MAT240 Problem Set 10

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December 7, 2021

Lemma 1.1 If an operator $F : \mathbb{C}[x] \to \mathbb{C}[x]$ has the specified properties of 1.1 then $F(x^n) = nx^{n-1}$.

Proof.

We will use induction on n.
Base Case: n = 1. $D(x) = 1 = 1x^0$. Thus the base case is true.
Induction Hypothesis: If $D(x^n) = nx^{n-1}$ then $D(x^{n+1}) = (n+1)x^n$.
inductive Step:
$D(x^{n+1}) = D(x \cdot x^n) = D(x)x^n + xD(x^n) = x^n + xnx^{n-1} = (n+1)x^n$
as desired.

□

Lemma 1.2 If an operator $F : \mathbb{C}[x, e] \to \mathbb{C}[x, e]$ has the specified properties of 1.3 then $F(e^n) = ne^n$.

Proof.

We will use induction on n.
Base Case: n = 1. $D(x) = 1 = 1x^0$. Thus the base case is true.
Induction Hypothesis: If $D(e^n) = ne^n$ then $D(e^{n+1}) = (n+1)e^{n+1}$.
inductive Step:
$D(e^{n+1}) = D(e \cdot e^n) = D(e)e^n + eD(e^n) = e^{n+1} + ne^{n+1} = (n+1)e^{n+1}$
as desired.

□

1. 1)

Proof.

Consider $D' : \mathbb{C}[x] \to \mathbb{C}[x]$ with the mentioned properties. Consider a polynomial $f \in \mathbb{C}[x]$ such that
$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Then $D'f$ is defined as
$Df(x) = D(a_0) + D(a_1x) + D(a_2x^2) + \cdots + D(a_nx^n) =$

$a_0D(1) + a_1D(x) + a_2D(x^2) + \cdots + a_nD(x^n)$. Using *Lemma 1.1* we know that this is equal to $a_1 + 2a_2x + \cdots + na_nx^{n-1}$. Thus $D'f = Df$; however since f was arbitrary $D = D'$ meaning that D is unique.

$\square$

1. 2)

Proof.

Considering that any polynomial of degree $n,\ n \geq 1$ differentiates into a polynomial of degree $n-1$, that means that no polynomial of degree $n \geq 1$ can be an Eigenvector. Thus, only constant polynomials can be Eigenvectors. Consider the polynomial $f(x) = \lambda,\ \lambda \in \mathbb{C}$. Then $Df = 0$ which is a scalar multiple of λ, thus 0 is an Eigenvalue with every constant polynomial being an Eigenvector because λ was arbitrary.

$\square$

1. 3)

Proof.

Clearly, all the Eigenvectors in U are still Eigenvectors in V with Eigenvalue 0. However, in V we have some new Eigenvalues and Eigenvectors. Because of *Lemma 1.1*, $D(e^n) = ne^n$ which is a scalar multiple of e^n by n. meaning that any natural number is an Eigenvalue with Eigenvector being of the form $f(x, e) = e^n,\ \forall n \in \mathbb{N}$. As for generalized Eigenvectors, every polynomial that does not have a factor of e will eventually differentiate into 0. Thus every polynomial of solely x is a generalized Eigenvector of Eigenvalue 0. Consider if you take $D^m(\lambda e^n) = \lambda n^m e^n$. Thus, e^n is a generalized Eigenvector with Eigenvalue n^m for any $n, m \in \mathbb{N}$.

$\square$

1. 4)

Proof.

$D^3 - 4D^2 + 5D - 2 = (D-2)(D-1)^2$. Thus, 1 and 2 are Eigenvalues. By performing the Jordan Basis algorithm, we find that the basis for the null space of $(D-2)$ is just e^2 while the basis for the null space of $(D-1)^2$ is just e. Thus the Jordan Basis for S is e, e^2.

$\square$

2.

Proof.

Let $f(x, e) = e^a$. Then $(D - a)f = 0$. However for $k \in \mathbb{N}$. There is no other function that maps to e^a from $(D - a)^k$ thus e^a is the only basis vector of W_a.

$\square$

3.

Proof.

The Jordan Canonical Form of the matrix is as follows.

$$A_J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let P be a invertible matrix defined as follows:

$$P = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that $P^{-1}AP = A_J$ as desired.

□

4. 1)

Proof.

Consider any object in τ

$\square$