

# **Advanced Control for Robotics: Homework #1**

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# 1 ODE and Its Simulation

## 1.1 Equation of Pendulum Motions

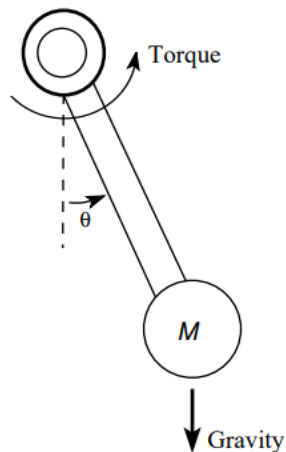


Figure 1: pendulum model

By applying the Newton's law of dynamics, a pendulum with no external force can be formulated as:

$$ml^2\ddot{\theta} + ml^2\alpha\dot{\theta} + mgl\sin\theta - T = 0. \quad (1)$$

in which,

$m$  is mass of the ball

$l$  is length of the rod

$\alpha$  is the damping constant

$g$  is the gravitational constant

$\theta$  is angle measured between the rod and the vertical axis

$T$  is torque of the joint, which is also the control input  $u$

to a system of two first order equation by letting  $x_1 = \theta$ ,  $x_2 = \dot{\theta}$ :

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{g}{l}\sin x_1 - \alpha x_2 + \frac{T}{ml^2}. \quad (2)$$

Written in standard state-space form:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \alpha x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} T \quad (3)$$

$$\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x} \quad (4)$$

## 1.2 Simulation of Pendulum

When assuming  $m = l = 1$  with proper unit, equation (3) can be simplified as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -g \sin x_1 - \alpha x_2 + T \end{bmatrix} \quad (5)$$

according the equation, we code the simulation as following:

```
1 import numpy as np
2 from scipy.integrate import odeint
3 import matplotlib.pyplot as plt
4
5 def pendulum(x, t, g, alpha, T):
6     '''
7     pendulum system vector-space function
8     '''
9     x1, x2 = x
10    dxdt = [x2, -g*np.sin(x1) - alpha*x2 + T ]
11    return dxdt
12
13 ## initial condition
14 g = 9.8 # gravitational constant
15
16 # damping constant alpha collection of two different cases
17 alpha_c = [0.3, 0.7]
18 T = 0 # the control input
19
20 # initial theta collection of two different cases
21 x1_0_c = [np.pi*3/4, np.pi/4]
22 x2_0 = 0 # initial omega
23
24 ## simulation setup
25 t = np.linspace(0, 9.9, 400)
26 # y = [] # the output collection of four cases
27
28 plt.subplots(2, 2, sharex='all', sharey='all', figsize=(14, 8))
29 # plt.figure()
30
31 for i in range(4):
32     '''
33     four cases
34     choose x1_0 with rem:
35         when i = 0 or 2, x1_0 is in the first case;
36         when i = 1 or 3, in another one
37     choose alpha with mod:
38         when i = 0 or 1, alpha is in the first case;
39         when i = 2 or 3, in another one
40     '''
41    x0 = [x1_0_c[i%2], x2_0]
42    alpha = alpha_c[i//2]
```

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```

```

## solve
y = odeint(pendulum, x0, t, args=(g, alpha, T))

## plot
plt.subplot(2, 2, i+1)
plt.plot(t, y[:, 0], label='x1:theta')
plt.plot(t, y[:, 1], label='x2:omega')
plt.title('x1_0={:.2f},x2_0={:.2f},alpha={:.2f},T={:.2f}'\
          .format(x0[0], x0[1], alpha, T))
plt.legend(loc='best')
plt.ylim(-6, 6)
if(i>=2):plt.xlabel('time')
plt.grid()

## save and show
plt.savefig(r'./HW1/img/pendulumSim.png')
plt.show()

```

and getting the output as:

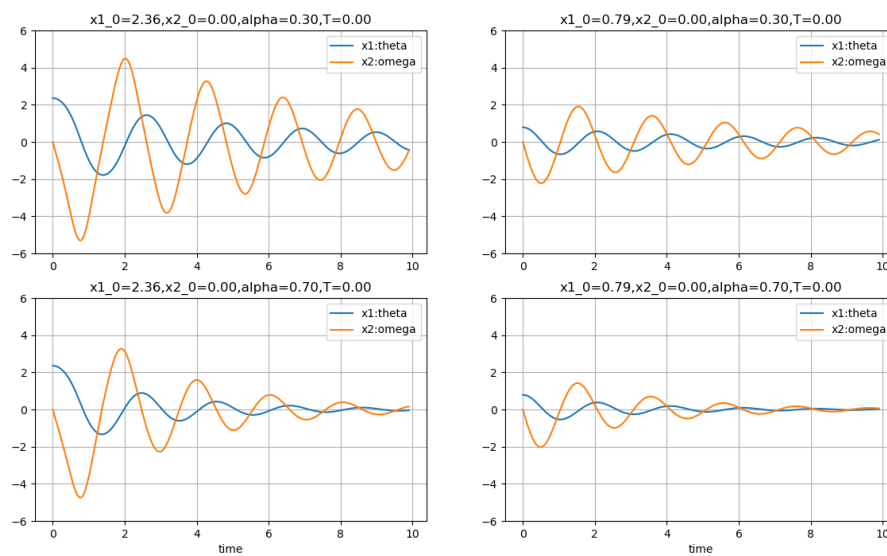


Figure 2: pendulum simulation output

## 2 Matrix calculus

### 2.1 Tutorial

$$\frac{\partial}{\partial X} f(X) = \begin{bmatrix} \frac{\partial f(X)}{\partial X_{11}} & \cdots & \frac{\partial f(X)}{\partial X_{1m}} \\ \vdots & \frac{\partial f(X)}{\partial X_{ij}} & \vdots \\ \frac{\partial f(X)}{\partial X_{n1}} & \cdots & \frac{\partial f(X)}{\partial X_{nm}} \end{bmatrix} \quad (6)$$

Derivative of scalar function  $f(X)$  can be calculated by taking derivatives of the scalar function with respect to each entry  $X_{ij}$  of the matrix  $X$  separately, showing as above equation (6).

