CSC240 Problem Set 4

Nicolas

February 17, 2022

Discussed $Q3$ with Aaron Ma and Adi Rao, discussed $Q4$ with Aaron Ma and Adi Rao.

1.

Proof.

We will prove this using structural induction.

Base Case: $\lambda \in R$. λ is trivially in S.

Induction Hypothesis: If $x, y \in R$ then $x, y \in S$.

Constructor Case 1: $b = pxqy$. $x, y \in S$ by induction hypothesis. Thus $pxq \in S$, making $pxqy \in S$ because $mn \in S$, $\forall m, n \in S$.

Constructor Case 2: $b = qxpy$. $x, y \in S$ by induction hypothesis. Thus, $qxp \in S$, making $qxpy$ also in S.

□

2.

Proof.

We will prove this using structural induction.

Base Case: $\lambda \in R$. λ is trivially in T because λ contains 0 p's and 0 q's.

Induction Hypothesis: If $x, y \in S$ then $x, y \in T$.

Constructor Case 1: $b = pxq$. x in T by induction hypothesis, meaning that $\exists n \in \mathbb{N}$ such that x has n p's and n q's. Thus pxq has $n + 1$ p's and $n + 1$ q's. Thus pxq in T.

Constructor Case 2: $b = qxp$. x in T by induction hypothesis, meaning that $\exists n \in \mathbb{N}$ such that x has n p's and n q's. Thus qxp has $n+1$ p's and $n+1$ q's. Thus qxp in T.

Constructor Case 3: $b = xy$. $x, y \in T$ by induction hypothesis, meaning that $existsn, m \in \mathbb{N}$ such that x has n p's and q's and y has m p's and q's. Thus xy has $m+n$ p's and $m+n$ q's. Hence, $xy \in T$. $\square$

Lemma: If $w \in T$ and $w \neq \lambda$ then $\exists x, y \in T.w = pxqy$ or $\exists x, y \in T.w = qxpy$.

Proof.

Consider any $w \in T$ with n p's and q's. $w \neq \lambda$ so it must begin with either p or q.

Case 1: w begins with p. Thus $w = pk$ where k is string with $2n-1$ bits. Let $c = 1$ represent a counter. Let $i = 1$ be an indexing variable that represents which bit of the binary string we are on.

If $k^i = p$ add 1 to the c and move to $i+1$.
If $k^i = q$ then subtract 1 from c and if $c > 0$ move to $i+1$ but if $c = 0$ then stop.

After this process the substring of k from k^1 to k^{i-1} must have the same number of p's and q's because $c = 0$. This process must terminate because there are n q's and only $n-1$ p's. Let us denote this substring as h. Thus $w = phqz$ where z is the substring of k^{i+1} to k^{2n-1} (note that h or z can be the empty string). But because we know that phq has the same number of p's and q's and w has the same number of p's and q's, thus z must have the same number of p's and q's. Thus $phqz = w$ and $h \in T$ and $z \in T$.

Case 2 :w begins with q. Thus $w = qk$ where k is string with $2n-1$ bits. Let $c = 1$ represent a counter. Let $i = 1$ be an indexing variable that represents which bit of the binary string we are on.

If $k^i = p$ add 1 to the c and move to $i+1$.
If $k^i = q$ then subtract 1 from c and if $c > 0$ move to $i+1$ but if $c = 0$ then stop.

After this process the substring of k from k^1 to k^{i-1} must have the same number of p's and q's because $c = 0$. This process must terminate because there are n p's and only $n-1$ q's. Let us denote this substring as h. Thus $w = phqz$ where z is the substring of k^{i+1} to k^{2n-1} (note that h or z can be the empty string). But because we know that phq has the same number of p's and q's and w has the same number of p's and q's, thus z must have the same number of p's and q's. Thus $phqz = w$ and $h \in T$ and $z \in T$.

$\square$

3.

Proof.

We will prove this using strong induction on the number of ps and qs of a string in T.

Base Case: $n = 0$. Let $x \in T$ such that there are n p's and q's. This string must be λ which is in R.

Induction Hypothesis: If $j < n$ and x is a string in T of length j, then $x \in R$.

Inductive Step: Let w be an arbitrary element of T with n p's and q's. We have two cases $\exists x, y \in T. w = pxqy$ or $\exists x, y \in T. w = qxpy$.

Case 1: $w = pxqy$. By induction hypothesis, $x, y \in R$ by induction hypothesis because x has some $j < n$ p's and q's and y has some $k < n$ p's and q's. Because $x, y \in R$, $pxqy \in \mathbb{R}$ by definition R.

Case 2: $w = qxpy$. By induction hypothesis, $x, y \in R$ by induction hypothesis because x has some $j < n$ p's and q's and y has some $k < n$ p's and q's. Because $x, y \in R$, $qxpy \in \mathbb{R}$ by definition R.

$\square$

4.

Proof.

Consider $ppqqqqpp \in T$. We claim that $ppqqqqpp$ cannot be in U. We know that every object in U must have the same amount of p's and q's. Thus only the substrings pq, qp, $ppqq$, and $qqpp$ are in U. The substring pq can be turned into $ppqq$ which is in U. However, $ppqq$ cannot be turned into $ppqqqqpp$ following the definition of U.

The substring qp can be transformed into $qqpp$. However, $qqpp$ cannot be transformed into $ppqqqqpp$ following the definition of U. $ppqq$ cannot be tranformed into $ppqqqqpp$ and $qqpp$ cannot be transformed into $ppqqqqpp$. Because we have shown that we cannot recursively derive $ppqqqqpp$ from anything in U, thus $U \neq T = R$.

$\square$