MAT240 Problem Set 3

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Lemma 1.0: $3n^2 + 3n + 6$ is divisible by 6 for all $n \in \mathbb{N}$.

Proof.

We will use induction.

Base Case: $n = 1$.
$3 + 3 + 6 = 12 = 6(2)$ Thus, the base case is true.
Induction Hypothesis: If $3n^2 + 3n + 6$ is divisible by 6, then
$3(n+1)^2 + 3(n+1) + 6$ is divisible by 6.
Inductive Step:

$$\begin{aligned} 3(n+1)^2 + 3(n+1) + 6 =& 3(n^2 + 2n + 1) + 3n + 3 + 6 \\ =& 3n^2 + 6n + 3 + 3n + 9 \\ =& 3n^2 + 3n + 6 + 6n + 6 \\ =& 6k + 6n + 6, \ k \in \mathbb{N} \\ =& 6(k + n + 1) \end{aligned}$$

Therefore, $3(n+1)^2 + 3(n+1) + 6$ is divisible by 6; thus,
$3n^2 + 3n + 6$ is divisible by 6 for all $n \in \mathbb{N}$.

$\square$

1. a)

Proof.

We will use induction.

Base Case: $n = 1$.
$1^3 + 5 = 6 = 6(1)$ Thus, the base case is true.
Induction Hypothesis: If $n^3 + 5n$ is divisible by 6, then
$(n+1)^3 + 5(n+1)$ is divisible by 6.

Inductive Step:

$$\begin{aligned}(n+1)^3+5(n+1) =& n^3+3n^2+3n+1+5n+5\\ =& n^3+5n+3n^2+3n+6\\ =& 6k+3n^2+3n+6,\ k\in\mathbb{Z}\\ =& 6k+6j,\ j\in\mathbb{Z},\ \text{Lemma 1.0}\end{aligned}$$

Then, $(n+1)^3+5(n+1)$ is divisible by 6; therefore, n^3+5n is divisible by 6 for all $n\in\mathbb{N}$.

$\square$

b)

Proof.

We will use induction.

Base Case: $n=1$.
$\sum_{k=1}^{1}\frac{k}{2^k}=\frac{1}{2}=2-\frac{3}{2}=2-\frac{n+2}{2^n}$. Thus, the base case is true.
Induction Hypothesis: If $\sum_{k=1}^{n}\frac{k}{2^k}=2-\frac{n+2}{2^n}$, then
$\sum_{k=1}^{n+1}\frac{k}{2^k}=2-\frac{(n+1)+2}{2^{n+1}}$
Inductive Step:

$$\begin{aligned}\sum_{k=1}^{n+1}\frac{k}{2^k} =& \sum_{k=1}^{n}\frac{k}{2^k}+\frac{n+1}{2^{n+1}}\\ =& 2-\frac{n+2}{2^n}+\frac{n+1}{2^{n+1}}\\ =& 2-\frac{2n+4}{2^{n+1}}+\frac{n+1}{2^{n+1}}\\ =& 2+\frac{n+1-2n-4}{2^{n+1}}\\ =& 2+\frac{-n-3}{2^{n+1}}\\ =& 2-\frac{(n+1)+2}{2^{n+1}}\end{aligned}$$

Thus, $\sum_{k=1}^{n+1}\frac{k}{2^k}=2-\frac{(n+1)+2}{2^{n+1}}$; therefore, $\sum_{k=1}^{n}\frac{k}{2^k}=2-\frac{n+2}{2^n}$ for all $n\in\mathbb{N}$.

$\square$

c)

Proof.

Consider:

$$\begin{aligned}
8 =& 3+5 \\
9 =& 3(3)+5(0) \\
10 =& 3(0)+5(2) \\
11 =& 3(2)+5 \\
12 =& 3(4)+5(0) \\
13 =& 3+5(2) \\
14 =& 3(3)+5 \\
15 =& 3(5)+5(0) \\
16 =& 3(2)+5(2) \\
17 =& 3(4)+5
\end{aligned}$$

Thus, integers from 8 to 17 can be expressed in the form $3a+5b,\ a,b \in \mathbb{N} \cup \{0\}$.

