MAT247 Problem Set 1

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1.

Proof.

Consider $\langle (0,1,0),(0,1,0)\rangle = 0$; however, $(0,1,0) \neq 0$, meaning that the function is not definite.

$\square$

2.

Proof.

If there exists an $x \in V$ such that $(x, x) > 0$, then $x^2(1, 1) > 0$ by linearity and symmetry. Because x^2 is positive (if $x^2 = 0$ then the inequality doesn't hold) then $(1, 1)$ must also be positive. Thus for any $y \in V \setminus \{0\}$, $(y, y) = y^2(1, 1) > 0$. And for 0, $(0, 0) = 0$. Thus, definiteness is achieved.

$\square$

3.

Proof.

$\Leftarrow$

Let $a, b \in \mathbb{R}$

$$\begin{aligned}
\|au+bv\|^2 &= \|bu+av\|^2 \\
\langle au+bv, au+bv\rangle &= \langle bu+av, bu+av\rangle \\
\langle au, au+bv\rangle + \langle bv, au+bv\rangle &= \langle bu, bu+av\rangle + \langle av, bu+av\rangle \\
\langle au, au\rangle + \langle au, bv\rangle + \langle bv, av\rangle + \langle bv, bv\rangle &= \langle bu, bu\rangle + \langle bu, av\rangle + \langle av, bu\rangle + \langle av, av\rangle \\
a\|u\|^2 + b\|v\|^2 &= b\|u\|^2 + a\|v\|^2
\end{aligned}$$

but because a, b were arbitrary, this holds for $a = 1$ and $b = 0$.
Then, we have $\|u\|^2 = \|v\|^2 \implies \|u\| = \|v\|$.

$\Rightarrow$

Let $a, b \in \mathbb{R}$

$$\begin{aligned}
\|u\| &= \|v\| \\
\|u\|^2 &= \|v\|^2 \\
\langle u, u\rangle &= \langle v, v\rangle \\
\langle au, au\rangle + \langle au, bv\rangle + \langle bv, av\rangle + \langle bv, bv\rangle &= \langle bu, bu\rangle + \langle bu, av\rangle + \langle av, bu\rangle + \langle av, av\rangle \\
\|au+bv\|^2 &= \|bu+av\|^2
\end{aligned}$$

As desired.

$\square$

4.

Proof.

Consider the $V = \mathbb{R}^4$ with the the Euclidean Inner-product. Let $u = (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d})$ and $v = (\sqrt{\frac{1}{a}}, \sqrt{\frac{1}{b}}, \sqrt{\frac{1}{c}}, \sqrt{\frac{1}{d}}),\ a, b, c, d \in \mathbb{R}$. By *Cauchy-Schwarz* $|\langle u, v\rangle| \leq \|u\| \, \|v\|$; however, $\langle u, v\rangle = 4$. Thus $4 \leq \|u\| \, \|v\|$. But because $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in \mathbb{R}^4$. Thus, $4 \leq \sqrt{a+b+c+d}\sqrt{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}} \implies 16 \leq (a+b+c+d)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d})$ as desired.

$\square$