

Find reflexion points on a a 3d surface

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1 Notations

- \underline{A} is the point of observation
- \underline{B} is the point observed
- \underline{C} is the local coordinate center of the 3d surface
- \underline{D} is the projection of \underline{B} on the 3d surface as seen from \underline{A}
- \underline{e} is the unit vector such that $\underline{AB} = \|\underline{AB}\| \underline{e} = l \underline{e}$
- \underline{E} is a point on (AB) parameterized by $\underline{AE} = kl \underline{e}$
- $\underline{n}(D)$ is the normal vector of the 3d surface at point \underline{D}

Point \underline{D} on the 3d surface is the reflexion point connecting A and B . It means that $\underline{n}(D)$ is in the same plane as (A, D, B) .

Point \underline{E} on is the projection, on line (A, B) of point \underline{D} along \underline{n} .

The idea is to look for \underline{E} , which is parameterized by k , and then derive \underline{D} from \underline{E} .

Hence both $\underline{n}(\underline{D})$ and $d_E = \|\underline{ED}\|$ are parametrized by k : $\underline{n}(k)$ and $d_E(k)$.

2 General equations

2.1 co-planarity

The point \underline{D} on the 3d surface is such that $\underline{n}(D)$ is in the same plane as (A, D, B) , which is written:

$$\begin{aligned} (\underline{DA} \wedge \underline{n}) \cdot (\underline{DB} \wedge \underline{n}) &= 0 \\ \Leftrightarrow (\underline{DA} \wedge \underline{n}) \wedge (\underline{DB} \wedge \underline{n}) &= 0 \\ \Leftrightarrow (\underline{EA} \wedge \underline{n}) \wedge (\underline{EB} \wedge \underline{n}) &= 0 \\ ((-kl)\underline{e} \wedge \underline{n}) \wedge ((1-k)l\underline{e} \wedge \underline{n}) &= 0 \\ k(1-k)l^2(\underline{e} \wedge \underline{n}) \wedge (\underline{e} \wedge \underline{n}) &= 0 \end{aligned} \tag{1}$$

Which is true by construction of E .

2.2 equal angles

Since it is a specular reflexion, angles (A, D, E) and (B, D, E) are equal, which means \underline{E} is necessarily standing on the bisector of angle (A, D, B) .

As such, the distance between \underline{E} and line (A, D) is equal to the distance between \underline{E} and line (B, D) , which is written:

$$d_{E,(A,D)} = \frac{\|\underline{AD} \wedge \underline{AE}\|}{\|\underline{AD}\|} = \frac{\|\underline{BD} \wedge \underline{BE}\|}{\|\underline{BD}\|} = d_{E,(B,D)}$$

Knowing that:

$$\begin{aligned} \|\underline{AD}\|^2 &= \|\underline{AE} + \underline{ED}\|^2 \\ &= k^2l^2 + d_E^2 + 2\underline{AE} \cdot \underline{ED} \\ &= k^2l^2 + d_E^2 + 2kl\underline{e} \cdot (-d_E)\underline{n} \\ &= k^2l^2 + d_E(k)^2 - 2kld_E(k)\underline{e} \cdot \underline{n}(k) \end{aligned}$$

Similarly:

$$\begin{aligned} \|\underline{BD}\|^2 &= \|\underline{BE} + \underline{ED}\|^2 \\ &= (1-k)^2l^2 + d_E^2 + 2\underline{BE} \cdot \underline{ED} \\ &= (1-k)^2l^2 + d_E^2 + 2(1-k)l(-\underline{e}) \cdot (-d_E)\underline{n} \\ &= (1-k)^2l^2 + d_E(k)^2 + 2(1-k)ld_E(k)\underline{e} \cdot \underline{n}(k) \end{aligned}$$

And the cross-products:

$$\begin{aligned} \|\underline{AD} \wedge \underline{AE}\|^2 &= \|\underline{ED} \wedge \underline{AE}\|^2 \\ &= \|(-d_E)\underline{n} \wedge kl\underline{e}\|^2 \\ &= k^2d_E(k)^2l^2\|\underline{n}(k) \wedge \underline{e}\|^2 \end{aligned}$$

