MAT240 Assignment 1

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1. a) $|Map(X_n, X_n)| = n^n$ if $n > 0$ or $|Map(X_n, X_n)| = 1$ if $n = 0$.

Proof.

Consider the finite, non-empty set $X_n = \{x_1, x_2, ..., x_n\}$ consisting of n elements and the map $f : X_n \to X_n$. $\forall x_i \in X_n, \exists f(x_i) \in X_n$, and because X_n has a cardinality of n, there are only n different values for $f(x_i)$. Then, there is a choice of n, n different times; thus, there are $\underbrace{n \cdot n \cdots n}_{n \text{ times}}$ different possible maps $f : X_n \to X_n$. Thus the cardinality of $Map(X_n, X_n)$ is $\underbrace{n \cdot n \cdots n}_{n \text{ times}}$, or simply n^n.

Considering the case where $n = 0$, X_n is empty, and thus we can easily say that there is one trivial map between the empty set and the empty set, so $|Map(\emptyset, \emptyset)| = 1$.

□

b) $|Bij(X_n, X_n)| = n!$

Proof.

Consider the finite, non-empty set $X_n = \{x_1, x_2, ..., x_n\}$ consisting of n elements and the bijection $f : X_n \to X_n$. Suppose $f(x_1) \in X_n$, because f is bijective $f(x_2) \in X_n \setminus \{f(x_1)\}$. Thus if we continue this process for n iterations we get:

$\forall x_i, i > 1, x_i \in X_n \setminus \{f(x_{i-1})\} \setminus \cdots \setminus \{f(x_1)\}$

Thus, there is a choice of $(n - i + 1)$, n different times (i ranges from 1 to n). This leaves us with $n \cdot (n-1) \cdots 2 \cdot 1$ different possible bijections $f : X_n \to X_n$. Therefore, the cardinality of $Bij(X_n, X_n)$ is $n \cdot (n-1) \cdots 2 \cdot 1$, or more concisely, $n!$! (This last "!" is an exclamation mark and not a second factorial.)

Taking into the consideration when $n = 0$ and $X_n = \emptyset$, there is only one bijection, the identity function. Thus $Bij(\emptyset, \emptyset) = 1 = 0!$

$\square$

2. a)

Proof.

$\square$

3. a)

Proof.

Consider g and g' are both inverses of f.

$$\forall x \in X,\ g(f(x)) = x = g'(f(x))$$

Since x was chosen arbitrarily, $f(x)$ is just an arbitrary object in Y.
Thus, $\forall y \in Y,\ g(y) = g'(y)$, and therefore $g = g'$

$\square$

b) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$ is not invertible because it is not a bijection. This is because $1 \neq -1$ but $f(1) = f(-1)$ (This is shown by the problem below.)

c)

Proof.

$\Rightarrow$
We will first prove that f is a surjection.
By assumption $\forall y \in Y,\ f(g(y)) = y$ and because $g(y)$ is just an arbitrary element of X, this shows that $\forall y \in Y,\ \exists x \in X$ st $f(x) = y$

We will now show that f is an injection.
Assume that f is not an injection.
Thus, $\exists x, x' \in X$ st $x \neq x'$ and $f(x) = f(x')$
Then $g(f(x)) = g(f(x')) = x = x'$ and thus we have reached a contradiction.

Thus we have proven that f is bijective.

$\Leftarrow$
Because $f : X \to Y$ is a bijective function, then
$\forall y \in Y, \exists! x \in X$ st $f(x) = y$
Thus we can construct a function $g : Y \to X$ that maps all $y \in Y$ to the unique $x \in X$ st $f(x) = y$
Thus, by construction $g(f(x)) = x$ and $f(g(y)) = y$

$\square$

d) It does not follow that $f \circ g = I_Y$

Proof.

Consider the function $f : \{a\} \to \{1, 2\}$ such that $f(a) = 1$ and the function $g : \{1, 2\} \to \{a\}$ such that $g(1) = a$ and $g(2) = a$
Thus $(g \circ f)(a) = a$ and $(g \circ f) = I_X$ But $(f \circ g)(2) = 1$ so $(f \circ g) \neq I_Y$
Thus, we have constructed a counterexample.

$\square$

e) It does follow that $f \circ g = I_Y$ now.

Proof.

If $g \circ f = I_X$ then f is injective because if we assume f is not injective, then we result in the contradiction where $g(f(x)) = g(f(x')) = x = x'$ when $x \neq x'$
Thus f is injective and surjective, making it a bijection, and by part c, this implies that it is invertible and there exists a g st $f \circ g = I_Y$

$\square$