MAT157 Problem Set 12

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1. a)

Proof.

Clearly this integral converges if $\lambda \geq 0$ because on $\forall x \in [0, 1]$, $(1 - x) \geq 0$. Thus $(1 - x)^\lambda$ is a well defined, continuous function. However, if $\lambda < 0$ then the integral does not converge. Assume for the sake of contradiction that $\int_0^1 (1 - x)^\lambda \, dx$ exists. Then, $\int_{-1}^0 (-x)^\lambda \, dx = \int_{-1}^0 \frac{1}{(-x)^{|\lambda|}} \, dx$ exists by symmetry. However, this implies that this integral converges, aha! a contradiction. a lil sneaky you...

$\square$

b)

Proof.

For this integral, $\lambda < 0$ diverges for similar reasons to the other integral. If $\lambda \in \mathbb{N}$ then this function is clearly converges because $\int_{-1}^0 (-x^2)^n \, dx$ converges. However, if $\lambda = \frac{p}{2n}$, $p, n \in \mathbb{N}$ with $2n$ as small as possible. If we assume that the integral converges, we must also assume that $\int_{-1}^0 (-x^2)^{\frac{p}{2n}} \, dx$ converges. However, we know that $\forall x \in [-1, 0)$, $(-x^2) < 0$. Thus $(-x^2)^{\frac{p}{2n}}$ doesn't exist. Because the function does not exist on that interval, the function cannot be convergent. Otherwise (odd rational denominator), this integral converges.

$\square$

2. a)

Proof.

Let $x \in [0, \infty)$.

$$\begin{aligned}
&\frac{1+x^2}{\sqrt{1+x^5}} \\
&= \frac{1}{\sqrt{1+x^5}} + \frac{x^2}{\sqrt{1+x^5}} \\
&\leq 1 + \frac{x^2}{\sqrt{1+x^5}} \\
&\leq 1 + \frac{x^2}{\sqrt{x^5}} \\
&=1 + \frac{x^2}{x^{\frac{2}{5}}} \\
&\leq 2
\end{aligned}$$

Therefore, $\forall x \in [0, \infty)$, $\frac{2}{1+x^2} > \frac{1}{\sqrt{1+x^5}}$. Thus if $f : [0, \infty) \to \mathbb{R}$ is RHS and $g : [0, \infty) \to \mathbb{R}$ (the function we want to integrate) is LHS, then $f > g > 0$. Because f is converges, g must also converge.

$\square$

b)

Proof.

Since $1 + x^2 \cos x$ is a surjection, then consider the first positive value of x such that $1 + x^2 \cos x = 0$ and call it M. Now consider the function $f : (1, M) \to \mathbb{R},\ x \mapsto \log(M - x)$. We claim that $\forall x \in (1, M),\ \frac{x}{1+x^2 \cos x} > \log(M - x)$. However, if $\exists x \in (0, M)$ such that $\frac{x}{1+x^2 \cos x} < \log(M - x)$ this contradicts the fact that log is **SLOW**. Thus, because $\int_1^M \log(M - x)\, dx = \infty$, $\int_1^M \frac{x}{1+x^2 \cos x\, dx} = \infty$, meaning that $\int_0^\infty \frac{x}{1+x^2 \cos x}\, dx = \infty$ and is therefore, divergent.

$\square$

c)

Proof.

Consider $f : (1, \infty) \to \mathbb{R},\ f(x) = \frac{1}{x^2} + 1$.
$\forall x \in (1, \infty) f(x) \geq 1 \geq \sin(\frac{1}{x^2}) > 0$. Thus because $\int_1^\infty f(x)\, dx$ converges, that implies that $\int_1^\infty \sin(\frac{1}{x^2})\, dx$ also converges.

$\square$

3. a)

Proof.

