx+D}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}dx+\int_{x_2}^{x_3} x\frac{A'x^3+B'x^2+C'x+D'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}dx+\int_{x_3}^{x_4} x\frac{-x^3+3x_4x^2-3x_4^2x+x_4^3}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}dx \\
&= \frac{\left[\frac{x^5}{5}-3x_0\frac{x^4}{4}+3x_0^2\frac{x^3}{3}-x_0^3\frac{x^2}{2}\right]_{x_0}^{x_1}}{(x_3-x_0)(x_2-x_0)(x_1-x_0)}+\frac{\left[A\frac{x^5}{5}+B\frac{x^4}{4}+C\frac{x^3}{3}+D\frac{x^2}{2}\right]_{x_1}^{x_2}}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}+\frac{\left[A'\frac{x^5}{5}+B'\frac{x^4}{4}+C'\frac{x^3}{3}+D'\frac{x^2}{2}\right]_{x_2}^{x_3}}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}+\frac{\left[-\frac{x^5}{5}+3x_4\frac{x^4}{4}-3x_4^2\frac{x^3}{3}+x_4^3\frac{x^2}{2}\right]_{x_3}^{x_4}}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} \\
&= \frac{12(x_1^5-x_0^5)-45x_0(x_1^4-x_0^4)+60x_0^2(x_1^3-x_0^3)-30x_0^3(x_1^2-x_0^2)}{60(x_3-x_0)(x_2-x_0)(x_1-x_0)} \\
&\quad+\frac{12A(x_2^5-x_1^5)+15B(x_2^4-x_1^4)+20C(x_2^3-x_1^3)+30D(x_2^2-x_1^2)}{60(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)} \\
&\quad+\frac{12A'(x_3^5-x_2^5)+15B'(x_3^4-x_2^4)+20C'(x_3^3-x_2^3)+30D'(x_3^2-x_2^2)}{60(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} \\
&\quad+\frac{-12(x_4^5-x_3^5)+45x_4(x_4^4-x_3^4)-60x_4^2(x_4^3-x_3^3)+30x_4^3(x_4^2-x_3^2)}{60(x_4-x_3)(x_4-x_2)(x_4-x_1)}
\end{aligned}$$

B.5. Deriv = 1 - Deg = 1

$$\begin{aligned}
\int_{x_0}^{x_2} x\partial_x b_{1,0}(x)dx &= \int_{x_0}^{x_1} x\frac{1}{x_1-x_0}dx-\int_{x_1}^{x_2} x\frac{1}{x_2-x_1}dx \\
&= \frac{x_1^2-x_0^2}{2(x_1-x_0)}-\frac{x_2^2-x_1^2}{2(x_2-x_1)} \\
&= \frac{x_1+x_0}{2}-\frac{x_2+x_1}{2} \qquad = -\frac{x_2-x_0}{2}
\end{aligned}$$

B.6. Deriv = 1 - Deg = 2

$$\begin{aligned}
\int_{x_0}^{x_3} x b_{2,0}(x)dx &= \int_{x_0}^{x_1} x\frac{2(x-x_0)}{(x_2-x_0)(x_1-x0)}dx+\int_{x_1}^{x_2} x\frac{-2x+(x_2+x_0)}{(x_2-x_0)(x_2-x_1)}+x\frac{-2x+(x_3+x_1)}{(x_2-x_1)(x_3-x_1)}dx+\int_{x_2}^{x_3} x\frac{-2(x_3-x)}{(x_3-x_2)(x_3-x_1)}dx \\
&= 2\frac{\left[\frac{x^3}{3}-x_0\frac{x^2}{2}\right]_{x_0}^{x_1}}{(x_2-x_0)(x_1-x0)}+\frac{\left[-2\frac{x^3}{3}+(x_2+x_0)\frac{x^2}{2}\right]_{x_1}^{x_2}}{(x_2-x_0)(x_2-x_1)}+\frac{\left[-2\frac{x^3}{3}+(x_3+x_1)\frac{x^2}{2}\right]_{x_1}^{x_2}}{(x_2-x_1)(x_3-x_1)}+2\frac{\left[\frac{x^3}{3}-x_3\frac{x^2}{2}\right]_{x_2}^{x_3}}{(x_3-x_2)(x_3-x_1)} \\
&= 2\frac{2(x_1^3-x_0^3)-3x_0(x_1^2-x_0^2)}{6(x_2-x_0)(x_1-x0)}+\frac{-4(x_2^3-x_1^3)+3(x_2+x_0)(x_2^2-x_1^2)}{6(x_2-x_0)(x_2-x_1)}+\frac{-4(x_2^3-x_1^3)+3(x_3+x_1)(x_2^2-x_1^2)}{6(x_2-x_1)(x_3-x_1)}+2\frac{2(x_3^3-x_2^3)-3x_3(x_3^2-x_2^2)}{6(x_3-x_2)(x_3-x_1)} \\
&= 2\frac{2(x_1^2+x_1x_0+x_0^2)-3x_0(x_1+x_0)}{6(x_2-x_0)}+\frac{-4(x_2^2+x_2x_1+x_1^2)+3(x_2+x_0)(x_2+x_1)}{6(x_2-x_0)}+\frac{-4(x_2^2+x_2x_1+x_1^2)+3(x_3+x_1)(x_2+x_1)}{6(x_3-x_1)}+2\frac{2(x_3^2+x_3x_2+x_2^2)-3x_3(x_3+x_2)}{6(x_3-x_1)} \\
&= \frac{2x_1^2-x_1x_0-x_0^2}{3(x_2-x_0)}+\frac{-x_2^2-x_2x_1-4x_1^2+3x_0x_2+3x_0x_1}{6(x_2-x_0)}+\frac{-4x_2^2-x_2x_1-x_1^2+3x_3x_2+3x_3x_1}{6(x_3-x_1)}+\frac{-x_3^2-x_3x_2+2x_2^2}{3(x_3-x_1)}
\end{aligned}$$

B.7. Deriv = 1 - Deg = 3

$$\begin{aligned}
\int_{x_0}^{x_4} x b_{3,0}(x)dx &= \int_{x_0}^{x_1} x\frac{3x^2-6x_0x+3x_0^2}{(x_3-x_0)(x_2-x_0)(x_1-x_0)}dx+\int_{x_1}^{x_2} x\frac{3Ax^2+2Bx+C}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}dx+\int_{x_2}^{x_3} x\frac{3A'x^2+2B'x+C'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}dx+\int_{x_3}^{x_4} x\frac{-3x^2+6x_4x-3x_4^2}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}dx \\
&= \frac{\left[3\frac{x^4}{4}-6x_0\frac{x^3}{3}+3x_0^2\frac{x^2}{2}\right]_{x_0}^{x_1}}{(x_3-x_0)(x_2-x_0)(x_1-x_0)}+\frac{\left[3A\frac{x^4}{4}+2B\frac{x^3}{3}+C\frac{x^2}{2}\right]_{x_1}^{x_2}}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}+\frac{\left[3A'\frac{x^4}{4}+2B'\frac{x^3}{3}+C'\frac{x^2}{2}\right]_{x_2}^{x_3}}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}+\frac{\left[-3\frac{x^4}{4}+6x_4\frac{x^3}{3}-3x_4^2\frac{x^2}{2}\right]_{x_3}^{x_4}}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} \\
&= \frac{9(x_1^4-x_0^4)-24x_0(x_1^3-x_0^3)+18x_0^2(x_1^2-x_0^2)}{12(x_3-x_0)(x_2-x_0)(x_1-x_0)}+\frac{9A(x_2^4-x_1^4)+8B(x_2^3-x_1^3)+6C(x_2^2-x_1^2)}{12(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}+\frac{9A'(x_3^4-x_2^4)+8B'(x_3^3-x_2^3)+6C'(x_3^2-x_2^3)}{12(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}+\frac{-9(x_4^4-x_3^4)+24x_4(x_4^3-x_3^3)-18x_4^2(x_4^2-x_3^2)}{12(x_4-x_3)(x_4-x_2)(x_4-x_1)} \\
&= \frac{9(x_1^2+x_0^2)(x_1+x_0)-24x_0(x_1^2+x_1x_0+x_0^2)+18x_0^2(x_1+x_0)}{12(x_3-x_0)(x_2-x_0)}+\frac{9A(x_2^4-x_1^4)+8B(x_2^3-x_1^3)+6C(x_2^2-x_1^2)}{12(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}+\frac{9A'(x_3^4-x_2^4)+8B'(x_3^3-x_2^3)+6C'(x_3^2-x_2^3)}{12(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}+\frac{-9(x_4^2+x_3^2)(x_4+x_3)+24x_4(x_4^2+x_4x_3+x_3^2)-18x_4^2(x_4+x_3)}{12(x_4-x_2)(x_4-x_1)} \\
&= \frac{3x_1^3-5x_1^2x_0+x_1x_0^2+x_0^3}{4(x_3-x_0)(x_2-x_0)}+\frac{9A(x_2^4-x_1^4)+8B(x_2^3-x_1^3)+6C(x_2^2-x_1^2)}{12(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}+\frac{9A'(x_3^4-x_2^4)+8B'(x_3^3-x_2^3)+6C'(x_3^2-x_2^3)}{12(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}+\frac{-x_4^3+5x_4x_3^2-x_4^2x_3-3x_3^3}{4(x_4-x_2)(x_4-x_1)}
\end{aligned}$$

B.8. Deriv = 2 - Deg = 2

$$\begin{aligned}
\int_{x_0}^{x_3} x b_{2,0}(x)dx &= \int_{x_0}^{x_1} x\frac{2}{(x_2-x_0)(x_1-x0)}dx+\int_{x_1}^{x_2} x\frac{-2}{(x_2-x_0)(x_2-x_1)}+x\frac{-2}{(x_2-x_1)(x_3-x_1)}dx+\int_{x_2}^{x_3} x\frac{2}{(x_3-x_2)(x_3-x_1)}dx \\
&= \frac{(x_1^2-x_0^2)}{(x_2-x_0)(x_1-x0)}-\frac{x_2^2-x_1^2}{(x_2-x_0)(x_2-x_1)}-\frac{x_2^2-x_1^2}{(x_2-x_1)(x_3-x_1)}+\frac{x_3^2-x_2^2}{(x_3-x_2)(x_3-x_1)} \\
&= \frac{x_1+x_0}{x_2-x_0}-\frac{x_2+x_1}{x_2-x_0}-\frac{x_2+x_1}{x_3-x_1}+\frac{x_3+x_2}{x_3-x_1}
\end{aligned}$$

B.9. Deriv = 2 - Deg = 3

$$\begin{aligned}
\int_{x_0}^{x_4} x b_{3,0}(x)dx &= \int_{x_0}^{x_1} x\frac{6x-6x_0}{(x_3-x_0)(x_2-x_0)(x_1-x_0)}dx+\int_{x_1}^{x_2} x\frac{6Ax+2B}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}dx+\int_{x_2}^{x_3} x\frac{6A'x+2B'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}dx+\int_{x_3}^{x_4} x\frac{-6x+6x_4}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}dx \\
&= \frac{\left[6\frac{x^3}{3}-6x_0\frac{x^2}{2}\right]_{x_0}^{x_1}}{(x_3-x_0)(x_2-x_0)(x_1-x_0)}+\frac{\left[6A\frac{x^3}{3}+2B\frac{x^2}{2}\right]_{x_1}^{x_2}}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}+\frac{\left[6A'\frac{x^3}{3}+2B'\frac{x^2}{2}\right]_{x_2}^{x_3}}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}+\frac{\left[-6\frac{x^3}{3}+6x_4\frac{x^2}{2}\right]_{x_3}^{x_4}}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} \\
&= \frac{12(x_1^3-x_0^3)-18x_0(x_1^2-x_0^2)}{6(x_3-x_0)(x_2-x_0)(x_1-x_0)}+\frac{12A(x_2^3-x_1^3)+6B(x_2^2-x_1^2)}{6(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}+\frac{12A'(x_3^3-x_2^3)+6B'(x_3^2-x_2^2)}{6(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}+\frac{-12(x_4^3-x_3^3)+18x_4(x_4^2-x_3^2)}{6(x_4-x_3)(x_4-x_2)(x_4-x_1)} \\
&= \frac{2x_1^2-x_1x_0-x_0^2}{(x_3-x_0)(x_2-x_0)}+\frac{2A(x_2^3-x_1^3)+B(x_2^2-x_1^2)}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}+\frac{2A'(x_3^3-x_2^3)+B'(x_3^2-x_2^2)}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}+\frac{x_4^2+x_4x_3-2x_3^2}{(x_4-x_2)(x_4-x_1)}
\end{aligned}$$

B.10. Deriv = 3 - Deg = 3

$$\begin{aligned}
\int_{x_0}^{x_4} x b_{3,0}(x)dx &= \int_{x_0}^{x_1} x\frac{6}{(x_3-x_0)(x_2-x_0)(x_1-x_0)}dx+\int_{x_1}^{x_2} x\frac{6A}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}dx+\int_{x_2}^{x_3} x\frac{6A'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}dx+\int_{x_3}^{x_4} x\frac{-6}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}dx \\
&= \frac{3(x_1^2-x_0^2)}{(x_3-x_0)(x_2-x_0)(x_1-x_0)}+\frac{3A(x_2^2-x_1^2)}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}+\frac{3A'(x_3^2-x_2^2)}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}+\frac{-3(x_4^2-x_3^2)}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} \\
&= \frac{3(x_1+x_0)}{(x_3-x_0)(x_2-x_0)}+\frac{3A(x_2^2-x_1^2)}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}+\frac{3A'(x_3^2-x_2^2)}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}-\frac{3(x_4+x_3)}{(x_4-x_2)(x_4-x_1)}
\end{aligned}$$

C. D0N2 - Exact formulations

C.1. Deg = 0, Surf

$$\int_{x_0}^{x_1}\|b_{0,0}(x)\|^2dx=\int_{x_0}^{x_1}dx=x_1-x_0$$

C.2. Deg = 0, Vol

$$\int_{x_0}^{x_1}x\|b_{0,0}(x)\|^2dx=\int_{x_0}^{x_1}x\,dx=\frac{x_1^2-x_0^2}{2}$$

C.3. Deg = 1, Surf

$$\begin{aligned}
\int_{x_0}^{x_2}\|b_{1,0}(x)\|^2dx &= \int_{x_0}^{x_1}\left(\tfrac{x-x_0}{x_1-x_0}\right)^2dx+\int_{x_1}^{x_2}\left(\tfrac{x_2-x}{x_2-x_1}\right)^2dx\\
&= \frac{\left[\frac{(x-x_0)^3}{3}\right]_{x_0}^{x_1}}{(x_1-x_0)^2}+\frac{\left[-\frac{(x_2-x)^3}{3}\right]_{x_1}^{x_2}}{(x_2-x_1)^2}\\
&= \frac{x_1-x_0}{3}+\frac{x_2-x_1}{3} \qquad = \frac{x_2-x_0}{3}
\end{aligned}$$

and

$$\begin{aligned}
\int_{x_1}^{x_2}b_{1,0}(x)\times b_{1,1}(x)dx &= \int_{x_1}^{x_2}\frac{x_2-x}{x_2-x_1}\,\frac{x-x_1}{x_2-x_1}dx\\
&= \frac{\left[-\frac{x^3}{3}+(x_2+x_1)\frac{x^2}{2}-x_2x_1x\right]_{x_1}^{x_2}}{(x_2-x_1)^2}\\
&= \frac{-2(x_2^3-x_1^3)+3(x_2+x_1)(x_2^2-x_1^2)-6x_2x_1(x_2-x_1)}{6(x_2-x_1)^2}\\
&= \frac{-2(x_2^2+x_2x_1+x_1^2)+3(x_2+x_1)(x_2+x_1)-6x_2x_1}{6(x_2-x_1)}\\
&= \frac{-2x_2^2-2x_2x_1-2x_1^2+3x_2^2+6x_2x_1+3x_1^2-6x_2x_1}{6(x_2-x_1)}\\
&= \frac{x_2^2-2x_2x_1+x_1^2}{6(x_2-x_1)} \qquad = \frac{x_2-x_1}{6}
\end{aligned}$$

