Study unit 6, sections 6.4 – 6.5

Activity 6-14

1. *Give 5 functions from A = {1, 2, 3, 4} to B = {a, b, c}.*

Think of building each function by filling in the template

{ (1,), (2,), (3,), (4,) }

with elements of B.

Some possibilities are:

f_1 = { (1, **a**), (2, **a**), (3, **a**), (4, **a**) }
f_2 = { (1, **b**), (2, **b**), (3, **b**), (4, **b**) }
f_3 = { (1, **c**), (2, **c**), (3, **c**), (4, **c**) }
f_4 = { (1, **a**), (2, **b**), (3, **a**), (4, **b**) }
f_5 = { (1, **c**), (2, **a**), (3, **b**), (4, **c**) }.

2. *Give all the functions from A = {a, b} to B = {5, 6, 7}.*

We can get the functions by filling in the template { (a,), (b,) }
with second co-ordinates chosen from B.

f_1 = { (a, 5), (b, 5) }
f_2 = { (a, 6), (b, 6) }
f_3 = { (a, 7), (b, 7) }
f_4 = { (a, 5), (b, 6) }
f_5 = { (a, 5), (b, 7) }
f_6 = { (a, 6), (b, 5) }
f_7 = { (a, 6), (b, 7) }
f_8 = { (a, 7), (b, 5) }
f_9 = { (a, 7), (b, 6) }.

3. *Give 3 functions from A × A to B if A = {a, b} and B = {5, 6, 7}.*

Each function has as domain the set

A × A = { (a, a) , (a, b) , (b, a) , (b, b) }.

To build such a function, just fill in the template

{ **(**(a, a)**,** **)**, **(**(a, b)**,** **)**, **(**(b, a)**,** **)**, **(**(b, b)**,** **)** }

with second co-ordinates chosen from B. Three examples are:

f_1 = { **(**(a, a)**, 5)**, **(**(a, b)**, 5)**, **(**(b, a)**, 5)**, **(**(b, b)**, 5)** }
f_2 = { **(**(a, a)**, 5)**, **(**(a, b)**, 6)**, **(**(b, a)**, 7)**, **(**(b, b)**, 5)** }
f_3 = { **(**(a, a)**, 7)**, **(**(a, b)**, 6)**, **(**(b, a)**, 5)**, **(**(b, b)**, 6)** }.

4. *Let R be a relation on A = {1, 2, 3, {1}, {2}} defined by*
R = { (1, {1}), (1, 3), (2, {1}), (2, {2}), ({1}, 3), ({2}, {1}) }.

(a) *Is R a function from A to A?*

First we have to ask: is R functional? (ie if (x, y) ∈ R and (x, z) ∈ R is y = z?) and is dom(R) = A?

No, R is not a function. We give a *counterexample*:

$(1, \{1\}) \in R$ and $(1, 3) \in R$,
so 1 appears twice as first co-ordinate, but with different second co-ordinates, namely {1} and 3 as partners, so R is not functional and thus not a function.
(We also have that $\text{dom}(R) \neq A$ since $3 \notin \text{dom}(R)$.)

(b) *Is ran(R) equal to the codomain of R?*
The codomain $A = \{1, 2, 3, \{1\}, \{2\}\}$ is not equal to the range of R
ie $\text{ran}(R) = \{ 3, \{1\}, \{2\} \} \neq A$.

5. *Consider the set* $\mathcal{P}(A) = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \}$.

Show that the relations f, g, and h described below are functional and have as domains $\mathcal{P}(A)$, $\mathcal{P}(A) \times \mathcal{P}(A)$, *and* $\mathcal{P}(A) \times \mathcal{P}(A)$ *respectively.*

(a) Let $f = \{(x, y) \mid x, y \in \mathcal{P}(A) \text{ and } y = x'\}$.

We can follow either of two approaches. The brute force approach involves writing out in list notation the set f, so that we can verify by inspection that f is functional and that each element of $\mathcal{P}(A)$ occurs as a first co-ordinate. This approach is suitable only for smallish sets and the set $\mathcal{P}(A)$ is just barely small enough. A more sophisticated approach would involve abstract reasoning with the help of variables.
Let us use the brute force approach here and the abstract approach for subsequent questions where the domain is bigger.
$F = \{(\emptyset, \{a, b, c\}), (\{a\}, \{b, c\}), (\{b\}, \{a, c\}), (\{c\}, \{a, b\}), (\{a, b\}, \{c\}), (\{a, c\}, \{b\}), (\{b, c\}, \{a\}), (\{a, b, c\}, \emptyset) \}$.

