CSC240 Problem Set 2

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1.

We will use proof by contrapostitive.

Assume NOT 2CR(C).

NOT 2CR(C) IMPLIES $\exists \alpha, \beta, \gamma \in \{1, ..., N\}$.

$\alpha \neq \beta$ AND $\alpha \neq \gamma$ AND ($\forall e \in C_\alpha(e \in C_\beta$ OR $e \in C_\gamma$

By definition, $i \in C_j$ IFF $M_{[i,j]} = 1$ IFF $j \in R_i$.

Let $\alpha \in \{1, ..., N\}$ such that $\exists \beta \in \{1, ..., N\}.\exists \gamma \in \{1, ..., N\}$.

$\alpha \neq \beta$ AND $\alpha \neq \gamma$ AND ($\forall e \in C_\alpha(e \in C_\beta$ OR $e \in C_\gamma$

Let $e := \alpha \in \{a, b, c\}$

Let $i \in \{1, ..., N\}$ be arbitrary.

Either $\alpha \in R_i$ OR NOT $\alpha \in R_i$.

Case 1: NOT $\alpha \in R_i$.

NOT $e \in R_i$.

Case 2: $\alpha \in R_i$.

By above $\alpha \in R_i$ IMPLIES $\beta \in R_i$ OR $\gamma \in R_i$

$\exists d \in S.d \in T$ AND NOT $d = e$, namely γ or β.

Thus either $e \notin R_i$ OR $\exists d \in S.d \in T$ AND NOT $d = e$

i was arbitrary so it holds for all R_i.

$\exists S \in \mathcal{P}_3(\{1, ..., N\}).\exists e \in S \forall T \in R.e \notin T$ OR $\exists d \in S.d \in T$ AND NOT $d = e$

NOT SEL($3, R$).

2.

1 Assume NOT 2CR(C).

2 To obtain a contradiction, assume $\forall\alpha \in \{1, ..., N\}.\forall\beta \in \{1, ..., N\}.\forall\gamma \in \{1, ..., N\}$.
[(NOT $(\alpha = \beta)$) AND (NOT $(\alpha = \gamma)$) AND ($\exists e \in C_\alpha$(NOT MEMBER(e, C_β) AND NOT MEMBER(e, C_γ)

3 $\exists S \in C.\exists X \in C.\exists Y \in C$.
[(NOT $(S = X)$) AND (NOT $(S = Y)$) AND ($\exists e \in S$(MEMBER(e, X) OR MEMBER(e, Y)))]
modus ponens: 1.

4 Let $\alpha \in \{1, ..., N\}$ such that $C_\alpha = S$.

5 let $\beta \in \{1, ..., N\}$ such that $C_\beta = X$.

6 Let $\gamma \in \{1, ..., N\}$ such that $C_\gamma = Y$.

7 This is a contradiction: 3,4,5,6.

8 Therefore $\exists\alpha \in \{1, ..., N\}.\exists\beta \in \{1, ..., N\}.\exists\gamma \in \{1, ..., N\}$.
[(NOT $(\alpha = \beta)$) AND (NOT $(\alpha = \gamma)$) AND ($\forall e \in C_\alpha$(MEMBER(e, C_β) OR MEMBER(e, C_γ)))]

9 Let $i \in \{1, ..., n\}$ be arbitrary.

10 MEMBER(i, C_n) IMPLIES $M_{[i,n]} = 1$: modus ponens: by assumption.

11 $M[i, n] = 1$ IMPLIES MEMBER(n, R_i): modus ponens: by assumption.

12 Since i is an arbitrary element of $\{1, ..., N\}$.

13 $\forall i \in \{1, ..., N\}$.MEMBER$(i, C_n)$ IMPLIES MEMBER(n, R_i): generalization 9.

14 $\exists\alpha \in \{1, ..., N\}.\exists\beta \in \{1, ..., N\}.\exists\gamma \in \{1, ..., N\}$.
[(NOT $(\alpha = \beta)$) AND (NOT $(\alpha = \gamma)$)($\exists e \in C_\alpha$(MEMBER(e, C_β) OR MEMBER(e, C_γ)))];
modus ponens: 8.

15 let $a \in \{1, ..., N\}$ be such that $\forall b \in \{1, ..., N\}.\exists c \in \{1, ..., N\}$.
[(NOT $(a = b)$) AND (NOT $(a = c)$) AND ($\forall e \in C_a$(MEMBER(e, C_b) OR MEMBER(e, C_c)))]
Instantiation: 8

16 Let $\{a, b, c\} \in \mathcal{P}_3(\{1, ..., N\})$.

17 Let $e := a \in \{a, b, c\}$

18 Let $i \in \{1, ..., n\}$ be arbitrary.

19 Either MEMBER(a, R_i) OR NOT MEMBER(a, R_i).

20 Case 1: Assume MEMBER(a, R_i)

21 MEMBER(a, R_i) IMPLIES MEMBER(i, C_a); modus ponens: by assumption.

22 MEMBER(i, C_a) IMPLIES $\exists b \in \{1, ..., n\}.\exists c \in \{1, ..., n\}$.
[(NOT $(a = b)$) AND (NOT $(a = c)$) AND (MEMBER(i, C_b)
OR MEMBER(i, C_c)))]; modus ponens: 15.

23 Either MEMBER(i, C_b) OR NOT MEMBER(i, C_b).

24 Case 1.a: Assume MEMBER(i, C_b).

25 MEMBER(i, C_b) IMPLIES MEMBER(b, R_i); modus ponens: 13.

26 Thus $\exists d \in R_i$.MEMBER$(d, \{a, b, c\})$ AND NOT $d = e$.

27 MEMBER(i, C_b) IMPLIES $\exists d \in R_i$.MEMBER$(d, \{a, b, c\})$ AND NOT $d = e$
direct proof: 25, 26.

28 Case 1.b: Assume NOT MEMBER(i, C_b)

29 NOT MEMBER(i, C_b) IMPLIES MEMBER(c, R_i); modus ponens: 13.

30 Thus $\exists d \in R_i$.MEMBER$(d, \{a, b, c\})$ AND NOT $d = e$.

31 NOT MEMBER(i, C_b) IMPLIES $\exists d \in R_i$.MEMBER$(d, \{a, b, c\})$ AND NOT $d = e$
direct proof: 29, 30.

32 MEMBER(a, R_i) IMPLIES $\exists d \in R_i$.MEMBER$(d, \{a, b, c\})$ AND NOT $d = e$;
direct proof: 24, 29.

33 Case 2: Assume NOT MEMBER(a, R_i).

34 NOT MEMBER(a, R_i) IMPLIES NOT MEMBER(e, R_i); modus ponens: 17.

35 NOT MEMBER(a, R_i) IMPLIES NOT MEMBER(e, R_i); direct proof: 34

36 Since i is an arbitrary element of $\{1, ..., n\}$

37 $\forall i \in \{1, ..., n\}.\exists d \in R_i$.MEMBER$(d, \{a, b, c\})$ AND NOT $d = e$ OR NOT MEMBER(e, R_i).
generalization: 18.

38 $\exists S \in \mathcal{P}_3(R).\exists e \in S.\forall T \in R.\exists d \in S$.MEMBER$(d, T)$ AND NOT $d = e$ OR NOT MEMBER(e, T).
Construction: 15, 16, 17.

39 NOT SEL(R)

40 SEL$(3, R)$ IMPLIES 2CF(C); indirect proof: 1, 40.