MAT240 Assignment 1

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1. a) $|Map(X_n, X_n)| = n^n$ if $n > 0$ or $|Map(X_n, X_n)| = 1$ if $n = 0$.

Proof.

Consider the finite, non-empty set $X_n = \{x_1, x_2, ..., x_n\}$ consisting of n elements and the map $f : X_n \to X_n$. $\forall x_i \in X_n, \exists f(x_i) \in X_n$, and because X_n has a cardinality of n, there are only n different values for $f(x_i)$. Then, there is a choice of n, n different times; thus, there are $\underbrace{n \cdot n \cdots n}_{n \text{ times}}$ different possible maps $f : X_n \to X_n$. Thus the cardinality of $Map(X_n, X_n)$ is $\underbrace{n \cdot n \cdots n}_{n \text{ times}}$, or simply n^n.

Considering the case where $n = 0$, X_n is empty, and thus we can easily say that there is one trivial map between the empty set and the empty set, so $|Map(\emptyset, \emptyset)| = 1$.

$\square$

b) $|Bij(X_n, X_n)| = n!$

Proof.

Consider the finite, non-empty set $X_n = \{x_1, x_2, ..., x_n\}$ consisting of n elements and the bijection $f : X_n \to X_n$. Suppose $f(x_1) \in X_n$, because f is bijective $f(x_2) \in X_n \setminus \{f(x_1)\}$. Thus if we continue this process for n iterations we get:

$\forall x_i, i > 1, x_i \in X_n \setminus \{f(x_{i-1})\} \setminus \cdots \setminus \{f(x_1)\}$

Thus, there is a choice of $(n - i + 1)$, n different times (i ranges from 1 to n). This leaves us with $n \cdot (n - 1) \cdots 2 \cdot 1$ different possible bijections $f : X_n \to X_n$. Therefore, the cardinality of $Bij(X_n, X_n)$ is $n \cdot (n - 1) \cdots 2 \cdot 1$, or more concisely, $n!$! (This last "!" is an exclamation mark and not a second factorial.)

Taking into the consideration when $n = 0$ and $X_n = \emptyset$, there is only one bijection, the identity function. Thus $Bij(\emptyset, \emptyset) = 1 = 0!$

$\square$

2. a)

Proof.

We will prove that composition of functions is associative:

$$\begin{aligned}
\forall x \in S, ((h \circ g) \circ f)(x) =& (h \circ g)(f(x)) \\
=& h(g(f(x))) \\
=& h((g \circ f)(x)) \\
=& (h \circ (g \circ f))(x)
\end{aligned}$$

And thus, $(h \circ g) \circ f = h \circ (g \circ f)$

$\square$

b) We will show all the bracketing of the compositions of 4 functions and prove that they are equivalent, and then we will show how many ways you can bracket 5 functions.

Proof.

Let us fix any $x \in S$.

$$\begin{aligned}
(((k \circ h) \circ g) \circ f)(x) =& ((k \circ h \circ g) \circ f)(x) \\
=& (k \circ h \circ g)(f(x)) \\
=& k(h(g(f(x)))) \\
((k \circ (h \circ g)) \circ f)(x) =& ((k \circ h \circ g) \circ f)(x) \\
=& (k \circ h \circ g)(f(x)) \\
=& k(h(g(f(x)))) \\
(k \circ ((h \circ g) \circ f))(x) =& (k \circ (h \circ g \circ f))(x) \\
=& k((h \circ g \circ f)(x)) \\
=& k(h(g(f(x)))) \\
(k \circ (h \circ (g \circ f)))(x) =& (k \circ (h \circ g \circ f))(x) \\
=& k((h \circ g \circ f)(x)) \\
=& k(h(g(f(x)))) \\
((k \circ h) \circ (g \circ f))(x) =& (k \circ h)((g \circ f)(x)) \\
=& k((h \circ g \circ f)(x)) \\
=& k(h(g(f(x))))
\end{aligned}$$

Thus all bracketing of the composition of 4 functions all agree with each other.

