MAT247 Problem Set 2

Nicolas

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1.

Proof.

$\Rightarrow$

Consider $L = v - \sum_{i=1}^{m} \langle v, e_i \rangle e_i$. Because our list is orthonormal, this is equal to $\langle v, e_i \rangle - \langle v, e_i \rangle = 0, \ \forall 1 \leq i \leq m$ Thus, this shows that L and all e_i are orthogonal. Which means that $\|L\|^2 = \|\langle v - \sum_{i=1}^{m} \langle v, e_i \rangle, e_i \rangle\|^2 = \|v\|^2$ by construction. However, we have an orthonormal list, so $\|\sum_{i=1}^{m} \langle v, e_i \rangle e_i\| = \sum_{i=1}^{m} |\langle v, e_i \rangle|^2$, which by assumption means that $L = 0$.

$\Leftarrow$

This direction is trivial because if $v \in \text{span}(e_1, e_2, ..., e_m)$ then that list is a ONB for the subspace defined by the span of the list (following from *Axler* 6.30).

$\square$

2.

Proof.

Applying *Gram-Schmidt Process* to the basis of vectors of U gives us the ONB $((\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0), (0, \frac{i}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}))$. Thus, to find the $u \in U$ that minimizes $\|u - (1,2,3,4)\|$ we simply need to find the projection of $(1,2,3,4)$ onto U. This is easy because we have a ONB. Thus $P_U(1,2,3,4) = \langle(1,2,3,4), (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0)\rangle(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0) + \langle(1,2,3,4), (0, \frac{i}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}})\rangle(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0)$. Thus, $u = \frac{4}{\sqrt{2}}(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0) + \frac{2i}{\sqrt{5}}(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0)$.

$\square$

3.

Proof.

Let $U = P(V)$. Consider $x \in \text{Image}(P)$ and $y \in \text{null}(P)$ Because P is linear $P(x + \alpha y) = P(x) + P(\alpha y) = P(x) = x, \alpha \in F$. This implies that $\|x\|^2 \leq \|x + \alpha y\|^2$ by assumption. But if this is true for all α, then $\langle x, y \rangle = 0$. Thus we have can make an orthogonal decomposition of every vector in V, with $x \in U = P(v)$ and $y \in U^{\perp}$, which by definition is an orthogonal projection onto U.

$\square$

4.

Proof.

$\subset$

Let $x \in U^\star = (U^\perp)^\perp$. Then $\langle x, u \rangle = 0, \forall u \in (U^\perp)^\perp)^\perp)$ which means by definition, $x \in (((U^\perp)^\perp)^\perp)^\perp) = (U^\star)^\star$

$\supset$

Consider $x \notin U^\star$. This means that $\langle x, u \rangle \neq 0,\ \forall u \in (U^\star)^\perp$ which means that $x \notin (U^\star)^\star$. Thus we have proven the contrapositive.

$\square$

5.

Proof.

Consider the functional $L(v) : V \to \mathbb{C},\ L(v) = \sum_{n\in\mathbb{N}} f(n)$. This functional is linear because $L(\lambda v) = \sum_{n\in\mathbb{N}} \lambda f(n) = \lambda \sum_{n\in\mathbb{N}} f(n)$. $L(u+v) = \sum_{n\in\mathbb{N}}(u+v)(n) = \sum_{n\in\mathbb{N}} u(n) + \sum_{n\in\mathbb{N}} v(n)$. However, this cannot be expressed as an inner product because inner products need require some form of a second vector.

$\square$