MAT157 Problem Set 13

Nicolas

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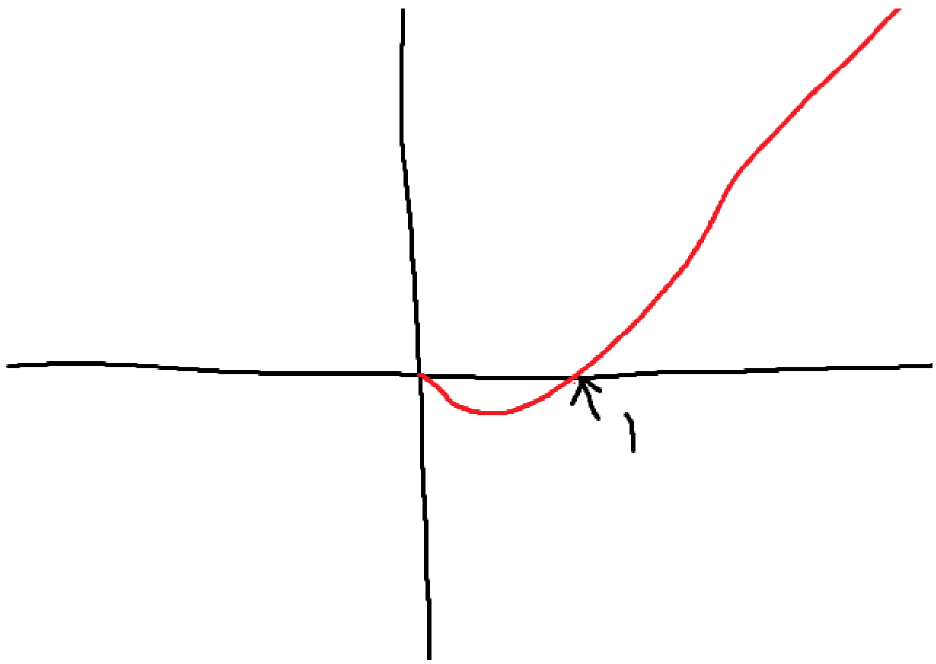
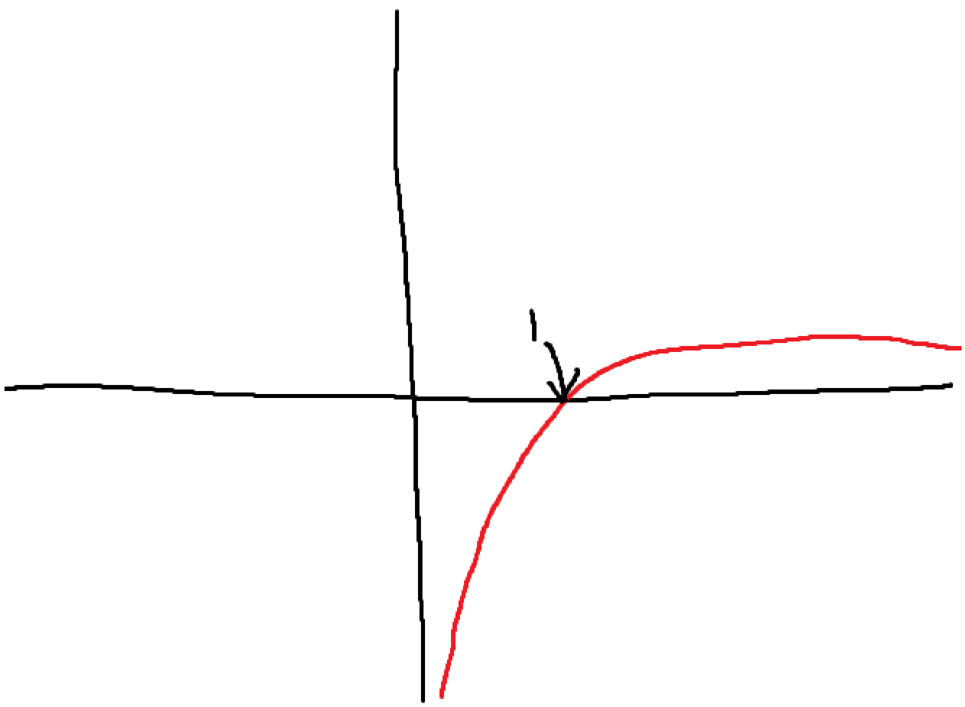
1.



