MAT240 Problem Set 3

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1.

Proof.

We will now show that $\mathbb{R}^2$ with the operators $\tilde{+}$ and $\tilde{\cdot}$ define a vector space over $\mathbb{R}$.

Commutativity

Consider $x, y \in \mathbb{R}^2$ with $x = (a, b)$ and $y = (u, v),\ a, b, u, v \in \mathbb{R}$. Then:

$$\begin{aligned} x \tilde{+} y =& (a + u + 1, y + v - 1) \\ =& (u + a + 1, v + y - 1) \\ =& y \tilde{+} x \end{aligned}$$

Thus, vector addition is commutative.

Associativity

Consider $x, y, z \in \mathbb{R}^2$ with $x = (a, b),\ y = (u, v)$, and $z = (\phi, \psi),\ a, b, u, v, \phi, \psi \in \mathbb{R}$. Then:

$$\begin{aligned} x \tilde{+} (y \tilde{+} z) =& (a, b) \tilde{+} (u + \phi + 1, v + \psi - 1) \\ =& (a + (u + \phi + 1) + 1, b + (v + \psi - 1) - 1) \\ =& ((a + u + 1) + \phi + 1, (b + v - 1) + \psi - 1) \\ =& (a + u + 1, b + v - 1) \tilde{+} (\phi, \psi) \\ =& (x \tilde{+} y) \tilde{+} z \end{aligned}$$

Now consider $\lambda, \iota \in \mathbb{R}$. Then:

$$\begin{aligned}(\lambda\iota)\tilde{\cdot}x =&(\lambda\iota a + \lambda\iota - 1, \lambda\iota b - \lambda\iota + 1)\\ =&(\lambda\iota a + \lambda\iota - \lambda + \lambda - 1, \lambda\iota b - \lambda\iota - \lambda + \lambda - 1)\\ =&(\lambda(\iota a + \iota - 1) + \lambda - 1, \lambda(\iota b - \iota - 1) - \lambda + 1)\\ =&\lambda\tilde{\cdot}(\iota a + \iota - 1, \iota b - \iota + 1)\\ =&\lambda\tilde{\cdot}(\iota\tilde{\cdot}x)\end{aligned}$$

Thus, vector addition and scalar multiplication is associative.

Additive Identity

There exists an element $0 \in \mathbb{R}^2$, namely $0 = (-1, 1)$, such that for all $x \in \mathbb{R}^2,\ x\tilde{+}0 = x$. Consider $x = (a, b),\ a, b \in \mathbb{R}$. Then:

$$\begin{aligned}x\tilde{+}0 =&(a - 1 + 1, b + 1 - 1)\\ =&(a, b)\\ =&x\end{aligned}$$

Additive Inverse

For all $x \in \mathbb{R}^2,\ x = (a, b),\ a, b \in \mathbb{R}$, there exists an additive inverse, namely $-x = (-a - 2, -b + 2)$, such that $x\tilde{+} - x = 0$. Then:

$$\begin{aligned}x\tilde{+} - x =&(a - a - 2 + 1, b - b + 2 - 1)\\ =&(-2 + 1, 2 - 1)\\ =&(-1, 1)\\ =&0\end{aligned}$$

Multiplicative Identity

There exists an element $1 \in \mathbb{R}$ such that for all $x \in \mathbb{R},\ 1\tilde{\cdot}x = x$. Consider $x = (a, b),\ a, b \in \mathbb{R}$. Then:

$$\begin{aligned}1\tilde{\cdot}x =&(1a + 1 - 1, 1b - 1 + 1)\\ =&(a, b)\\ =&x\end{aligned}$$

