Algoma
UNIVERSITY
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Linear Algebra

Math 1057

Instructor

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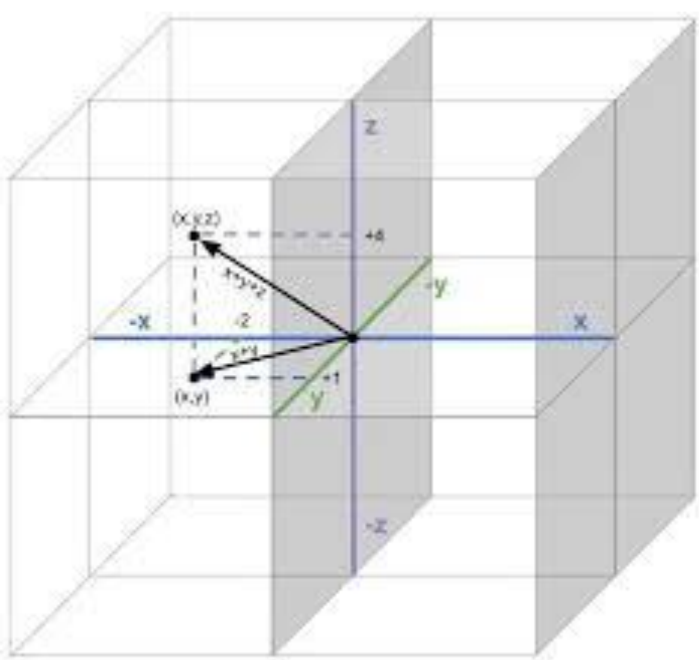

10 Invertible Matrices

Lecture 10

Invertible Matrices

10.1 Inverse of a Matrix

The inverse of a **square** matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ generalizes the notion of the reciprocal of a non-zero number $a \in \mathbb{R}$. Formally speaking, the inverse of a non-zero number $a \in \mathbb{R}$ is the unique number $c \in \mathbb{R}$ such that $ac = ca = 1$. The inverse of $a \neq 0$, usually denoted by $a^{-1} = \frac{1}{a}$, can be used to solve the equation $ax = b$:

$$ax = b \;\Rightarrow\; a^{-1}ax = a^{-1}b \;\Rightarrow\; x = a^{-1}b.$$

This motivates the following definition.

Definition 10.1: A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is called **invertible** if there exists a matrix $\mathbf{C} \in \mathbb{R}^{n\times n}$ such that $\mathbf{AC} = \mathbf{I}_n$ and $\mathbf{CA} = \mathbf{I}_n$.

If $\mathbf{A}$ is invertible then can it have more than one inverse? Suppose that there exists $\mathbf{C}_1, \mathbf{C}_2$ such that $\mathbf{AC}_i = \mathbf{C}_i\mathbf{A} = \mathbf{I}_n$. Then

$$\mathbf{C}_2 = \mathbf{C}_2(\mathbf{AC}_1) = (\mathbf{C}_2\mathbf{A})\mathbf{C}_1 = \mathbf{I}_n\mathbf{C}_1 = \mathbf{C}_1.$$

Thus, if $\mathbf{A}$ is invertible, it can have only one inverse. This motivates the following definition.

Definition 10.2: If $\mathbf{A}$ is invertible then we denote **the** inverse of $\mathbf{A}$ by $\mathbf{A}^{-1}$. Thus, $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$.

Example 10.3. Given $\mathbf{A}$ and $\mathbf{C}$ below, show that $\mathbf{C}$ is the inverse of $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 \\ -1 & 2 & -2 \\ -2 & 6 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -14 & -3 & -6 \\ -5 & -1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$$

Solution. Compute $\mathbf{AC}$:

$$\mathbf{AC} = \begin{bmatrix} 1 & -3 & 0 \\ -1 & 2 & -2 \\ -2 & 6 & 1 \end{bmatrix} \begin{bmatrix} -14 & -3 & -6 \\ -5 & -1 & -2 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute $\mathbf{CA}$:

$$\mathbf{CA} = \begin{bmatrix} -14 & -3 & -6 \\ -5 & -1 & -2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ -1 & 2 & -2 \\ -2 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, by definition $\mathbf{C} = \mathbf{A}^{-1}$. □

Theorem 10.4: Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and suppose that $\mathbf{A}$ is invertible. Then for any $\mathbf{b} \in \mathbb{R}^n$ the matrix equation $\mathbf{Ax} = \mathbf{b}$ has a unique solution given by $\mathbf{A}^{-1}\mathbf{b}$.