$$f(x) = x \log x$$
$$f'(x) = 1 + \log x$$
$$f''(x) = \frac{1}{x}$$

f is increasing on $(\frac{1}{e}, \infty)$ because f' is positive on that ray. Similarly, f is decreasing on $(0, \frac{1}{e})$ because f' is negative on that interval. It so happens that f has a local minima which also happens to be a global minima at $\frac{1}{e}$ with $f(\frac{1}{e}) = -\frac{1}{e}$. f is always concave up (or as the normies call "convex") because f'' is always positive. f has a zero at 1 because $\log 1 = 0$.

2. a)



$$
\begin{aligned}
f(x) &= \frac{\log x}{x} \\
f'(x) &= -\frac{\log x - 1}{x^2} \\
f''(x) &= \frac{2\log x - 3}{x^3}
\end{aligned}
$$

f is increasing on $(0, e)$ because f' is positive on that interval. Similarly f decreasing on (e, ∞) because f' is negative on that ray. f has a local maxima/ global maxima at e with $f(e) = \frac{1}{e}$. f is concave down on $(0, e^{\frac{3}{2}})$ because f'' is negative on that interval. f is concave up on $(e^{\frac{3}{2}}, \infty)$ because f'' is positive on that ray. f has a zero at 1 because $\log 1 = 0$. As x goes to ∞, $f(x)$ also goes to ∞ because both $x \mapsto x$ and $x \mapsto \log x$ go to ∞. As x goes to ∞, $f(x)$ goes to 0 because $\log x$ goes to ∞, while $\frac{1}{x}$ goes to 0. However it goes to 0 faster than $\log x$ goes to ∞, which can be verified using *l'hopital's rule.*

b)

Proof.

$$\frac{\log(x^y)}{\log(y^x)} = \frac{y \log x}{x \log y} = \frac{\log x}{x} \frac{y}{\log y} = \frac{f(x)}{f(y)}$$

ta-da!

$\square$

c)

$$\pi^e < e^\pi$$

Proof.

Consider for the sake of contradiction that $e^\pi \leq \pi^e$. Because log is an increasing function $\log e^\pi \leq \log \pi^e$. This implies that

$$1 \leq \frac{\log \pi^e}{\log e^\pi} = \frac{f(\pi)}{f(e)}$$

However, if we take a look at the sketch of the graph, f is decreasing on (e, ∞). Thus $f(\pi) < f(e)$. However

$$1 \leq \frac{f(\pi)}{f(e)} \implies f(e) \leq f(\pi) \lightning$$

Aha, a contradiction. Bam, *proof by picture*!

$\square$

3. a)

Proof.

We will prove this using strong induction on n.

Base Case: $n = 1$. $\varphi(1) = \sum_{k=1}^{1} f(k) - \int_1^1 f(x)\,dx = f(1)$

$\varphi(2) = \sum_{k=1}^{2} f(k) - \int_1^2 f(x)dx = f(1) + f(2) - \int_1^2 f(x)\,dx$

However, because f is decreasing, we know that $f(1) > \int_1^2 f(x)\,dx > f(2)$. Because the constant functions of $x \mapsto f(1) > f(x),\ \forall x \in (1,2)$, and $f(x) > x \mapsto f(2),\ \forall x \in (1,2)$. Thus, if we know

$$f(1) > \int_1^2 f(x)\,dx > f(2)$$

It easily follows that

$$f(2) < f(1) + f(2) - \int_1^2 f(x)\,dx < f(1)$$

And $f(1) = \varphi(1)$, making the base case true.

Induction Hypothesis: If $j < n$, then $\varphi(j+1) < \varphi(j)$.

