Study Unit 6, Sections 6.1 to 6.3

Activity 6-7

1. *Let X = {a,b,c}. Write down all strict partial orders on X. Which of them are linear?*

Strict partial orders on X are irreflexive, antisymmetric and transitive. The strict partial orders on X are:
∅, { (a,b) }, { (a,c) }, { (b,a) }, { (b,c) }, { (c,a) }, { (c,b) },
{ (a,b), (a,c) }, { (a,b), (c,b) }, { (a,c), (b,c) }, { (b,a), (b,c) }, { (b,a), (c,a) }, { (c,a), (c,b) },
{ (a,b), (b,c), (a,c) }, { (b,a), (a,c), (b,c) }, { (b,c), (c,a), (b,a) }, { (a,c), (c,b), (a,b) }, { (c,a), (a,b), (c,b) }, and
{ (c,b), (b,a), (c,a) }.
All the relations containing three elements satisfy trichotomy and are therefore linear.
Note: How do we know we have found all the strict partial orders on X? Well, we were systematic. We wrote down those with one pair then we listed all the ways to add another pair without losing properties like transitivity. Lastly we listed the relations containing 3 pairs, and each of these can be viewed as a 2-step journey together with the contraction of that journey required by transitivity. It is not possible to have more than 3 pairs without losing irreflexivity or antisymmetry.

2. *In each of the following cases, determine whether or not R is some sort of order relation on the given set X (weak partial, weak total, strict partial, or strict total). Justify your answer.*

To determine whether R is some sort of order relation on X, we have to examine the relevant properties of R in each case.

(a) X = { ∅, {0}, {2} } and R = { (∅, {0}), (∅, {2}) }:

Reflexivity:
R is not reflexive so we provide a counterexample:
(∅, ∅) ∉ R (We also have that ({0}, {0}) ∉ R and ({2}, {2}) ∉ R.)

Irreflexivity:
For all x ∈ X we have that (x, x) ∉ R:
(∅, ∅) ∉ R, ({0}, {0}) ∉ R and ({2}, {2}) ∉ R.
Hence R is irreflexive.

Antisymmetry:
R is antisymmetric because the mirror images of (∅, {0}) and (∅, {2}) are not in R.

Transitivity:
R is transitive because it does not contain any members with first co-ordinates
{0} and {2}, so there are no 2-step journeys to worry about.

(Does this proof bother you? If so, remember the definition of transitivity, '**if** (x, y) $\in$R and (y, z) $\in$R, **then** ...'. When the **if** part does not apply, we have no more work to do!)

Trichotomy:
There is no ordered pair comparing {0} and {2}.
So R does not satisfy trichotomy.
R is a strict partial order on X because R is irreflexive, antisymmetric and transitive.

(b) *X = {∅, {∅}, {{∅}}} and R = $\subseteq$ (R is the relation of all ordered pairs where each first co-ordinate is a subset of the second co-ordinate, and R $\subseteq$ X $\times$ X.)*
For example, ∅ $\subseteq$ {∅}, so (∅, {∅})$\in$ $\subseteq$ or (∅, {∅})$\in$R.
Because X has only three elements we can describe R in list notation.
$\subseteq$ = { (∅, ∅), (∅, {∅}), (∅, {{∅}}), ({∅}, {∅}), ({{∅}}, {{∅}}) }.

Note: ({∅}, {{∅}})$\notin$ $\subseteq$ because {∅} is not a subset of {{∅}}. *{∅} is not a subset of {{∅}} because each element of {∅} is not an element of {{∅}} – the only element of {∅} is ∅ and the only element of {{∅}} is {∅}. We see that ∅ $\in$ {∅} but ∅ $\notin$ {{∅}} thus {∅} is not a subset of {{∅}}.*

Reflexivity:
It is clear that for every x $\in$ X, (x, x) $\in$ R: (∅, ∅), ({∅}, {∅}) and ({{∅}}, {{∅}}) are all members of R. So R (or $\subseteq$, if you prefer) is reflexive on X.

Antisymmetry:
By inspection of R it is clear that for all x, y $\in$ X, if x ≠ y and (x, y) $\in$ R then (y, x) $\notin$ R.
Or to say the same thing in different words, whenever both x $\subseteq$ y and y $\subseteq$ x, then x = y.
So R (or $\subseteq$, if you prefer) is antisymmetric.

Transitivity:
A typical 2-step journey is (∅, {∅}) followed by ({∅}, {∅}), and its corresponding 1-step journey (∅, {∅}) is also in R, playing a double role.
Clearly every 2-step journey in R can be performed in a single step.
Can you find them all?
Therefore R (or $\subseteq$, if you prefer) is transitive.

Irreflexivity:
There are several x in X such that (x, x) $\in$ $\subseteq$, for example x = ∅, where (∅, ∅) $\in$ $\subseteq$.
So $\subseteq$ is not irreflexive.

Trichotomy:
$\subseteq$ does not satisfy trichotomy, because it is not the case that for all x, y$\in$ X, if x ≠ y then (x, y) $\in$ $\subseteq$ or (y, x) $\in$ $\subseteq$, for example, there is no ordered pair containing both {∅} and {{∅}}.

Because $\subseteq$ is reflexive, antisymmetric and transitive, it is a weak partial order.

(c) $X = \mathbb{Z}$ and $R = \leq$:

Reflexivity:
For all $x \in \mathbb{Z}$, $(x, x) \in \leq$, because $x \leq x$ for every integer x.
So R is reflexive.

