

## **Study Unit 6, Sections 6.1 – 6.3**

### **Activity 6-10**

1. Let  $X = \{a, b, c\}$ . Write down all equivalence relations on  $X$ .

Equivalence relations on  $X$  must be reflexive, symmetric and transitive:

$$\begin{aligned}R_1 &= \{(a, a), (b, b), (c, c)\}, \\R_2 &= \{(a, a), (b, b), (c, c), (a, b), (b, a)\}, \\R_3 &= \{(a, a), (b, b), (c, c), (a, c), (c, a)\}, \\R_4 &= \{(a, a), (b, b), (c, c), (b, c), (c, b)\},\end{aligned}$$

$$R_5 = \{(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (c, a), (b, c), (c, b)\},$$

2. In each of the following cases, determine whether or not the given relation is an equivalence relation. If it is, describe the equivalence class(es) of  $R$ . Justify your reasoning.

- (a)  $X = \{a, b, c\}$  and  $R = \{(c, c), (b, b), (a, a)\}$ :

#### **Reflexivity:**

$R$  is reflexive because for every  $x \in X$ , we have  $(x, x) \in R$ , as we can see by inspecting  $R$ . The ordered pairs  $(a, a)$ ,  $(b, b)$  and  $(c, c)$  are all present in  $R$ .

#### **Symmetry:**

$R$  is also symmetric, because there is no pair  $(x, y) \in R$ ,  $x \neq y$ , such that  $(y, x) \notin R$ , in other words, for every  $(x, y) \in R$  it is also the case that  $(y, x) \in R$ , since the first co-ordinate is equal to the second co-ordinate in all the ordered pairs belonging to  $R$ . Each pair in  $R$ , namely,  $(a, a)$ ,  $(b, b)$  and  $(c, c)$ , plays a double role; each plays the part of  $(x, y)$  as well as  $(y, x)$ .

#### **Transitivity:**

Is it the case that for all  $x, y, z \in X$ , if  $(x, y) \in R$  and  $(y, z) \in R$ , then  $(x, z) \in R$ ?

This is surely the case, because the only 2-step journeys are trivial ones like ‘From  $b$  go to  $b$ , and then go from  $b$  to  $b$ ’.

In fact, each pair in  $R$  plays a triple role; it plays the parts of  $(x, y)$ ,  $(y, z)$  and  $(x, z)$ .

To illustrate, let’s consider one specific example, say  $(b, b)$  in this triple role:

$$\begin{array}{ccc}x y & y z & x z \\ \text{ie } (b,b) & \text{and } (b,b) & \text{and } (b,b).\end{array}$$

This means that  $R$  is transitive.

$R$  is the equality (or identity) relation on  $X$ .

*What are the equivalence classes of  $R$ ?*

Because  $X$  has only three elements we can consider each element individually:

$$[a] = \{y \mid (a, y) \in R\}$$

$= \{a\}$ .

Similarly,  $[b] = \{b\}$  and  $[c] = \{c\}$ .

(b)  $X = \{a, b, c\}$  and  $R = X \times X$ :

It is easy to describe R in list notation because X has only three elements.

$R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$ .

### Reflexivity:

R is reflexive, because  $(a, a)$ ,  $(b, b)$  and  $(c, c)$  are all contained in R.

### Symmetry:

R is symmetric, because for each pair  $(x, y)$ , its mirror image  $(y, x)$  is also in R. This can be checked by inspecting R. We find  $(a, b) \in R$  and  $(b, a) \in R$ ;  $(a, c) \in R$  and  $(c, a) \in R$ ; and  $(b, c) \in R$  and  $(c, b) \in R$ . Furthermore the ordered pairs  $(a, a)$ ,  $(b, b)$  and  $(c, c)$  each play a double role, being itself and its own mirror image.

### Transitivity:

Scrutinising R very carefully we see that for all  $x, y, z \in X$ , if  $(x, y) \in R$  and  $(y, z) \in R$  then  $(x, z) \in R$ .

For example, among the various 2-step journeys found in R is  $(b, c)$  followed by  $(c, a)$ .

Since  $(b, a)$  is in R, the 2-step journey can be contracted to a single step.

We also have  $(a, a)$  and  $(a, b)$  in R, and  $(a, b)$  the single-step journey is there, playing a double role.

Similarly we find  $(b, a)$ ,  $(a, b)$  and  $(b, b)$ .

Can you spot **all** the other 2-step journeys?

**All** must be tested, and the associated single-step journeys must be found to be present, before we can confirm transitivity.

This can be done; thus R is transitive.

We can now conclude that R is an equivalence relation.