Inductive Step: $\varphi(n) = f(n) + \varphi(n-1) - \int_{n-1}^{n} f(x)\,dx$

$\varphi(n+1) = f(n+1) + \varphi(n) - \int_n^{n+1} f(x)\,dx$

By the induction hypothesis $\varphi(n) < \varphi(n-1)$. Because f is decreasing, $f(n) > \int_n^{n+1} f(x)\,dx > f(n+1)$, and which implies that $f(n+1) - \int_n^{n+1} f(x)\,dx < 0$. Thus

$$\varphi(n) > f(n+1) + \varphi(n) - \int_n^{n+1} f(x)\,dx = \varphi(n+1)$$

As desired.

$\square$

b)

Proof.

We will prove this using strong induction on n.

Base Case: $n = 2$

$$f(2) = \sum_{k=2}^{2} f(k) < \int_1^2 f(x)dx < f(1) = \sum_{k=1}^{1} f(k)$$

Therefore, the base is true.

Induction Hypothesis: If $j < n$, then

$$\sum_{k=2}^{j} f(k) < \int_1^j f(x)dx < \sum_{k=1}^{j-1} f(k)$$

Inductive Step: Because $x \mapsto f(n+1) < f$ on the interval $(n-1, n)$, then $\int_{n-1}^{n} f(x)\,dx > f(n+1)$. Thus by the induction hypothesis

$$\sum_{k=2}^{n-1} f(k) < \int_1^{n-1} f(x)\,dx$$

Thus, it follows by above that

$$\sum_{k=2}^{n} f(k) = f(n+1) + \sum_{k=2}^{n-1} f(k) < \int_1^{n-1} f(x)\,dx + \int_{n-1}^{n} f(x)\,dx = \int_1^n f(x)\,dx$$

Because $x \mapsto f(n-1) > f$ on the interval $(n-1, n)$, then $\int_{n-1}^{n} f(x)\,dx < f(n-1)$. Thus, it follows by above that

$$\int_1^n f(x)\,dx = \int_1^{n-1} f(x)\,dx + \int_{n-1}^{n} f(x)\,dx < \sum_{k=1}^{n-2} f(k) + f(n-1) = \sum_{k=1}^{n-1} f(k)$$

As desired.

□

c)

Proof.

We will prove this using strong induction on n.

Base Case: $n = 2$

$$f(2) < \sum_{k=1}^{2} f(k) - \int_1^2 f(x)\,dx$$
$$= f(1) + f(2) - \int_1^2 f(x)\,dx$$
$$=\varphi(2)$$

We know this because $\int_1^2 f(x)\,dx < f(1)$. Similarly, we know that $\int_1^2 f(x)\,dx > f(2)$, implying

$$\varphi(2) = f(1) + f(2) - \int_1^2 f(x)\,dx$$
$$= f(1)$$

Therefore, the base case is true.

Induction Hypothesis: If $j < n$, then $f(j) < \varphi(j) < f(1)$

Inductive Step: $\varphi(n) < f(1)$ is true because we have proven that $\varphi(n+1) < \varphi(n)$. Thus, we only have to prove that $f(n) < \varphi(n)$. Notice that because f is decreasing $f(n) - \int_1^n f(x)\,dx > 0$. Thus

$$f(n) < \sum_{k=1}^{n-1} f(k) < \sum_{k=1}^{n-1} f(k) + f(n) - \int_1^n f(x)\,dx = \varphi(n)$$

As desired.

$\square$

4. a)

Proof.

Notice that $f(x) = x^x \exp(-x^\lambda) = \exp(x\log(x) - x^\lambda)$. Notice that because the exponential is continuous

$$\lim_{x\to\infty} f(x) = \exp(\lim_{x\to\infty} x\log x - x^\lambda)$$

however

$$\lim_{x\to\infty} x\log x - x^\lambda = \lim_{x\to\infty} \frac{\frac{1}{x^\lambda} - \frac{1}{x\log x}}{\frac{1}{x^{\lambda+1}\log x}} \to \frac{0}{0}$$

$$\frac{d}{dx}\left(\frac{1}{x^\lambda} - \frac{1}{x\log x}\right) = -\frac{\lambda}{x^{-\lambda-1}} - \frac{1}{x^2\log x} - \frac{1}{x^2\log^2 x}$$

$$\frac{d}{dx}\left(\frac{1}{x^{\lambda+1}\log x}\right) = -\frac{\lambda}{x^{-\lambda-2}\log x} + \frac{1}{x^{-\lambda-2}\log^2 x} + \frac{x^\lambda \log x}{x^{2\lambda+2}\log^2 x}$$

□

b)

Proof.

