MAT157 Assignment 1

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1. a) $sin(\frac{11}{24}\pi) = \sqrt{\frac{\sqrt{\frac{\frac{\sqrt{3}}{2}+1}{2}}+1}{2}}$

Proof.

Consider the double angle formula for cosine.
$cos(2\theta) = 2cos^2(\theta) - 1 = 1 - 2sin^2(\theta)$. If $\theta = \frac{\pi}{12}$ and $\alpha = \frac{\pi}{24}$, then we can write $cos(\frac{\pi}{6}) = 2cos^2(\frac{\pi}{12}) - 1$. By rearrangement:

$$\begin{aligned} cos(\frac{\pi}{12}) =& \sqrt{\frac{cos(\frac{\pi}{6}) + 1}{2}} \\ =& \sqrt{\frac{\frac{\sqrt{3}}{2} + 1}{2}} \\ cos(\frac{\pi}{24}) =& \sqrt{\frac{cos(\frac{\pi}{12}) + 1}{12}} \\ =& \sqrt{\frac{\sqrt{\frac{\frac{\sqrt{3}}{2}+1}{2}} + 1}{2}} \end{aligned}$$

Because $\frac{\pi}{12}$ is in the first quadrant, we can take the positive root for $cos(\frac{\pi}{12})$.

Consider $sin(\frac{11}{24}\pi)$. $\frac{11}{24}\pi = \frac{\pi}{2} - \frac{\pi}{24}$; thus, $sin(\frac{11}{24}\pi) = sin(\frac{\pi}{2} - \frac{\pi}{24})$.
With the compound angle formula, we can write:
$sin(\frac{\pi}{2} - \frac{\pi}{24}) = sin(\frac{\pi}{2})cos(-\frac{\pi}{24}) + cos(\frac{\pi}{2})sin(-\frac{\pi}{24})$
Thus we can use our known trigonometric values, our own formula, and the fact that cosine is an even function to evaluate the statement to be:

$$\sqrt{\frac{\sqrt{\frac{\frac{\sqrt{3}}{2}+1}{2}}+1}{2}}+0(sin(-\frac{\pi}{24}))$$
$$=\sqrt{\frac{\sqrt{\frac{\frac{\sqrt{3}}{2}+1}{2}}+1}{2}}$$

$\square$

b) $sin(\alpha)sin(\beta)sin(\gamma) =$
$\frac{1}{4}sin(\alpha-\beta+\gamma)-\frac{1}{4}sin(\alpha-\beta-\gamma)-\frac{1}{4}sin(\alpha+\beta+\gamma)+\frac{1}{4}sin(\alpha+\beta-\gamma)$

Proof.

By the product formula: $sin(\alpha)sin(\beta)=\frac{1}{2}(cos(\alpha-\beta)-cos(\alpha+\beta))$.
Then,

$$\begin{aligned} sin(\alpha)sin(\beta)sin(\gamma) =& \frac{1}{2}(cos(\alpha-\beta)-cos(\alpha+\beta))(sin(\gamma)) \\ =& \frac{1}{2}(cos(\alpha-\beta)sin(\gamma)-cos(\alpha+\beta)sin(\gamma)) \end{aligned}$$

Using the other product formula:
$cos(\alpha)sin(\beta)=\frac{1}{2}(sin(\alpha+\beta)-sin(\alpha-\beta))$.
Substituting that formula leads us to

$$\begin{aligned} &\frac{1}{2}(cos(\alpha-\beta)sin(\gamma)-cos(\alpha+\beta)sin(\gamma)) \\ =&\frac{1}{2}(\frac{1}{2}(sin(\alpha-\beta+\gamma)-sin(\alpha-\beta-\gamma))-\frac{1}{2}(sin(\alpha+\beta+\gamma)-sin(\alpha+\beta-\gamma))) \\ =&\frac{1}{4}sin(\alpha-\beta+\gamma)-\frac{1}{4}sin(\alpha-\beta-\gamma)-\frac{1}{4}sin(\alpha+\beta+\gamma)+\frac{1}{4}sin(\alpha+\beta-\gamma) \end{aligned}$$

$\square$

c) $sin(\omega_1 t)+sin(\omega_2 t)=2sin(\frac{t(\omega_1+\omega_2)}{2})cos(\frac{t(\omega_1-\omega_2)}{2})$

Proof.

