Vectors and tensors analysis by Penrose's graphical notation

Apr 9, 2023

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1 Vectors analysis

1.1 What you need to know

In this section, an understanding of the following concepts is required:

- Scalar and vector
- Dot product and cross product
- Grad, div, curl, $\triangle$
- Einstein notation
- Kronecker delta
- Levi-Civita symbol and its contraction formula
- Display of cross product with Levi-Civita symbol

When using index variables, we follow Einstein notation, and we do not distinguish between tangent and cotangent spaces. All indices are placed in the lower position.

1.2 Scalars, Vectors, and Operations

1.2.1 Scalars and vectors

A scalar is represented in a box in graphical notation as shown as

$$\text{scalar } f = \boxed{f}\ .$$

A vector is in a box with a branch:

$$\text{vector } \boldsymbol{v} = \boxed{\boldsymbol{v}}$$

This branch means Kronecker delta as we will see in 2.2.1.

1.2.2 Scalars multiples

Scalar multiplication is represented by placing the scalar coefficient next to the symbol for the vector or scalar being multiplied:

$$fg\boldsymbol{v} = \boxed{f}\,\boxed{g}\,\boxed{\boldsymbol{v}}$$

1.2.3 Dot product

The dot product of two vectors u and v is represented by connecting their branches.

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i \delta_{ij} v_j = \boxed{\boldsymbol{u}}\!-\!\!-\!\!-\boxed{\boldsymbol{v}}$$

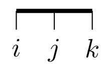
Branch connection of vectors means dot product, i.e., $\boldsymbol{u} \cdot \boldsymbol{v} = u_i \delta_{ij} v_j$.

1.2.4 Levi-Civita symbol of 3 components

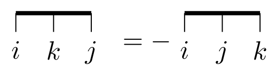
In the previous section, we displayed scalars and vectors in a square to represent their components. However, we will now display indices without a square.

The three component Levi-Civita symbol, denoted by ϵ_{ijk}, is represented graphically as follows:

$$\epsilon_{ijk} =$$



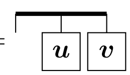
In this diagram, the thick line represents anti-symmetry. Note that exchanging the branches of the diagram flips the sign of the Levi-Civita symbol.



1.2.5 Cross product

The cross product of vectors $\boldsymbol{u} \times \boldsymbol{v}$ is represented graphically as follows, with the remaining branch indicating that the result is a vector:

$$\boldsymbol{u} \times \boldsymbol{v} =$$

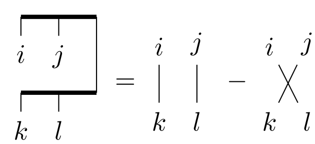


1.3 Formulae on dot and cross products of vectors

1.3.1 Contraction formula of Levi-Civita symbol

$$\epsilon_{ijm}\epsilon klm = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

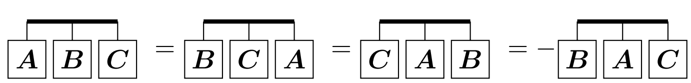
Here we use this formula without proof in the following sections. However, we will provide more details on the contraction formula in 2.



1.3.2 Scalar triple product

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = -\boldsymbol{B} \cdot (\boldsymbol{A} \times \boldsymbol{C})$$

This can be represented graphically by connecting the branches of a vector to the cross product of the remaining two, as shown below:

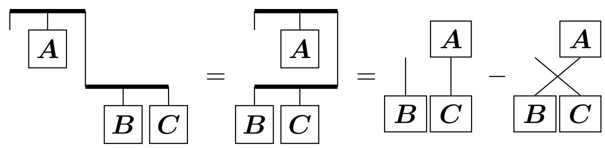


Note that odd permutations of the vectors result in a sign change, while even permutations do not.

1.3.3 Vector triple product

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$$

This can be represented by connecting the branch of the cross product to the other cross product, as shown below:

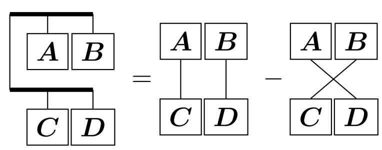


Note that after even permutations, the Levi-Civita symbols can be contracted more easily.

1.3.4 Scalar quadruple product

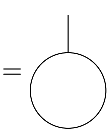
$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = \det \begin{pmatrix} \boldsymbol{A} \cdot \boldsymbol{C} & \boldsymbol{B} \cdot \boldsymbol{C} \\ \boldsymbol{A} \cdot \boldsymbol{D} & \boldsymbol{B} \cdot \boldsymbol{D} \end{pmatrix}$$

It is often written with determinant.



1.4 Derivative

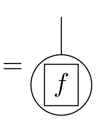
In Penrose graphical notation, the derivative operator ∇ (nabla) is represented by drawing a branch out of a circle and placing what you want to differentiate inside the circle.

$\nabla =$ 

1.4.1 Gradient

$$\operatorname{grad} f = \nabla f$$

Place the scalar in the circle.

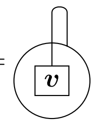
$\operatorname{grad} f = \nabla f =$ 

1.4.2 Divergent

$$\operatorname{div} \boldsymbol{v} = \nabla \cdot \boldsymbol{v}$$

Connect branches of the vector and the derivative circle.

$$\operatorname{div} \boldsymbol{v} = \nabla \cdot \boldsymbol{v} =$$



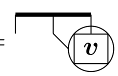
The branch represents Kronecker delta so the notation means $\partial_i \delta_{ij} v_j$.

1.4.3 Curl

$$\operatorname{curl} \boldsymbol{v} = \nabla \times \boldsymbol{v}$$

In penrose graphical notation, this is represented as the following:

$$\operatorname{curl} \boldsymbol{v} = \nabla \times \boldsymbol{v} =$$



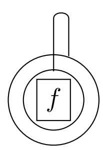
The Levi-Civita symbol ϵ_{ijk} is used to indicate the anti-symmetry. Note that the branch of what you differentiate should be placed immediately to the left of the derivative.

