Algoma UNIVERSITY

School of Computer Science & Technology - Algoma

Linear Algebra

Math 1057

Instructor

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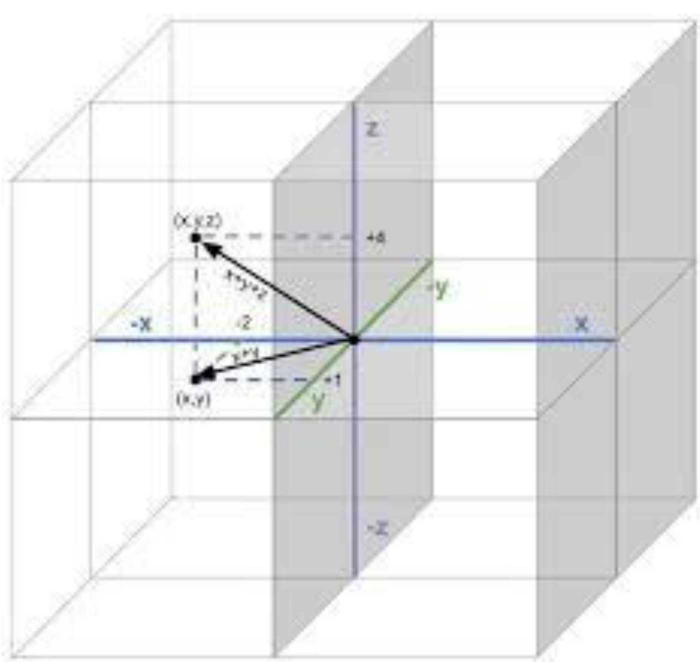

11 Determinants

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Example 11.2. Compute the determinant of $\mathbf{A}$.

(i) $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 8 & 2 \end{bmatrix}$

(ii) $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$

(iii) $\mathbf{A} = \begin{bmatrix} -110 & 0 \\ 568 & 0 \end{bmatrix}$

Solution. For (i):

$$\det(\mathbf{A}) = \begin{vmatrix} 3 & -1 \\ 8 & 2 \end{vmatrix} = (3)(2) - (8)(-1) = 14$$

For (ii):

$$\det(\mathbf{A}) = \begin{vmatrix} 3 & 1 \\ -6 & -2 \end{vmatrix} = (3)(-2) - (-6)(1) = 0$$

For (iii):

$$\det(\mathbf{A}) = \begin{vmatrix} -110 & 0 \\ 568 & 0 \end{vmatrix} = (-110)(0) - (568)(0) = 0$$

□

As in the 2×2 case, the solution of a 3×3 linear system $\mathbf{Ax} = \mathbf{b}$ can be shown to be

$$x_1 = \frac{\text{Numerator}_1}{D},\ x_2 = \frac{\text{Numerator}_2}{D},\ x_3 = \frac{\text{Numerator}_3}{D}$$

where

$$D = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Notice that the terms of D in the parenthesis are determinants of 2×2 submatrices of $\mathbf{A}$:

$$D = a_{11}(\underbrace{a_{22}a_{33} - a_{23}a_{32}}_{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}) - a_{12}(\underbrace{a_{21}a_{33} - a_{23}a_{31}}_{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}) + a_{13}(\underbrace{a_{21}a_{32} - a_{22}a_{31}}_{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}).$$

Let

$$\mathbf{A}_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{A}_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \quad \text{and } \mathbf{A}_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

Then we can write

$$D = a_{11}\det(\mathbf{A}_{11}) - a_{12}\det(\mathbf{A}_{12}) + a_{13}\det(\mathbf{A}_{13}).$$

The matrix $\mathbf{A}_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$ is obtained from $\mathbf{A}$ by deleting the 1st row and the 1st column:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \mathbf{a_{22}} & \mathbf{a_{23}} \\ a_{31} & \mathbf{a_{32}} & \mathbf{a_{33}} \end{bmatrix} \longrightarrow \mathbf{A}_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}.$$

Similarly, the matrix $\mathbf{A}_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$ is obtained from $\mathbf{A}$ by deleting the 1st row and the 2nd column:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \mathbf{a_{21}} & a_{22} & \mathbf{a_{23}} \\ \mathbf{a_{31}} & a_{32} & \mathbf{a_{33}} \end{bmatrix} \longrightarrow \mathbf{A}_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}.$$

Finally, the matrix $\mathbf{A}_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ is obtained from $\mathbf{A}$ by deleting the 1st row and the 3rd column:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \mathbf{a_{21}} & \mathbf{a_{22}} & a_{23} \\ \mathbf{a_{31}} & \mathbf{a_{32}} & a_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

Notice also that the sign in front of the coefficients a_{11}, a_{12}, and a_{13}, alternate. This motivates the following definition.