Distributive Property

Consider $x, y \in \mathbb{R}^2$ with $x = (a, b)$ and $y = (u, v)$, $a, b, u, v \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. Then:

$$\begin{aligned}
\lambda \tilde{\cdot} (x \tilde{+} y) =& \lambda \tilde{\cdot} (a + u + 1, b + v - 1) \\
=& (\lambda(a + u + 1) + \lambda - 1, \lambda(b + v - 1) - \lambda + 1) \\
=& (\lambda a + \lambda u + \lambda + \lambda - 1, \lambda b + \lambda v - \lambda - \lambda + 1) \\
=& (\lambda a + \lambda - 1 + \lambda u + \lambda - 1 + 1, \lambda b - \lambda + 1 + \lambda v - \lambda + 1 - 1) \\
=& (\lambda a + \lambda - 1, \lambda b - \lambda + 1) \tilde{+} (\lambda u + \lambda - 1, \lambda v - \lambda + 1) \\
=& \lambda \tilde{\cdot} x \tilde{+} \lambda \tilde{\cdot} y
\end{aligned}$$

Now consider $\iota \in \mathbb{R}$. Then:

$$\begin{aligned}
(\lambda + \iota) \tilde{\cdot} x =& ((\lambda + \iota)a + (\lambda + \iota) - 1, (\lambda + \iota)b - (\lambda + \iota) + 1) \\
=& (\lambda a + \iota a + \lambda + \iota - 1, \lambda b + \iota b - \lambda - \iota + 1) \\
=& (\lambda a + \lambda - 1 + \iota a + \iota - 1 + 1, \lambda b - \lambda + 1 + \iota b - \iota b + 1) \\
=& (\lambda a + \lambda - 1, \lambda b - \lambda + 1) \tilde{+} (\iota a + \iota - 1, \iota b - \iota + 1) \\
=& \lambda \tilde{\cdot} x \tilde{+} \iota \tilde{\cdot} x
\end{aligned}$$

Thus, vector addition and scalar multiplication are linked through the distributive property.

Therefore, $\mathbb{R}^2$ with the operators $\tilde{+}$ and $\tilde{\cdot}$ define a vector space over $\mathbb{R}$.

$\square$

2. a)

Proof.

These are all of the linear subspaces of $(\mathbb{F}_5)^2$:

$$\begin{aligned}
&\{(x, y) : x, y \in \mathbb{F}_5\}\\
&\{(x, 0) : x \in \mathbb{F}_5\}\\
&\{(0, x) : x \in \mathbb{F}_5\}\\
&\{(x, x) : x \in \mathbb{F}_5\}\\
&\{(x, 2x) : x \in \mathbb{F}_5\}\\
&\{(x, 3x) : x \in \mathbb{F}_5\}\\
&\{(x, 4x) : x \in \mathbb{F}_5\}\\
&\{(0, 0)\}
\end{aligned}$$

There are 8.

□

b)

Proof.

These are all of the linear subspaces of $(\mathbb{F}_2)^3$:

$$\{(x, y, z) : x, y, z \in \mathbb{F}_2\}$$
$$\{(x, y, 0) : x, y \in \mathbb{F}_2\}$$
$$\{(x, 0, y) : x, y \in \mathbb{F}_2\}$$
$$\{(0, x, y) : x, y \in \mathbb{F}_2\}$$
$$\{(x, 0, 0) : x \in \mathbb{F}_2\}$$
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$$\{(x, y, x) : x \in \mathbb{F}_2\}$$
$$\{(x, x, 0) : x \in \mathbb{F}_2\}$$
$$\{(0, x, x) : x \in \mathbb{F}_2\}$$
$$\{(x, x, y) : x, y \in \mathbb{F}_2\}$$
$$\{(x, y, y) : x, y \in \mathbb{F}_2\}$$
$$\{(x, x, x) : x \in \mathbb{F}_2\}$$
$$\{(x, y, x + y) : x, y \in \mathbb{F}_2\}$$
$$\{(0, 0, 0)\}$$

There are 16.

$\square$

3. a)

Proof.

We will use $(\mathbb{F}_5)^2_T$ to denote the set of affine linear subspaces of $(\mathbb{F}_5)^2$ modelled on T.