And:

$$\begin{aligned} \|\underline{BD} \wedge \underline{BE}\|^2 &= \|\underline{ED} \wedge \underline{BE}\|^2 \\ &= \|(-d_E)\underline{n} \wedge (1-k)l(-\underline{e})\|^2 \\ &= (1-k)^2d_E(k)^2l^2\|\underline{n}(k) \wedge \underline{e}\|^2 \end{aligned}$$

So in the end:

$$\begin{aligned} d_{E,(A,D)}^2 &= d_{E,(B,D)}^2 \\ \Leftrightarrow \|\underline{AD} \wedge \underline{AE}\|^2 \|\underline{BD}\|^2 &= \|\underline{BD} \wedge \underline{BE}\|^2 \|\underline{AD}\|^2 \\ \Leftrightarrow k^2d_E^2l^2\|\underline{n} \wedge \underline{e}\|^2 [(1-k)^2l^2 + d_E^2 + 2(1-k)ld_E\underline{e} \cdot \underline{n}] &= (1-k)^2d_E^2l^2\|\underline{n} \wedge \underline{e}\|^2 [k^2l^2 + d_E^2 - 2kld_E\underline{e} \cdot \underline{n}] \end{aligned}$$

So assuming that:

$$\begin{cases} \|\underline{n}(k) \wedge \underline{e}\| \neq 0 \\ l \neq 0 \\ d_E(k) \neq 0 \end{cases}$$

So in the end:

$$\begin{aligned} d_{E,(A,D)}^2 &= d_{E,(B,D)}^2 \\ \Leftrightarrow k^2[(1-k)^2l^2 + d_E^2 + 2(1-k)ld_E\underline{e} \cdot \underline{n}] &= (1-k)^2[k^2l^2 + d_E^2 - 2kld_E\underline{e} \cdot \underline{n}] \\ \Leftrightarrow k^2[d_E^2 + 2(1-k)ld_E\underline{e} \cdot \underline{n}] &= (1-k)^2[d_E^2 - 2kld_E\underline{e} \cdot \underline{n}] \\ \Leftrightarrow k^2d_E^2 + 2k^2(1-k)ld_E\underline{e} \cdot \underline{n} &= (1-k)^2d_E^2 - 2(1-k)^2kld_E\underline{e} \cdot \underline{n} \\ \Leftrightarrow 2kd_E^2 - d_E^2 + 2k(1-k)ld_E\underline{e} \cdot \underline{n}(k+1-k) &= 0 \\ \Leftrightarrow (2k-1)d_E + 2k(1-k)l\underline{e} \cdot \underline{n} &= 0 \end{aligned}$$

Hence:

$$(2) \Leftrightarrow (2k-1)d_E(k) + 2k(1-k)l\underline{e} \cdot \underline{n}(k) = 0$$

2.3 equal angles 2

Deriving with a different method to double-check the formula. This time we use the scalar product:

$$(\underline{DA} \cdot \underline{n})^2 \|DB\|^2 = (\underline{DB} \cdot \underline{n})^2 \|DA\|^2$$

With:

$$\begin{cases} \underline{DA} \cdot \underline{n} = d - kl \underline{e} \cdot \underline{n} \\ \underline{DB} \cdot \underline{n} = d + (1-k)l \underline{e} \cdot \underline{n} \\ \|\underline{DA}\|^2 = d^2 + k^2 l^2 - 2dkl \underline{e} \cdot \underline{n} \\ \|\underline{DB}\|^2 = d^2 + (1-k)^2 l^2 + 2d(1-k)l \underline{e} \cdot \underline{n} \end{cases}$$

