Study unit 8

Activity 8-3

1. Let X be the set {2, 7}.

(a) *Give 3 binary operations on X, both in list notation and in tabular form.*
Let us compile the table first and then the set of ordered pairs in each case.
There are many possible examples, of which we give just three.

+	**2**	**7**
2	2	2
7	2	2

This table represents the very simple operation that does not care what you feed it, because it has made up its mind to constantly spit out the value 2.

Note: Although we have chosen to call the operation '+', it has *no* connection with ordinary addition whatsoever.
In list notation, + = **{ (**(2, 2), 2**)**, **(**(2, 7), 2**)**, **(**(7, 2), 2**)**, **(**(7, 7), 2**)** }.

Our next example is:

∗	**2**	**7**
2	7	2
7	2	7

In list notation, ∗ = { **(**(2, 2), 7**)**, **(**(2, 7), 2**)**, **(**(7, 2), 2**)**, **(**(7, 7), 7**)** }.

Our final example is:

☐	**2**	**7**
2	7	7
7	2	7

In list notation: ☐ = {**(**(2, 2), 7**)**, **(**(2, 7), 7**)**, **(**(7, 2), 2**)**, **(**(7, 7), 7**)** }.

(b) *Check these operations for commutativity and associativity.*

Commutativity:
+ and * are commutative, whereas ☐ is not.
A quick way to see this is to look at the tables and see which are symmetric about the diagonal from the top left to the bottom right. In the case of ☐, the triangle above the diagonal is not a mirror image of the triangle below, because, for example, 2 ☐ 7 = 7 whereas 7 ☐ 2 = 2.

Associativity:
+ must be associative, because it always spits out 2,
so for all x, y, and z in X, x + (y + z) = 2 = (x + y) + z.
To see that * is associative we have to check all the various cases (8 of them), and if you do, you will find that everything works out.

☐ fails to be associative. A counterexample is provided by the values x=2, y=2, and z=2:
x ☐ (y ☐ z) = 2 ☐ (2 ☐ 2) = 2 ☐ 7 = 7, but (x ☐ y) ☐ z = (2 ☐ 2) ☐ 2 = 7 ☐ 2 = 2.

2. *Give 2 binary operations on X = {a, b, c} and check them for commutativity and associativity.*

Bear in mind there are many examples of such operations, and we will choose two more or less at random.
First, an example using list notation. Just fill in the template
{ ((a, a),), ((a, b),), ((a, c),), ((b, a),), ((b, b),), ((b, c),), ((c, a),), ((c, b),), ((c, c),) }
to get, for instance,
{ ((a, a), **b**) , ((a, b), **b**) , ((a, c), **b**) , ((b, a), **b**) , ((b, b), **b**) , ((b, c), **b**), ((c, a), **b**) , ((c, b), **b**) , ((c, c), **b**) }.

This operation is commutative and associative - do you agree? Think about it.

Finally, the following is an example of an operation that fails to be either commutative or associative:

*	**a**	**b**	**c**
a	b	c	b
b	a	b	b
c	b	b	b

Commutativity fails, because a * b = c whereas b * a = a.
Associativity fails, because a * (a * a) = a * b = c whereas (a * a) * a = b * a = a.

3. *Consider the dot operation, "•", defined in section 8.1. Let us compare the dot operation on {a, b, c, d} with ordinary multiplication.*

•	**a**	**b**	**c**	**d**
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

(a) *We know that ordinary multiplication on R is commutative. Examine x • y and y • x for each x, y ∈ A. What do you conclude?*

a • b = b = b • a
a • c = c = c • a
a • d = d = d • a
b • a = b = a • b

b • c = d = c • b
b • d = c = d • b
c • d = b = d • c
The *dot* operation is commutative.

(b) *We know that R has an identity for multiplication, namely 1. This means that 1 • x = x = x • 1 for all x $\in$ R. Does A have an element that behaves similarly?*
a • a = a
a • b = b
a • c = c
a • d = d
b • a = b
c • a = c
d • a = d It appears that the element a is an identity element for the *dot* operation.