Vectors and tensors analysis by Penrose's graphical notation

Apr 9, 2023

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1 Vectors analysis

1.1 What you have to know

In this section, understandings of

- scalar and vector
- dot product and cross product
- grad, div, rot, $\triangle$
- Einstein notation
- Kronecker delta
- Levi-Civita symbol and its contraction formula
- display of cross product with Levi-Civita symbol

is required.

When we use index variables, they follow Einstein notation. We do not distinguish tangent/cotangent spaces. All the indices are put on lower position.

1.2 Scalars, vectors, and operations

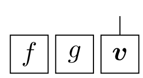
1.2.1 Scalars and vectors

Scalars and vectors are displayed in graphical notation as the following figure. We can distinguish them by the branch. This branch means Kronecker delta (cf. **??**).

scalar f vector $\boldsymbol{v}$

1.2.2 Scalars multiple

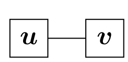
Scalar multiple is represented by putting scalars aside:



$fg\boldsymbol{v}$

1.2.3 Dot product

Branch connection of vectors means dot product, i.e., $\boldsymbol{u} \cdot \boldsymbol{v} = u_i \delta_{ij} v_j$.

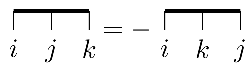


$\boldsymbol{u} \cdot \boldsymbol{v}$

1.2.4 Levi-Civita symbol of 3 components

Above, scalars and vectors are displayed in a square. We determine to show indices without a square.

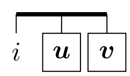
In this notation, we draw three component Levi-Civita symbol as the follows. The thick line represents anti-symmetry. Exchange of branches flips the sign.



$$\epsilon_{ijk} = -\epsilon_{ikj}$$

1.2.5 Cross product

Cross product of vectors $\boldsymbol{u} \times \boldsymbol{v}$ are displayed as the follows. The remaining branch supposes it a vector.



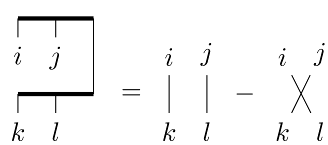
$$\epsilon_{ijk} u_j v_k$$

1.3 Formulae on dot and cross products of vectors

1.3.1 Contraction formula of Levi-Civita symbol

$$\epsilon_{ijm}\epsilon klm = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

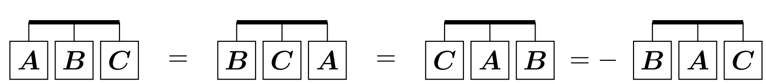
Here we use this formula without proof. We will see details of contraction formula in 2.



1.3.2 Scalar triple product

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = -\boldsymbol{B} \cdot (\boldsymbol{A} \times \boldsymbol{C})$$

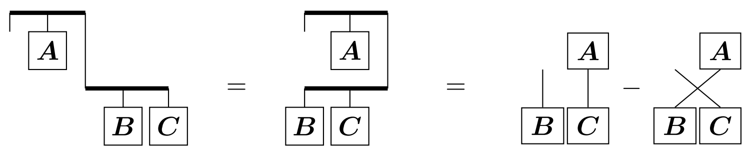
You can display by connecting branches of a vector and the cross product of the remaining. It is obvious that odd permutation exchanges the sign but even permutation does not.



1.3.3 Vector triple product

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$$

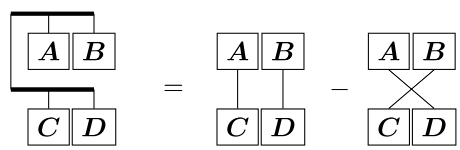
You can display it by connecting the branch of cross product to that of another cross product. It will be easy to contract Levi-Civita symbols after even permutation.



1.3.4 Vector quadruple product

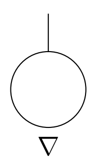
$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = \det \begin{pmatrix} \boldsymbol{A} \cdot \boldsymbol{C} & \boldsymbol{B} \cdot \boldsymbol{C} \\ \boldsymbol{A} \cdot \boldsymbol{D} & \boldsymbol{B} \cdot \boldsymbol{D} \end{pmatrix}$$

It is often written with determinant.



1.4 Derivative

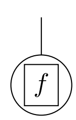
In Penrose graphical notation, the derivative ∇ (nabla) is represented by drawing a branch out of a circle. Put what you differentiate in the circle.



1.4.1 Gradient

$$\operatorname{grad} f = \nabla f$$

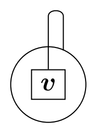
Put a scalar in a derivative circle.



1.4.2 Divergent

$$\operatorname{div} \boldsymbol{v} = \nabla \cdot \boldsymbol{v}$$

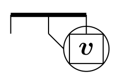
Connect branches of the vector and the circle. The branch represents Kronecker delta so that the notation means $\partial_i \delta_{ij} v_j$.



1.4.3 Rotation

$$\operatorname{rot} \boldsymbol{v} = \nabla \times \boldsymbol{v}$$

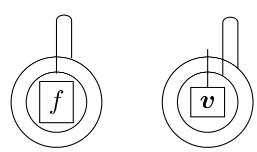
You can display as well as a simple cross product, but it is necessary to stick the branch of what you differentiate immediately right of the derivative.



1.4.4 Laplacian

$$\triangle A = \nabla^2 A$$

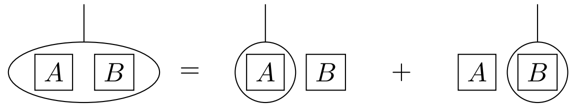
Connect the branches of derivatives, i.e., $\partial_i \delta_{ij} \partial_j A$. You can operate both on a scalar and a vector.



