MAT157 Problem Set 8

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1.

Proof.

If we take $X = \begin{bmatrix} 2 & \frac{1}{3} \\ 1 & \frac{2}{3} \end{bmatrix}$ with our original transformation $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ then $AX = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$ thus, we now know that $\frac{dx_1}{dt} = -3x_1$ and $\frac{dx_2}{dt} = -1x_2$. Thus, $e^{-3}t = ax_1 + bx_2$ and $e^{-t} = cx_1 + dx_2$. Therefore, $x_1 = \frac{e^{-3t}+e^{-t}-x_2(b+d)}{a+c}$ and $x_1 = \frac{e^{-3t}+e^{-t}-x_1(a+c)}{b+d}$

$\square$

2.

Proof.

Consider our basis for $\mathbb{R}^2$ is the standard basis. $A(1,0) = (\frac{2}{3}, \frac{1}{3})$ and $A(0,1) = (\frac{1}{2}, \frac{1}{2})$. Thus we can write the matrix of this transformation as:

$$A = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix}$$

With this, $A(\frac{3}{2}, 1) = (\frac{3}{2}, 1)$ and $A(-1,1) = (\frac{-1}{6}, \frac{1}{6})$, thus our Eigenvalues are 1 and $\frac{1}{6}$. These are the only Eigenvalues because the dimension of $\mathbb{R}^2$ is 2. Thus we can write the diagonal matrix as such:

$$D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$$

We can also make our change of basis matrices as such:

$$P = \begin{bmatrix} \frac{3}{2} & -1 \\ 1 & 1 \end{bmatrix}$$
$$P^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{-2}{5} & \frac{3}{5} \end{bmatrix}$$

And as one can verify $PDP^{-1} = A$ Now using the fact that $A^n = PD^nP^{-1}$ we can simply calculate A^n by abusing the diagonal matrix D because D^n has indices $d_{ij}^n = (d_{ij})^n$. Thus:

$$A^n = \begin{bmatrix} \frac{3}{2} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & \frac{1}{6}^n \end{bmatrix} \begin{bmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{-2}{5} & \frac{3}{5} \end{bmatrix}$$

Thus:

$$A(\frac{1}{2}, \frac{1}{2}) = (\frac{7}{12}, \frac{5}{12}) \approx (0.5833, 0.4166)$$
$$A^2(\frac{1}{2}, \frac{1}{2}) = (\frac{43}{72}, \frac{29}{72}) \approx (0.5833, 0.4027)$$
$$A^3(\frac{1}{2}, \frac{1}{2}) = (\frac{259}{432}, \frac{173}{432}) \approx (0.5995, 0.4027)$$

Thus, we will handwavily say this converges to $(0.6, 0.4)$ because we don't know what our metric is.

$\square$

3.

Proof.

Every $\lambda \in \mathbb{F}$ is an Eigenvalue because if we consider the vector $(x_0, \lambda x_0, \lambda^2 x_0, ...)$ for any $x_0 \in \mathbb{F}$. Then $T(x_0, \lambda x_0, \lambda^2 x_0, ...) = (\lambda x_0, \lambda^2 x_0, \lambda^3 x_0, ...) = \lambda(x_0, \lambda x_0, \lambda^2 x_0, ...)$. Thus, the Eigenvectors of these Eigenvalues are precisely the vectors of the form $(x_0, \lambda x_0, \lambda^2 x_0, ...), \ \forall x_0, \lambda \in \mathbb{F}$.

$\square$

4.

Proof.

Because we know V is finite dimensional, we also know that all linear operators of V have a diagonal matrix consisting of its Eigenvalues. Thus, we know that if we multiply two diagonal matrices, $S = [s_{ij}]$ and $T = [t_{ij}]$. Then $ST = [st_{ij}]$ has $st_{ii} = s_{ii}t_{ii} = t_{ii}s_{ii} = ts_{ii}$ for all $1 \leq i \leq \dim V$. And since on non-diagonal coordinates, the indexes are all zero. Since ST and TS are also diagonal matrices, then their diagonals are the Eigenvalues which are the same. This falls apart for infinite-dimensional vector spaces because we do not know if all linear operators have a diagonal matrix.

$\square$