So:

$$\begin{aligned} & (\underline{DA} \cdot \underline{n})^2 \|DB\|^2 = (\underline{DB} \cdot \underline{n})^2 \|DA\|^2 \\ \Leftrightarrow & (d - kl \underline{e} \cdot \underline{n})^2 (d^2 + (1-k)^2 l^2 + 2d(1-k)l \underline{e} \cdot \underline{n}) = (d + kl \underline{e} \cdot \underline{n} + l \underline{e} \cdot \underline{n})^2 (d^2 + k^2 l^2 - 2dkl \underline{e} \cdot \underline{n}) \\ \Leftrightarrow & (d - kl \underline{e} \cdot \underline{n})^2 ((1-k)^2 l^2 + 2d(1-k)l \underline{e} \cdot \underline{n} - k^2 l^2 + 2dkl \underline{e} \cdot \underline{n}) = (l^2 \underline{e} \cdot \underline{n}^2 + 2l \underline{e} \cdot \underline{n} (d - kl \underline{e} \cdot \underline{n})) (d^2 + k^2 l^2 - 2dkl \underline{e} \cdot \underline{n}) \\ \Leftrightarrow & (d - kl \underline{e} \cdot \underline{n})^2 (l^2 - 2kl^2 + 2dl \underline{e} \cdot \underline{n}) = l \underline{e} \cdot \underline{n} (l \underline{e} \cdot \underline{n} + 2(d - kl \underline{e} \cdot \underline{n})) (d^2 + k^2 l^2 - 2dkl \underline{e} \cdot \underline{n}) \\ \Leftrightarrow & (d - kl \underline{e} \cdot \underline{n}) \left[(d - kl \underline{e} \cdot \underline{n}) (l^2 - 2kl^2 + 2dl \underline{e} \cdot \underline{n}) - 2l \underline{e} \cdot \underline{n} (d^2 + k^2 l^2 - 2dkl \underline{e} \cdot \underline{n}) \right] = l^2 \underline{e} \cdot \underline{n}^2 (d^2 + k^2 l^2 - 2dkl \underline{e} \cdot \underline{n}) \\ \Leftrightarrow & (d - kl \underline{e} \cdot \underline{n}) \left[(d - kl \underline{e} \cdot \underline{n}) (l^2 - 2kl^2 + 2dl \underline{e} \cdot \underline{n}) - 2l \underline{e} \cdot \underline{n} (d^2 + k^2 l^2 - 2dkl \underline{e} \cdot \underline{n}) \right] = l^2 \underline{e} \cdot \underline{n}^2 (d^2 + k^2 l^2 - 2dkl \underline{e} \cdot \underline{n}) \\ \Leftrightarrow & (d - kl \underline{e} \cdot \underline{n}) \left[d(l^2 - 2kl^2) - kl \underline{e} \cdot \underline{n} (l^2 - 2kl^2 + 2dl \underline{e} \cdot \underline{n}) - 2k^2 l^3 \underline{e} \cdot \underline{n} + 4dkl^2 \underline{e} \cdot \underline{n}^2 \right] = l^2 \underline{e} \cdot \underline{n}^2 (d^2 + k^2 l^2 - 2dkl \underline{e} \cdot \underline{n}) \\ \Leftrightarrow & (d - kl \underline{e} \cdot \underline{n}) \left[d(l^2 - 2kl^2) - kl \underline{e} \cdot \underline{n} (l^2 + 2dl \underline{e} \cdot \underline{n}) + 4dkl^2 \underline{e} \cdot \underline{n}^2 \right] = l^2 \underline{e} \cdot \underline{n}^2 (d^2 + k^2 l^2 - 2dkl \underline{e} \cdot \underline{n}) \\ \Leftrightarrow & (d - kl \underline{e} \cdot \underline{n}) \left[d(l^2 - 2kl^2) - kl^3 \underline{e} \cdot \underline{n} + 2dkl^2 \underline{e} \cdot \underline{n}^2 \right] = l^2 \underline{e} \cdot \underline{n}^2 (d^2 + k^2 l^2 - 2dkl \underline{e} \cdot \underline{n}) \\ \Leftrightarrow & (d - kl \underline{e} \cdot \underline{n}) \left[d(1-2k) - kl \underline{e} \cdot \underline{n} + 2dk \underline{e} \cdot \underline{n}^2 \right] = \underline{e} \cdot \underline{n}^2 (d^2 + k^2 l^2 - 2dkl \underline{e} \cdot \underline{n}) \\ \Leftrightarrow & d^2(1-2k) - kdl \underline{e} \cdot \underline{n} + 2d^2 k \underline{e} \cdot \underline{n}^2 - kdl(1-2k) \underline{e} \cdot \underline{n} + k^2 l^2 \underline{e} \cdot \underline{n}^2 - 2dk^2 l \underline{e} \cdot \underline{n}^3 = \underline{e} \cdot \underline{n}^2 (d^2 + k^2 l^2 - 2dkl \underline{e} \cdot \underline{n}) \\ \Leftrightarrow & k^2 (2dl \underline{e} \cdot \underline{n} + l^2 \underline{e} \cdot \underline{n}^2 - 2dl \underline{e} \cdot \underline{n}^3 - l^2 \underline{e} \cdot \underline{n}^2) + k(-2d^2 - dl \underline{e} \cdot \underline{n} + 2d^2 \underline{e} \cdot \underline{n}^2 - dl \underline{e} \cdot \underline{n} + 2dl \underline{e} \cdot \underline{n}^3) + d^2 - d^2 \underline{e} \cdot \underline{n}^2 = 0 \\ \Leftrightarrow & k^2 2dl \underline{e} \cdot \underline{n} (1 - \underline{e} \cdot \underline{n}^2) + k 2d(-d - l \underline{e} \cdot \underline{n} + d \underline{e} \cdot \underline{n}^2 + l \underline{e} \cdot \underline{n}^3) + d^2 (1 - \underline{e} \cdot \underline{n}^2) = 0 \\ \Leftrightarrow & k^2 2l \underline{e} \cdot \underline{n} (1 - \underline{e} \cdot \underline{n}^2) - k 2(d - d \underline{e} \cdot \underline{n}^2 + l \underline{e} \cdot \underline{n} - l \underline{e} \cdot \underline{n}^3) + d(1 - \underline{e} \cdot \underline{n}^2) = 0 \\ \Leftrightarrow & k^2 2l \underline{e} \cdot \underline{n} (1 - \underline{e} \cdot \underline{n}^2) - k 2(d + l \underline{e} \cdot \underline{n}) (1 - \underline{e} \cdot \underline{n}^2) + d(1 - \underline{e} \cdot \underline{n}^2) = 0 \\ \Leftrightarrow & k^2 2l \underline{e} \cdot \underline{n} - k 2(d + l \underline{e} \cdot \underline{n}) + d = 0 \\ \Leftrightarrow & d(1-2k) + 2lk^2 \underline{e} \cdot \underline{n} - 2kl \underline{e} \cdot \underline{n} = 0 \\ \Leftrightarrow & (2k-1)d_E(k) + 2k(1-k)l \underline{e} \cdot \underline{n}(k) = 0 \end{aligned}$$