By inspection it is clear that $\text{dom}(f) = \mathcal{P}(A)$ and that every element of $\mathcal{P}(A)$ occurs exactly once as a first co-ordinate, so $f: \mathcal{P}(A) \to \mathcal{P}(A)$.

(b) Let $g = \{ ((u, v), y) \mid (u, v) \in \mathcal{P}(A) \times \mathcal{P}(A) \text{ and } y = u \cup v \}$.

This time we use abstract reasoning.

$\text{Dom}(g) \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$, because g is a relation from $\mathcal{P}(A) \times \mathcal{P}(A)$ to $\mathcal{P}(A)$.

Is it the case that $\mathcal{P}(A) \times \mathcal{P}(A) \subseteq \text{dom}(g)$?

Yes, because if $(u, v) \in \mathcal{P}(A) \times \mathcal{P}(A)$, then $u \cup v$ is a subset of A, ie for each element (u, v) of the set
$\mathcal{P}(A) \times \mathcal{P}(A)$ we can find an element y of $\mathcal{P}(A)$ such that $y = u \cup v$.

Since $\text{dom}(g) \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$ and $\mathcal{P}(A) \times \mathcal{P}(A) \subseteq \text{dom}(g)$, it follows that

$\text{dom}(g) = \mathcal{P}(A) \times \mathcal{P}(A)$.

Is g functional?
Suppose $(x, y) \in g$ and $(x, z) \in g$.

Then x = (u, v) for some u $\subseteq$ A and some v $\subseteq$ A,
and y = u $\cup$ v = z, so g is indeed functional.

Since dom(g) = $\mathcal{P}$(A) $\times$ $\mathcal{P}$(A) and g is functional, it follows that

g : $\mathcal{P}$(A) $\times$ $\mathcal{P}$(A) $\to$ $\mathcal{P}$(A), ie g is a function from $\mathcal{P}$(A) $\times$ $\mathcal{P}$(A) to $\mathcal{P}$(A).

(c) Let h = { ((u, v), y) | (u, v) $\in$ $\mathcal{P}$(A) $\times$ $\mathcal{P}$(A) and y = u $\cap$ v }.

Dom(h) $\subseteq$ $\mathcal{P}$(A) $\times$ $\mathcal{P}$(A) because h is a relation from $\mathcal{P}$(A) $\times$ $\mathcal{P}$(A) to $\mathcal{P}$(A).

Is it the case that $\mathcal{P}$(A) $\times$ $\mathcal{P}$(A) $\subseteq$ dom(h)?

Yes, for if (u, v) $\in$ $\mathcal{P}$(A) $\times$ $\mathcal{P}$(A) then u $\cap$ v is a subset of A,

so we can find an element y of $\mathcal{P}$(A) such that ((u, v), y) $\in$ h, namely y = u $\cap$ v.

Since dom(h) $\subseteq$ $\mathcal{P}$(A) $\times$ $\mathcal{P}$(A) and $\mathcal{P}$(A) $\times$ $\mathcal{P}$(A) $\subseteq$ dom(h), it follows that dom(h) = $\mathcal{P}$(A) $\times$ $\mathcal{P}$ (A).

Is h functional?
If (x, y) $\in$ h and (x, z) $\in$ h,
then x has the form (u, v) for some subsets u and v of A, and y = u $\cap$ v = z.

Since dom(h) = $\mathcal{P}$(A) $\times$ $\mathcal{P}$(A) and h is functional, it follows that

h: $\mathcal{P}$(A) $\times$ $\mathcal{P}$(A) $\to$ $\mathcal{P}$(A), ie h is a function from $\mathcal{P}$(A) $\times$ $\mathcal{P}$(A) to $\mathcal{P}$(A).

6. *For each of the following relations from X to Y, determine whether or not the relation may be regarded as a function from X to Y, then determine the range of the relation.*

A relation R from X to Y is a function iff R is **functional** and **dom(R) = X**.
In each of the following examples we investigate whether or not R (or S) has these two properties, then we determine ran(R) or ran(S).

