MAT157 Problem Set 6

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Lemma 1.0 If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$, then
$\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$.

Proof.

By assumption
$\forall\epsilon_1 > 0 \exists\delta_1 > 0 \forall x \in \mathbb{R} : 0 < |x-a| < \delta_1 \implies |f(x) - L_1| < \epsilon_1$ and
$\forall\epsilon_2 > 0 \exists\delta_2 > 0 \forall x \in \mathbb{R} : 0 < |x-a| < \delta_2 \implies |g(x) - L_2| < \epsilon_2$.
Choose $\delta = \min\{\delta_1, \delta_2\}$ such that $\epsilon_1 < \frac{\epsilon}{2}$ and $\epsilon_2 < \frac{\epsilon}{2}$, then
$|(f+g)(x) - (L_1 + L_2)| = |f(x) + g(x) - L_1 - L_2| \leq$
$|f(x) - L_1| + |g(x) - L_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\square$

1. a)

Proof.

By assumption,
$\forall\epsilon > 0 \exists\delta_1 > 0 \forall x \in \mathbb{R} : 0 < |x-0| < \delta_1 \implies |\frac{\sin x}{x} - 1| < \epsilon$. Thus,
consider $\delta = \min\{1, \delta_1\}$. Then,
$\epsilon > |\frac{\sin x}{x} - 1| \geq |x||\frac{\sin x}{x} - 1| \geq |\sin x - x|$. Thus,
$\lim_{x\to 0} \sin x - x = 0$. By *Lemma 1.0*,
$\lim_{x\to 0} \sin x - x + \lim_{x\to a} x = \lim_{x\to 0} \sin x$. Therefore, $0 + 0 = 0$.
Therefore, $\lim_{x\to 0} \sin x = 0$.

If $\lim_{x\to 0} \sin x = 0$, then $\lim_{x\to 0} 1 - \sin^2 x = 1$. Thus,
$\lim_{x\to a} \cos^2 x = 1$. Thus,
$\forall\epsilon > 0 \exists\delta' > 0 : 0 < |x-a| < \delta \implies |\cos^2 x - 1| < \epsilon$. Notice if
$\delta = \min\{\delta', \frac{\pi}{2}\}$, then $\cos x + 1 > 1$. Thus,
$|\cos x - 1| < |(\cos x - 1)(\cos x + 1)| = |\cos^2 x - 1| < \epsilon$. Therefore,
$\lim_{x\to 0} \cos x = 1$.

$\square$

2. a)

Proof.

Consider a is a limit point of either A_1 or A_2. Thus either,
$\forall \delta_1 > 0 \exists x \in A_1 : 0 < |x - a| < \delta_1$ or
$\forall \delta_2 > 0 \exists x \in A_2 : 0 < |x - a| < \delta_2$. Because $\forall x \in A_1 \cup A_2,\ x \in A$.
Then for any $\delta > 0$ we can simply choose the $x \in A_1$ or the $x \in A_2$ where $0 < |x - a| < \delta$.

$\square$

b)

Proof.

$\Rightarrow$

Suppose that $\lim_{x \to a} f(x) = \ell$, then
$\forall \epsilon > 0 \exists \delta' > 0 \forall x \in A : 0 < |x - a| < \delta' \implies |f(x) - \ell| < \epsilon$. By assumption, $\forall \delta_1 > 0 \exists x \in A_1 : 0 < |x - a| < \delta_1$ and
$\forall \delta_2 > 0 \exists x \in A_2 : 0 < |x - a| < \delta_2$. For $f|_{A_1}$ and $\forall \epsilon > 0$, take $\delta := \delta_1 < \delta'$. We can do this because δ_1 is any positive real. Thus,
$\forall x \in A_1 : 0 < |x - a| < \delta_1 < \delta' \implies |f|_{A_1}(x) - \ell| < \epsilon$. Similarly, for $f|_{A_2}$ and $\epsilon > 0$, take $\delta := \delta_2 < \delta'$. Thus,
$\forall x \in A_2 : 0 < |x - a| < \delta_2 < \delta' \implies |f|_{A_2}(x) - \ell| < \epsilon$

