MAT157 Problem Set 5

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Lemma 1.1: If $x > y > z$, then $0 < |x - y| < |x - z|$ and if $x > y > z$, then $|z - x| > |z - y| > 0$

Proof.

$x - y < x - z$ because $y > z$, but because $x > y > z$ then
$x - z > x - y > 0$ which is the same as $|x - z| > |x - y| > 0$.

$z - x < z - y$ because $x > y$, but because $x > y > z$ then
$z - x < z - y < 0$ which is the the same as $|z - x| > |z - y| > 0$.

$\square$

1. a)

Proof.

We will prove this by cases where either $x \in \mathbb{Q}$ or $x \notin \mathbb{Q}$.
Let $\epsilon > 0$ and let $\delta = \epsilon$.
Case 1: $x \in \mathbb{Q}$

$$|f(x) - 3| = |x + 1 - 3| = |x - 2| = \delta < \epsilon$$

Case 2: $x \notin \mathbb{Q}$

$$|f(x) - 3| = |5 - x - 3| = |2 - x| = |x - 2| < \delta = \epsilon$$

Thus, for all x such that $|x - 2| < \delta \implies |f(x) - 3| < \epsilon$

$\square$

b)

Proof.

Assume for the sake of contradiction that $\lim_{x \to a} f(x)$ does exists.
Then there exists some $\ell \in \mathbb{R}$ such that

$\forall \epsilon > 0 \exists \delta > 0 \forall x \in \mathbb{R} : 0 < |x-a| < \delta \implies |f(x) - \ell| < \epsilon$. There are 3 cases: $1.\ell \neq f(a),\ \forall a \in \mathbb{R}$, $2.\ell = f(a)$ and $a \in \mathbb{Q}$, and $3.\ell = f(a)$ and $a \notin \mathbb{Q}$.

Case 1: $\ell \neq f(a)$ Choose $\epsilon := |f(a) - \ell|$. Either $\ell > f(a)$ or $\ell < f(a)$. If $\ell > f(a)$, then any δ-interval around a will contain $f(a-x),\ \forall 0 < x < \delta$. Which is less than $f(a)$ because f is strictly increasing on the rationals. Thus, by *Lemma 1.1*, $|f(a-x) - \ell| > |f(a) - \ell| = \epsilon$. Because δ was arbitrary, this is a contradiction. If $\ell < f(a)$, then any δ-interval around a will contain $f(a+x),\ \forall 0 < x < \delta$. Which is greater than $f(a)$ because f is strictly increasing on the rationals. Thus, by *Lemma 1.1*, $|f(a+x) - \ell| > |f(a) - \ell| = \epsilon$. Because δ was arbitrary, this is a contradiction.

Case 2: $\ell = f(a)$ and $a \in \mathbb{Q}$. There are two cases here again. Either $a > 2$ or $a < 2$. If $a > 2$ then $\forall x > 2 \in \mathbb{Q} \forall y > 2 \notin \mathbb{Q}, f(x) > f(y)$. This is because f is strictly increasing on the rationals and strictly decreasing on the irrationals and $f(a) > 3,\ \forall a > 2 \in \mathbb{Q}$ and $f(a) < 3,\ \forall a > 2 \notin \mathbb{Q}$. Thus, choose $\epsilon := |f(c) - \ell|$ for some $c \notin \mathbb{Q}$ such that $2 < c < a$. Thus for any δ-interval around a there exists $f(a+x),\ \forall x \notin \mathbb{Q}$ such that $0 < x < \delta$ with $f(a+x) < f(c) < \ell$. By *Lemma 1.1*, $|f(a+x) - \ell| > |f(c) - \ell| = \epsilon$. Because δ was arbitrary, $\exists \epsilon > 0 \forall \delta > 0 \forall x \in \mathbb{R} : 0 < |x-a| < \delta$ and $|f(x) - \ell| \geq \epsilon$. This is a contradiction. On the other side, if $a < 2$ then $\forall x < 2 \in \mathbb{Q} \forall y < 2 \notin \mathbb{Q}, f(x) < f(y)$. Thus, choose $\epsilon := |f(c) - \ell|$ for some $c \notin \mathbb{Q}$ such that $2 > c > a$. Thus for any δ-interval around a there exists $f(a-x),\ \forall x \notin \mathbb{Q}$ such that $0 < x < \delta$ with $f(a-x) < f(c) < \ell$. By *Lemma 1.1*, $|f(a-x) - \ell| > |f(c) - \ell| = \epsilon$. Because δ was arbitrary, $\exists \epsilon > 0 \forall \delta > 0 \forall x \in \mathbb{R} : 0 < |x-a| < \delta$ and $|f(x) - \ell| \geq \epsilon$. This is a contradiction.