Proof: Let $\mathbf{b} \in \mathbb{R}^n$ be arbitrary. Then multiplying the equation $\mathbf{Ax} = \mathbf{b}$ by $\mathbf{A}^{-1}$ from the left we obtain that

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{Ax} &= \mathbf{A}^{-1}\mathbf{b} \\ \Rightarrow \mathbf{I}_n\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\ \Rightarrow \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b}. \end{aligned}$$

Therefore, with $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ we have that

$$\mathbf{Ax} = \mathbf{A}(\mathbf{A}^{-1}\mathbf{b}) = \mathbf{AA}^{-1}\mathbf{b} = \mathbf{I}_n\mathbf{b} = \mathbf{b}$$

and thus $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ is a solution. If $\tilde{\mathbf{x}}$ is another solution of the equation, that is, $\mathbf{A}\tilde{\mathbf{x}} = \mathbf{b}$, then multiplying both sides by $\mathbf{A}^{-1}$ we obtain that $\tilde{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{b}$. Thus, $\mathbf{x} = \tilde{\mathbf{x}}$. □

Example 10.5. Use the result of **Example 10.3.** to solve the linear system $\mathbf{Ax} = \mathbf{b}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 \\ -1 & 2 & -2 \\ -2 & 6 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}.$$

Solution. We showed in Example 10.3 that

$$\mathbf{A}^{-1} = \begin{bmatrix} -14 & -3 & -6 \\ -5 & -1 & -2 \\ 2 & 0 & 1 \end{bmatrix}.$$

Therefore, the unique solution to the linear system $\mathbf{Ax} = \mathbf{b}$ is

$$\mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} -14 & -3 & -6 \\ -5 & -1 & -2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Verify:

$$\begin{bmatrix} 1 & -3 & 0 \\ -1 & 2 & -2 \\ -2 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$$

□

The following theorem summarizes the relationship between the matrix inverse and matrix multiplication and matrix transpose.

Theorem 10.6: Let $\mathbf{A}$ and $\mathbf{B}$ be invertible matrices. Then:

(1) The matrix $\mathbf{A}^{-1}$ is invertible and its inverse is $\mathbf{A}$:

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}.$$

(2) The matrix $\mathbf{AB}$ is invertible and its inverse is $\mathbf{B}^{-1}\mathbf{A}^{-1}$:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

(3) The matrix $\mathbf{A}^T$ is invertible and its inverse is $(\mathbf{A}^{-1})^T$:

$$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T.$$

Proof: To prove **(2)** we compute

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{ABB}^{-1}\mathbf{A}^{-1} = \mathbf{AI}_n\mathbf{A}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}_n.$$

To prove **(3)** we compute

$$\mathbf{A}^T(\mathbf{A}^{-1})^T = (\mathbf{A}^{-1}\mathbf{A})^T = \mathbf{I}_n^T = \mathbf{I}_n.$$

□

10.2 Computing the Inverse of a Matrix

If $\mathbf{A} \in M_{n\times n}$ is invertible, how do we find $\mathbf{A}^{-1}$? Let $\mathbf{A}^{-1} = \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n\end{bmatrix}$ and we will find expressions for $\mathbf{c}_i$. First note that $\mathbf{AA}^{-1} = \begin{bmatrix}\mathbf{Ac}_1 & \mathbf{Ac}_2 & \cdots & \mathbf{Ac}_n\end{bmatrix}$. On the other hand, we also have $\mathbf{AA}^{-1} = \mathbf{I}_n = \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n\end{bmatrix}$. Therefore, we want to find $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ such that

$$\underbrace{\begin{bmatrix}\mathbf{Ac}_1 & \mathbf{Ac}_2 & \cdots & \mathbf{Ac}_n\end{bmatrix}}_{\mathbf{AA}^{-1}} = \underbrace{\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n\end{bmatrix}}_{\mathbf{I}_n}.$$

To find $\mathbf{c}_i$ we therefore need to solve the linear system $\mathbf{Ax} = \mathbf{e}_i$. Here the image vector "$\mathbf{b}$" is $\mathbf{e}_i$. To find $\mathbf{c}_1$ we form the augmented matrix $\begin{bmatrix}\mathbf{A} & \mathbf{e}_1\end{bmatrix}$ and find its RREF:

$$\begin{bmatrix}\mathbf{A} & \mathbf{e}_1\end{bmatrix} \sim \begin{bmatrix}\mathbf{I}_n & \mathbf{c}_1\end{bmatrix}.$$

