CMPS 101
Homework Assignment 2
Solutions

1. p.52: 3.1-1
   Let $f(n)$ and $g(n)$ be asymptotically non-negative functions. Using the basic definition of Θ-notation, prove that $f(n) + g(n) = \Theta(\max(f(n), g(n))$.

   Proof:
   Since $f(n)$ and $g(n)$ are asymptotically non-negative, there exists a positive constant n_0 such that $f(n) \geq 0$ and $g(n) \geq 0$ for all $n \geq n_0$. For such n we have

$$\begin{aligned} 0 &\leq \max(f(n), g(n)) \\ &\leq \min(f(n), g(n)) + \max(f(n), g(n)) \\ &\leq 2 \cdot \max(f(n), g(n)). \end{aligned}$$

   But $f(n) + g(n) = \min(f(n), g(n)) + \max(f(n), g(n))$, so for all $n \geq n_0$ we have

$$0 \leq 1 \cdot \max(f(n), g(n)) \leq f(n) + g(n) \leq 2 \cdot \max(f(n), g(n)).$$

   Thus $f(n) + g(n) = \Theta(\max(f(n), g(n)))$, as required. ■

2. p. 53: 3.1-4
   Determine whether the following statements are true or false.

   a. $2^{n+1} = O(2^n)$
      Solution: True since $2^{n+1} = 2 \cdot 2^n = \text{const} \cdot 2^n = O(2^n)$. ■
   b. $2^{2n} = O(2^n)$
      Solution: False since $2^{2n} = (2^2)^n = 4^n = \omega(2^n)$ and since $\omega(2^n) \cap O(2^n) = \emptyset$. ■

3. p.61: 3-2abcdef
   Indicate, for each pair of expressions (A, B) in the table below, whether A is O, o, Ω, ω, or Θ of B. Assume that $k \geq 1$, $\varepsilon > 0$, and $c > 1$ are constants. Place 'yes' or 'no' in each of the empty cells below, and justify your answers.

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ε	yes	yes	no	no	no
b.	n^k	c^n	yes	yes	no	no	no
c.	$\sqrt{n}$	$n^{\sin n}$	no	no	no	no	no
d.	2^n	$2^{n/2}$	no	no	yes	yes	no
e.	$n^{\lg c}$	$c^{\lg n}$	yes	no	yes	no	yes
f.	$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes

Justifications:

a. Applying l'Hopital's rule $\lceil k \rceil$ times, we get $\lim\limits_{n\to\infty} \frac{(\lg n)^k}{n^{\varepsilon}} = 0$, whence $\lg^k n = o(n^{\varepsilon})$.

b. Again, by $\lceil k \rceil$ applications of l'Hopital's rule, $\lim\limits_{n\to\infty} \frac{n^k}{c^n} = 0$, whence $n^k = o(c^n)$.

c. Let $c > 0$, and observe that $n^{sin\, n} > c\sqrt{n}$ for $n = 2\pi k + \frac{\pi}{2}$, where k is a sufficiently large positive integer. Therefore the inequality $n^{\sin n} \le c\sqrt{n}$ is false for arbitrarily large n, and hence $n^{\sin(n)} \neq O(\sqrt{n})$. Also note that $n^{\sin n} < c\sqrt{n}$ for $n = 2\pi k + \frac{3\pi}{2}$, where k is sufficiently large, and therefore the inequality $n^{\sin n} \ge c\sqrt{n}$ is false for arbitrarily large n, whence $n^{\sin(n)} \neq \Omega(\sqrt{n})$.

d. Observe $\lim\limits_{n\to\infty}(2^n/2^{n/2}) = \lim\limits_{n\to\infty}(\sqrt{2})^n = \infty$, and therefore $2^n = \omega(2^{n/2})$.

e. The identity $x^{\log_b y} = y^{\log_b x}$ implies that the two functions are equal: $n^{\lg c} = c^{\lg n}$.

f. Note $\lg(n^n) = n\lg(n)$, and Stirling's formula implies $\lg(n!) = \Theta(n\lg(n))$.

4. p.62: 3-4deh

Let $f(n)$ and $g(n)$ be asymptotically positive functions (i.e. $f(n) > 0$ and $g(n) > 0$ for sufficiently large n.) Prove or disprove the following statements.

d. $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$. **False**

Counter-Example:

Let $f(n) = 2n$ and $g(n) = n$. Then $2^{g(n)} = 2^n$ and $2^{f(n)} = 2^{2n} = 4^n = \omega(2^n)$, so $2^{f(n)} = \omega(2^{g(n)})$, and therefore $2^{f(n)} \neq O(2^{g(n)})$. ∎

e. $f(n) = O((f(n))^2)$. **False**

Counter-Example:

Let $f(n) = 1/n$. Then $f(n) = \omega((f(n))^2)$ since $\lim\limits_{n\to\infty} \frac{f(n)}{f(n)^2} = \lim\limits_{n\to\infty} \frac{1}{f(n)} = \lim\limits_{n\to\infty} n = \infty$, and hence $f(n) \neq O((f(n))^2)$. ∎

h. $f(n) + o(f(n)) = \Theta(f(n))$. **True**

Proof:

In the above formula, $o(f(n))$ stands for some anonymous function $h(n)$ in the class $o(f(n))$, whence $\lim\limits_{n\to\infty} \frac{h(n)}{f(n)} = 0$. Thus $\lim\limits_{n\to\infty} \frac{f(n)+h(n)}{f(n)} = \lim\limits_{n\to\infty}\left(1 + \frac{h(n)}{f(n)}\right) = 1$, and $f(n) + h(n) = \Theta(f(n))$, as claimed. ∎

5. Let $g(n)$ be an asymptotically non-negative function. Prove that $o\big(g(n)\big) \cap \Omega\big(g(n)\big) = \emptyset$.

Proof:
Assume to get a contradiction that $f(n) \in o(g(n)) \cap \Omega(g(n))$. Then since $f(n) = \Omega(g(n))$ we have

(1) $\quad \exists c_1 > 0,\ \exists n_1 > 0, \forall n \geq n_1: \quad 0 \leq c_1 g(n) \leq f(n)$

and since $f(n) = o(g(n))$ we have

(2) $\quad \forall c_2 > 0,\ \exists n_2 > 0, \forall n \geq n_2: \quad 0 \leq f(n) < c_2 g(n)$

Let $c_2 = c_1$. Then $c_2 > 0$, and by (2) there exists $n_2 > 0$ such that $0 \leq f(n) < c_1 g(n)$ for all $n \geq n_2$. Pick any $m \geq \max(n_1,\ n_2)$. Then by (1) and (2) we have $0 \leq c_1 g(m) \leq f(m) < c_1 g(m)$, and hence $c_1 g(m) < c_1 g(m)$, a contradiction. Our assumption was therefore false, and no such function $f(n)$ can exist. We conclude that $o\big(g(n)\big) \cap \Omega(g(n)) = \emptyset$. ■

6. Prove that $3^{2^n} = o\big(2^{3^n}\big)$.
Proof:
Observe that $\ln\left(\frac{3^{2^n}}{2^{3^n}}\right) = 2^n(\ln 3) - 3^n(\ln 2) \ \rightarrow\ -\infty$, since $3^n = \omega(2^n)$. Therefore $\frac{3^{2^n}}{2^{3^n}} \rightarrow e^{-\infty} = 0$, whence $3^{2^n} = o\big(2^{3^n}\big)$. ■