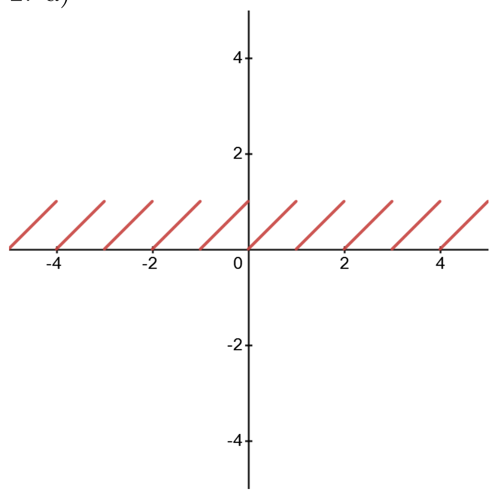
Case 3: $\ell = f(a)$ and $a \notin \mathbb{Q}$. There are two cases here again. Either $a > 2$ or $a <$. If $a > 2$ then $\forall x > 2 \in \mathbb{Q} \forall y > 2 \notin \mathbb{Q}, f(x) > f(y)$. This is because f is strictly increasing on the rationals and strictly decreasing on the irrationals and $f(a) > 3,\ \forall a > 2 \in \mathbb{Q}$ and $f(a) < 3,\ \forall a > 2 \notin \mathbb{Q}$. Thus, choose $\epsilon := |f(c) - \ell|$ for some $c \notin \mathbb{Q}$ such that $2 < c < a$. Thus for any δ-interval around a there exists $f(a+x)$ such that $0 < x < \delta$ and $x = -a + z,\ \exists z \in \mathbb{Q}$ with

$f(a+x) > f(c) > \ell$. By *Lemma 1.1*, $|f(a+x) - \ell| > |f(c) - \ell| = \epsilon$. Because δ was arbitrary,
$\exists \epsilon > 0 \forall \delta > 0 \forall x \in \mathbb{R} : 0 < |x-a| < \delta$ and $|f(x) - \ell| \geq \epsilon$. This is a contradiction. On the other side, if $a < 2$ then
$\forall x < 2 \in \mathbb{Q} \forall y < 2 \notin \mathbb{Q}, f(x) < f(y)$. Thus, choose $\epsilon := |f(c) - \ell|$ for some $c \notin \mathbb{Q}$ such that $2 > c > a$. Thus for any δ-interval around a there exists $f(a-x)$, $\forall x \notin \mathbb{Q}$ such that $0 < x < \delta$ and
$x = -a + z$, $\exists z \in \mathbb{Q}$ with $f(a-x) < f(c) < \ell$. By *Lemma 1.1*,
$|f(a-x) - \ell| > |f(c) - \ell| = \epsilon$. Because δ was arbitrary,
$\exists \epsilon > 0 \forall \delta > 0 \forall x \in \mathbb{R} : 0 < |x-a| < \delta$ and $|f(x) - \ell| \geq \epsilon$. This is a contradiction.

Thus $\lim_{x \to a} f(x)$ does not exist if $a \neq 2$.