We will need to do this for each $\mathbf{c}_2, \ldots, \mathbf{c}_n$ so we might as well form the combined augmented matrix $\begin{bmatrix}\mathbf{A} & \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n\end{bmatrix}$ and find the RREF all at once:

$$\begin{bmatrix}\mathbf{A} & \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n\end{bmatrix} \sim \begin{bmatrix}\mathbf{I}_n & \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n\end{bmatrix}.$$

In summary, to determine if $\mathbf{A}^{-1}$ exists and to simultaneously compute it, we compute the RREF of the augmented matrix

$$\begin{bmatrix}\mathbf{A} & \mathbf{I}_n\end{bmatrix},$$

that is, $\mathbf{A}$ augmented with the $n \times n$ identity matrix. If the RREF of $\mathbf{A}$ is $\mathbf{I}_n$, that is

$$\begin{bmatrix}\mathbf{A} & \mathbf{I}_n\end{bmatrix} \sim \begin{bmatrix}\mathbf{I}_n & \mathbf{c}_1 & \mathbf{c}_2 \cdots & \mathbf{c}_n\end{bmatrix}$$

then

$$\mathbf{A}^{-1} = \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2 \cdots & \mathbf{c}_n\end{bmatrix}.$$

If the RREF of $\mathbf{A}$ is not $\mathbf{I}_n$ then $\mathbf{A}$ is not invertible.

Example 10.7. Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$ if it exists.

Solution. Form the augmented matrix $\begin{bmatrix}\mathbf{A} & \mathbf{I}_2\end{bmatrix}$ and row reduce:

$$\begin{bmatrix}\mathbf{A} & \mathbf{I}_2\end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}$$

Add rows R_1 and R_2:

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Perform the operation $\xrightarrow{-3R_2+R_1}$:

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{-3R_2+R_1} \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Thus, $\texttt{rref}(\mathbf{A}) = \mathbf{I}_2$, and therefore $\mathbf{A}$ is invertible. The inverse is

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$$

Verify:

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

□

Example 10.8. Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 0 \\ -2 & 0 & -7 \end{bmatrix}$ if it exists.

Solution. Form the augmented matrix $\begin{bmatrix}\mathbf{A} & \mathbf{I}_3\end{bmatrix}$ and row reduce:

$$\begin{bmatrix}1 & 0 & 3 & 1 & 0 & 0\\ 1 & 1 & 0 & 0 & 1 & 0\\ -2 & 0 & -7 & 0 & 0 & 1\end{bmatrix} \xrightarrow{-R_1+R_2,\, 2R_1+R_2} \begin{bmatrix}1 & 0 & 3 & 1 & 0 & 0\\ 0 & 1 & -3 & -1 & 1 & 0\\ 0 & 0 & -1 & 2 & 0 & 1\end{bmatrix}$$

$-R_3$:

$$\begin{bmatrix}1 & 0 & 3 & 1 & 0 & 0\\ 0 & 1 & -3 & -1 & 1 & 0\\ 0 & 0 & -1 & 2 & 0 & 1\end{bmatrix} \xrightarrow{-R_3} \begin{bmatrix}1 & 0 & 3 & 1 & 0 & 0\\ 0 & 1 & -3 & -1 & 1 & 0\\ 0 & 0 & 1 & -2 & 0 & -1\end{bmatrix}$$

$3R_3 + R_2$ and $-3R_3 + R_1$:

$$\begin{bmatrix}1 & 0 & 3 & 1 & 0 & 0\\ 0 & 1 & -3 & -1 & 1 & 0\\ 0 & 0 & 1 & -2 & 0 & -1\end{bmatrix} \xrightarrow{3R_3+R_2,\, -3R_3+R_1} \begin{bmatrix}1 & 0 & 0 & 7 & 0 & 3\\ 0 & 1 & 0 & -7 & 1 & -3\\ 0 & 0 & 1 & -2 & 0 & -1\end{bmatrix}$$

Therefore, $\texttt{rref}(\mathbf{A}) = \mathbf{I}_3$, and therefore $\mathbf{A}$ is invertible. The inverse is

$$\mathbf{A}^{-1} = \begin{bmatrix}7 & 0 & 3\\ -7 & 1 & -3\\ -2 & 0 & -1\end{bmatrix}$$

Verify:

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix}1 & 0 & 3\\ 1 & 1 & 0\\ -2 & 0 & -7\end{bmatrix} \begin{bmatrix}7 & 0 & 3\\ -7 & 1 & -3\\ -2 & 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

□

Example 10.9. Find the inverse of $\mathbf{A} = \begin{bmatrix}1 & 0 & 1\\ 1 & 1 & -2\\ -2 & 0 & -2\end{bmatrix}$ if it exists.