Now consider $n \geq 18$. Then, $n-8 \geq 10$. Thus we can choose the largest (well-ordering principle) $k \in \mathbb{N} \cup \{0\}$ such that $10k \leq n-8$. If we take the difference between the two. Then $n-8-10k = j,\ j \in \{0,1,2,3,4,5,6,7,8,9\}$. Notice that if we add 8 to both sides we get $n-10k = z,\ z \in \{8,9,10,11,12,13,14,15,16,17\}$. But we know that this set of numbers can be expressed in the form $3a+5b$. Thus, we can rewrite the equation as $n = 3a+5b+10k = 3a+5b+5(2k) = 3a+5(b+2k)$. Therefore, all numbers greater than 8 can be expressed in the form $3a+5b,\ a,b \in \mathbb{N} \cup \{0\}$. Hence, you are able to make any sum of rubles greater than equal to 8 with a non-negative integer sum of 3 ruble and 5 ruble bills.

Alternatively, assume that you cannot express $n \geq 8$ in the form $3a+5b,\ a,b \in \mathbb{N} \cup \{0\}$. This contradicts contradicts the *Chicken McNugget Theorem.*

□

2. a)

Proof.

The largest sum you can make out of $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}$ is the sum of the largest $\frac{1}{n_1}$, $\frac{1}{n_2}$, and $\frac{1}{n_3}$ individually. This is trivial because for all $x \in \mathbb{R}$, if $y > z$, then $x + y > x + z$. Since $f : \mathbb{N} \to \mathbb{R}, f(n) = \frac{1}{n}$ is a monotone decreasing function for all $n > 0$, then the largest value it can take is at $n = 1$; thus, $f(1) = 1$. Therefore, the supremum of S is 3 and it is in S, so it is also the maximum of S.

0 is a lower bound for S because a sum of 3 positive numbers is always greater than 0. We will now show that 0 is the greatest lower bound by assuming that there exists some $\epsilon > 0$ such that ϵ is a lower bound. But if we choose n_1, n_2, and n_3 arbitrarily large such that $\frac{1}{n_1}, \frac{1}{n_2}, \frac{1}{n_3} < \frac{\epsilon}{3}$. Thus $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. Thus we created an element that is smaller than ϵ, thus ϵ cannot be a lower bound, making 0 the greatest lower bound. But since a sum of positive numbers can never be 0, 0 is not a minimum of S.

□

b)

Proof.

Consider $f : [\frac{1}{2}, 2] \to \mathbb{R},\ f(x) = x + \frac{1}{x}$. Notice that $T = f((\frac{1}{2}, 2])$. We can choose $a := 1 \in (\frac{1}{2}, 2]$. $f(a) = 2$. Suppose that 2 is a lower bound to T. We will show this by proving that $f(1 + \psi) > 2,\ \forall \psi \in (0, 1]$ and $f(1 - \phi) > 2,\ \forall \phi \in (0, \frac{1}{2})$. If $\psi \in (0, 1]$, then $\frac{\psi}{1+\psi} < \psi$ because the denominator on the left is greater than 1. But we can rewrite the inequality as:

$$\begin{aligned} \psi >& \frac{\psi + 1 - 1}{\psi} \\ =& \frac{1 + \psi}{1 + \psi} - \frac{1}{1 + \psi} \\ =& 1 - \frac{1}{1 + \psi} \\ \psi + \frac{1}{1 + \psi} >& 1 \end{aligned}$$

Thus $2 = 1 + 1 < 1 + \psi + \frac{1}{1+\psi},\ \forall \psi \in (0, 1]$; thus,
$f(1 + \psi) > 2,\ \forall \psi \in (0, 1]$. Now if $\phi \in (0, \frac{1}{2})$, then $\phi < \frac{\phi}{1-\phi}$. This is because the denominator on the right is in $(0, 1)$. Then:

$$\begin{aligned}
-\phi >& \frac{-\phi}{1-\phi} \\
>& \frac{-\phi + 1 - 1}{1-\phi} \\
& \frac{1-\phi}{1-\phi} - \frac{1}{1-\phi} \\
>& 1 - \frac{1}{1-\phi} \\
-\phi + \frac{1}{1-\phi} >& 1
\end{aligned}$$

Thus $2 = 1 + 1 < 1 - \psi + \frac{1}{1-\psi},\ \forall \psi \in (0, \frac{1}{2})$; thus,
$f(1 + \psi) > 2,\ \forall \psi \in (0, 1]$. Therefore, 2 is less than equal to all elements in T, but 2 is the greatest element less than equal to all elements in T, making it a infimum. 2 is in T, so 2 is also a minimum.

