Study Unit 7

Activity 7-12

1. *Write down the bijective (one-to-one correspondence) functions from X to Y in each case:*

(a) $X = \{ \emptyset, \{113\} \}$ and $Y = \{ \{1\} \}$:
To obtain a bijective function from X to Y we must fill in the template
$\{ (\emptyset, \quad), (\{113\}, \quad) \}$
in such a way that different pairs get different elements of Y (for injectivity) and all elements of Y are used up (for surjectivity).
This is not possible, since Y has only one element and there are two pairs needing *different* second co-ordinates.
So there is no bijective function from X to Y in this case.

(b) $X = \{ \emptyset, \{113\} \}$ and $Y = \{ \{1\}, \{2\} \}$:
We need to fill in the template
$\{ (\emptyset, \quad), (\{113\}, \quad) \}$
so that we use, once only, each element of Y. This is possible in 2 ways:
$h_1 = \{ (\emptyset, \{1\}) , (\{113\}, \{2\}) \}$
and $h_2 = \{ (\emptyset, \{2\}) , (\{113\}, \{1\}) \}$.

(c) $X = \{ \emptyset, \{113\} \}$ and $Y = \{ \{1\}, \{2\}, \{7\} \}$:
There is no way to fill in the template $\{ (\emptyset, \quad), (\{113\}, \quad) \}$ so that we use each member of Y exactly once, because there are 3 elements in Y and only 2 gaps to be filled in.
So there are no bijective functions from X to Y in this case.

2. *Check the following functions for injectivity (one-to-one), surjectivity (onto) and bijectivity (a one-to-one correspondence) (ie functions that are both injective and surjective), and give the inverse function of each:*

(a) $f: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f(x) = x + 1$:
f is **injective**, because

if	$f(u) = f(v)$
then	$u + 1 = v + 1$
ie	$u = v$.

f is **surjective**, because

$$\begin{aligned}
\text{ran}(f) &= \{y \mid \text{for some } x \in \mathbb{Z}, (x, y) \in f\} \\
&= \{y \mid \text{for some } x \in \mathbb{Z}, y = x + 1\} \\
&= \{y \mid \text{for some } x \in \mathbb{Z}, x = y - 1\} \\
&= \{y \mid y - 1 \text{ is an integer}\} = \mathbb{Z}
\end{aligned}$$

Since $f: \mathbb{Z} \to \mathbb{Z}$ is **bijective**, f^{-1} is a function from $\mathbb{Z}$ to $\mathbb{Z}$.

We can determine the inverse function f^{-1}:

$(y, x) \in f^{-1}$ iff $(x, y) \in f$
iff $y = x + 1$
iff $x = y - 1$

so $f^{-1}: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f^{-1}(y) = y - 1$.

Note: We can also write this as
$f^{-1}: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f^{-1}(x) = x - 1$ or
$f^{-1}: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f^{-1}(z) = z - 1$, etc.
It is important to note that it does not matter which variable we use.

(b) $f: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f(x) = x^2$:

f is **not injective**, because we can find a counterexample:
If we choose $u = 2$ and $v = -2$, then $u \neq v$ but
$f(u) = f(2) = 4$ and $f(v) = f(-2) = 4$
ie $f(u) = f(v)$.

f is **not surjective**, because we can find a counterexample:
If we choose $y = -9$,
then there is no $x \in \mathbb{Z}$ such that $x^2 = y$ (since $x^2 \geq 0$ for every integer x),
so $-9 \notin \text{ran}(f)$.
Hence $\text{ran}(f) \neq \mathbb{Z}$. Since $f: \mathbb{Z} \to \mathbb{Z}$ is **not bijective**, f^{-1} is not a function from $\mathbb{Z}$ to $\mathbb{Z}$.

