CMPS 101
Fall 2018
Midterm Exam 2 **Solutions**

1. (20 Points) Use the Master Theorem to find tight asymptotic bounds for the following recurrences.

   a. (10 Points) $T(n) = 5T(n/2) + n^2 \log(n)$

   Solution:
   Let $\epsilon = \frac{1}{2}(\log_2(5) - 2)$. Then $4 < 5 \Rightarrow 2 < \log_2(5) \Rightarrow \epsilon > 0$. We have $2\epsilon = \log_2(5) - 2$, and hence $\log_2(5) - \epsilon = 2 + \epsilon$. Therefore

   $$\lim_{n\to\infty}\left(\frac{n^2 \log(n)}{n^{\log_2(5)-\epsilon}}\right) = \lim_{n\to\infty}\left(\frac{n^2 \log(n)}{n^{2+\epsilon}}\right) = \lim_{n\to\infty}\left(\frac{\log(n)}{n^{\epsilon}}\right) = 0,$$

   and hence

   $$n^2 \log(n) = o\left(n^{\log_2(5)-\epsilon}\right) \subseteq O\left(n^{\log_2(5)-\epsilon}\right).$$

   Case 1 of the Master Theorem gives us $T(n) = \Theta(n^{\log_2(5)})$. ■

   b. (10 Points) $T(n) = 16T(n/8) + n^{4/3}$

   Solution:
   Observe that $\log_8(16) = 4/3$ since $8^{4/3} = 2^4 = 16$. Thus $n^{4/3} = n^{\log_8(16)} = \Theta(n^{\log_8(16)})$, and by case 2 we have $T(n) = \Theta\left(n^{4/3} \log n\right)$. ■

2. (20 Points) Let G be a connected graph with n vertices and m edges. Use induction on m to prove that $m \geq n - 1$. (Hint: you may use the following fact without proof. If an edge e is removed from G, then the resulting graph $G - e$ is either connected, or has exactly two connected components H_1 and H_2.)

Proof:

I. Let $m = 0$. Then G, being connected, can have only one vertex. Therefore $m \geq n - 1$ reduces to $0 \geq 0$, and the base case is satisfied.

II. Let $m > 0$. Assume for any connected graph G' with $|E(G')| < m$ that $|E(G')| \geq |V(G')| - 1$. We must show that $m \geq n - 1$. Pick any edge $e \in E(G)$ and remove it. By the above hint, we have two cases to consider.

<u>Case 1</u>: $G - e$ is connected.
In this case the induction hypothesis gives $m - 1 = |E(G - e)| \geq |V(G - e)| - 1 = n - 1$, whence $m \geq n > n - 1$, and therefore $m \geq n - 1$ as required.

<u>Case 2</u>: $G - e$ is disconnected.
Following the hint, $G - e$ consists of two connected components H_1 and H_2, each having fewer than m edges. The induction hypothesis now guarantees $|E(H_i)| \geq |V(H_i)| - 1$ for $i = 1, 2$. Since no vertices were removed, $n = |V(H_1)| + |V(H_2)|$, and therefore

$$\begin{aligned} m &= |E(H_1)| + |E(H_2)| + 1 \\ &\geq (|V(H_1)| - 1) + (|V(H_2)| - 1) + 1 \qquad \text{(by the induction hypothesis)} \\ &= (|V(H_1)| + |V(H_2)|) - 1 \\ &= n - 1 \end{aligned}$$

In this case also, $m \geq n - 1$.

The result now follows for all connected graphs by induction. ■

3. (20 Points) Let G be a graph with n vertices, m edges and k connected components. Prove $m \geq n - k$. (Hint: Use the result of problem 2).