Using the product formula:
$sin(\alpha)cos(\beta)=\frac{1}{2}(sin(\alpha+\beta)+sin(\alpha-\beta))$. Setting $\alpha:=\frac{t(\omega_1+\omega_2)}{2}$ and $\beta:=\frac{t(\omega_1-\omega_2)}{2}$, we are given $sin(\frac{t(\omega_1+\omega_2)}{2})cos(\frac{t(\omega_1-\omega_2)}{2})=$
$\frac{1}{2}(sin(\frac{t(\omega_1+\omega_2)}{2}+\frac{t(\omega_1-\omega_2)}{2})+sin(\frac{t(\omega_1+\omega_2)}{2}-\frac{t(\omega_1-\omega_2)}{2})$

From here we can simplify terms:

$$\begin{aligned} sin(\frac{t(\omega_1+\omega_2)}{2})cos(\frac{t(\omega_1-\omega_2)}{2}) =& \frac{1}{2}(sin(\omega_1 t)+sin(\omega_2 t)) \\ 2sin(\frac{t(\omega_1+\omega_2)}{2})cos(\frac{t(\omega_1-\omega_2)}{2}) =& sin(\omega_1 t)+sin(\omega_2 t) \end{aligned}$$

Thus we have written $sin(\omega_1 t)+sin(\omega_2 t)$ as the product of two trigonometric functions.

□

2. a)

Proof.

Consider that $\sqrt{243}$ is rational.

$$\begin{aligned}\sqrt{243} =& \frac{p}{q}\\ 243 =& \frac{p^2}{q^2}\\ 243q^2 =& p^2\\ 3^5q^2 =& p^2\end{aligned}$$

Because p is raised to the power of 2, then it has to have an even amount of factors of 3, but because 3^5q^2 has a even amount plus 5 factors of 3 (because an even number plus an odd number is always odd), 3^5q^2 will never have an even amount of factors of 3, contradicting the fundamental theorem of arithmetic.

□

b)

Proof.

Consider that $\sqrt[11]{11}$ is rational.

$$\begin{aligned}\sqrt[11]{11} =& \frac{p}{q}\\ 11 =& \frac{p^{11}}{q^{11}}\\ 11q^{11} =& p^{11}\end{aligned}$$

Because p is raised to the power of 11, then it has to have a multiple of 11 factors of 11, but because $11q^{11}$ has a multiple of 11 plus 1 factors of 11 (never a multiple of 11), this contradicts the fundamental theorem of arithmetic.

□

c)

Proof.

First we will show that $\sqrt{35}$ is irrational. Assume that $\sqrt{35}$ is rational first.

$$\begin{aligned}\sqrt{35} =& \frac{p}{q} \\ 35 =& \frac{p^2}{q^2} \\ 35q^2 =& p^2 \\ 5 \cdot 7q^2 =& p^2\end{aligned}$$

Thus, $5 \cdot 7q^2$ will always have a odd amount of factors of 5, while p^2 will always have an even amount of factors of 5, contradicting the fundamental theorem of arithmetic.

Assume $\sqrt{2} + \sqrt{5} + \sqrt{7}$ is rational. Then:

$$\begin{aligned}\sqrt{2} + \sqrt{5} + \sqrt{7} =& \frac{p}{q} \\ \sqrt{5} + \sqrt{7} =& \frac{p}{q} - \sqrt{2} \\ (\sqrt{5} + \sqrt{7})^2 =& (\frac{p}{q} - \sqrt{2})^2 \\ 12 + 2\sqrt{35} =& \frac{p^2}{q^2} - 2\sqrt{2}\frac{p}{q} + 2 \\ 10 + 2\sqrt{35} =& \frac{p^2}{q^2} - 2\sqrt{2}\frac{p}{q} \\ \frac{p^2}{q^2} - 10 - 2\sqrt{35} =& 2\sqrt{2}\frac{p}{q}\end{aligned}$$

Because $\frac{p^2}{q^2} - 10$ is clearly rational, we can rewrite it as $\frac{m}{n}$.

$$\begin{aligned}\frac{m}{n} - 2\sqrt{35} =& 2\sqrt{2}\frac{p}{q} \\ (\frac{m}{n} - 2\sqrt{35})^2 =& (2\sqrt{2}\frac{p}{q})^2 \\ \frac{m^2}{n^2} + 140 - 4\frac{m}{n}\sqrt{35} =& 8\frac{p^2}{q^2}\end{aligned}$$

Because $\frac{m^2}{n^2} + 140$ is clearly rational, we can rewrite it as $\frac{a}{b}$.

$$4\sqrt{35} = 8\frac{p^2}{q^2} - \frac{a}{b}$$
$$\sqrt{35} = 8\frac{p^2}{4q^2} - \frac{a}{4b}$$

This shows that $\sqrt{35}$ is rational, clearly a contradiction.

$\square$

3. 1) $\bigcup_{n\in\mathbb{N}} X_n = (-2, 2]$ and $\bigcap_{n\in\mathbb{N}} X_n = [-1, 1]$

Proof.

