Study Unit 4

Activity 4-12

1. *Determine whether or not for V, W, Z $\subseteq$ U, if V $\subseteq$ W, then V $\cup$ Z $\subseteq$ W $\cup$ Z and V $\cap$ Z $\subseteq$ W $\cap$ Z. Provide either a proof or a counterexample, whichever is appropriate.*

Let us first try to prove that if V $\subseteq$ W then V $\cup$ Z $\subseteq$ W $\cup$ Z.

Suppose V $\subseteq$ W.
Let x $\in$ V $\cup$ Z,
then x $\in$ V or x $\in$ Z
ie x $\in$ W or x $\in$ Z (V $\subseteq$ W, so if x $\in$ V then x $\in$ W)
ie x $\in$ W $\cup$ Z.
Therefore V $\cup$ Z $\subseteq$ W $\cup$ Z.

Let us now consider whether it is the case that if V $\subseteq$ W, then V $\cap$ Z $\subseteq$ W $\cap$ Z.
Suppose V $\subseteq$ W.
Let x $\in$ V $\cap$ Z,
then x $\in$ V and x $\in$ Z
ie x $\in$ W and x $\in$ Z (because V $\subseteq$ W)
ie x $\in$ W $\cap$ Z.
We can conclude that if V $\subseteq$ W then V $\cap$ Z $\subseteq$ W $\cap$ Z.

2. *Is it the case that, for all subsets X, Y, W $\subseteq$ U, if X = Y and Y = W, then X = W, and if X $\subset$ Y and Y $\subset$ W, then X $\subset$ W? Justify your answer.*

In general, what does 'A $\subset$ B' mean? It means that A is a subset of (but not equal to) B, ie A is a proper subset of B.
First we have to attempt to prove that if X = Y and Y = W, then X = W.
If X = Y and Y = W then we know that X has precisely the same elements as Y and that Y has exactly the same elements as W.
Therefore X and W contain exactly the same elements and hence X = W.

Also, if X $\subset$ Y and Y $\subset$ W we can try to prove that X $\subset$ W as follows:
Suppose X $\subset$ Y and Y $\subset$ W.
Let x $\in$ X,
then x $\in$ Y (since X $\subset$ Y)
ie x $\in$ W (because Y $\subset$ W).
So X $\subseteq$ W.
But Y has at least one element not in X (since X $\subset$ Y)
and W has at least one element not in Y (since Y $\subset$ W),

so W has at least two elements not in X
ie $X \neq W$,
so $X \subset W$.

What does this tell us about $\subseteq$? We now know that if $X \subseteq Y$ and $Y \subseteq W$, ie ($X \subset Y$ or $X = Y$) and ($Y \subset W$ or $Y = W$), then ($X \subset W$ or $X = W$) (from the above proofs)
ie $X \subseteq W$.
In other words, the fact that $=$ and $\subset$ satisfy transitive laws for sets tells us that $\subseteq$ is also transitive.

3. *Is it the case that, for all subsets X of U, $X \cup \emptyset = X$? Justify your answer.*
We know that the set we obtain when we determine the union of two sets Y and Z contains all the elements of Y and all the elements of Z. So when we form the union of a subset X and $\emptyset$, the new set,
$X \cup \emptyset$, contains all the elements of X and all the elements of $\emptyset$.

But since $\emptyset$ has no elements, $X \cup \emptyset$ will only contain the elements of X.
Therefore $X \cup \emptyset = X$ for all subsets X of U.
In the style of our other proofs, we may also argue as follows:
Let $x \in X \cup \emptyset$,
then $x \in X$ or $x \in \emptyset$
ie $x \in X$ (because it is impossible for x to reside in $\emptyset$).
Thus $X \cup \emptyset \subseteq X$.

Conversely, let $x \in X$,
then $x \in X$ or $x \in \emptyset$
ie $x \in X \cup \emptyset$.
Thus $X \subseteq X \cup \emptyset$.
It follows that $X \cup \emptyset = X$.

