Study Unit 5

Activity 5-8

1. *Let P and R be relations on A = { 1, 2, 3, {1}, {2} } given by*
P = { (1, {1}), (1, 2) } and
R = { (1, {1}), (1, 3), (2, {1}), (2, {2}), ({1}, 3), ({2}, {1}) }. Investigate the following:

(a) **Irreflexivity:**
Is it the case that for all $x \in A$, $(x, x) \notin R$?
Yes, R is irreflexive, since the first and second co-ordinates differ from each other in each ordered pair of R. We do not have all the following as elements of R: (1, 1), (2, 2), (3, 3), ({1}, {1}), ({2}, {2}).

(b) **Reflexivity:**
Is it the case that for all $x \in A$, $(x, x) \in R$?
No, R is not reflexive, we give a counterexample:
$(1, 1) \notin R$.

(c) **Symmetry:**
If $(x, y) \in R$, is it the case that $(y, x) \in R$?
No, R is not symmetric. We give a counterexample:
$(1, \{1\}) \in R$, but its mirror image, $(\{1\}, 1) \notin R$.

(d) **Antisymmetry:**
If $x \neq y$ and $(x, y) \in R$, is it the case that $(y, x) \notin R$?
Yes, R is antisymmetric, since no member of R has its mirror image also living in R.
For each ordered pair (x, y) living in R, we have that $(y, x) \notin R$.

(e) **Transitivity:**
If $(x, y) \in R$ and $(y, z) \in R$, is it the case that $(x, z) \in R$?
No, R is not transitive. We give a counterexample:
$(2, \{1\}) \in R$, and $(\{1\}, 3) \in R$, but $(2, 3) \notin R$.

(f) **Trichotomy**:
Is it the case for all $x, y \in A$, if $x \neq y$ then $(x, y) \in R$ or $(y, x) \in R$?
No, R does not satisfy trichotomy. We give a counterexample:
$(2, 3) \in R$ and also $(3, 2) \in R$.

This means that $2 \neq 3$ but we cannot compare the elements 2 and 3 of A in terms of R because these 2 elements do not appear together in any ordered pair of R.

(g) **R ○ R:** $(x, w) \in R \circ R$ iff for some y there exist pairs $(x, y) \in R$ and $(y, w) \in R$.
R = { (1, {1}), (1, 3), (2, {1}), (2, {2}), ({1}, 3), ({2}, {1}) }

We have that $(1, \{1\}) \in R$ and $(\{1\}, 3) \in R$, hence $(1, 3) \in R \circ R$.

Also: $(2, \{1\}) \in R$ and $(\{1\}, 3) \in R$, hence $(2, 3) \in R \circ R$.

$(2, \{2\}) \in R$ and $(\{2\}, \{1\}) \in R$, hence $(2, \{1\}) \in R \circ R$.

$(\{2\}, \{1\}) \in R$ and $(\{1\}, 3) \in R$, hence $(\{2\}, 3) \in R \circ R$.

Thus $R \circ R = \{ (1, 3), (2, 3), (2, \{1\}), (\{2\}, 3) \}$.

(h) **$R \circ P$:** $(x, w) \in R \circ P$ iff for some y there exist pairs $(x, y) \in P$ and $(y, w) \in R$.

We have that $(1, \{1\}) \in P$ and $(\{1\}, 3) \in R$, hence $(1, 3) \in R \circ P$.

Also: $(1, 2) \in P$ and $(2, \{1\}) \in R$, hence $(1, \{1\}) \in R \circ P$.

$(1, 2) \in P$ and $(2, \{2\}) \in R$, hence $(1, \{2\}) \in R \circ P$.

Thus $R \circ P = \{ (1, 3), (1, \{1\}), (1, \{2\}) \}$.

(i) **T:** T is a subset of R, so T is also a relation on A.
We have $(a, B) \in T$ iff $a \in B$.
In each ordered pair in T, the first co-ordinate must be a member of the second co-ordinate.
We have $T = \{ (1, \{1\}), (2, \{2\}) \}$.

