MAT157 Problem Set 10

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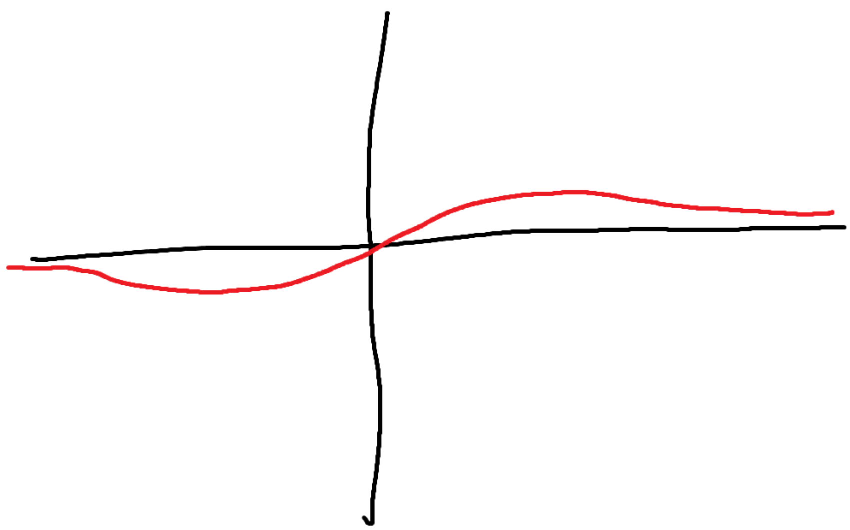

$f : \mathbb{R} \to \mathbb{R},\ f(x) = \dfrac{x}{1+x^2}$

$f' : \mathbb{R} \to \mathbb{R},\ f'(x) = \dfrac{-(x^2-1)}{(x^2+1)^2}$

$f'' : \mathbb{R} \to \mathbb{R},\ f''(x) = \dfrac{2x(x^2-3)}{(x^2+1)^2}$

$f(0) = 0,$ so f has a root at $x = 0$

$f'(1) = f'(-1) = 0,$ so f' has roots at $x = 1$ and $x = -1$

This means that f is incresing on the interval $(-1, 1)$ because f' is positive on that interval.

This means that f is decreasing on the set $(-\infty, -1) \cup (1, \infty)$ because f' is negative on that set.

We also may conclude that $f(1)$ is a local/global maxima, whereas $f(-1)$ is a local/global minima.

$f''(0) = f''(\sqrt{3}) = f''(-\sqrt{3}) = 0,$ so f'' has roots at $x = 0, \sqrt{3}$ and $-\sqrt{3}$.

This means that f is concave up on the set $(-1, 0) \cup (1, \infty)$ because f'' is positive on that set.

This means that f is concave down on the set $(-\infty, -1) \cup (0, 1)$ because f'' is negative on that set.

$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$

Proof.

Let $\varepsilon > 0$ Choose $n > \frac{1}{\varepsilon}$. Then for every $x > n$,
$|f(x)| = \frac{x}{1+x^2} < \frac{n}{1+n^2} < \frac{n}{n^2} = \frac{1}{n} < \varepsilon$ as desired. The proof is similar for $x \to -\infty$.

$\square$

2. a)

Proof.

Let $f : [a, b] \to \mathbb{R}$ and P be an arbitrary partition of $[a, b]$. Let $[t_{i-1}, t_i]$ be a interval induced by the partition P. If $f(x) \geq 0, \forall x \in [t_{i-1}, t_i]$ then $f = |f|$, meaning $\sup f([t_{i-1}, t_i]) - \inf f([t_{i-1}, t_i]) = \sup |f|([t_{i-1}, t_i]) - \inf |f|([t_{i-1}, t_i])$. Now, consider $f(x) \leq 0, \forall x \in [t_{i-1}, t_i]$ the $-f = |f|$, meaning $\sup f([t_{i-1}, t_i]) - \inf f([t_{i-1}, t_i]) = \sup |f|([t_{i-1}, t_i]) - \inf |f|([t_{i-1}, t_i])$ as well. If we consider our last alternative, $f(x)$ is neither all positive nor all negative (nor all zero, because that case is trivial) $\forall x \in [t_{i-1}, t_i]$. Consider $M := \sup f([t_{i-1}, t_i]) > 0$ and $m := \inf f([t_{i-1}, t_i]) < 0$. Thus $M - m = M + |m|$. Now consider $M' := \sup |f|([t_{i-1}, t_i]) > 0$ and $m' := \inf |f|([t_{i-1}, t_i]) \geq 0$. Thus, $M' - m' \leq M' = \max\{M, |m|\}$. Thus, $M' - m \leq \max\{M, |m|\} < M - m$. Now, we have proven that for any arbitrary interval induced by the partition P on f, $\sup |f|([t_{i-1}, t_i]) - \inf |f|([t_{i-1}, t_i]) \leq \sup f([t_{i-1}, t_i]) - \inf f([t_{i-1}, t_i])$. Thus, because this holds true for all intervals of any partitions, it simply follows that $U(|f|, P) - L(|f|, P) = \sum_{i=0}^{n} M_i'(t_{i-1} - t_i) - \sum_{i=0}^{n} m_i'(t_{i-1} - t_i) \leq \sum_{i=0}^{n} M_i(t_{i-1} - t_i) - \sum_{i=0}^{n} m_i(t_{i-1} - t_i) = U(f, P) - L(f, P)$ as desired.