4. *Is it the case that, for all subsets V and W of U, $V \cap W = \emptyset$ iff $V = \emptyset$ or $W = \emptyset$? Justify your answer.* This claim has two parts. These are
if $V = \emptyset$ or $W = \emptyset$ then $V \cap W = \emptyset$, and
if $V \cap W = \emptyset$ then $V = \emptyset$ or $W = \emptyset$.
Both these parts must hold for the claim to be true.
Let us consider the first part. Suppose $V = \emptyset$ or $W = \emptyset$.
We know that, when the intersection of the two sets V and W is formed, the set $V \cap W$ contains the elements common to both V and W.
In this case at least one of V or W is empty so there is no element common to V and W.
Therefore $V \cap W$ is also empty.

Looking at the second part of the claim, we have to decide whether it is necessarily the case whenever
$V \cap W = \emptyset$, it follows that $V = \emptyset$ or $W = \emptyset$

Well, if $V \cap W = \emptyset$ all we know is that V and W have no elements in common.
It is not necessarily the case that one of them is empty.

Consider the example $V = \{1, 2\}$ and $W = \{3, 4\}$.
It is clear that $V \cap W = \emptyset$ although neither V nor W is empty.
We can conclude that if $V = \emptyset$ or $W = \emptyset$ then $V \cap W = \emptyset$,
but if $V \cap W = \emptyset$, then it is not necessarily the case that $V=\emptyset$ or $W=\emptyset$.
Therefore it is not the case that, for all subsets V and W, $V \cap W = \emptyset$ iff $V = \emptyset$ or $W = \emptyset$.

5. *Is it the case that for every subset X of U there exists a subset Y of U such that $X \cup Y = \emptyset$? Justify your answer.*

No. We give a counterexample.

We know that the set $X \cup Y$ contains all the elements of X as well as those of Y.
So if U is the set $\{1, 2\}$ and X is the subset $\{1\}$, then there is no subset Y of U such that $X \cup Y = \emptyset$.
To see this, note that there are just four possible values for Y,
namely $Y = \emptyset$, $Y = \{1\}$, $Y = \{2\}$, and $Y = \{1, 2\}$.
In each case, $X \cup Y$ contains at least the element 1, so $X \cup Y \neq \emptyset$.

6. *Is it the case that for every subset X of U there is some subset Y such that $X \cap Y = U$? Justify your answer.*

No. We give a counterexample.

We know that the intersection $X \cap Y$ contains the elements that are common to X and Y.
So if $X \cap Y = U$ it must be the case that X and Y have all the elements of U in common. It is not usually the case.
Take $U = \{1, 2\}$ and $X = \{1\}$.
Then there is no subset Y of U such that $X \cap Y = U$.
To see this, note that Y can have four possible values,
namely $Y = \emptyset$, $Y = \{1\}$, $Y = \{2\}$, and $Y = \{1, 2\}$.
In none of these four cases does $X \cap Y$ contain the element 2.

7. *Using "if and only if" statements, determine the following*:

(a) *Is it the case that $X + Y = Y + X$ for all $X, Y \subseteq U$?*

$x \in X + Y$
iff $x \in X$ or $x \in Y$ but not both
iff $x \in Y$ or $x \in X$ but not both
iff $x \in Y + X$. We conclude that $X + Y = Y + X$.

(b) *Is it case that $X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$ for all $X, Y, Z \subseteq U$?*

$x \in X \cap (Y + Z)$
iff $x \in X$ and $x \in (Y + Z)$
iff $x \in X$ and ($x \in Y$ or $x \in Z$ but not both)
iff x is in X and in exactly one of Y or Z
iff either ($x \in X$ and $x \in Y$) or ($x \in X$ and $x \in Z$) but not both
iff $x \in X \cap Y$ or $x \in X \cap Z$ but not both
iff $x \in (X \cap Y) + (X \cap Z)$.
Therefore $X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$.