Task 1

1.- To create $G'_{(4,8)}$, we must first get $G'_{(4,7)}$. This matrix has the following column order:

$$[P_1 \; P_2 \; D_1 \; P_3 \; D_2 \; D_3 \; D_4]$$

If we represent the position of the parity bits in binary, we get what data bits are affected by each parity bits, thus:

- P_1:= Position $1 \rightarrow (0001)_{(2)}$; So it affects pos. $3(0011)_{(2)}$, $5(0101)_{(2)}$ and $7(0111)_{(2)}$
- P_2:= Position $2 \rightarrow (0010)_{(2)}$; So it affects pos. $3(0011)_{(2)}$, $6(0110)_{(2)}$ and $7(0111)_{(2)}$
- P_3:= Position $4 \rightarrow (0100)_{(2)}$; So it affects pos. $5(0101)_{(2)}$, $6(0110)_{(2)}$ and $7(0111)_{(2)}$

In order to get the values of the parity bits, we need to perform the XOR of the values in the affected positions. Knowing that the data matrix is I_4, we have the following set of operations:

$$P_1 \rightarrow D_1 \oplus D_2 \oplus D_4 = (1000) \oplus (0100) \oplus (0001) \rightarrow (1101)$$
$$P_2 \rightarrow D_1 \oplus D_3 \oplus D_4 = (1000) \oplus (0010) \oplus (0001) \rightarrow (1011)$$
$$P_3 \rightarrow D_2 \oplus D_3 \oplus D_4 = (0100) \oplus (0010) \oplus (0001) \rightarrow (0111)$$

Combining the data matrix (I_4) and the parity bits in the correct order, we get that $G'_{(4,7)}$ is:

$$G'_{(4,7)} = \begin{array}{c} \begin{array}{ccccccc} p_1 & p_2 & d_1 & p_3 & d_2 & d_3 & d_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

To get $G'_{(4,8)}$ from this, we need to add one more parity bit, P_4, in the last position of the matrix. P_4 has the value (1110)

$$G'_{(4,8)} = \begin{array}{c} \begin{array}{cccccccc} p_1 & p_2 & d_1 & p_3 & d_2 & d_3 & d_4 & p_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

2: To get $H'_{(4,8)}$, we must first get $H'_{(3,7)}$. To obtain this, we must use the formulas $G := (I_k | -A^T)$ and $H := (A | I_{n-k})$ where:

- $n \to$ lenght of the full word
- $k \to$ lenght of the source word
- $I \to$ Identity matrix
- $A \to$ Parity matrix

From the previously calculated $G'_{(3,7)}$, we can get that $-A^T$ is:

$$-A^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ so then, we get that } A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

And I_{n-k} is I_{7-4}; or I_3, wich looks like this:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therfore, applying $H := (A | I_3)$ and placing the bits accordingly, we get

$$H'_{(3,7)} = \begin{array}{c} \begin{array}{ccccccc} d_1 & d_2 & p_1 & d_3 & p_2 & p_3 & p_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

To get $H'_{(4,8)}$, we only need to fill the fourth row with ones, and the last column with three rows of ceros. Thus:

$$\begin{array}{c} \begin{array}{cccccccc} d_1 & d_2 & p_1 & d_3 & p_2 & p_3 & p_4 & d_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} = H'_{(4,8)}$$

Task 2

In order to convert G' into G, we can use row reduction. Knowing G' looks like this:

$$G'_{(4,8)} = \begin{pmatrix} 1&1&1&0&0&0&0&1 \\ 1&0&0&1&1&0&0&1 \\ 0&1&0&1&0&1&0&1 \\ 1&1&0&1&0&0&1&0 \end{pmatrix}$$ we can apply the following transformations

1º Add the first row to the second row, and then the first to the fourth

$$\begin{pmatrix} 1&1&1&0&0&0&0&1 \\ 1&0&0&1&1&0&0&1 \\ 0&1&0&1&0&1&0&1 \\ 1&1&0&1&0&0&1&0 \end{pmatrix} \to \begin{pmatrix} 1&1&1&0&0&0&0&1 \\ 0&1&1&1&1&0&0&0 \\ 0&1&0&1&0&1&0&1 \\ 1&1&0&1&0&0&1&0 \end{pmatrix} \to \begin{pmatrix} 1&1&1&0&0&0&0&1 \\ 0&1&1&1&1&0&0&0 \\ 0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1 \end{pmatrix} \to$$

2º Add the second row to the first and third ones

$$\to \begin{pmatrix} 1&0&0&1&1&0&0&1 \\ 0&1&1&1&1&0&0&0 \\ 0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1 \end{pmatrix} \to \begin{pmatrix} 1&0&0&1&1&0&0&1 \\ 0&1&1&1&1&0&0&0 \\ 0&0&1&0&1&1&0&1 \\ 0&0&1&1&0&0&1&1 \end{pmatrix}$$