*What are the equivalence classes of R?*

$$\begin{aligned} [a] &= \{y \mid (a, y) \in R\} \\ &= \{a, b, c\}. \end{aligned}$$

We do not even bother to work out  $[b]$  and  $[c]$ , because b and c are both in  $[a]$ , so we know that  $[a] = [b] = [c] = X$ .

In other words, R says ‘All the elements of X are equivalent to one another’, and there is only one equivalence class.

(c)  $X = P(Y)$  where  $Y = \{1, 2, 3\}$  and

R consists of all pairs  $(C, D)$  such that  $C \cap \{2\} = D \cap \{2\}$ :

We can use a brute force approach to this problem, because X and R are small sets:

$P(Y) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$  and

we can work out what R is by noting that the only possible outcome of  $C \cap \{2\}$  is  $\emptyset$ , if  $2 \notin C$ ,  
and  $C \cap \{2\}$  is  $\{2\}$ , if  $2 \in C$ .

So all subsets of Y that do not contain the member 2 are related to one another by R, and all subsets of Y that do contain the element 2 are related to one another by R. Thus  
 $R = \{ (\emptyset, \emptyset), (\emptyset, \{1\}), (\emptyset, \{3\}), (\emptyset, \{1, 3\}), (\{1\}, \emptyset), (\{1\}, \{1\}), (\{1\}, \{3\}), (\{1\}, \{1, 3\}), (\{3\}, \emptyset), (\{3\}, \{1\}), (\{3\}, \{3\}), (\{3\}, \{1, 3\}), (\{1, 3\}, \emptyset), (\{1, 3\}, \{1\}), (\{1, 3\}, \{3\}), (\{1, 3\}, \{1, 3\}), (\{2\}, \{2\}), (\{2\}, \{1, 2\}), (\{2\}, \{2, 3\}), (\{2\}, Y), (\{1, 2\}, \{2\}), (\{1, 2\}, \{1, 2\}), (\{1, 2\}, \{2, 3\}), (\{1, 2\}, Y), (\{2, 3\}, \{2\}), (\{2, 3\}, \{1, 2\}), (\{2, 3\}, \{2, 3\}), (\{2, 3\}, Y), (Y, \{2\}), (Y, \{1, 2\}), (Y, \{2, 3\}), (Y, Y) \}.$

By inspection R is reflexive on X, symmetric, and transitive. So R is an equivalence relation.

The equivalence classes of R are

$$\begin{aligned} [\emptyset] &= \{ \emptyset, \{1\}, \{3\}, \{1, 3\} \} \text{ and} \\ [\{2\}] &= \{ \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\} \}. \end{aligned}$$

*Note:* Of course we could call  $[\emptyset]$  by the name  $[\{1\}]$  instead, or  $[\{2\}]$  by the name  $[\{2, 3\}]$  and so on.

3. Let R be the relation on  $\mathbb{Z}$  such that  $(x, y) \in R$  iff  $x - y$  is a multiple of 4.

We can define R as follows:  $(x, y) \in R$  iff  $x - y = 4k$  for some integer k.

(a)

#### Reflexivity:

Is it true that  $(x, x) \in R$  for all  $x \in \mathbb{Z}$ ?

Yes, for all  $x \in \mathbb{Z}$  we have  $x - x = 0 = 4 \cdot 0$ , which is a multiple of 4,  
thus R is reflexive on  $\mathbb{Z}$ .

#### Irreflexivity:

Is it the case that for all  $x \in \mathbb{Z}$ ,  $(x, x) \notin R$ ?

No, there is no integer x such that  $(x, x) \notin R$ .

We give a counterexample:

$(1, 1) \in R$  since  $1 - 1 = 0 = 4 - 0$ ,

thus R is not irreflexive.

#### Symmetry:

If  $(x, y) \in R$ , is it true that  $(y, x) \in R$ ?

Suppose  $(x, y) \in R$ , ie  $x - y$  is a multiple of 4,

ie  $x - y = 4k$  for some  $k \in \mathbb{Z}$ .

ie  $y - x = -(x - y) = -4k = 4(-k)$ .

So  $y - x$  is a multiple of 4, hence  $(y, x) \in R$ .

Thus R is symmetric.

#### Antisymmetry:

No, if  $(x, y) \in R$ , it is not necessarily true that  $(y, x) \notin R$ .

We give a counterexample:

$(5, 1) \in R$  since  $5 - 1 = 4$  which is a multiple of 4, but  $1 - 5 = -4$  which is also a multiple of 4, so  $(1, 5) \in R$ .

Thus  $R$  is not antisymmetric.

### Transitivity:

If  $(x, y) \in R$  and  $(y, z) \in R$ , is it true that  $(x, z) \in R$ ?