$\square$

Proof.

Consider a new function $j : W \to X$ and the composition of the 5 functions f, g, h, k, j, with all their different bracketing.

$$
\begin{gathered}
(((j \circ k) \circ h) \circ g) \circ f,\ ((j \circ (k \circ h)) \circ g) \circ f,\ (j \circ ((k \circ h) \circ g)) \circ f \\
(j \circ (k \circ (h \circ g))) \circ f,\ ((j \circ k) \circ (h \circ g)) \circ f,\ j \circ (((k \circ h) \circ g) \circ f) \\
j \circ ((k \circ (h \circ g)) \circ f),\ j \circ (k \circ ((h \circ g) \circ f)),\ j \circ (k \circ (h \circ (g \circ f))) \\
j \circ ((k \circ h) \circ (g \circ f)), (j \circ k) \circ ((h \circ g) \circ f), (j \circ k) \circ (h \circ (g \circ f)) \\
((j \circ k) \circ h) \circ (g \circ f), ((j \circ k) \circ h) \circ (g \circ f)
\end{gathered}
$$

Thus there are 14 ways to bracket 5 functions.

$\square$

c) We will prove this using induction.

Proof.

Base Case: $n = 3$, which is already proven from part a.

Induction Step: $k = n + 1$. We will prove this through cases of how to bracket $n + 1$ function compositions. If $f : X_0 \to X_{n+1}$, then because the composition of functions is a binary operation, either we have:

Case 1: $f = f_{n+1} \circ f_c$, where $f_c : X_0 \to X_n$ is a composition of n functions with arbitrary bracketing. Because compositions of n functions is well-defined without bracketing by the induction hypothesis, we can simply write $f_c = f_n \circ f_{n-1} \circ \cdots \circ f_1$. Thus:

$$
\begin{aligned}
\forall x \in X_0,\ (f_{n+1} \circ (f_n \circ f_{n-1} \circ \cdots \circ f_1))(x) =& f_{n+1}(f_n \circ f_{n-1} \circ \cdots \circ f_1)(x)) \\
=& f_{n-1}(f_n(f_{n-1} ... f_1(x)))
\end{aligned}
$$

Case 2: $f = f_c \circ f_1$, where $f_c : X_1 \to X_{n+1}$ is a composition of n functions with arbitrary bracketing. Because compositions of n

functions is well-defined without bracketing by the induction hypothesis, we can simply write $f_c = f_{n+1} \circ f_n \circ \cdots \circ f_2$. Thus:

$$\begin{aligned}\forall x \in X_0,\ ((f_{n+1} \circ f_n \circ \cdots \circ f_2) \circ f_1)(x) =& (f_{n+1} \circ f_n \circ \cdots \circ f_2)(f_1(x)) \\ =& f_{n-1}(f_n(f_{n-1} \ldots f_1(x)))\end{aligned}$$

Case 3: There exists way bracketing such that $f = f_a \circ f_b$, where $f_a : X_b \to X_{n+1}$ and $f_b : X_0 \to X_b$ are arbitrary composition of less than n functions. Because f_a and f_b are of compositions of less than n functions, it is also well defined without bracketing by the induction hypothesis (strong induction). Thus without confusion we can substitute $f_a = f_{n+1} \circ f'_a$, where $f'_a : X_b \to X_n$. Thus, $f = (f_{n+1} \circ f'_a) \circ f_b$. By the associativity of the composition of 3 functions we can rewrite each as $f = f_{n+1} \circ (f'_a \circ f_b)$. Since $(f'_a \circ f_b) : X_0 \to X_n$ is an arbitrary composition of less than n functions, it reduces this to just case 1.

□

3. a)

Proof.