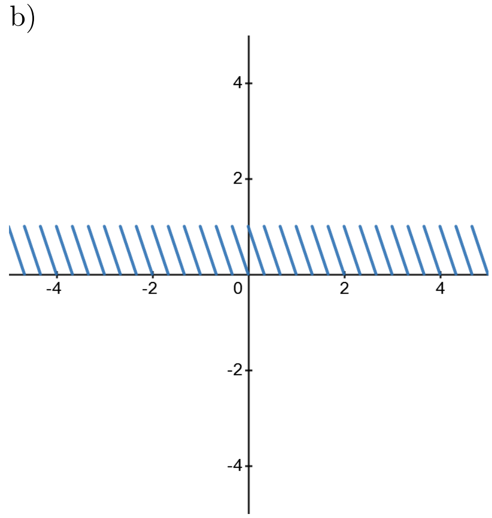
$\square$

2. a)



Clearly, $f : \mathbb{R} \to [0, 1)$ and f is a surjection. Consider any interval $[n, n+1),\ \forall n \in \mathbb{Z}$. f is strictly increasing on this interval because for any $x < y \in [n, n+1)$, the fractional part of y is always greater than the fractional part of x. This functions is also the same as $g : \mathbb{R} \to [0, 1),\ g(x) := x \bmod 1$. Because $f(x) = f(x+1)$ for all $x \in \mathbb{R}$, f is 1-periodic.

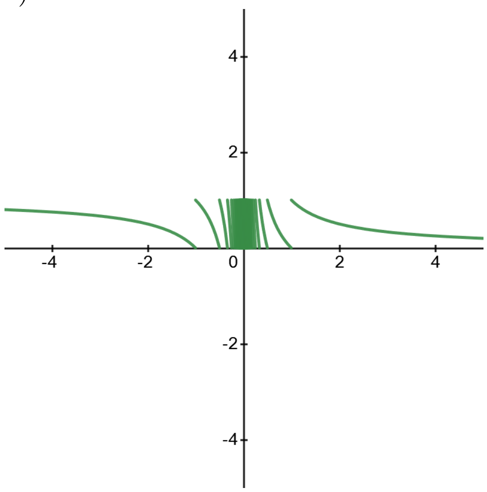
b)



This function is almost the same as the first, except it is horizontally compressed by a factor of 3 and flipped vertically because it is multiplied by -3. Because it is horizontally compressed by a factor of 3, then the function is strictly decreasing on the interval

$[a\frac{n}{3}, a\frac{n}{3}+\frac{1}{3}), \forall n \in \mathbb{Z}$ and $\forall a \in \{0,1,2\}$. Because $f(x) = f(x+\frac{1}{3})$ for all $x \in \mathbb{R}$, f is $\frac{1}{3}$-periodic.

c)



This function is very different from the first two because it is not just a scalar multiple of the first function. If $x > 1$ then $f(\frac{1}{x}) < 1$ and strictly decreasing. But because $\frac{1}{x} < 1$ then $x = f(\frac{1}{x}),\ \forall x > 1$. A similar thing happens if $x < -1$. If $x \in (-1,1) \setminus \{0\}$, then as x gets closer to zero, $\frac{1}{x}$ grows reciprocally. Because $g : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ g(x) := \frac{1}{x}$ is strictly decreasing, all intervals of $(\frac{1}{n}, \frac{1}{n+1}],\ \forall n \in \mathbb{N} : -1 < \frac{1}{n} < 1$ are strictly decreasing, and as $\frac{1}{n}$ gets closer to 0, these intervals get arbitrarily small. That is why you see this area near zero where the lines almost do not look separated.

3.

Proof.