C.4. DEG = 1, VOL

$$
\begin{aligned}
\int_{x_0}^{x_2} x\|b_{1,0}(x)\|^2 dx &= \int_{x_0}^{x_1} x\left(\frac{x-x_0}{x_1-x_0}\right)^2 dx + \int_{x_1}^{x_2} x\left(\frac{x_2-x}{x_2-x_1}\right)^2 dx \\
&= \frac{\left[\frac{x^4}{4}-2x_0\frac{x^3}{3}+x_0^2\frac{x^2}{2}\right]_{x_0}^{x_1}}{(x_1-x_0)^2} + \frac{\left[\frac{x^4}{4}-2x_2\frac{x^3}{3}+x_2^2\frac{x^2}{2}\right]_{x_1}^{x_2}}{(x_2-x_1)^2} \\
&= \frac{3(x_1^4-x_0^4)-8x_0(x_1^3-x_0^3)+6x_0^2(x_1^2-x_0^2)}{12(x_1-x_0)^2} + \frac{3(x_2^4-x_1^4)-8x_2(x_2^3-x_1^3)+6x_2^2(x_2^2-x_1^2)}{12(x_2-x_1)^2} \\
&= \frac{3(x_1^2+x_0^2)(x_1+x_0)-8x_0(x_1^2+x_1x_0+x_0^2)+6x_0^2(x_1+x_0)}{12(x_1-x_0)} + \frac{3(x_2^2+x_1^2)(x_2+x_1)-8x_2(x_2^2+x_2x_1+x_1^2)+6x_2^2(x_2+x_1)}{12(x_2-x_1)} \\
&= \frac{3x_1^3+3x_1^2x_0+3x_0^2x_1+3x_0^3-8x_0x_1^2-8x_1x_0^2-8x_0^3+6x_0^2x_1+6x_0^3}{12(x_1-x_0)} + \frac{3x_2^3+3x_2^2x_1+3x_2x_1^2+3x_1^3-8x_2^3-8x_2^2x_1-8x_2x_1^2+6x_2^3+6x_2^2x_1}{12(x_2-x_1)} \\
&= \frac{3x_1^3-5x_1^2x_0+x_0^2x_1+x_0^3}{12(x_1-x_0)} + \frac{x_2^3+x_2^2x_1-5x_2x_1^2+3x_1^3}{12(x_2-x_1)}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{x_1}^{x_2} x b_{1,0}(x)\times b_{1,1}(x) dx &= \int_{x_1}^{x_2} x\frac{x_2-x}{x_2-x_1}\frac{x-x_1}{x_2-x_1} dx \\
&= \frac{\left[-\frac{x^4}{4}+(x_2+x_1)\frac{x^3}{3}-x_2x_1\frac{x^2}{2}\right]_{x_1}^{x_2}}{(x_2-x_1)^2} \\
&= \frac{-3(x_2^4-x_1^4)+4(x_2+x_1)(x_2^3-x_1^3)-6x_2x_1(x_2^2-x_1^2)}{12(x_2-x_1)^2} \\
&= \frac{-3(x_2^2+x_1^2)(x_2+x_1)+4(x_2+x_1)(x_2^2+x_2x_1+x_1^2)-6x_2x_1(x_2+x_1)}{12(x_2-x_1)} \\
&= \frac{-3x_2^3-3x_2x_1^2-3x_2^2x_1-3x_1^3+4x_2^3+4x_2^2x_1+4x_2x_1^2+4x_2^2x_1+4x_2x_1^2+4x_1^3-6x_2^2x_1-6x_2x_1^2}{12(x_2-x_1)} \\
&= \frac{x_2^2-x_1^2}{12}
\end{aligned}
$$

C.5. DEG = 2, SURF

$$
\begin{aligned}
\int_{x_0}^{x_3} \|b_{2,0}(x)\|^2 dx &= \int_{x_0}^{x_1}\left(\frac{(x-x_0)^2}{(x_2-x_0)(x_1-x_0)}\right)^2 dx + \int_{x_1}^{x_2}\left(\frac{(x-x_0)(x_2-x)}{(x_2-x_0)(x_2-x_1)}+\frac{(x-x_1)(x_3-x)}{(x_2-x_1)(x_3-x_1)}\right)^2 dx + \int_{x_2}^{x_3}\left(\frac{(x_3-x)^2}{(x_3-x_2)(x_3-x_1)}\right)^2 dx \\
&= \frac{\left[\frac{(x-x_0)^5}{5}\right]_{x_0}^{x_1}}{(x_2-x_0)^2(x_1-x_0)^2} + \int_{x_1}^{x_2}\frac{(x-x_0)^2(x_2-x)^2}{(x_2-x_0)^2(x_2-x_1)^2} + 2\frac{(x-x_0)(x-x_1)(x_2-x)(x_3-x)}{(x_2-x_0)(x_2-x_1)^2(x_3-x_1)} + \frac{(x-x_1)^2(x_3-x)^2}{(x_2-x_1)^2(x_3-x_1)^2} dx + \frac{\left[\frac{(x_3-x)^5}{5}\right]_{x_2}^{x_3}}{(x_3-x_2)^2(x_3-x_1)^2} \\
&= \frac{(x_1-x_0)^5}{5(x_2-x_0)^2(x_1-x_0)^2} \\
&\quad + \int_{x_1}^{x_2}\frac{x^4-2x^3(x_2+x_0)+x^2(x_2^2+4x_2x_0+x_0^2)-2xx_2x_0(x_2+x_0)+x_2^2x_0^2}{(x_2-x_0)^2(x_2-x_1)^2} dx \\
&\quad +2\int_{x_1}^{x_2}\frac{x^4-x^3(x_0+x_1+x_2+x_3)+x^2(x_0x_1+x_0x_2+x_0x_3+x_1x_2+x_1x_3+x_2x_3)-x(x_0x_1x_2+x_0x_1x_3+x_0x_2x_3+x_1x_2x_3)+x_0x_1x_2x_3}{(x_2-x_0)(x_2-x_1)^2(x_3-x_1)} dx \\
&\quad + \int_{x_1}^{x_2}\frac{x^4-2x^3(x_3+x_1)+x^2(x_3^2+4x_3x_1+x_1^2)-2xx_3x_1(x_3+x_1)+x_3^2x_1^2}{(x_2-x_1)^2(x_3-x_1)^2} dx \\
&\quad - \frac{(x_3-x_2)^5}{5(x_3-x_2)^2(x_3-x_1)^2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_{x_0}^{x_3} \|b_{2,0}(x)\|^2 dx \quad = \quad & \frac{(x_1-x_0)^3}{5(x_2-x_0)^2} \\
& + \frac{\left[\frac{x^5}{5} - 2(x_2+x_0)\frac{x^4}{4} + (x_2^2+4x_2x_0+x_0^2)\frac{x^3}{3} - 2x_2x_0(x_2+x_0)\frac{x^2}{2} + x_2^2x_0^2x\right]_{x_1}^{x_2}}{(x_2-x_0)^2(x_2-x_1)^2} \\
& + 2\frac{\left[\frac{x^5}{5} - (x_0+x_1+x_2+x_3)\frac{x^4}{4} + (x_0x_1+x_0x_2+x_0x_3+x_1x_2+x_1x_3+x_2x_3)\frac{x^3}{3} - (x_0x_1x_2+x_0x_1x_3+x_0x_2x_3+x_1x_2x_3)\frac{x^2}{2} + x_0x_1x_2x_3x\right]_{x_1}^{x_2}}{(x_2-x_0)(x_2-x_1)^2(x_3-x_1)} \\
& + \frac{\left[\frac{x^5}{5} - 2(x_3+x_1)\frac{x^4}{4} + (x_3^2+4x_3x_1+x_1^2)\frac{x^3}{3} - 2x_3x_1(x_3+x_1)\frac{x^2}{2} + x_3^2x_1^2x\right]_{x_1}^{x_2}}{(x_2-x_1)^2(x_3-x_1)^2} \\
& - \frac{(x_3-x_2)^3}{5(x_3-x_1)^2} \\
= \quad & \frac{(x_1-x_0)^3}{5(x_2-x_0)^2} \\
& + \frac{6(x_2^4+x_2^3x_1+x_2^2x_1^2+x_2x_1^3+x_1^4) - 15(x_2+x_0)(x_2^2+x_1^2)(x_2+x_1) + 10(x_2^2+4x_2x_0+x_0^2)(x_2^2+x_2x_1+x_1^2) - 30x_2x_0(x_2+x_0)(x_2+x_1) + 30x_2^2x_0^2}{30(x_2-x_0)^2(x_2-x_1)} \\
& + 2\frac{12(x_2^4+x_2^3x_1+x_2^2x_1^2+x_2x_1^3+x_1^4) - 15(x_0+x_1+x_2+x_3)(x_2^2+x_1^2)(x_2+x_1) + 20(x_0x_1+x_0x_2+x_0x_3+x_1x_2+x_1x_3+x_2x_3)(x_2^2+x_2x_1+x_1^2) - 30(x_0x_1x_2+x_0x_1x_3+x_0x_2x_3+x_1x_2x_3)(x_2+x_1) + 60x_0x_1x_2x_3}{60(x_2-x_0)(x_2-x_1)(x_3-x_1)} \\
& + \frac{6(x_2^4+x_2^3x_1+x_2^2x_1^2+x_2x_1^3+x_1^4) - 15(x_3+x_1)(x_2^2+x_1^2)(x_2+x_1) + 10(x_3^2+4x_3x_1+x_1^2)(x_2^2+x_2x_1+x_1^2) - 30x_3x_1(x_3+x_1)(x_2+x_1) + 30x_3^2x_1^2}{30(x_2-x_1)(x_3-x_1)^2} \\
& - \frac{(x_3-x_2)^3}{5(x_3-x_1)^2} \\
= \quad & \frac{(x_1-x_0)^3}{5(x_2-x_0)^2} \\
& + \frac{x_2^4+x_2^3x_1+x_2^2x_1^2-9x_2x_1^3+6x_1^4-5x_2^3x_0+25x_2x_1^2x_0-5x_2^2x_1x_0-15x_1^3x_0+10x_2^2x_0^2-20x_2x_1x_0^2+10x_1^2x_0^2}{30(x_2-x_0)^2(x_2-x_1)} \\
& + \frac{-3x_2^4+2x_2^3x_1+2x_2^2x_1^2+2x_2x_1^3-3x_1^4+5x_2^3x_0-5x_2x_1^2x_0-5x_2^2x_1x_0+5x_1^3x_0+5x_3x_2^3-5x_3x_2x_1^2-5x_3x_2^2x_1+5x_3x_1^3-10x_3x_2^2x_0+20x_3x_2x_1x_0-10x_3x_1^2x_0}{30(x_2-x_0)(x_2-x_1)(x_3-x_1)} \\
& + \frac{6x_2^4-9x_2^3x_1+x_2^2x_1^2+x_2x_1^3+x_1^4-15x_3x_2^3-5x_3x_2x_1^2+25x_3x_2^2x_1-5x_3x_1^3+10x_3^2x_2^2-20x_3^2x_2x_1+10x_3^2x_1^2}{30(x_2-x_1)(x_3-x_1)^2} \\
& - \frac{(x_3-x_2)^3}{5(x_3-x_1)^2} \\
= \quad & \frac{(x_1-x_0)^3}{5(x_2-x_0)^2} \\
& + \frac{(x_2^2+3x_2x_1+6x_1^2)(x_2-x_1)^2 - 5x_0(x_2+3x_1)(x_2-x_1)^2 + 10x_0^2(x_2-x_1)^2}{30(x_2-x_0)^2(x_2-x_1)} \\
& + \frac{(-3x_2^2-4x_2x_1-3x_1^2)(x_2-x_1)^2 + 5x_0(x_2+x_1)(x_2-x_1)^2 + 5x_3(x_2+x_1)(x_2-x_1)^2 - 10x_3x_0(x_2-x_1)^2}{30(x_2-x_0)(x_2-x_1)(x_3-x_1)} \\
& + \frac{(6x_2^2+3x_2x_1+x_1^2)(x_2-x_1)^2 - 5x_3(3x_2+x_1)(x_2-x_1)^2 + 10x_3^2(x_2-x_1)^2}{30(x_2-x_1)(x_3-x_1)^2} \\
& - \frac{(x_3-x_2)^3}{5(x_3-x_1)^2} \\
= \quad & \frac{(x_1-x_0)^3}{5(x_2-x_0)^2} \\
& + (x_2-x_1)\frac{x_2^2+3x_2x_1+6x_1^2-5x_0(x_2+3x_1)+10x_0^2}{30(x_2-x_0)^2} + (x_2-x_1)\frac{-3x_2^2-4x_2x_1-3x_1^2+5(x_3+x_0)(x_2+x_1)-10x_3x_0}{30(x_2-x_0)(x_3-x_1)} + (x_2-x_1)\frac{6x_2^2+3x_2x_1+x_1^2-5x_3(3x_2+x_1)+10x_3^2}{30(x_3-x_1)^2} \\
& - \frac{(x_3-x_2)^3}{5(x_3-x_1)^2}
\end{aligned}
$$

And:

