CSC240 Problem Set 2

Nicolas

February 17, 2022

Discussed Q1 and Q2 with Aaron Ma, Vishnu Nittoor, and Kary Ishwaran.
1.
We start with:

$$
\begin{aligned}
&(\forall i \in \mathbb{Z}.g(i,n)) \text{ IMPLIES } [(\exists x \in \mathbb{Z}.e(i,x)) \text{ AND } \forall j \in \mathbb{Z}.(g(j,i) \text{ IMPLIES } \forall y \in \mathbb{Z}.e(j,y))] \\
&= \exists a \in \mathbb{Z}.(g(a,n) \text{ IMPLIES } [(\exists x \in \mathbb{Z}.e(i,x)) \text{ AND } \forall j \in \mathbb{Z}.(g(j,i) \text{ IMPLIES } \forall y \in \mathbb{Z}.e(j,y))] \\
&= \exists a \in \mathbb{Z}.(g(a,n) \text{ IMPLIES } \exists x \in \mathbb{Z}.(e(i,x) \text{ AND } \forall j \in \mathbb{Z}.(g(j,i) \text{ IMPLIES } \forall y \in \mathbb{Z}.e(j,y)) \\
&= \exists a \in \mathbb{Z}.(g(a,n) \text{ IMPLIES } \exists x \in \mathbb{Z}.(e(i,x) \text{ AND } \forall j \in \mathbb{Z}.\forall y \in \mathbb{Z}.(g(j,i) \text{ IMPLIES } e(j,y)) \\
&= \exists a \in \mathbb{Z}.(g(a,n) \text{ IMPLIES } \exists x \in \mathbb{Z}.\forall j \in \mathbb{Z}.(e(i,x) \text{ AND } \forall y \in \mathbb{Z}.(g(j,i) \text{ IMPLIES } e(j,y)) \\
&= \exists a \in \mathbb{Z}.(g(a,n) \text{ IMPLIES } \exists x \in \mathbb{Z}.\forall j \in \mathbb{Z}.\forall y \in \mathbb{Z}.(e(i,x) \text{ AND } (g(j,i) \text{ IMPLIES } e(j,y)) \\
&= \exists a \in \mathbb{Z}.\exists x \in \mathbb{Z}.(g(a,n) \text{ IMPLIES } \forall j \in \mathbb{Z}.\forall y \in \mathbb{Z}.(e(i,x) \text{ AND } (g(j,i) \text{ IMPLIES } e(j,y)) \\
&= \exists a \in \mathbb{Z}.\exists x \in \mathbb{Z}.\forall j \in \mathbb{Z}.(g(a,n) \text{ IMPLIES } \forall y \in \mathbb{Z}.(e(i,x) \text{ AND } (g(j,i) \text{ IMPLIES } e(j,y)) \\
&= \exists a \in \mathbb{Z}.\exists x \in \mathbb{Z}.\forall j \in \mathbb{Z}.\forall y \in \mathbb{Z}.(g(a,n) \text{ IMPLIES } (e(i,x) \text{ AND } (g(j,i) \text{ IMPLIES } e(j,y))
\end{aligned}
$$

Steps:
1. $(\forall x \in D.p(x))$ IMPLIES E is transformed into
$\exists x \in D.(p(x)$ IMPLIES $E)$.
2. $(\exists x \in D.p(x))$ AND E is transformed into $\exists x \in D.(p(x)$ AND $E)$.
3. $(E$ IMPLIES $(\forall x \in D.q(x)$ is transformed into
$\forall x \in D.(E$ IMPLIES $q(x))$.
4. E AND $(\forall x \in D.p(x))$ is transformed into $\forall x \in D.(E$ AND $p(x))$.
5. Same step as 4.
6. E IMPLIES $(\exists x \in D.q(x))$ is transformed into
$\exists x \in D.(E$ IMPLIES $q(x))$.
7. E IMPLIES $(\forall x \in D.q(x))$ is transformed into
$\forall x \in D.(E$ IMPLIES $q(x))$.
8. Same step as 7.

2. a)

Proof.

For all $i,\ 1 \leq i \leq 6$ define $f_i : \{T, F, N\} \to \{T, F, N\}$

$$\begin{aligned}
f_1(P) &:= P,\ f_1(T) = T,\ f_1(F) = F,\ f_1(N) = N. \\
f_2(P) &:= \text{GNOT } P,\ f_2(T) = F,\ f_2(F) = T,\ f_2(N) = N. \\
f_3(P) &:= \text{GROT } P,\ f_3(T) = N,\ f_3(F) = T,\ f_3(N) = F. \\
f_4(P) &:= \text{GROT (GROT } P),\ f_4(T) = F,\ f_4(F) = N,\ f_4(N) = T. \\
f_5(P) &:= \text{GNOT (GROT } P),\ f_5(T) = N,\ f_5(F) = F,\ f_5(N) = T. \\
f_5(P) &:= \text{GNOT (GROT } P),\ f_6(T) = N,\ f_6(F) = N,\ f_6(N) = F.
\end{aligned}$$

We know these are all of these bijective functions because there are $n!$ bijections between a set of n elements and itself, and this set has 3 elements.

□

b)

Proof.

Let us define $\mathcal{F}, \mathcal{T}, \mathcal{N} : \{T, F, N\} \to \{T, F, N\}$
$\mathcal{F}(P) =$
(GROT P GAND GROT (GROT P)) GAND GROT (GROT (GROT P))
$\mathcal{T}(P) =$
(GROT P GOR GROT (GROT P)) GOR GROT (GROT (GROT P))
$\mathcal{N}(P) =$ GROT ($T(P)$) GAND GROT ($T(P)$)

Notice that $\forall P \in \{T, F, N\},\ \mathcal{T}(P) = T,\ \mathcal{F}(P) = F,\ \mathcal{N}(P) = N$ as desired.

□

c)

Proof.

Consider a function $f : \{T, F, N\} \to \{T, F, N\}$. Consider
$f(T) = x_0,\ f(F) = x_1,\ f(N) = x_2,\ x_0, x_1, x_2 \in \{T, F, N\}$. Then
$f(P) =$ ((GROT ($\mathcal{N}(P)$ GOR P)) GAND ($\mathcal{T}$ GAND x_0))
GOR ((GROT ($\mathcal{N}(P)$ GOR GROT (P))) GAND ($\mathcal{T}$ GAND x_1))
GOR ((GROT ($\mathcal{N}(P)$ GOR GROT (GROT (P)))) GAND ($\mathcal{T}$ GAND x_2)).

Because x_0, x_1, x_2 were arbitrary, any arbitrary function is expressible. Notice that we are using (GROT ($\mathcal{N}(P)$ GOR P)) as a sort of indicator function.

$\square$

d)

Proof.

GROT is not expressible using only GAND , GOR , and GNOT . Because there is no univariable expression of compositions of GAND , GOR , GNOT such that $f(N) = F$.

$\square$

e)

Proof.

Because $\{T, F, N\} \times \{T, F, N\}$ has 9 elements and $\{T, F, N\}$ has 3 elements, we can easily conclude combinatorially, that there are 3^9 different functions from $\{T, F, N\} \times \{T, F, N\} \to \{T, F, N\}$.

$\square$

f)

Proof.

For any arbitrary function $\{T, F, N\} \times \{T, F, N\}$, we can similarly define a new indicator function defined to be the GAND of the indicators between each coordinate in $\{T, F, N\}$.

$\square$