3 Planar

If the 3d surface is a plane, then \underline{n} is constant and does not depend on k .

The equations become:

$$(2k - 1)d_E(k) + 2k(1 - k)l\underline{e} \cdot \underline{n} = 0 \quad (2)$$

In that case:

$$\begin{aligned} d_E(k) &= \underline{DE} \cdot \underline{n} \\ &= (\underline{DC} + \underline{CA} + \underline{AE}) \cdot \underline{n} \\ &= \underline{CA} \cdot \underline{n} + kl\underline{e} \cdot \underline{n} \end{aligned}$$

Which means:

$$\begin{aligned} (2) &\Leftrightarrow 2k(\underline{CA} \cdot \underline{n} + kl\underline{e} \cdot \underline{n}) - \underline{CA} \cdot \underline{n} - kl\underline{e} \cdot \underline{n} + 2k(1 - k)l\underline{e} \cdot \underline{n} = 0 \\ &\Leftrightarrow k^2(2l\underline{e} \cdot \underline{n} - 2l\underline{e} \cdot \underline{n}) + k(2\underline{CA} \cdot \underline{n} - l\underline{e} \cdot \underline{n} + 2l\underline{e} \cdot \underline{n}) - \underline{CA} \cdot \underline{n} = 0 \\ &\Leftrightarrow k(2\underline{CA} \cdot \underline{n} + l\underline{e} \cdot \underline{n}) - \underline{CA} \cdot \underline{n} = 0 \end{aligned}$$