$$
\begin{aligned}
\int_{x_1}^{x_3} b_{2,0}(x)\times b_{2,1}(x)dx &= \int_{x_1}^{x_2}\left(\frac{(x-x_0)(x_2-x)}{(x_2-x_0)(x_2-x_1)}+\frac{(x-x_1)(x_3-x)}{(x_2-x_1)(x_3-x_1)}\right)\frac{(x-x_1)^2}{(x_3-x_1)(x_2-x_1)}dx+\int_{x_2}^{x_3}\left(\frac{(x-x_1)(x_3-x)}{(x_3-x_1)(x_3-x_2)}+\frac{(x-x_2)(x_4-x)}{(x_3-x_2)(x_4-x_2)}\right)\frac{(x_3-x)^2}{(x_3-x_2)(x_3-x_1)}dx\\
&= \int_{x_1}^{x_2}\frac{(x-x_0)(x_2-x)(x-x_1)^2}{(x_3-x_1)(x_2-x_1)^2(x_2-x_0)}+\frac{(x-x_1)^3(x_3-x)}{(x_3-x_1)^2(x_2-x_1)^2}dx\\
&\quad+\int_{x_2}^{x_3}\frac{(x-x_1)(x_3-x)^3}{(x_3-x_2)^2(x_3-x_1)^2}+\frac{(x-x_2)(x_4-x)(x_3-x)^2}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)}dx\\
&= \int_{x_1}^{x_2}\frac{-x^4+2x_1x^3-x_1^2x^2+(x_2+x_0)x^3-2(x_2+x_0)x_1x^2+(x_2+x_0)x_1^2x-x_2x_0x^2+2x_2x_1x_0x-x_2x_1^2x_0}{(x_3-x_1)(x_2-x_1)^2(x_2-x_0)}dx\\
&\quad+\int_{x_1}^{x_2}\frac{x_3x^3-3x_3x_1x^2+3x_3x_1^2x-x_3x_1^3-x^4+3x_1x^3-3x_1^2x^2+x_1^3x}{(x_3-x_1)^2(x_2-x_1)^2}dx\\
&\quad+\int_{x_2}^{x_3}\frac{x_3^3x-3x_3^2x^2+3x_3x^3-x^4-x_3^3x_1+3x_3^2x_1x-3x_3x_1x^2+x_1x^3}{(x_3-x_2)^2(x_3-x_1)^2}dx\\
&\quad+\int_{x_2}^{x_3}\frac{-x^4+2x_3x^3-x_3^2x^2+(x_4+x_2)x^3-2(x_4+x_2)x_3x^2+(x_4+x_2)x_3^2x-x_4x_2x^2+2x_4x_3x_2x-x_4x_3^2x_2}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)}dx\\
&= \int_{x_1}^{x_2}\frac{-x^4+(x_2+2x_1+x_0)x^3-(x_2x_0+2x_2x_1+2x_1x_0+x_1^2)x^2+(x_2x_1^2+x_1^2x_0+2x_2x_1x_0)x-x_2x_1^2x_0}{(x_3-x_1)(x_2-x_1)^2(x_2-x_0)}dx\\
&\quad+\int_{x_1}^{x_2}\frac{-x^4+(x_3+3x_1)x^3-(3x_3x_1+3x_1^2)x^2+(3x_3x_1^2+x_1^3)x-x_3x_1^3}{(x_3-x_1)^2(x_2-x_1)^2}dx\\
&\quad+\int_{x_2}^{x_3}\frac{-x^4+(3x_3+x_1)x^3-(3x_3^2+3x_3x_1)x^2+(3x_3^2x_1+x_3^3)x-x_3^3x_1}{(x_3-x_2)^2(x_3-x_1)^2}dx\\
&\quad+\int_{x_2}^{x_3}\frac{-x^4+(x_4+2x_3+x_2)x^3-(2x_4x_3+2x_3x_2+x_4x_2+x_3^2)x^2+(x_4x_3^2+2x_4x_3x_2+x_3^2x_2)x-x_4x_3^2x_2}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)}dx\\
&= \frac{\left[-12x^5+15(x_2+2x_1+x_0)x^4-20(x_2x_0+2x_2x_1+2x_1x_0+x_1^2)x^3+30(x_2x_1^2+x_1^2x_0+2x_2x_1x_0)x^2-60x_2x_1^2x_0x\right]_{x_1}^{x_2}}{60(x_3-x_1)(x_2-x_1)^2(x_2-x_0)}\\
&\quad+\frac{\left[-4x^5+5(x_3+3x_1)x^4-20(x_3x_1+x_1^2)x^3+10(3x_3x_1^2+x_1^3)x^2-20x_3x_1^3x\right]_{x_1}^{x_2}}{20(x_3-x_1)^2(x_2-x_1)^2}\\
&\quad+\frac{\left[-4x^5+5(3x_3+x_1)x^4-20(x_3^2+x_3x_1)x^3+10(3x_3^2x_1+x_3^3)x^2-20x_3^3x_1x\right]_{x_2}^{x_3}}{20(x_3-x_2)^2(x_3-x_1)^2}\\
&\quad+\frac{\left[-12x^5+15(x_4+2x_3+x_2)x^4-20(2x_4x_3+2x_3x_2+x_4x_2+x_3^2)x^3+30(x_4x_3^2+2x_4x_3x_2+x_3^2x_2)x^2-60x_4x_3^2x_2x\right]_{x_2}^{x_3}}{60(x_4-x_2)(x_3-x_2)^2(x_3-x_1)}\\
&= \frac{-12(x_2^4+x_2^3x_1+x_2^2x_1^2+x_2x_1^3+x_1^4)+15(x_2+2x_1+x_0)(x_2^2+x_1^2)(x_2+x_1)-20(x_2x_0+2x_2x_1+2x_1x_0+x_1^2)(x_2^2+x_2x_1+x_1^2)+30(x_2x_1^2+x_1^2x_0+2x_2x_1x_0)(x_2+x_1)-60x_2x_1^2x_0}{60(x_3-x_1)(x_2-x_1)(x_2-x_0)}\\
&\quad+\frac{-4(x_2^4+x_2^3x_1+x_2^2x_1^2+x_2x_1^3+x_1^4)+5(x_3+3x_1)(x_2^2+x_1^2)(x_2+x_1)-20(x_3x_1+x_1^2)(x_2^2+x_2x_1+x_1^2)+10(3x_3x_1^2+x_1^3)(x_2+x_1)-20x_3x_1^3}{20(x_3-x_1)^2(x_2-x_1)}\\
&\quad+\frac{-4(x_3^4+x_3^3x_2+x_3^2x_2^2+x_3x_2^3+x_2^4)+5(3x_3+x_1)(x_3^2+x_2^2)(x_3+x_2)-20(x_3^2+x_3x_1)(x_3^2+x_3x_2+x_2^2)+10(3x_3^2x_1+x_3^3)(x_3+x_2)-20x_3^3x_1}{20(x_3-x_2)(x_3-x_1)^2}\\
&\quad+\frac{-12(x_3^4+x_3^3x_2+x_3^2x_2^2+x_3x_2^3+x_2^4)+15(x_4+2x_3+x_2)(x_3^2+x_2^2)(x_3+x_2)-20(2x_4x_3+2x_3x_2+x_4x_2+x_3^2)(x_3^2+x_3x_2+x_2^2)+30(x_4x_3^2+2x_4x_3x_2+x_3^2x_2)(x_3+x_2)-60x_4x_3^2x_2}{60(x_4-x_2)(x_3-x_2)(x_3-x_1)}\\
&= \frac{3x_2^4-7x_2^3x_1+3x_2^2x_1^2+3x_2x_1^3-2x_1^4-5x_2^3x_0+15x_2^2x_1x_0-15x_2x_1^2x_0+5x_1^3x_0}{60(x_3-x_1)(x_2-x_1)(x_2-x_0)}\\
&\quad+\frac{-4x_2^4+11x_2^3x_1-9x_2^2x_1^2+x_2x_1^3+x_1^4+5x_3x_2^3-15x_3x_2^2x_1+15x_3x_2x_1^2-5x_3x_1^3}{20(x_3-x_1)^2(x_2-x_1)}\\
&\quad+\frac{x_3^4+x_3^3x_2-9x_3^2x_2^2+11x_3x_2^3-4x_2^4-5x_3^3x_1+15x_3^2x_2x_1-15x_3x_2^2x_1+5x_2^3x_1}{20(x_3-x_2)(x_3-x_1)^2}\\
&\quad+\frac{-2x_3^4+3x_3^3x_2+3x_3^2x_2^2-7x_3x_2^3+3x_2^4+5x_4x_3^3+15x_4x_3x_2^2-15x_4x_3^2x_2-5x_4x_2^3}{60(x_4-x_2)(x_3-x_2)(x_3-x_1)}\\
&= \frac{(3x_2+2x_1)(x_2-x_1)^3-5x_0(x_2-x_1)^3}{60(x_3-x_1)(x_2-x_1)(x_2-x_0)}+\frac{-(4x_2+x_1)(x_2-x_1)^3+5x_3(x_2-x_1)^3}{20(x_3-x_1)^2(x_2-x_1)}\\
&\quad+\frac{(4x_2+x_3)(x_3-x_2)^3-5x_1(x_3-x_2)^3}{20(x_3-x_2)(x_3-x_1)^2}+\frac{-(2x_3+3x_2)(x_3-x_2)^3+5x_4(x_3-x_2)^3}{60(x_4-x_2)(x_3-x_2)(x_3-x_1)}
\end{aligned}
$$

Thus

$$\begin{aligned}
\int_{x_1}^{x_3} b_{2,0}(x)\times b_{2,1}(x)dx &= \frac{(3x_2+2x_1-5x_0)(x_2-x_1)^2}{60(x_3-x_1)(x_2-x_0)} + \frac{(5x_3-4x_2-x_1)(x_2-x_1)^2}{20(x_3-x_1)^2} \\
&\quad + \frac{(4x_2+x_3-5x_1)(x_3-x_2)^2}{20(x_3-x_1)^2} + \frac{(5x_4-2x_3-3x_2)(x_3-x_2)^2}{60(x_4-x_2)(x_3-x_1)}
\end{aligned}$$

And

$$\begin{aligned}
\int_{x_2}^{x_3} b_{2,0}(x)\times b_{2,2}(x)dx &= \int_{x_2}^{x_3} \frac{(x_3-x)^2}{(x_3-x_2)(x_3-x_1)}\frac{(x-x_2)^2}{(x_4-x_2)(x_3-x_2)}dx \\
&= \int_{x_2}^{x_3} \frac{(x^2-2x_3x+x_3^2)(x^2-2x_2x+x_2^2)}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)}dx \\
&= \int_{x_2}^{x_3} \frac{x^4-2x_3x^3+x_3^2x^2-2x_2x^3+4x_3x_2x^2-2x_3^2x_2x+x_2^2x^2-2x_3x_2^2x+x_3^2x_2^2}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)}dx \\
&= \int_{x_2}^{x_3} \frac{x^4-2(x_3+x_2)x^3+(x_3^2+4x_3x_2+x_2^2)x^2-2(x_3^2x_2+x_3x_2^2)x+x_3^2x_2^2}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)}dx \\
&= \frac{\left[6x^5-15(x_3+x_2)x^4+10(x_3^2+4x_3x_2+x_2^2)x^3-30(x_3^2x_2+x_3x_2^2)x^2+30x_3^2x_2^2x\right]_{x_2}^{x_3}}{30(x_4-x_2)(x_3-x_2)^2(x_3-x_1)} \\
&= \frac{6(x_3^4+x_3^3x_2+x_3^2x_2^2+x_3x_2^3+x_2^4)-15(x_3+x_2)(x_3^2+x_2^2)(x_3+x_2)+10(x_3^2+4x_3x_2+x_2^2)(x_3^2+x_3x_2+x_2^2)-30(x_3^2x_2+x_3x_2^2)(x_3+x_2)+30x_3^2x_2^2}{30(x_4-x_2)(x_3-x_2)(x_3-x_1)} \\
&= \frac{x_3^4-4x_3^3x_2+6x_3^2x_2^2-4x_3x_2^3+x_2^4}{30(x_4-x_2)(x_3-x_2)(x_3-x_1)} \\
&= \frac{(x_3-x_2)^3}{30(x_4-x_2)(x_3-x_1)}
\end{aligned}$$

C.6. DEG = 2, VOL

$$\begin{aligned}
\int_{x_0}^{x_3} x\|b_{2,0}(x)\|^2 dx &= \int_{x_0}^{x_1} x\left(\frac{(x-x_0)^2}{(x_2-x_0)(x_1-x_0)}\right)^2 dx + \int_{x_1}^{x_2} x\left(\frac{(x-x_0)(x_2-x)}{(x_2-x_0)(x_2-x_1)} + \frac{(x-x_1)(x_3-x)}{(x_2-x_1)(x_3-x_1)}\right)^2 dx + \int_{x_2}^{x_3} x\left(\frac{(x_3-x)^2}{(x_3-x_2)(x_3-x_1)}\right)^2 dx \\
&= \frac{\left[\frac{x^6}{6}-4x_0\frac{x^5}{5}+6x_0^2\frac{x^4}{4}-4x_0^3\frac{x^3}{3}+x_0^4\frac{x^2}{2}\right]_{x_0}^{x_1}}{(x_2-x_0)^2(x_1-x_0)^2} + \int_{x_1}^{x_2} \frac{x(x-x_0)^2(x_2-x)^2}{(x_2-x_0)^2(x_2-x_1)^2} + 2\frac{x(x-x_0)(x-x_1)(x_2-x)(x_3-x)}{(x_2-x_0)(x_2-x_1)^2(x_3-x_1)} + \frac{x(x-x_1)^2(x_3-x)^2}{(x_2-x_1)^2(x_3-x_1)^2}dx + \frac{\left[\frac{x^6}{6}-4x_3\frac{x^5}{5}+6x_3^2\frac{x^4}{4}-4x_3^3\frac{x^3}{3}+x_3^4\frac{x^2}{2}\right]_{x_2}^{x_3}}{(x_3-x_2)^2(x_3-x_1)^2} \\
&= \frac{5(x_1^5+x_1^4x_0+x_1^3x_0^2+x_1^2x_0^3+x_1x_0^4+x_0^5)-24x_0(x_1^4+x_1^3x_0+x_1^2x_0^2+x_1x_0^3+x_0^4)+45x_0^2(x_1^3+x_1^2x_0+x_1x_0^2+x_0^3)-40x_0^3(x_1^2+x_1x_0+x_0^2)+15x_0^4(x_1+x_0)}{30(x_2-x_0)^2(x_1-x_0)} \\
&\quad + \int_{x_1}^{x_2} \frac{x^5-2x^4(x_2+x_0)+x^3(x_2^2+4x_2x_0+x_0^2)-2x^2x_2x_0(x_2+x_0)+x_2^2x_0^2x}{(x_2-x_0)^2(x_2-x_1)^2}dx \\
&\quad +2\int_{x_1}^{x_2} \frac{x^5-x^4(x_0+x_1+x_2+x_3)+x^3(x_0x_1+x_0x_2+x_0x_3+x_1x_2+x_1x_3+x_2x_3)-x^2(x_0x_1x_2+x_0x_1x_3+x_0x_2x_3+x_1x_2x_3)+xx_0x_1x_2x_3}{(x_2-x_0)(x_2-x_1)^2(x_3-x_1)}dx \\
&\quad + \int_{x_1}^{x_2} \frac{x^5-2x^4(x_3+x_1)+x^3(x_3^2+4x_3x_1+x_1^2)-2x^2x_3x_1(x_3+x_1)+x_3^2x_1^2x}{(x_2-x_1)^2(x_3-x_1)^2}dx \\
&\quad + \frac{5(x_3^5+x_3^4x_2+x_3^3x_2^2+x_3^2x_2^3+x_3x_2^4+x_2^5)-24x_3(x_3^4+x_3^3x_2+x_3^2x_2^2+x_3x_2^3+x_2^4)+45x_3^2(x_3^3+x_3^2x_2+x_3x_2^2+x_2^3)-40x_3^3(x_3^2+x_3x_2+x_2^2)+15x_3^4(x_3+x_2)}{30(x_3-x_2)(x_3-x_1)^2} \\
&= \frac{5x_1^5-19x_1^4x_0+26x_1^3x_0^2-14x_1^2x_0^3+x_1x_0^4+x_0^5}{30(x_2-x_0)^2(x_1-x_0)} \\
&\quad + \frac{\left[10x^6-24(x_2+x_0)x^5+15(x_2^2+4x_2x_0+x_0^2)x^4-40x_2x_0(x_2+x_0)x^3+30x_2^2x_0^2x^2\right]_{x_1}^{x_2}}{60(x_2-x_0)^2(x_2-x_1)^2} \\
&\quad +2\frac{\left[10x^6-12(x_0+x_1+x_2+x_3)x^5+15(x_0x_1+x_0x_2+x_0x_3+x_1x_2+x_1x_3+x_2x_3)x^4-20(x_0x_1x_2+x_0x_1x_3+x_0x_2x_3+x_1x_2x_3)x^3+30x_0x_1x_2x_3x^2\right]_{x_1}^{x_2}}{60(x_2-x_0)(x_2-x_1)^2(x_3-x_1)} \\
&\quad + \frac{\left[10x^6-24(x_3+x_1)x^5+15(x_3^2+4x_3x_1+x_1^2)x^4-40x_3x_1(x_3+x_1)x^3+30x_3^2x_1^2x^2\right]_{x_1}^{x_2}}{60(x_2-x_1)^2(x_3-x_1)^2} \\
&\quad + \frac{x_3^5+x_3^4x_2-14x_3^3x_2^2+26x_3^2x_2^3-19x_3x_2^4+5x_2^5}{30(x_3-x_2)(x_3-x_1)^2}
\end{aligned}$$

