MAT240 Problem Set 6

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1.

Proof.

Consider any vector in $\mathbb{R}^3$ that can be expressed in terms of its standard basis: $v = a(1,0,0) + b(0,1,0) + c(0,0,1),\ a,b,c \in \mathbb{R}$. By the linearity of T, $Tv = aT(1,0,0) + bT(0,1,0) + cT(0,0,1)$. Thus the range of T is all that can be produced from any linear combination of $(a,0,a,0),(0,b,0,b),(c,c,c,c)$. For any $c \in \mathbb{R}$, there exists a linear combination of $(a,0,a,0)$ and $(0,b,0,b)$ that can result in (c,c,c,c); thus, we can just remove it from our list. Thus, $\text{range}(T) = \{(x,y,x,y) : x,y \in \mathbb{R}\}$. A basis that can span this subspace is $(1,0,1,0),(0,1,0,1)$.

For any vector $v \in \mathbb{R}^3$,
$Tv = aT(1,0,0) + bT(0,1,0) + cT(0,0,1) = (a+c, b+c, a+c, b+c)$.
Thus, the set of vectors that map to 0 are the vectors where $a + c = 0$ and $b + c = 0$. Thus, $a = b$. Therefore, the $\text{null}(T) = \{(x,x,y) : x + y = 0,\ x,y \in \mathbb{R}\}$. A simple basis that spans this subspace is just the vector $(1,1,-1)$.

$\square$

2.

Proof.

We will prove this using double inclusion.

$\subseteq$

Consider any $x \in \text{null}(S) \cap \text{null}(T)$. Thus, $Sx = 0$ and $Tx = 0$. Then $Sx + Tx = 0 + 0 = 0$ and $Sx - Tx = 0 - 0 = 0$. Thus $x \in \text{null}(S + T) \cap \text{null}(S - T)$.

$\supseteq$

Consider any $x \in \text{null}(S + T) \cap \text{null}(S - T)$. Thus, $Sx + Tx = 0$ and $Sx - Tx = 0$. Then, $Tx = -Tx = 0$. Therefore, $Sx + 0 = 0$, making $Sx = 0$. Henceforth, $x \in \text{null}(S) \cap \text{null}(T)$.

$\square$

3. a)

Proof.

$$\begin{aligned}
Q^2 &= (I_v - P)^2 \\
&= I_v^2 - I_v P - I_v P + P^2 \\
&= I_v - P - P + P \\
&= I_v - P \\
&= Q
\end{aligned}$$

□

b)

Proof.

$$\begin{aligned}
PQ &= P(I_v - P) \\
&= PI_v - P^2 \\
&= P - P^2 \\
&= P - P \\
&= 0 \\
QP &= (I_v - P)P \\
&= I_v P - P^2 \\
&= P - P^2 \\
&= P - P \\
&= 0
\end{aligned}$$

□

c)

Proof.

Consider $w \in \text{range}(P) \setminus \{0\} : w = Pv,\ v \in V$. Assume for the sake of contradiction that $Pw = 0$, but this means that $PPv = Pv = w = 0$. But this contradicts the fact that w is non-zero.

Thus no non-zero element in the range of P maps to 0. Therefore, $N \cap R = \{0\}$.

$\square$

d)

Proof.

$$\begin{aligned} Pv + Qv &= Pv + (I_v - P)v \\ &= Pv + I_v v - Pv \\ &= Pv + v - Pv \\ &= v + Pv - Pv, \text{ vector addition associative and commutative} \\ &= v + 0 \\ &= v \end{aligned}$$

$\square$

e)

Proof.
We have already shown that P either maps a non-zero vector in V to 0 or it maps it to a vector, v, with the property that $Pv \neq 0$. Because we know that if w is in the range of P, then Pw is also in the range of P; thus, $P(\text{range}(P)) \subseteq \text{range}(P)$. Now consider any $w \in \text{range}(P)$ and $P|_R$, which is P restricted to the range of P. Assume for the sake of contradiction that $P|_R$ is not the identity map. Then consider any $P|_R v \in \text{range}(P)$. Thus, $P|_R P|_R v \neq P|_R v$. But this is a contradiction because $P = P^2$. Thus, $P|_R = I_R$. Thus, for any $v \in V$, either, $Pv = 0$ or $Pv = v$. $\{v \in V : Pv = 0\} = N$ and $\{v \in V : Pv = P\} = \text{range}(P)$. Therefore, $V = N \oplus R$ (we know this sum is direct because we already have proven that $N \cap R$ is non-zero disjoint).