$T_1 = \{(x, y) : x, y \in \mathbb{F}_5\}$
$(\mathbb{F}_5)^2_{T_1} = \{(0, 0)\}$
$|(\mathbb{F}_5)^2_{T_1}| = 1$
All elements are equivalent because $T = (\mathbb{F}_5)^2$. Thus, for all $x, y \in \mathbb{F}_5,\ y - x \in T_1$.

$T_2 = \{(x, 0) : x \in \mathbb{F}_5\}$
$(\mathbb{F}_5)^2_{T_2} = \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4)\}$
$|(\mathbb{F}_5)^2_{T_2}| = 5$
For any two elements $x = (u, v)$ and $y = (z, v),\ u, v, z \in \mathbb{F}_5$, $y - x = (s, 0),\ s \in \mathbb{F}_5$. Since s is arbitrary, $(s, 0) \in T_2$; thus every element with an equivalent second coordinate are equivalent.

$T_3 = \{(0, x) : x \in \mathbb{F}_5\}$
$(\mathbb{F}_5)^2_{T_3} = \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0)\}$
$|(\mathbb{F}_5)^2_{T_3}| = 5$
Symmetrically to the previous affine linear subspace, every element with an equivalent first coordinate are equivalent.

$T_4 = \{(x, x) : x \in \mathbb{F}_5\}$
$(\mathbb{F}_5)^2_{T_4} = \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4)\}$
$|(\mathbb{F}_5)^2_{T_4}| = 5$
Consider $x = (0, b)$ and $y = (a, a + b),\ a, b \in \mathbb{F}_5$, then $y - x = (a, a) \in T_4$. Thus, $\{(0, b) : b \in \mathbb{F}_5\}$ are the set of equivalence classes because all elements in $(\mathbb{F}_5)^2$ can be written in the form $(a, a + b),\ a, b \in \mathbb{F}_5$.

$T_5 = \{(x, 2x) : x \in \mathbb{F}_5\}$
$(\mathbb{F}_5)^2_{T_5} = \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4)\}$
$|(\mathbb{F}_5)^2_{T_5}| = 5$
Consider $y = (a, b),\ a, b \in \mathbb{F}_5$. Since $T_5 = \{(0, 0), (1, 2), (2, 4), (3, 1), (4, 3)\}$ we can choose $x = (0, c)$ such that $b - c = 0$ if $a = 0$, $b - c = 2$ if $a = 1$, $b - c = 4$ if $a = 2$, $b - c = 1$ if $a = 3$, and $b - c = 3$ if $a = 4$. Thus, if we choose x

correctly then $y - x \in T_5$; thus, $\{(0,b) : b \in \mathbb{F}_5\}$ defines the set of equivalence classes.

$T_6 = \{(x, 3x) : x \in \mathbb{F}_5\}$
$(\mathbb{F}_5)^2_{T_6} = \{(0,0), (0,1), (0,2), (0,3), (0,4)\}$
$|(\mathbb{F}_5)^2_{T_6}| = 5$
Consider $y = (a,b),\ a, b \in \mathbb{F}_5$. Since
$T_6 = \{(0,0), (1,3), (2,1), (3,4), (4,2)\}$ we can choose $x = (0,c)$ such that $b - c = 0$ if $a = 0$, $b - c = 3$ if $a = 1$, $b - c = 1$ if $a = 2$, $b - c = 4$ if $a = 3$, and $b - c = 2$ if $a = 4$. Thus, if we choose x correctly then $y - x \in T_5$; thus, $\{(0,b) : b \in \mathbb{F}_5\}$ defines the set of equivalence classes.