2. *Let $A = \{a, b\}$. For each of the specifications given below, find suitable examples of relations on $\mathcal{P}(A)$.*

First of all let us write down $\mathcal{P}(A)$:

$\mathcal{P}(A) = \boldsymbol{\{} \emptyset, \{a\}, \{b\}, \{a,b\} \boldsymbol{\}}$,

and $\mathcal{P}(A) \times \mathcal{P}(A) = \boldsymbol{\{} (\emptyset, \emptyset), (\emptyset, \{a\}), (\emptyset, \{b\}), (\emptyset, \{a,b\}), (\{a\}, \emptyset), (\{a\}, \{a\}), (\{a\}, \{b\}), (\{a\}, \{a,b\}), (\{b\}, \emptyset), (\{b\}, \{a\}), (\{b\}, \{b\}), (\{b\}, \{a,b\}), (\{a,b\}, \emptyset), (\{a,b\}, \{a\}), (\{a,b\}, \{b\}), (\{a,b\},\{a,b\}) \boldsymbol{\}}$.

(a) R is reflexive on $\mathcal{P}(A)$, symmetric, and transitive:

Two examples of relations that satisfy this specification are $\boldsymbol{\{} (\emptyset, \emptyset), (\{a\}, \{a\}), (\{b\}, \{b\}), (\{a,b\}, \{a,b\}) \boldsymbol{\}}$ and $\boldsymbol{\{} (\emptyset, \emptyset), (\{a\}, \{a\}), (\{b\}, \{b\}), (\{a,b\}, \{a,b\}), (\emptyset, \{a\}), (\{a\}, \emptyset) \}$.

(b) R is reflexive on $\mathcal{P}(A)$ and symmetric, but not transitive:

$\boldsymbol{\{} (\emptyset, \emptyset), (\{a\}, \{a\}), (\{b\}, \{b\}), (\{a,b\}, \{a,b\}), (\emptyset, \{a\}), (\{a\}, \emptyset), (\{a\}, \{a,b\}), (\{a,b\}, \{a\}) \boldsymbol{\}}$

This relation is not transitive because it contains both $(\emptyset, \{a\})$ and $(\{a\}, \{a,b\})$ but not $(\emptyset, \{a,b\})$. Another counterexample to transitivity is that both $(\{a,b\}, \{a\})$ and $(\{a\}, \emptyset)$ belong to the relation, but not $(\{a,b\}, \emptyset)$.

(c) R is reflexive on $\mathcal{P}(A)$, transitive, but is not symmetric and not antisymmetric:

{ ($\emptyset$, $\emptyset$), ({a}, {a}), ({b}, {b}), ({a,b}, {a,b}), ($\emptyset$, {a}), ($\emptyset$, {b}), ({a}, {b}), ({a,b}, $\emptyset$), ({a,b}, {a}), ({a,b}, {b}), ({b}, {a}) **}**

(d) R is simultaneously symmetric and antisymmetric:
{ ($\emptyset$, $\emptyset$), ({a}, {a}), ({b}, {b}), ({a,b}, {a,b}) **}**

(e) R is irreflexive, antisymmetric, transitive:
{ ($\emptyset$, {a}), ($\emptyset$, {b}), ($\emptyset$, {a,b}), ({a}, {b}), ({a}, {a,b}), ({b}, {a,b}).}

3. *Prove that if R is a relation on X, then R is transitive iff $R \circ R \subseteq R$.*

First we attempt to prove that if R is transitive, then we prove that $R \circ R \subseteq R$.

Assume R is transitive.

Suppose $(x, z) \in R \circ R$, then, according to the definition of composition, there exists some
$y \in X$ such that $(x, y) \in R$ and $(y, z) \in R$.
Because R is transitive, it follows that $(x, z) \in R$.
This completes the proof that if R is transitive, then $R \circ R \subseteq R$.

Now we have to prove the converse.
Assume $R \circ R \subseteq R$.

Suppose $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R \circ R$ according to the definition of composition.
Because $R \circ R \subseteq R$, it follows that $(x, z) \in R$.
Therefore if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.
Hence R is transitive.

We can now conclude that if R is a relation on X, then R is transitive iff $R \circ R \subseteq R$.