Study unit 6, sections 6.1 – 6.3

Activity 6.4

For each of the following relations, determine whether or not the relation is a weak partial order (reflexive, antisymmetric and transitive) on the given set:

(a) Let A = {a, b, {a, b} }. The relation S on A is defined by (c, B) $\in$ S iff c $\in$ B.
We see that a $\in$ {a, b} and b $\in$ {a, b} and thus S = { (a, {a, b}), (b, {a, b}) }.

Reflexivity:
Is it the case that for all x $\in$ A, (x, x) $\in$ S?
No, we give a counterexample: (a, a) $\notin$ S. (It is also the case that (b, b) $\notin$ S and ({a, b}, {a, b}) $\notin$ S.)

S is not reflexive therefore we can say that it is not a weak partial order.
(You can test whether or not S is antisymmetric and transitive.)

(b) Define R $\subseteq$ $\mathbb{Z} \times \mathbb{Z}$ by x R y iff x + y is even.

If x + y is even then we can say that x + y = 2k for some integer k.

Reflexivity:
Is it the case that for all x $\in$ $\mathbb{Z}$, (x, x) $\in$ R?
x + x = 2x, ie x + x is an even number for any x $\in$ $\mathbb{Z}$.
Thus R is reflexive on $\mathbb{Z}$.

Antisymmetry:
If (x, y) $\in$ R, is it the case that (y, x) $\notin$ R?
Suppose (x, y) $\in$ R
then x + y = 2k
ie y + x = 2k, but this means that (y, x) $\in$ R.
Thus R is not antisymmetric. R is actually symmetric.
Because R is not antisymmetric it is not a weak partial order.

For interest's sake, let's test whether R is transitive:
Transitivity:
If (x, y) $\in$ R and (y, z) $\in$ R, is it the case that (x, z) $\in$ R?
Suppose (x, y) $\in$ R and (y, z) $\in$ R
then x + y = 2k and y + z = 2m for some k, m $\in$ $\mathbb{Z}$.
ie x = 2k – y and z = 2m – y
ie x + z = 2k – y + 2m – y
ie x + z = 2 (k + m – y)
ie x + z = 2t for some integer t
ie (x, z) $\in$ R. Thus R is transitive.

(c) Define R on $\mathbb{Z} \times \mathbb{Z}$ by (a, b) R (c, d) if either a < c or else (a = c and b $\leq$ d).

Reflexivity:

Is it the case that for all (a, b) $\in \mathbb{Z} \times \mathbb{Z}$, (a, b) R (a, b)?

For any (a, b) $\in \mathbb{Z} \times \mathbb{Z}$ it is not the case that a < a but

(a = a and b $\leq$ b)

ie (a, b) R (a, b).

Thus R is reflexive.

Antisymmetry:

If (a, b) $\neq$ (c, d) and ((a, b), (c, d)) $\in$ R, is it the case that ((c, d), (a, b)) $\notin$ R?

Suppose (a, b) $\neq$ (c, d) and (a, b) R (c, d), then a < c or else (a = c and b $\leq$ d).

Firstly, we do not have c < a and secondly we do not have that c = a and d $\leq$ b, thus ((c, d), (a, b)) $\notin$ R.

(In the second case we cannot have c = a and d = b because we assumed that (a, b) $\neq$ (c, d).

Furthermore, we cannot have c = a and d < b because by our assumption, b $\leq$ d.)

We can safely say that ((c, d), (a, b)) $\notin$ R, thus R is antisymmetric.

Transitivity:

If ((a, b), (c, d)) $\in$ R and ((c, d), (e, f)) $\in$ R, is it the case that ((a, b), (e, f)) $\in$ R?

Suppose ((a, b), (c, d)) $\in$ R and ((c, d), (e, f)) $\in$ R

ie a < c or else (a = c and b $\leq$ d), and c < e or else (c = e and d $\leq$ f).

We can look at the following cases:

a < c and c < e,

a < c and c = e and d $\leq$ f,

a = c and b $\leq$ d and c < e, or

a = c and b $\leq$ d and c = e and d $\leq$ f.

In the first three cases we have a < e and in the last case we have a = e and b $\leq$ f.

We can deduce that ((a, b), (e, f)) $\in$ R, thus R is transitive.

Because R is reflexive, antisymmetric and transitive we can say that R is a weak partial order.