$T_7 = \{(x, 4x) : x \in \mathbb{F}_5\}$
$(\mathbb{F}_5)^2_{T_7} = \{(0,0), (0,1), (0,2), (0,3), (0,4)\}$
$|(\mathbb{F}_5)^2_{T_7}| = 5$
Consider $y = (a,b),\ a, b \in \mathbb{F}_5$. Since
$T_6 = \{(0,0), (1,4), (2,3), (3,2), (4,1)\}$ we can choose $x = (0,c)$ such that $b - c = 0$ if $a = 0$, $b - c = 3$ if $a = 1$, $b - c = 1$ if $a = 2$, $b - c = 4$ if $a = 3$, and $b - c = 2$ if $a = 4$. Thus, if we choose x correctly then $y - x \in T_5$; thus, $\{(0,b) : b \in \mathbb{F}_5\}$ defines the set of equivalence classes.

$T_8 = \{(0,0)\}$
$(\mathbb{F}_5)^2_{T_8} = \{(x,y) : x, y \in \mathbb{F}_5\}$
$|(\mathbb{F}_5)^2_{T_8}| = 25$
This because the only elements that have a difference equal to the zero vector, are identical elements.

Thus, there are $\sum_{n=1}^{8} |(\mathbb{F}_5)^2_{T_n}| = 56$ total affine linear subspaces of $(\mathbb{F}_5)^2$ modelled on $(T_n)_{n=1}^{8}$.

□

b)

Proof.

We will use $(\mathbb{F}_2)^3_T$ to denote the set of affine linear subspaces of $(\mathbb{F}_2)^3$ modelled on T.

$T_1 = \{(x,y,z) : x, y, z \in \mathbb{F}_2\}$
$(\mathbb{F}_2)^3_{T_1} = \{(0,0)\}$

$|(\mathbb{F}_5)^2_{T_1}| = 1$
All elements are equivalent because $T = (\mathbb{F}_5)^2$. Thus, for all $x, y \in \mathbb{F}_5,\ y - x \in T_1$.

$T_2 = \{(x, y, 0) : x, y \in \mathbb{F}_2\}$
$(\mathbb{F}_2)^3_{T_2} = \{(0, 0, 0), (0, 0, 1)\}$
$|(\mathbb{F}_5)^2_{T_2}| = 2$
There is a dichotomy between vectors; either the last coordinate is 1 or the last coordinate is 0. If you take the difference between any two vector with last coordinate 0, you get a vector with last coordinate being 0; thus, being in the set. If you take the difference between any two vectors with last coordinate 1, you get a vector with last coordinate being 0; thus, being in the set as well.

$T_3 = \{(x, 0, y) : x, y \in \mathbb{F}_2\}$
$(\mathbb{F}_2)^3_{T_3} = \{(0, 0, 0), (0, 1, 0)\}$
$|(\mathbb{F}_5)^2_{T_3}| = 2$
Again we can take the difference between any two vectors with second coordinate 0, and get a vector with second coordinate 0. We can similarly do the same to any two vectors with second coordinate 1, creating a vector with second coordinate 0.

$T_4 = \{(0, x, y) : x, y \in \mathbb{F}_2\}$
$(\mathbb{F}_2)^3_{T_4} = \{(0, 0, 0), (0, 1, 0)\}$
$|(\mathbb{F}_5)^2_{T_4}| = 2$
$T_16 = \{(0, 0, 0)\}$
Again we can take the difference between any two vectors with first coordinate 0, and get a vector with second coordinate 0. We can similarly do the same to any two vectors with second coordinate 1, creating a vector with second coordinate 0.

$T_5 = \{(x, 0, 0) : x \in \mathbb{F}_2\}$
$(\mathbb{F}_2)^3_{T_5} = \{(0, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$
$|(\mathbb{F}_5)^2_{T_5}| = 4$
$(0, 0, 0)$ and $(1, 0, 0)$ are trivially equal. $(1, 1, 0) - (0, 1, 0) = (1, 0, 0)$, $(0, 1, 1) - (1, 1, 1) = (1, 0, 0)$, and $(1, 0, 1) - (0, 0, 1) = (1, 0, 0)$.