3º Add the third row to the second and fourth ones

$$\begin{pmatrix} 1&0&0&1&1&0&0&1 \\ 0&1&1&1&1&0&0&0 \\ 0&0&1&0&1&1&0&1 \\ 0&0&1&1&0&0&1&1 \end{pmatrix} \to \begin{pmatrix} 1&0&0&1&1&0&0&1 \\ 0&1&0&1&0&1&0&1 \\ 0&0&1&0&1&1&0&1 \\ 0&0&1&1&0&0&1&1 \end{pmatrix} \to \begin{pmatrix} 1&0&0&1&1&0&0&1 \\ 0&1&0&1&0&1&0&1 \\ 0&0&1&0&1&1&0&1 \\ 0&0&0&1&1&1&1&0 \end{pmatrix}$$

4º Lastly, add the fourth row to the first and second ones

$$\begin{pmatrix} 1&0&0&1&1&0&0&1 \\ 0&1&0&1&0&1&0&1 \\ 0&0&1&0&1&1&0&1 \\ 0&0&0&1&1&1&1&0 \end{pmatrix} \to \begin{pmatrix} 1&0&0&0&0&1&1&1 \\ 0&1&0&1&0&1&0&1 \\ 0&0&1&0&1&1&0&1 \\ 0&0&0&1&1&1&1&0 \end{pmatrix} \to \begin{pmatrix} 1&0&0&0&0&1&1&1 \\ 0&1&0&0&1&0&1&1 \\ 0&0&1&0&1&1&0&1 \\ 0&0&0&1&1&1&1&0 \end{pmatrix}$$

This way, we get that:

$$G_{(4,8)} = \begin{array}{c} \begin{matrix} d_1 & d_2 & d_3 & d_4 & p_1 & p_2 & p_3 & p_4 \end{matrix} \\ \begin{pmatrix} 1&0&0&0&0&1&1&1 \\ 0&1&0&0&1&0&1&1 \\ 0&0&1&0&1&1&0&1 \\ 0&0&0&1&1&1&1&0 \end{pmatrix} \end{array}$$

To get $H_{(4,8)}$ from $G_{(4,8)}$, we use the same formulas as in task 1. We now have:

$$I_{8-4} = I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ (for our data columns)}$$

and

$$-A^T = A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \text{ (for our parity columns)}$$

Thus, we have that $H_{(4,8)}$ is $H := (A \mid I_{8-4})$ or:

$$H_{(4,8)} = \begin{array}{c} \begin{array}{cccccccc} p_1 & p_2 & p_3 & p_4 & d_1 & d_2 & d_3 & d_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Task 3

1.- $\vec{a} = (0100)$

$$\left((0100) \cdot \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \right) \bmod 2 = (10011001) \bmod 2 =$$

$$= \underline{(10011001)}$$

2.- $\vec{a} = (1001)$

$$\left((1001) \cdot \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \right) \bmod 2 = (10011221) \bmod 2 =$$

$$= \underline{(10011001)}$$

3.- $\vec{a} = (0011)$

$$\left((0011) \cdot \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \right) \bmod 2 = (0011\ 22\ 11) \bmod 2 =$$

$$= \underline{(00111111)}$$

4.- $\vec{a} = (1101)$

$$\left((1101) \cdot \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \right) \bmod 2 = (1101\ 2232) \bmod 2 =$$

$$= \underline{(11010010)}$$

Task 4

1:- $\vec{x} = (11001101)$ ~~codeword is (1100)~~

$$\left(\begin{pmatrix} 0&1&1&1&1&0&0&0 \\ 1&0&1&1&0&1&0&0 \\ 1&1&0&1&0&0&1&0 \\ 1&1&1&0&0&0&0&1 \end{pmatrix} \cdot (11001101)^T\right) \bmod 2 = (2223) \bmod 2 = (0001)$$

Not transmited properly!

Check $p_4 \to$ Odd

So, knowing that the syndrome is (0001) and in H, we can correct the word to (11001100), giving us that the original word was <u>(1100)</u>

2:- $\vec{x} = (10011001)$

$$\left(\begin{pmatrix} 0&1&1&1&1&0&0&0 \\ 1&0&1&1&0&1&0&0 \\ 1&1&0&1&0&0&1&0 \\ 1&1&1&0&0&0&0&1 \end{pmatrix} \cdot (10011001)^T\right) \bmod 2 = (2222) \bmod 2 = (0000)$$

Since $p_4 \to 0$ and has even parity, we know it was transmited correctly, and the word was <u>(1001)</u>

3:- $\vec{x} = (11011011)$

$$\left(\begin{pmatrix} 0&1&1&1&1&0&0&0 \\ 1&0&1&1&0&1&0&0 \\ 1&1&0&1&0&0&1&0 \\ 1&1&1&0&0&0&0&1 \end{pmatrix} \cdot (11011011)^T\right) \bmod 2 = (3243) \bmod 2 = (1001)$$

Since $p_4 \to 1$ and the syndrome is not in H, we know that we have <u>multiple errors</u>

4:- $\vec{x} = (11010101)$

$$\left(\begin{pmatrix} 0&1&1&1&1&0&0&0 \\ 1&0&1&1&0&1&0&0 \\ 1&1&0&1&0&0&1&0 \\ 1&1&1&0&0&0&0&1 \end{pmatrix} \cdot (11010101)^T\right) \bmod 2 = (2333) \bmod 2 = (0111)$$

Since $p_4 \to 1$ and the syndrome is in H, we can correct the word to (01010101), giving us that the original word was <u>(0101)</u>