MAT157 Problem Set 8

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1.

Proof. Because g is continuous and unbounded above and below, $f - g$ is continuous and unbounded above and below. Thus, we can choose x_0 such that $(f - g)(x_0) = 0$ (if we could not do this than there would have to be a infimum or supremum to $(f - g)(\mathbb{R})$ which is a contradiction). Thus, if $(f - g)(x_0) = 0$ that means that $f(x_0) - g(x_0) = 0$, meaning $f(x_0) = g(x_0)$ as desired.

$\square$

2. a)

Proof.

Because the equation of the tangent line for f at a is $g(a) = f(a) + f'(a)(x - a)$. $f(2) = 8$ and $f'(x) = 3x^2$; thus, $f'(2) = 12$. Thus $g(2) = f(2) + f'(2)(x - 2) = 8 + 12(x - 2)$ is the equation of the tangent line.

$\square$

b)

Proof.

If $f(x - 3) = (x - 2)^2$ then $f(x) = ((x + 3) - 2)^2 = (x + 1)^2 = x^2 + 2x + 1$. Then $f'(x) = 2x + 2$, making $f'(x^2 + 5) = 2(x^2 + 5) + 2 = 2x^2 + 12$.

$\square$

3.

Proof.

Let $g_{a_0}(x) = \frac{1}{x-a_0}, \forall a_0, ..., a_n$. Because these functions are all continuous (without their trivial holes), then the function $f : \mathbb{R} - \{a_0, ..., a_n\},\ f(x) = g_{a_0}(x) + \cdots + g_{a_n}(x)$ is continuous. Thus, if we take $f|_{(-\infty,a_0)}$. $f|_{(-\infty,a_0)}(x) \neq 0,\ \forall x \in (-\infty, a_0)$. This is because $\frac{1}{x-y} < 0, \forall x < y$ and in each function g_{a_0}, and a_0 is greater than x. Similarly we see that for $f|_{(a_n,\infty)},\ f|_{(a_n,\infty)}(x) > 0,\ \forall x \in (a_n, \infty)$. Now consider any interval (c, d) where $c = a_i$ and $d = a_{i+1},\ \forall i \in \{0, ..., n-1\}$. Now we will claim that $f|_{(c,d)}$ is not bounded above or below.

Consider $f|_{(c,d)}$. Every $g_{a_i}|_{(c,d)}(x)$ is bounded above and below for all $a_i \geq d$ or $a_i < c$ because $g_{a_i}|_{(c,d)}((c, d))$ is bounded above by $g_{a_i}(c)$ and below by $g_{a_i}(d)$. Thus, we know that $f|_{(c,d)} = \sum_{i<c \text{ or } i \geq d} g_i|_{(c,d)} + g_c|_{(c,d)}$. However, we know that $\sum_{i<c \text{ or } i \geq d} g_i|_{(c,d)}$ is strictly decreasing and bounded on (c, d) and $g_i|_{(c,d)}$ is strictly decreasing and not bounded above or below. Thus, $f|_{(c,d)}$ is strictly decreasing and not bounded above or below. This means that $f|_{(c,d)}$ is a bijection to $\mathbb{R}$, meaning that it has exactly one zero.

Thus we have shown that $f|_{(c,d)}$ has exactly one zero. And there are n intervals of the form (c, d) for our set $\{a_0, ..., a_n\}$, and we know that $f|_{(-\infty,a_0)}$ and $f|_{(a_n,\infty)}$ have no zeroes. Thus f has exactly n zeroes as desired.

□

Lemma 4.1: If $f : A \to \mathbb{R}$ is continuous and A has the Heine-Borel property, then f is uniformly continuous. (Heine-Cantor Theorem in the 1-dimensional Euclidean Metric).

Proof.

Let $\epsilon > 0$, let δ_α be the δ such that if $0 < |x - \alpha| < \delta \implies |f(x) - f(\alpha)| < \frac{\epsilon}{2}$. Because f is continuous, there exists a $\delta_\alpha > 0, \ \forall \alpha \in A$. Consider the sets $U_\alpha = (\alpha - \delta_\alpha, \alpha + \delta_\alpha), \ \forall \alpha \in A$. Notice that $\bigcup_{\alpha \in A} U_\alpha$ covers A. Because A has the Heine-Borel property, then $\bigcup_{\alpha \in A} U_\alpha$ has finite subcover, $\bigcup_{i=1}^{n} U_i$ with a corresponding finite set of δs, Δ. Consider the set $\Delta' = \{\frac{\delta}{2} : \delta \in \Delta\}$

Let $\epsilon > 0$, choose $\delta = \min(\Delta')$. If $0 < |x - y| < \delta$ then there exists $1 \leq i \leq n$ such that $x \in U_i$. Let x_i be the center of U_i, then $|x - x_i| < \frac{\delta_i}{2}$. Thus, $|x_i - y| = |x_i + x - x - y| \leq |x_i - x| + |x - y| \leq \frac{\delta_i}{2} + \delta \leq \delta_i$. This implies that $|y - x_i| \leq \delta_i$. Which means that $|f(x_i) - f(x)| < \frac{\epsilon}{2}$ and $|f(x_i) - f(y)| < \frac{\epsilon}{2}$. Thus, $|f(x) - f(y)| \leq |f(x) - f(x_i)| + |f(x_i) - f(y)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ as desired. $\square$

4. a)

Proof.

