MAT157 Problem Set 9

Nicolas

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1.

$$\frac{d^2 \sin^3(x^4)}{dx^2} = 6x^2 \sin(x^4)(4x^4 + 3\sin(2x^4) + 12x^4 + 12x^4 \cos(2x^4))$$

2.

$$\frac{d}{dx}\Big(\frac{1}{1+\sin^2(x)}\Big)^3 = -3(1+\sin^2(x))^{-4}(2\sin(x)\cos(x))$$

2.

Proof.

Consider the derivative of $f(x)$ at $a = 2$. We will use the limit definition.

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{f(2+h)}{h} \\
&= \lim_{h\to 0} \frac{|(2+h)^2 - 4| - |2^2 - 4|}{h} \\
&= \lim_{h\to 0} \frac{|4 + 4h + h^2 - 4| - |0|}{h} \\
&= \lim_{h\to 0} \frac{|4h + h^2|}{h} \\
&= \lim_{h\to 0} \frac{|h||4+h|}{h}
\end{aligned}$$

However, if $h > 0$ than, $\lim_{h\to 0} \frac{|h||4+h|}{h} = 4$, but if $h < 0$ then $\lim_{h\to 0} \frac{|h||4+h|}{h} = -4$ Because the left and right side limit are not equal, the limit does not exist.

□

3.

Proof.

$g'(y) = \frac{1}{f'(g(y))}$

$g''(y)$

$$\begin{aligned}
g''(y) &= \frac{d}{dy}\frac{1}{f'(g(y))} \\
&= -\frac{1}{(f'(g(y)))^2}\frac{d}{dy}f'(g(y)) \\
&= -\frac{1}{(f'(g(y)))^2}f''(g(y))\frac{d}{dy}g(y) \\
&= -\frac{1}{(f'(g(y)))^2}f''(g(y))g'(y) \\
&= -\frac{1}{(f'(x))^2}f''(x)\frac{1}{f'(x)} \\
&= -\frac{f''(x)}{(f'(x))^3}
\end{aligned}$$

$g'''(y)$

$$\begin{aligned}
g'''(y) &= \frac{d}{dy}\frac{-f''(g(y))}{(f'(g(y)))^3} \\
&= \frac{-f'''(g(y))g'(y)}{(f'(g(y)))^3} + 3\frac{f''(g(y))}{(f'(g(y)))^4}f''(g(y))g'(y) \\
&= \frac{-f'''(x)}{(f'(x))^4} + 3\frac{(f''(x))^2}{(f'(x))^5}
\end{aligned}$$

$g''''(y)$

$$
\begin{aligned}
g''''(y) &= \frac{d}{dy}\left(\frac{-f'''(g(y))}{(f'(g(y)))^4} + 3\frac{(f''(g(y)))^2}{(f'(g(y)))^5}\right) \\
&= \frac{d}{dy}\frac{-f'''(g(y))}{(f'(g(y)))^4} + 3\frac{d}{dy}\frac{(f''(g(y)))^2}{(f'(g(y)))^5} \\
&= \frac{-f''''(g(y))g'(y)}{(f'(g(y)))^5} + 4\frac{f''(g(y))}{(f'(g(y)))^5}f''(g(y))g'(y) \\
&\quad + 6\frac{f''(g(y))f'''(g(y))g'(y)}{(f'(g(y)))^5} - 5\frac{(f''(g(y)))^2}{(f'(g(y))^6}f''(g(y))g'(y) \\
&= \frac{-f''''(x)}{(f'(x))^6} + 4\frac{(f''(x))^2}{(f'(x))^6}f''(x) + 6\frac{f''(x)f'''(x)}{(f'(x))^6} - 5\frac{(f''(x))^3}{(f'(x))^7}
\end{aligned}
$$

$\square$

4.

Proof.

Consider that the Max$f([a,b]) > 0$. Thus, by EVT, then there exists an $x \in [a,b]$ such that $f(x) =$Max$f([a,b])$. However, $x \neq a$ and $x \neq b$ because $f(a) = f(b) = 0 <$ Max$f([a,b])$. Because the maximum of $f|_{[a,b]}$ is not on the boundary, and $f|_{[a,b]}$ is differentiable, then $f'(x) = 0$. Thus, $x^2 f''(x) = (x^2+1)$Max$f([a,b])$. Thus, let $\omega := f''(x) > 0$. Thus
$\forall \epsilon > 0, \exists \delta_1 > 0, \forall y \in [a,b] : 0 < |y-x| < \delta_1 \implies |\frac{f'(y)-f'(x)}{y-x} - \omega| < \epsilon.$
Therefore, for every $y \in (x, x+\delta_1)$, $f'(y) > 0$. Let $\gamma := f'(y) > 0$. Then
$\forall \epsilon > 0, \exists \delta_2 > 0, \forall z \in [a,b] : 0 < |z-x| < \delta_2 \implies |\frac{f(z)-f(x)}{z-x} - \gamma| < \epsilon.$
Therefore, for every $z \in (x, x+\delta_2)$, $f(z) >$ Max$f([a,b])$, a contradiction. Thus, the Max$f([a,b])$ cannot be greater than 0.

Consider the Min$f([a,b]) < 0$. Thus, by EVT¡ then there exists an $x \in [a,b]$ such that $f(x) =$ Min$f([a,b])$. Similarly to the other case $f'(x) = 0$. Thus, $f''(x) < 0$, meaning that $f'(y) < 0$ for any $y \in (x, x+\delta_1)$. Therefore, for every $z \in (x, x+\delta_2)$, $f(z) <$ Min$f([a,b])$, a contradiction. Thus, the Min$f([a,b])$ cannot be less than 0.

If the Max$f([a,b])$ cannot be greater than 0 and the Min$f([a,b])$ cannot be less than 0. $f(x) = 0, \; \forall x \in [a,b]$.

□

5. a)

Proof.

By assumption, we can say that $\forall y \in J, \forall \varepsilon' > 0, \exists \delta > 0, \forall x \in J : 0 < |x - y| < \delta \implies |\frac{f(x)-f(y)}{x-y} - f'(y)| < \varepsilon'$. Thus, through the triangle inequality, $|\frac{f(x)-f(y)}{x-y}| - |f'(y)| < \varepsilon'$. From, this we get that $|\frac{f(x)-f(y)}{x-y}| < \varepsilon' + |f'(y)|$. If we choose $M := \varepsilon' + \sup |f'(J)|$, this implies that $|f(x) - f(y)| < M|x - y|$. But because x, y and ε' were arbitrary, this holds for all $x, y \in J$ and $\varepsilon' > 0$.

Now let $\varepsilon > 0$ and choose $M' := \varepsilon + \sup |f'(J)|$ and $\delta := \frac{\varepsilon}{M}$. Thus, for every $x, y \in J$, if $0 < |x - y| < \delta$ then $|f(x) - f(y)| < M'|x - y| < M'\delta = \epsilon$ as desired.

□

b)

Proof.

The function $f : (0, \infty),\ f(x) = \sqrt{x}$ is uniformly continuous because $\forall \varepsilon > 0$ let $\delta := \varepsilon^2$. Thus, $\forall x, y \in (0, \infty) : 0 < |x - y| < \delta$ then, $|\sqrt{x} - \sqrt{y}|^2 \leq |\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}| = |x - y| < \varepsilon^2$. Thus, $|\sqrt{x} - \sqrt{y}| < \varepsilon$ as desired.

However, $f'(x) = \frac{1}{\sqrt{x}}$ which is not bounded on the interval $(0, \infty)$. Thus, we have found a example.

□