(a) X = Y = $\mathbb{Z}$ and R = {(x, y) | y = x}.
Determine dom(R):

Dom(R) = {x | for some y $\in$ Y, (x, y) $\in$ R}
= {x | for some y $\in$ $\mathbb{Z}$, y = x}
= {x | x is an integer}
= $\mathbb{Z}$.

Next we investigate functionality.
Suppose (x, y) $\in$ R and (x, z) $\in$ R
ie y = x and z = x
ie y = z.
So every x in $\mathbb{Z}$ that appears as a first co-ordinate does so in only one pair.

Hence R is functional.

Because R is functional and dom(R) = $\mathbb{Z}$, it follows that R is a function, so we may write R: $\mathbb{Z} \to \mathbb{Z}$.

(R is in fact a very important function, namely the **identity** function on Z. Informally, R is the function that instructs us, no matter in which city we find ourselves, not to go anywhere else but to stay just where we are.)

Determine ran(R):
ran(R) = {y | for some x $\in$ X, (x, y) $\in$ R}
= {y | for some x $\in \mathbb{Z}$, y = x}
= {y | y is an integer}
=$\mathbb{Z}$.

(b) X = Y = $\mathbb{Z}$ and R = {(x, y) | y = x + 1}.

Determine dom(R):
Note: We show that dom(R) $\subseteq$ Z and $\mathbb{Z} \subseteq$ dom(R), ie dom(R) = $\mathbb{Z}$.

By definition we know that R $\subseteq \mathbb{Z} \times \mathbb{Z}$, so dom(R) $\subseteq \mathbb{Z}$.

But we also have that $\mathbb{Z} \subseteq$ dom(R) since for **every** x in $\mathbb{Z}$ there is an **integer** y of the form x + 1, so we have a pair of the form (x, x + 1) in R and therefore x $\in$ dom(R).

Since dom(R) $\subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq$ dom(R), it follows that dom(R) = $\mathbb{Z}$.

Next we look at functionality.
Suppose (x, y) $\in$ R and (x, z) $\in$ R
ie y = x + 1 and z = x + 1
ie y = x + 1 = z
ie y = z
Hence R is functional.

Since R is functional and dom(R) = $\mathbb{Z}$, R is a function from Z to Z.

Note: R is called the successor function because it tells us to go from x to x + 1.

Determine ran(R):
Note: We show that ran(R) $\subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq$ ran(R), ie ran(R) = $\mathbb{Z}$.

Now, we know that ran(R) $\subseteq \mathbb{Z}$, because R $\subseteq \mathbb{Z} \times \mathbb{Z}$.
It is also the case that $\mathbb{Z} \subseteq$ ran(R) because
for **every** integer y we can find an **integer** x such that y = x + 1,

(just take x to be y – 1, then x + 1 = (y – 1) + 1 = y then (y – 1, y) $\in$ R, so y $\in$ ran(R).
Since ran(R) $\subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq$ ran(R), it follows that ran(R) = $\mathbb{Z}$.

(c) X = Y = $\mathbb{Z}$ and R = {(x, y) | y = 3 – x}.
Determine dom(R):

dom(R) = {x | for some y $\in$ Y, (x, y) $\in$ R}
= {x | for some y $\in$ Z, y = 3 – x}
= {x | 3 – x is an integer}
= $\mathbb{Z}$

Now for functionality.
Suppose (x, y) $\in$ R and (x, z) $\in$ R
ie y = 3 – x and z = 3 – x
ie y = 3 – x = z
ie y = z.
So R is functional.
Since R is functional and dom(R) = $\mathbb{Z}$, R: $\mathbb{Z} \to \mathbb{Z}$.
Determine ran(R):

ran(R) = {y | for some x $\in$ X, (x, y) $\in$ R}
= {y | for some x $\in \mathbb{Z}$, y = 3 – x}
= {y | for some x $\in \mathbb{Z}$, x = 3 – y}
= {y | 3 – y is an integer}
= $\mathbb{Z}$

(d) X = Y = $\mathbb{Z}$ and R = {(x, y) | y = $\sqrt{x}$ }, where the notation $\sqrt{x}$ refers to the positive square root of x.
Determine dom(R):

dom(R) ={x | for some y $\in$ Y, (x, y) $\in$ R}
= {x | for some y $\in \mathbb{Z}$, y = $\sqrt{x}$ }

Now, we know that dom(R) $\subseteq \mathbb{Z}$ since R $\subseteq \mathbb{Z} \times \mathbb{Z}$.