Proof:
Let the connected components of G be $H_1, H_2, H_3, \ldots H_k$. Suppose component H_i has n_i vertices and m_i edges (for $i = 1, 2, \ldots, k$). Then by the result of problem 2, we have $m_i \geq n_i - 1$ (for $i = 1, 2, \ldots, k$). Summing these inequalities we get

$$m = \sum_{i=1}^{k} m_i \geq \sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i \; - \; \sum_{i=1}^{k} 1 \; = \; n - k,$$

i.e. $m \geq n - k$, as required. ■

4. (20 Points) Let G be a connected graph with n vertices and m edges. Suppose also that $m = n$. Prove that G contains exactly one cycle. (Hint: Use the result of problem 2, and also Lemma 1 which says that if T is a tree then $|E(T)| = |V(T)| - 1$.)

Proof:
G contains at least one cycle since if not, G is a tree and hence $m = n - 1$ (by Lemma 1), contrary to the hypothesis $m = n$. It remains to show that G contains at most one cycle.

Assume, to get a contradiction, that G contains two distinct cycles, call them C_1 and C_2. It is not possible that either $E(C_1) \subsetneq E(C_2)$ or $E(C_2) \subsetneq E(C_1)$ since removing an edge from a cycle destroys it. Therefore both $E(C_1) - E(C_2) \neq \emptyset$ and $E(C_1) - E(C_2) \neq \emptyset$. Pick $e_1 \in E(C_1) - E(C_2)$ and $e_2 \in E(C_2) - E(C_1)$. Then $G - e_1$ is a connected graph with the cycle C_2 still intact. Therefore $G - e_1 - e_2$ is also connected. Applying the result of problem 2 to this graph, we have $|E(G - e_1 - e_2)| \geq |V(G - e_1 - e_2)| - 1$, which says $m - 2 \geq n - 1$, and hence $m \geq n + 1 > n$, contrary to the hypothesis $m = n$. This contradiction shows our assumption was false, and therefore G must contain exactly one cycle. ■

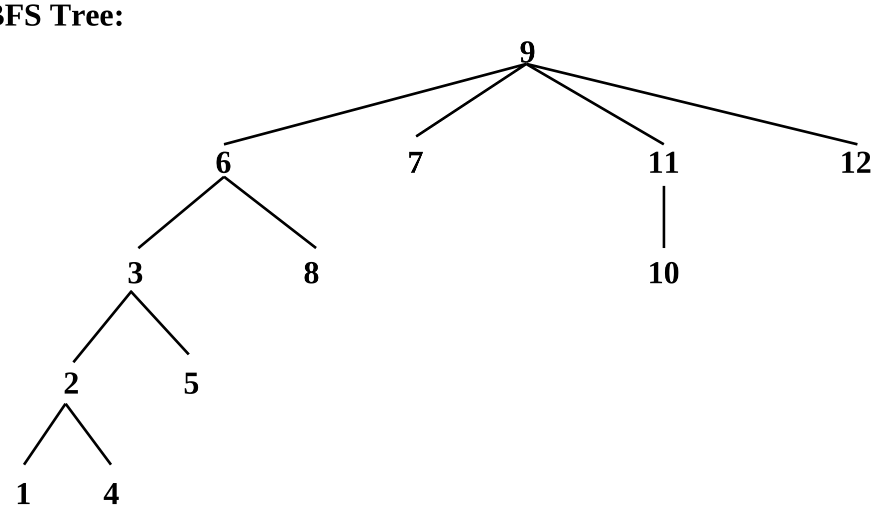
5. (20 Points) Run BFS on the graph pictured on the back page of this exam, using source vertex $s = 9$. Pseudo-code for BFS is also included on the back page. Execute the for loop on lines 12-17 of BFS in increasing order by vertex label.

   a. (14 Points) Fill in the following table, determine the order in which vertices enter the Queue, and draw the BFS Tree.
   Solution:

	Adjacency List	Color	Distance	Parent
1	**2 4**	**b**	**4**	**2**
2	**1 3 4 5**	**b**	**3**	**3**
3	**2 5 6**	**b**	**2**	**6**
4	**1 2**	**b**	**4**	**2**
5	**2 3 8**	**b**	**3**	**3**
6	**3 7 8 9**	**b**	**1**	**9**
7	**6 9**	**b**	**1**	**9**
8	**5 6**	**b**	**2**	**6**
9	**6 7 11 12**	**b**	**0**	**NIL**
10	**11**	**b**	**2**	**11**
11	**9 10 12**	**b**	**1**	**9**
12	**11 9**	**b**	**1**	**9**

