PUMP II Problem Set 2

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1. a)

Proof.

Let $q(x) := 0$ and $r(x) := P_1(x)$
Thus $P_1 = 0(P_2) + P_1$
And by assumption $deg(r) = deg(P_1) < deg(P_2)$

$\square$

b)

Proof.

Let $P_1(x) = a_0 + a_1 x + \ldots + a_n x^n$ and $P_2(x) = b_0 + b_1 x + \ldots + b_m x^m$

Case 1: $n = m$
$P_1(x) =$
$\frac{a_n}{b_m} P_2(x) + (a_0 - \frac{a_n}{b_m} b_0) + (a_1 - \frac{a_n}{b_m} b_1)x + \ldots + (a_{m-1} - \frac{a_n}{b_m} b_{m-1})x^{m-1}$
Thus $q(x) := \frac{a_n}{b_m}$ and
$r(x) := (a_0 - \frac{a_n}{b_m} b_0) + (a_1 - \frac{a_n}{b_m} b_1)x + \ldots + (a_{m-1} - \frac{a_n}{b_m} b_{m-1})x^{m-1}$
and $deg(r) = deg(m) - 1 < deg(m)$ as desired.

Case 2: $n > m$
We will induct on n
Base Case: $n = 1$ and $m = 0$

$$P_1(x) = \frac{1}{b_0} P_1(x) P_2(x)$$
$$q(x) := \frac{1}{b_0} P_1(x) \text{ and } r(x) := 0$$

Thus the base case is true.
Inductive Step: $k = n + 1$
Let $q(x) = \frac{a_{n+1}}{b_m} x^{n+1-m} + q'(x)$ where $q'(x)$ is some polynomial.

$$P_1(x) = (\frac{a_{n+1}}{b_m} x^{n+1-m} + q'(x))P_2(x) + r(x)$$

$$P_1(x) - (\frac{a_{n+1}}{b_m} x^{n+1-m})P_2(x) = q'(x)P_2(x) + r(x)$$

Because $P_1(x)$ and $(\frac{a_{n+1}}{b_m} x^{n+1-m})P_2(x)$ have the same degree and leading coefficient, their difference, at most, is of degree n.
Thus we can set some polynomial

$$P_1'(x) := P_1(x) - (\tfrac{a_{n+1}}{b_m} x^{n+1-m})P_2(x)$$
$$P_1'(x) = q'(x)P_2(x) + r(x)$$

But because $deg(P_1') \leq n$ by the induction hypothesis, $q'(x)$ and $r(x)$ exists.

□

c)

Proof.

$$P_1 = qP_2 + r$$
$$P_1 = \tilde{q}P_2 + \tilde{r}$$

Taking the difference of these equations leaves us with

$$0 = (q - \tilde{q})P_2 + r - \tilde{r}$$

Because $deg((q - \tilde{q})P_2) > deg(r)$ and $deg((q - \tilde{q})P_2) > deg(\tilde{r})$ then $(q - \tilde{q})P_2 \notin span(r, \tilde{r})$
Thus to equal the zero polynomial, $(q - \tilde{q})P_2 = 0$ and since $P_2 \neq 0$ then $q - \tilde{q} = 0$, leaving us with $q = \tilde{q}$
Because $(q - \tilde{q})P_2 = 0$, then $r - \tilde{r} = 0$ and thus $r = \tilde{r}$ as desired.

□

d)

Proof.

$\Rightarrow$

Using Theorem 1, we can set $P_2(x) = x - \alpha$ and thus any polynomial of positive degree, P_1 can be written

$$\begin{aligned} P_1(x) &= q(x)(x-\alpha) + r(x) \\ P_1(\alpha) &= q(\alpha)(\alpha-\alpha) + r(\alpha) \\ 0 &= r(\alpha) \end{aligned}$$

Because $deg(r) < deg(x-\alpha) = 1$, r must be a constant, and since it maps α to zero, it must be the zero function. Then $P_1(x) = q(x)(x-\alpha)$ and thus $(x-\alpha)$ is a factor of P_1.

$\Leftarrow$

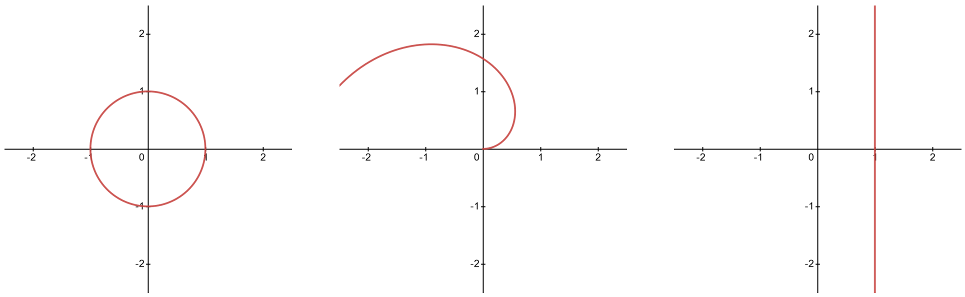
By assumption $P(x) = q(x)(x-\alpha)$

$$\begin{aligned} P(\alpha) &= q(\alpha)(\alpha-\alpha) \\ &= q(\alpha)(0) \\ &= 0 \end{aligned}$$

And thus, α is a root of P.

