CMPS 101
Homework Assignment 5
Solutions

1. Define $T(n)$ defined by the recurrence formula

$$T(n) = \begin{cases} 6 & 1 \le n < 3 \\ 2T(\lfloor n/3 \rfloor) + n & n \ge 3 \end{cases}$$

   a. Use the iteration method to write a summation formula for $T(n)$.
   Solution:

$$\begin{aligned} T(n) &= n + 2T(\lfloor n/3 \rfloor) \\ &= n + 2(\lfloor n/3 \rfloor + 2T(\lfloor \lfloor n/3 \rfloor /3 \rfloor)) \\ &= n + 2\lfloor n/3 \rfloor + 2^2 T(\lfloor n/3^2 \rfloor) \\ &= n + 2\lfloor n/3 \rfloor + 2^2\lfloor n/3^2 \rfloor + 2^3 T(\lfloor n/3^3 \rfloor) \quad \text{etc..} \end{aligned}$$

   After substituting the recurrence into itself k times, we get

$$T(n) = \sum_{i=0}^{k-1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor \quad + \quad 2^k T(\lfloor n/3^k \rfloor).$$

   This process terminates when the recursion depth k is chosen so that $1 \le \lfloor n/3^k \rfloor < 3$, which is equivalent to $1 \le n/3^k < 3$, whence $3^k \le n < 3^{k+1}$, so $k \le \log_3(n) < k+1$, and hence $k = \lfloor \log_3(n) \rfloor$. With this value of k we have $T\left(\left\lfloor \frac{n}{3^k} \right\rfloor\right) = T(1 \text{ or } 2) = 6$. Therefore

$$T(n) = \sum_{i=0}^{\lfloor \log_3(n) \rfloor - 1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor \ + \ 6 \cdot 2^{\lfloor \log_3(n) \rfloor}.$$

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   b. Use the summation in (a) to show that $T(n) = O(n)$
   Solution:
   Using the above summation, we have

$$\begin{aligned} T(n) &\le n\left(\sum_{i=0}^{\lfloor \log_3(n) \rfloor - 1} (2/3)^i\right) + 6 \cdot 2^{\log_3(n)} && \text{since } \lfloor x \rfloor \le x \text{ for any } x \\ &\le n\left(\sum_{i=0}^{\infty} (2/3)^i\right) + 6n^{\log_3(2)} && \text{adding } \infty\text{-many positive terms} \\ &= n\left(\frac{1}{1-(2/3)}\right) + 6n^{\log_3(2)} \\ &= 3n + 6n^{\log_3(2)} = O(n) \end{aligned}$$

   The last equation follows since $2 < 3 \Rightarrow \log_3(2) < 1 \Rightarrow n^{\log_3(2)} = o(n)$. It follows that $T(n) = O(n)$. ■

   c. Use the Master Theorem to show that $T(n) = \Theta(n)$
   Solution:
   Let $\varepsilon = 1 - \log_3(2) > 0$. Then $\log_3(2) + \varepsilon = 1$, and $n = n^{\log_3(2)+\varepsilon} = \Omega(n^{\log_3(2)+\varepsilon})$. Also for any c in the range $2/3 \le c < 1$, and any positive n, we have $2(n/3) = (2/3)n \le cn$, so the regularity condition holds. By case (3) of the Master Theorem $T(n) = \Theta(n)$. ■

2. Use the Master theorem to find asymptotic solutions to the following recurrences.

a. $T(n) = 7T(n/4) + n$

Solution:

$4 < 7 \;\Rightarrow\; 1 < \log_4(7) \;\Rightarrow\; \log_4(7) - 1 > 0$. Let $\varepsilon = \log_4(7) - 1$. Then $\varepsilon > 0$, and $1 = \log_4(7) - \varepsilon$, whence $n = n^{\log_4(7)-\varepsilon} = O(n^{\log_4(7)-\varepsilon})$. By case (1) we have $T(n) = \Theta(n^{\log_4(7)})$.

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b. $T(n) = 9T(n/3) + n^2$

Solution:

Observe that $n^2 = n^{\log_3(9)} = \Theta(n^{\log_3(9)})$, and therefore $T(n) = \Theta(n^2\log(n))$ by case (2).