$\Leftarrow$

Suppose that $\lim_{x \to a} f|_{A_1}(x) = \ell$ and $\lim_{x \to a} f|_{A_2}(x) = \ell$. Then
$\forall \epsilon_1 > 0 \exists \delta_1 > 0 \forall x \in A_1 : 0 < |x - a| < \delta_1 \implies |f|_{A_1}(x) - \ell| < \epsilon_1$ and
$\forall \epsilon_2 > 0 \exists \delta_2 > 0 \forall x \in A_2 : 0 < |x - a| < \delta_2 \implies |f|_{A_2}(x) - \ell| < \epsilon_2$.
Notice that $\forall x \in A_1$ or $\forall x \in A_2$, $x \in A$. Thus if we choose $\delta := \min\{\delta_1, \delta_2\}$, then for all $x \in A$, if
$0 < |x - a| < \delta < \min\{\delta_1, \delta_2\} \implies |f(x) - \ell| < \epsilon$.

$\square$

c)

Proof.

Consider $A_1 = \mathbb{Q} \cup \{a\}$ and $A_2 = \bar{\mathbb{Q}} \cup \{a\}$. If $a \neq 2$, then
$\lim_{x \to a} f|_{A_1}(x) \neq \lim_{x \to a} f|_{A_2}(x)$. This is because $\forall x, y : x > 2$ and $y > 2,\ f|_{A_1}(x) > f|_{A_2}(y)$ and $\forall x, y : x < 2$ and $y < 2,\ f|_{A_1}(x) < f|_{A_2}(y)$ However, if $a = 2$, then
$\lim_{x \to a} f|_{A_1}(x) = \lim_{x \to a} f|_{A_2}(x) = 3$

$\square$

3.

Proof.

Consider that $\lim_{x\to\infty} f(x) = \ell$. Then, $\forall \epsilon > 0 \exists S \in \mathbb{R} : x > S \implies |f(x) - \ell| < \epsilon$. Then, $\forall \epsilon > 0 \forall T > 0$ if $x > S$ then $x + T > S$. Thus, $f(x+T) < \epsilon$. Thus if $f(x) < \frac{\epsilon}{2}$ and $f(x+T) < \frac{\epsilon}{2}$, then $|f(x) - f(x+T)| = |f(x) - f(x+T) - \ell + \ell| \leq |f(x) - \ell| + |f(x+T) - \ell| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Therefore, $\forall \epsilon > 0, \forall T > 0,\ |f(x) - f(x+T)| < \epsilon$. Now consider for the sake of contradiction that $f(x) \neq f(x+T)$. Then either there exists some $c \in \mathbb{R}$ such that $f(x) < c < f(x+T)$ or $f(x) > c > f(x+T)$. Consider the first case, then this suggests that if $\epsilon := |f(x) - c|,\ |f(x) - f(x+T)| > |f(x) - c| = \epsilon$; a contradiction. A similar thing happens for the second case. Thus $\forall T > 0,\ f(x) = f(x+T)$. Now because f is defined on $\mathbb{R}$ then we can choose some $x > S \in \mathbb{R}$ arbitrarily, and thus $f(x) \in \mathbb{R}$. If we choose $\ell := f(x)$, then because $f(x) = f(x+T) = f(x-T),\ \forall T > 0$. Because T is any positive real, and any real other than x can be written as $x + T$ or $x - T$, $\forall x, y \in \mathbb{R},\ f(x) = f(y) = \ell$. Therefore, $f(x) = \ell,\ \forall x \in \mathbb{R}$.

$\square$

4. a)

Proof.