$\square$

2. a)



(a) $r = 1$ (b) $r = \theta$ (c) $r = sec(\theta)$

b)

Proof.

$$
\begin{aligned}
d^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \\
&= x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2 \\
&= r_1^2cos^2(\theta_1) - 2r_1cos(\theta_1)r_2cos(\theta_2) + r_2^2cos^2(\theta_2) + r_1^2sin^2(\theta_1) \\
&\quad - 2r_1sin(\theta_1)r_2sin(\theta_2) + r_2^2sin^2(\theta_2) \\
&= r_1^2(cos^2(\theta_1) + sin^2(\theta_1)) + r_2^2(cos^2(\theta_2) + sin^2(\theta_2)) \\
&\quad - 2r_1r_2(cos(\theta_1)cos(\theta_2) + sin(\theta_1)sin(\theta_2)) \\
&= r_1^2 + r_2^2 - 2r_1r_2cos(\theta_1 - \theta_2)
\end{aligned}
$$

$\square$

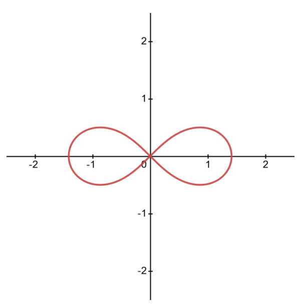
c)

Proof.

$$
\begin{aligned}
1 &= d_1 d_2 \\
1 &= d_1^2 d_2^2 \\
1 &= (r^2 + 1^2 - 2rcos(\theta))(r^2 + (-1)^2 - 2rcos(\theta - \pi)) \\
1 &= (r^2 + 1 - 2rcos(\theta))(r^2 + 1 + 2rcos(\theta)) \\
0 &= r^4 + r^2 + 2r^3cos(\theta) + r^2 + 2rcos(\theta) - 2r^3cos(\theta) \\
&\quad - 2rcos(\theta) - 4r^2cos^2(\theta) \\
0 &= r^4 + 2r^2 - 4r^2cos^2(\theta) \\
0 &= r^2 + 2 - 4cos^2 \\
2(2cos(\theta) - 1) &= r^2 \\
2cos(2\theta) &= r^2
\end{aligned}
$$

□

d)



(a) $r^2 = 2cos(2\theta)$

e)

$$
\begin{aligned}
r^2 &= 2cos(2\theta) \\
r^2 &= 2cos^2(arccos(\frac{x}{r})) - 2sin^2(arcsin(\frac{y}{r})) \\
r^2 &= 2\frac{x^2}{r^2} - 2\frac{y^2}{r^2} \\
r^4 &= 2x^2 - 2y^2 \\
(x^2 + y^2)^2 &= 2x^2 - 2y^2
\end{aligned}
$$

3. a)

Proof.

$$\begin{aligned}
f_{\vec{u}}(\vec{v}+\vec{w}) &= \begin{pmatrix} u_2(v_3+w_3) - u_3(v_2+w_2) \\ u_3(v_1+w_1) - u_1(v_3+w_3) \\ u_1(v_2+w_2) - u_2(v_1+w_1) \end{pmatrix} \\
&= \begin{pmatrix} u_2v_3 + u_2w_3 - u_3v_2 - u_3w_2 \\ u_3v_1 + u_3w_1 - u_1v_3 - u_1w_3 \\ u_1v_2 + u_1w_2 - u_2v_1 - u_2w_1 \end{pmatrix} \\
&= \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} + \begin{pmatrix} u_2w_3 - u_3w_2 \\ u_3w_1 - u_1w_3 \\ u_1w_2 - u_2w_1 \end{pmatrix} \\
&= f_{\vec{u}}(\vec{v}) + f_{\vec{u}}(\vec{w})
\end{aligned}$$

$$\begin{aligned}
f_{\vec{u}}(\lambda\vec{v}) &= \begin{pmatrix} u_2(\lambda v_3) - u_3(\lambda v_2) \\ u_3(\lambda v_1) - u_1(\lambda v_3) \\ u_1(\lambda v_2) - u_2(\lambda v_1) \end{pmatrix} \\
&= \begin{pmatrix} \lambda(u_2v_3 - u_3v_2) \\ \lambda(u_3v_1 - u_1v_3) \\ \lambda(u_1v_2 - u_2v_1) \end{pmatrix} \\
&= \lambda \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} \\
&= \lambda f_{\vec{u}}(\vec{v})
\end{aligned}$$