Consider g and g' are both inverses of f.

$$\forall x \in X,\ g(f(x)) = x = g'(f(x))$$

Since x was chosen arbitrarily, $f(x)$ is just an arbitrary object in Y.
Thus, $\forall y \in Y,\ g(y) = g'(y)$, and therefore $g = g'$

$\square$

b) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$ is not invertible because it is not a bijection. This is because $1 \neq -1$ but $f(1) = f(-1)$ (This is shown by the problem below.)

c)

Proof.

$\Rightarrow$
We will first prove that f is a surjection.
By assumption $\forall y \in Y,\ f(g(y)) = y$ and because $g(y)$ is just an arbitrary element of X, this shows that $\forall y \in Y,\ \exists x \in X$ st $f(x) = y$

We will now show that f is an injection.
Assume that f is not an injection.
Thus, $\exists x, x' \in X$ st $x \neq x'$ and $f(x) = f(x')$
Then $g(f(x)) = g(f(x')) = x = x'$ and thus we have reached a contradiction.

Thus we have proven that f is bijective.

$\Leftarrow$
Because $f : X \to Y$ is a bijective function, then
$\forall y \in Y, \exists! x \in X$ st $f(x) = y$
Thus we can construct a function $g : Y \to X$ that maps all $y \in Y$ to the unique $x \in X$ st $f(x) = y$
Thus, by construction $g(f(x)) = x$ and $f(g(y)) = y$

$\square$

d) It does not follow that $f \circ g = I_Y$

Proof.

Consider the function $f : \{a\} \to \{1, 2\}$ such that $f(a) = 1$ and the function $g : \{1, 2\} \to \{a\}$ such that $g(1) = a$ and $g(2) = a$
Thus $(g \circ f)(a) = a$ and $(g \circ f) = I_X$ But $(f \circ g)(2) = 1$ so $(f \circ g) \neq I_Y$
Thus, we have constructed a counterexample.

$\square$

e) It does follow that $f \circ g = I_Y$ now.

Proof.

If $g \circ f = I_X$ then f is injective because if we assume f is not injective, then we result in the contradiction where $g(f(x)) = g(f(x')) = x = x'$ when $x \neq x'$
Thus f is injective and surjective, making it a bijection, and by part c, this implies that it is invertible and there exists a g st $f \circ g = I_Y$

$\square$

4. a) I have italicized the elements in the domain of $f_i : \{1,2,3,4\} \to \{1,2,3,4\}$ that are fixed points.

$$f_1 = \begin{pmatrix} \mathit{1} & \to & 1 \\ \mathit{2} & \to & 2 \\ \mathit{3} & \to & 3 \\ \mathit{4} & \to & 4 \end{pmatrix} \qquad f_2 = \begin{pmatrix} \mathit{1} & \to & 1 \\ \mathit{2} & \to & 2 \\ 3 & \to & 4 \\ 4 & \to & 3 \end{pmatrix} \qquad f_3 = \begin{pmatrix} \mathit{1} & \to & 1 \\ 2 & \to & 3 \\ 3 & \to & 4 \\ 4 & \to & 2 \end{pmatrix}$$

$$f_4 = \begin{pmatrix} \mathit{1} & \to & 1 \\ 2 & \to & 3 \\ 3 & \to & 2 \\ 4 & \to & 4 \end{pmatrix} \qquad f_5 = \begin{pmatrix} \mathit{1} & \to & 1 \\ 2 & \to & 4 \\ 3 & \to & 2 \\ 4 & \to & 3 \end{pmatrix} \qquad f_6 = \begin{pmatrix} \mathit{1} & \to & 1 \\ 2 & \to & 4 \\ \mathit{3} & \to & 3 \\ 4 & \to & 2 \end{pmatrix}$$

$$f_7 = \begin{pmatrix} 1 & \to & 2 \\ 2 & \to & 1 \\ \mathit{3} & \to & 3 \\ \mathit{4} & \to & 4 \end{pmatrix} \qquad f_8 = \begin{pmatrix} 1 & \to & 2 \\ 2 & \to & 1 \\ 3 & \to & 4 \\ 4 & \to & 3 \end{pmatrix} \qquad f_9 = \begin{pmatrix} 1 & \to & 2 \\ 2 & \to & 3 \\ 3 & \to & 1 \\ 4 & \to & 4 \end{pmatrix}$$