Hence:

$$
\begin{aligned}
\int_{x_0}^{x_3} x\|b_{2,0}(x)\|^2dx &= \frac{(5x_1+x_0)(x_1-x_0)^4}{30(x_2-x_0)^2(x_1-x_0)} \\
&+\frac{10(x_2^5+x_2^4x_1+x_2^3x_1^2+x_2^2x_1^3+x_2x_1^4+x_1^5)-24(x_2+x_0)(x_2^4+x_2^3x_1+x_2^2x_1^2+x_2x_1^3+x_1^4)+15(x_2^2+4x_2x_0+x_0^2)(x_2^3+x_2^2x_1+x_2x_1^2+x_1^3)-40x_2x_0(x_2+x_0)(x_2^2+x_2x_1+x_1^2)+30x_2^2x_0^2(x_2+x_1)}{60(x_2-x_0)^2(x_2-x_1)} \\
&+2\frac{10(x_2^5+x_2^4x_1+x_2^3x_1^2+x_2^2x_1^3+x_2x_1^4+x_1^5)-12(x_0+x_1+x_2+x_3)(x_2^4+x_2^3x_1+x_2^2x_1^2+x_2x_1^3+x_1^4)+15(x_0x_1+x_0x_2+x_0x_3+x_1x_2+x_1x_3+x_2x_3)(x_2^3+x_2^2x_1+x_2x_1^2+x_1^3)-20(x_0x_1x_2+x_0x_1x_3+x_0x_2x_3+x_1x_2x_3)(x_2^2+x_2x_1+x}{60(x_2-x_0)(x_2-x_1)(x_3-x_1)} \\
&+\frac{10(x_2^5+x_2^4x_1+x_2^3x_1^2+x_2^2x_1^3+x_2x_1^4+x_1^5)-24(x_3+x_1)(x_2^4+x_2^3x_1+x_2^2x_1^2+x_2x_1^3+x_1^4)+15(x_3^2+4x_3x_1+x_1^2)(x_2^3+x_2^2x_1+x_2x_1^2+x_1^3)-40x_3x_1(x_3+x_1)(x_2^2+x_2x_1+x_1^2)+30x_3^2x_1^2(x_2+x_1)}{60(x_2-x_1)(x_3-x_1)^2} \\
&+\frac{(x_3+5x_2)(x_3-x_2)^4}{30(x_3-x_2)(x_3-x_1)^2} \\
&= \frac{(5x_1+x_0)(x_1-x_0)^3}{30(x_2-x_0)^2} \\
&+\frac{x_2^5+x_2^4x_1+x_2^3x_1^2+x_2^2x_1^3-14x_2x_1^4+10x_1^5-4x_0(x_2^4+x_2^3x_1+x_2^2x_1^2-9x_2x_1^3+6x_1^4)+5x_0^2(x_2^3+x_2^2x_1-5x_2x_1^2+3x_1^3)}{60(x_2-x_0)^2(x_2-x_1)} \\
&+\frac{-2x_2^5+x_2^4x_1+x_2^3x_1^2+x_2^2x_1^3+x_2x_1^4-2x_1^5+x_0(3x_2^4-2x_2^3x_1-2x_2^2x_1^2-2x_2x_1^3+3x_1^4)+x_3(3x_2^4-2x_2^3x_1-2x_2^2x_1^2-2x_2x_1^3+3x_1^4)-5x_3x_0(x_2^3-x_2^2x_1-x_2x_1^2+x_1^3)}{30(x_2-x_0)(x_2-x_1)(x_3-x_1)} \\
&+\frac{10x_2^5-14x_2^4x_1+x_2^3x_1^2+x_2^2x_1^3+x_2x_1^4+x_1^5-4x_3(6x_2^4-9x_2^3x_1+x_2^2x_1^2+x_2x_1^3+x_1^4)+5x_3^2(3x_2^3-5x_2^2x_1+x_2x_1^2+x_1^3)}{60(x_2-x_1)(x_3-x_1)^2} \\
&+\frac{(x_3+5x_2)(x_3-x_2)^3}{30(x_3-x_1)^2} \\
&= \frac{(5x_1+x_0)(x_1-x_0)^3}{30(x_2-x_0)^2} \\
&+(x_2-x_1)\frac{x_2^3+3x_2^2x_1+6x_2x_1^2+10x_1^3-4x_0(x_2^2+3x_2x_1+6x_1^2)+5x_0^2(x_2+3x_1)}{60(x_2-x_0)^2} \\
&+(x_2-x_1)\frac{-2x_2^3-3x_2^2x_1-3x_2x_1^2-2x_1^3+(x_3+x_0)(3x_2^2+4x_2x_1+3x_1^2)-5x_3x_0(x_2+x_1)}{30(x_2-x_0)(x_3-x_1)} \\
&+(x_2-x_1)\frac{10x_2^3+6x_2^2x_1+3x_2x_1^2+x_1^3-4x_3(6x_2^2+3x_2x_1+x_1^2)+5x_3^2(3x_2+x_1)}{60(x_3-x_1)^2} \\
&+\frac{(x_3+5x_2)(x_3-x_2)^3}{30(x_3-x_1)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\int_{x_1}^{x_3} x b_{2,0}(x) \times b_{2,1}(x)\, dx &= \int_{x_1}^{x_2} x \left(\frac{(x-x_0)(x_2-x)}{(x_2-x_0)(x_2-x_1)} + \frac{(x-x_1)(x_3-x)}{(x_2-x_1)(x_3-x_1)} \right) \frac{(x-x_1)^2}{(x_3-x_1)(x_2-x_1)}\, dx + \int_{x_2}^{x_3} x \left(\frac{(x-x_1)(x_3-x)}{(x_3-x_1)(x_3-x_2)} + \frac{(x-x_2)(x_4-x)}{(x_3-x_2)(x_4-x_2)} \right) \frac{(x_3-x)^2}{(x_3-x_2)(x_3-x_1)}\, dx \\
&= \int_{x_1}^{x_2} x \frac{(x-x_0)(x_2-x)(x-x_1)^2}{(x_3-x_1)(x_2-x_1)^2(x_2-x_0)} + x \frac{(x-x_1)^3(x_3-x)}{(x_3-x_1)^2(x_2-x_1)^2}\, dx \\
&\quad + \int_{x_2}^{x_3} x \frac{(x-x_1)(x_3-x)^3}{(x_3-x_2)^2(x_3-x_1)^2} + x \frac{(x-x_2)(x_4-x)(x_3-x)^2}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)}\, dx \\
&= \int_{x_1}^{x_2} x \frac{-x^4+2x_1x^3-x_1^2x^2+(x_2+x_0)x^3-2(x_2+x_0)x_1x^2+(x_2+x_0)x_1^2x-x_2x_0x^2+2x_2x_1x_0x-x_2x_1^2x_0}{(x_3-x_1)(x_2-x_1)^2(x_2-x_0)}\, dx \\
&\quad + \int_{x_1}^{x_2} x \frac{x_3x^3-3x_3x_1x^2+3x_3x_1^2x-x_3x_1^3-x^4+3x_1x^3-3x_1^2x^2+x_1^3x}{(x_3-x_1)^2(x_2-x_1)^2}\, dx \\
&\quad + \int_{x_2}^{x_3} x \frac{x_3^3x-3x_3^2x^2+3x_3x^3-x^4-x_3^3x_1+3x_3^2x_1x-3x_3x_1x^2+x_1x^3}{(x_3-x_2)^2(x_3-x_1)^2}\, dx \\
&\quad + \int_{x_2}^{x_3} x \frac{-x^4+2x_3x^3-x_3^2x^2+(x_4+x_2)x^3-2(x_4+x_2)x_3x^2+(x_4+x_2)x_3^2x-x_4x_2x^2+2x_4x_3x_2x-x_4x_3^2x_2}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)}\, dx \\
&= \int_{x_1}^{x_2} \frac{-x^5+(x_2+2x_1+x_0)x^4-(x_2x_0+2x_2x_1+2x_1x_0+x_1^2)x^3+(x_2x_1^2+x_1^2x_0+2x_2x_1x_0)x^2-x_2x_1^2x_0x}{(x_3-x_1)(x_2-x_1)^2(x_2-x_0)}\, dx \\
&\quad + \int_{x_1}^{x_2} \frac{-x^5+(x_3+3x_1)x^4-(3x_3x_1+3x_1^2)x^3+(3x_3x_1^2+x_1^3)x^2-x_3x_1^3x}{(x_3-x_1)^2(x_2-x_1)^2}\, dx \\
&\quad + \int_{x_2}^{x_3} \frac{-x^5+(3x_3+x_1)x^4-(3x_3^2+3x_3x_1)x^3+(3x_3^2x_1+x_3^3)x^2-x_3^3x_1x}{(x_3-x_2)^2(x_3-x_1)^2}\, dx \\
&\quad + \int_{x_2}^{x_3} \frac{-x^5+(x_4+2x_3+x_2)x^4-(2x_4x_3+2x_3x_2+x_4x_2+x_3^2)x^3+(x_4x_3^2+2x_4x_3x_2+x_3^2x_2)x^2-x_4x_3^2x_2x}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)}\, dx \\
&= \frac{\left[-10x^6+12(x_2+2x_1+x_0)x^5-15(x_2x_0+2x_2x_1+2x_1x_0+x_1^2)x^4+20(x_2x_1^2+x_1^2x_0+2x_2x_1x_0)x^3-30x_2x_1^2x_0x^2\right]_{x_1}^{x_2}}{60(x_3-x_1)(x_2-x_1)^2(x_2-x_0)} \\
&\quad + \frac{\left[-10x^6+12(x_3+3x_1)x^5-45(x_3x_1+x_1^2)x^4+20(3x_3x_1^2+x_1^3)x^3-30x_3x_1^3x^2\right]_{x_1}^{x_2}}{60(x_3-x_1)^2(x_2-x_1)^2} \\
&\quad + \frac{\left[-10x^6+12(3x_3+x_1)x^5-45(x_3^2+x_3x_1)x^4+20(3x_3^2x_1+x_3^3)x^3-30x_3^3x_1x^2\right]_{x_2}^{x_3}}{60(x_3-x_2)^2(x_3-x_1)^2} \\
&\quad + \frac{\left[-10x^6+12(x_4+2x_3+x_2)x^5-15(2x_4x_3+2x_3x_2+x_4x_2+x_3^2)x^4+20(x_4x_3^2+2x_4x_3x_2+x_3^2x_2)x^3-30x_4x_3^2x_2x^2\right]_{x_2}^{x_3}}{60(x_4-x_2)(x_3-x_2)^2(x_3-x_1)} \\
&= \frac{-10(x_2^5+x_2^4x_1+x_2^3x_1^2+x_2^2x_1^3+x_2x_1^4+x_1^5)+12(x_2+2x_1+x_0)(x_2^4+x_2^3x_1+x_2^2x_1^2+x_2x_1^3+x_1^4)-15(x_2x_0+2x_2x_1+2x_1x_0+x_1^2)(x_2^3+x_2^2x_1+x_2x_1^2+x_1^3)+20(x_2x_1^2+x_1^2x_0+2x_2x_1x_0)(x_2^2+x_2x_1+x_1^2)-30x_2x_1^2x_0(x_2+x_1)}{60(x_3-x_1)(x_2-x_1)(x_2-x_0)} \\
&\quad + \frac{-10(x_2^5+x_2^4x_1+x_2^3x_1^2+x_2^2x_1^3+x_2x_1^4+x_1^5)+12(x_3+3x_1)(x_2^4+x_2^3x_1+x_2^2x_1^2+x_2x_1^3+x_1^4)-45(x_3x_1+x_1^2)(x_2^3+x_2^2x_1+x_2x_1^2+x_1^3)+20(3x_3x_1^2+x_1^3)(x_2^2+x_2x_1+x_1^2)-30x_3x_1^3(x_2+x_1)}{60(x_3-x_1)^2(x_2-x_1)} \\
&\quad + \frac{-10(x_3^5+x_3^4x_2+x_3^3x_2^2+x_3^2x_2^3+x_3x_2^4+x_2^5)+12(3x_3+x_1)(x_3^4+x_3^3x_2+x_3^2x_2^2+x_3x_2^3+x_2^4)-45(x_3^2+x_3x_1)(x_3^3+x_3^2x_2+x_3x_2^2+x_2^3)+20(3x_3^2x_1+x_3^3)(x_3^2+x_3x_2+x_2^2)-30x_3^3x_1(x_3+x_2)}{60(x_3-x_2)(x_3-x_1)^2} \\
&\quad + \frac{-10(x_3^5+x_3^4x_2+x_3^3x_2^2+x_3^2x_2^3+x_3x_2^4+x_2^5)+12(x_4+2x_3+x_2)(x_3^4+x_3^3x_2+x_3^2x_2^2+x_3x_2^3+x_2^4)-15(2x_4x_3+2x_3x_2+x_4x_2+x_3^2)(x_3^3+x_3^2x_2+x_3x_2^2+x_2^3)+20(x_4x_3^2+2x_4x_3x_2+x_3^2x_2)(x_3^2+x_3x_2+x_2^2)-30x_4x_3^2x_2(x_3+x}{60(x_4-x_2)(x_3-x_2)(x_3-x_1)} \\
&= \frac{2x_2^5-4x_2^4x_1+x_2^3x_1^2+x_2^2x_1^3+x_2x_1^4-x_1^5-x_0(3x_2^4-7x_2^3x_1+3x_2^2x_1^2+3x_2x_1^3-2x_1^4)}{60(x_3-x_1)(x_2-x_1)(x_2-x_0)} + \frac{-10x_2^5+26x_2^4x_1-19x_2^3x_1^2+x_2^2x_1^3+x_2x_1^4+x_1^5+3x_3(4x_2^4-11x_2^3x_1+9x_2^2x_1^2-x_2x_1^3-x_1^4)}{60(x_3-x_1)^2(x_2-x_1)} \\
&\quad + \frac{x_3^5+x_3^4x_2+x_3^3x_2^2-19x_3^2x_2^3+26x_3x_2^4-10x_2^5-3x_1(x_3^4+x_3^3x_2-9x_3^2x_2^2+11x_3x_2^3-4x_2^4)}{60(x_3-x_2)(x_3-x_1)^2} + \frac{-x_3^5+x_3^4x_2+x_3^3x_2^2+x_3^2x_2^3-4x_3x_2^4+2x_2^5-x_4(3x_2^4-7x_3x_2^3+3x_3^2x_2^2+3x_3^3x_2-2x_3^4)}{60(x_4-x_2)(x_3-x_2)(x_3-x_1)} \\
&= (x_2-x_1)^2 \frac{2x_2^2+2x_2x_1+x_1^2-x_0(3x_2+2x_1)}{60(x_3-x_1)(x_2-x_0)} + (x_2-x_1)^2 \frac{(-10x_2^2-4x_2x_1-x_1^2)+3x_3(4x_2+x_1)}{60(x_3-x_1)^2} \\
&\quad + (x_3-x_2)^2 \frac{x_3^2+4x_3x_2+10x_2^2-3x_1(x_3+4x_2)}{60(x_3-x_1)^2} + (x_3-x_2)^2 \frac{(-x_3^2-2x_3x_2-2x_2^2)+x_4(2x_3+3x_2)}{60(x_4-x_2)(x_3-x_1)}
\end{aligned}
$$

And

$$\begin{aligned}
\int_{x_2}^{x_3} x b_{2,0}(x)\times b_{2,2}(x)dx &= \int_{x_2}^{x_3} x \frac{(x_3-x)^2}{(x_3-x_2)(x_3-x_1)}\frac{(x-x_2)^2}{(x_4-x_2)(x_3-x_2)}dx \\
&= \int_{x_2}^{x_3} x \frac{(x^2-2x_3x+x_3^2)(x^2-2x_2x+x_2^2)}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)}dx \\
&= \int_{x_2}^{x_3} x \frac{x^4-2x_3x^3+x_3^2x^2-2x_2x^3+4x_3x_2x^2-2x_3^2x_2x+x_2^2x^2-2x_3x_2^2x+x_3^2x_2^2}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)}dx \\
&= \int_{x_2}^{x_3} \frac{x^5-2(x_3+x_2)x^4+(x_3^2+4x_3x_2+x_2^2)x^3-2(x_3^2x_2+x_3x_2^2)x^2+x_3^2x_2^2x}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)}dx \\
&= \frac{[10x^6-24(x_3+x_2)x^5+15(x_3^2+4x_3x_2+x_2^2)x^4-40(x_3^2x_2+x_3x_2^2)x^3+30x_3^2x_2^2x^2]_{x_2}^{x_3}}{60(x_4-x_2)(x_3-x_2)^2(x_3-x_1)} \\
&= \frac{10(x_3^5+x_3^4x_2+x_3^3x_2^2+x_3^2x_2^3+x_3x_2^4+x_2^5)-24(x_3+x_2)(x_3^4+x_3^3x_2+x_3^2x_2^2+x_3x_2^3+x_2^4)+15(x_3^2+4x_3x_2+x_2^2)(x_3^3+x_3^2x_2+x_3x_2^2+x_2^3)-40(x_3^2x_2+x_3x_2^2)(x_3^2+x_3x_2+x_2^2)+30x_3^2x_2^2(x_3+x_2)}{60(x_4-x_2)(x_3-x_2)(x_3-x_1)} \\
&= \frac{x_3^5-3x_3^4x_2+2x_3^3x_2^2+2x_3^2x_2^3-3x_3x_2^4+x_2^5}{60(x_4-x_2)(x_3-x_2)(x_3-x_1)} \\
&= \frac{(x_3+x_2)(x_3-x_2)^3}{60(x_4-x_2)(x_3-x_1)}
\end{aligned}$$