Definition 11.3: Let $\mathbf{A}$ be a 3×3 matrix. Let $\mathbf{A}_{jk}$ be the 2×2 matrix obtained from $\mathbf{A}$ by deleting the jth row and kth column. Define the **cofactor** of a_{jk} to be the number $C_{jk} = (-1)^{j+k} \det \mathbf{A}_{jk}$. Define the **determinant** of $\mathbf{A}$ to be

$$\det \mathbf{A} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.$$

This definition of the determinant is called the **expansion of the determinant along the first row**. In the cofactor $C_{jk} = (-1)^{j+k} \det \mathbf{A}_{jk}$, the expression $(-1)^{j+k}$ will evaluate to either 1 or -1, depending on whether $j + k$ is even or odd. For example, the cofactor of a_{12} is

$$C_{12} = (-1)^{1+2} \det \mathbf{A}_{12} = -\det \mathbf{A}_{12}$$

and the cofactor of a_{13} is

$$C_{13} = (-1)^{1+3} \det \mathbf{A}_{13} = \det \mathbf{A}_{13}.$$

We can also compute the cofactor of the other entries of $\mathbf{A}$ in the obvious way. For example, the cofactor of a_{23} is

$$C_{23} = (-1)^{2+3} \det \mathbf{A}_{23} = -\det \mathbf{A}_{23}.$$

A helpful way to remember the sign $(-1)^{j+k}$ of a cofactor is to use the matrix

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$

This works not just for 3×3 matrices but for any square $n \times n$ matrix.

Example 11.4. Compute the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 3 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

Solution. From the definition of the determinant

$$\begin{aligned}
\det \mathbf{A} &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\
&= (4)\det \mathbf{A}_{11} - (-2)\det \mathbf{A}_{12} + (3)\det \mathbf{A}_{13} \\
&= 4\begin{vmatrix} 3 & 5 \\ 0 & 6 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 \\ 1 & 6 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} \\
&= 4(3\cdot 6 - 5\cdot 0) + 2(2\cdot 6 - 1\cdot 5) + 3(2\cdot 0 - 1\cdot 3) \\
&= 72 + 14 - 9 \\
&= 77
\end{aligned}$$

□

We can compute the determinant of a matrix $\mathbf{A}$ by expanding along any row or column. For example, the expansion of the determinant for the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

along the 3rd row is

$$\det \mathbf{A} = a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

And along the 2nd column:

$$\det \mathbf{A} = -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}.$$

The punchline is that any way you choose to expand (row or column) you will get the same answer. If a particular row or column contains zeros, say entry a_{jk}, then the computation of the determinant is simplified if you expand along either row j or column k because $a_{jk}C_{jk} = 0$ and we need not compute C_{jk}.

Example 11.5. Compute the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 3 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

Solution. In Example 11.4, we computed $\det(\mathbf{A}) = 77$ by expanding along the 1st row.

Notice that $a_{32} = 0$. Expanding along the 3rd row:

$$\begin{aligned}
\det \mathbf{A} &= (1) \det \mathbf{A}_{31} - (0) \det \mathbf{A}_{32} + (6) \det \mathbf{A}_{33} \\
&= \begin{vmatrix} -2 & 3 \\ 3 & 5 \end{vmatrix} + 6 \begin{vmatrix} 4 & -2 \\ 2 & 3 \end{vmatrix} \\
&= 1(-2 \cdot 5 - 3 \cdot 3) + 6(4 \cdot 3 - (-2) \cdot 2) \\
&= -19 + 96 \\
&= 77
\end{aligned}$$

□

11.2 Determinants of $n \times n$ Matrices

Using the 3×3 case as a guide, we define the determinant of a general $n \times n$ matrix as follows.

Definition 11.6: Let $\mathbf{A}$ be a $n \times n$ matrix. Let $\mathbf{A}_{jk}$ be the $(n-1) \times (n-1)$ matrix obtained from $\mathbf{A}$ by deleting the jth row and kth column, and let $C_{jk} = (-1)^{j+k} \det \mathbf{A}_{jk}$ be the (j,k)**-cofactor** of $\mathbf{A}$. The determinant of $\mathbf{A}$ is defined to be

$$\det \mathbf{A} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

The next theorem tells us that we can compute the determinant by expanding along any row or column.