$T_6 = \{(0, x, 0) : x \in \mathbb{F}_2\}$
$(\mathbb{F}_2)^3_{T_6} = \{(0, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$
$|(\mathbb{F}_5)^2_{T_6}| = 4$

$(0,0,0)$ and $(0,1,0)$ are trivially equal. $(1,1,0)-(1,0,0)=(0,1,0)$, $(0,1,1)-(0,0,1)=(0,1,0)$, and $(1,0,1)-(1,1,1)=(0,1,0)$.

$T_7 = \{(0,0,x) : x \in \mathbb{F}_2\}$
$(\mathbb{F}_2)^3_{T_7} = \{(0,0,0),(0,1,0),(0,0,1),(1,1,1)\}$
$|(\mathbb{F}_5)^2_{T_7}| = 4$
$(0,0,0)$ and $(0,0,1)$ are trivially equal. $(1,0,1)-(1,0,0)=(0,0,1)$, $(0,1,1)-(0,1,0)=(0,0,1)$, and $(1,1,1)-(1,1,0)=(0,0,1)$.

$T_8 = \{(x,0,x) : x \in \mathbb{F}_2\}$
$(\mathbb{F}_2)^3_{T_8} = \{(0,0,0),(0,1,0),(0,0,1),(1,1,1)\}$
$|(\mathbb{F}_5)^2_{T_8}| = 4$
$(0,0,0)$ and $(1,0,1)$ are trivially equal. $(1,1,1)-(0,1,0)=(1,0,1)$, $(0,1,1)-(1,1,0)=(1,0,1)$, and $(1,0,0)-(0,0,1)=(1,0,1)$.

$T_9 = \{(x,y,x) : x,y \in \mathbb{F}_2\}$
$(\mathbb{F}_2)^3_{T_9} = \{(0,0,0),(1,1,0),(1,0,0)\}$
$|(\mathbb{F}_5)^2_{T_9}| = 3$
$(0,0,0)$, $(1,0,1)$, $(0,1,0)$, and $(1,0,1)$ are trivially equal.
$(1,0,0)-(0,0,1)=(1,0,1)$ and $(0,1,1)-(1,1,0)=(1,0,1)$.

$T_{10} = \{(x,x,0) : x \in \mathbb{F}_2\}$
$(\mathbb{F}_2)^3_{T_{10}} = \{(0,0,0),(1,0,0),(1,0,1),(1,1,1)\}$
$|(\mathbb{F}_5)^2_{T_{10}}| = 4$
$(0,0,0)$ and $(1,1,0)$ are trivially equal. $(1,0,0)-(0,1,0)=(1,1,0)$, $(1,1,1)-(0,0,1)=(1,1,0)$, and $(0,1,1)-(1,0,1)=(1,1,0)$.

$T_{11} = \{(0,x,x) : x \in \mathbb{F}_2\}$
$(\mathbb{F}_2)^3_{T_{11}} = \{(0,0,0),(1,1,1),(1,1,0),(0,0,1)\}$
$|(\mathbb{F}_5)^2_{T_{11}}| = 4$
$(0,0,0)$ and $(0,1,1)$ are trivially equal. $(1,1,1)-(1,0,0)=(0,1,1)$, $(1,1,0)-(1,0,1)=(0,1,1)$ and $(0,0,1)-(0,1,0)=(0,1,1)$.

$T_{12} = \{(x,x,y) : x,y \in \mathbb{F}_2\}$
$(\mathbb{F}_2)^3_{T_{12}} = \{(0,0,0),(1,0,1),(1,0,0)\}$
$|(\mathbb{F}_5)^2_{T_{12}}| = 3$
$(0,0,0)$, $(1,1,1)$, $(1,1,0)$ and $(0,0,1)$ are trivially equal.
$(1,0,1)-(0,1,0)=(1,1,1)$ and $(1,0,0)-(0,1,1)=(1,1,1)$.