(c) $f: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f(x) = 3 - x$:

f is **injective** because
if $f(u) = f(v)$
then $3 - u = 3 - v$
ie $u = v$.

f is **surjective** because

$$\begin{aligned}
\text{ran}(f) &= \{y \mid \text{for some } x \in \mathbb{Z}, (x, y) \in f\} \\
&= \{y \mid \text{for some } x \in \mathbb{Z}, y = 3 - x\} \\
&= \{y \mid \text{for some } x \in \mathbb{Z}, x = 3 - y\} \\
&= \{y \mid 3 - y \text{ is an integer}\} \\
&= \mathbb{Z}
\end{aligned}$$

Since $f: \mathbb{Z} \to \mathbb{Z}$ is **bijective**, f^{-1} is a function from $\mathbb{Z}$ to $\mathbb{Z}$.

$(y, x) \in f^{-1}$ iff $(x, y) \in f$

iff $y = 3 - x$
iff $x = 3 - y$.
So $f^{-1}: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f^{-1}(y) = 3 - y$.

(d) $f: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f(x) = 4x + 5$:

f is **injective** because
if $f(u) = f(v)$
then $4u + 5 = 4v + 5$
ie $4u = 4v$
ie $u = v$.

f is **not surjective** because, if we choose y to be even, say y = 8,
then y cannot be written in the form 4x + 5 (Can you remember why not?)
ie $y \notin \text{ran}(f)$.
Hence $\text{ran}(f) \neq \mathbb{Z}$ (although $\text{ran}(f) \subseteq \mathbb{Z}$).
So $f: \mathbb{Z} \to \mathbb{Z}$ is **not bijective**.

Note: If f were defined to be a function from $\mathbb{R}$ to $\mathbb{R}$, then f would be bijective (can you show this?) with an inverse calculated as follows:

$(y, x) \in f^{-1}$ iff $(x, y) \in f$
iff $y = 4x + 5$
iff $y - 5 = 4x$
iff $(y - 5)/4 = x$.
Hence $f^{-1}: \mathbb{R} \to \mathbb{R}$ is defined by the rule $f^{-1}(y) = (y - 5)/4$.

1. Consider an identity function $i_C: C \to C$.

(a) Prove that $i_C: C \to C$ is bijective.
A function is **bijective** if it is **injective** and **surjective**.

$i_C: C \to C$ is defined by the rule $i_C(x) = x$ (i_C is the identity function on C):

i_C is **injective**, because
if $i_C(u) = i_C(v)$
then $u = v$.

i_C is **surjective**, because
$\text{ran}(i_C) = \{y \mid \text{for some } x \in C, (x, y) \in i_C\}$
$= \{y \mid \text{for some } x \in C, y = x\}$
$= \{y \mid y \in C\} = C$
i_C is injective and surjective, thus it is a bijective function.

(c) *Prove that i_C is an equivalence relation on C.*

In order to prove that i_C is an equivalence relation, we have to prove that i_C **is reflexive, symmetric and transitive.**

Reflexivity:
Is it the case that for all $x \in C$, $(x, x) \in i_C$?
Yes, for any $x \in C$, $x = x$,
ie $(x, x) \in i_C$.
Thus i_C is reflexive.

Symmetry:
If $(x, y) \in i_C$, is it the case that $(y, x) \in i_C$?
Suppose $(x, y) \in i_C$,
then $y = x$
ie $x = y$,
therefore $(y, x) \in i_C$.
Thus i_C is symmetric.

Transitivity:
If $(x, y) \in i_C$ and $(y, z) \in i_C$, is it the case that $(x, z) \in i_C$?
*(Assume $(x, y) \in i_C$ and $(y, z) \in i_C$ then **use** this information to prove that $(x, z) \in i_C$.)*

Suppose $(x, y) \in i_C$,
then $y = x$ ①
and
suppose $(y, z) \in i_C$,
then $z = y$. ②

From ① and ② it follows that:
$z = y = x$
ie $z = x$,
therefore $(x, z) \in i_C$.
Thus i_C is transitive.

i_C is reflexive, symmetric and transitive, thus it is an equivalence relation.