Especially when you operate on a scalar, you get immediately $\operatorname{div} \operatorname{grad} f = \triangle f$.

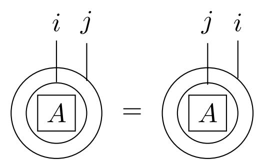
1.4.5 Leibnitz rule

$$\nabla(AB) = \nabla(A)B + A\nabla(B)$$



1.4.6 Symmetry of second derivatives

A class C^2 function satisfies symmetry of second derivatives. In graphical notation, it is exchange of in/out circles.



From now on all the quantities are C^∞ unless otherwise noted.

1.5 Formulae on derivatives

All the following formulae are proved only through the five operations:

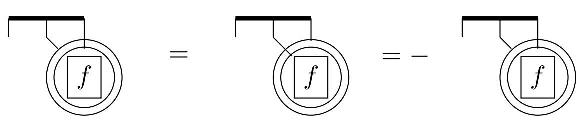
- Anti-symmetry of Levi-Civita symbol
- Even permutation of Levi-Civita symbol indices
- Contraction of Levi-Civita symbols
- Leibnitz rule
- Symmetry of second derivatives

All the proof is on the same way in Einstein notation.

1.5.1 rot grad $= 0$

$$\text{rot grad} f = \nabla \times \nabla f = 0$$

First symmetry of derivatives, second anti-symmetry of Levi-Civita symbol.

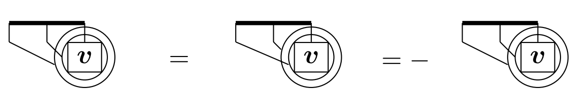


Both hand sides makes $A = -A$, thus the value is zero.

1.5.2 div rot $= 0$

$$\text{div rot } \boldsymbol{v} = \nabla \cdot (\nabla \times \boldsymbol{v}) = 0$$

Proved as well as 1.5.1. First symmetry of derivatives, second anti-symmetry of Levi-Civita symbol.

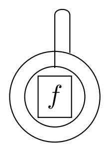


Both hand sides makes $A = -A$, thus the value is zero.

1.5.3 $\operatorname{div}\operatorname{grad} = \triangle$

$$\operatorname{div}\operatorname{grad} f = \nabla \cdot \nabla f = \triangle f$$

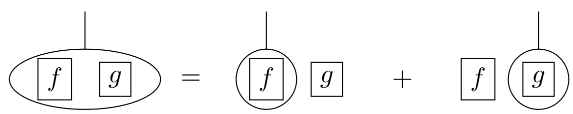
We noted in 1.4.4, but we repeat.



1.5.4 $\operatorname{grad}(fg) = g \operatorname{grad} f + f \operatorname{grad} g$

$$\begin{aligned} \operatorname{grad}(fg) &= (\operatorname{grad} f)g + f(\operatorname{grad} g) \\ \nabla(fg) &= (\nabla f)g + f(\nabla g) \end{aligned}$$

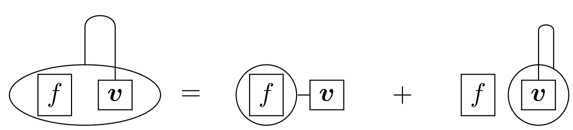
Expand by Leibnitz rule.



1.5.5 $\operatorname{div}(fv) = \operatorname{grad} f \cdot v + f \operatorname{div} v$

$$\begin{aligned} \operatorname{div}(f\boldsymbol{v}) &= \operatorname{grad} f \cdot \boldsymbol{v} + f \operatorname{div} \boldsymbol{v} \\ \nabla(f\boldsymbol{v}) &= \nabla f \cdot \boldsymbol{v} + f\nabla \cdot \boldsymbol{v} \end{aligned}$$

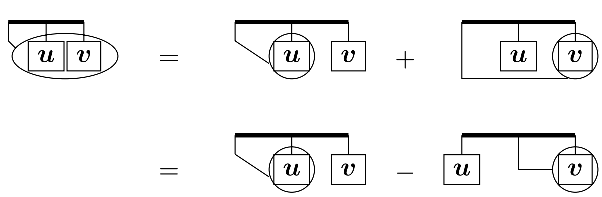
Expand by Leibnitz rule.



1.5.6 $\operatorname{div}(u \times v) = \operatorname{rot} u \cdot v - u \cdot \operatorname{rot} v$

$$\begin{aligned} \operatorname{div}(\boldsymbol{u} \times \boldsymbol{v}) &= \operatorname{rot} \boldsymbol{u} \cdot \boldsymbol{v} - \boldsymbol{u} \cdot \operatorname{rot} \boldsymbol{v} \\ \nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) &= (\nabla \times \boldsymbol{u})\boldsymbol{v} - \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v}) \end{aligned}$$

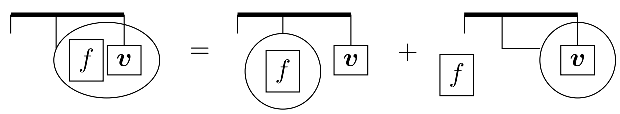
Use Leibnitz rule and anti-symmetry. Follow the rule in 1.4.3; stick the branch of what you differentiate immediately right of the derivative.