Is it also the case that $\mathbb{Z} \subseteq$ dom(R)? Alas, no. Let us find a counterexample:
Take the integer 2. As we saw in study unit 2, $\sqrt{2}$ is irrational, so there can be no integer y equal to $\sqrt{2}$, and hence 2 $\notin$ dom(R).
(Other counterexamples are -1, -2, -3, etc.)
As far as dom(R) is concerned, therefore, we can do no better than to describe dom(R) as {x | for some y $\in \mathbb{Z}$, y = $\sqrt{x}$ }.

Equivalently, saying the same thing in different words,
dom(R) = {x | x = y^2 for some y $\in \mathbb{Z}$ }.
Note: The advantage of the latter description is that it suggests a way to generate the members of dom(R) in the following way: Start with y = 1, form y^2, then take y = 2 and form y^2, and so on.

Now for functionality.
Suppose $(x, y) \in R$ and $(x, z) \in R$
ie $y = \sqrt{x}$ and $z = \sqrt{x}$
ie $y = \sqrt{x} = z$.
Thus y = z and R is functional.

However, we saw earlier that $\text{dom}(R) \neq \mathbb{Z}$, so R is **not** a function from $\mathbb{Z}$ to $\mathbb{Z}$.

Determine ran(R):

$\text{ran}(R) = \{y \mid \text{for some } x \in X, (x, y) \in R\}$
$\quad = \{y \mid \text{for some } x \in \mathbb{Z}, y = \sqrt{x}\}$.

We know that $\text{ran}(R) \subseteq \mathbb{Z}$ since $R \subseteq \mathbb{Z} \times \mathbb{Z}$.

Is it the case that $\mathbb{Z} \subseteq \text{ran}(R)$?
Well, is it the case that every integer is the square root of some other integer?
Only for integers living in $\mathbb{Z}^{\geq}$, since for every $y \in \mathbb{Z}^{\geq}$, y is the square root of the integer y^2,
ie $y = \sqrt{x}$ for some integer x (just take $x = y^2$), ie $y \in \text{ran}(R)$.
But no negative integer can be written in the form $\sqrt{x}$, because $\sqrt{x}$ refers to the positive square root of x.
As a counterexample, $\sqrt{4} = 2$, whereas -2 can only be indicated by $-\sqrt{4}$, so $-2 \notin \text{ran}(R)$.
Thus $\text{ran}(R) \neq \mathbb{Z}$.

(e) $X = Y = \mathbb{Z}$ and $R = \{(x, y) \mid y^2 = x\}$.
You may have been strongly tempted to say that this relation is the same as the one we dealt with in (d), but in fact it is not. The present relation contains pairs like (4, 2) **as well as (4, −2)**, since $(-2)^2 = 4$. With every integer x in its domain, the present relation associates both the positive square root $\sqrt{x}$ and the negative square root $-\sqrt{x}$, so we could in fact have described the relation by
$R = \{(x, y) \mid x \in Z \text{ and } y = \sqrt{x} \text{ or } y = -\sqrt{x}\}$.

Determine dom(R):

$\text{dom}(R) = \{x \mid \text{for some } y \in Y, (x, y) \in R\}$
$\quad = \{x \mid \text{for some } y \in Z, y^2 = x\}$.
$\quad = \{x \mid +\sqrt{x} \text{ and } -\sqrt{x} \text{ are integers}\}$

Is $\text{dom}(R) = \mathbb{Z}$?
Well, we know that $\text{dom}(R) \subseteq \mathbb{Z}$, but just as in the previous question, integers like 2 or −5 do not belong to dom(R), so $\text{dom}(R) \neq \mathbb{Z}$.

Now for functionality.
Suppose $(x, y) \in R$ and $(x, z) \in R$.
Is it necessarily the case that $y = z$?

No! Take $x = 4$, as a c*ounterexample*, then we may think of y as 2 and z as −2. It is clear that $y \neq z$.

(Informally, R is not functional because it often gives us conflicting instructions: if we are in city number 4, say, R tells us to go directly to city 2 and also to go directly to city −2, and we cannot do both at the same time.)