C.7. DEG = 3, SURF - TO BE FINISHED AND CHECKED

$$\begin{aligned}
\int_{x_0}^{x_4}\|b_{3,0}(x)\|^2dx &= \int_{x_0}^{x_1}\left(\frac{x^3-3x^2x_0+3xx_0^2-x_0^3}{(x_3-x_0)(x_2-x_0)(x_1-x_0)}\right)^2dx+\int_{x_1}^{x_2}\left(\frac{x^3A+x^2B+xC+D}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}\right)^2dx+\int_{x_2}^{x_3}\left(\frac{x^3A'+x^2B'+xC'+D'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}\right)^2dx+\int_{x_3}^{x_4}\left(\frac{-x^3+3x^2x_4-3xx_4^2+x_4^3}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}\right)^2dx \\
&= \int_{x_0}^{x_1}\frac{x^6-6x_0x^5+15x_0^2x^4-20x_0^3x^3+15x_0^4x^2-6x_0^5x+x_0^6}{(x_3-x_0)^2(x_2-x_0)^2(x_1-x_0)^2}dx \\
&\quad+\int_{x_1}^{x_2}\frac{A^2x^6+2ABx^5+(2AC+B^2)x^4+(2AD+2BC)x^3+(2BD+C^2)x^2+2CDx+D^2}{((x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0))^2}dx \\
&\quad+\int_{x_2}^{x_3}\frac{A'^2x^6+2A'B'x^5+(2A'C'+B'^2)x^4+(2A'D'+2B'C')x^3+(2B'D'+C'^2)x^2+2C'D'x+D'^2}{((x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0))^2}dx \\
&\quad+\int_{x_3}^{x_4}\frac{x^6-6x_4x^5+15x_4^2x^4-20x_4^3x^3+15x_4^4x^2-6x_4^5x+x_4^6}{(x_4-x_3)^2(x_4-x_2)^2(x_4-x_1)^2}dx \\
&= \frac{[x^7-7x_0x^6+21x_0^2x^5-35x_0^3x^4+35x_0^4x^3-21x_0^5x^2+7x_0^6x]_{x_0}^{x_1}}{7(x_3-x_0)^2(x_2-x_0)^2(x_1-x_0)^2} \\
&\quad+\frac{[30A^2x^7+70ABx^6+42(2AC+B^2)x^5+105(AD+BC)x^4+70(2BD+C^2)x^3+210CDx^2+210D^2x]_{x_1}^{x_2}}{210((x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0))^2} \\
&\quad+\frac{[30A'^2x^7+70A'B'x^6+42(2A'C'+B'^2)x^5+105(A'D'+B'C')x^4+70(2B'D'+C'^2)x^3+210C'D'x^2+210D'^2x]_{x_2}^{x_3}}{210((x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0))^2} \\
&\quad+\frac{[x^7-7x_4x^6+21x_4^2x^5-35x_4^3x^4+35x_4^4x^3-21x_4^5x^2+7x_4^6x]_{x_3}^{x_4}}{7(x_4-x_3)^2(x_4-x_2)^2(x_4-x_1)^2} \\
&= \frac{x_1^6-6x_1^5x_0+15x_1^4x_0^2-20x_1^3x_0^3+15x_1^2x_0^4-6x_1x_0^5+x_0^6}{7(x_3-x_0)^2(x_2-x_0)^2(x_1-x_0)} \\
&\quad+\frac{[30A^2x^7+70ABx^6+42(2AC+B^2)x^5+105(AD+BC)x^4+70(2BD+C^2)x^3+210CDx^2+210D^2x]_{x_1}^{x_2}}{210((x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0))^2} \\
&\quad+\frac{[30A'^2x^7+70A'B'x^6+42(2A'C'+B'^2)x^5+105(A'D'+B'C')x^4+70(2B'D'+C'^2)x^3+210C'D'x^2+210D'^2x]_{x_2}^{x_3}}{210((x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0))^2} \\
&\quad+\frac{x_4^6-6x_4^5x_3+15x_4^4x_3^2-20x_4^3x_3^3+15x_4^2x_3^4-6x_4x_3^5+x_3^6}{7(x_4-x_3)(x_4-x_2)^2(x_4-x_1)^2}
\end{aligned}$$

Hence:

$$
\begin{aligned}
\int_{x_0}^{x_4} \|b_{3,0}(x)\|^2 dx \quad = \quad & \frac{(x_1-x_0)^5}{7(x_3-x_0)^2(x_2-x_0)^2} \\
& + \frac{\left[30A^2x^7+70ABx^6+42(2AC+B^2)x^5+105(AD+BC)x^4+70(2BD+C^2)x^3+210CDx^2+210D^2x\right]_{x_1}^{x_2}}{210((x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0))^2} \\
& + \frac{\left[30A'^2x^7+70A'B'x^6+42(2A'C'+B'^2)x^5+105(A'D'+B'C')x^4+70(2B'D'+C'^2)x^3+210C'D'x^2+210D'^2x\right]_{x_2}^{x_3}}{210((x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0))^2} \\
& + \frac{(x_4-x_3)^5}{7(x_4-x_2)^2(x_4-x_1)^2}
\end{aligned}
$$

And:

$$
\begin{aligned}
\int_{x_1}^{x_4} b_{3,0}(x) \times b_{3,1}(x) dx \quad = \quad & \int_{x_1}^{x_2} \frac{x^3-3x_1x^2+3x_1^2x-x_1^3}{(x_4-x_1)(x_3-x_1)(x_2-x_1)} \frac{Ax^3+Bx^2+Cx+D}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)} dx \\
& + \int_{x_2}^{x_3}
\end{aligned}
$$

To be refined / simplified, apparently, the reduced expression with A, B, C, D is not numerically accurate.... (check)

C.8. DEG = 3, VOL - TO DO

D. D1N2 - Exact formulations

D.1. Deg = 1, Surf

$$
\begin{aligned}
\int_{x_0}^{x_2} \|\partial_x b_{1,0}(x)\|^2 dx &= \int_{x_0}^{x_1} \left(\frac{1}{x_1-x_0}\right)^2 dx + \int_{x_1}^{x_2} \left(\frac{-1}{x_2-x_1}\right)^2 dx \\
&= \frac{x_1-x_0}{(x_1-x_0)^2} + \frac{x_2-x_1}{(x_2-x_1)^2} \qquad = \frac{1}{x_1-x_0} + \frac{1}{x_2-x_1}
\end{aligned}
$$

and

$$
\int_{x_1}^{x_2} \partial_x b_{1,0}(x) \times \partial_x b_{1,1}(x) dx = \int_{x_1}^{x_2} \frac{-1}{x_2-x_1} \frac{1}{x_2-x_1} dx \quad = \frac{-1}{x_2-x_1}
$$

D.2. Deg = 1, Vol

$$
\begin{aligned}
\int_{x_0}^{x_2} x\|\partial_x b_{1,0}(x)\|^2 dx &= \int_{x_0}^{x_1} x\left(\frac{1}{x_1-x_0}\right)^2 dx + \int_{x_1}^{x_2} x\left(\frac{-1}{x_2-x_1}\right)^2 dx \\
&= \frac{x_1^2-x_0^2}{2(x_1-x_0)^2} + \frac{x_2^2-x_1^2}{2(x_2-x_1)^2} \qquad = \frac{x_1+x_0}{2(x_1-x_0)} + \frac{x_2+x_1}{2(x_2-x_1)}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{x_1}^{x_2} x\partial_x b_{1,0}(x) \times \partial_x b_{1,1}(x) dx &= \int_{x_1}^{x_2} x\frac{-1}{x_2-x_1} \frac{1}{x_2-x_1} dx \\
&= -\frac{x_2^2-x_1^2}{2(x_2-x_1)^2} \qquad = -\frac{x_2+x_1}{2(x_2-x_1)}
\end{aligned}
$$

D.3. Deg = 2, Surf

$$
\begin{aligned}
\int_{x_0}^{x_3} \|\partial_x b_{2,0}(x)\|^2 dx &= \int_{x_0}^{x_1} \left(\frac{2(x-x_0)}{(x_2-x_0)(x_1-x_0)}\right)^2 dx + \int_{x_1}^{x_2} \left(\frac{-2(x_3+x_2-x_1-x_0)x+2(x_3x_2-x_1x_0)}{(x_3-x_1)(x_2-x_1)(x_2-x_0)}\right)^2 dx + \int_{x_2}^{x_3} \left(\frac{-2(x_3-x)}{(x_3-x_2)(x_3-x_1)}\right)^2 dx \\
&= \frac{\left[4(x-x_0)^3\right]_{x_0}^{x_1}}{3(x_2-x_0)^2(x_1-x_0)^2} + \frac{\left[4(x_3+x_2-x_1-x_0)^2x^3-12(x_3+x_2-x_1-x_0)(x_3x_2-x_1x_0)x^2+12(x_3x_2-x_1x_0)^2x\right]_{x_1}^{x_2}}{3(x_3-x_1)^2(x_2-x_1)^2(x_2-x_0)^2} + \frac{\left[4(x-x_3)^3\right]_{x_2}^{x_3}}{3(x_3-x_2)^2(x_3-x_1)^2} \\
&= \frac{4(x_1-x_0)}{3(x_2-x_0)^2} \\
&\quad + \frac{4x_2^4-4x_2^3x_1-4x_2x_1^3+4x_1^4+4(x_3^2+x_0^2)(x_2^2-2x_2x_1+x_1^2)-4x_3(x_2^3-3x_2x_1^2+2x_1^3)+4x_0(3x_2^2x_1-2x_2^3-x_1^3)+4x_3x_0(x_2^2-2x_2x_1+x_1^2)}{3(x_3-x_1)^2(x_2-x_1)(x_2-x_0)^2} \\
&\quad + \frac{4(x_3-x_2)}{3(x_3-x_1)^2} \\
&= \frac{4(x_1-x_0)}{3(x_2-x_0)^2} \\
&\quad + 4(x_2-x_1)\frac{x_2^2+x_2x_1+x_1^2+x_3^2+x_3x_0+x_0^2-x_3(x_2+2x_1)-x_0(2x_2+x_1)}{3(x_3-x_1)^2(x_2-x_0)^2} \\
&\quad + \frac{4(x_3-x_2)}{3(x_3-x_1)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\int_{x_1}^{x_3} \partial_x b_{2,0}(x) \times \partial_x b_{2,1}(x) dx &= \int_{x_1}^{x_2} \frac{-2(x_3+x_2-x_1-x_0)x+2(x_3x_2-x_1x_0)}{(x_3-x_1)(x_2-x_1)(x_2-x_0)} \frac{2(x-x_1)}{(x_3-x_1)(x_2-x_1)} dx + \int_{x_2}^{x_3} \frac{-2(x_4+x_3-x_2-x_1)x+2(x_4x_3-x_2x_1)}{(x_4-x_2)(x_3-x_2)(x_3-x_1)} \frac{2(x-x_3)}{(x_3-x_2)(x_3-x_1)} dx \\
&= \int_{x_1}^{x_2} 4\frac{-(x_3+x_2-x_1-x_0)x^2+(x_3x_2+x_3x_1+x_2x_1-x_1^2-2x_1x_0)x-x_3x_2x_1+x_1^2x_0}{(x_3-x_1)^2(x_2-x_1)^2(x_2-x_0)} dx \\
&\quad + \int_{x_2}^{x_3} 4\frac{-(x_4+x_3-x_2-x_1)x^2+(2x_4x_3+x_3^2-x_3x_2-x_3x_1-x_2x_1)x-x_4x_3^2+x_3x_2x_1}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)^2} dx
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\int_{x_1}^{x_3} \partial_x b_{2,0}(x) \times \partial_x b_{2,1}(x)\, dx &= \frac{-4(x_3+x_2-x_1-x_0)(x_2^2+x_2x_1+x_1^2)+6(x_3x_2+x_3x_1+x_2x_1-x_1^2-2x_1x_0)(x_2+x_1)-12x_3x_2x_1+12x_1^2x_0}{3(x_3-x_1)^2(x_2-x_1)(x_2-x_0)} \\
&\quad + \frac{-4(x_4+x_3-x_2-x_1)(x_3^2+x_3x_2+x_2^2)+6(2x_4x_3+x_3^2-x_3x_2-x_3x_1-x_2x_1)(x_3+x_2)-12x_4x_3^2+12x_3x_2x_1}{3(x_4-x_2)(x_3-x_2)(x_3-x_1)^2} \\
&= \frac{-4x_3x_2^2-4x_2^3+4x_2^2x_1+4x_2^2x_0-4x_3x_2x_1-4x_2^2x_1+4x_2x_1^2+4x_2x_1x_0-4x_3x_1^2-4x_2x_1^2+4x_1^3+4x_1^2x_0+6x_3x_2^2+6x_3x_2x_1+6x_2^2x_1-6x_2x_1^2-12x_2x_1x_0+6x_3x_2x_1+6x_3x_1^2+6x_2x_1^2-6x_1^3-12x_1^2x_0-12x_3x_2x_1+12}{3(x_3-x_1)^2(x_2-x_1)(x_2-x_0)} \\
&\quad + \frac{-4x_4x_3^2-4x_3^3+4x_3^2x_2+4x_3^2x_1+-4x_4x_3x_2-4x_3^2x_2+4x_3x_2^2+4x_3x_2x_1+-4x_4x_2^2-4x_3x_2^2+4x_2^3+4x_2^2x_1+12x_4x_3^2+6x_3^3-6x_3^2x_2-6x_3^2x_1-6x_3x_2x_1+12x_4x_3x_2+6x_3^2x_2-6x_3x_2^2-6x_3x_2x_1-6x_2^2x_1-12x_4x_3^2-}{3(x_4-x_2)(x_3-x_2)(x_3-x_1)^2} \\
&= \frac{-4x_2^3+6x_2^2x_1-2x_1^3+x_0(4x_2^2-8x_2x_1+4x_1^2)+x_3(-4x_2x_1+2x_1^2+2x_2^2)}{3(x_3-x_1)^2(x_2-x_1)(x_2-x_0)} \\
&\quad + \frac{2x_3^3-6x_3x_2^2+4x_2^3+x_4(-4x_3^2+8x_3x_2-4x_2^2)-x_1(2x_3^2-4x_3x_2+2x_2^2)}{3(x_4-x_2)(x_3-x_2)(x_3-x_1)^2} \\
&= 2(x_2-x_1)\frac{x_3-2x_2-x_1+2x_0}{3(x_3-x_1)^2(x_2-x_0)} \\
&\quad +2(x_3-x_2)\frac{-2x_4+x_3+2x_2-x_1}{3(x_4-x_2)(x_3-x_1)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\int_{x_2}^{x_3} \partial_x b_{2,0}(x) \times \partial_x b_{2,2}(x)\, dx &= \int_{x_2}^{x_3} \frac{2(x-x_3)}{(x_3-x_2)(x_3-x_1)} \frac{2(x-x_2)}{(x_4-x_2)(x_3-x_2)}\, dx \\
&= \int_{x_2}^{x_3} 4\frac{x^2-(x_3+x_2)x+x_3x_2}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)}\, dx \\
&= \frac{\left[4x^3-6(x_3+x_2)x^2+12x_3x_2x\right]_{x_2}^{x_3}}{3(x_4-x_2)(x_3-x_2)^2(x_3-x_1)} \\
&= \frac{4(x_3^2+x_3x_2+x_2^2)-6(x_3+x_2)(x_3+x_2)+12x_3x_2}{3(x_4-x_2)(x_3-x_2)(x_3-x_1)} \quad = -2\frac{x_3-x_2}{3(x_4-x_2)(x_3-x_1)}
\end{aligned}
$$