$T_{13} = \{(x,y,y) : x,y \in \mathbb{F}_2\}$
$(\mathbb{F}_2)^3_{T_{13}} = \{(0,0,0),(1,1,0),(0,1,0)\}$

$|(\mathbb{F}_5)^2_{T_{13}}| = 3$
$(0,0,0)$, $(1,1,1)$, $(1,0,0)$ and $(0,1,1)$ are trivially equal.
$(1,1,1) - (0,0,1) = (1,1,1)$ and $(0,1,0) - (0,1,0) = (1,1,1)$.

$T_{14} = \{(x,x,x) : x, y \in \mathbb{F}_2\}$
$(\mathbb{F}_2)^3_{T_{14}} = \{(0,0,0), (1,1,1), (1,1,0), (0,0,1)\}$
$|(\mathbb{F}_5)^2_{T_{14}}| = 4$
$(0,0,0)$ and $(1,1,1)$ are trivially equal. $(1,0,1) - (0,1,0) = (1,1,1)$, $(1,1,0) - (0,0,1) = (1,1,1)$ and $(1,0,0) - (0,1,1) = (1,1,1)$.

$T_{15} = \{(x,y,x+y) : x, y \in \mathbb{F}_2\}$
$(\mathbb{F}_2)^3_{T_{15}} = \{(0,0,0), (1,1,1), (0,1,0)\}$
$|(\mathbb{F}_5)^2_{T_{15}}| = 3$
$(0,0,0)$, $(1,0,1)$, $(1,1,0)$ and $(0,1,1)$ are trivially equal.
$(1,1,1) - (0,0,1) = (1,1,0)$ and $(1,0,0) - (0,1,0) = (1,1,0)$.

$T_16 = \{(0,0,0)\}$
$(\mathbb{F}_2)^3_{T_16} = \{(x,y,z) : x, y, z \in \mathbb{F}_2\}$
$|(\mathbb{F}_2)^3_{T_16}| = 8$
This because the only elements that have a difference equal to the zero vector, are identical elements.

Thus, there are $\sum_{n=1}^{1} 6|(\mathbb{F}_2)^3_{T_n}| = 59$ total affine linear subspaces of $(\mathbb{F}_2)^3$ modelled on $(T_n)_{n=1}^{16}$.

□

4. a)

Proof.

We will first show that the sum is a direct sum. Suppose that there exists $f \in V_e \setminus \{0\}$ and $g \in V_o \setminus \{0\}$ such that $f + g = 0$. Thus, $f = -g$, then $f(x) = f(-x) = -g(-x) = -g(x)$ This is a contradiction because if g is odd and non-zero, than $-g(-x) = g(x) \neq -g(x)$. Therefore, there is only one way to express the zero-vector in terms of vectors from V_e and V_o, namely the sum of the zero vectors in each respective subspace. Thus, according to **1.44** in *Axler's Linear Algebra Done Right*, $V_e + V_o$ is a direct sum

We will now show that any vector in $\mathbb{R}^{\mathbb{R}}$ can be expressed as a sum of even and odd functions. Consider $f \in \mathbb{R}^{\mathbb{R}}$. Consider the function $g : \mathbb{R} \to \mathbb{R},\ g(x) = \frac{f(x)+f(-x)}{2}$. Notice that $g(-x) = \frac{f(-x)+f(x)}{2} = \frac{f(x)+f(-x)}{2} = g(x)$. Thus, g is even. Consider the function $h : \mathbb{R} \to \mathbb{R},\ h(x) = \frac{f(x)-f(-x)}{2}$. Notice that $h(-x) = \frac{f(-x)-f(x)}{2} = \frac{-f(x)+f(-x)}{2} = -h(x)$. Thus, h is odd. Take the sum of $g + h = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = \frac{2f(x)}{2} = f(x) = f$.

Thus, we have shown that $V = V_e \oplus V_o$.

$\square$

b)

Proof.

$\exp(x) = \frac{\exp(x)+\exp(-x)}{2} + \frac{\exp(x)-\exp(-x)}{2}$.

$\square$