We only need to consider $a = 1$ because for any $\lambda > 0$, $\lim_{x\to\lambda} \sqrt[n]{x} = \sqrt[n]{\lambda} \lim_{x\to 1} \sqrt[n]{x}$. If $a \neq 1$ then if we choose $\delta := \min\{\frac{|a-1|}{2}, 1\}$, then $\forall x \in (a-\delta, a+\delta)$, $|\sqrt[n]{x} - \sqrt[n]{a}| < |\sqrt[n]{x} - 1| < \epsilon$. This is because if $a < 1$, then $0 < \sqrt[n]{a} < 1$, so $|\sqrt[n]{x} - \sqrt[n]{a}| < |\sqrt[n]{x} - 1| < \epsilon$. If $a > 1$, then $\sqrt[n]{a} > 1$, so $|\sqrt[n]{x} - \sqrt[n]{a}| < |\sqrt[n]{x} - 1| < \epsilon$. Consider were working on the interval $[0, \infty)$. Consider the function $f : [0, \infty) \to \mathbb{R}$, $f(x) = |x-1| + 1$ and $g : (0, \infty) \to \mathbb{R}$, $g(x) = -|x-1| + 1$. Now consider for the sake of contradiction that on the interval $[0, 1]$, $\sqrt[n]{x} \leq x$. But because $h_n : [0, \infty) \to \mathbb{R}$, $h(x) = x^n$ is always strictly increasing. $h_n(\sqrt[n]{x}) = x \leq x^n = h_n(x)$. Which is a contradiction because if $x \in [0, 1]$, then $x^n \leq x, \forall n \in \mathbb{N}$. Thus on the interval $[0, 1]$, $g(x) = x \leq \sqrt[n]{x} \leq f(x)$. Now on the interval $[1, \infty)$, $\sqrt[n]{x} \leq x$. This is because $x \leq x^n$, $\forall x > 1$. Thus, on the interval $[1, \infty)$, $g(x) \leq \sqrt[n]{x} \leq x = f(x)$. We can now conclude that $\forall x \in [0, \infty)$, $g(x) \leq \sqrt[n]{x} \leq f(x)$. Now let $\delta := \epsilon$, now if $0 < |x-1| < \delta$, then $|f(x) - 1| = ||x-1| + 1 - 1| = |x-1| < \delta = \epsilon$ and $|g(x) - 1| = |-|x-1| + 1 - 1| = |x-1| < \delta = \epsilon$. Thus $\lim_{x\to 1} g(x) = 1$ and $\lim_{x\to 1} f(x) = 1$. Thus, by the *Squeeze Theorem* $\lim_{x\to 1} g(x) = 1 \leq \lim_{x\to 1} \sqrt[n]{x} \leq 1 = \lim_{x\to 1} f(x)$. Therefore, $\forall a > 0$, $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$.

$\square$

b)

Proof.

We only need to consider $a = 1$ or $a = 0$ because for any $\lambda > 0$, $\lim_{x\to\lambda} \frac{\sqrt[n]{x} - \sqrt[n]{\lambda}}{x-\lambda} = \frac{\sqrt[n]{\lambda}}{\lambda} \lim_{x\to 1} \frac{\sqrt[n]{x} - a}{x-a}$. This is because if $a > 1$ then $\sqrt[n]{x} - \sqrt[n]{a} < \sqrt[n]{x} - 1$ and $x - a < x - 1$. Thus, if $\sqrt[n]{x} - \sqrt[n]{a} < \sqrt[n]{x} - 1$ then $|\frac{\sqrt[n]{x} - \sqrt[n]{a}}{x-a}| < |\frac{\sqrt[n]{x} - \sqrt[n]{a}}{x-a}| < |\frac{\sqrt[n]{x} - 1}{x-1}|$ If $a < 1$ then $|\frac{\sqrt[n]{x} - \sqrt[n]{a}}{x-a}| < |\frac{\sqrt[n]{x} - \sqrt[n]{a}}{x-0}| < |\frac{\sqrt[n]{x} - 0}{x-0}|$. We claim that if $\lim_{x\to a} \frac{\sqrt[n]{x} - \sqrt[n]{a}}{x-a} = a^{\frac{1}{n}-1}$. Let $a = 1$ and take $\delta' := \frac{2}{\epsilon}$. Thus if $0 < |x-1| < \delta'$ then $|\frac{\sqrt[n]{x} - 1}{x-1}| \leq |\sqrt[n]{x} - 1||\frac{1}{x-1}| < |\sqrt[n]{x} - 1|\frac{1}{\delta'} < 2\frac{1}{\delta'} = 2\frac{\epsilon}{2} = \epsilon$. Thus, if $a > 1$, then choose $\delta := \min\{1, \delta'\}$. Thus, if $0 < |x-a| < \delta$, then