We will show that $f(x) = \frac{x}{\sqrt{1-\cos x}}$ is not bounded, thus it is not integrable.

Consider

$$\begin{aligned}
\lim_{x\to 2\pi} \frac{x}{\sqrt{1-\cos x}} &= \lim_{x\to 2\pi} x\frac{1}{\sqrt{\cos x - 1}} \\
&= \lim_{x\to 2\pi} x\exp\Big(\log\Big(\frac{1}{\cos x - 1}\Big)\Big) \\
&= \lim_{x\to 2\pi} x\exp\Big(-\frac{1}{2}\log\Big(\cos x - 1\Big)\Big)
\end{aligned}$$

But because the exponential function is continuous we can rewrite this as

$$\lim_{x\to 2\pi} x\cdot\exp\Big(-\frac{1}{2}\lim_{x\to 2\pi}\log\Big(\cos x - 1\Big)\Big)$$

However, the logarithm is also continuous.

$$\lim_{x \to 2\pi} x \cdot \exp\left(-\frac{1}{2}\log\left(\lim_{x \to 2\pi} \cos x - 1\right)\right)$$

However, as x approaches 2π, $\cos x$ approaches 0, which means that $\log x$ goes to $-\infty$. Thus $\exp x$ goes to ∞ Which means the function is unbounded and thus, not integrable.

$\square$

5.

Proof.

We will prove using strong induction that
$a_{2n-1} > a_{2n+1} > 0, \ \forall n \in \mathbb{N}$.

Base Case: $n = 0$

$$\begin{aligned} a_3 &= \int_0^{3\pi} f(x) \sin x \, dx \\ &= \int_0^{\pi} f(x) \sin x \, dx + \int_{\pi}^{2\pi} f(x) \sin x \, dx + \int_{2\pi}^{3\pi} f(x) \sin x \, dx \end{aligned}$$

but we know the last two terms are less than 0 but greater than a_1 because f is decreasing and sin is negative on odd intervals of π. Thus

$$a_1 > \int_0^{\pi} f(x) \sin x \, dx + \int_{\pi}^{2\pi} f(x) \sin x \, dx + \int_{2\pi}^{3\pi} f(x) \sin x \, dx = a_3 > 0$$

Therefore, the base case is true.

Induction hypothesis: If $j < n$, then $a_{2j-1} > a_{2j+1} > 0$.

Inductive Step:

$$\begin{aligned} a_{2n+1} &= \int_0^{(2n+1)\pi} \\ &= \int_0^{(2n-1)\pi} f(x) \sin x \, dx + \int_{(2n-1)\pi}^{(2n)\pi} f(x) \sin x \, dx + \int_{(2n)\pi}^{(2n+1)\pi} f(x) \sin x \, dx \end{aligned}$$

However, the last two terms are less than 0 but greater than a_{2n+1} because f is decreasing, thus it easily follows that

$$a_{(2n+1)} < a_{(2n-1)}$$

We can similarly prove that $a_{2n-1} < a_{2n} < a_{2n+1}, \ \forall n \in \mathbb{N}$. Thus, consider $(a_n)_{n \in \mathbb{N}}$ and the subsequences $(a_{2n})_{n \in \mathbb{N}}$ and $(a_{2n-1})_{n \in \mathbb{N}}$. Because the odd subsequence is decreasing we know that

$$M := \lim_{x \to \infty} (a_{2n-1})$$

exists by *monotone convergence theorem.* The even sequence is also bounded above by a_1, thus we know it also converges

$$m := \lim x \to \infty(a_{2n})$$

because $\lim_{x\to\infty} f(x) = 0$, then the even and odd subsequences must converge somewhere in between a_1 and 0 because you can choose an n such that their difference is arbitrarily small.

$\square$