$$f_{10} = \begin{pmatrix} 1 & \to & 2 \\ 2 & \to & 3 \\ 3 & \to & 4 \\ 4 & \to & 1 \end{pmatrix} \qquad f_{11} = \begin{pmatrix} 1 & \to & 2 \\ 2 & \to & 4 \\ 3 & \to & 1 \\ 4 & \to & 3 \end{pmatrix} \qquad f_{12} = \begin{pmatrix} 1 & \to & 2 \\ 2 & \to & 4 \\ \mathit{3} & \to & 3 \\ 4 & \to & 1 \end{pmatrix}$$

$$f_{13} = \begin{pmatrix} 1 & \to & 3 \\ 2 & \to & 1 \\ 3 & \to & 2 \\ \mathit{4} & \to & 4 \end{pmatrix} \qquad f_{14} = \begin{pmatrix} 1 & \to & 3 \\ 2 & \to & 1 \\ 3 & \to & 4 \\ 4 & \to & 2 \end{pmatrix} \qquad f_{15} = \begin{pmatrix} 1 & \to & 3 \\ \mathit{2} & \to & 2 \\ 3 & \to & 1 \\ \mathit{4} & \to & 4 \end{pmatrix}$$

$$f_{16} = \begin{pmatrix} 1 & \to & 3 \\ \mathit{2} & \to & 2 \\ 3 & \to & 4 \\ 4 & \to & 1 \end{pmatrix} \qquad f_{17} = \begin{pmatrix} 1 & \to & 3 \\ 2 & \to & 4 \\ 3 & \to & 1 \\ 4 & \to & 2 \end{pmatrix} \qquad f_{18} = \begin{pmatrix} 1 & \to & 3 \\ 2 & \to & 4 \\ 3 & \to & 2 \\ 4 & \to & 1 \end{pmatrix}$$

$$f_{19} = \begin{pmatrix} 1 & \to & 4 \\ 2 & \to & 1 \\ 3 & \to & 2 \\ 4 & \to & 3 \end{pmatrix} \qquad f_{20} = \begin{pmatrix} 1 & \to & 4 \\ 2 & \to & 1 \\ \mathit{3} & \to & 3 \\ 4 & \to & 2 \end{pmatrix} \qquad f_{21} = \begin{pmatrix} 1 & \to & 4 \\ \mathit{2} & \to & 2 \\ 3 & \to & 1 \\ 4 & \to & 3 \end{pmatrix}$$

$$f_{22} = \begin{pmatrix} 1 & \to & 4 \\ \mathit{2} & \to & 2 \\ \mathit{3} & \to & 3 \\ 4 & \to & 1 \end{pmatrix} \qquad f_{23} = \begin{pmatrix} 1 & \to & 4 \\ 2 & \to & 3 \\ 3 & \to & 1 \\ 4 & \to & 2 \end{pmatrix} \qquad f_{24} = \begin{pmatrix} 1 & \to & 4 \\ 2 & \to & 3 \\ 3 & \to & 2 \\ 4 & \to & 1 \end{pmatrix}$$

b) Let $M = \{f_1, f_2, \cdots, f_{24}\}$ and let $i : M \to M,\ i(f) =$ the inverse of f. Again, I will italicize the elements of the domain that are fixed points.