$\Rightarrow$

Let A be closed. For the sake of contradiction assume that there exists a limit point of A, a such that $a \notin A$. Because a is a limit point of A:

$$\forall \epsilon > 0 \exists x \in A : 0 < |x - a| < \epsilon$$

Consider we choose some $b \in A$ such that $b := x$. Since A is closed, then A^c is open:

$$\forall c \in A^c \exists \epsilon > 0 \forall y \in \mathbb{R} : |c - y| < \epsilon \implies y \in A^c$$

Because we are assuming that $a \in A^c$, then we can choose $c := a$ and $y := b$. By a being a limit point of A, $|b - a| = |a - b| < \epsilon$ meaning that $|c - y| < \epsilon$. This implies that $y = b$ is in A^c, but we specifically chose b such that $b \in A$. Hence, this a contradiction.

Therefore, we have shown that if A is closed, then A contains all of its limit points.

$\Leftarrow$

We will prove the contrapositive. Assume that A is not closed, then A^c is not open. Thus:

$$\exists x \in A^c \forall \epsilon > 0 \exists y \in R : |x - y| < \epsilon \text{ and } y \notin A^c$$

If $y \notin A^c$ then $y \in A$. Thus, $\forall \epsilon > 0 : |x - y| < \epsilon$, and because $y \in A$, x is a limit point of A. However, by assumption, $x \in A^c$. Thus, A does not contain all of its limit points.

Therefore, if A contains all of its limits points, then A is closed.

$\square$

4. a)

Proof.

Because $f(x) \leq g(x) \leq h(x),\ \forall x \in A$, then $f(x) - \ell \leq g(x) - \ell \leq h(x) - \ell$. For any $\epsilon > 0$ we can choose a $\delta_1 > 0$ such that if $|x - a| < \delta_1 \implies |f(x) - \ell| < \epsilon$. We can also choose a $\delta_2 > 0$ such that if $|x - a| < \delta_2 \implies |h(x) - \ell| < \epsilon$. Suppose we chose $\delta := \min\{\delta_1, \delta_2\}$. Thus if $|x - a| < \delta \implies$

$$-\epsilon < f(x) - \ell \leq g(x) - \ell \leq h(x) - \ell < \epsilon$$
$$-\epsilon < g(x) - \ell < \epsilon$$
$$|g(x) - \ell| < \epsilon$$

Therefore, $\lim_{x\to a} g(x) = \ell$.

□

b)

Proof.

By assumption,
$\forall \epsilon_1 > 0 \exists \delta_1 > 0 \forall x \in A : 0 < |x - a| < \delta_1 \implies |f(x) - \ell| < \epsilon_1$ and
$\forall \epsilon_2 > 0 \exists \delta_2 > 0 \forall x \in A : 0 < |x - a| < \delta_2 \implies |g(x) - m| < \epsilon_2$.

Suppose that $\max\{\ell, m\} = \ell$.
$\forall \epsilon > 0 \exists \delta' := \min\{\delta_1, \delta_2\} \forall x \in A : |x - a| < \delta' \implies |f(x) - \ell| < \epsilon$ and $|g(x) - m| < \epsilon$.
Assume for the sake of contradiction
$\exists \epsilon > 0 \forall 0 < \delta \leq \delta' : f(x) < g(x),\ \forall x \in (a - \delta, a + \delta)$. Let $\kappa := \min(g(x) - f(x))$ on $(a - \delta, a + \delta)$. Notice that $\kappa > 0$ because $g(x) > f(x),\ \forall x \in (a - \delta, a + \delta)$. Choose ϵ' such that $\epsilon' < \min\{\epsilon, \kappa\}$. This implies that:

$$\begin{aligned}
|f(x) - \ell| &< \frac{\epsilon'}{2} \\
|g(x) - m| &< \frac{\epsilon'}{2} \\
|g(x) - | + |f(x) - \ell| &< \epsilon' \\
|g(x) - f(x) + \ell - m| &\leq |f(x) - \ell| + |g(x) - m| \\
|\kappa + \ell - m| &\leq |g(x) - f(x) + \ell - m| \\
-\epsilon < \kappa + \ell - m &< \epsilon' \\
\kappa - \epsilon + \ell &< m,\ \kappa - \epsilon' > 0
\end{aligned}$$