So finally:

$$(2) \Leftrightarrow k = \frac{\underline{CA} \cdot \underline{n}}{2\underline{CA} \cdot \underline{n} + l\underline{e} \cdot \underline{n}}$$

4 Cylinder

Consider a cylinder of axes (O, \underline{z}) , with \underline{z} unit vector and radius r . The normal vector associated to any point E (not on the axis) is:

$$\underline{n}(E) = \frac{(\underline{OE} \wedge \underline{z}) \wedge \underline{z}}{\|(\underline{OE} \wedge \underline{z}) \wedge \underline{z}\|} = \frac{(\underline{OE} \wedge \underline{z}) \wedge \underline{z}}{\|\underline{OE} \wedge \underline{z}\|}$$

Given that E in on the (A, B) line:

$$\underline{OE} = \underline{OA} + kl \underline{e}$$

So:

$$(\underline{OE} \wedge \underline{z}) \wedge \underline{z} = (\underline{OA} \wedge \underline{z}) \wedge \underline{z} + kl(\underline{e} \wedge \underline{z}) \wedge \underline{z}$$

Which entails:

$$\begin{aligned} \underline{e} \cdot \underline{n}(k) &= \frac{1}{\|\underline{OE} \wedge \underline{z}\|} [\underline{e} \cdot ((\underline{OE} \wedge \underline{z}) \wedge \underline{z})] \\ &= -\frac{1}{\|\underline{OE} \wedge \underline{z}\|} [(\underline{OE} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z})] \end{aligned}$$

And:

$$d_E = r - \|\underline{OE} \wedge \underline{z}\|$$

So the equation becomes:

$$\begin{aligned} (2) \quad &\Leftrightarrow (2k-1)(r - \|\underline{OE} \wedge \underline{z}\|) + 2k(1-k)l \underline{e} \cdot \underline{n} = 0 \\ &\Leftrightarrow (2k-1)(r - \|\underline{OE} \wedge \underline{z}\|) \|\underline{OE} \wedge \underline{z}\| - 2k(1-k)l [(\underline{OE} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z})] = 0 \\ &\Leftrightarrow (2k-1)r \|\underline{OE} \wedge \underline{z}\| - (2k-1) \|\underline{OE} \wedge \underline{z}\|^2 - 2k(1-k)l [(\underline{OE} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z})] = 0 \end{aligned}$$

Observing that:

$$\underline{OE} \wedge \underline{z} = \underline{OA} \wedge \underline{z} + kl \underline{e} \wedge \underline{z}$$

And:

$$\|\underline{OE} \wedge \underline{z}\|^2 = \|\underline{OA} \wedge \underline{z} + kl \underline{e} \wedge \underline{z}\|^2 = (\underline{OA} \wedge \underline{z})^2 + 2kl(\underline{OA} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z}) + k^2 l^2 (\underline{e} \wedge \underline{z})^2$$

