**Study Unit 10**

**Activity 10-10**

1. *Prove each of the following by direct proof, contrapositive, and contradiction (reductio ad absurdum) respectively. Which strategy worked best?*

(a) If $x^2 - 3x + 2 < 0$ then $x > 0$.

**Direct proof:**
Suppose $x^2 - 3x + 2 < 0$,
then $(x - 1)(x - 2) < 0$
The result is < 0, so one factor must be < 0 and the other > 0.
ie either $x - 1 < 0$ and simultaneously $x - 2 > 0$
OR else $x - 1 > 0$ and simultaneously $x - 2 < 0$
ie either $x < 1$ and simultaneously $x > 2$ (which is impossible)
OR else $x > 1$ and simultaneously $x < 2$ ie $1 < x < 2$
We can conclude that $1 < x < 2$
Thus $x > 0$.

**Contrapositive:**
Suppose $x \leq 0$,
then $x^2 \geq 0$ and $-3x \geq 0$ (A minus times a minus, remember?)
thus $x^2 - 3x + 2 \geq 0$ (The sum of non-negative numbers is also $\geq 0$).

**Contradiction (*Reductio ad absurdum*):**
Suppose $x^2 - 3x + 2 < 0$,
then $(x - 1)(x - 2) < 0$
and this means that one of the factors is < 0 and the other is > 0.

Now there are just 2 possibilities for x: either $x > 0$ or $x \leq 0$.
The former is the good possibility, so let us try to eliminate $x \leq 0$.

Suppose, just for the moment, that $x \leq 0$ (bad possibility).
Then $x - 1 < 0$ and $x - 2 < 0$, but this contradicts the original fact that one of these factors must be > 0, so we can discard the bad possibility.
Thus we conclude that $x > 0$.

Which worked best? Well, contrapositive was the shortest, but the one that worked best is the one you feel most comfortable with. You decide. (Most beginners feel safest with direct proof; after a year or two of practice, many people grow to love *reductio ad absurdum* (contradiction), perhaps because so many of the opinions you meet in daily life lead to some absurd conclusion!)

(b) If $x^2 - x - 6 > 0$ then $x \neq 1$:

**Direct proof:**
Suppose $x^2 - x - 6 > 0$,
then $(x - 3)(x + 2) > 0$
ie either $x - 3 < 0$ and $x + 2 < 0$ (both factors are negative)
OR
$x - 3 > 0$ and $x + 2 > 0$ (both factors are positive)
ie either $x < 3$ and $x < -2$, ie $x < -2$
OR
$x > 3$ and $x > -2$, ie $x > 3$
ie $x < -2$ or $x > 3$
Thus $x \neq 1$.

**Contrapositive:**
Suppose $x = 1$,
then $x^2 - x - 6 = 1 - 1 - 6 = -6$
ie $x^2 - x - 6 \leq 0$

**Contradiction:**
Suppose $x^2 - x - 6 > 0$.

Now there are just 2 possibilities for x: either $x = 1$ or $x \neq 1$.
The latter is the good possibility, so let us eliminate $x = 1$.

Suppose, just for the moment, that $x = 1$.
Then $x^2 - x - 6 = 1-1-6 = -6$
ie $x^2 - x - 6 < 0$. But this contradicts our starting assumption.

Hence we conclude that $x \neq 1$.

The easiest? Direct proof, for most people.

(c) *If a + b is odd, exactly one of a and b is odd.*

**Direct proof:**
Suppose a + b is odd (assume $a, b \in \mathbb{Z}$),
then $a + b = 2n + 1$ for some integer n.
Now there are exactly 2 cases: a is either even or odd.
*Case 1:* Suppose a is even,
then $a = 2k$ for some integer k
so $b = (a + b) - a$
$= 2n + 1 - 2k$
$= 2(n - k)+1$
Thus b is odd.

*Case 2:* Suppose a is odd,

then $a = 2k + 1$ for some integer k.

Now $b = (a + b) - a$

$$= 2n + 1 - (2k + 1)$$
$$= 2(n - k)$$

Thus b is even.

(Note that we have considered the two cases; not in order to eliminate one of them, as we would in a proof by contradiction, but in order to show that **in both cases** exactly one of a and b is odd.)

**Contrapositive:**

Suppose a and b are both even or both odd,
then either $a = 2m$ and $b = 2k$ for some integers m and k,
or else $a = 2m + 1$ and $b = 2k + 1$ for some $m, k \in Z$.
So either $a + b = 2m + 2k = 2(m + k)$
which means that $a + b$ is even,
or else $a + b = (2m + 1) + (2k + 1) = 2m + 2k + 2 = 2(m + k + 1)$
which also means that $a + b$ is even.