Consider the primative, F, of the function $f(x) = \frac{1}{\sin^2 x}$. As you can verify $F(x) = -\frac{1}{\tan x}$. Thus, by **FTC** $\lim_{x\to 0^+} \int_{Ax}^{Bx} \frac{1}{\sin^2 t}\,dt = \lim_{x\to 0^+} F(x)]_{Ax}^{Bx} = \lim_{x\to 0^+} F(Bx) - F(Ax)$. Thus, we must simply evaluate the $\lim_{x\to 0^+} \frac{1}{\tan(Ax)} - \frac{1}{\tan(Bx)} = \lim_{x\to 0^+} \frac{\tan(Bx)-\tan(Ax)}{\tan(Ax)\tan(Bx)}$ As x arbitrarily small, the numerator is always positive, while the denominator goes to zero and is positive. Thus, $\int_{Ax}^{Bx} \frac{1}{\sin^2 t}\,dt = \infty$.

□

b)

Proof.

$\lim_{x\to 0^+} x \int_{Ax}^{Bx} f(t)\,dt = \lim_{x\to 0^+} \frac{\int_{Ax}^{Bx} f(t)\,dt}{x^{-1}}$. This is indeterminate form $\frac{\infty}{\infty}$ so we can apply *l'hopital's rule.* $\frac{d}{dx}\int_{Ax}^{Bx} f(t)\,dt = \frac{d}{dx}(\int_0^{Bx} f(t)\,dt - \int_0^{Ax} f(t)\,dt) = \frac{B}{\sin^2(Bx)} - \frac{A}{\sin^2(Ax)}$.

Thus, $\lim_{x\to 0^+} x \int_{Ax}^{Bx} f(t)\,dt = \frac{\frac{B}{\sin^2(Bx)} - \frac{A}{\sin^2(Ax)}}{-(0)^{-2}} = 0$ as desired.

□

4. a)

Proof.

Let $M_i = f(t_i)$ and $m_i = f(t_{i-1})$. Notice that $t_i - t_{i-1} = a + i\frac{b-a}{n} - a - (i-1)\frac{b-a}{n}$. Also notice that $\sum_{i=1}^{n} M_i - m_i =$ $f(t_1) - f(t_0) + f(t_2) - f(t_1) + \cdots + f(t_n) - f(t_{n-1}) = |f(b) - f(a)|$ because the series telescopes.

$$\begin{aligned} U(f, P_n) - L(f, P_n) &= \sum_{i=1}^{n} (t_i - t_{i-1}) M_i - m_i \\ &= \frac{b-a}{n} \sum_{i=1}^{n} M_i - m_i \\ &= \frac{b-a}{n} |f(b) - f(a)| \end{aligned}$$

As desired.

$\square$

b)

Proof.

Let $\varepsilon > 0$ let $N > \frac{\varepsilon}{\sup\{M_i - m_i | i \in \{1,...,n\}\} \cdot (b-a)}$. Thus for every $n > N$, $U(f, P_n) - L(f, P_n) = \frac{b-a}{n} \sum_{i=1}^{n} M_i - m_i < \frac{b-a}{N} \sup\{M_i - m_i | i \in \{1, ..., n\}\} < \varepsilon$. Thus, the limits are of upper and lower Riemann sums are equal, thus the whole function is integrable.

$\square$

c)

Proof.

Consider $\lim_{n\to\infty} n^d \sum_{i=0}^{n} \frac{1}{(i+n)^{d+1}} = \lim_{x\to\infty} \sum_{i=0}^{n} \frac{n^d}{(i+n)^{d+1}}$ However, as x goes to infinity, we are left with $\frac{\infty}{\infty}$ which is an indeterminate form. Notice, that the expression is in indeterminate form for any $d \in \mathbb{N}$. Thus we can apply *l'hopital's rule* and differentiate in respect to n, d times. Thus $\frac{d}{dn^d} n^d = d!$ and $\frac{d}{dn^d}(a+n)^{d+1} = (d+1)!(a+n),\ a \in \mathbb{N}$. Thus, $\lim_{n\to\infty} \frac{d}{dn^d} \frac{d!}{(d+1)!(a+n)} = \lim_{n\to\infty} \frac{1}{(d+1)(a+n)} = 0$. Thus, $\lim_{n\to\infty} n^d \sum_{i=0}^{n} \frac{1}{(i+n)^{d+1}} = \lim_{n\to\infty} \sum_{i=1}^{n} 0 = 0$.