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c. $T(n) = 6T(n/5) + n^2$

Solution:

Observe $6 < 25 \;\Rightarrow\; \log_5(6) < 2 \;\Rightarrow\; 2 - \log_5(6) > 0$. Let $\varepsilon = 2 - \log_5(6)$. Then $\log_5(6) + \varepsilon = 2$, and $n^2 = \Omega(n^{\log_5(6)+\varepsilon})$. Also for any c in the range $6/25 \le c < 1$, and for any positive n, we have $6(n/5)^2 = (6/25)n^2 \le cn^2$, so the regularity condition holds. Therefore $T(n) = \Theta(n^2)$ by case (3) of the Master Theorem. ■

d. $T(n) = 6T(n/5) + n\log(n)$

Solution:

Observe $\log_5(6) > 1$, so letting $\varepsilon = \frac{\log_5(6)-1}{2}$, we have $\varepsilon > 0$ and $1 + \varepsilon = \log_5(6) - \varepsilon$. Therefore by l'Hopital's rule

$$\lim_{n\to\infty}\frac{n\log(n)}{n^{\log_5(6)-\varepsilon}} = \lim_{n\to\infty}\frac{n\log(n)}{n^{1+\varepsilon}} = lim_{n\to\infty}\frac{\log(n)}{n^{\varepsilon}} = 0,$$

showing that $n\log(n) = o(n^{\log_5(6)-\varepsilon}) \subseteq O(n^{\log_5(6)-\varepsilon})$. Case (1) now gives $T(n) = \Theta(n^{\log_5(6)})$.

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e. $T(n) = 7T(n/2) + n^2$

Solution:

Observe that $7 > 4 \;\Rightarrow\; \log_2(7) > 2$, so upon setting $\varepsilon = \log_2(7) - 2$ we have $\varepsilon > 0$. It follows that $2 = \log_2(7) - \varepsilon$, and $n^2 = n^{\log_2(7)-\varepsilon} = O(n^{\log_2(7)-\varepsilon})$. Case 1 of the Master Theorem now gives $T(n) = \Theta(n^{\log_2(7)})$. ■

f. $S(n) = aS(n/4) + n^2$ (Note: your answer will depend on the parameter a.)

Solution:

We have three cases to consider corresponding to the three cases of the Master Theorem:

Case 1:
$a > 16 \;\Rightarrow\; \log_4(a) > 2$, so letting $\varepsilon = \log_4(a) - 2 > 0$, we have $n^2 = O(n^{\log_4(a)-\varepsilon})$, and hence $S(n) = \Theta(n^{\log_4(a)})$.

Case 2:
$a = 16 \;\Rightarrow\; \log_4(a) = 2 \;\Rightarrow\; n^2 = \Theta(n^{\log_4(a)})$, whence $S(n) = \Theta(n^2\log(n))$.

Case 3:
$1 \le a < 16 \;\Rightarrow\; \log_4(a) < 2$, so if $\varepsilon = 2 - \log_4(a) > 0$ then $n^2 = \Omega(n^{\log_4(a)+\varepsilon})$. Further, for any c in the range $a/16 \le c < 1$ we have $a(n/4)^2 = (a/16)n^2 \le cn^2$, showing that the regularity condition holds. Therefore $S(n) = \Theta(n^2)$. ■

3. Let $T(n)$ satisfy the recurrence $T(n) = aT(n/b) + f(n)$, where $f(n)$ is a polynomial satisfying $\deg(f) > \log_b(a)$. Prove that case (3) of the Master Theorem applies, and in particular that the regularity condition necessarily holds.

   Proof:
   Let $d = \deg(f)$ and replace $f(n)$ by the asymptotically equivalent function n^d. The Master Theorem tells us to compare the polynomials n^d and $n^{\log_b(a)}$. Let $\varepsilon = d - \log_b(a)$, which is positive since $d > \log_b(a)$. Therfore $d = \log_b(a) + \varepsilon$, and hence

   $$n^d = \Omega(n^d) = \Omega\left(n^{log_b(a)+\varepsilon}\right)$$

   verifying the first hypothesis of case (3). To establish the regularity condition, observe $d > \log_b(a) \Rightarrow b^d > a \Rightarrow a/b^d < 1$. Pick any c in the range $a/b^d \le c < 1$. Then for any $n \ge 1$ we have $a(n/b)^d = (a/b^d)n^d \le cn^d$, which verifies the regularity condition. ■

4. p.75: 4.3-2
   The recurrence $T(n) = 7T(n/2) + n^2$ describes the running time of an algorithm *A*. A competing algorithm *B* has a running time given by $S(n) = aS(n/4) + n^2$. What is the largest integer value for *a* such that *B* is a faster algorithm than *A* (asymptotically speaking)? In other words, find the largest integer *a* such that $S(n) = o(T(n))$.