Because f is bounded than there exists $M_f \in \mathbb{R} : |M_f| > |f(x)|, \ \forall x \in A$. Similarly, there exists $M_g \in \mathbb{R} : |M_g| > g(x), \ \forall x \in A$. We can simply choose M_f and M_g such that they are non-zero. Let δ_f be the δ such that $|f(x) - f(y)| < \frac{\epsilon}{2|M_g|}$ and let δ_g be the δ such that $|g(x) - g(y)| < \frac{\epsilon}{2|M_f|}$.

Let $\epsilon > 0$. Choose $\delta = \min\{\delta_f, \delta_g\}$. If $0 < |x - y| < \delta$, then:

$$\begin{aligned}
|f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
&\leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| \\
&= |f(x)||g(x) - g(y)| + |g(x)||f(x) - f(y)| \\
&< |M_f||g(x) - g(y)| + |M_g||f(x) - f(y)| \\
&< |M_f|\frac{\epsilon}{2|M_f|} + |M_g|\frac{\epsilon}{2|M_g|} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

As desired.

$\square$

b)

Proof.

f

Let $\epsilon > 0$. Choose $\delta = \epsilon$. If $0 < |x - y| < \delta$ then $|f(x) - f(y)| = |x - y| < \delta = \epsilon$ as desired.

g

Because $g(x) = -g(x+1), \ \forall x \in \mathbb{R}$, then $g(x) = g(x+2), \ \forall x \in \mathbb{R}$. Thus g is 2-periodic and we can describe every aspect of g by taking on $g|_{[x,x+2]}, \ x \in \mathbb{R}$. However, $g|_{[x,x+2]}$ is continuous and its domain has the Heine-Borel property because it is closed and bounded, thus it must be uniformly continuous by *Lemma 4.1*. Note for the next section because g is continuous and $g(x) = -g(x+1)$, then there is some point in $a \in (x, x+1)$ where there exists a δ-interval around a such that $f(a) - f(x) \neq 0, \ \forall x \in (x, x+1) - \{a\}$

fg
Consider for the sake of contradiction that fg is uniformly continuous. Thus,
$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x, y \in \mathbb{R} : 0 < |x-y| < \delta \implies |xg(x) - yg(y)| < \epsilon$.
However, if fix x and y such that $x \neq y$ and $g(x) \neq g(y)$ we choose M sufficiently large such that $M = 2N,\ N \in \mathbb{N}$ with $M > \frac{\epsilon}{|g(y)-g(x)|} + \frac{|xg(y)-x(gx)|}{|g(y)-g(x)} + \frac{|g(y)||x-y|}{|g(y)-g(x)|}$.
$|(x-M)-(y-M)| = |x-y+M-M| < \delta$, but:

$$
\begin{aligned}
&|(x-M)g(x-M) - (y-M)g(y-M)| \\
&= |(x-M)g(x-M) - (x-M)g(y-M) + (x-M)g(y-M) - (y-M)g(y-M)| \\
&\geq |(x-M)g(x-M) - (x-M)g(y-M)| - |(y-M)g(y-M) - (x-M)g(y-M)| \\
&= |x-M||g(x-M) - g(y-M)| - |g(y-M)||(y-M) - (x-M)| \\
&= |x-M||g(x) - g(y)| - |g(y)||x-y| \\
&= |xg(x) - xg(y) - Mg(x) + Mg(y)| - |g(y)||x-y| \\
&\geq |Mg(y) - Mg(x)| - |xg(y) - xg(x)| - |g(y)||x-y| \\
&= |M||g(y) - g(x)| - |xg(y) - xg(x)| - |g(y)||x-y| \\
&> \epsilon
\end{aligned}
$$

This is a contradiction, thus fg is not uniformly continuous.

$\square$

5. a)

Proof.

If we define $g : [a, b] \to \mathbb{R},\ g(x) = \begin{cases} f(x) \text{ if } x \in (a, b] \\ \limsup_{x \to a^+} f(x) \text{ otherwise} \end{cases}$.
Then because g is continuous on a closed interval, g is bounded.
Because $f = g|_{(a,b]}$, that means that f is also bounded.

$\square$

b)

Proof.

Let $\ell = \limsup_{x \to a^+} f(x)$. Let $\epsilon > 0$. If $0 < |x - a| < \delta$, then

$\square$