MAT157 Problem Set 7

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1.

Proof.

Consider $\forall x,\ g : [x, x+T] \to \mathbb{R},\ x \mapsto f(x)$. Because f is periodic, all elements in the $f(\mathbb{R})$ are in $g([x, x+T])$. Similarly, all elements of $g([x, x+T])$ are in $f(\mathbb{R})$. Thus, $f(\mathbb{R}) = g([x, x+T])$. However, because f is continuous and $[x, x+T]$ is a bounded interval, f must also be bounded through the boundedness theorem. Likewise, due to the extreme value theorem, because $[x, x+T]$ is closed and continuous, $g([x, x+T])$ also take on a minimum and a maximum.

□

2.

Proof.

Let $\epsilon > 0$.

If $\inf\{|a-y| : y \in S\} = 0$, then let $\delta = \epsilon$. If $0 < |x-a| < \delta$ then:

$$\begin{aligned}
|\inf\{|x-y|\} - \inf\{|a-y|\}| &= |\inf\{|x-y|\}| \\
&\leq |\inf\{|x-y| + |a-y|\}| \\
&= \inf\{|x-a|\} \\
&\leq \delta \\
&< \epsilon
\end{aligned}$$

If $\inf\{|a-y| : y \in S\} > 0$, then let $c =$inf$\{|a-y| : y \in S\}$, and let $\delta = \frac{\epsilon}{2}$. Notice that if $x \in (a-\delta, a+\delta)$, then $\inf\{|x-y|\} \leq c+\delta$ or $\inf\{|x-y|\} \leq c-\delta$ If $0 < |x-a| < \delta$ then:

$$\begin{aligned}
|\inf\{|x-y|\} - c| &= \inf\{|x-y| + |a-y|\} \\
&\leq |c \pm \delta - c| \\
&= \delta \\
&= \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

Thus, f is continuous for all $a \in \mathbb{R}$, making f continuous.

□

3.

Proof.

$\subseteq$

Consider any open $U \subseteq \mathbb{R}$. Consider any $u \in U$. For $a \in A$ such that $f(a) = u$, then $a \in f^{-1}(U)$. Now consider the set of intervals $(u-1, u+1),\ \forall u \in U$. Clearly, all of these intervals cover U; however, because f is continuous, then for each 1-interval of U, there exists a single or multiple corresponding δ-interval around $a \in A$, such that $f(a) = u$. Thus, if we take the infinite union of these open intervals, V, V is open and $f(V) = U$ and $f^{-1}(U) = V \cap A$.

$\supseteq$

We will prove the contrapostitve. Assume that f is not continuous. Then there exists an $\epsilon > 0, \forall \delta > 0, \exists a \in A$ such that $0 < |x - a| < \delta$ and $|f(x) - f(a)| \geq \epsilon$. Thus if $f^{-1}((f(a) - \epsilon, f(a) + \epsilon))$, surely a is in this set. However, because f is not continuous at a, there exists no corresponding δ interval around a there is no δ-interval around a such that $(a - \delta, a + \delta) \subseteq f^{-1}((f(a) - \epsilon, f(a) + \epsilon))$, making $f^{-1}((f(a) - \epsilon, f(a) + \epsilon))$ not open. Therefore, there exists a open set of U that does not have a corresponding open set V such that $f^{-1}(U) = V \cap A$.

$\square$

4.a)

Proof.

If A has the Heine-Borel property, then for any cover of A, there exists a finite supcover. Consider $\bigcup_{i\in\mathbb{R}} U_i,\ U_i = (i-1, i+1),\ i \in \mathbb{R}$ is a cover of A, and thus, A must have a finite subcover. A single open set in the cover of A is bounded, thus if we take the finite union of the open sets the cover A (the finite subcover), the union of those open sets is also bounded. Since those open sets cover A, A must be bounded.

□

b)

Proof.

We will prove the contrapostive.

Consider A is not closed. Then, A has a limit point, $c \notin A$. Then we can construct a cover of A:
$\bigcup_{n\in\mathbb{N}} U_n,\ U_n = (-\infty, c-\frac{1}{n}) \cup (c+\frac{1}{n}, \infty)$ Clearly this covers A but if we take any finite subcover, of $\bigcup_{n\in\mathbb{N}} U_n$ with largest open set being $U_N,\ N \in \mathbb{N}$, then $(c-\frac{1}{N}, c+\frac{1}{N})$ is not covered. But since c is a limit point, there are points in A that are always going to be some ϵ-close to c which are not going to be covered by our finite subcover. Therefore, A does not have the Heine-Borel property.