$$i = \begin{pmatrix} f_{\mathit{1}} & \to & f_1 \\ f_{\mathit{2}} & \to & f_2 \\ f_{\mathit{3}} & \to & f_3 \\ f_4 & \to & f_5 \\ f_5 & \to & f_4 \\ f_{\mathit{6}} & \to & f_6 \\ f_{\mathit{7}} & \to & f_7 \\ f_{\mathit{8}} & \to & f_8 \\ f_9 & \to & f_{13} \\ f_{10} & \to & f_{19} \\ f_{11} & \to & f_{14} \\ f_{12} & \to & f_{20} \\ f_{13} & \to & f_9 \\ f_{14} & \to & f_{11} \\ f_{\mathit{15}} & \to & f_{15} \\ f_{16} & \to & f_{21} \\ f_{\mathit{17}} & \to & f_{17} \\ f_{18} & \to & f_{23} \\ f_{19} & \to & f_{10} \\ f_{20} & \to & f_{12} \\ f_{21} & \to & f_{16} \\ f_{\mathit{22}} & \to & f_{22} \\ f_{23} & \to & f_{18} \\ f_{\mathit{24}} & \to & f_{24} \end{pmatrix}$$

5. There are 203 distinct partitions of the set $\{1, 2, 3, 4, 5, 6\}$.

Proof.

$$\{\{1,2,3,4,5,6\}\}$$
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$$\{\{1,3,4\},\{2,5\},\{6\}\}$$
$$\{\{1,3,4\},\{2,6\},\{5\}\}$$
$$\{\{1,3,5\},\{2,4,6\}\}$$
$$\{\{1,3,5\},\{2\},\{4,6\}\}$$
$$\{\{1,3,5\},\{2\},\{4\},\{6\}\}$$
$$\{\{1,3,5\},\{2,4\},\{6\}\}$$
$$\{\{1,3,5\},\{2,6\},\{4\}\}$$
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$$\{\{1,3,6\},\{2\},\{4\},\{5\}\}$$
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$$\{\{1,4,5\},\{2\},\{3\},\{6\}\}$$

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$$\{\{1,5,6\},\{2\},\{3\},\{4\}\}$$
$$\{\{1,5,6\},\{2,3\},\{4\}\}$$
$$\{\{1,5,6\},\{2,4\},\{3\}\}$$
$$\{\{1,2,3,4\},\{5,6\}\}$$
$$\{\{1,2,3,4\},\{5\},\{6\}\}$$
$$\{\{1,2,3,5\},\{4,6\}\}$$
$$\{\{1,2,3,5\},\{4\},\{6\}\}$$
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$$\{\{1,2,3,6\},\{4\},\{5\}\}$$
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$$\{\{1,2,4,5\},\{3\},\{6\}\}$$
$$\{\{1,2,4,6\},\{3,5\}\}$$
$$\{\{1,2,4,6\},\{3\},\{5\}\}$$
$$\{\{1,2,5,6\},\{3,4\}\}$$
$$\{\{1,2,5,6\},\{3\},\{4\}\}$$
$$\{\{1,3,4,5\},\{2,6\}\}$$

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$$\{\{1,3,5,6\},\{2\},\{4\}\}$$
$$\{\{1,4,5,6\},\{2,3\}\}$$
$$\{\{1,4,5,6\},\{2\},\{3\}\}$$
$$\{\{1,2,3,4,5\},\{6\}\}$$
$$\{\{1,2,3,4,6\},\{5\}\}$$
$$\{\{1,2,3,5,6\},\{4\}\}$$
$$\{\{1,2,4,5,6\},\{3\}\}$$
$$\{\{1,3,4,5,6\},\{2\}\}$$

Thus, these are all the 203 distinct partitions of the set $\{1,2,3,4,5,6\}$.

$\square$

6. $|\mathcal{P}(X)| = 2^n$

Proof.

Consider X is a set with $n \in \mathbb{N}$ elements. Then for every element $x \in \mathcal{P}(X)$ and for every element $y \in X$, there are two choices: either $y \in x$ or $y \notin x$. For n objects in X, this gives us $\underbrace{2 \cdot 2 \cdots 2}_{n \text{ times}}$ distinct objects in $\mathcal{P}(X)$. Hence, the cardinality of $\mathcal{P}(X)$ is 2^n

$\square$