PUMP II Problem Set 1

Nick Coballe, Krishna Cheemalapati, Daniel Xu

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lemma. 1.0 $\binom{n}{k} = 0$

Proof.

By Pascal's inequality:

$$\begin{aligned}
\binom{n}{k-1} &= \binom{n+1}{k} - \binom{n}{k} \\
\binom{n}{-1} &= \binom{n+1}{0} - \binom{n}{0} \\
&= \frac{(n+1)!}{0!(n+1-0)!} - \frac{n!}{0!(n-0)!} \\
&= 1 - 1 \\
&= 0
\end{aligned}$$

□

1. a)

$\mathcal{P}(A \cup B) = \{A \cup B, A, B, \{1\}, \{2\}, \{cat\}, \{dog\}, \{1, dog\}, \{1, cat\},$
$\{2, dog\}, \{2, cat\}, \{1, 2, cat\}, \{1, 2, dog\}, \{1, cat, dog\}, \{2, cat, dog\}, \emptyset\}$

$\mathcal{P}(\mathcal{P}(B)) = \{\mathcal{P}(B), \{B\}, \{cat\}, \{dog\}, \{\emptyset\}, \{B, \{cat\}\}, \{B, \{dog\}\}, \{B, \emptyset\},$
$\{B\{cat\}, \{dog\}\}, \{B, \{cat\}, \{\emptyset\}\}, \{B, \{cat\}, \emptyset\}, \{\{cat\}, \{dog\}\}, \{\{cat\}, \emptyset\},$
$\{\{dog\}, \emptyset\}, \{\{cat\}, \{dog\}, \emptyset\}, \emptyset\}$

b)

Proof.

The total number of ways that n objects in a set can be arranged into $s = \{1, 2, 3, ..., n\}$ sized subsets is equal to $\sum\limits_{k=0}^{n} \binom{n}{k}$ Thus it is sufficient to prove that $\sum\limits_{k=0}^{n} \binom{n}{k} = 2^n$. We prove this statement through induction on n.
Base Case: $n = 0$

$$\sum_{k=0}^{0} \binom{0}{k} = \frac{0!}{0!(0-0)!} = 1 = 2^0$$

Thus the base case is true.
Inductive Step: $j = n + 1$

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{n+1}{k} &= \sum_{k=0}^{n+1} \left(\binom{n}{k} + \binom{n}{k-1} \right), \text{ via Pascal's inequality} \\
&= \sum_{k=0}^{n+1} \binom{n}{k} + \sum_{k=0}^{n} \binom{n}{k-1} \\
&= \sum_{k=0}^{n} \binom{n}{k} + \binom{n}{n+1} + \sum_{k=0}^{n} \binom{n}{k-1}, \text{ via reindexing} \\
&= 2^n + 2^n + 0 + 0, \text{ by lemma 1.1} \\
&= 2^{n+1}
\end{aligned}$$

□

c)

Proof.

We will induct on n.

$$|\mathcal{P}(S)| = 2^n$$

Base case: $n = 0, S = \emptyset$

$$|\mathcal{P}(S)| = 1 = 2^0$$

Thus the base case is true.
Inductive Step: $j = n + 1$ Let A be a set s.t. $A \cap S = \emptyset, |A| = 1$
We can construct some set V s.t. $V = \{x : \forall y \in \mathcal{P}(S), x = y \cup A\}$
V and $\mathcal{P}(S)$ are equal in size but disjoint.

$$\mathcal{P}(A \cup S) = \{T : T \in \mathcal{P}(S) \text{ or } T \in V\}$$
$$|V| = |\mathcal{P}(S)| = 2^n$$

Thus $|\mathcal{P}(A \cup S)| = |V| + |\mathcal{P}(S)| = 2^n + 2^n = 2^{n+1}$

$\square$

lemma. ***2.0*** If x^2 is a multiple of 7, then x is a multiple of 7.

Proof.

Take the contrapositive.

$$\begin{aligned} x &= 7n + a, n \in \mathbb{Z}, a \in \{1, 2, 3, 4, 5, 6\} \\ x^2 &= 49k^2 + 14ak + a^2 \\ &= 7(7k^2 + 2ak) + a^2 \end{aligned}$$

a^2 can never be a multiple of 7, and thus x^2 is not a multiple of 7.

$\square$

lemma. ***2.1*** If $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$, then $x + y \notin \mathbb{Q}$.

Proof.

Assume $x + y \in \mathbb{Q}$.

$$\begin{aligned} \frac{p}{q} + x &= \frac{m}{n}, p, q, m, n \in \mathbb{Z} \\ x &= \frac{m}{n} - \frac{p}{q} \end{aligned}$$

Thus x must be in $\mathbb{Q}$, a contradiction.