Scalar function  $f(X)$  project matrix variable  $X \in \mathbb{R}^{n \times m}$  to a scalar  $y \in \mathbb{R}^1$ , so its derivative is the partial derivative, except that its results are arranged in form of a matrix, who has the same shape as  $X$ .

For instance, let's say  $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ ,  $f(X) = [1 \ 1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ . So  $y = f(X) = X_{11} + X_{12} + X_{21} + X_{22}$ . And the partial derivative of  $f(X)$  is

$$\frac{\partial}{\partial X} f(X) = \begin{bmatrix} \frac{\partial f(X)}{\partial X_{11}} & \frac{\partial f(X)}{\partial X_{12}} \\ \frac{\partial f(X)}{\partial X_{21}} & \frac{\partial f(X)}{\partial X_{22}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (7)$$

### 2.2 Derivative of Trace

$$\begin{aligned} \frac{\partial}{\partial X} \text{tr}(AX) &= \frac{\partial}{\partial X} \text{tr} \left( \begin{bmatrix} A_{11}X_{11} & \cdots & A_{1m}X_{m1} \\ \vdots & A_{ij}X_{ji} & \vdots \\ A_{n1}X_{1n} & \cdots & A_{nm}X_{mn} \end{bmatrix} \right) \\ &= \frac{\partial}{\partial X} (A_{11}X_{11} + \cdots + A_{ij}X_{ji} + \cdots + A_{nm}X_{mn}) \\ &= \begin{bmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & A_{ji} & \vdots \\ A_{1m} & \cdots & A_{mn} \end{bmatrix} = A^T \end{aligned} \quad (8)$$

in which,  $\frac{\partial}{\partial X_{ij}} (A_{11}X_{11} + \cdots + A_{ij}X_{ji} + \cdots + A_{nm}X_{mn}) = A_{ji}$

### 2.3 Derivation

According to *The Matrix Cookbook* equation (81), we have

$$\frac{\partial x^T Q x}{\partial x} = (Q + Q^T)x \quad (9)$$

and we can derive that

$$\frac{\partial \text{tr}(xx^T)}{\partial x} = \frac{\partial}{\partial x} (x_1^2 + x_2^2 + \cdots + x_n^2) = \begin{bmatrix} 2x_1 \\ 2x_2 \\ \vdots \\ 2x_n \end{bmatrix} = 2x \quad (10)$$

comprehensive above, we get

$$\begin{aligned}\frac{\partial}{\partial x} f(x) &= \frac{\partial x^T Q x}{\partial x} + \frac{\partial \text{tr}(xx^T)}{\partial x} \\ &= (Q + Q^T)x + 2x\end{aligned}\tag{11}$$

### 3 Inner product

#### 3.1 Angle between Two Vectors

The inner product of two vectors is  $\langle x, y \rangle = \|x\| \|y\| \cos \theta$ , so the angle  $\theta$  equal to  $\arccos \frac{\langle x, y \rangle}{\|x\| \|y\|}$

#### 3.2 Compute the Angle

Using the way to calculate the angle above, we get

$$\theta = \arccos \frac{\langle A, B \rangle}{\|A\| \|B\|}\tag{12}$$

and we find that

$$\langle A, B \rangle = \text{tr}(A^T B) = \text{tr}\begin{pmatrix} -1 & 2 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix} = 0\tag{13}$$

so, the angle between A and B is  $\frac{\pi}{2}$ .

### 4 Some linear algebra

#### 4.1 Condition

Take row reduction to  $Ax = b$ , if any row come up with the situation that left hand side of the equation are zeros, while the right side is not, then equation  $Ax = b$  has no solution, else it has at least one solution.

#### 4.2 Compute

$A$  has two linearly independent columns, so  $\text{rank}(A) = 2$ . Knowing  $a_3 + a_1 = a_2$  and  $a_4 - a_3 = a_1$ , so we can get  $a_3 = a_2 - a_1$  and  $a_4 = a_1 + a_3 = a_1 + a_2 - a_1 = a_2$ , so

$$A = [a_1, a_2, a_3, a_4] = [a_1, a_2, a_2 - a_1, a_2]\tag{14}$$

We can easily find two independent vectors satisfying  $Ax = 0$

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}\tag{15}$$

So,  $\text{Null}(A) = \{x_1, x_2\}$

## 5 Gradient Flow

### 5.1 State Space Form

Let  $x_1 = \omega, x_2 = \dot{\omega}, x = [x_1, x_2]^T$ , we get

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\nabla l(x_1) - Ax_2 \end{bmatrix} \quad (16)$$

$$y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (17)$$

### 5.2 Characterize the Equilibrium

The equilibrium point satisfies  $\dot{x} = 0$ , i.e.

$$\begin{cases} x_2 = 0 \\ -\nabla l(x_1) - Ax_2 = 0 \end{cases} \Rightarrow \begin{cases} \dot{\omega} = 0 \\ \nabla l(\omega) = 0 \end{cases} \quad (18)$$

### 5.3 Simulation