MAT157 Problem Set 2

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1. a) $\forall x \in F,\ \exists y \in F : \forall z \in F,\ z \neq x,\ xy = 1 \implies yz \neq 1$
The negation:
$\exists x \in F,\ \forall y \in F : \exists z \in F, z \neq x, xy = 1$ and $yz = 1$
Plain English:
There exists an element x that for all y such that there exists z with $z \neq x$, $xy = 1$ and $yz = 1$

b) Let $a(x)$ be the angle sum of polygon x and let H be the set of all hyperbolic octagons.
$\forall x \in H,\ a(x) > \pi$
The negation:
$\exists x \in H,\ a(x) \leq \pi$
Plain English:
There exists a hyperbolic octagon with an angle sum less than or equal to π.

c) Let Q be the set of all flavors of quarks, $c(x)$ be the charge of quark x, and $m(x)$ be the mass of quark x.
$\forall x, y \in Q, c(x) = c(y)$ and $m(x) = m(y)$
The negation:
$\exists x, y \in Q, c(x) \neq c(y)$ or $m(x) \neq m(y)$
Plain English:
There exists two flavors of quarks that do not have the same charge or the same mass.

d) Let S be the set of students in this class, H be the set of all homework assignments, L be the set of all lectures, $h(x, y)$ be student x does homework y, $l(x, y)$ be student x goes to lecture y, and $f(x)$ be the percentage that student x gets on the final exam.
$\forall x \in S : h(x, y), \forall y \in H$ and $l(x, z), \forall z \in L \implies f(x) \geq 50$

The negation:
$\exists x \in S : (h(x, y), \forall y \in H$ and $l(x, z), \forall z \in L)$ and $f(x) < 50$
Plain English:
There exists a student who does all the homework and all the assignments, but scores less than 50 percent on the final exam.

2. a)

Proof.

$\Rightarrow$
If $a = b$ then $|a - b| = |a - a| = 0 < \epsilon$.

$\Leftarrow$
We will take the contrapositive. If $a \neq b$ then $\exists x \in \mathbb{R} \setminus \{0\}$ such that $b = a + x$. Thus, $|a - b| = |a - (a + x)| = |x| > \epsilon$

□

b)

Proof.

Because the distance between a and b is a positive real number, $\forall \epsilon > 0$ we can choose $q \in \mathbb{N}$ arbitrarily large such that $q(b - a) > \epsilon$. Thus we can choose q such that $(qb - qa) > 10$. Using the ceiling function, $\lceil qa \rceil - qa < 1$. If we take $\lceil qa \rceil + 1$, the distance $(\lceil qa \rceil + 1) - qa < 2$. Note that $\lceil qa \rceil + 1$ is a rational number, greater that qa and less than qb because it's distance from qa is less than 10. Thus, $qa < \lceil qa \rceil + 1 < qb$. If we take $x = \frac{\lceil qa \rceil + 1}{q}$, then $a < x < b$.

□

3. a)

Proof.

$\Rightarrow$
We will prove this using the contrapositive: If $\exists a \in F$ such that $a \cdot a + 1 = 0$, then F^c is not a field. For the sake of contradiction, assume that F^c is a field. Then, take product of two non-zero elements, $(a, 1) \cdot (-a, 1)$. We can rewrite this as $-a^2 - ai + ai + i^2$. But this is simply $-1(a^2 + 1)$, which is 0. This is a contradiction because two non-zero elements cannot have a product of zero.

$\Leftarrow$
We will show that every non-zero element has a well-defined multiplicative inverse. Consider the arbitrary element in F^c, (a, b). Then, (a, b) has multiplicative inverse, (c, d) where $c = \frac{a}{a^2+b^2}$ and $d = \frac{-b}{a^2+b^2}$ (it is very easy to find c and d by multiplying $\frac{1}{a+bi}$ by its conjugate). This inverse is only well-defined, if $a^2 + b^2 \neq 0$. Because (a, b) is non-zero, we have 3 cases:
Case 1: $a = 0,\ b \neq 0$. Thus we have $a^2 + b^2 = 0 + b^2$, and since b^2 is non-zero, $0 + b^2$ is not zero.
Case 2: $a \neq 0,\ b = 0$. This is the same as case 1 except reversed.
Case 3: $a, b \neq 0$, Thus if $a^2 + b^2 = 0$, then $\frac{a^2}{b^2} + 1 = 0$, which contradicts our assumption that there does not exists an element in F such that $a \cdot a + 1 = 0$.

$\square$

b)

Proof.

$\Rightarrow$
We will prove this using the contrapositive: If $\exists a \in F$ such that $a^2 + a + 1 = 0$, then F' is not a field. For the sake of contradiction, assume that F' is a field. Then, take product of two non-zero elements, $(a + 1, 1) \cdot (-a, 1)$. We can rewrite this as $-a^2 + ai - a + i - ai + i^2$. But this is simply $(-a^2 - a) + (i^2 + i) = 1 - 1$, which is 0. This is a contradiction because two non-zero elements cannot have a product of zero.