Determine ran(R):

$$\begin{aligned}\mathrm{ran}(R) &= \{y \mid \text{for some } x \in X, (x, y) \in R\}\\ &= \{y \mid \text{for some } x \in \mathbb{Z}, y^2 = x\}\\ &= \{y \mid y^2 \text{ is an integer}\}\\ &= \mathbb{Z}.\end{aligned}$$

(f) $X = Y = \mathbb{R}$ and $S = \{(x, y) \mid x^2 + y^2 = 1\}$.

If you wish, you may visualise S as the circle with its centre at the origin and with a radius of one unit.

Determine dom(S):

$$\begin{aligned}\mathrm{dom}(S) &= \{x \mid \text{for some } y \in Y, (x, y) \in S\}\\ &= \{x \mid \text{for some } y \in R, x^2 + y^2 = 1\}\\ &= \{x \mid -1 \leq x \leq 1\}.\end{aligned}$$

Is S functional?

No. We can find a counterexample. Take $x = 0$, for example.

Then $(0, 1) \in S$ because $0^2 + 1^2 = 1$, and $(0, -1) \in S$ because $0^2 + (-1)^2 = 1$.

So the element 0 of the domain is used more than once as a first co-ordinate.

We can conclude that S is not a function since it is not functional and $\mathrm{dom}(S) \neq \mathbb{R}$.

Determine ran(S):

$$\begin{aligned}\mathrm{ran}(S) &= \{y \mid \text{for some } x \in X, (x, y) \in S\}\\ &= \{y \mid \text{for some } x \in R, x^2 + y^2 = 1\}\\ &= \{y \mid -1 \leq y \leq 1\}.\end{aligned}$$

7. *Is the relation R on $\mathbb{Z}^+$, which consists of all pairs (x, y) such that $y = x - 1$, a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$?*

Well, R certainly seems functional but the problem lies with dom(R).

Recall that $R \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$. Is $1 \in \mathrm{dom}(R)$, ie can we find an appropriate second co-ordinate inside $\mathbb{Z}^+$ to match with 1?

No, because the only second co-ordinate that is suitable is 0, and $0 \notin \mathbb{Z}^+$. So $\mathrm{dom}(R) \neq \mathbb{Z}^+$.

So R is not a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$.

8. *Let A = {a, b, c}. Consider all the equivalence relations on A. (See Activity 6-**10**(1)). How many relations are also functions from A to A?*

We use brute force and then abstract reasoning in our answer. We recommend the latter.

Brute force:
From the previous example we have all the equivalence relations on A. Now we can inspect them to see which are functional:

R_1 = { (a, a), (b, b), (c, c) }
R_2 = { (a, a), (b, b), (c, c), (a, b), (b, a) }
R_3 = { (a, a), (b, b), (c, c), (b, c), (c, b) }
R_4 = { (a, a), (b, b), (c, c), (a, c), (c, a) }
R_5 = { (a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b), (a, c), (c, a) }

Clearly R_1 is the only function from A to A.

Abstract reasoning:
If R is any equivalence relation on A, then R is reflexive on A and therefore dom(R) = A.
But if R is any equivalence relation other than the identity relation {(a, a), (b, b), (c, c)}, then R will fail to be functional.
To see this, note that every equivalence relation on A must contain the pairs (a, a), (b, b) and (c, c). Any additional pair such as, for example, (a, b) will result in a member of A being used more than once as a first co-ordinate.

9. *Let A = {a, b, c}. (See Activity 6-**7**(1))*
(In the answers to the questions that follow, we do everything twice, first using brute force and then using abstract reasoning. We recommend the latter.)

(a) *How many weak partial orders on A (reflexive, antisymmetric and transitive relations) are also functions from A to A?*

Brute force:
Here are all the weak partial orders on A. We can inspect them to see which are functional:

