CMPS 101
Algorithms and Abstract Data Types
Fall 2018
Midterm Exam 1 Solutions

1. (20 Points) Determine whether the following statements are true or false. Prove or disprove each statement accordingly.

   a. (10 Points) If $a < b$, then $a^n = \mathrm{o}(b^n)$. **True**

   Proof:
   Since $\frac{a}{b} < 1$ we have $\lim_{n\to\infty}\left(\frac{a^n}{b^n}\right) = \lim_{n\to\infty}\left(\frac{a}{b}\right)^n = 0$, whence $a^n = \mathrm{o}(b^n)$. ■

   b. (10 Points) If $a > b$, then $a^{\log_b(n)} = \omega(n)$. **True**

   Proof:
   Since $a > b$, we have $\log_b(a) > \log_b(b) = 1$, and $\log_b(a) - 1 > 0$. Therefore

$$\lim_{n\to\infty}\left(\frac{a^{\log_b(n)}}{n}\right) = \lim_{n\to\infty}\left(\frac{n^{\log_b(a)}}{n}\right) = \lim_{n\to\infty}\left(n^{\log_b(a)-1}\right) = \infty,$$

   and hence $a^{\log_b(n)} = \omega(n)$. ■

2. (20 Points) Use Stirling's formula to prove that $\log(n!) = \Theta(n\log(n))$.

   Proof:
   Taking log (any base) of both sides of Stirling's formula, and using the laws of logarithms, we get

$$\begin{aligned}
\log(n!) &= \log\left(\sqrt{2\pi n}\cdot(n/e)^n\cdot\left(1+\Theta(1/n)\right)\right)\\
&= \log\sqrt{2\pi n} + \log(n/e)^n + \log\left(1+\Theta(1/n)\right)\\
&= \frac{1}{2}\log(2\pi) + \frac{1}{2}\log(n) + n\log(n) - n\log(e) + \log\left(1+\Theta(1/n\,)\right).
\end{aligned}$$

   Therefore

$$\frac{\log(n!)}{n\log(n)} = \frac{\log(2\pi)}{2n\log(n)} + \frac{1}{2n} + 1 - \frac{\log(e)}{\log(n)} + \frac{\log\left(1+\Theta(1/n)\right)}{n\log(n)},$$

   and hence

$$\lim_{n\to\infty}\left(\frac{\log(n!)}{n\log(n)}\right) = 1.$$

   Thus $\log(n!) = \Theta(n\log(n))$, as claimed. ■

3. (20 Points) Consider the following algorithm that wastes time.

<u>WasteTime(n)</u> (pre: $n \geq 1$)
1. if $n = 1$
2. waste 2 units of time
3. else
4. WasteTime($\lceil n/2 \rceil$)
5. WasteTime($\lfloor n/2 \rfloor$)
6. waste 5 units of time

a. (10 Points) Write a recurrence relation for the number of units of time $T(n)$ wasted by this algorithm.

Solution:

$$T(n) = \begin{cases} 2 & n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + 5 & n \geq 2 \end{cases}$$

b. (10 Points) Show that $T(n) = 7n - 5$ is the solution to this recurrence. (Hint: you may use without proof the fact that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.)

Proof:
First observe that if $T(n) = 7n - 5$, then $T(1) = 7 - 5 = 2$. Second, if $n \geq 2$ we have

$$\begin{aligned} \text{RHS} &= T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + 5 \\ &= (7\lceil n/2 \rceil - 5) + (7\lfloor n/2 \rfloor - 5) + 5 \\ &= 7(\lceil n/2 \rceil + \lfloor n/2 \rfloor) - 5 \\ &= 7n - 5 = T(n) = \text{LHS}, \end{aligned}$$

showing that $T(n) = 7n - 5$ solves the recurrence. ■

4. (20 Points) Let $T(n)$ be defined by the recurrence formula

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & n \geq 2 \end{cases}$$

a. (4 Points) Determine the values $T(2), T(3), T(4)$ and $T(5)$.

Solution:

$T(2) = T(1) + 2^2 = 1 + 4 = 5$

$T(3) = T(1) + 3^3 = 1 + 9 = 10$

$T(4) = T(2) + 4^2 = 5 + 16 = 21$

$T(5) = T(2) + 5^2 = 5 + 25 = 30$

b. (16 Points) Prove that $T(n) \leq \frac{4}{3}n^2$ for all $n \geq 1$. (Hint: use strong induction.)

Proof:

Base Step

Observe that $T(1) = 1 \leq 4/3 = (4/3) \cdot 1^2$, which establishes the base case.

Induction Step (IId)

Let $n > 1$ be chosen arbitrarily. Assume for all k in the range $1 \leq k < n$ that $T(k) \leq (4/3)k^2$. We must show as a consequence that $T(n) \leq (4/3)n^2$. Observe

$$\begin{aligned} T(n) &= T(\lfloor n/2 \rfloor) + n^2 && \text{by the recurrence formula for } T(n) \\ &\leq (4/3)\lfloor n/2 \rfloor^2 + n^2 && \text{by the induction hypothesis with } k = \lfloor n/2 \rfloor \\ &\leq (4/3)(n/2)^2 + n^2 && \text{since } \lfloor x \rfloor \leq x \text{ for any } x \\ &= n^2/3 + n^2 \\ &= (4/3)n^2, \end{aligned}$$

as required. It follows from the second principle of mathematical induction that $T(n) \leq \dfrac{4}{3}n^2$ for all $n \geq 1$. ■

5. (20 Points) Prove that for all $n \geq 1$: if T is a tree on n vertices, then T has $n-1$ edges. (Hint: you may use the following fact without proof: removing an edge from a tree results in exactly two trees, each with fewer vertices than the original.)

Proof:

I. There is only one tree with a single vertex and it has no edges. Therefore if T is a tree on 1 vertex, then T has $0 = 1 - 1$ edges. The base case is therefore satisfied.

II. Let $n > 1$ be chosen arbitrarily, and assume for all k in the range $1 \leq k < n$ that if T' is a tree on k vertices, then T' has $k - 1$ edges. We must show that if T is a tree on n vertices, then T has $n - 1$ edges.

Assume T is a tree on n vertices. Pick any edge $e \in E(T)$ and remove it. By the above hint, this removal results in two trees T_1 and T_2, each with fewer than n vertices. Say T_1 has $k_1 < n$ vertices and T_2 has $k_2 < n$ vertices. By the induction hypothesis T_1 has $k_1 - 1$ edges and T_2 has $k_2 - 1$ edges. Since no vertices were removed, we also have $k_1 + k_2 = n$. Therefore the number of edges in our original tree T must be

$$\begin{aligned}(\text{\# of edges in } T_1) + (\text{\# of edges in } T_2) + 1 &= (k_1 - 1) + (k_2 - 1) + 1 \\ &= (k_1 + k_2) - 1 \\ &= n - 1,\end{aligned}$$

as required. The result follows for all $n \geq 1$ by the 2nd Principle of Mathematical Induction. ■