We will prove the first statement by showing that for every $n > 1,\ X_n \subseteq X_1$. The set $X_n = \{x : -1 - \frac{1}{n} < x \leq 1 + \frac{1}{n}\}$. If we let $n > 1$, then $1 + \frac{1}{n} < 2$ and conversely, $-1 - \frac{1}{n} > -2$. Thus, for every element of $x \in X_n,\ -2 < x < 2$; therefore, $\forall x \in X_n,\ x \in X_1$, making $X_n \subseteq X_1,\ \forall n > 1$. Furthermore, the union of all X_n and X_1 must just be X_1.

We will prove the second by proving that $x \in [-1, 1] \iff x \in X_n,\ \forall n \in \mathbb{N}$. Because $[-1, 1]$ is an interval, we can simply prove that the boundary of $(-1, 1]$ is in all X_n and then prove that if $y \notin [-1, 1]$, then there exists an $n \in \mathbb{N}$ such that $y \notin X_n$. We will take the boundaries, namely 1 and -1. Because $\frac{1}{n}$ is just some positive rational number, then $-1 - \frac{1}{n} < -1 < 1 < 1 + \frac{1}{n},\ \forall n \in \mathbb{N}$ is trivially true. Now take some objects outside of the interval, namely $a := 1 + \epsilon$ and $b := -1 - \epsilon,\ \forall \epsilon > 0$. $\forall \epsilon,\ \exists n \in \mathbb{N}$ such that $1 + \frac{1}{n} < a$ and similarly $\exists m \in \mathbb{N}$ such that $b < -1 - \frac{1}{m}$. Thus for every object, $x \in [-1, 1],\ x \in X_n,\ \forall n \in \mathbb{N}$ and for every object, $y \notin [-1, 1], \exists m \in \mathbb{N}$ st $y \notin X_m$.

□

2) a)

Proof.

$\subseteq$

Consider some $x \in f(\bigcup_{i\in I} S_i)$. Then, $x = f(y)$ for some $y \in \bigcup_{i\in I} S_i$. Because $\bigcup_{i\in I} f(S_i)$ is the union of all $f(S_i)$ for some $i \in I$, x is also in $\bigcup_{i\in I} f(S_i)$.

$\supseteq$

Consider some $x \in \bigcup_{i\in I} f(S_i)$. Then, there exists some $i \in I$ such that $x = f(y)$ for some $y \in S_i$. Because $f(\bigcup_{i\in I} S_i)$ is the image of f on $\bigcup_{i\in I} S_i$, x is also in $f(\bigcup_{i\in I} S_i)$

□

b)

Proof.

Consider the indexing set $I := \{1, 2\}$ and the sets $S_i \subseteq \{1, 2, 3\}$ such that $i \in I$. $S_1 := \{1, 2\}$ and $S_2 := \{2, 3\}$. Take the function $f : \{1, 2, 3\} \to \{0, 1\}, \;\; f(x) = 1$ if x is odd and 0 otherwise. Thus $\bigcap_{i\in I} f(S_i) = \{1\}$, but $f(\bigcap_{i\in I} S_i) = \{0\}$; therefore, $\bigcap_{i\in I} f(S_i) \neq f(\bigcap_{i\in I} S_i)$, so we have constructed an example.

$\square$

4. a)

Proof.

By assumption, $\exists n, m \in \mathbb{Z}$ such that $a - a' = nk$ and $b - b' = mk$. We can rewrite these formulas as $a = nk + a'$ and $b = mk + b'$. We will now prove that addition is well-defined:

$$\begin{aligned} a + b =& (nk + a') + (mk + b') \\ =& a' + b' + k(n + m) \\ (a + b) - (a' + b') =& k(n + m) \end{aligned}$$

Thus, $[a + b]_k = [a' + b']_k$. We will now prove that multiplication is well defined:

$$\begin{aligned} ab =& (nk + a')(mk + b') \\ =& nmk^2 + ka' + mkb' + a'b' \\ =& k(nmk + na' + mb') + a'b' \\ (ab) - (a'b') =& k(nmk + na' + mb') \end{aligned}$$

Thus, $[ab]_k = [a'b']_k$

$\square$

b)

Proof.