$\Leftarrow$

We will show that every non-zero element has a well-defined multiplicative inverse. Consider the arbitrary element in F', (a,b). Then, we will do a series of steps to find out what the inverse of (a,b) is:

$$\begin{aligned}
(a+bi)\frac{1}{a+bi} &= 1\\
(a+bi)\frac{(b+ai)}{(a+bi)(b+ai)} &= 1\\
(a+bi)\frac{(b+ai)}{ab+a^2i+b^2i+abi^2} &= 1\\
(a+bi)\frac{(b+ai)}{ab+a^2i+b^2i-ab-abi} &= 1\\
(a+bi)\frac{(b+ai)}{a^2i+b^2i-abi} &= 1\\
(a+bi)\frac{(b+ai)}{i(a^2+b^2-ab)} &= 1\\
(a+bi)\frac{(b+ai)(1+i)}{i(a^2+b^2-ab)(1+i)} &= 1\\
(a+bi)\frac{(b+bi+ai+ai^2)}{(a^2+b^2-ab)(i+i^2)} &= 1\\
(a+bi)\frac{(b+bi+ai-a-ai)}{(a^2+b^2-ab)(-1)} &= 1\\
(a+bi)\frac{(b-a)+(b)i}{(-a^2+-b^2+ab)} &= 1
\end{aligned}$$

Thus there is an inverse to (a,b), namely, (c,d) where $c = \frac{b-a}{-a^2+-b^2+ab}$ and $d = \frac{b}{-a^2+-b^2+ab}$. This inverse is only well defined, if $-a^2+-b^2+ab \neq 0$. Because (a,b) is non-zero, we have 3 cases:

Case 1: $a=0,\ b\neq 0$. Thus we have $-a^2+-b^2+ab = 0+-b^2 = 0$, and since b^2 is non-zero, $0-b^2+0$ is not zero.

Case 2: $a\neq 0,\ b=0$. This is the same as case 1 except reversed.

Case 3: $a,b\neq 0$, Thus if $-a^2-b^2+ab=0$, then $\frac{a^2}{b^2}+\frac{a}{-b}+1=0$, which contradicts our assumption that there does not exists an element in F such that $a^2+a+1=0$.

$\square$

c) If $F \in \{\mathbb{Z}_3, \mathbb{Z}_7\}$, then F^c is a field and F' is not a field, and if $F \in \{\mathbb{Z}_2, \mathbb{Z}_5\}$, then F' is a field and F^c is a field.

Proof.

If $F \in \{\mathbb{Z}_3, \mathbb{Z}_7\}$, then $\forall a \in F,\ a \cdot a + 1 \neq 0$; thus, F^c is a field by part a. We can show this exhaustively by making $f : \mathbb{Z}_3 \to \mathbb{Z}_3$ such that $f(x) := x \cdot x + 1$ and $g : \mathbb{Z}_7 \to \mathbb{Z}_7$ such that $g(x) := x \cdot x + 1$.

$$\begin{aligned}
f(0) &\neq 0 \\
f(1) &\neq 0 \\
f(2) &\neq 0 \\
g(0) &\neq 0 \\
g(1) &\neq 0 \\
g(2) &\neq 0 \\
g(3) &\neq 0 \\
g(4) &\neq 0 \\
g(5) &\neq 0 \\
g(6) &\neq 0
\end{aligned}$$

If $F \in \{\mathbb{Z}_3, \mathbb{Z}_7\}$, then $\exists a \in F,\ a^2 + a + 1 = 0$, namely, $a = 1 \in \mathbb{Z}_3$ and $a = 2 \in \mathbb{Z}_7$; thus, F' is not a field.

If $F \in \{\mathbb{Z}_2, \mathbb{Z}_5\}$, then $\forall a \in F,\ a^2 + a + 1 \neq 0$; thus, F' is a field by part b. We can show this exhaustively by making $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ such that $f(x) := x^2 + x + 1$ and $g : \mathbb{Z}_5 \to \mathbb{Z}_5$ such that $g(x) := x^2 + x + 1$.

$$\begin{aligned}
f(0) &\neq 0 \\
f(1) &\neq 0 \\
g(0) &\neq 0 \\
g(1) &\neq 0 \\
g(2) &\neq 0 \\
g(3) &\neq 0 \\
g(4) &\neq 0
\end{aligned}$$

If $F \in \{\mathbb{Z}_2, \mathbb{Z}_5\}$, then $\exists a \in F,\ a \cdot a + 1 = 0$, namely, $a = 1 \in \mathbb{Z}_2$ and $a = 2 \in \mathbb{Z}_5$; thus, F^c is not a field.

$\square$

4. a)

Proof.

We will consider every other case to be true and show that they all lead to contradictions:
Case 1: $xy = 0$. This contradicts the fact that no non-zero elements can have a product of 0.
Case 2: $xy = x$. This contradicts the fact that the multiplicative identity is unique.
Case 3: $xy = y$. This also contradicts the fact that the multiplicative identity is unique.
Thus $xy = 1$ must be true.