Suppose  $(x, y) \in R$ ,

then  $x - y = 4k$  for some  $k \in \mathbb{Z}$  ①

and suppose  $(y, z) \in R$ ,

then  $y - z = 4m$  for some  $m \in \mathbb{Z}$ . ②

① + ②, then  $(x - y) + (y - z) = 4k + 4m$

i.e.  $x - z = 4(k + m)$ , which is a multiple of 4, so  $(x, z) \in R$ .

Thus  $R$  is transitive.

### Trichotomy:

Is it true that for all positive integers  $x, y$  if  $x \neq y$ , then either  $(x, y) \in R$  or  $(y, x) \in R$ ?

No, it is not true! We show this by using a counterexample:

Choose  $1, 2 \in \mathbb{Z}$ ,

then  $2 - 1 = 1$  and  $1 - 2 = -1$  and these are not multiples of 4,

so  $(2, 1) \notin R$  and  $(1, 2) \notin R$ .

Thus  $R$  does not satisfy trichotomy.

(b) *What kind of relation is  $R$ ?*

Since  $R$  is reflexive on  $\mathbb{Z}$ , symmetric and transitive, it follows that  $R$  is an equivalence relation.

(c)  *$R$  is an equivalence relation, so we can give the equivalence classes of  $R$ :*

$$[x] = \{y \mid (x, y) \in R\}$$

We know  $(v, w) \in R$  iff  $v - w = 4k$ , therefore  $[x] = \{y \mid x - y = 4k \text{ for some } k \in \mathbb{Z}\}$

$$[0] = \{y \mid 0 - y = 4k \text{ for some } k \in \mathbb{Z}\}$$

$$= \{y \mid y = -4k\}$$

$$= \{\dots, -8, -4, 0, 4, 8, \dots\}$$

$$[1] = \{y \mid 1 - y = 4k \text{ for some } k \in \mathbb{Z}\}$$

$$= \{y \mid y = -4k + 1\}$$

$$= \{\dots, -3, 1, 5, 9, \dots\}$$

$$[2] = \{y \mid 2 - y = 4k \text{ for some } k \in \mathbb{Z}\}$$

$$= \{y \mid y = -4k + 2\}$$

$$= \{\dots, -2, 2, 6, 10, \dots\}$$

$$[3] = \{y \mid 3 - y = 4k \text{ for some } k \in \mathbb{Z}\}$$

$$= \{y \mid y = -4k + 3\}$$

$$= \{\dots -1, 3, 7, 11, \dots\}$$

$[-4]$ ,  $[4]$ , etc. are identical to  $[0]$ ,  
similarly  $[-3]$ ,  $[5]$ , etc. are the same as  $[1]$ ,  
similarly  $[-2]$ ,  $[6]$ , etc. are the same as  $[2]$ ,  
and similarly  $[-1]$ ,  $[7]$ , etc. are the same as  $[3]$ .

So  $R$  has four different equivalence classes namely,  $[0]$ ,  $[1]$ ,  $[2]$  and  $[3]$ .

4. Suppose  $\mathbb{Q}^+$  is the set of all positive quotients  $n/m$ , where  $n, m \in \mathbb{Z}^+$   
ie  $\mathbb{Q}^+$  is the set of positive rational numbers.

Let  $R$  be the relation on  $\mathbb{Q}^+$ , defined by  $(x, y) \in R$  iff  $y = (a \cdot x) / b$  for some  $a, b \in \mathbb{Z}^+$ .

Prove that  $R$  is an equivalence relation and describe the equivalence classes of  $R$ .

We can get the 'feel' of a relation by writing down some of its members. Let us do this with  $R$ :

The members of  $R$  are ordered pairs of positive rational numbers, such as  $(1/2, 3/5)$ ,  
ie  $x = 1/2$  and  $y = 3/5$ .

Does this pair meet the entrance requirement for  $R$ , namely that  $y = (a \cdot x)/b$ ?

Yes, where  $a = 6$  and  $b = 5$ .

$$\begin{array}{cccc} y & a & x & b \end{array}$$

(Test it to check:  $3/5 = (6 \cdot (1/2)) / 5$ .)

Another member of  $R$  is  $(4, 5)$ .

(It meets the requirement  $y = (a \cdot x) / b$ , because  $5 = (5 \cdot 4) / 4$ , where  $a = 5$  and  $b = 4$ .)

Now we want to prove that  $R$  is an equivalence relation:

In this kind of proof, we often need to determine whether a certain ordered pair belongs to  $R$ . Say, for example, we need to show that  $(x, y) \in R$ . We must then demonstrate that **x and y meet the requirements of the definition of R**, ie that  $y = a \cdot x / b$ . (Make sure you use the appropriate sequence for  $x$  and  $y$ !)