This is a contradiction because $\max\{\ell, m\} = \ell$. Thus,
$\forall \epsilon > 0 \exists 0 < \delta \leq \delta' : f(x) \geq g(x),\ \forall x \in (a-\delta, a+\delta)$. If $|x-a| < \delta$ then $|\max\{f(x), g(x)\} - \max\{\ell, m\}| = |f(x) - \ell| < \epsilon$

We can make a similar case if $\max\{\ell, m\} = m$, where we simply says that there is some ϵ-interval where $g(x) \geq f(x)$ for all x in the interval.

Therefore, $\lim_{x\to a} \max\{f(x), g(x)\} = \max\{\ell, m\}$.

$\square$

5.

Proof.

$\Rightarrow$

Consider $\forall \epsilon > 0 \exists \delta > 0 \forall x \in A : 0 < |x - a| < \delta \implies |f(x) - \ell| < \epsilon$ and $U \subseteq \mathbb{R}$ is open and $\ell \in U$. We will claim that there exists some $\epsilon > 0$ such that $(\ell - \epsilon, \ell + \epsilon) \subseteq U$. Consider this is false, and $\{\ell\} \subseteq U$, but $(\ell - \epsilon, \ell + \epsilon) \not\subseteq U$, $\forall \epsilon > 0$, then $\{\ell\} \notin U^c$, but ℓ is a limit point of U^c. This means that U^c does not contain all of its limit points, making it not closed, meaning that U is not open, a contradiction. Therefore, we can follow along with the existence of an $\epsilon > 0$ such that $(\ell - \epsilon, \ell + \epsilon) \subseteq U$. Because $\forall \epsilon > 0 \exists \delta > 0 \forall x \in A : 0 < |x - a| < \delta \implies |f(x) - \ell| < \epsilon$ we can simply choose $V := (a - \delta, a + \delta)$. Thus, V is open and $f((V \cap A) \setminus \{a\}) \subseteq U$.

Therefore, if $\lim_{x \to a} f(x) = \ell$ then $\forall U \subseteq \mathbb{R} \exists V \subseteq \mathbb{R} : U$ is open and $\ell \in U \implies V$ is open and $a \in V$ and $f((V \cap A) \setminus \{a\}) \subseteq U$.

$\Leftarrow$

Consider that $\forall U \subseteq \mathbb{R} \exists V \subseteq \mathbb{R} : U$ is open and $\ell \in U \implies V$ is open and $a \in V$ and $f((V \cap A) \setminus \{a\}) \subseteq U$. Consider $U := (\ell - \epsilon, \ell + \epsilon)$, $\forall \epsilon > 0$. Then there exists some V such that $a \in V$ and V is open and $f((V \cap A) \setminus \{a\}) \subseteq U$. We will claim that there exists some $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq V$. Consider that this is false, and $\{a\} \subseteq V$, but $(a - \delta, a + \delta) \not\subseteq V$, $\forall \delta > 0$, then $\{a\} \notin V^c$, but a is a limit point of V^c. This means that V^c does not contain all of its limit points, meaning V is not open, a contradiction. Thus, $\exists \delta > 0$ such that $(a - \delta, a + \delta) \subseteq V$. Because our choice of ϵ was arbitrary, we can now choose this δ to prove that $\forall \epsilon > 0 \exists \delta > 0 \forall x \in \mathbb{R} : 0 < |x - a| < \delta \implies |f(x) - \ell| < \epsilon$.

Therefore, if $\forall U \subseteq \mathbb{R} \exists V \subseteq \mathbb{R} : U$ is open and $\ell \in U \implies V$ is open and $a \in V$ and $f((V \cap A) \setminus \{a\}) \subseteq U$ then $\lim_{x \to a} f(x) = \ell$.

$\square$