$\square$

lemma. ***2.3*** 0 is the only rational that when multiplied with an irrational results in a rational.

Proof.

Assume there is a rational, $x \in \mathbb{Q}$ s.t. $x \neq 0$ and x multiplied by an irrational is rational.

$$\begin{aligned} yx &= \frac{m}{n}, y \notin \mathbb{Q}, x \in \mathbb{Q}, m, n \in \mathbb{Z} \\ y\frac{p}{q} &= \frac{m}{n}, p, q \in \mathbb{Z} \\ y &= \frac{mq}{np} \end{aligned}$$

Thus, y would be rational, a contradiction.

$\square$

2. a)

Proof.

Assume that $x \in \mathbb{Q}$

$$
\begin{aligned}
&x^2 = 7 \\
&\frac{p^2}{q^2} = 7, p \in \mathbb{N}, q \in \mathbb{Z} \\
&p^2 = 7q^2
\end{aligned}
$$

From lemma 2.0 if p^2 is a multiple of 7, then p is a multiple of 7. Thus we can write $p = 7n, n \in \mathbb{Z}, n < p$.

$$
\begin{aligned}
(7n)^2 &= 7q^2 \\
49n^2 &= 7q^2 \\
7n^2 &= q^2
\end{aligned}
$$

Thus we can do the same thing we did to q as we did to p, where $k < q, k \in \mathbb{Z}$. Thus we get to the stage where $n^2 = 7k^2$. We then get to the equality that we started, and we can repeat the steps we've taken; however, since $n < p$ then we need to be able to descend infinitely on the naturals, which we cannot, and thus we reach a contradiction.

□

Proof.

Assume that $x \in \mathbb{Q}$

$$
\begin{aligned}
7\frac{p^2}{q^2} - 2 &= 0, p, q \in \mathbb{Q} \\
\frac{p^2}{q^2} &= \frac{2}{7}
\end{aligned}
$$

Therefore $p^2 = 2$ and $q^2 = 7$, but we have already proven that these equalities do no have integer solutions; thus, we have reached a contradiction.

□

b)

Proof.

$u = \frac{x}{x^2-7y^2}, v = \frac{-y}{x^2-7y^2}$
Because $x, y, 7 \in \mathbb{Q}$, and $\mathbb{Q}$ being a field, $u, v \in \mathbb{Q}$.

$\square$

c)

Proof.

Assume $\sqrt{2} \in \mathbb{Q}(\sqrt{7})$

$$\sqrt{2} = (u + \sqrt{7}v), u, v \in \mathbb{Q}$$
$$2 = u^2 + 2\sqrt{7}uv + 7v^2$$

Because 2 is rational, and the sum of a rational and an irrational is always irrational by lemma 2.2, either $u = 0$ or $v = 0$ for $2\sqrt{7}uv = 0$ (by lemma 2.3).
Case 1: $u = 0$

$$7v^2 = 2$$
$$v^2 = \frac{2}{7}$$

We have already proven that this equation has no solution in $\mathbb{Q}$, thus a contradiction.
Case 2: $v = 0$

$$u^2 = 2$$

We have already proven that this equation has no solutions in $\mathbb{Q}$, thus a contradiction.
Case 3: $u = v = 0$

$$2 = 0$$

This is clearly a contradiction.

$\square$

3. a)

The fallacy that the student is employing is that every subset of students of size n have the same grade. It is only true that the *first* n students have the same grade.

b) i) We need to prove $P(1)$.

ii) We need to prove $P(2)$ and $P(3)$.

iii) We have to prove that $P(n) \Rightarrow P(n+1)$.

4. a)

$\neg(\forall \epsilon > 0, \exists \delta > 0 : 0 < |x-a| < \delta \Rightarrow 0 < |f(x) - f(a)| < \epsilon)$
Is the same as:
$\exists \epsilon > 0, \forall \delta > 0 : 0 < |x-a| < \delta \wedge |f(x) - f(a)| \geq \epsilon$

b) i)

$\emptyset$, because $|f(x) - f(a)| \geq 0$

ii)

$\{f : f$ is continuous at $x\}$, because if $|f(x) - f(a)| = \frac{\epsilon}{2}$ then
$|f(x) - f(a)| < \epsilon$

iii)

$\{f : f$ is a function$\}$, because if $\delta = 0$, then the condition is vacuously satisfied.

iv)

$f : D \to C$
$\{f : \exists \delta > 0, \exists x \in D$ s.t. f is constant on the interval $(x - \delta, x + \delta)\}$