So

$$\begin{aligned} (2) \quad &\Leftrightarrow (2k-1)r \|\underline{OE} \wedge \underline{z}\| - (2k-1)(\underline{OA} \wedge \underline{z})^2 - (2k-1)2kl(\underline{OA} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z}) - (2k-1)k^2 l^2 (\underline{e} \wedge \underline{z})^2 \\ &\quad - 2k(1-k)l(\underline{OA} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z}) - 2k^2(1-k)l^2 (\underline{e} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z}) \\ &\Leftrightarrow (2k-1)r \|\underline{OE} \wedge \underline{z}\| - (2k-1)(\underline{OA} \wedge \underline{z})^2 + 2kl(\underline{OA} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z})(k-1-2k+1) + k^2 l^2 (\underline{e} \wedge \underline{z})^2 (2k-2-2k+1) \\ &\Leftrightarrow (2k-1)r \|\underline{OE} \wedge \underline{z}\| - (2k-1)(\underline{OA} \wedge \underline{z})^2 + 2kl(\underline{OA} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z})(-k) + k^2 l^2 (\underline{e} \wedge \underline{z})^2 (-1) \\ &\Leftrightarrow (2k-1)r \|\underline{OE} \wedge \underline{z}\| = (2k-1)(\underline{OA} \wedge \underline{z})^2 + 2k^2 l(\underline{OA} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z}) + k^2 l^2 (\underline{e} \wedge \underline{z})^2 \\ &\Leftrightarrow (2k-1)^2 r^2 (\underline{OA} \wedge \underline{z})^2 + (2k-1)^2 r^2 2kl(\underline{OA} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z}) + (2k-1)^2 r^2 k^2 l^2 (\underline{e} \wedge \underline{z})^2 \\ &\quad = [(2k-1)(\underline{OA} \wedge \underline{z})^2 + k^2 l (2(\underline{OA} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z}) + l(\underline{e} \wedge \underline{z})^2)]^2 \\ &\Leftrightarrow (2k-1)^2 r^2 (\underline{OA} \wedge \underline{z})^2 + (2k-1)^2 r^2 2kl(\underline{OA} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z}) + (2k-1)^2 r^2 k^2 l^2 (\underline{e} \wedge \underline{z})^2 \\ &\quad = (2k-1)^2 (\underline{OA} \wedge \underline{z})^4 + k^4 l^2 (2(\underline{OA} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z}) + l(\underline{e} \wedge \underline{z})^2)^2 + 2(2k-1)(\underline{OA} \wedge \underline{z})^2 k^2 l (2(\underline{OA} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z}) + l(\underline{e} \wedge \underline{z})^2) \\ &\Leftrightarrow (2k-1)^2 A + k(2k-1)^2 B + k^2(2k-1)^2 C = k^4 D + k^2(2k-1)E \\ &\Leftrightarrow k^4(4C-D) + k^3(4B-4C-2E) + k^2(4A-4B+C+E) + k(-4A+B) + A = 0 \end{aligned}$$

Where:

$$\begin{cases} A = (r^2 - (\underline{OA} \wedge \underline{z})^2)(\underline{OA} \wedge \underline{z})^2 \\ B = 2r^2 l(\underline{OA} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z}) \\ C = r^2 l^2 (\underline{e} \wedge \underline{z})^2 \\ D = l^2 (2(\underline{OA} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z}) + l(\underline{e} \wedge \underline{z})^2)^2 \\ E = 2l(\underline{OA} \wedge \underline{z})^2 (2(\underline{OA} \wedge \underline{z}) \cdot (\underline{e} \wedge \underline{z}) + l(\underline{e} \wedge \underline{z})^2) \end{cases}$$

4.1 Particular case: E on axis

If we know E is on the axis, then n is not defined and $d_E = r$. The equation becomes an equation on n only:

$$(\underline{DA} \cdot \underline{n})^2 \|DB\|^2 = (\underline{DB} \cdot \underline{n})^2 \|DA\|^2$$

With:

$$\begin{cases} \underline{DA} \cdot \underline{n} = r - kl \underline{e} \cdot \underline{n} \\ \underline{DB} \cdot \underline{n} = r + (1-k)l \underline{e} \cdot \underline{n} \\ \|\underline{DA}\|^2 = r^2 + k^2 l^2 - 2rkl \underline{e} \cdot \underline{n} \\ \|\underline{DB}\|^2 = r^2 + (1-k)^2 l^2 + 2r(1-k)l \underline{e} \cdot \underline{n} \end{cases}$$

Which translates into:

$$\begin{aligned} & (\underline{DA} \cdot \underline{n})^2 \|DB\|^2 = (\underline{DB} \cdot \underline{n})^2 \|DA\|^2 \\ \Leftrightarrow & (2k-1)r + 2k(1-k)l \underline{e} \cdot \underline{n}(k) = 0 \\ \Leftrightarrow & \underline{e} \cdot \underline{n}(k) = \frac{(1-2k)r}{2k(1-k)l} \end{aligned}$$