P1
Like the integers, the integers modulo k is also associative under addition.

$$\begin{aligned} \forall x, y, z \in \mathbb{Z},\ [x]_k + ([y]_k + [z]_k) =& [x]_k + [y + z]_k \\ =& [x + y + z]_k \\ =& [x + y]_k + [z]_k \\ =& ([x]_k + [y]_k) + [z]_k \end{aligned}$$

P2
There exists a 0-element, namely $[0]_k$.
$\forall x \in \mathbb{Z},\ [x]_k + [0]_k = [0]_k + [x]_k = [x]_k$

P3

Addition under the integers modulo k also has inverses.
$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ such that $[x]_k + [y]_k = [0]_k$. We can choose that element $y := -x$, and thus: $[x]_k + [-x]_k = [x - x]_k = [0]_k$ P4

Inherited from the integers, the integers modulo k is also commutative under addition.
$\forall x, y \in \mathbb{Z},\ [x]_k + [y]_k = [x + y]_k = [y + x]_k = [y]_k + [x]_k$

P5
Integers modulo k is similarly associative under multiplication.

$$\begin{aligned}\forall x, y, z \in \mathbb{Z},\ [x]_k \cdot ([y]_k \cdot [z]_k) =& [x]_k + [yz]_k \\ =& [xyz]_k \\ =& [xy]_k \cdot [z]_k \\ =& ([x]_k \cdot [y]_k) \cdot [z]_k\end{aligned}$$

P6
There exists a e-element, namely $[1]_k$.
$\forall x \in \mathbb{Z},\ [x]_k \cdot [1]_k = [1x]_k = [1]_k \cdot [x]_k = [x]_k$

P7
This is the problematic axiom.

P8
Modular multiplication is also commutative.
$\forall x, y \in \mathbb{Z},\ [x]_k \cdot [y]_k = [xy]_k = [yx]_k = [y]_k \cdot [x]_k$

P9
Finally, the distributive property holds true.

$$\begin{aligned}\forall x, y, z \in \mathbb{Z},\ [x]_k \cdot ([y]_k + [z]_k) =& [x]_k \cdot [y + z]_k \\ =& [x(y + z)]_k \\ =& [xy + xz]_k \\ =& [xy]_k + [xz]_k\end{aligned}$$

$\square$

c)

Proof.

We will prove this by showing that each non-zero element of $\mathbb{Z}_{13}$ has a multiplicative inverse.

$$\begin{aligned}
[1]_{13} \cdot [1]_{13} &= [1]_{13} \\
[2]_{13} \cdot [7]_{13} &= [14]_{13} = [1]_{13} \\
[3]_{13} \cdot [9]_{13} &= [27]_{13} = [1]_{13} \\
[4]_{13} \cdot [10]_{13} &= [40]_{13} = [1]_{13} \\
[5]_{13} \cdot [8]_{13} &= [40]_{13} = [1]_{13} \\
[6]_{13} \cdot [11]_{13} &= [66]_{13} = [1]_{13} \\
[7]_{13} \cdot [2]_{13} &= [14]_{13} = [1]_{13} \\
[8]_{13} \cdot [5]_{13} &= [40]_{13} = [1]_{13} \\
[9]_{13} \cdot [3]_{13} &= [27]_{13} = [1]_{13} \\
[10]_{13} \cdot [4]_{13} &= [40]_{13} = [1]_{13} \\
[11]_{13} \cdot [6]_{13} &= [66]_{13} = [1]_{13} \\
[12]_{13} \cdot [12]_{13} &= [144]_{13} = [1]_{13}
\end{aligned}$$

□

d)

Proof.

There exists an $a \in \{1, 2, 3, \cdots, 12\}$ that satisfies the equation, namely $a = 2$.

$$\begin{aligned}
\frac{1}{\frac{[4]_{13}}{[3]_{13}} + \frac{[2]_{13}}{[7]_{13}}} - \frac{[3]_{13}}{[10]_{13}} =& \frac{1}{([4]_{13} \cdot [3]_{13}^{-1}) + ([2]_{13} \cdot [7]_{13}^{-1})} - [3]_{13}[10]_{13}^{-1} \\
=& \frac{1}{([4]_{13} \cdot [9]_{13}) + ([2]_{13} \cdot [2]_{13})} - [3]_{13}[4]_{13} \\
=& \frac{1}{[10]_{13} + [4]_{13}} - [12]_{13} \\
=& \frac{1}{[1]_{13}} - [12]_{13} \\
=& [1]_{13} + [-12]_{13} \\
=& [2]_{13}
\end{aligned}$$

□

5. a) We will prove these statements using contradiction.

Proof.

Assume all functions $f : A \to B$ are not bijective. Then, consider $A := \{1\}$ and $f : A \to A,\ f(x) = x$. f is injective because there is only one element in the domain. Now let $B := f(A)$ and thus, $f(A) \subseteq B \subseteq A$. Now consider the map $h : A \to B,\ h(x) = f(x)$. We have already proven that this function is injective and because $h(A) = B,\ h$ is surjective. Thus h is bijective, a contradiction.

□

b)

Proof.

Assume all functions from $h : A \to B$ are not bijective. Consider $A := \{1\}$; $B := \{1\}$; $f : A \to B,\ f(1) = 1$; and $g : B \to A,\ g(1) = 1$. These functions are both clearly injective. f is also clearly bijective, contradicting the assumption that all functions $h : A \to B$ are not bijective.

□