$$
\begin{aligned}
\int_{x_0}^{x_3} x\|\partial_x b_{2,0}(x)\|^2 dx &= \int_{x_0}^{x_1} x\left(\frac{2(x-x_0)}{(x_2-x_0)(x_1-x_0)}\right)^2 dx+\int_{x_1}^{x_2} x\left(\frac{-2(x_3+x_2-x_1-x_0)x+2(x_3x_2-x_1x_0)}{(x_3-x_1)(x_2-x_1)(x_2-x_0)}\right)^2 dx+\int_{x_2}^{x_3} x\left(\frac{-2(x_3-x)}{(x_3-x_2)(x_3-x_1)}\right)^2 dx\\
&= \int_{x_0}^{x_1} 4\frac{x^3-2x_0x^2+x_0^2x}{(x_2-x_0)^2(x_1-x_0)^2}dx\\
&\quad+\int_{x_1}^{x_2} 4\frac{(x_3+x_2-x_1-x_0)^2x^3-2(x_3+x_2-x_1-x_0)(x_3x_2-x_1x_0)x^2+(x_3x_2-x_1x_0)^2x}{(x_3-x_1)^2(x_2-x_1)^2(x_2-x_0)^2}dx\\
&\quad+\int_{x_2}^{x_3} 4\frac{x^3-2x_3x^2+x_3^2x}{(x_3-x_2)^2(x_3-x_1)^2}dx\\
&= \frac{\left[3x^4-8x_0x^3+6x_0^2x^2\right]_{x_0}^{x_1}}{3(x_2-x_0)^2(x_1-x_0)^2}\\
&\quad+\frac{\left[3(x_3+x_2-x_1-x_0)^2x^4-8(x_3+x_2-x_1-x_0)(x_3x_2-x_1x_0)x^3+6(x_3x_2-x_1x_0)^2x^2\right]_{x_1}^{x_2}}{3(x_3-x_1)^2(x_2-x_1)^2(x_2-x_0)^2}\\
&\quad+\frac{\left[3x^4-8x_3x^3+6x_3^2x^2\right]_{x_2}^{x_3}}{3(x_3-x_2)^2(x_3-x_1)^2}\\
&= \frac{3(x_1^3+x_1^2x_0+x_1x_0^2+x_0^3)-8x_0(x_1^2+x_1x_0+x_0^2)+6x_0^2(x_1+x_0)}{3(x_2-x_0)^2(x_1-x_0)}\\
&\quad+\frac{3(x_3+x_2-x_1-x_0)^2(x_2^3+x_2^2x_1+x_2x_1^2+x_1^3)-8(x_3^2x_2+x_3x_2^2-x_3x_2x_1-x_3x_2x_0-x_3x_1x_0-x_2x_1x_0+x_1^2x_0+x_1x_0^2)(x_2^2+x_2x_1+x_1^2)+6(x_3x_2-x_1x_0)^2(x_2+x_1)}{3(x_3-x_1)^2(x_2-x_1)(x_2-x_0)^2}\\
&\quad+\frac{3(x_3^3+x_3^2x_2+x_3x_2^2+x_2^3)-8x_3(x_3^2+x_3x_2+x_2^2)+6x_3^2(x_3+x_2)}{3(x_3-x_2)(x_3-x_1)^2}\\
&= \frac{3x_1^3-5x_1^2x_0+x_1x_0^2+x_0^3}{3(x_2-x_0)^2(x_1-x_0)}\\
&\quad+\frac{x_3^2x_2^3+3x_2^5+3x_2^3x_0^2-2x_3x_2^4+2x_3x_2^3x_0-3x_2^4x_1-6x_2^4x_0+8x_2^3x_1x_0+x_3^2x_2^2x_1-5x_2^2x_1x_0^2-2x_3x_2^2x_1x_0-5x_3^2x_2x_1^2-3x_2x_1^4+x_2x_1^2x_0^2+8x_3x_2x_1^3-2x_3x_2x_1^2x_0+3x_3^2x_1^3+3x_1^5+x_1^3x_0^2-6x_3x_1^4+2x_3x_1^3x_0-2x_1^4x_0}{3(x_3-x_1)^2(x_2-x_1)(x_2-x_0)^2}\\
&\quad+\frac{x_3^3+x_3^2x_2-5x_3x_2^2+3x_2^3}{3(x_3-x_2)(x_3-x_1)^2}\\
&= \frac{(3x_1+x_0)(x_1-x_0)^2}{3(x_2-x_0)^2(x_1-x_0)}\\
&\quad+\frac{3(x_2^5-x_2^4x_1-x_2x_1^4+x_1^5)+x_3^2(x_2^3+x_2^2x_1-5x_2x_1^2+3x_1^3)+x_0^2(3x_2^3-5x_2^2x_1+x_2x_1^2+x_1^3)-2x_3(x_2^4-4x_2x_1^3+3x_1^4)-2x_0(3x_2^4-4x_2^3x_1+x_1^4)+2x_3x_0(x_2^3-x_2^2x_1-x_2x_1^2+x_1^3)}{3(x_3-x_1)^2(x_2-x_1)(x_2-x_0)^2}\\
&\quad+\frac{(x_3+3x_2)(x_3-x_2)^2}{3(x_3-x_2)(x_3-x_1)^2}\\
&= \frac{(3x_1+x_0)(x_1-x_0)}{3(x_2-x_0)^2}\\
&\quad+(x_2-x_1)\frac{3(x_2+x_1)(x_2^2+x_1^2)+x_3^2(x_2+3x_1)+x_0^2(3x_2+x_1)-2x_3(x_2^2+2x_2x_1+3x_1^2)-2x_0(3x_2^2+2x_2x_1+x_1^2)+2x_3x_0(x_2+x_1)}{3(x_3-x_1)^2(x_2-x_0)^2}\\
&\quad+\frac{(x_3+3x_2)(x_3-x_2)}{3(x_3-x_1)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\int_{x_1}^{x_3} x\partial_x b_{2,0}(x)\times\partial_x b_{2,1}(x)dx &= \int_{x_1}^{x_2} x\frac{-2(x_3+x_2-x_1-x_0)x+2(x_3x_2-x_1x_0)}{(x_3-x_1)(x_2-x_1)(x_2-x_0)}\frac{2(x-x_1)}{(x_3-x_1)(x_2-x_1)}dx+\int_{x_2}^{x_3} x\frac{-2(x_4+x_3-x_2-x_1)x+2(x_4x_3-x_2x_1)}{(x_4-x_2)(x_3-x_2)(x_3-x_1)}\frac{2(x-x_3)}{(x_3-x_2)(x_3-x_1)}dx\\
&= \int_{x_1}^{x_2} 4\frac{-(x_3+x_2-x_1-x_0)x^3+(x_3x_2+x_3x_1+x_2x_1-x_1^2-2x_1x_0)x^2+(x_1^2x_0-x_3x_2x_1)x}{(x_3-x_1)^2(x_2-x_1)^2(x_2-x_0)}dx\\
&\quad+\int_{x_2}^{x_3} 4\frac{-(x_4+x_3-x_2-x_1)x^3+(2x_4x_3+x_3^2-x_3x_2-x_3x_1-x_2x_1)x^2+(x_3x_2x_1-x_4x_3^2)x}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)^2}dx\\
&= \frac{\left[-3(x_3+x_2-x_1-x_0)x^4+4(x_3x_2+x_3x_1+x_2x_1-x_1^2-2x_1x_0)x^3+6(x_1^2x_0-x_3x_2x_1)x^2\right]_{x_1}^{x_2}}{3(x_3-x_1)^2(x_2-x_1)^2(x_2-x_0)}\\
&\quad+\frac{\left[-3(x_4+x_3-x_2-x_1)x^4+4(2x_4x_3+x_3^2-x_3x_2-x_3x_1-x_2x_1)x^3+6(x_3x_2x_1-x_4x_3^2)x^2\right]_{x_2}^{x_3}}{3(x_4-x_2)(x_3-x_2)^2(x_3-x_1)^2}
\end{aligned}
$$

So

$$\begin{aligned}
\int_{x_1}^{x_3} x\partial_x b_{2,0}(x)\times\partial_x b_{2,1}(x)dx &= \frac{-3(x_3+x_2-x_1-x_0)(x_2^3+x_2^2x_1+x_2x_1^2+x_1^3)+4(x_3x_2+x_3x_1+x_2x_1-x_1^2-2x_1x_0)(x_2^2+x_2x_1+x_1^2)+6(x_1^2x_0-x_3x_2x_1)(x_2+x_1)}{3(x_3-x_1)^2(x_2-x_1)(x_2-x_0)} \\
&+\frac{-3(x_4+x_3-x_2-x_1)(x_3^3+x_3^2x_2+x_3x_2^2+x_2^3)+4(2x_4x_3+x_3^2-x_3x_2-x_3x_1-x_2x_1)(x_3^2+x_3x_2+x_2^2)+6(x_3x_2x_1-x_4x_3^2)(x_3+x_2)}{3(x_4-x_2)(x_3-x_2)(x_3-x_1)^2} \\
&= \frac{-3x_2^4+4x_2^3x_1-x_1^4+x_3(x_2^3-x_2^2x_1-x_2x_1^2+x_1^3)+x_0(3x_2^3-5x_2^2x_1+x_2x_1^2+x_1^3)}{3(x_3-x_1)^2(x_2-x_1)(x_2-x_0)} \\
&+\frac{x_3^4-4x_3x_2^3+3x_2^4-x_4(x_3^3+x_3^2x_2-5x_3x_2^2+3x_2^3)-x_1(x_3^3-x_3^2x_2-x_3x_2^2+x_2^3)}{3(x_4-x_2)(x_3-x_2)(x_3-x_1)^2} \\
&= (x_2-x_1)\frac{-(3x_2^2+2x_2x_1+x_1^2)+x_3(x_2+x_1)+x_0(3x_2+x_1)}{3(x_3-x_1)^2(x_2-x_0)} \\
&+(x_3-x_2)\frac{x_3^2+2x_3x_2+3x_2^2-x_4(x_3+3x_2)-x_1(x_3+x_2)}{3(x_4-x_2)(x_3-x_1)^2}
\end{aligned}$$

And

$$\begin{aligned}
\int_{x_2}^{x_3} x\partial_x b_{2,0}(x)\times\partial_x b_{2,2}(x)dx &= \int_{x_2}^{x_3} x\frac{2(x-x_3)}{(x_3-x_2)(x_3-x_1)}\frac{2(x-x_2)}{(x_4-x_2)(x_3-x_2)}dx \\
&= \int_{x_2}^{x_3} 4\frac{x^3-(x_3+x_2)x^2+x_3x_2x}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)}dx \\
&= \frac{\left[3x^4-4(x_3+x_2)x^3+6x_3x_2x^2\right]_{x_2}^{x_3}}{3(x_4-x_2)(x_3-x_2)^2(x_3-x_1)} \\
&= -\frac{x_3^3-x_3^2x_2-x_3x_2^2+x_2^3}{3(x_4-x_2)(x_3-x_2)(x_3-x_1)} \qquad = -\frac{(x_3+x_2)(x_3-x_2)}{3(x_4-x_2)(x_3-x_1)}
\end{aligned}$$

D.5. DEG = 3, SURF - TO BE FINISHED AND CHECKED

$$\begin{aligned}
\int_{x_0}^{x_4}\|\partial_x b_{3,0}(x)\|^2dx &= \int_{x_0}^{x_1}\left(\frac{3(x-x_0)^2}{(x_3-x_0)(x_2-x_0)(x_1-x_0)}\right)^2dx+\int_{x_1}^{x_2}\left(\frac{3Ax^2+2Bx+C}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}\right)^2dx+\int_{x_2}^{x_3}\left(\frac{3A'x^2+2B'x+C'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}\right)^2dx+\int_{x_3}^{x_4}\left(\frac{-3(x_4-x)^2}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}\right)^2dx \\
&= \frac{\left[9(x-x_0)^5\right]_{x_0}^{x_1}}{5(x_3-x_0)^2(x_2-x_0)^2(x_1-x_0)^2}dx \\
&+\int_{x_1}^{x_2}\frac{9A^2x^4+12ABx^3+2(2B^2+3AC)x^2+4BCx+C^2}{((x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0))^2}dx \\
&+\int_{x_1}^{x_2}\frac{9A'^2x^4+12ABx^3+2(2B'^2+3AC)x^2+4BCx+C'^2}{((x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0))^2}dx \\
&+\frac{\left[-9(x_4-x)^5\right]_{x_3}^{x_4}}{5(x_4-x_3)^2(x_4-x_2)^2(x_4-x_1)^2} \\
&= \frac{9(x_1-x_0)^3}{5(x_3-x_0)^2(x_2-x_0)^2} \\
&+\frac{\left[27A^2x^5+45ABx^4+10(2B^2+3AC)x^3+30BCx^2+15C^2x\right]_{x_1}^{x_2}}{15((x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0))^2} \\
&+\frac{\left[27A'^2x^5+45ABx^4+10(2B'^2+3AC)x^3+30BCx^2+15C'^2x\right]_{x_1}^{x_2}}{15((x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0))^2} \\
&+\frac{9(x_4-x_3)^3}{5(x_4-x_2)^2(x_4-x_1)^2}
\end{aligned}$$

and

$$\begin{aligned}
\int_{x_1}^{x_4}\partial_x b_{3,0}(x)\times\partial_x b_{3,1}dx &= \int_{x_1}^{x_2}\frac{3Ax^2+2Bx+C}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}\frac{3(x-x_1)^2}{(x_4-x_1)(x_3-x_1)(x_2-x_1)}dx \\
&+\int_{x_2}^{x_3}\frac{3A'x^2+2B'x+C'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}\frac{3A(1)x^2+2B(1)x+C(1)}{(x_5-x_2)(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)}dx \\
&+\int_{x_3}^{x_4}\frac{-3(x_4-x)^2}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}\frac{3A(1)'x^2+2B(1)'x+C(1)'}{(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)(x_4-x_1)}
\end{aligned}$$

and

$$\begin{aligned}
\int_{x_2}^{x_4} \partial_x b_{3,0}(x) \times \partial_x b_{3,2} dx \quad &= \quad \int_{x_2}^{x_3} \frac{3A^/ x^2+2B^/ x+C^/}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} \frac{3(x-x_2)^2}{(x_5-x_2)(x_4-x_2)(x_3-x_2)} dx \\
&\quad + \int_{x_3}^{x_4} \frac{-3(x_4-x)^2}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} \frac{3A(2)x^2+2B(2)x+C(2)}{(x_6-x_3)(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)}
\end{aligned}$$

and

$$\begin{aligned}
\int_{x_3}^{x_4} \partial_x b_{3,0}(x) \times \partial_x b_{3,3} dx \quad &= \quad \int_{x_3}^{x_4} \frac{-3(x_4-x)^2}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} \frac{3(x-x_3)^2}{(x_6-x_3)(x_5-x_3)(x_4-x_3)} dx \\
&= \int_{x_3}^{x_4} \frac{-9(x-x_3)^2(x_4-x)^2}{(x_6-x_3)(x_5-x_3)(x_4-x_3)^2(x_4-x_2)(x_4-x_1)} dx \\
&= \int_{x_3}^{x_4} -9 \frac{x^4-2(x_4+x_3)x^3+(x_3^2+4x_4x_3+x_4^2)x^2-2(x_4x_3^2+x_4^2x_3)x+x_4^2x_3^2}{(x_6-x_3)(x_5-x_3)(x_4-x_3)^2(x_4-x_2)(x_4-x_1)} dx \\
&- \frac{\left[18x^5-45(x_4+x_3)x^4+30(x_3^2+4x_4x_3+x_4^2)x^3-90(x_4x_3^2+x_4^2x_3)x^2+90x_4^2x_3^2x\right]_{x_3}^{x_4}}{10(x_6-x_3)(x_5-x_3)(x_4-x_3)^2(x_4-x_2)(x_4-x_1)} \\
&- \frac{18(x_4^4+x_4^3x_3+x_4^2x_3^2+x_4x_3^3+x_3^4)-45(x_4+x_3)(x_4^3+x_4^2x_3+x_4x_3^2+x_3^3)+30(x_3^2+4x_4x_3+x_4^2)(x_4^2+x_4x_3+x_3^2)-90(x_4x_3^2+x_4^2x_3)(x_4+x_3)+90x_4^2x_3^2}{10(x_6-x_3)(x_5-x_3)(x_4-x_3)(x_4-x_2)(x_4-x_1)} \\
&- \frac{3x_4^4-12x_4^3x_3+18x_4^2x_3^2-12x_4x_3^3+3x_3^4}{10(x_6-x_3)(x_5-x_3)(x_4-x_3)(x_4-x_2)(x_4-x_1)} \\
&= -3 \frac{(x_4-x_3)^3}{10(x_6-x_3)(x_5-x_3)(x_4-x_2)(x_4-x_1)}
\end{aligned}$$