1.5.7 rot $(fv) = \text{grad}\, f \times v + f \text{rot}\, v$

$$\text{rot}\,(f\boldsymbol{v}) = \text{grad}\, f \times \boldsymbol{v} + f\ \text{rot}\, \boldsymbol{v}$$
$$\nabla \times (f\boldsymbol{v}) = \nabla f \times \boldsymbol{v} + f \nabla \times \boldsymbol{v}$$

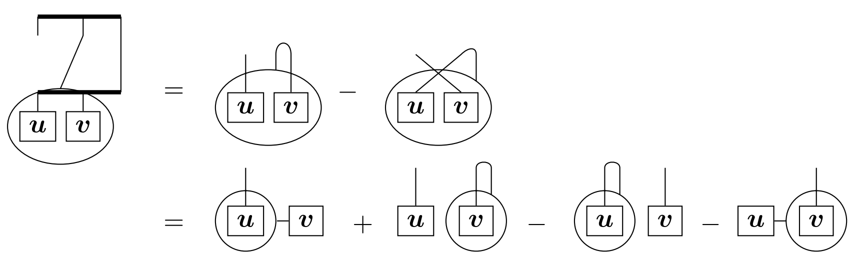
Expand by Leibnitz rule.



1.5.8 rot $(u \times v) = (v \cdot \text{grad})u + u\, \text{div}\, v - v\, \text{div}\, u - (u \cdot \text{grad})v$

$$\text{rot}\,(\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{v} \cdot \text{grad})\boldsymbol{u} + \boldsymbol{u}\ \text{div}\, \boldsymbol{v} - \boldsymbol{v}\ \text{div}\, \boldsymbol{u} - (\boldsymbol{u} \cdot \text{grad})\boldsymbol{v}$$
$$\nabla \times (\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} + \boldsymbol{u}(\nabla \cdot \boldsymbol{v}) - \boldsymbol{v}(\nabla \cdot \boldsymbol{u}) - (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}$$

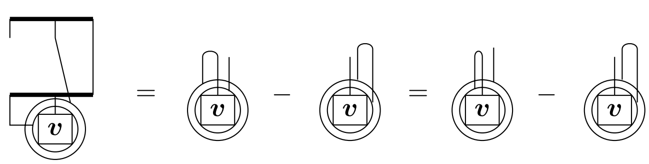
Contraction of Levi-Civita symbols and Leibnitz rule.



1.5.9 rot rot $= \text{grad}\, \text{div} - \triangle$

$$\text{rot}\, \text{rot}\, \boldsymbol{v} = \text{grad}\, \text{div}\, \boldsymbol{v} - \triangle \boldsymbol{v}$$
$$\nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - (\nabla \cdot \nabla)\boldsymbol{v}$$

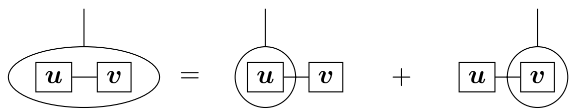
Levi-Civita contraction and symmetry fo derivatives.



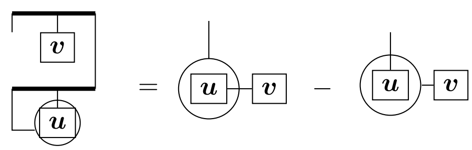
1.5.10 $\text{grad}(u \cdot v) = v \times \text{rot}\, u + (v \cdot \text{grad})u + u \times \text{rot}\, v + (u \cdot \text{grad})v$

$$\text{grad}(\boldsymbol{u} \cdot \boldsymbol{v}) = \boldsymbol{v} \times \text{rot}\, \boldsymbol{u} + (\boldsymbol{v} \cdot \text{grad})\boldsymbol{u} + \boldsymbol{u} \times \text{rot}\, \boldsymbol{v} + (\boldsymbol{u} \cdot \text{grad})\boldsymbol{v}$$
$$\nabla(\boldsymbol{u} \cdot \boldsymbol{v}) = \boldsymbol{v} \times (\nabla \times \boldsymbol{u}) + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} + \boldsymbol{u} \times (\nabla \times \boldsymbol{v}) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}$$

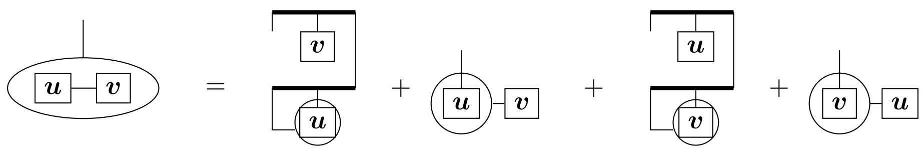
First expand by Leibnitz rule.



We cannot write this operation by the normal way, then consider what Levi-Civita contraction makes them. The first term of R.H.S. is from the following contraction.



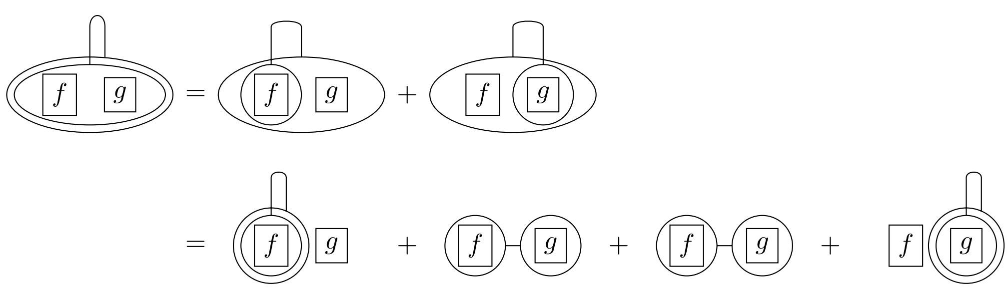
As well in the second term, we get the following notation.



