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COS 529: Advanced CV
November 22, 2025

Assignment 4: Problem Set

Question 1

We show that $Q(X)$ is a stationary distribution of the Markov chain by verifying the detailed balance condition. It suffices to prove that

$$Q(Y)\,P(Y \to Y') = Q(Y')\,P(Y' \to Y), \quad \forall\, Y, Y' \in U. \tag{1.1}$$

We consider three cases according to the Hamming distance between Y and Y'.

Case 1. Y and Y' differ in at least two coordinates.

In one transition, the chain updates only a single coordinate. Therefore it is impossible to move from Y to Y' (or vice versa) in one step, so

$$P(Y \to Y') = P(Y' \to Y) = 0,$$

and (1.1) holds trivially.

Case 2. $Y = Y'$.

Both sides of (1.1) reduce to

$$Q(Y)\,P(Y \to Y),$$

so the equality holds.

Case 3. Y and Y' differ in exactly one coordinate.

Assume they differ only at index k. That is,

$$Y_i = Y'_i \text{ for all } i \neq k, \qquad Y_k \neq Y'_k.$$

Then

$$Y = (Y_{-k}, Y_k), \qquad Y' = (Y'_{-k}, Y'_k), Y_{-k} = Y'_{-k} = Z_{-k}.$$

By the definition of the transition,

$$\begin{aligned} Q(Y)\,P(Y \to Y') &= Q(Y)\,\frac{1}{N}\,Q\Big(X_k = Y'_k \mid X_{-k} = z_{-k}\Big) \\ &= \frac{1}{N}\,Q\Big(X_{-k} = Z_{-k}, X_k = Y_k\Big)\,\frac{Q\Big(X_{-k} = Z_{-k}, X_k = Y'_k\Big)}{Q\Big(X_{-k} = Z_{-k}\Big)}. \end{aligned} \tag{1.2}$$

Similarly,

$$\begin{aligned} Q(Y')\,P(Y' \to Y) &= Q(Y')\,\frac{1}{N}\,Q\Big(X_k = Y_k \mid X_{-k} = Z_{-k}\Big) \\ &= \frac{1}{N}\,Q\Big(X_{-k} = Z_{-k}, X_k = Y'_k\Big)\,\frac{Q\Big(X_{-k} = Z_{-k}, X_k = Y_k\Big)}{Q\Big(X_{-k} = Z_{-k}\Big)}. \end{aligned} \tag{1.3}$$

The right-hand sides of (1.2) and (1.3) are identical. Hence

$$Q(Y)\,P(Y \to Y') = Q(Y')\,P(Y' \to Y),$$

and the detailed balance condition (1.1) holds.

Since the chain satisfies detailed balance with respect to Q, it follows that Q is a stationary distribution of the Markov chain.

Question 2

2.1 **Adjoint Operation** The inverse of $T = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \in SE(3)$ is

$$T^{-1} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}. \tag{2.1}$$

Any $\xi \in \mathbb{R}^6$ can be written as two vectors $\psi, \rho \in \mathbb{R}^3$:

$$\xi = \begin{bmatrix} \psi \\ \rho \end{bmatrix}.$$

The $\wedge$ operator gives

$$\xi^\wedge = \begin{bmatrix} \psi^\wedge & \rho \\ 0 & 0 \end{bmatrix}. \tag{2.2}$$

Now compute the adjoint map:

$$\begin{aligned} Ad_T(\xi^\wedge) &= T\xi^\wedge T^{-1} \\ &= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \psi^\wedge & \rho \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R\psi^\wedge R^T & -R\psi^\wedge R^T t + R\rho \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{2.3}$$

To extract the 6×6 adjoint matrix $A = \mathrm{Ad}_T$, we apply the $\vee$ operator to (2.3). Since cross product is rotation invariant:

$$R(x \times y) = (Rx) \times (Ry),$$

we have

$$-R\psi^\wedge R^T t = -R(\psi \times (R^T t)) = -(R\psi) \times t = -(R\psi)^\wedge t = t \times R\psi = (t^\wedge R\psi). \tag{2.4}$$

This holds for any $t \in R^3$. Applying $(\cdot)^\vee$ to (2.3) and using (2.4), we obtain

$$(T\xi^\wedge T^{-1})^\vee = \begin{bmatrix} R\psi \\ t^\wedge R\psi + R\rho \end{bmatrix}. \tag{2.5}$$

Thus A satisfies

$$A \begin{bmatrix} \psi \\ \rho \end{bmatrix} = \begin{bmatrix} R\psi \\ t^\wedge R\psi + R\rho \end{bmatrix}.$$

Comparing block-wise, we identify

$$A = \begin{bmatrix} R & 0 \\ t^\wedge R & R \end{bmatrix} = \begin{bmatrix} R & 0 \\ t \times R & R \end{bmatrix}.$$

This is exactly the standard adjoint representation of $SE(3)$.

2.2 **Gradient** From the lecture, we know that for any $T \in SE(3)$,

$$Te^{\xi^\wedge} T^{-1} = e^{T\xi^\wedge T^{-1}} = e^{\mathrm{Ad}_T(\xi)^\wedge} = e^{(\mathrm{Ad}_T \xi)^\wedge}. \tag{2.6}$$

We have already computed $\mathrm{Ad}_T \in \mathbb{R}^{6\times 6}$ in the previous problem.

Consider now the function

$$L(\xi) = \log\Big(Te^{\xi^\wedge} T_1 T_2^{-1}\Big)^\vee = \log\Big(e^{(\mathrm{Ad}_T \xi)^\wedge}\, TT_1T_2^{-1}\Big)^\vee, \qquad \xi \in \mathfrak{se}(3). \tag{2.7}$$

The BCH approximation for small $\Delta\xi$ states that

$$f(\Delta\xi) = \log\Big(e^{(\Delta\xi)^\wedge} e^{\xi^\wedge}\Big)^\vee = J_l^{-1}(\xi)\,\Delta\xi + \xi, \qquad \|\Delta\xi\| \text{ small}. \tag{2.8}$$

To match this form, write

$$e^{(\mathrm{Ad}_T\xi)^\wedge} T T_1 T_2^{-1} = e^{(\mathrm{Ad}_T\xi)^\wedge} \exp\left(z^\wedge\right),$$

where

$$z = \log(T T_1 T_2^{-1})^\vee.$$

Applying (2.8) with $\Delta\xi = \mathrm{Ad}_T\xi$ and base point z,

$$L(\xi) = \log\Big(e^{(\mathrm{Ad}_T\xi)^\wedge} e^{z^\wedge}\Big)^\vee \approx J_l^{-1}(z)\,(\mathrm{Ad}_T\xi) + z. \tag{2.9}$$

Since the linear approximation (2.9) is exact to first order,

$$L(\xi) = J_l^{-1}(z)\,\mathrm{Ad}_T\,\xi + z + O(\|\xi\|^2), \qquad z = \log(T T_1 T_2^{-1})^\vee.$$

Thus,

$$\left.\frac{\partial L}{\partial \xi}\right|_{\xi=0} = J_l^{-1}\Big(\log(T T_1 T_2^{-1})^\vee\Big)\,\mathrm{Ad}_T. \tag{2.10}$$