   Solution:
   We seek the largest integer a for which $S(n) = o(T(n))$. Using parts (e) and (f) of the previous problem, we find that $S(n) = o(T(n))$ in cases 2 and 3 since $4 < 7 \Rightarrow 2 < \log_2(7)$ and hence $n^2 = o(n^{\log_2(7)})$, and $n^2\log(n) = o(n^{\log_2(7)})$. In case 1 we have $S(n) = o(T(n))$ if and only if $n^{\log_4(a)} = o(n^{\log_2(7)})$, i.e. if and only if $\log_4(a) < \log_2(7)$. This is equivalent to $a < 4^{\log_2(7)} = 7^{\log_2(4)} = 7^2 = 49$. The largest such integer is $a = 48$. ■

5. Show that the number vertices of odd degree in any graph must be even. (Hint: Use the Handshake Lemma mentioned in the Graph Theory handout.)

   Proof:
   The handshake lemma says: $\sum_{x\in V(G)} \deg(x) = 2|E(G)|$. Let $E = \{x \in V(G) \mid \deg(x) \text{ is even}\}$ and $O = \{y \in V(G) \mid \deg(y) \text{ is odd}\}$. The handshake lemma can then be written as:

   $$\sum_{x\in E} \deg(x) + \sum_{y\in O} \deg(y) = 2|E(G)|$$

   The right hand side of the above equation is obviously even and the first term on the left hand side is also even, being the sum of even numbers. Therefore the second term on the left hand side is even as well. Observe that the sum of an odd number of odd numbers is necessarily odd, while the sum of an even number of odd numbers is even. It follows that the sum

   $$\sum_{y\in O} \deg(y)$$

   contains an even number of terms. Therefore *G* contains an even number of odd-degree vertices. ■

6. Show that any connected graph G satisfies $|E(G)| \geq |V(G)| - 1$. Hint: use induction on the number of edges.

Proof:
Let $m = |E(G)|$ and $n = |V(G)|$. We proceed by induction on m.

I. Let $m = 0$. Then being connected, G can have only one vertex, hence $n = 1$. Therefore $m \geq n - 1$ reduces to $0 \geq 0$, and the base case is verified.

II. Let $m > 0$. Assume for any connected graph G' with fewer than m edges: $|E(G')| \geq |V(G')| - 1$. We must show that $|E(G)| \geq |V(G)| - 1$, i.e. $m \geq n - 1$. Remove an edge $e \in E(G)$ and let $G - e$ denote the resulting subgraph. We have two cases to consider.

Case 1: $G - e$ is connected. Note that $G - e$ has n vertices and $m - 1$ edges, so the induction hypothesis gives $m - 1 \geq n - 1$. Certainly then $m \geq n - 1$, as was claimed.

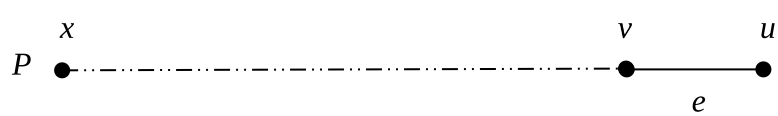
Case 2: $G - e$ is disconnected. In this case $G - e$ consists of two connected components. (**See the claim and proof below.) Call them H_1 and H_2. Observe that each component contains fewer than m edges. Suppose H_i has m_i edges and n_i vertices ($i = 1,\ 2$). The induction hypothesis gives $m_i \geq n_i - 1$ ($i = 1,\ 2$). Also $n = n_1 + n_2$ since no vertices were removed. Therefore

$$m \;=\; m_1 + m_2 + 1 \;\geq\; (n_1 - 1) + (n_2 - 1) + 1 \;=\; n_1 + n_2 - 1 \;=\; n - 1,$$

and therefore $m \geq n - 1$ as required. ■

(Note to grader: the following claim and proof are not necessary for full credit.)

Claim**: Let G be a connected graph and $e \in E(G)$, and suppose that $G - e$ is disconnected. (Such an edge e is called a *bridge*). Then $G - e$ has exactly two connected components.
Proof:
Since $G - e$ is disconnected, it has at least two components. We must show that it also has at most two components. Let e have end vertices u, and v. Let C_u and C_v be the connected components of $G - e$ that contain u and v respectively. Choose $x \in V(G)$ arbitrarily, and let P be an x-u path in G (note P exists since G is connected.) Either P includes the edge e, or it does not. If P does not contain e, then P remains intact after the removal of e, and hence P is an x-u path in $G - e$, whence $x \in C_u$. If on the other hand P does contain the edge e, then e must be the last edge along P from x to u.



In this case $P - e$ is an x-v path in $G - e$, whence $x \in C_v$. Since x was arbitrary, every vertex in $G - e$ belongs to either C_u or C_v, and therefore $G - e$ has at most two connected components. ■