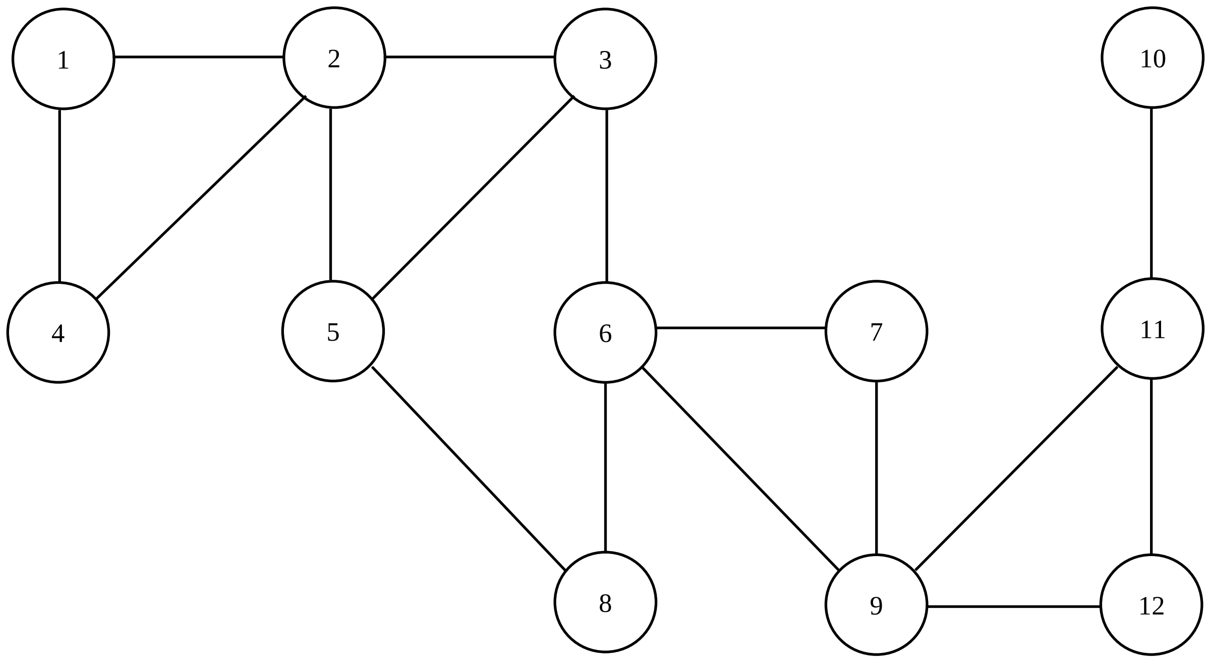
Queue	**9 6 7 11 12 3 8 10 2 5 1 4**

BFS Tree:



   b. (6 Points) Determine the shortest 9-4 path found by BFS, and the shortest 9-10 path found by BFS. Find a shortest 9-5 path ***not*** found by BFS.
   Solution:
   Shortest 9-4 path found by BFS: **9 6 3 2 4**
   Shortest 9-10 path found by BFS: **9 11 12**
   Shortest 9-5 path ***not*** found by BFS: **9 6 8 5**

Problem 5 refers to the following graph (you may tear off this page and keep it)



The following pseudo-code is included for reference.

```
BFS(G, s)
for all  x ∈ V(G) − {s}
    color[x] = white
    d[x] = ∞
    p[x] = NIL
color[s] = gray
d[s] = 0
p[s] = NIL
Q = ∅
Enqueue(Q, s)
while  Q ≠ ∅
    x = Dequeue(Q)
    for all y ∈ adj[x]
        if color[y] == white
            color[y] = gray
            d[y] = d[x] + 1
            p[y] = x
            Enqueue(Q, y)
    color[x] = black
```