To find the supremum we will show that f is strictly increasing on the interval $(1, 2]$ and strictly decreasing on the interval $[\frac{1}{2}, 1)$. Then, we can take the $max\{f(q)\},\ q \in \partial[\frac{1}{2}, 2]$ (boundary of the interval).

To show that f is strictly increasing on the interval $(1, 2]$, we will take $1 < a < b \leq 2$. Because $a, b > 1$, then $ab > 1$. Thus:

$$\begin{aligned}
b - a >& \frac{b-a}{ab} \\
>& \frac{b}{ab} - \frac{a}{ab} \\
>& \frac{1}{a} - \frac{1}{b} \\
b + \frac{1}{b} >& a\frac{1}{a}
\end{aligned}$$

Thus f is strictly increasing on the interval $(1, 2]$.

To show that f is strictly decreasing on the interval $[\frac{1}{2}, 1)$, we will take $\frac{1}{2} \leq a < b < 1$. Because $0 < a < b < 1$, then $ab < 1$. Thus:

$$\begin{aligned}
b - a &< \frac{b-a}{ab} \\
&< \frac{b}{ab} - \frac{a}{ab} \\
&< \frac{1}{a} - \frac{1}{b} \\
b + \frac{1}{b} &< a + \frac{1}{a}
\end{aligned}$$

Thus f is strictly decreasing on the interval $(\frac{1}{2}, 1)$. Because f strictly increases on the interval $(1, 2]$ and strictly decreases on the interval $(\frac{1}{2}, 1)$ then we can just take the maximum between the boundary $f(\frac{1}{2}) = \frac{3}{2}$ and $f(2) = \frac{3}{2}$. But because in the actual set T, $f(\frac{1}{2})$ is not included in the set, but $f(x),\ x \in (\frac{1}{2}, 1)$ is in the T, then $\forall x \in (\frac{1}{2}, 1),\ f(x) < \frac{3}{2}$ (because the function is decreasing on this interval). Since 2 is in T, then we can take $f(2)$ as the maximum of T because it is equal to $\frac{3}{2}$. Because 2 is the maximum of T, 2 is also the supremum. (Consider that there is some supremum less than 2, well it would be less than 2, contradicting the fact that it is a supremum; thus, 2 is the least upper bound).

$\square$

Lemma 3.0: There exists $a, b \in \mathbb{Z}$ such that $0 < a + \sqrt{2}b < \frac{1}{n},\ \forall n \in \mathbb{N}$.

Proof.

We will use induction.

Base Case: $n = 1$, $0 < \sqrt{2}(1) - 1 < \frac{1}{1}$
$n = 2$, $0 < \sqrt{2}(1) - 1 < \frac{1}{2}$ Thus, both base cases are true.
Induction Hypothesis: If $\exists a, b \in \mathbb{Z} : 0 < a + \sqrt{2}b < \frac{1}{n}$, then $\exists c, d \in \mathbb{Z} : 0 < c + \sqrt{2}d < \frac{1}{n^2}$.
Inductive Step: By the induction hypothesis $0 < a + \sqrt{2}b < \frac{1}{n}$, then $0 < a^2 + 2b^2 + 2b\sqrt{2} < \frac{1}{n^2}$ because $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$ is an increasing function on the interval $(0, +\infty)$. Thus if $c := a^2 + 2b^2$ and $d := 2b$, then $0 < c + \sqrt{2}d < \frac{1}{n^2}$. Thus there exists $a, b \in \mathbb{Z}$ such that $0 < a + \sqrt{2}b < \frac{1}{n},\ \forall n \in \{2^{2^k} : k \in \mathbb{N}\}$. If we wanted $0 < a + b\sqrt{2} < \frac{1}{n},\ \forall n \in \mathbb{N}$, then for any n we can choose k sufficiently large such that $n < 2^{2^k}$. Thus, there exists $0 < c + d\sqrt{2} < \frac{1}{2^{2^k}} < \frac{1}{n}$. This is because if $n < 2^{2^k}$ then $\frac{1}{n} > \frac{1}{2^{2^k}}$.
Therefore, there exists $a, b \in \mathbb{Z}$ such that $0 < a + \sqrt{2}b < \frac{1}{n},\ \forall n \in \mathbb{N}$.

□

3.

Proof.

Clearly, 0 is a lower bound to S; thus, we will show that 0 is the greatest lower bound to S.