Since:

$$\underline{n} = -(\cos(\theta)\underline{n}_{out} + \sin(\theta)\underline{e}_1)$$

We can write:

$$\cos(\theta)\underline{n}_{out} \cdot \underline{e} + \sin(\theta)\underline{e}_1 \cdot \underline{e} = -\frac{(1-2k)r}{2k(1-k)l}$$

or:

$$\cos(\theta - \beta) = -\frac{(1-2k)r}{2k(1-k)l}$$

5 Sphere

Consider a sphere of center O and radius r . The normal vector associated to any point E (not on O) is:

$$\underline{n}(E) = -\frac{\underline{OE}}{\|\underline{OE}\|}$$

Given that E in on the (A, B) line:

$$\begin{cases} \underline{OE} = \underline{OA} + kl \underline{e} \\ \|\underline{OE}\|^2 = \|\underline{OA}\|^2 + k^2 l^2 + 2kl \underline{OA} \cdot \underline{e} \end{cases}$$

Which entails:

$$\underline{e} \cdot \underline{n}(k) = -\frac{1}{\|\underline{OE}\|} [\underline{OA} \cdot \underline{e} + kl]$$

And:

$$d_E = r - \|\underline{OE}\|$$

So the equation becomes:

$$\begin{aligned} (2) \quad &\Leftrightarrow (2k-1)(r - \|\underline{OE}\|) + 2k(1-k)l \underline{e} \cdot \underline{n} = 0 \\ &\Leftrightarrow (2k-1)(r - \|\underline{OE}\|) \|\underline{OE}\| - 2k(1-k)l [\underline{OA} \cdot \underline{e} + kl] = 0 \\ &\Leftrightarrow (2k-1)r \|\underline{OE}\| - (2k-1) \|\underline{OE}\|^2 - 2k(1-k)l \underline{OA} \cdot \underline{e} - 2k^2 l^2 (1-k) = 0 \\ &\Leftrightarrow (2k-1)r \|\underline{OE}\| - (2k-1) \|\underline{OA}\|^2 - (2k-1)k^2 l^2 - 2(2k-1)kl \underline{OA} \cdot \underline{e} - 2k(1-k)l \underline{OA} \cdot \underline{e} - 2k^2 l^2 (1-k) = 0 \\ &\Leftrightarrow (2k-1)r \|\underline{OE}\| - (2k-1) \|\underline{OA}\|^2 - k^2 l^2 (2k-1+2-2k) - 2kl \underline{OA} \cdot \underline{e} (2k-1+1-k) = 0 \\ &\Leftrightarrow (2k-1)r \|\underline{OE}\| - (2k-1) \|\underline{OA}\|^2 - k^2 l^2 - 2k^2 l \underline{OA} \cdot \underline{e} = 0 \\ &\Leftrightarrow (2k-1)^2 r^2 \|\underline{OE}\|^2 = [(2k-1) \|\underline{OA}\|^2 + k^2 (l^2 + 2l \underline{OA} \cdot \underline{e})]^2 \\ &\Leftrightarrow (2k-1)^2 r^2 \|\underline{OA}\|^2 + (2k-1)^2 r^2 k^2 l^2 + (2k-1)^2 r^2 2kl \underline{OA} \cdot \underline{e} \\ &\quad = (2k-1)^2 \|\underline{OA}\|^4 + 2(2k-1) \|\underline{OA}\|^2 k^2 (l^2 + 2l \underline{OA} \cdot \underline{e}) + k^4 (l^2 + 2l \underline{OA} \cdot \underline{e})^2 \\ &\Leftrightarrow (2k-1)^2 (r^2 - \|\underline{OA}\|^2) \|\underline{OA}\|^2 + (2k-1)^2 r^2 k^2 l^2 + (2k-1)^2 r^2 2kl \underline{OA} \cdot \underline{e} \\ &\quad = 2(2k-1)l \|\underline{OA}\|^2 k^2 (l + 2 \underline{OA} \cdot \underline{e}) + k^4 l^2 (l + 2 \underline{OA} \cdot \underline{e})^2 \\ &\Leftrightarrow (4k^2 - 4k + 1)A + (4k^2 - 4k + 1)k^2 C + (4k^2 - 4k + 1)kB = (2k-1)k^2 E + k^4 D \\ &\Leftrightarrow k^4 [4C - D] + k^3 [4B - 4C - 2E] + k^2 [4A - 4B + C + E] + k [-4A + B] + A = 0 \end{aligned}$$