S_1 = { (a,a), (b,b), (c,c) }
S_2 = { (a,a), (b,b), (c,c), (a,b) }
S_3 = { (a,a), (b,b), (c,c), (a,c) }
S_4 = { (a,a), (b,b), (c,c), (b,a) }
S_5 = { (a,a), (b,b), (c,c), (b,c) }
S_6 = { (a,a), (b,b), (c,c), (c,a) }
S_7 = { (a,a), (b,b), (c,c), (c,b) }
S_8 = { (a,a), (b,b), (c,c), (a,b), (a,c) }
S_9 = { (a,a), (b,b), (c,c), (a,b), (c,b) }
S_{10} = { (a,a), (b,b), (c,c), (b,a), (b,c) }
S_{11} = { (a,a), (b,b), (c,c), (b,a), (c,a) }
S_{12} = { (a,a), (b,b), (c,c), (c,a), (c,b) }
S_{13} = { (a,a), (b,b), (c,c), (a,c), (b,c) }
S_{14} = { (a,a), (b,b), (c,c), (a,b), (b,c), (a,c) }
S_{15} = { (a,a), (b,b), (c,c), (a,c), (c,b), (a,b) }
S_{16} = { (a,a), (b,b), (c,c), (b,a), (a,c), (b,c) }
S_{17} = { (a,a), (b,b), (c,c), (b,c), (c,a), (b,a) }
S_{18} = { (a,a), (b,b), (c,c), (c,a), (a,b), (c,b) }

S_{19} = { (a,a), (b,b), (c,c), (c,b), (b,a), (c,a) }

Clearly S_1 is the only function from A to A.
(Incidentally, how do we know that we have found all the weak partial orders on A? Well, because we were systematic: S_1 is the smallest possible chap, with just 3 elements; then we wrote down those with 4 elements; then 5; and finally those with 6.)

Abstract reasoning:
Every weak partial order is reflexive, so every element of A already appears in an ordered pair as first co-ordinate.
The moment a weak partial order has more than just the pairs needed for reflexivity, ie has more pairs than the identity relation on A, some element of A will occur more than once as a first co-ordinate, so the relation will not be functional.
Thus the identity relation on A is the only weak partial order that is also a function from A to A.

(b) *How many strict partial orders on A (irreflexive, antisymmetric and transitive relations) are also functions from A to A?*

Brute force:
Here are all the strict partial orders on A, so that we can inspect them to see which are functional:
T_1 = { }
T_2 = { (a,b) }
T_3 = { (a,c) }
T_4 = { (b,a) }
T_5 = { (b,c) }
T_6 = { (c,a) }
T_7 = { (c,b) }
T_8 = { (a,b), (a,c) }
T_9 = { (a,b), (c,b) }
T_{10} = { (b,a), (b,c) }
T_{11} = { (b,a), (c,a) }
T_{12} = { (c,a), (c,b) }
T_{13} = { (a,c), (b,c) }
T_{14} = { (a,b), (b,c), (a,c) }
T_{15} = { (a,c), (c,b), (a,b) }
T_{16} = { (b,a), (a,c), (b,c) }
T_{17} = { (b,c), (c,a), (b,a) }
T_{18} = { (c,a) ,(a,b), (c,b) }
T_{19} = { (c,b), (b,a), (c,a) }

Clearly none of the relations are functions from A to A. Many of the relations are functional, but none have A as domain.
How do we know we have found all the strict partial orders on A? Easy. Each of them is just the corresponding weak partial order with the reflexive pairs thrown away.

Abstract reasoning:

If a strict partial order T is to be a function from A to A, then it must have A as its domain and be functional.

So every member of A must occur as first co-ordinate of exactly one pair in T.

This means that T must be obtainable by filling in the gaps in the template

{ (a,   ), (b,   ), (c,   ) }

in such a way that the result is irreflexive, antisymmetric, and transitive.

Let us try to fill the gaps:

To assign to **a** the value **a** would violate irreflexivity.

So suppose we start by assigning to **a** the value **b**:

{ (a, **b**), (b,   ), (c,   ) }.

Then we cannot in the next pair assign to **b** the value **a** (because that would violate antisymmetry), nor can we assign to **b** the value **b** (because that would violate irreflexivity), so we must assign to **b** the value **c**: { (a, **b**), (b, **c**), (c,   ) }.

To maintain transitivity, we are now forced to add the pair (a, c) even before we can think about filling the gap in the pair (c,). But then we have two different pairs starting with **a** and this violates functionality.

Our last hope is to start by assigning to **a** the value **c**: { (a, **c**), (b,   ), (c,   ) }.

But then, reasoning as before, we find that to **c** we must assign the value **b**, so that transitivity demands the inclusion of the pair (a, b), and the relation loses functionality.

From this we conclude that there is no way to fill in the template so as to produce a relation that is both a strict partial order and a function.