1.5.11 $\triangle(fg) = (\triangle f)g + 2\ \mathrm{grad}\, f \cdot \mathrm{grad}\, g + f(\triangle g)$

$$\triangle(fg) = (\triangle f)g + 2\ \mathrm{grad}\, f \cdot \mathrm{grad}\, g + f(\triangle g)$$
$$\nabla \cdot \nabla(fg) = (\nabla \cdot \nabla f)g + 2(\nabla f) \cdot (\nabla g) + f(\nabla \cdot \nabla g)$$

Leibnitz rule twice.

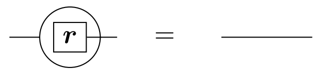


1.6 Derivative formula on a position vector

When you differentiate a position vector with a nabla, you can contract as the follows:

$$\frac{\partial}{\partial r_i} r_j \boldsymbol{e}_j = \delta_{ij} \boldsymbol{e}_j$$

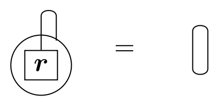
Displaying this in graphical notation, the position vector and the circle disappear, and both ends are connected.



1.6.1 $\operatorname{div} r = 3$

$$\operatorname{div} \boldsymbol{r} = \nabla \cdot \boldsymbol{r} = \delta_{ii} = 3$$

R.H.S. is dimension of the space; $\operatorname{div} \boldsymbol{r} = 4$ in 4-dim, n in n-dim.

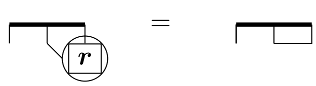


The ring of R.H.S. represents connection of both ends of Kronecker delta, i.e. δ_{ii}. In 3-dim space summed up through $i = 1, 2, 3$.

1.6.2 $\operatorname{rot} r = 0$

$$\operatorname{rot} \boldsymbol{r} = \nabla \times \boldsymbol{r} = 0$$

Contract the derivative of position.



R.H.S. represents $\epsilon_{ijk}\delta_{jk} = \epsilon_{ijj}$ so that zero.

2 Levi-Civita symbol contraction

2.1 What you have to know

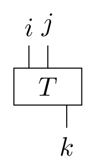
In this section, knowledge of

- Upper/lower index
- Levi-Civita symbol
- Einstein notation
- Kronecker delta

is required.

2.2 Display of tensors

In Penrose graphical notation, tensors are drawn as a box that has as many branches as the rank. Direction of branches corresponds to upper/lower indices.



$T^{ij}{}_{k}$

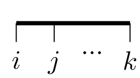
2.2.1 Kronecker delta

Kronecker delta is displayed as a branch whose both ends are open.

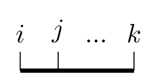
$\delta^{i}{}_{j}$

2.2.2 Levi-Civita symbol

Horizontal thick line represents anti-symmetry, i.e. Levi-Civita symbol.

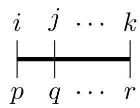


$\epsilon_{ij\ldots k}$



$\epsilon^{ij\ldots k}$

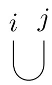
Penrose's paper draws product of Levi-Civita symbols with the same rank as the following figure. We also use this notation.



$\epsilon^{ij\ldots k}\epsilon_{pq\ldots r}$

2.2.3 Metric tensor

Bending Kronecker delta, you can represent metric tensor.



g^{ij}

g_{ij}

2.3 Contraction of Levi-Civita symbols

Contraction of Levi-Civita symbols is often defined with a determinant of Kronecker delta:

$$\epsilon^{ij\cdots k}\epsilon_{pq\cdots r} = \begin{vmatrix} \delta^i_p & \delta^i_q & \cdots & \delta^i_r \\ \delta^j_p & \delta^j_q & \cdots & \delta^j_r \\ \vdots & \vdots & \ddots & \ddots \\ \delta^k_p & \delta^k_q & \cdots & \delta^k_r \end{vmatrix}$$

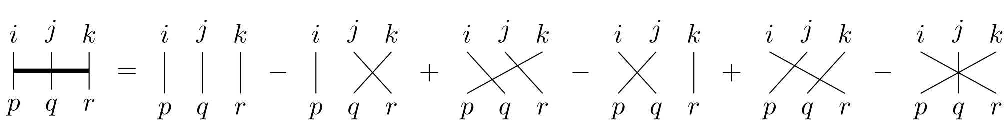
Since it would be complicated to express expansions in general dimensions in graphical notation, here we will discuss contraction formulas in three and four dimensions, which are frequently used in relativity theory. You can calculate in the same way in other dimensions.

2.3.1 Contraction in 3-dim.

Writing down with Kronecker delta,

$$\begin{aligned} \epsilon^{ijk}\epsilon_{pqr} &= \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ \delta^k_p & \delta^k_q & \delta^k_r \end{vmatrix} \\ &= \delta^i_p\delta^j_q\delta^k_r - \delta^i_p\delta^j_r\delta^k_q + \delta^i_q\delta^j_r\delta^k_p - \delta^i_q\delta^j_p\delta^k_r + \delta^i_r\delta^j_p\delta^k_q - \delta^i_r\delta^j_q\delta^k_p . \end{aligned}$$

Draw all the connections between the upper and lower ends, and give the sign corresponding to the permutation.

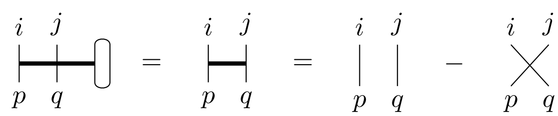


Graphical notation of Levi-Civita symbol plays a great role when some of the branches are contracted.