D.6. DEG = 3, VOL - TO BE DONE

E. D2N2 - Exact formulations

E.1. Deg = 2, Surf

$$\begin{aligned}
\int_{x_0}^{x_3} \|\partial_x^2 b_{2,0}(x)\|^2 dx &= \int_{x_0}^{x_1} \left(\frac{2}{(x_2-x_0)(x_1-x_0)}\right)^2 dx + \int_{x_1}^{x_2} \left(\frac{-2(x_3+x_2-x_1-x_0)}{(x_3-x_1)(x_2-x_1)(x_2-x_0)}\right)^2 dx + \int_{x_2}^{x_3} \left(\frac{2}{(x_3-x_2)(x_3-x_1)}\right)^2 dx \\
&= \int_{x_0}^{x_1} \frac{4}{(x_2-x_0)^2(x_1-x_0)^2} dx + \int_{x_1}^{x_2} \frac{4(x_3+x_2-x_1-x_0)^2}{(x_3-x_1)^2(x_2-x_1)^2(x_2-x_0)^2} dx + \int_{x_2}^{x_3} \frac{4}{(x_3-x_2)^2(x_3-x_1)^2} dx \\
&= \frac{4}{(x_2-x_0)^2(x_1-x_0)} + \frac{4(x_3+x_2-x_1-x_0)^2}{(x_3-x_1)^2(x_2-x_1)(x_2-x_0)^2} + \frac{4}{(x_3-x_2)(x_3-x_1)^2}
\end{aligned}$$

and

$$\begin{aligned}
\int_{x_1}^{x_3} \partial_x^2 b_{2,0}(x) \times \partial_x^2 b_{2,1}(x) dx &= \int_{x_1}^{x_2} \frac{-2(x_3+x_2-x_1-x_0)}{(x_3-x_1)(x_2-x_1)(x_2-x_0)} \frac{2}{(x_3-x_1)(x_2-x_1)} dx + \int_{x_2}^{x_3} \frac{-2(x_4+x_3-x_2-x_1)}{(x_4-x_2)(x_3-x_2)(x_3-x_1)} \frac{2}{(x_3-x_2)(x_3-x_1)} dx \\
&= \int_{x_1}^{x_2} \frac{-4(x_3+x_2-x_1-x_0)}{(x_3-x_1)^2(x_2-x_1)^2(x_2-x_0)} dx + \int_{x_2}^{x_3} \frac{-4(x_4+x_3-x_2-x_1)}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)^2} dx \\
&= \frac{-4(x_3+x_2-x_1-x_0)}{(x_3-x_1)^2(x_2-x_1)(x_2-x_0)} + \frac{-4(x_4+x_3-x_2-x_1)}{(x_4-x_2)(x_3-x_2)(x_3-x_1)^2}
\end{aligned}$$

and

$$\begin{aligned}
\int_{x_2}^{x_3} \partial_x^2 b_{2,0}(x) \times \partial_x^2 b_{2,2}(x) dx &= \int_{x_2}^{x_3} \frac{2}{(x_3-x_2)(x_3-x_1)} \frac{2}{(x_4-x_2)(x_3-x_2)} dx \\
&= \int_{x_2}^{x_3} \frac{4}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)} dx \qquad = \frac{4}{(x_4-x_2)(x_3-x_2)(x_3-x_1)}
\end{aligned}$$

E.2. Deg = 2, Vol

$$\begin{aligned}
\int_{x_0}^{x_3} x\|\partial_x^2 b_{2,0}(x)\|^2 dx &= \int_{x_0}^{x_1} x\left(\frac{2}{(x_2-x_0)(x_1-x_0)}\right)^2 dx + \int_{x_1}^{x_2} x\left(\frac{-2(x_3+x_2-x_1-x_0)}{(x_3-x_1)(x_2-x_1)(x_2-x_0)}\right)^2 dx + \int_{x_2}^{x_3} x\left(\frac{2}{(x_3-x_2)(x_3-x_1)}\right)^2 dx \\
&= \int_{x_0}^{x_1} \frac{4x}{(x_2-x_0)^2(x_1-x_0)^2} dx + \int_{x_1}^{x_2} \frac{4(x_3+x_2-x_1-x_0)^2 x}{(x_3-x_1)^2(x_2-x_1)^2(x_2-x_0)^2} dx + \int_{x_2}^{x_3} \frac{4x}{(x_3-x_2)^2(x_3-x_1)^2} dx \\
&= \frac{2(x_1+x_0)}{(x_2-x_0)^2(x_1-x_0)} + \frac{2(x_3+x_2-x_1-x_0)^2(x_2+x_1)}{(x_3-x_1)^2(x_2-x_1)(x_2-x_0)^2} + \frac{2(x_3+x_2)}{(x_3-x_2)(x_3-x_1)^2}
\end{aligned}$$

and

$$\begin{aligned}
\int_{x_1}^{x_3} x\partial_x^2 b_{2,0}(x) \times \partial_x^2 b_{2,1}(x) dx &= \int_{x_1}^{x_2} x\frac{-2(x_3+x_2-x_1-x_0)}{(x_3-x_1)(x_2-x_1)(x_2-x_0)} \frac{2}{(x_3-x_1)(x_2-x_1)} dx + \int_{x_2}^{x_3} x\frac{-2(x_4+x_3-x_2-x_1)}{(x_4-x_2)(x_3-x_2)(x_3-x_1)} \frac{2}{(x_3-x_2)(x_3-x_1)} dx \\
&= \int_{x_1}^{x_2} \frac{-4(x_3+x_2-x_1-x_0)x}{(x_3-x_1)^2(x_2-x_1)^2(x_2-x_0)} dx + \int_{x_2}^{x_3} \frac{-4(x_4+x_3-x_2-x_1)x}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)^2} dx \\
&= \frac{-2(x_3+x_2-x_1-x_0)(x_2+x_1)}{(x_3-x_1)^2(x_2-x_1)(x_2-x_0)} + \frac{-2(x_4+x_3-x_2-x_1)(x_3+x_2)}{(x_4-x_2)(x_3-x_2)(x_3-x_1)^2}
\end{aligned}$$

and

$$\begin{aligned}
\int_{x_2}^{x_3} x\partial_x^2 b_{2,0}(x) \times \partial_x^2 b_{2,2}(x) dx &= \int_{x_2}^{x_3} x\frac{2}{(x_3-x_2)(x_3-x_1)} \frac{2}{(x_4-x_2)(x_3-x_2)} dx \\
&= \int_{x_2}^{x_3} \frac{4x}{(x_4-x_2)(x_3-x_2)^2(x_3-x_1)} dx \\
&= \frac{2(x_3+x_2)}{(x_4-x_2)(x_3-x_2)(x_3-x_1)}
\end{aligned}$$

$$\begin{aligned}
\int_{x_0}^{x_4}\|\partial_x^2 b_{3,0}(x)\|^2 dx &= \int_{x_0}^{x_1}\left(\frac{6(x-x_0)}{(x_3-x_0)(x_2-x_0)(x_1-x_0)}\right)^2 dx+\int_{x_1}^{x_2}\left(\frac{6Ax+2B}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}\right)^2 dx+\int_{x_2}^{x_3}\left(\frac{6A'x+2B'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}\right)^2 dx+\int_{x_3}^{x_4}\left(\frac{6(x_4-x)}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}\right)^2 dx\\
&= \int_{x_0}^{x_1}\frac{36(x-x_0)^2}{(x_3-x_0)^2(x_2-x_0)^2(x_1-x_0)^2}dx+\int_{x_1}^{x_2}\frac{36A^2x^2+24ABx+4B^2}{((x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0))^2}dx+\int_{x_2}^{x_3}\frac{36A'^2x^2+24A'B'x+4B'^2}{((x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0))^2}dx+\int_{x_3}^{x_4}\frac{36(x_4-x)^2}{(x_4-x_3)^2(x_4-x_2)^2(x_4-x_1)^2}dx\\
&= \frac{12(x_1-x_0)}{(x_3-x_0)^2(x_2-x_0)^2}+\frac{\left[12A^2x^3+12ABx^2+4B^2x\right]_{x_1}^{x_2}}{((x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0))^2}+\frac{\left[12A'^2x^3+12A'B'x^2+4B'^2x\right]_{x_2}^{x_3}}{((x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0))^2}+\frac{12(x_4-x_3)}{(x_4-x_2)^2(x_4-x_1)^2}\\
&= \frac{12(x_1-x_0)}{(x_3-x_0)^2(x_2-x_0)^2}\\
&\quad+\frac{12A^2(x_2^2+x_2x_1+x_1^2)+12AB(x_2+x_1)+4B^2}{((x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_0))^2(x_2-x_1)}\\
&\quad+\frac{12A'^2(x_3^2+x_3x_2+x_2^2)+12A'B'(x_3+x_2)+4B'^2}{((x_4-x_2)(x_4-x_1)(x_3-x_1)(x_3-x_0))^2(x_3-x_2)}\\
&\quad+\frac{12(x_4-x_3)}{(x_4-x_2)^2(x_4-x_1)^2}
\end{aligned}$$

And

$$\begin{aligned}
\int_{x_1}^{x_4}\partial_x^2 b_{3,0}(x)\times\partial_x^2 b_{3,1}(x)dx &= \int_{x_1}^{x_2}\frac{6Ax+2B}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}\frac{6(x-x_1)}{(x_4-x_1)(x_3-x_1)(x_2-x_1)}dx+\int_{x_2}^{x_3}\frac{6A'x+2B'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}\frac{6A(1)x+2B(1)}{(x_5-x_2)(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)}dx+\int_{x_3}^{x_4}\frac{6(x_4-x}{(x_4-x_3)(x_4-x_2}\\
&= \int_{x_1}^{x_2}12\frac{3Ax^2-(3Ax_1-B)x-Bx_1}{(x_4-x_1)^2(x_3-x_1)^2(x_3-x_0)(x_2-x_1)^2(x_2-x_0)}dx\\
&\quad+\int_{x_2}^{x_3}4\frac{9A'A(1)x^2+3(A(1)B'+A'B(1))x+B'B(1)}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)^2(x_3-x_2)^2(x_3-x_1)^2(x_3-x_0)}dx\\
&\quad+\int_{x_3}^{x_4}12\frac{-3A'(1)x^2+(3A'(1)x_4-B'(1))x+B'(1)x_4}{(x_5-x_3)(x_5-x_2)(x_4-x_3)^2(x_4-x_2)^2(x_4-x_1)^2}dx\\
&= 6\frac{2A(x_2^2+x_2x_1+x_1^2)-(3Ax_1-B)(x_2+x_1)-2Bx_1}{(x_4-x_1)^2(x_3-x_1)^2(x_3-x_0)(x_2-x_1)(x_2-x_0)}+2\frac{6A'A(1)(x_3^2+x_3x_2+x_2^2)+3(A(1)B'+A'B(1))(x_3+x_2)+2B'B(1)}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)^2(x_3-x_2)(x_3-x_1)^2(x_3-x_0)}-6\frac{2A'(1)(x_4^2+x_4x_3+x_3^2)-(3A'(1)x_4-B'(1))(x_4+x_3)-2B'(1)x_4}{(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)^2(x_4-x_1)^2}\\
&= 6\frac{A(2x_2^2-x_2x_1-x_1^2)+B(x_2-x_1)}{(x_4-x_1)^2(x_3-x_1)^2(x_3-x_0)(x_2-x_1)(x_2-x_0)}+2\frac{6A'A(1)(x_3^2+x_3x_2+x_2^2)+3(A(1)B'+A'B(1))(x_3+x_2)+2B'B(1)}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)^2(x_3-x_2)(x_3-x_1)^2(x_3-x_0)}-6\frac{-A'(1)(x_4^2+x_4x_3-2x_3^2)-B'(1)(x_4-x_3)}{(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)^2(x_4-x_1)^2}\\
&= 6\frac{A(2x_2+x_1)+B}{(x_4-x_1)^2(x_3-x_1)^2(x_3-x_0)(x_2-x_0)}+2\frac{6A'A(1)(x_3^2+x_3x_2+x_2^2)+3(A(1)B'+A'B(1))(x_3+x_2)+2B'B(1)}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)^2(x_3-x_2)(x_3-x_1)^2(x_3-x_0)}+6\frac{A'(1)(x_4+2x_3)+B'(1)}{(x_5-x_3)(x_5-x_2)(x_4-x_2)^2(x_4-x_1)^2}
\end{aligned}$$

and

$$\begin{aligned}
\int_{x_2}^{x_4}\partial_x^2 b_{3,0}(x)\times\partial_x^2 b_{3,2}(x)dx &= \int_{x_2}^{x_3}\frac{6A'x+2B'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}\frac{6(x-x_2)}{(x_5-x_2)(x_4-x_2)(x_3-x_2)}dx+\int_{x_3}^{x_4}\frac{6(x_4-x)}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}\frac{6A(2)x+2B(2)}{(x_6-x_3)(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)}dx\\
&= \int_{x_2}^{x_3}\frac{36A'x^2+12(B'-3A'x_2)x-12B'x_2}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)(x_3-x_2)^2(x_3-x_1)(x_3-x_0)}dx+\int_{x_3}^{x_4}\frac{-36A(2)x^2+12(3A(2)x_4-B(2))x+12B(2)x_4}{(x_6-x_3)(x_5-x_3)(x_5-x_2)(x_4-x_3)^2(x_4-x_2)^2(x_4-x_1)}dx\\
&= 6\frac{2A'(x_3^2+x_3x_2+x_2^2)+(B'-3A'x_2)(x_3+x_2)-2B'x_2}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}-6\frac{2A(2)(x_4^2+x_4x_3+x_3^2)+(B(2)-3A(2)x_4)(x_4+x_3)-2B(2)x_4}{(x_6-x_3)(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)^2(x_4-x_1)}\\
&= 6\frac{A'(2x_3^2-x_3x_2-x_2^2)+B'(x_3-x_2)}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}+6\frac{A(2)(x_4^2+x_4x_3-2x_3^2)+B(2)(x_4-x_3)}{(x_6-x_3)(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)^2(x_4-x_1)}\\
&= 6\frac{A'(2x_3+x_2)+B'}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)(x_3-x_1)(x_3-x_0)}+6\frac{A(2)(x_4+2x_3)+B(2)}{(x_6-x_3)(x_5-x_3)(x_5-x_2)(x_4-x_2)^2(x_4-x_1)}
\end{aligned}$$

and

$$\begin{aligned}
\int_{x_3}^{x_4}\partial_x^2 b_{3,0}(x)\times\partial_x^2 b_{3,3}(x)dx &= \int_{x_3}^{x_4}\frac{6(x_4-x)}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}\frac{6(x-x_3)}{(x_6-x_3)(x_5-x_3)(x_4-x_3)}dx\\
&= \int_{x_3}^{x_4}36\frac{-x^2+(x_4+x_3)x-x_4x_3}{(x_6-x_3)(x_5-x_3)(x_4-x_3)^2(x_4-x_2)(x_4-x_1)}dx\\
&= 6\frac{-2(x_4^2+x_4x_3+x_3^2)+3(x_4+x_3)^2-6x_4x_3}{(x_6-x_3)(x_5-x_3)(x_4-x_3)(x_4-x_2)(x_4-x_1)}\\
&= 6\frac{x_4-x_3}{(x_6-x_3)(x_5-x_3)(x_4-x_2)(x_4-x_1)}
\end{aligned}$$

$$\begin{aligned}
\int_{x_0}^{x_4} x\|\partial_x^2 b_{3,0}(x)\|^2 dx &= \int_{x_0}^{x_1} x\left(\frac{6(x-x_0)}{(x_3-x_0)(x_2-x_0)(x_1-x_0)}\right)^2 dx + \int_{x_1}^{x_2} x\left(\frac{6Ax+2B}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}\right)^2 dx + \int_{x_2}^{x_3} x\left(\frac{6A'x+2B'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}\right)^2 dx + \int_{x_3}^{x_4} x\left(\frac{6(x_4-x)}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}\right)^2 dx \\
&= \int_{x_0}^{x_1} \frac{36x(x-x_0)^2}{(x_3-x_0)^2(x_2-x_0)^2(x_1-x_0)^2} dx + \int_{x_1}^{x_2} \frac{36A^2x^3+24ABx^2+4B^2x}{((x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0))^2} dx + \int_{x_2}^{x_3} \frac{36A'^2x^3+24A'B'x^2+4B'^2x}{((x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0))^2} dx + \int_{x_3}^{x_4} \frac{36x(x_4-x)^2}{(x_4-x_3)^2(x_4-x_2)^2(x_4-x_1)^2} dx \\
&= 3\frac{3(x_1^3+x_1^2x_0+x_1x_0^2+x_0^3)-8x_0(x_1^2+x_1x_0+x_0^2)+6x_0^2(x_1+x_0)}{(x_3-x_0)^2(x_2-x_0)^2(x_1-x_0)} + \frac{9A^2(x_2^3+x_2^2x_1+x_2x_1^2+x_1^3)+8AB(x_2^2+x_2x_1+x_1^2)+2B^2(x_2+x_1)}{(x_4-x_1)^2(x_3-x_1)^2(x_3-x_0)^2(x_2-x_1)(x_2-x_0)^2} + \frac{9A'^2(x_3^3+x_3^2x_2+x_3x_2^2+x_2^3)+8A'B'(x_3^2+x_3x_2+x_2^2)+2B'^2(x_3+x_2)}{(x_4-x_2)^2(x_4-x_1)^2(x_3-x_2)(x_3-x_1)^2(x_3-x_0)^2} + 3\frac{3(x_4^3+x \ldots}{} \\
&= 3\frac{(3x_1+x_0)(x_1-x_0)}{(x_3-x_0)^2(x_2-x_0)^2} + \frac{9A^2(x_2^3+x_2^2x_1+x_2x_1^2+x_1^3)+8AB(x_2^2+x_2x_1+x_1^2)+2B^2(x_2+x_1)}{(x_4-x_1)^2(x_3-x_1)^2(x_3-x_0)^2(x_2-x_1)(x_2-x_0)^2} + \frac{9A'^2(x_3^3+x_3^2x_2+x_3x_2^2+x_2^3)+8A'B'(x_3^2+x_3x_2+x_2^2)+2B'^2(x_3+x_2)}{(x_4-x_2)^2(x_4-x_1)^2(x_3-x_2)(x_3-x_1)^2(x_3-x_0)^2} + 3\frac{(x_4+3x_3)(x_4-x_3)}{(x_4-x_2)^2(x_4-x_1)^2}
\end{aligned}$$