In order to be an equivalence relation,  $R$  must be reflexive on  $\mathbb{Q}^+$ , symmetric and transitive. This gives our 'agenda'.

### Reflexivity:

**Goal:** to show that for every  $x \in \mathbb{Q}^+$ ,  $(x, x) \in R$ .

Let us relate the definition of reflexivity to the definition of the specific relation  $R$  on  $\mathbb{Q}^+$ , ie to show that  $(x, x) \in R$ , we must show that  $x = (a \cdot x) / b$  for some  $a, b \in \mathbb{Z}^+$ .)

For all  $x \in \mathbb{Q}^+$ , we know that  $x = x$ ,

ie  $x = 1 \cdot x / 1$  and  $1 \in \mathbb{Z}^+$ .

Thus  $(x, x) \in R$ , and so  $R$  is reflexive on  $\mathbb{Q}^+$ .

### Symmetry:

**Goal:** We assume that  $(x, y) \in R$ , ie  $y = (a \cdot x) / b$ , and we want to **use** this to demonstrate that  $(y, x) \in R$ .

Suppose  $(x, y) \in R$ ,

then  $y = (a \cdot x) / b$  for some  $a, b \in \mathbb{Z}^+$

ie  $b \cdot y = a \cdot x$

ie  $(b \cdot y) / a = x$

ie  $x = (b \cdot y) / a$ .

Thus  $(y, x) \in R$  and so  $R$  is symmetric.

### Transitivity:

**Goal:** We assume that  $(x, y) \in R$ , ie  $y = (a \cdot x) / b$  and that  $(y, z) \in R$ ,

ie  $z = (c \cdot y) / d$ , and then set out to **use** these facts to prove that  $(x, z) \in R$ .

Suppose  $(x, y) \in R$ , then  $y = (a \cdot x) / b$  ① for some  $a, b \in \mathbb{Z}^+$ , and

suppose  $(y, z) \in R$ , then  $z = (c \cdot y) / d$  ② for some  $c, d \in \mathbb{Z}^+$ .

Substitute ① into ②, then

$$z = c \cdot (a \cdot x / b) / d$$

$$\text{ie } z = (ca \cdot x) / bd$$

$$\text{ie } z = (e \cdot x) / f \quad \text{where } e = ca \text{ and } f = bd \text{ for some } e, f \in \mathbb{Z}^+.$$

Thus  $(x, z) \in R$  and so  $R$  is transitive.

Since  $R$  is reflexive on  $\mathbb{Q}^+$ , symmetric and transitive,  $R$  is an equivalence relation.

Next we look at the equivalence classes of  $R$ :

**Note:** Remember that equivalence classes are determined by considering sets of the following format:

$$[x] = \{y \mid (x, y) \in R\} \text{ for all } x \in \mathbb{Q}^+.$$

In this case it means that:

$$[x] = \{y \mid y = (a \cdot x) / b\}$$

Consider  $x = 1$  in the above equation.

$$[1] = \{y \mid y = (a \cdot 1) / b\}$$

$$= \{y \mid y = a/b\}$$

This is the set of all positive rational numbers, for example,  $[1] = [2] = [1/2] = [3/4] = \dots$  etc.

In this example, each equivalence class is equal to every other equivalence class, so there is only one equivalence class in  $R$ .

5. Prove that if  $R$  is a relation on  $\mathbb{Z}^+$ , then  $R$  is symmetric iff  $R = R^{-1}$ .

Let us first try to prove that if  $R$  is symmetric, then  $R = R^{-1}$ .

Assume  $R$  is symmetric. We want to show that  $R = R^{-1}$ .

Suppose  $(x, y) \in R$ ,

then  $(y, x) \in R$  because  $R$  is symmetric.

So  $(x, y) \in R^{-1}$  according to the definition of  $R^{-1}$ .

Hence  $R \subseteq R^{-1}$ .

Conversely, suppose  $(x, y) \in R^{-1}$ ,  
then  $(y, x) \in R$ , and because  $R$  is symmetric,  $(x, y) \in R$ .  
Hence  $R^{-1} \subseteq R$ .

Since  $R \subseteq R^{-1}$  and  $R^{-1} \subseteq R$ , we can conclude that  $R = R^{-1}$ .

Next we assume that  $R = R^{-1}$ . Now we need to show that  $R$  is symmetric, ie that if  $(x, y) \in R$ , then  $(y, x) \in R$ .

Suppose  $(x, y) \in R$ ,  
then  $(y, x) \in R^{-1}$ .  
But because  $R = R^{-1}$ ,  $(y, x) \in R$ .  
Therefore  $R$  is symmetric.

This completes the proof that if  $R$  is a relation on  $Z^+$ , then  $R$  is symmetric iff  $R = R^{-1}$ .