When an pair of upper and lower ends is connected, as well as in 1.3.1, the rank of the symbol decreases. Note that all the three numbers in i, j, k are different, as in p, q, r.

$$
\begin{aligned}
\epsilon^{ijk}\epsilon_{pqk} &= \begin{vmatrix} \delta^i_p & \delta^i_q & 0 \\ \delta^j_p & \delta^j_q & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \delta^i_p & \delta^i_q \\ \delta^j_p & \delta^j_q \end{vmatrix} \\
&= \delta^i_p\delta^j_q - \delta^i_q\delta^j_p
\end{aligned}
$$

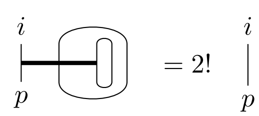
In graphical notation, delete the contracted branches.



Two pairs contractions needs a factor 2! corresponding to permutations of contracted two indices.

$$
\epsilon^{ijk}\epsilon_{pjk} = 2! \begin{vmatrix} \delta^i_p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\delta^i_p
$$

This is as well in graphical notation.

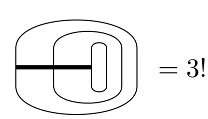


The reason will be intuitive why the factor 2! is required. Consider the example where two pairs $(j, q), (k, r)$ in a pair of three indices $(i, j, k), (p, q, r)$ are contracted. According to Einstein notation, you have to sum up the following two cases: the inner connected line is δ^j_q, the outer δ^k_r; exchanged.

When you contract all the three indices, you have to set a factor 3!.

$$
\epsilon^{ijk}\epsilon_{ijk} = 3! \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6
$$

You will be understand intuitively considering permutation of the three pairs: $(i, p), (j, q), (k, r)$.



2.3.2 Contraction in 4-dim.

We use Euclid metric from now on. In Minkowski metric, the metric tensor

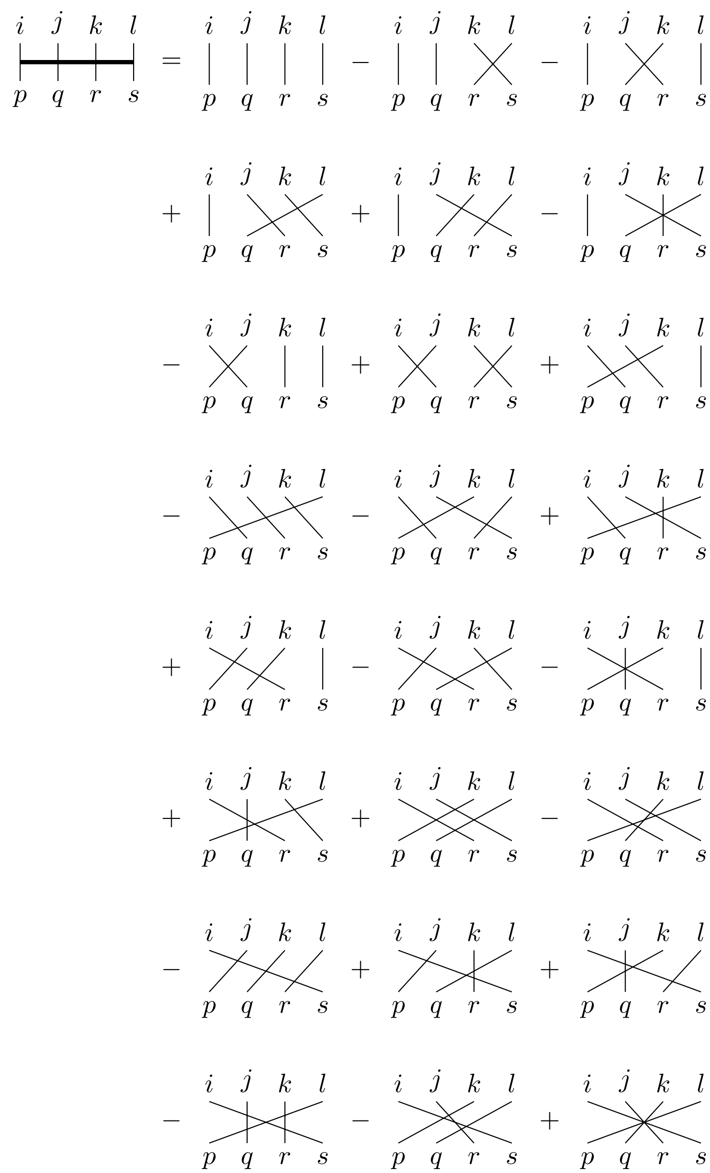
$$g_{ij} = \begin{cases} 1 & i = j = 0 \\ -1 & i = j \neq 0 \\ 0 & i \neq j \end{cases}$$

are multiplied to all the cotangent indices, so that every results in this section is negative.

Computing the troublesome expansion with Kronecker delta, we get the following terms:

$$\begin{aligned}
\epsilon^{ijkl}\epsilon_{pqrs} =& \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r & \delta^i_s \\ \delta^j_p & \delta^j_q & \delta^j_r & \delta^j_s \\ \delta^k_p & \delta^k_q & \delta^k_r & \delta^k_s \\ \delta^l_p & \delta^l_q & \delta^l_r & \delta^l_s \end{vmatrix} \\
=& \delta^i_p\delta^j_q\delta^k_r\delta^l_s - \delta^i_p\delta^j_q\delta^k_s\delta^l_r - \delta^i_p\delta^j_r\delta^k_q\delta^l_s + \delta^i_p\delta^j_r\delta^k_s\delta^l_q + \delta^i_p\delta^j_s\delta^k_q\delta^l_r - \delta^i_p\delta^j_s\delta^k_r\delta^l_q \\
& - \delta^i_q\delta^j_p\delta^k_r\delta^l_s + \delta^i_q\delta^j_p\delta^k_s\delta^l_r + \delta^i_q\delta^j_r\delta^k_p\delta^l_s - \delta^i_q\delta^j_r\delta^k_s\delta^l_p - \delta^i_q\delta^j_s\delta^k_p\delta^l_r + \delta^i_q\delta^j_s\delta^k_r\delta^l_p \\
& + \delta^i_r\delta^j_p\delta^k_q\delta^l_s - \delta^i_r\delta^j_p\delta^k_s\delta^l_q - \delta^i_r\delta^j_q\delta^k_p\delta^l_s + \delta^i_r\delta^j_q\delta^k_s\delta^l_p + \delta^i_r\delta^j_s\delta^k_p\delta^l_q - \delta^i_r\delta^j_s\delta^k_q\delta^l_p \\
& - \delta^i_s\delta^j_p\delta^k_q\delta^l_r + \delta^i_s\delta^j_p\delta^k_r\delta^l_q + \delta^i_s\delta^j_q\delta^k_p\delta^l_r - \delta^i_s\delta^j_q\delta^k_r\delta^l_p - \delta^i_s\delta^j_r\delta^k_p\delta^l_q + \delta^i_s\delta^j_r\delta^k_q\delta^l_p
\end{aligned}$$