$\square$

5.

Proof.

Consider the function

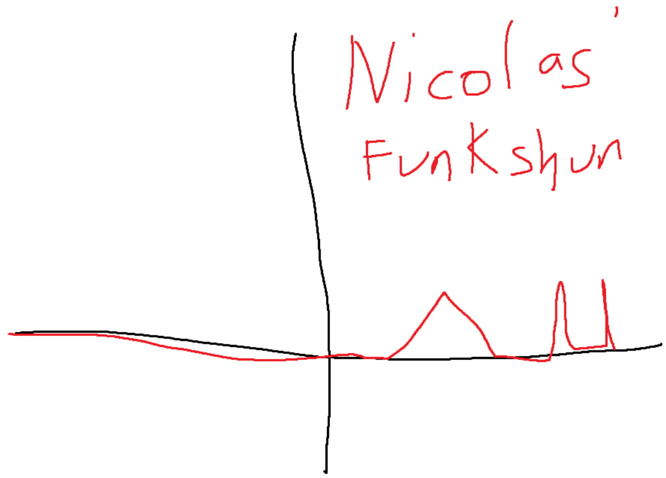
$$f:\mathbb{R}\to\mathbb{R},\ f(x)=\begin{cases}\frac{1}{n^2}(x-10n-\frac{1}{n^2}) & \text{if } 0<10n-x<\frac{1}{n^2},\ \forall n\in\mathbb{N}\\ -\frac{1}{n^2}(x-10n-\frac{1}{n^2}) & \text{if } 0>x-10n>-\frac{1}{n^2},\ \forall n\in\mathbb{N}\\ 0 & \text{otherwise}\end{cases}$$

Clearly, we can see that f is non-negative everywhere, and if $0<10n-x<\frac{1}{n^2},\ \forall n\in\mathbb{N}$ then f is locally the same as a degree 1 polynomial, which is continuous, and similarly for $0>x-10n>-\frac{1}{n^2}$. If x is otherwise, then f is locally the zero function which is continuous.

Now consider $\lim_{x\to\infty}\int_0^x f(t)\,dt$. Let $\varepsilon>0$ choose $N>S$ such that the S'th partial sum of $|L-\sum_{n=1}^{S}\frac{1}{n^2}|<\varepsilon$. We know this converges because $\int_1^\infty \frac{1}{x^2}\,dx$ converges. Thus if $x>N$, we know that $\int_0^x f(t)\,dt=\int_0^{11} f(t)\,dt+\int_{11}^{21} f(t)+\cdots+\int_c^x f(t)\,dt\geq\sum_{n=1}^{N}\frac{1}{n^2}$. However, we know that there exists $N'>x>N$ such that $\sum_{n=1}^{N}\frac{1}{n^2}\leq\int_0^x f(t)\,dt<\sum_{n=1}^{N'}\frac{1}{n^2}$. Since $N'>N$, this implies that $|L-\sum_{n=1}^{N'}\frac{1}{n^2}|<\varepsilon$ which gives us $|L-\int_0^x f(t)\,dt|<\varepsilon$ as desired.

$\lim_{x\to\infty} f(x)$ does not exist because if $\varepsilon=\frac{1}{2}$ no matter how we choose N there exists $x>y>N$ such that $f(x)=1$ and $f(y)=0$. Thus if $|L-f(x)|<\varepsilon$ that means that $|L-f(y)|\geq\varepsilon$, implying the limit does not exist.

This is the graph of my function (* *not to scale*):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