Antisymmetry:
For any $x, y \in \mathbb{Z}$, if $x \neq y$ and $(x, y) \in \leq$ then $x \leq y$.
Therefore it is not the case that $y \leq x$. So $(y, x) \notin \leq$.
So $\leq$ is antisymmetric.
Another way to say the same thing is that if $x \leq y$ and, at the same time, $y \leq x$, then it must be the case that $x = y$, because if the value of x appears to the left of the value of y on the number line and the value of y appears to the left of that of x, then they must lie on the same position.

Transitivity:
If $(x, y) \in \leq$ and $(y, z) \in \leq$ for any $x, y, z \in \mathbb{Z}$, then $x \leq y$ and $y \leq z$,
ie x appears to the left of y and y appears to the left of z.
Thus x appears to the left of z, ie $x \leq z$, ie $(x, z) \in$ v. Therefore $\leq$ is transitive.

Irreflexivity:
$\leq$ is not irreflexive, because we can find values of x such that $(x, x) \in \leq$,
for example $x = 113$, or if you prefer simple values, $x = 2$.

Trichotomy: For all $x, y \in \mathbb{Z}$, if $x \neq y$ then either $x > y$ or $y > x$,
because the one must appear to the left of the other on the number line.
Therefore either $(x, y) \in \leq$ or $(y, x) \in \leq$. Hence $\leq$ satisfies trichotomy.
Because $\leq$ is reflexive, antisymmetric and transitive, it is a weak partial ordering,
and because $\leq$ is a weak partial ordering satisfying trichotomy, it is also a weak total (linear) ordering.

(d) $X = \mathbb{Z}$ and $R = >$:

Reflexivity:
There are values of x such that $(x, x) \notin >$, since if $x = 113$, say, then it is not the case that $x > x$. So $>$ is not reflexive on $\mathbb{Z}$.

Antisymmetry:
If $(x, y) \in >$, then $x > y$, ie x lies to the right of y on the number line.
Thus it cannot be the case that $y > x$.
So $(y, x) \notin >$, and $>$ is therefore antisymmetric.

Transitivity:

Suppose $(x, y) \in >$ and $(y, z) \in >$, ie $x > y$ and $y > z$.
This means that x lies to the right of y and y to the right of z on the number line.
Therefore x lies to the right of z, ie $x > z$, ie $(x, z) \in >$.
Thus $>$ is transitive.

Irreflexivity:
For all $x \in \mathbb{Z}$ it is not the case that x lies to the right of itself, ie it will never be the case that $x > x$.
So $(x, x) \notin >$ for all $x \in \mathbb{Z}$, and hence $>$ is irreflexive.

Trichotomy:
For all $x, y \in \mathbb{Z}$, if $x \neq y$ then either x lies to the right of y (ie $x > y$) or x lies to the left of y (ie $y > x$). So either $(x, y) \in >$ or $(y, x) \in >$. Therefore $>$satisfies trichotomy.
We can conclude that $>$ is a strict partial order relation because it is irreflexive, antisymmetric and transitive. What is more, $>$ is a strict linear ordering because it satisfies trichotomy as well.
Note: Any linear order is also a partial order, but not vice versa.

(e) *$X = \mathbb{Z}^+$ and R is defined by: x R y iff x divides y with zero remainder, ie $y = kx$ for some $k \in \mathbb{Z}^+$.*
(x R y is another way of saying $(x, y) \in R$.)

This means that x is a factor of y and y is a multiple of x.
Let us synthesize some ordered pairs that belong to R:
How about (2, 6), (3, 6), (5, 35) and (4, 24)?
All of these meet the requirement that $y = kx$ for some $k \in \mathbb{Z}^+$.

Reflexivity:
For each $x \in \mathbb{Z}^+$ we have that $x = 1x$ and $1 \in \mathbb{Z}^+$, so $(x, x) \in R$.
R is therefore reflexive on $\mathbb{Z}^+$.

Antisymmetry:
Suppose $x \neq y$ and $(x, y) \in R$.
Can (y, x) qualify to belong to R?
If $(x, y) \in R$, $y = kx$ ① for some $k \in \mathbb{Z}^+$.

Does it ever happen that $(y, x) \in R$, ie $x = my$ ② for some $m \in \mathbb{Z}^+$?

Substitute ② into ①:

$y = kx = k(my) = (km)y$, ie $y = (km)y$, which means $km=1$.
Hence $k = m = 1$, so $x = y$.
But we specifically assumed that $x \neq y$,
so it can never happen that $(y, x) \in R$, which means that $(y, x) \notin R$.
Therefore R is antisymmetric.

Transitivity:
Suppose $(x, y) \in R$ and $(y, z) \in R$.
Then $y = kx$ for some $k \in \mathbb{Z}^+$.
and $z = my$ for some $m \in \mathbb{Z}^+$.
Hence $z = my = m(kx) = (mk)x$,
ie $(x, z) \in R$.
Thus R is transitive.

Irreflexivity:
Since we can find values of x such that $(x, x) \in R$,
for example $x = 113$, where $(113, 113) \in R$,
R cannot be irreflexive.

Trichotomy:
R does not satisfy trichotomy.
Take $x = 2$ and $y = 3$, then there do not exist some $k, m \in Z^+$ such that $3 = k(2)$ or $2 = m(3)$, so neither
$(2, 3) \in R$ nor $(3, 2) \in R$.
Thus R is a weak partial order on $\mathbb{Z}^+$.