Solution. Form the augmented matrix $\begin{bmatrix}\mathbf{A} & \mathbf{I}_3\end{bmatrix}$ and row reduce:

$$\begin{bmatrix}1 & 0 & 1 & 1 & 0 & 0\\ 1 & 1 & -2 & 0 & 1 & 0\\ -2 & 0 & -2 & 0 & 0 & 1\end{bmatrix} \xrightarrow{-R_1+R_2,\, 2R_1+R_2} \begin{bmatrix}1 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & -3 & -1 & 1 & 0\\ 0 & 0 & 0 & 2 & 0 & 1\end{bmatrix}$$

We need not go further since the $\texttt{rref}(\mathbf{A})$ is not $\mathbf{I}_3$ ($\text{rank}(\mathbf{A}) = 2$). Therefore, $\mathbf{A}$ is not invertible. □

10.3 Invertible Linear Mappings

Let $\mathsf{T}_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^n$ be a matrix mapping with standard matrix $\mathbf{A}$ and suppose that $\mathbf{A}$ is invertible. Let $\mathsf{T}_{\mathbf{A}^{-1}} : \mathbb{R}^n \to \mathbb{R}^n$ be the matrix mapping with standard matrix $\mathbf{A}^{-1}$. Then the standard matrix of the composite mapping $\mathsf{T}_{\mathbf{A}^{-1}} \circ \mathsf{T}_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^n$ is

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n.$$

Therefore, $(\mathsf{T}_{\mathbf{A}^{-1}} \circ \mathsf{T}_{\mathbf{A}})(\mathbf{x}) = \mathbf{I}_n\mathbf{x} = \mathbf{x}$. Let's unravel $(\mathsf{T}_{\mathbf{A}^{-1}} \circ \mathsf{T}_{\mathbf{A}})(\mathbf{x})$ to see this:

$$(\mathsf{T}_{\mathbf{A}^{-1}} \circ \mathsf{T}_{\mathbf{A}})(\mathbf{x}) = \mathsf{T}_{\mathbf{A}^{-1}}(\mathsf{T}_{\mathbf{A}}(\mathbf{x})) = \mathsf{T}_{\mathbf{A}^{-1}}(\mathbf{A}\mathbf{x}) = \mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{x}.$$

Similarly, the standard matrix of $(\mathsf{T}_{\mathbf{A}} \circ \mathsf{T}_{\mathbf{A}^{-1}})$ is also $\mathbf{I}_n$. Intuitively, the linear mapping $\mathsf{T}_{\mathbf{A}^{-1}}$ *undoes* what $\mathsf{T}_{\mathbf{A}}$ does, and conversely. Moreover, since $\mathbf{A}\mathbf{x} = \mathbf{b}$ always has a solution, $\mathsf{T}_{\mathbf{A}}$ is onto. And, because the solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ is unique, $\mathsf{T}_{\mathbf{A}}$ is one-to-one.

The following theorem summarizes equivalent conditions for matrix invertibility.

Theorem 10.10: Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. The following statements are equivalent:

(a) $\mathbf{A}$ is invertible.
(b) $\mathbf{A}$ is row equivalent to $\mathbf{I}_n$, that is, $\texttt{rref}(\mathbf{A}) = \mathbf{I}_n$.
(c) The equation $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the trivial solution.
(d) The linear transformation $\mathsf{T}_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is one-to-one.
(e) The linear transformation $\mathsf{T}_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is onto.
(f) The matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ is always solvable.
(g) The columns of $\mathbf{A}$ span $\mathbb{R}^n$.
(h) The columns of $\mathbf{A}$ are linearly independent.
(i) $\mathbf{A}^T$ is invertible.

Proof: This is a summary of all the statements we have proved about matrices and matrix mappings specialized to the case of square matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$. Note that for non-square matrices, one-to-one does not imply ontoness, and conversely.

Example 10.11. Without doing any arithmetic, write down the inverse of the dilation matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}.$$

Example 10.12. Without doing any arithmetic, write down the inverse of the rotation matrix

$$\mathbf{A} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

After this lecture you should know the following:

- how to compute the inverse of a matrix
- properties of matrix inversion and matrix multiplication
- relate invertibility of a matrix with properties of the associated linear mapping (1-1, onto)
- the characterizations of invertible matrices Theorem 10.10