Study Unit 6, sections 6-1-6-3

Activity 6.5

For each of the following relations, determine whether or not the relation is a strict partial order (irreflexive, antisymmetric and transitive) on the given set:

Let A = {a, {a}, {b}} and let S on A be the relation S = { (a, {a}), (a, {b})}.

Irreflexivity:

Is it the case that for all $x \in A$, $(x, x) \notin S$?
Yes. $(a, a) \notin S$, $(\{a\}, \{a\}) \notin S$ and $(\{b\}, \{b\}) \notin S$.

Antisymmetry:

If $(x, y) \in S$, is it the case that $(y, x) \notin S$?
Yes. (a, {a}) and (a, {b}) are the elements of S but $(\{a\}, a), (\{b\}, a) \notin S$.

Transitivity:

If $(x, y) \in S$ and $(y, z) \in S$, is it the case that $(x, z) \in S$?
No two ordered pairs in S are such that $(x, y) \in S$ and $(y, z) \in S$, so we need not find the pair (x, z) in S. S is thus transitive.
(We cannot prove that S is not transitive. Such a proof actually has a special name: it is vacuously true that S is transitive.)

Because S is irreflexive, antisymmetric and transitive we can say that S is a strict partial order.

(b) Define $R \subseteq (\mathbb{Z} \times \mathbb{Z}) \times (\mathbb{Z} \times \mathbb{Z})$ by (a, b) R (c, d) iff $a < c$.

Irreflexivity:

Is it the case that for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, $((a, b), (a, b)) \notin R$?
Yes. It is not true that $a < a$ therefore $((a, b), (a, b)) \notin R$.

Antisymmetry:

If $(a, b) \neq (c, d) \in R$ and $((a, b), (c, d)) \in R$, is it the case that $((c, d), (a, b)) \notin R$?

Yes. Suppose $(a, b) \neq (c, d)$ and (a, b) R (c, d), then $a < c$.

This means that it is not possible that $c < a$, therefore $((c, d), (a, b)) \notin R$.

Transitivity:

If $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$, is it the case that $((a, b), (e, f)) \in R$?

Suppose $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$

then $a < c$ and $c < e$

ie $a < e$

We can deduce that $((a, b), (e, f)) \in R$, thus R is transitive.

Because R is irreflexive, antisymmetric and transitive we can say that R is a strict partial order.