□

c)

Proof.

Since A is bounded, then we can create a close interval $[a, b]$ such that $A \subseteq [a, b]$. Consider any cover of $[a, b]$, $\bigcup_{i\in I} U_i$ where I is any indexing set. Let $S := \{x \in [a, b] : [a, x] \text{ has finite subcover}\}$. Clearly, S is non-empty because there has to exist at least one open set in $\bigcup_{i\in I} U_i$ such that a is in that open set. Notice that S is also bounded above by b. Because S is non-empty, then S has a supremum. Thus, there must be some open set, U, that covers $\sup(S)$. Because the set is open, we can choose some element $x < \sup(S),\ x \in U$. However, x is not an upperbound for S, thus through completeness, we can choose y, such that $x < y < \sup(S)$ and $y \in S$. Since y is in S, then there is a finite subcover of $[a, y]$.

Now consider $\sup(S) < b$. Then, we can choose some element greater than $\sup(S)$ in U, which will also be in S, contradicting the fact that $\sup(S)$ is the smallest upper bound. If $\sup(S)$ cannot be less than b, but S is bounded above by b, that implies that $\sup(S) = b$. Because we know that $[a, y], \forall y < \sup(S)$ has finite subcover and $b \in U$, then $[a, y] \cup U$ is a finite subcover that covers $[a, b]$. Thus, $[a, b]$ has the Heine-Borel Property.

Now we will show that A has the Heine-Borel property. Notice that if A is closed, then A^c is open. Consider any cover of A, $\bigcup_{i \in I} U_i$. Then $A^c \cup \bigcup_{i \in I} U_i$ is an open cover that covers $[a, b]$. We already have shown that this has finite subcover, $\bigcup_{i=1}^{n} U_i$. Thus if we take the difference of this cover with A^c, we get a finite subcover that covers A as desired.

Thus, if A is closed and bounded, it has the Heine-Borel property.

$\square$

5. a)

Proof.

Consider $f(x) = \begin{cases} 0 \text{ if } x = 0 \\ 1 \text{ otherwise} \end{cases}$. For any $a \in \mathbb{R}$, f is locally bounded by 2. However, f is not continuous at 0 because $\lim_{x \to 0} f(x) = 1 \neq f(0)$.

$\square$

b)

Proof.

Consider $f(x) = \begin{cases} q \text{ if } x \in \mathbb{Q} \text{ with } x = \frac{p}{q} \text{ in reduced form} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$.
Consider any $x_0 \in \mathbb{R}$ with any δ-interval around x_0. Consider f is bounded on this δ-interval. Then it has some upperbound $C > 0$. Because the rationals are dense, then there exists $\frac{m}{n}, \frac{s}{t} \in (x_0 - \delta, x_0 + \delta) : \frac{m}{n} < x < \frac{s}{t}$. Choose a prime number p such that $p > \max\{C, n\}$ such that $\frac{m}{n} + \frac{1}{p} < \frac{s}{t}$. Thus, $\frac{P+n}{np} \in (x_0 - \delta, x_0 + \delta)$. But notice that $n < P$, and thus, $P + n$ does not divide np, so this rational is in reduced form. Thus, $f(\frac{P+n}{np}) = np > C$. Thus, f is nowhere locally bounded.

$\square$

c)

Proof.

If f is locally bounded on $[a, b]$ then there exists δ-intervals around x for all $x \in \mathbb{R}$ such that f is locally bounded by some C. Let our open cover of $[a, b]$ be $\bigcup_{i \in \mathbb{R}} U_i$ such that U_i is the corresponding locally-bounded δ-interval around i for all $i \in \mathbb{R}$. Clearly, this covers $[a, b]$. Since $[a, b]$ is closed and bounded, it has the Heine-Borel property. Which allows us to find finite subcover of $[a, b]$. Thus, we have a finite set of n δ-intervals that cover $[a, b]$. Induced by this we also have a finite set $\{C_1, C_2, ..., C_n\}$ of bounds of f for each δ-interval. If we take the max of $\{C_1, C_2, ..., C_n\}$, call it C'; thus $|f(x)| < C'$, $\forall x \in [a, b]$, meaning that f is bounded.

$\square$