And

$$\begin{aligned}
\int_{x_1}^{x_4} x\partial_x^2 b_{3,0}(x)\times\partial_x^2 b_{3,1}(x)dx &= \int_{x_1}^{x_2} x\frac{6Ax+2B}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}\,\frac{6(x-x_1)}{(x_4-x_1)(x_3-x_1)(x_2-x_1)} dx + \int_{x_2}^{x_3} x\frac{6A'x+2B'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}\,\frac{6A(1)x+2B(1)}{(x_5-x_2)(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)} dx + \int_{x_3}^{x_4} x\frac{\ldots}{(x_4-x_3)\ldots} \\
&= \int_{x_1}^{x_2} 12\frac{3Ax^3-(3Ax_1-B)x^2-Bx_1x}{(x_4-x_1)^2(x_3-x_1)^2(x_3-x_0)(x_2-x_1)^2(x_2-x_0)} dx \\
&\quad + \int_{x_2}^{x_3} 4\frac{9A'A(1)x^3+3(A(1)B'+A'B(1))x^2+B'B(1)x}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)^2(x_3-x_2)^2(x_3-x_1)^2(x_3-x_0)} dx \\
&\quad + \int_{x_3}^{x_4} 12\frac{-3A'(1)x^3+(3A'(1)x_4-B'(1))x^2+B'(1)x_4x}{(x_5-x_3)(x_5-x_2)(x_4-x_3)^2(x_4-x_2)^2(x_4-x_1)^2} dx \\
&= \frac{3A(3x_2^3-x_2^2x_1-x_2x_1^2-x_1^3)+2B(2x_2^2-x_2x_1-x_1^2)}{(x_4-x_1)^2(x_3-x_1)^2(x_3-x_0)(x_2-x_1)(x_2-x_0)} \\
&\quad + \frac{9A'A(1)(x_3^3+x_3^2x_2+x_3x_2^2+x_2^3)+4(A(1)B'+A'B(1))(x_3^2+x_3x_2+x_2^2)+2B'B(1)(x_3+x_2)}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)^2(x_3-x_2)(x_3-x_1)^2(x_3-x_0)} \\
&\quad + \frac{3A'(1)(x_4^3+x_4^2x_3+x_4x_3^2-3x_3^3)+2B'(1)(x_4^2+x_4x_3-2x_3^2)}{(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)^2(x_4-x_1)^2} \\
&= \frac{3A(3x_2^2+2x_2x_1+x_1^2)+2B(2x_2+x_1)}{(x_4-x_1)^2(x_3-x_1)^2(x_3-x_0)(x_2-x_0)} \\
&\quad + \frac{9A'A(1)(x_3^3+x_3^2x_2+x_3x_2^2+x_2^3)+4(A(1)B'+A'B(1))(x_3^2+x_3x_2+x_2^2)+2B'B(1)(x_3+x_2)}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)^2(x_3-x_2)(x_3-x_1)^2(x_3-x_0)} \\
&\quad + \frac{3A'(1)(x_4^2+2x_4x_3+3x_3^2)+2B'(1)(x_4+2x_3)}{(x_5-x_3)(x_5-x_2)(x_4-x_2)^2(x_4-x_1)^2}
\end{aligned}$$

and

$$\begin{aligned}
\int_{x_2}^{x_4} x\partial_x^2 b_{3,0}(x)\times\partial_x^2 b_{3,2}(x)dx &= \int_{x_2}^{x_3} x\frac{6A'x+2B'}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}\,\frac{6(x-x_2)}{(x_5-x_2)(x_4-x_2)(x_3-x_2)} dx + \int_{x_3}^{x_4} x\frac{6(x_4-x)}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}\,\frac{6A(2)x+2B(2)}{(x_6-x_3)(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)} dx \\
&= \int_{x_2}^{x_3} \frac{36A'x^3+12(B'-3A'x_2)x^2-12B'x_2x}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)(x_3-x_2)^2(x_3-x_1)(x_3-x_0)} dx + \int_{x_3}^{x_4} \frac{-36A(2)x^3+12(3A(2)x_4-B(2))x^2+12B(2)x_4x}{(x_6-x_3)(x_5-x_3)(x_5-x_2)(x_4-x_3)^2(x_4-x_2)^2(x_4-x_1)} dx \\
&= \frac{3A'(3x_3^3-x_3^2x_2-x_3x_2^2-x_2^3)+2B'(2x_3^2-x_3x_2-x_2^2)}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} + \frac{3A(2)(x_4^3+x_4^2x_3+x_4x_3^2-3x_3^3)+2B(2)(x_4^2+x_4x_3-2x_3^2)}{(x_6-x_3)(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)^2(x_4-x_1)} \\
&= \frac{3A'(3x_3^2+2x_3x_2+x_2^2)+2B'(2x_3+x_2)}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)(x_3-x_1)(x_3-x_0)} + \frac{3A(2)(x_4^2+2x_4x_3+3x_3^2)+2B(2)(x_4+2x_3)}{(x_6-x_3)(x_5-x_3)(x_5-x_2)(x_4-x_2)^2(x_4-x_1)}
\end{aligned}$$

and

$$\begin{aligned}
\int_{x_3}^{x_4} x\partial_x^2 b_{3,0}(x)\times\partial_x^2 b_{3,3}(x)dx &= \int_{x_3}^{x_4} x\frac{6(x_4-x)}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}\,\frac{6(x-x_3)}{(x_6-x_3)(x_5-x_3)(x_4-x_3)} dx \\
&= \int_{x_3}^{x_4} 36\frac{-x^3+(x_4+x_3)x^2-x_4x_3x}{(x_6-x_3)(x_5-x_3)(x_4-x_3)^2(x_4-x_2)(x_4-x_1)} dx \\
&= 3\frac{x_4^3-x_4^2x_3-x_4x_3^2+x_3^3}{(x_6-x_3)(x_5-x_3)(x_4-x_3)(x_4-x_2)(x_4-x_1)} \\
&= 3\frac{x_4^2-x_3^2}{(x_6-x_3)(x_5-x_3)(x_4-x_2)(x_4-x_1)}
\end{aligned}$$

F. D3N2 - EXACT FORMULATIONS

F.1. DEG = 3, SURF

$$
\begin{aligned}
\int_{x_0}^{x_4} \|\partial_x^3 b_{3,0}(x)\|^2 dx &= \int_{x_0}^{x_1} \left(\frac{6}{(x_3-x_0)(x_2-x_0)(x_1-x_0)}\right)^2 dx + \int_{x_1}^{x_2} \left(\frac{6A}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}\right)^2 dx + \int_{x_2}^{x_3} \left(\frac{6A^{\prime}}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}\right)^2 dx + \int_{x_3}^{x_4} \left(\frac{-6}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}\right)^2 dx \\
&= \frac{36}{(x_3-x_0)^2(x_2-x_0)^2(x_1-x_0)} + \frac{36A^2}{(x_4-x_1)^2(x_3-x_1)^2(x_3-x_0)^2(x_2-x_1)(x_2-x_0)^2} + \frac{36A^{\prime 2}}{(x_4-x_2)^2(x_4-x_1)^2(x_3-x_2)(x_3-x_1)^2(x_3-x_0)^2} + \frac{36}{(x_4-x_3)(x_4-x_2)^2(x_4-x_1)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\int_{x_1}^{x_4} \partial_x^3 b_{3,0}(x) \times \partial_x^3 b_{3,1}(x) dx &= \int_{x_1}^{x_2} \frac{6A}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)} \frac{6}{(x_4-x_1)(x_3-x_1)(x_2-x_1)} dx \\
&\quad \int_{x_2}^{x_3} \frac{6A^{\prime}}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} \frac{6A(1)}{(x_5-x_2)(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)} dx \\
&\quad \int_{x_3}^{x_4} \frac{-6}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} \frac{6A^{\prime}(1)}{(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)(x_4-x_1)} dx \\
&= \int_{x_1}^{x_2} \frac{36A}{(x_4-x_1)^2(x_3-x_1)^2(x_3-x_0)(x_2-x_1)^2(x_2-x_0)} dx + \int_{x_2}^{x_3} \frac{36A^{\prime}A(1)}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)^2(x_3-x_2)^2(x_3-x_1)^2(x_3-x_0)} dx + \int_{x_3}^{x_4} \frac{-36A^{\prime}(1)}{(x_5-x_3)(x_5-x_2)(x_4-x_3)^2(x_4-x_2)^2(x_4-x_1)^2} dx \\
&= \frac{36A}{(x_4-x_1)^2(x_3-x_1)^2(x_3-x_0)(x_2-x_1)(x_2-x_0)} + \frac{36A^{\prime}A(1)}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)^2(x_3-x_2)(x_3-x_1)^2(x_3-x_0)} + \frac{-36A^{\prime}(1)}{(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)^2(x_4-x_1)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{x_2}^{x_4} \partial_x b_{3,0}(x) \times \partial_x b_{3,2} dx &= \int_{x_2}^{x_3} \frac{6A^{\prime}}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} \frac{6}{(x_5-x_2)(x_4-x_2)(x_3-x_2)} dx + \int_{x_3}^{x_4} \frac{-6}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} \frac{6A(2)}{(x_6-x_3)(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)} \\
&= \frac{36A^{\prime}}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)(x_3-x_2)^2(x_3-x_1)(x_3-x_0)} + \frac{-36A(2)}{(x_6-x_3)(x_5-x_3)(x_5-x_2)(x_4-x_3)^2(x_4-x_2)^2(x_4-x_1)}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{x_3}^{x_4} \partial_x^3 b_{3,0}(x) \times \partial_x^3 b_{3,3}(x) dx &= \int_{x_3}^{x_4} \frac{-6}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} \frac{6}{(x_6-x_3)(x_5-x_3)(x_4-x_3)} dx \\
&= \frac{-36}{(x_6-x_3)(x_5-x_3)(x_4-x_3)^2(x_4-x_2)(x_4-x_1)}
\end{aligned}
$$

F.2. DEG = 3, VOL

$$
\begin{aligned}
\int_{x_0}^{x_4} x\|\partial_x^3 b_{3,0}(x)\|^2 dx &= \int_{x_0}^{x_1} x \left(\frac{6}{(x_3-x_0)(x_2-x_0)(x_1-x_0)}\right)^2 dx + \int_{x_1}^{x_2} x \left(\frac{6A}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)}\right)^2 dx + \int_{x_2}^{x_3} x \left(\frac{6A^{\prime}}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)}\right)^2 dx + \int_{x_3}^{x_4} x \left(\frac{-6}{(x_4-x_3)(x_4-x_2)(x_4-x_1)}\right)^2 dx \\
&= \frac{18(x_1+x_0)}{(x_3-x_0)^2(x_2-x_0)^2(x_1-x_0)} + \frac{18A^2(x_2+x_1)}{(x_4-x_1)^2(x_3-x_1)^2(x_3-x_0)^2(x_2-x_1)(x_2-x_0)^2} + \frac{18A^{\prime 2}(x_3+x_2)}{(x_4-x_2)^2(x_4-x_1)^2(x_3-x_2)(x_3-x_1)^2(x_3-x_0)^2} + \frac{18(x_4+x_3)}{(x_4-x_3)(x_4-x_2)^2(x_4-x_1)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\int_{x_1}^{x_4} x\partial_x^3 b_{3,0}(x) \times \partial_x^3 b_{3,1}(x) dx &= \int_{x_1}^{x_2} x \frac{6A}{(x_4-x_1)(x_3-x_1)(x_3-x_0)(x_2-x_1)(x_2-x_0)} \frac{6}{(x_4-x_1)(x_3-x_1)(x_2-x_1)} dx \\
&\quad \int_{x_2}^{x_3} x \frac{6A^{\prime}}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} \frac{6A(1)}{(x_5-x_2)(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)} dx \\
&\quad \int_{x_3}^{x_4} x \frac{-6}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} \frac{6A^{\prime}(1)}{(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)(x_4-x_1)} dx \\
&= \int_{x_1}^{x_2} \frac{36Ax}{(x_4-x_1)^2(x_3-x_1)^2(x_3-x_0)(x_2-x_1)^2(x_2-x_0)} dx + \int_{x_2}^{x_3} \frac{36A^{\prime}A(1)x}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)^2(x_3-x_2)^2(x_3-x_1)^2(x_3-x_0)} dx + \int_{x_3}^{x_4} \frac{-36A^{\prime}(1)x}{(x_5-x_3)(x_5-x_2)(x_4-x_3)^2(x_4-x_2)^2(x_4-x_1)^2} dx \\
&= \frac{18A(x_2+x_1)}{(x_4-x_1)^2(x_3-x_1)^2(x_3-x_0)(x_2-x_1)(x_2-x_0)} + \frac{18A^{\prime}A(1(x_3+x_2)}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)^2(x_3-x_2)(x_3-x_1)^2(x_3-x_0)} + \frac{-18A^{\prime}(1)(x_4+x_3)}{(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)^2(x_4-x_1)^2}
\end{aligned}
$$

and

$$\begin{aligned}\int_{x_2}^{x_4} x\partial_x b_{3,0}(x) \times \partial_x b_{3,2} dx &= \int_{x_2}^{x_3} x \frac{6A^{/}}{(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_3-x_0)} \frac{6}{(x_5-x_2)(x_4-x_2)(x_3-x_2)} dx + \int_{x_3}^{x_4} x \frac{-6}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} \frac{6A(2)}{(x_6-x_3)(x_5-x_3)(x_5-x_2)(x_4-x_3)(x_4-x_2)} \\ &= \frac{18A^{/}(x_3+x_2)}{(x_5-x_2)(x_4-x_2)^2(x_4-x_1)(x_3-x_2)^2(x_3-x_1)(x_3-x_0)} + \frac{-18A(2)(x_4+x_3)}{(x_6-x_3)(x_5-x_3)(x_5-x_2)(x_4-x_3)^2(x_4-x_2)^2(x_4-x_1)}\end{aligned}$$

and

$$\begin{aligned}\int_{x_3}^{x_4} \partial_x^3 b_{3,0}(x) \times \partial_x^3 b_{3,3}(x) dx &= \int_{x_3}^{x_4} x \frac{-6}{(x_4-x_3)(x_4-x_2)(x_4-x_1)} \frac{6}{(x_6-x_3)(x_5-x_3)(x_4-x_3)} dx \\ &= \frac{-18(x_4+x_3)}{(x_6-x_3)(x_5-x_3)(x_4-x_3)^2(x_4-x_2)(x_4-x_1)}\end{aligned}$$

$$\begin{aligned}
A^2 &= (-((x_3-x_1)(x_4-x_1)+(x_2-x_0)(x_4-x_1)+(x_3-x_0)(x_2-x_0)))^2 \\
&= ((x_3-x_1)(x_4-x_1)+(x_2-x_0)(x_4-x_1)+(x_3-x_0)(x_2-x_0))^2 \\
&= (x_3-x_1)^2(x_4-x_1)^2+(x_2-x_0)^2(x_4-x_1)^2+(x_3-x_0)^2(x_2-x_0)^2+2(x_3-x_1)(x_4-x_1)(x_3-x_0)(x_2-x_0)+2(x_3-x_1)(x_4-x_1)(x_3-x_0)(x_2-x_0)+2(x_2-x_0)(x_4-x_1)(x_3-x_0)(x_2 \\
&=
\end{aligned}$$