$\square$

4. a)

Proof.

$$L = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\square$

b)

Proof.

$$L^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L^5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Because for $k = 5$, the matrix is the 0-matrix, every $k \geq 5$ is also the 0-matrix $(0 \cdot 0 = 0)$.

$\square$

Proof.

L

Considering that the only vectors that get mapped to the 0-vector, are vectors in the form $(a, 0, 0, 0, 0),\ a \in \mathbb{R}$. That means that the vector $(1, 0, 0, 0, 0)$ spans null(L). Thus, the dimension of null(L) is 1. Since $\mathbb{R}^5$ has a dimension of 5, then the dimension of the range$(L) = 4$. This is because the dim $\mathbb{R}^5 =$ dim null(L) + dim range(L).

L^2

Considering the only vectors that get mapped to the 0-vector, are vectors in the form $(a, b, 0, 0, 0),\ a, b \in \mathbb{R}$. That means that the vectors $(1, 0, 0, 0, 0)$ and $(0, 1, 0, 0, 0)$ span null(L^2). Thus, the dimension of null(L^2) is 2. Thus the dimension of the range$(L^2) = 3$.

L^3

The only vectors that get mapped to the 0-vector, are vectors in the form $(a, b, c, 0, 0),\ a, b, c \in \mathbb{R}$. That means that the vectors $(1, 0, 0, 0, 0), (0, 1, 0, 0, 0)$ and $(0, 0, 1, 0, 0)$ span null(L^3). Thus, the dimension of null(L^3) is 3. Thus the dimension of the range$(L^3) = 2$.

L^4

The only vectors that get mapped to the 0-vector are vectors in the form $(a, b, c, d, 0),\ a, b, c \in \mathbb{R}$. That means that the vectors $(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)$ and $(0, 0, 0, 1, 0)$ span null(L^4). Thus, the dimension of the range$(L^4) = 1$.

$L^k,\ k \geq 5$

Because for $k \geq 5$, $L^k = 0$, then that means that $\mathbb{R} =$ null(L^K). Thus the dimension of null$(L^k) = 5$ and the dimension of the range$(L^k) = 0$.

$\square$

5. a)

Proof.

Because $e_i([k]) = \begin{cases} 1 \text{ if } [k] = i \\ 0 \text{ otherwise} \end{cases}$, then we can break δe_i into cases for $n \geq 2$.

Case 1: $[k] = i$. Then $\delta e_i([k]) = e_i([k]) - e_i([k+1]) = 1 - 0 = 1$.
Case 2: $[k] = i + 1$. Then $\delta e_i([k]) = e_i([k]) - e_i([k+1]) = 0 - 1 = -1$
Case 3: $[k] \neq i$ and $[k] \neq i + 1$. Then
$\delta e_i([k]) = e_i([k]) - e_i([k+1]) = 0 - 0 = 0$

Thus, we can write $\delta e_i = f_i$ such that $f_i = \begin{cases} 1 \text{ if } [k] = i \\ -1 \text{ if } [k] = i + 1 \\ 0 \text{ otherwise} \end{cases}$.

Thus, the matrix of δ is as follows:

$$\delta = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix}$$

However, if $n = 1$ then $e_1([k]) = 1,\ \forall k \in \mathbb{Z}_n$. Thus $\delta e_1 = 1 - 1 = 0$.
Thus, for the special case where $n = 1$, then $\delta = [0]$.

$\square$

b)

Proof.

Again, because $e_i = \begin{cases} 1 \text{ if } k = i \\ 0 \text{ otherwise} \end{cases}$, then we can break De_i into cases.

