CSC240 Problem Set 4

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Lemma 1: If b is a binary string consisting of n trailing 1s, we can apply f some amount of times to such that b is mapped to a string with n trailing 0s.

Proof.

Let $h : \{0,1\}^+ \to \{0,1\}^+,\ h(x) = x \cdot 1$. Our predicate will be $H(n)$ = "If b is a binary string consisting of n trailing 1s, we can apply f some amount of times to such that b is mapped to a string with n trailing 0s." We will use strong induction on n.

Base Case: $n = 1$ $f(h(b)) = f(b \cdot 1) = v \cdot 0$ for some $v \in \{0,1\}^+$. Thus, $H(1)$.

Induction Hypothesis: If $j < n$ then $H(j)$.

Inductive Step: Consider $g^{(n)}(b) = g^{(n-1)}(b \cdot 1)$ but by the induction f can be applied some amount of times such that b is mapped to $c \cdot 1$ followed by $(n-1)$ 0s for some $c \in \{0,1\}^+$. If we apply f again, we will get $h^{(n-1)}(k)$ for some $k \in \{0,1\}^+$. But by induction hypothesis again, f can be applied some amount of times such that $h^{(n-1)}(k)$ is mapped to v followed by $(n-1)$ trailing 0s for some $v \in \{0,1\}^+$. $\square$

1. a)

Proof.

Let $g : \{0,1\}^+ \to \{0,1\}^+,\ g(x) = x \cdot 0$. Our predicate will be $Z(n) = \forall x \in \{0,1\}.\exists v \in \{0,1\}^+.\exists i \in \mathbb{N}.f^{(i)}(g^{(n)}(x \cdot 1)) = g^{(n+1)}(v)$.

Base Case: $n = 1$ Thus $f^{(2)}(g^{(1)}(x \cdot 1)) = f^{(2)}(x \cdot 1 \cdot 0) = f(m \cdot 01) = v \cdot 00$. Thus $Z(1)$.

Induction Hypothesis: If $j < n$, then $Z(j)$.

Inductive Step: Consider $f(g^{(n)}(b \cdot 1)) = m \cdot 1 \cdot w$ where m is some binary string and w is a binary string consisting of only $(n-1)$ 1 bits. However, we know that if we apply f n times, we can sequentially turn every 1 in w into a 0 (using *Lemma 1*). Thus $f^{(n+1)}(g^{(n)}(x \cdot 1) = g^{(n+1)}(v)$, Thus $Z(n)$ as desired.

Let $b \in \{0,1\}^+$ be arbitrary. We have two cases: $b^1 = 1$ or $b^1 = 0$.

Case 1: $b^1 = 1$. This string has at most $|b|$ 1's. If the last bit is 0, then we can apply f so b's rightmost 1 bit turns into consecutive 0s. If the last bit is 1 then $f(b)^{|b|} = 1$ and $f(b)^{|b+1|} = 0$, so we can use the above. Notice that applying f to b will keep the first bit, while adding consecutive 0s between b^1 the rest of b. However, we know that the the rest of b will become 0 because we can make any binary string with 1 and some amount of 0s following at the end turn into all 0s, giving us a binary string s consisting of a single 1 followed by consecutive 0s. Apply f to this will give us 0 followed by 1s, which we know we can apply f finite amount of times to convert all of the consecutive 1s into 0s (by *Lemma 1* again).

Case 2: $b^1 = 0$. We can apply the same process as case 1, but we do not have to consider changing the leading 1 into a 0 because applying f multiple times will only add leading 0s to b.
Because b was arbitrary, this holds for all $b \in \{0,1\}^+$ as desired

$\square$

2. a)

Proof.

Our predicate is $P(n) =$ "Any horsie on the border of an $n \times n$ chess board can move to the bottom left square". For clarity, we will refer to the squares on the chessboard as the set of $\{1, ..., n\} \times \{1, ..., n\}$, with the bottom left square being denoted as $(1, 1)$. We will prove this using strong induction induction on n.

Base Case: $n = 3$. We will prove the base case by cases.
Case 1: The horsie starts on square $(1, 1)$. The horsie is already on $(1, 1)$.
Case 2: The horsie starts on square $(1, 2)$. The horsie can move to $(3, 1)$ to $(2, 3)$ to $(1, 1)$.
Case 3: The horsie starts on square $(1, 3)$. The horsie can move to $(3, 2)$ to $(1, 1)$.
Case 4: The horsie starts on square $(2, 1)$. The horsie can move to $(1, 3)$ to $(3, 2)$ to $(1, 1)$.
Case 5: The horsie starts on square $(2, 3)$. The horsie can move to $(1, 1)$.
Case 6: The horsie starts on square $(3, 1)$. The horsie can move to $(2, 3)$ to $(1, 1)$.
Case 7: The horsie starts on square $(3, 2)$. The horise can move to $(1, 1)$.
Case 8: The horsie starts on square $(3, 3)$. The horsie can move to $(2, 1)$ to $(1, 3)$ to $(3, 2)$ to $(1, 1)$.

These are all the cases, thus $P(3)$.

Induction Hypothesis: If $j < n$ then $P(j)$.

Inductive Step:
There are two disjoint cases: The horsie starts on $(x, y) \in \{1, ..., n-1\} \times \{1, ..., n-1\}$ or the horsie starts on (n, i) or (i, n) for $1 \leq i \leq n$.

Case 1: If the horsie starts on the (x, y) then the horsie is on square that is the border of an $y \times y$ chessboard. However, we know that $y < n$, thus by the induction hypothesis $P(y)$ which means that the horsie can get to $(1, 1)$ using squares in $y \times y$. This implies that $P(n)$.

Case 2: If the horsie starts on (n, i) or (i, n), then because $n \geq 3$ the horsie can always move downwards if its on (i, n) and it can always move left if its on (i, n), thus, the horsie can move to a square $(x, y) \in \{1, ..., n-1\} \times \{1, ..., n-1\}$, which reduces the problem to case 1. Using the induction hypothesis again, $P(y)$ so we can get to $(1, 1)$ only using squares in $y \times y$. We can avoid the case where $y = 2$ because if that is the case, then it can always move to a square in the 3×3 chessboard, thus $P(n)$.

Because $P(n)$ is true both cases, $P(n)$.

$\square$

b)

Proof.

The statement is false. Consider the horsie starts on $(2, 2)$ which is on the border of a 2×2 chessboard. The horsie cannot move anywhere because there is not enough space on the chessboard. Thus, it cannot get to $(1, 1)$, thus NOT $P(2)$ as desired.

$\square$