□

b)

Proof.

$$\begin{aligned}
f_{\vec{u}+\vec{v}}(\vec{w}) &= \begin{pmatrix}(u_2+v_2)w_3-(u_3+v_3)w_2\\(u_3+v_3)w_1-(u_1+v_1)w_3\\(u_1+v_1)w_2-(u_2+v_2)w_1\end{pmatrix}\\
&= \begin{pmatrix}u_2w_3+v_2w_3-u_3w_2-v_3w_2\\u_3w_1+v_3w_1-u_1w_3-v_1w_3\\u_1w_2+v_1w_2-u_2w_1-v_2w_1\end{pmatrix}\\
&= \begin{pmatrix}u_2w_3-u_3w_2\\u_3w_1-u_1w_3\\u_1w_2-u_2w_1\end{pmatrix}+\begin{pmatrix}v_2w_3-v_3w_2\\v_3w_1-v_1w_3\\v_1w_2-v_2w_1\end{pmatrix}\\
&= f_{\vec{u}}(\vec{w})+f_{\vec{v}}(\vec{w})
\end{aligned}$$

$$\begin{aligned}
f_{\lambda\vec{u}}(\vec{v}) &= \begin{pmatrix}(\lambda u_2)v_3-(\lambda u_3)v_2\\(\lambda u_3)v_1-(\lambda u_1)v_3\\(\lambda u_1)v_2-(\lambda u_2)v_1\end{pmatrix}\\
&= \begin{pmatrix}\lambda(u_2v_3-u_3v_2)\\\lambda(u_3v_1-u_1v_3)\\\lambda(u_1v_2-u_2v_1)\end{pmatrix}\\
&= \lambda\begin{pmatrix}(u_2v_3-u_3v_2)\\(u_3v_1-u_1v_3)\\(u_1v_2-u_2v_1)\end{pmatrix}\\
&= \lambda f_{\vec{u}}(\vec{v})
\end{aligned}$$

$\square$

c)

Proof.

$$
\begin{aligned}
(f_{\vec{u}} \circ f_{\vec{v}})(\vec{w}) - (f_{\vec{v}} \circ f_{\vec{u}})(\vec{w}) &= f_{\vec{u}}(f_{\vec{v}}(\vec{w})) - f_{\vec{v}}(f_{\vec{u}}(\vec{w})) \\
&= \begin{pmatrix} u_2v_1w_3 - u_2v_2w_1 - u_3v_3w_1 + u_3w_1w_3 \\ u_3v_2w_3 - u_3v_3w_2 - u_1w_1w_1 + u_1v_2w_1 \\ u_1v_3w_1 - u_1v_1w_3 - u_2v_2w_3 + u_2v_3w_2 \end{pmatrix} \\
&\quad - \begin{pmatrix} u_1v_2w_2 - u_2v_2w_1 - u_3v_3w_1 + u_1v_3w_3 \\ u_2v_3w_3 - u_3v_3w_2 - u_1v_1w_2 + u_2v_1w_1 \\ u_3v_1w_1 - u_1v_1w_3 - u_2v_2w_3 + u_3v_2w_2 \end{pmatrix} \\
&= \begin{pmatrix} u_2v_1w_3 + u_2v_1w_3 - u_1v_2w_2 - u_1v_3w_3 \\ u_3v_2w_3 + u_1v_2w_1 - u_2v_3w_3 - u_2v_1w_1 \\ u_1v_3w_1 + u_2v_3w_2 - u_3v_1w_1 - u_3v_2w_2 \end{pmatrix} \\
&= (f_{\vec{u}\times\vec{v}}(\vec{w})
\end{aligned}
$$

□

d)

Proof.

$\Rightarrow$
If

$$
f_{\vec{u}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \vec{0}
$$

$$
\begin{pmatrix} u_2(0) - u_3(0) \\ u_3(1) - u_1(0) \\ u_1(0) - u_2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

Then $u_3 = 0$ and $u_2 = 0$ and

$$f_{\vec{u}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} u_2(0) - u_3(1) \\ u_3(0) - u_1(0) \\ u_1(1) - u_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

then $u_1 = 0$ and thus $\vec{u} = \vec{0}$

$\Leftarrow$

$$\begin{aligned} f_{\vec{u}}(\vec{v}) &= \begin{pmatrix} 0w_3 - 0w_2 \\ 0w_1 - 0w_3 \\ 0w_2 - 0w_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Thus $f_{\vec{u}}(\vec{v}) = 0$

$\square$