MAT157 Problem Set 11

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1. a)

Proof.

Consider $g(x) := \int_0^x \sin(t^2)\,dt$. Then it is easy to see that $F(x) = g(x^3)$. Thus by *chain rule*, $F'(x) = 3g'(x^3)x^2$. By **FTC** $g'(x) = \frac{d}{dx}\int_0^x \sin(t^2)\,dt = \sin(t^2)$. Thus $F'(x) = 3\sin(x^6)x^2$.

□

b)

Proof.

Consider $g(x) := \int_0^x \cos(t^2)\,dt$. Thus, $F(x) = g(x^2) - g(x)$ and $F'(x) = g'(x^2)2x - g'(x)$. Then, $g'(x) = \cos(x^2)$ by **FTC**. Therefore, $F'(x) = \cos(x^4)2x - \cos(x^2)$.

□

c)

Proof.

Just by *chain rule* and **FTC** $F'(x) = \cos(\int_0^x \sin(\int_0^y \frac{1}{1+t^4}\,dt)\,dy)\sin(\int_0^y \frac{1}{1+t^4}\,dt)$.

□

2.

Proof. $\square$

3.

Proof.

Consider $f := \frac{1}{x^3}$ and $g := \frac{1}{x^3} - \frac{1}{\pi^2}$. Sine's output is always less than or equal to one so $f(x) \leq \frac{\sin x}{x^3}$ and $\frac{\sin x}{x^3} - g(x) = 0$ at $x = \pi$ and it is positive else where; thus, $\forall x \in [\frac{\pi}{2}, \pi],\ g(x) \leq \frac{\sin x}{x^3} \leq f(x)$. Since the integral of f and g are quite easily to evaluate (simply using powerrule for integration) we get
$\frac{1}{\pi^2} = \int_{\pi/2}^{\pi} g(x)\, dx \leq \frac{\sin x}{x^3}\, dx \leq \frac{3}{2\pi^2} = \int_{\pi/2}^{\pi} f(x)\, dx$, using the monotonicity of the integral to achieve the desired result.

□

4. a)

Proof.

It is clear that $\sup f(A) - \inf f(A)$ is an upper bound for $\sup\{|f(x) - f(y)| : x, y \in A\}$, so we will show that $\sup f(A) - \inf f(A)$ is the least upper bound. Consider $L = \sup f(A)$ and $l = \inf f(A)$. The supremum is always greater than or equal to the infimum; thus, $\sup f(A) - \inf f(A) = |L - l|$. Now consider, for the sake of contradiction, that there exists some $\varepsilon > 0$ such that $|L - l| - \varepsilon$ is an upper bound for $\{|f(x) - f(y)| x, y \in A\}$. Because $\sup f(A)$ exists, then for all $\varepsilon' > 0, \exists x \in A : |\sup f(A) - f(x)| < \varepsilon'$ and similarly we can find an $f(x)$ ε'-close to the $\inf f(A)$. Thus, if we choose $\varepsilon' = \frac{\varepsilon}{2}$. Then we can choose $x_0, x_1 \in A$ such that $|f(x_0) - L| < \varepsilon'$ and $|f(x_1) - l| < \varepsilon'$. This means that $f(x_0) > L - \varepsilon'$ and $f(x_1) > l + \varepsilon$. Thus, $|f(x_0) - f(x_1)| > |L - \varepsilon' - l - \varepsilon'| > |L - l - \varepsilon| > |L - l| - \varepsilon$ a contradiction.

□

b)

Proof.

Because f is integrable, then forall $\varepsilon > 0$ there exists a partition, P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. Thus, if we give the g the same partition, for any $[t_i, t_{i+1}] \subseteq [a, b]$ $\sup\{g(x)|x \in [t_i, t_{i+1}\} - \inf\{g(x)|x \in [t_i, t_{i+1}]\}$ is less than $\sup\{f(x)|x \in [t_i, t_{i+1}\} - \inf\{f(x)|x \in [t_i, t_{i+1}]\}$ by assumption; however, since t_i, t_{i+1} was arbitrary, this hold for all $t \in P$. Thus it follows if the each Riemann summand for g is less than f, then $U(g, P) - L(g, P) < \varepsilon$. Thus, g is integrable as desired.

□

c)

Proof.

We can see that if $|f(x) - f(y)| \geq 1$ then $|\frac{1}{f(x)} - \frac{1}{f(y)}| \leq |f(x) - f(y)|$. However, if $|f(x) - f(y)| = \varepsilon < 1$ then without loss of generality, $f(x) = f(y) + \varepsilon$.

□