Where:

$$\begin{cases} A = (r^2 - \|\underline{OA}\|^2) \|\underline{OA}\|^2 \\ B = 2lr^2 \underline{OA} \cdot \underline{e} \\ C = r^2 l^2 \\ D = l^2 (l + 2 \underline{OA} \cdot \underline{e})^2 \\ E = 2l \|\underline{OA}\|^2 (l + 2 \underline{OA} \cdot \underline{e}) \end{cases}$$

Which is the same equation as for the cylinder, but with cross products replaced by norms.

6 Torus

Consider a torus of axes (O, \underline{z}) , with \underline{z} unit vector. It has major radius r_a and minor radius r_b .

For any point E in space, we have:

$$\underline{e}_a(E) = -\frac{(\underline{OE} \wedge \underline{z}) \wedge \underline{z}}{\|(\underline{OE} \wedge \underline{z}) \wedge \underline{z}\|} = -\frac{(\underline{OE} \wedge \underline{z}) \wedge \underline{z}}{\|\underline{OE} \wedge \underline{z}\|}$$

Hence, the normal vector associated to any point E is:

$$\underline{n}(E) = -\frac{\underline{OE} - r_a \underline{e}_a}{\|\underline{OE} - r_a \underline{e}_a\|}$$

Given that E in on the (A, B) line:

$$\underline{OE} = \underline{OA} + kl \underline{e}$$

So:

$$(\underline{OE} \wedge \underline{z}) \wedge \underline{z} = (\underline{OA} \wedge \underline{z}) \wedge \underline{z} + kl(\underline{e} \wedge \underline{z}) \wedge \underline{z}$$

Which entails:

$$\begin{aligned} \underline{e} \cdot \underline{n}(k) &= -\frac{1}{\|\underline{OE} \wedge r_a \underline{e}_a\|} [\underline{e} \cdot \underline{OE} - r_a \underline{e} \cdot \underline{e}_a] \\ &= -\frac{1}{\|\underline{OE} \wedge r_a \underline{e}_a\|} [\underline{e} \cdot \underline{OA} + kl - r_a \underline{e} \cdot \underline{e}_a] \\ &= -\frac{1}{\|\underline{OE} \wedge r_a \underline{e}_a\|} \left[\underline{e} \cdot \underline{OA} + kl + \frac{r_a}{\|\underline{OE} \wedge \underline{z}\|} \underline{e} \cdot ((\underline{OE} \wedge \underline{z}) \wedge \underline{z}) \right] \end{aligned}$$

And:

$$d_E = r_b - \|\underline{OE} - r_a \underline{e}_a\|$$

Where:

$$\begin{aligned} \|\underline{OE} - r_a \underline{e}_a\|^2 &= \|\underline{OE}\|^2 + r_a^2 - 2r_a \underline{OE} \cdot \underline{e}_a \\ &= \|\underline{OA}\|^2 + k^2 l^2 + 2kl \underline{OA} \cdot \underline{e} + r_a^2 - 2r_a \underline{OE} \cdot \underline{e}_a \\ &= \|\underline{OA}\|^2 + k^2 l^2 + 2kl \underline{OA} \cdot \underline{e} + r_a^2 - 2r_a \underline{OA} \cdot \underline{e}_a - 2kl r_a \underline{e} \cdot \underline{e}_a \end{aligned}$$