For the sake of contradiction assume that $inf(S) = \epsilon > 0$. But, by *Lemma 3.0*, there exists $a, b \in \mathbb{Z}$ such that $0 < a + \sqrt{2}b < \frac{1}{n},\ \forall n \in \mathbb{N}$. Because we can choose n arbitrarily large, we can choose n such that $\frac{1}{n} < \epsilon$. Thus, there exists $a, b \in \mathbb{Z}$ such that $0 < a + \sqrt{2}b < \frac{1}{n} < \epsilon$. This contradicts the fact that ϵ is a lower bound of S. Therefore, the $inf(S) = 0$.

□

4.

Proof.

Trivially, $A(0) = 0$, $A(1) = 0$, $A(2) = 0$, and $A(3) = 1$. We want to prove that $A(n) = \frac{(n-1)(n-2)}{2}$, $\forall n \in \mathbb{N}$. Notice that we can show the ways we can make n as a sum of n_1, n_2, n_3 by listing the ordered pairs in the set $\{1, 2, ...n-1\} \times \{1, 2, ...n-2\}$. For example, (a, b) in the set corresponds to $n_1 := a,\ n_2 := b,\ n_3 := n - a + b$. Although, if $a + b \geq n$ then there will $n_3 < 1$. We want to ensure that each $n_i > 0$; thus, we will only permit elements (a, b) such that $a + b \leq n$. Consider that $1 + j < n$ for all $j < n - 2$; thus, there are $n - 2$ elements (a, b) where $a = 1$. $2 + j < n$ for all $j < n - 3$; thus, there are $n - 3$ elements (a, b) where $a = 2$. If we continue this process we will get a series of $(n-1) + (n-2) + \cdots + 2 + 1$. We can rewrite this as $\sum_{i=0}^{n-1} i$. But we know that $\sum_{i=0}^{n} i = \frac{n(n-1)}{2}$; thus, by substituting $n - 1$ into n we get $\frac{(n-1)(n-2)}{2}$ elements as desired.

$\square$

5. a)

Proof.

Because $\binom{3^n}{\ell} = \frac{3^n(3^n-1)\cdots(3^n-\ell+1)}{\ell!}$ is an integer and the numerator has at least n multiples of 3 while $\ell!$ has at most $n-1$ multiples of 3, $\ell!$ cannot divide out enough prime factors of 3; thus, $\binom{3^n}{\ell}$ must be divisible by 3.

$\square$

b)

Proof.

We will use induction on k.

Base Case: $k = 0$. $\binom{3^n}{1}$ is divisible by 3 via question 5 a. Thus, the base case is true.
Induction Hypothesis: If $\binom{3^n+k}{k+1}$ is divisible by 3, then $\binom{3^n+k+1}{k+1+1}$ is divisible by 3.
Inductive Step:

$$\begin{aligned}\binom{3^n+k+1}{k+1+1} =& \binom{3^n+k}{k+1} + \binom{3^n+k}{k+1+1}\\ =&3i+3j,\ i,j\in\mathbb{Z},\ \text{by the induction hypothesis}\\ =&3(i+j)\end{aligned}$$

Thus $\binom{3^n+k}{k+1}$ is divisible by 3.

$\square$

c)

Proof.

We will start with $\binom{3^n+k}{k+1}$.

We will induct on k.
Base Case: $k = 0$. $\binom{3^n}{0} = 1 \bmod 3$. Thus the base case is true.
Induction Hypthesis: If $\binom{3^n+k}{k} = 1 \bmod 3$, then $\binom{3^n+k+1}{k+1} = 1 \bmod 3$.

Inductive Step:

$$\begin{aligned}\binom{3^n+k+1}{k+1} =& \binom{3^n+k}{k+1} + \binom{3^n+k}{k}\\ =& 0+1, \text{ by the induction hypothesis and 5 b}\\ =& 1 \bmod 3\end{aligned}$$

Now we will do $\binom{3^n+k}{3^n}$.

We will induct on k.
Base Case: $k=0$. $\binom{3^n}{3^n} = 1 \bmod 3$. Thus, the base case is true.
Induction hypothesis: If $\binom{3^n+k}{3^n} = 1 \bmod 3$, then $\binom{3^n+k+1}{3^n} = 1 \bmod 3$.
Inductive Step:

$$\begin{aligned}\binom{3^n+k+1}{3^n} =& \binom{3^n+k}{3^n} + \binom{3^n+k}{3^n-1}\\ =& 0+1, \text{ by the induction hypothesis and 5 b}\\ =& 1 \bmod 3\end{aligned}$$

$\square$