Case 1: $k = 1 = i$. Then $De_i(k) = e_i(k) = 1$.
Case 2: $k = 1 \neq i$. Then $De_i(k) = e_i(k) = 0$.
Case 3: $i = k > 1$. Then $De_i(k) = e_i(k) - e_i(k-1) = 1 - 0 = 1$.
Case 4: $i + 1 = k > 1$. Then
$De_i(k) = e_i(k) - e_i(k-1) = 0 - 1 = -1$.
Case 5: Otherwise. Then $De_i(k) = e_i(k) - e_i(k-1) = 0 - 0 = 0$.

Thus, we can write $De_i = g_i$ such that $g_i = \begin{cases} 1 \text{ if } k = 1 = i \\ 0 \text{ if } k = 1 \neq i \\ 1 \text{ if } i = k > 1 \\ -1 \text{ if } i + 1 = k > 1 \\ 0 \text{ otherwise} \end{cases}$.

Then, the matrix of D is as follows:

$$D = \begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix}$$

□

c)

Proof.

Both maps are not invertible because the maps are not bijective.

For example consider $f, g \in \mathbb{R}^X$ such that $f(x) = 1$ and $g(x) = 2$. Clearly $f \neq g$ but $\delta f(x) = f(x) - f(x-1) = 1 - 1 = 0,\ \forall x \in \mathbb{Z}_n$ and $\delta g(x) = g(x) - g(x-1) = 2 - 2 = 0,\ \forall x \in \mathbb{Z}_n$. Because $f \neq g$ and $\delta f = \delta g$, then δ is not injective, and thus, it is not invertible.

Again consider $f, g \in \mathbb{R}^Y$ such that $f(x) = 0$ and $g(x) = 1$. Clearly $f \neq g$. For $x = 1$, then $Df(x) = 1$ and $Dg(x) = 1$. For $x > 1$, then $Df(x) = f(x) - f(x-1) = 1 - 1 = 0,\ \forall x \in Y \setminus \{1\}$ and $Dg(x) = g(x) - g(x-1) = 0 - 0 = 0,\ \forall x \in Y \setminus \{1\}$. Because $f \neq g$ and $Df = Dg$, then D is not injective, and thus, it is not invertible.

□

d)

Proof.

First we will prove that δ is linear. Consider $\lambda \in \mathbb{R}$ and $f \in X$, then $\delta(\lambda f)(x) = \lambda f(x) - \lambda f(x-1) = \lambda(f(x) - f(x-1)) = \lambda \delta f(x),\ \forall x \in X$. Now consider $f, g \in X$, then
$\delta(f+g)(x) = (f+g)(x) - (f+g)(x-1) =$
$f(x) + g(x) - f(x-1) - g(x-1) = \delta f(x) - \delta g(x),\ \forall x \in X$. Thus, δ is linear.

δ

Thus, $\delta^2 f(x) = \delta(f(x) - f(x-1)) = \delta f(x) - \delta f(x-1) =$
$f(x) - f(x-1) - f(x-1) + f(x_2)$. Thus $f(x)$ stays in the formula, whereas $f(x-1)$ gets subtracted iteratively, and $f(x-i)$ gets added or subtracted recursively based on its predecessor. Thus, we can express the recursive formula for
$\delta^k f(x) = f(x) - kf(x-1) + (\sum_{i_1=1}^{k-1} i_1) f(x-2) -$
$(\sum_{i_2=1}^{k-2} \sum_{i_0=1}^{i_2} i_1) f(x-3) - \cdots + \sum_{j=1}^{k-k} \cdots \sum_{i=1}^{i_2} i_1) f(x-k)$.

D

Because D is defined almost identically to δ we can define D similarly, but with a few difference because Y is not cyclic. Most notably, it does not make sense to define D^k for $k > n$. Also we have to define an explicit case for $Df(x)$ when $x = 1$. Thus, the formula for $D^k f(x) = \begin{cases} 1 \text{ if } x = 1 \\ \delta^k f(x) = f(x) - kf(x-1) + (\sum_{i_1=1}^{k-1} i_1) f(x-2) \\ -(\sum_{i_2=1}^{k-2} \sum_{i_0=1}^{i_2} i_1) f(x-3) \\ - \cdots + \sum_{j=1}^{k-k} \cdots \sum_{i=1}^{i_2} i_1) f(x-k) \text{ otherwise} \end{cases}$

$\square$