**Contradiction:**

Suppose $a + b$ is odd.

There are exactly two possibilities:
either it is the case that exactly one of a and b is odd, or this is not the case.
The former is the good possibility, so we eliminate the latter.

Suppose, just for a moment, that it is **not** the case that exactly one of a and b is odd,
then it may be that a and b are both odd, or else it may be that a and b are both even.

Suppose that a and b are both odd,
then we may write $a = 2m+1$ and $b = 2k + 1$ for some integers m and k.
Now $a + b = 2m + 2k + 2 = 2(m + k + 1)$, ie an even number, contradicting the fact that $a + b$ is odd.

On the other hand, suppose that a and b are both even, ie $a = 2m$ and $b = 2k$, for some integers m and k. Then $a + b = 2(m + k)$, ie an even number, also contradicting the fact that $a + b$ is odd.
So exactly one of a and b must be odd.
(Part of this proof is rather similar to the proof by contrapositive.)

The easiest method? Maybe proof by contrapositive.

(d) *If x is even then $x^2 + 4x + 2$ is even (assume $x \in \mathbb{Z}$).*

**Direct Proof:**
Suppose x is even. Then $x = 2k$ for some integer k.
So $x^2 + 4x + 2 = (2k)^2 + 4(2k) + 2$
$= 4k^2 + 8k + 2$
$= 2(2k^2 + 4k + 1)$
Thus $x^2 + 4x + 2$ is even.

**Contrapositive:**
Suppose $x^2 + 4x + 2$ is odd.

So $x^2 + 4x + 2 = 2m + 1$ for some integer m.
ie $x^2 + 4x + 4 = 2m + 2 + 1$ (completing the square)
ie $(x + 2)(x + 2) = 2(m + 1) + 1$
This means that $x + 2$ must be odd. (Product of two integers is odd, means both integers must be odd.)

Write this as $x + 2 = 2k + 1$ for some integer k.
ie $x = 2(k – 1) + 1$, which means that x is odd.
We can conclude that, if x is even then $x^2 + 4x + 2$ is even.

**Contradiction:**
Suppose x is even.
Then $x = 2k$ for some integer k.

There are two possibilities: either $x^2 + 4x + 2$ is even (the good possibility) or it is odd.
Let's eliminate the bad possibility. Assume $x^2 + 4x + 2$ is odd.
Then $x^2 + 4x + 2 = 2m + 1$ for some integer m.
ie $x^2 + 4x + 4 = 2m + 2 + 1$ (completing the square)
ie $(x + 2)(x + 2) = 2(m + 1) + 1$
This means that $x + 2$ must be odd. (Product of two integers is odd, means both integers must be odd.)
Write this as $x + 2 = 2k + 1$ for some integer k.
ie $x = 2(k – 1) + 1$, which means that x is odd.
This contradicts our initial assumption.
We can conclude that $x^2 + 4x + 2$ is even.

(e) *If n is a multiple of 3 then $n^3 + n^2$ is a multiple of 3.*

**Direct Proof:**
Suppose n is a multiple of 3.
Then $n = 3k$ for some integer k.
So $n^3 + n^2 = (3k)^3 + (3k)^2$
$= 3(9k^3) + 3(3k^2)$
$= 3(9k^3 + 3k^2)$

It follows that $n^3 + n^2$ is a multiple of 3.

**Contrapositive:**
Suppose $n^3 + n^2$ is a not a multiple of 3
Then $n(n^2 + n)$ can be written as $3k + 1$, or $3k + 2$ for some integer k.
Let's look at the first alternative:
$n(n^2 + n) = 3k + 1$
This means that both n and $(n^2 + n)$ are not multiples of 3.
We can conclude that n is not a multiple of 3.

**Contradiction:**
Suppose n is a multiple of 3.
Then $n = 3k$ for some integer k.

Now there are two possibilities: Either $n^3 + n^2$ is a multiple of 3 (the good possibility) or $n^3 + n^2$ is not a multiple of 3.

Let's eliminate the bad possibility, so we assume that $n^3 + n^2$ is not a multiple of 3
then $n(n^2 + n)$ can be written as $3k + 1$, or $3k + 2$ for some integer k.
Let's look at the first alternative:
$n(n^2 + n) = 3k + 1$
This means that both n and $(n^2 + n)$ are not multiples of 3.

We can conclude that n is not a multiple of 3.
But this contradicts our initial assumption.
We can conclude that $n^3 + n^2$ is a multiple of 3.

2. *Provide a counterexample to show that the following is not true for all integers $x > 0$:*
If $x > 0$, then $x^2 - 3x + 1 < 0$.
Choose $x = 3$.
Then $(3)^2 – 3(3) + 1 = 9 – 9 + 1$, which is greater than 0.