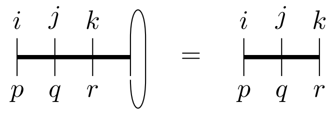
Display in graphical notation is also hazardous.



Contraction decreases the rank.

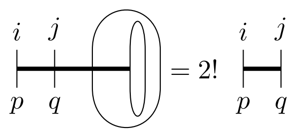
$$\epsilon^{ijkl}\epsilon_{pqrl} = \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r & 0 \\ \delta^j_p & \delta^j_q & \delta^j_r & 0 \\ \delta^k_p & \delta^k_q & \delta^k_r & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ \delta^k_p & \delta^k_q & \delta^k_r \end{vmatrix}$$

Delete the contracted part as well as in 3-dim.

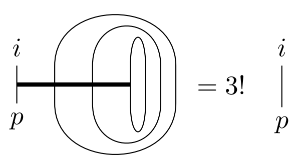


When multiple indices are contracted, put the factor $2!, 3!, 4!$.

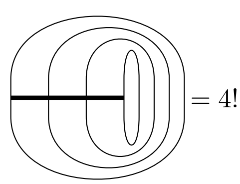
$$\epsilon^{ijkl}\epsilon_{pqkl} = 2!\begin{vmatrix}\delta^i_p & \delta^i_q & 0 & 0\\ \delta^j_p & \delta^j_q & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{vmatrix} = 2!\begin{vmatrix}\delta^i_p & \delta^i_q\\ \delta^j_p & \delta^j_q\end{vmatrix}$$



$$\epsilon^{ijkl}\epsilon_{pjkl} = 3!\begin{vmatrix}\delta^i_p & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{vmatrix} = 3!\delta^i_p$$



$$\epsilon^{ijkl}\epsilon_{ijkl} = 4!$$



3 Differential form analysis

3.1 What you have to know

In this section, knowledge of

- differential form
- Levi-Civita symbol
- wedge product
- exterior product
- interior product

is required.

All the indices follow Einstein notation. Derivative symbol ∂_i differentiates only the immediate following quantity, i.e., $\partial_i A^j B^k = \partial_i (A^j) B^k$.

We write n-ord symmetric group as $\mathfrak{S}_n$. The sign of $P \in \mathfrak{S}_n$ is $\mathrm{sgn}(P)$. k-ranked Levi-Civita symbol is defined as the follows:

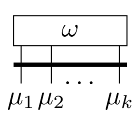
$$\epsilon^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k} \equiv \det \begin{pmatrix} \delta^{\mu_1}_{\nu_1} & \cdots & \delta^{\mu_1}_{\nu_k} \\ \vdots & \ddots & \vdots \\ \delta^{\mu_k}_{\nu_1} & \cdots & \delta^{\mu_k}_{\nu_k} \end{pmatrix} = \sum_{P \in \mathfrak{S}_n} \mathrm{sgn}(P) \delta^{P(\mu_1)}_{\nu_1} \cdots \delta^{P(\mu_k)}_{\nu_k}$$

3.2 Differential forms and operations

3.2.1 k-form

$$\omega \equiv \omega_{\mu_1 \cdots \mu_k} \, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k} = \omega_{\mu_1 \cdots \mu_k} \epsilon^{\mu_1 \cdots \mu_k}_{\mu'_1 \cdots \mu'_k} \, \mathrm{d}x^{\mu'_1} \otimes \cdots \otimes \mathrm{d}x^{\mu'_k} \, .$$

Using representation of Levi-Civita symbol, k-form is drawn as the follows:



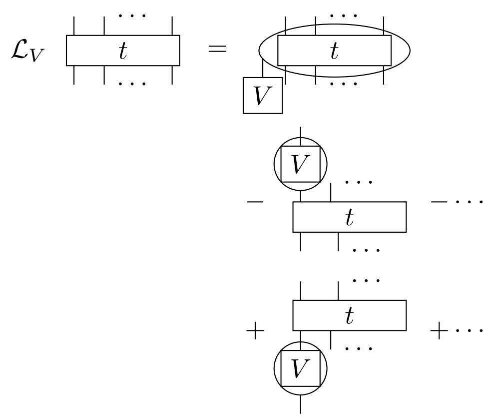
The thick line means Levi-Civita symbol. The rectangle labelled ω is the factor $\omega_{\mu'_1 \cdots \mu'_k}$. That is, the entire picture represents $\omega_{\mu'_1 \cdots \mu'_k} \epsilon^{\mu'_1 \cdots \mu'_k}_{\mu_1 \cdots \mu_k}$, thus k-form.