$|\frac{\sqrt[n]{x}-\sqrt[n]{a}}{x-a} - a^{\frac{1}{n}-1}| < |\frac{\sqrt[n]{x}-\sqrt[n]{a}}{x-a} - a^{\frac{1}{n}-1}| < |\frac{\sqrt[n]{x}-1}{x-1} - a^{\frac{1}{n}-1}| < \epsilon$. Let $a = 0$ and take $\delta' := \min\{\frac{1}{\epsilon}, 1$. Then if $0 < |x-a| < \delta'$ then $|\frac{\sqrt[n]{x}}{x}| \leq |\sqrt[n]{x}||\frac{1}{x}| \leq |\frac{1}{x}| = \frac{1}{\delta'} = \epsilon$. Thus if $a < 1$ choose $\delta := \min\{1, \delta'\}$. If $0 < |x-a| < \delta$ then,
$|\frac{\sqrt[n]{x}-\sqrt[n]{a}}{x-a} - a^{\frac{1}{n}-1}| < |\frac{\sqrt[n]{x}-\sqrt[n]{a}}{x-0} - a^{\frac{1}{n}-1}| < |\frac{\sqrt[n]{x}-0}{x-0} - a^{\frac{1}{n}-1}| < \epsilon$. Thus, $\lim_{x\to a} \frac{\sqrt[n]{x}-\sqrt[n]{a}}{x-a} = a^{\frac{1}{n}-1}$.

$\square$

5. a)

Proof.

If $\delta_1 < \delta_2$, then $0 < |x-a| < \delta_1 < \delta_2$. Thus, $\{f(x) : x \in A, 0 < |x-a| < \delta_1\} \subseteq \{f(x) : x \in A, 0 < |x-a| < \delta_2\}$. Since both subsets are bounded, then $\sup\{f(x) : x \in A, 0 < |x-a| < \delta_2\} \geq \sup\{f(x) : x \in A, 0 < |x-a| < \delta_1\}$. Therefore, $g(\delta_1) \leq g(\delta_2)$.

$\square$

b)

Proof.

Consider the set $S := \bigcup_{\delta>0} g(\delta)$. Because $f(A)$ is bounded then, S is bounded. Because $\emptyset \neq S \subseteq \mathbb{R}$ and S is bounded then S has a infimum. Let $\ell := \inf(S)$. We will claim that $\lim_{\delta' \to 0^+} g(\delta') = \ell$. Choose δ arbitrarily. There are two cases: $\ell = \min_{\delta>0} g(\delta)$ or $\min_{\delta>0} g(\delta)$ does not exist.
Case 1: $\ell = \min_{\delta>0} g(\delta)$. By assumption, there exists a δ such that $g(\delta) - \ell = 0 < \epsilon$. Thus, for all $\delta' < \delta$, $g(\delta') \leq g(\delta)$. But because $g(\delta)$ is the infimum, $g(\delta') = g(\delta)$. Thus, for all $0 < \delta' < \delta \implies g(\delta') - \ell = 0 < \epsilon$.
Case 2: $\min_{\delta>0} g(\delta)$ does not exist. For any $\epsilon > 0$ the define $S' \subseteq S$ such that $S' := \{x \in S : g(x) < \ell + \epsilon\}$ If this set was empty, than ℓ would not be an infimum; thus, S' is non-empty. Using the *Axiom of Choice* we can choose a δ such that $|g(\delta) - \ell| < \epsilon$. Thus, for all $0 < \delta' < \delta$, $|g(\delta') - \ell| \leq |g(\delta) - \ell| < \epsilon$.

$\square$

c)

Proof.

Because the maximum value of $sin(\frac{1}{x})$ over $\mathbb{R}$ is 1 and for any $\delta > 0$ and for n arbitrary large, there exists $0 < \frac{2}{\pi(1+4n)} < \delta$. Since $\sin(\frac{1}{\frac{2}{\pi(1+4n)}}) = \sin(\frac{\pi(1+4n}{2}) = 1$. Thus, $\forall\delta > 0$, $g(\delta) = 1$. Because $g(\delta) = 1$, $\forall\delta > 0$, the infimum of $\bigcup_{\delta>0} g(\delta) = 1 = \min(\bigcup_{\delta>0} g(\delta))$. Because $\sin(\mathbb{R})$ is bounded, we can choose δ such that $g(\delta) = 1$. Thus for all $0 < \delta' < \delta$, $|g(\delta') - 1| = |g(\delta) = 1| = 0 < \epsilon$.

$\square$