□

b)

Proof.

We will consider every other case to be true and show that they will lead to contradictions:
Case 1: $xx = 0$ and $yy = 0$. This contradicts the fact that no non-zero elements can have a product of 0.
Case 2: $xx = 1$ and $yy = 1$. This contradicts the fact that we already know that x and y are multiplicative inverses of each other, and multiplicative inverses are unique.
Case 3: $xx = x$ and $yy = y$. This contradicts the fact that the multiplicative identity is unique.
Thus $xx = y$ and $yy = x$ must be true.

□

c)

Proof.

We will consider every other case to be true and show that they will lead to contradictions:
Case 1: $x + y = 0$. Multiplying both side by x gives us the equation $xx + xy = 0$, and thus, $y + 1 = 0$. If instead we multiply the original equation by y gives us the equation $xy + yy = 0$, and thus, $x + 1 = 0$. Using the transitive property of the equality relation, we can say that $x + 1 = y + 1$. Hence, $x = y$ which is a contradiction because x and y are distinct.

Case 2: $x + y = x$. This contradicts the fact that the additive identity is unique.
Case 3: $x + y = y$. This contradicts the fact that the additive identity is unique.
Thus $x + y = 1$ must be true.

□

d)

Proof.

We will consider every other case to be true and show that they will lead to contradictions:
Case 1: $x + x = 1$. By the transitive property, we can say that $x + x = x + y$, and if we apply the additive inverse of x to both sides, we get $x = y$, a contradiction.
Case 2: $x + x = x$. This contradicts the fact that additive identities are unique.
Case 3: $x + x = y$. If we multiply x on both sides, we get $xx + xx = xy$. That leaves us with $y + y = 1$. This allows us to use the transitive property to say $y + y = x + y$, and by applying the additive inverse of y we get $y = x$, a contradiction.
Thus $x + x = 0$ must be true, and in a similar fashion, we can show that $y + y = 1$ must be true.

Because x and y are their own additive inverses, $x + 1 = y$ and $y + 1 = x$ is trivially true through rearrangement. $1 + 1 = 0$ because 1 needs an additive inverse, and we have already shown it cannot be y or x; thus, it has to be 1.

□

5. a)

Proof.

Consider the sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers and notice that for $I_{n+1} \subseteq I_n$, $a_n \leq a_{n+1}$. Because (a_n) is bounded above by at most b_1, then by definition the set of all a_n has a supremum. Consider $x := Sup(\{a_n\})$, if $\exists n \in \mathbb{N}$ such that $a_n > x$ then it contradicts x being an upper bound. If $\exists \epsilon > 0,\ \forall n > N \in \mathbb{N}:\ |x - a_n| \geq \epsilon$, then it contradicts that x is the lowest upper bound of $\{a_n\}$. Thus $\forall \epsilon > 0,\ \exists n > N \in \mathbb{N} : |x - a_n| < \epsilon$. Because (a_n) is non-decreasing then $\forall n > N \in \mathbb{N},\ |x - a_n| \leq |x - a_N| < \epsilon$. Thus we can say that (a_n) converges to $Sup(\{a_n\})$. Similarly, it is easy to prove that $(b_n)_{n\in\mathbb{N}}$ converges to $Inf(\{b_n\})$. Because (a_n) and (b_n) converge, then $\bigcap_{n\in\mathbb{N}} I_n$ is a defined interval. We know that $Sup(\{a_n\}) \in \bigcap_{n\in\mathbb{N}} I_n$ and $Inf(\{b_n\}) \in \bigcap_{n\in\mathbb{N}} I_n$ due to the least upper bound property of $\mathbb{R}$; thus, $\bigcap_{n\in\mathbb{N}} I_n$ contains all of its boundaries and has at least 1 element, making it closed and non-empty. (This makes heavy use of the monotone convergence theorem).

□

b)

Proof.

Consider $I_n = [a_n, b_n]$, where $a_1 = -10$ and $b_1 = 10$
$a_{n+1} \in \{x | x \in \mathbb{Q} \text{ and } \pi - \frac{1}{n+1} < x < \pi \text{ and } x > a_n\}$ and
$b_{n+1} \in \{x | x \in \mathbb{Q} \text{ and } \pi < x < \pi + \frac{1}{n+1} \text{ and } x < b_n\}$ (These sets are non-empty because of 2b; thus, we can invoke the Axiom of Choice). As n becomes arbitrarily large, the distance $|\pi - x| < \epsilon,\ \forall x \in I_n$. If we choose any object $x \in I_n$, then either $x > \pi$ or $x < \pi$. If we consider the case where $x > \pi$ then $\exists \epsilon > 0$ such that $x = \pi + \epsilon$. Therefore, we can find some $m > n$ such that $\pi < b_m < x$; thus, x cannot be in $\bigcap_{n\in\mathbb{N}} I_n$. We can make a similar argument if $x < \pi$. Because $\pi \notin \mathbb{Q}$, $\bigcap_{n\in\mathbb{N}} I_n = \emptyset$. Thus, we have provided a counterexample.

□