In graphical notation, it is difficult to display basis; we do not write $\mathrm{d}x$ and other symbols. Interpret as banches whose upper ends are empty connect basis for tangent space ∂_μ, and lower empty ends for cotangent space $\mathrm{d}x^\mu$.

3.2.2 Lie derivative

$$
\begin{aligned}
&\mathcal{L}_V(t^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_k}\partial_{\mu_1}\otimes\cdots\partial_{\mu_l}\otimes \mathrm{d}x^{\nu_1}\otimes\cdots\otimes \mathrm{d}x^{\nu_k})\\
&= V^\lambda\partial_\lambda t^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_k}\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_l}\otimes \mathrm{d}x^{\nu_1}\otimes\cdots\otimes \mathrm{d}x^{\nu_k}\\
&\quad - t^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_k}\partial_{\mu_1}V^\lambda\partial_\lambda\otimes\cdots\otimes\partial_{\mu_l}\otimes \mathrm{d}x^{\nu_1}\otimes\cdots\otimes \mathrm{d}x^{\nu_k}\\
&\quad - \cdots\\
&\quad - t^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_k}\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_l}V^\lambda\partial_\lambda\otimes \mathrm{d}x^{\nu_1}\otimes\cdots\otimes \mathrm{d}x^{\nu_k}\\
&\quad + t^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_k}\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_l}\otimes\partial_\lambda V^{\nu_1}\,\mathrm{d}x^\lambda\otimes\cdots\otimes \mathrm{d}x^{\nu_k}\\
&\quad + \cdots\\
&\quad + t^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_k}\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_l}\otimes \mathrm{d}x^{\nu_1}\otimes\cdots\otimes\partial_\lambda V^{\nu_k}\,\mathrm{d}x^\lambda
\end{aligned}
$$

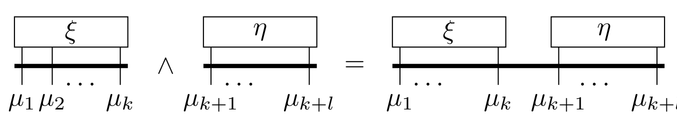
In graphical notation, this is drawn as the following figure.



3.2.3 Wedge product

$$
\begin{aligned}
&(\xi_{\mu_1\cdots\mu_k}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge \mathrm{d}x^{\mu_k})\wedge(\eta_{\mu_{k+1}\cdots\mu_{k+l}}\,\mathrm{d}x^{\mu_{k+1}}\wedge\cdots\wedge \mathrm{d}x^{\mu_{k+l}})\\
&= \xi_{\mu_1\cdots\mu_k}\eta_{\mu_{k+1}\cdots\mu_{k+l}}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge \mathrm{d}x^{\mu_{k+l}}\\
&= \xi_{\mu_1\cdots\mu_k}\eta_{\mu_{k+1}\cdots\mu_{k+l}}\epsilon^{\mu_1\cdots\mu_k\mu_{k+1}\cdots\mu_{k+l}}_{\mu'_1\cdots\mu'_k\mu'_{k+1}\cdots\mu'_{k+l}}\,\mathrm{d}x^{\mu'_1}\otimes\cdots\otimes \mathrm{d}x^{\mu'_{k+l}}
\end{aligned}
$$

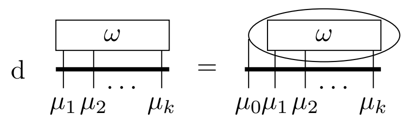
In graphical notation, connect each thick line that pierces the differential form.



3.2.4 Exterior product

$$
\begin{aligned}
\mathrm{d}\,(\omega_{\mu_1\cdots\mu_k}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge \mathrm{d}x^{\mu_k}) &= \partial_{\mu_0}\omega_{\mu_1\cdots\mu_k}\,\mathrm{d}x^{\mu_0}\wedge \mathrm{d}x^{\mu_1}\wedge\cdots\wedge \mathrm{d}x^{\mu_k}\\
&= \partial_{\mu_0}\omega_{\mu_1\cdots\mu_k}\epsilon^{\mu_0\mu_1\cdots\mu_k}_{\mu'_0\mu'_1\cdots\mu'_k}\,\mathrm{d}x^{\mu'_0}\otimes\cdots\otimes \mathrm{d}x^{\mu'_k}
\end{aligned}
$$

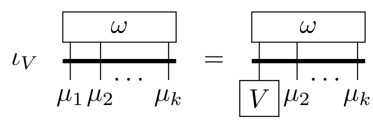
Surround the rectangle of factor by a circle and stick a branch into the beginning of the thick line.



3.2.5 Interior product

$$\iota_V(\omega_{\mu_1\cdots\mu_k}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_k}) = \omega_{\mu_1\cdots\mu_k}\epsilon^{\mu_1\cdots\mu_k}_{\mu'_1\cdots\mu'_k}V^{\mu'_1}\,\mathrm{d}x^{\mu'_2}\otimes\cdots\otimes\mathrm{d}x^{\mu'_k}$$

Put V at the first branch.



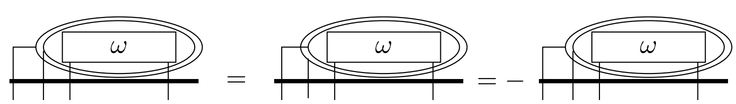
3.3 Formulae with differential forms

All the following formulae are proved in the same way as Einstein notation.

3.3.1 Poincaré lemma

$$\mathrm{d}^2 = 0$$

Use symmetry of the second derivatives and anti-symmetry of Levi-Civita symbol.



