MAT240 Problem Set 9

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1. a)

Proof. $\square$

3. 1)

Proof.

Consider two stochastic matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $\mathbb{F}^{n\times n}$ with the property that $a_{1j} + a_{2j} + \cdots + a_{nj} = 1$ and $b_{1j} + b_{2j} + \cdots + b_{nj} = 1,\ \forall j,\ 1 \le j \le n$ and $a_{ij}, b_{ij} \in \mathbb{R}^+$. If we take the product of A and B, $C = [c_{ij}]$ such that:

$$[c_{ij}] = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Thus, if we take the sum of the j-th columns of C, we get:

$$\begin{aligned} c_{1j} + c_{2j} + \cdots + c_{nj} &= \sum_{k=1}^{n} a_{1k}b_{kj} + \sum_{k=1}^{n} a_{2k}b_{kj} + \cdots + \sum_{k=1}^{n} a_{nk}b_{kj} \\ &= b_{1j}\sum_{k=1}^{n} a_{kj} + b_{2j}\sum_{k=1}^{n} a_{kj} + \cdots + b_{nj}\sum_{k=1}^{n} a_{kj} \end{aligned}$$

However, we know that $\sum_{k=1}^{n} a_{kj} = 1,\ \forall j, 1 \le j \le n$. Thus:

$$\begin{aligned} &= b_{1j} + b_{2j} + \cdots b_{nj} \\ &= 1 \end{aligned}$$

Because we chose the column arbitrarily, each column in C adds to 1. Because each entry in C is a sum of non-negative real numbers (product of two non-negatives is non-negative), each entry in C is also non-negative. Thus, the product of two stochastic matrices is indeed another stochastic matrix.

□

3. 2)

Proof.

4.

Proof.

Let P be a pentagon consisting of vertices $\{1, 2, 3, 4, 5\}$.
Then:

$$P_\Gamma = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This matrix has a Eigenvalues 2, $-\frac{1+\sqrt{5}}{2}$ and $\frac{1+\sqrt{5}}{2}$. This is because $P(1,1,1,1,1) = (2,2,2,2,2)$,
$P(-1, \frac{1+\sqrt{5}}{2}, -\frac{1+\sqrt{5}}{2}, 1, 0) =$
$(\frac{1+\sqrt{5}}{2}, -(\frac{1+\sqrt{5}}{2})^2, (\frac{1+\sqrt{5}}{2})^2, -(\frac{1+\sqrt{5}}{2}), 0)$, and
$P(-1, -\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, 1, 0) =$
$(-\frac{1+\sqrt{5}}{2}, (\frac{1+\sqrt{5}}{2})^2, -(\frac{1+\sqrt{5}}{2})^2, (\frac{1+\sqrt{5}}{2}), 0)$. We know that theses are all the Eigenvalues of because the null space of $A - \lambda I$ has dimension 3.

□

□

5.

Proof.

Consider the map

$A : \mathbb{F}^X \oplus \mathbb{F}^Y \to F^{X \sqcup Y},\ A(f,g) = h,\ h(x) = \begin{cases} f(x) \text{ if } x \in X \\ g(x) \text{ if } x \in Y \end{cases}$. This is indeed a map because X and Y are disjoint. This map is surjective because if we consider any function $h : X \sqcup Y \to \mathbb{F}$, there exists an element in $F^X \oplus F^Y,\ (f,g)$ such that if $\forall x \in X,\ h(x) = f(x)$ and if $\forall x \in Y,\ h(x) = g(x)$. This map is injective because consider $A(f,g) = A(f',g')$. Then, $\forall x \in X \sqcup Y,\ A(f,g)(x) = A(f',g')(x)$. If $x \in X$, then $A(f,g)(x) = f(x) = A(f',g')(x) = f'(x)$. Similarly, if $x \in Y$, then $A(f,g)(x) = g(x) = A(f',g')(x) = g'(x)$. Thus, if $A(f,g) = A(f',g')$, then $f = f'$ and $g = g'$. Thus we have shown that there exists a isomorphism between $\mathbb{F}^X \oplus \mathbb{F}^Y$ and $F^{X \sqcup Y}$.

□