Theorem 11.7: Let $\mathbf{A}$ be a $n \times n$ matrix. Then $\det \mathbf{A}$ may be obtained by a cofactor expansion along any row or any column of $\mathbf{A}$:

$$\det \mathbf{A} = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn}.$$

We obtain two immediate corollaries.

Corollary 11.8: If $\mathbf{A}$ has a row or column containing all zeros then $\det \mathbf{A} = 0$.

Proof. If the jth row contains all zeros then $a_{j1} = a_{j2} = \cdots = a_{jn} = 0$:

$$\det \mathbf{A} = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn} = 0.$$

Corollary 11.9: For any square matrix $\mathbf{A}$ it holds that $\det \mathbf{A} = \det \mathbf{A}^T$.

Sketch of the proof. Expanding along the jth row of $\mathbf{A}$ is equivalent to expanding along the jth column of $\mathbf{A}^T$.

Example 11.10. Compute the determinant of

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 & -2 \\ 1 & 2 & -2 & -1 \\ 0 & 0 & 2 & 1 \\ -1 & -3 & 1 & 0 \end{bmatrix}$$

Solution. The third row contains two zeros, so expand along this row:

$$\begin{aligned}
\det \mathbf{A} &= 0 \det \mathbf{A}_{31} - 0 \det \mathbf{A}_{32} + 2 \det \mathbf{A}_{33} - \det \mathbf{A}_{34} \\
&= 2 \begin{vmatrix} 1 & 3 & -2 \\ 1 & 2 & -1 \\ -1 & -3 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 3 & 0 \\ 1 & 2 & -2 \\ -1 & -3 & 1 \end{vmatrix} \\
&= 2 \left(1 \begin{vmatrix} 2 & -1 \\ -3 & 0 \end{vmatrix} - 3 \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix} \right) \\
&\quad - \left(1 \begin{vmatrix} 2 & -2 \\ -3 & 1 \end{vmatrix} - 3 \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} \right) \\
&= 2((0-3) - 3(0-1) - 2(-3+2)) - ((2-6) - 3(1-2)) \\
&= 5
\end{aligned}$$

□

Example 11.11. Compute the determinant of

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 & -2 \\ 1 & 2 & -2 & -1 \\ 0 & 0 & 2 & 1 \\ -1 & -3 & 1 & 0 \end{bmatrix}$$

Solution. Expanding along the second row:

$$\begin{aligned}
\det \mathbf{A} &= -\det \mathbf{A}_{21} + 2\det \mathbf{A}_{22} - (-2)\det \mathbf{A}_{23} - 1\det \mathbf{A}_{24} \\
&= -\begin{vmatrix} 3 & 0 & -2 \\ 0 & 2 & 1 \\ -3 & 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 & -2 \\ 0 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} \\
&\quad + 2\begin{vmatrix} 1 & 3 & -2 \\ 0 & 0 & 1 \\ -1 & -3 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \\ -1 & -3 & 1 \end{vmatrix} \\
&= -1(-3-12) + 2(-1-4) + 2(0) - (0) \\
&= 5
\end{aligned}$$

□

11.3 Triangular Matrices

Below we introduce a class of matrices for which the determinant computation is trivial.

Definition 11.12: A square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is called **upper triangular** if $a_{jk} = 0$ whenever $j > k$. In other words, all the entries of $\mathbf{A}$ below the diagonal entries a_{ii} are zero. It is called **lower triangular** if $a_{jk} = 0$ whenever $j < k$.

For example, a 4×4 upper triangular matrix takes the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Expanding along the first column, we compute

$$\det \mathbf{A} = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ 0 & a_{33} & a_{34} \\ 0 & 0 & a_{44} \end{vmatrix} = a_{11}\left(a_{22}\begin{vmatrix} a_{33} & a_{34} \\ 0 & a_{44} \end{vmatrix}\right) = a_{11}a_{22}a_{33}a_{44}.$$

The general $n \times n$ case is similar and is summarized in the following theorem.

Theorem 11.13: The determinant of a triangular matrix is the product of its diagonal entries.

After this lecture you should know the following:

- how to compute the determinant of any sized matrix
- that the determinant of $\mathbf{A}$ is equal to the determinant of $\mathbf{A}^T$
- the determinant of a triangular matrix is the product of its diagonal entries