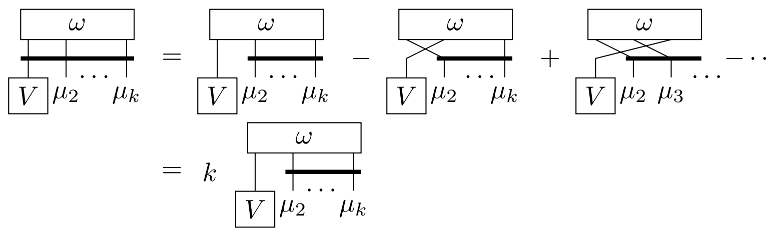
L.H.S. and R.H.S. differs only the sign thus the value must be zero.

3.3.2 Another representation of interior product

$$\iota_V\omega = k\omega_{\mu_1\cdots\mu_k}\epsilon^{\mu_2\cdots\mu_k}_{\mu'_2\cdots\mu'_k}V^{\mu'_1}\,\mathrm{d}x^{\mu'_2}\otimes\cdots\otimes\mathrm{d}x^{\mu'_k}$$

You can display interior product ι_V. Note that the indices of Levi-Civita symbol starts from μ_2, μ'_2.

First put the branch of ω to V off the thick line.

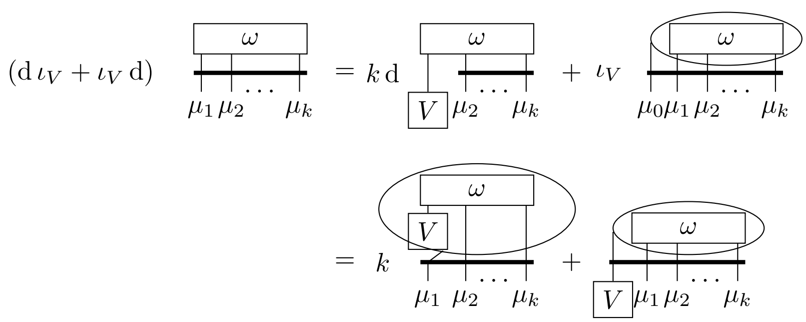


Remember $\omega_{\mu_1 \cdots \mu_k}$ is anti-symmetric for its indices. The positive terms in R.H.S. of the first line can be transformed into the first term by even permutation, and negative terms by odd permutation, so that all the terms are equivalent. There are k terms, thus we get the second line.

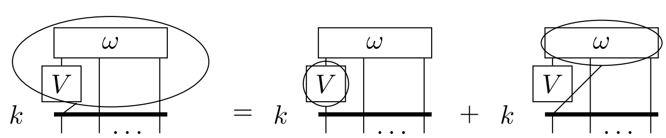
3.3.3 Cartan formula

$$(\mathrm{d}\iota_V + \iota_V \,\mathrm{d})\omega = \mathcal{L}_V \omega$$

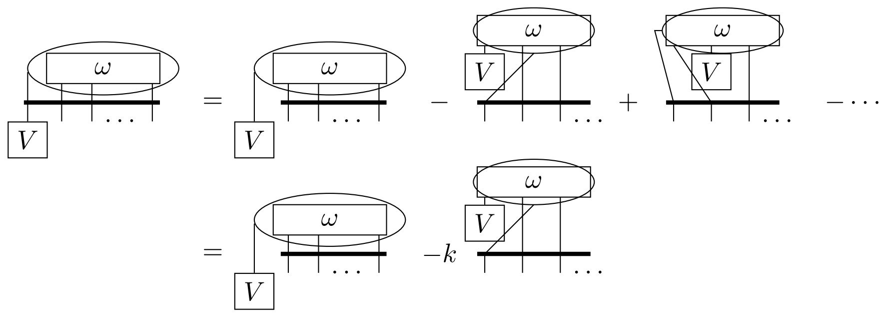
First draw down the L.H.S. We use another representation of interior product (cf. 3.3.2).



Apply Leibnitz rule to the first term.

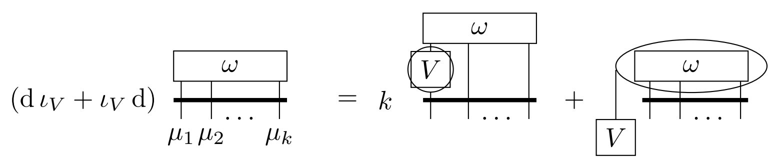


Put V off the thick line for the second term.

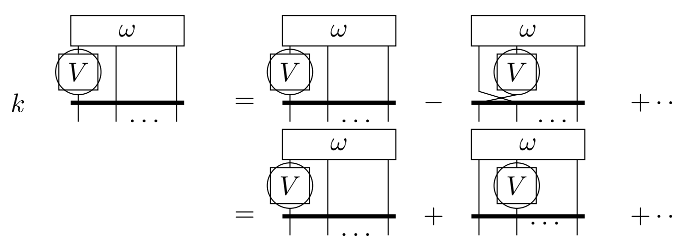


Focus on the second and the following terms in the first line. The positive terms can be transformed by even permutation, and negative terms by odd permutation, into the second term; all terms except the first are equivalent, thus the second line.

The questioned formula, therefore, is equivalent to the following figure:



The first term in R.H.S. is expanded according to anti-symmetry for indices of $\omega_{\mu_1 \cdots \mu_k}$.



Returning to the former figure, you will find Lie derivative of k-form.

References

[1] Roger Penrose, *Applications of Negative Dimensional Tensors*, Academic Press, 1971.

[2] E. Landau, E. M. Lifschitz, *The Classical Theory of Fields* Butterworth-Heinemann, 1980.

[3] M. Nakahara *Geometry, Topology and Physics, Second Edition* CRC Press, 2003