$\square$

b)

Proof.

It follows from part a) and *Riemann's Criterion* that because $\forall \varepsilon > 0, U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) < \varepsilon$, then $|f|$ is also integrable.

$\square$

c)

Proof.

Consider $f', g' : \mathbb{R} \to \mathbb{R},\ f'(x) = \frac{f(x)+|f(x)|}{2}, g'(x) = \frac{f(x)-|f(x)|}{2}$. We can very easily show that $f = f_+$ and that $g = f_-$. However, because f_+ is the sum of two integral functions (f and $|f|$) and multiplication by a scalar $\frac{1}{2}$, f_+ must also be integrable. It follows similarly for f_-.

$\square$

d)

Proof.

Consider $f', g' : \mathbb{R} \to \mathbb{R}$, $f'(x) := \frac{f(x)+g(x)+|f(x)-g(x)|}{2}$, $g'(x) := \frac{f(x)+g(x)-|f(x)-g(x)|}{2}$. It is easy to see that $\max = f'$ and $\min = g'$ (consider $x > y$ and vice-versa). Clearly, since max is sum of three integrable functions (f, g and $|f - g|$) with a product of a scalar, max must be integrable. Similarly, min is also integrable.

$\square$

3.

Proof.

We will use induction on n.
Base Case: $n = 2$: $1^k \leq \frac{2^{k+1}}{k+1} \leq 1^k + 2^k$. As desired.
Induction Hypothesis: If $S_k(n-1) \leq \frac{n^{k+1}}{k+1} \leq S_k(n)$, then $S_k(n) \leq \frac{(n+1)^{k+1}}{k+1} \leq S_k(n+1)$.
Induction Step:

$$\begin{aligned} S_k(n) &= S_k(n-1) + n^k \\ &\leq \frac{n^{k+1}}{k+1} + n^k \\ &= \frac{n^k(n + (k+1))}{k+1} \\ &< \frac{(n+1)^k(n+1)}{k+1} = \frac{(n+1)^{k+1}}{k+1} \end{aligned}$$

□

4. a)

Proof.

Let $\varepsilon > 0$ and consider any finite set of closed intervals $J_1, ..., J_N$ such that $S \subseteq \bigcup_{i=1}^{N} J_i,\ \sum_{i=1}^{N}(d_i - c_i) < \varepsilon',\ \varepsilon' \leq \min\{\frac{\varepsilon}{b-a}, \frac{\varepsilon}{|\sup f([a,b]) - \inf f([a,b])|}\}$. Now, take a finite set of points $x_0, ..., x_m$ such that the $\delta-$intervals (in respect to $\frac{\varepsilon'}{2}$) around each point is a finite cover of $[a, b]$. We will used the partition $P := \{c_1, d_1, ..., c_n, d_n\} \cup \{x_0, ..., x_m\} \cup \{a, b\}$. Thus, the distance between any two Riemann summands in the upper and lower Riemann sums is at most ε'. Thus, $U(f, P) - L(f, P) \leq \sum_{i=0}^{2n+m+3} \varepsilon'(t_i - t_{i-1}) = \varepsilon'(b - a) < \varepsilon$. Because our choice of ε was arbitrary, this allows us to use the *Riemann Criterion* to prove that f is integrable.

$\square$

b)

Proof.

Let $\varepsilon > 0$ and consider any finite set of closed intervals $J_1, ..., J_N$ such that $S \subseteq \bigcup_{i=1}^{N} J_i,\ \sum_{i=1}^{N}(d_i - c_i) < \varepsilon',\ \varepsilon' \leq \min\{\frac{\varepsilon}{b-a}, \frac{\varepsilon}{\sup\{|f(x)-g(x)|:\forall x \in [a,b]\}}\}$. Now choose a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon'$. If we take the common refinement between P and our set $\{c_1, d_1, ...c_n, d_n\}$ as our partition for g, P' then we know that $U(f) - U(g, P') < \varepsilon' < \varepsilon$. Thus we can choose a partition that makes difference between the upper integral of f and upper Riemann sum of g in respect to P' less than ε. We can do a similar thing to the lower integral of f and the lower Riemann sum of g in respect to P'. Since can make the difference between the upper and lower Riemann sums and $\int_{[a,b]} f$ less than ε, meaning we can choose a partition that makes $U(g, P) - L(g, P) < \varepsilon$, making it integrable following the *Riemann Criterion*.

The converse holds if we just replace f with g.

$\square$

5. a)

Proof.

We will first prove that $\sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k}p(k) = 0$ for any polynomial of the form $p(x) = \lambda x^n$. We will induct on n.
Base Case: $n = 1$, $\sum_{k=0}^{1} \binom{n}{k}(-1)^{1-k}p(k) = 0$ as desired.
Induction Hypothesis: If $\sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k}p(k) = 0$, then $\sum_{k=0}^{n+1} \binom{n+1}{k}(-1)^{n+1-k}p(k) = 0$.
Inductive Step:

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{n+1}{k}(-1)^{n+1-k}p(k) &= \lambda(k-1)^{n+1} \\
&= \lambda(k-1)^{n}(k-1) \\
&= \sum_{k=0}^{n+1} \binom{n}{k}(-1)^{1-k}p(k)(k-1) \\
&= 0
\end{aligned}$$

□