1.4.4 Laplacian

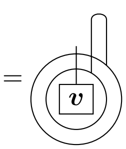
$$\triangle A = \nabla^2 A$$

Connect the branches of derivatives, i.e., $\partial_i \delta_{ij} \partial_j A$. You can operate both on a scalar and a vector.

$$\triangle f = \nabla^2 f =$$



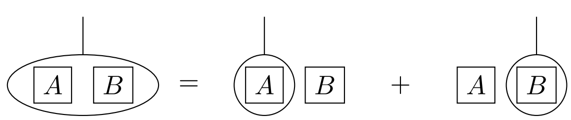
$$\triangle \boldsymbol{v} = \nabla^2 \boldsymbol{v} =$$



Especially when you operate on a scalar, you get immediately $\operatorname{div} \operatorname{grad} f = \triangle f$.

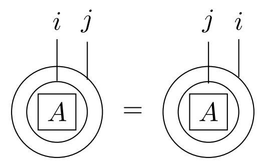
1.4.5 Leibniz rule

$$\nabla(AB) = \nabla(A)B + A\nabla(B)$$



1.4.6 Symmetry of second derivatives

A function belonging to class C^2 satisfies symmetry of second derivatives, which means that the order of differentiation does not matter. In Penrose graphical notation, this symmetry can be represented as an exchange of the in and out circles.



From now on all the quantities are assumed to be smooth, unless otherwise noted.

1.5 Formulae on derivatives

All of the formulae on derivatives presented below are proven using only five operations:

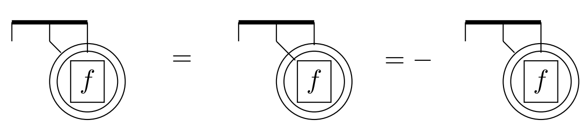
- anti-symmetry of Levi-Civita symbol
- even permutation of Levi-Civita symbol indices
- contraction of Levi-Civita symbols
- the Leibniz rule
- symmetry of second derivatives

The proofs for these formulae are done in a similar way as in Einstein notation.

1.5.1 curl grad = 0

$$\text{curl grad}\, f = \nabla \times \nabla f = 0$$

First symmetry of derivatives, second anti-symmetry of Levi-Civita symbol.

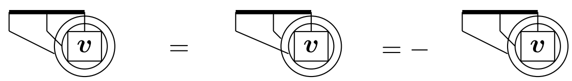


We have shown that curl grad $f = -$curl grad f, which means that the value of curl grad f is zero.

1.5.2 div curl $= 0$

$$\operatorname{div}\operatorname{curl}\boldsymbol{v} = \nabla\cdot(\nabla\times\boldsymbol{v}) = 0$$

This can be proved using the same method as in 1.5.1. First, we apply the symmetry of derivatives and then the anti-symmetry of the Levi-Civita symbol.

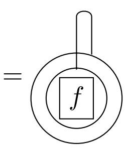


We get $\operatorname{div}\operatorname{curl}\boldsymbol{v} = -\operatorname{div}\operatorname{curl}\boldsymbol{v}$, thus the value is zero.

1.5.3 div grad $= \triangle$

$$\operatorname{div}\operatorname{grad} f = \nabla\cdot\nabla f = \triangle f$$

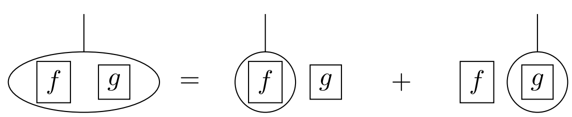
We noted in 1.4.4, but we repeat.

$\triangle f = \nabla^2 f =$ 

1.5.4 $\operatorname{grad}(fg) = g\operatorname{grad} f + f\operatorname{grad} g$

$$\begin{aligned}\operatorname{grad}(fg) &= (\operatorname{grad} f)g + f(\operatorname{grad} g)\\ \nabla(fg) &= (\nabla f)g + f(\nabla g)\end{aligned}$$

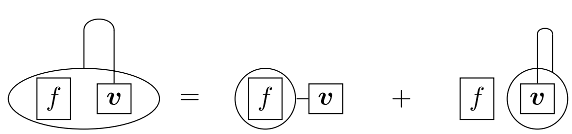
Expand by Leibniz rule.



1.5.5 $\operatorname{div}(fv) = \operatorname{grad} f\cdot v + f\operatorname{div} v$

$$\begin{aligned}\operatorname{div}(f\boldsymbol{v}) &= \operatorname{grad} f\cdot\boldsymbol{v} + f\operatorname{div}\boldsymbol{v}\\ \nabla(f\boldsymbol{v}) &= \nabla f\cdot\boldsymbol{v} + f\nabla\cdot\boldsymbol{v}\end{aligned}$$

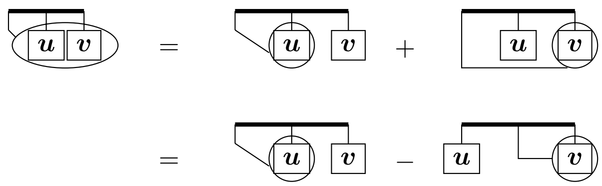
Expand by Leibniz rule.



1.5.6 $\operatorname{div}(u \times v) = \operatorname{curl} u \cdot v - u \cdot \operatorname{curl} v$

$$\operatorname{div}(\boldsymbol{u} \times \boldsymbol{v}) = \operatorname{curl} \boldsymbol{u} \cdot \boldsymbol{v} - \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v}$$
$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = (\nabla \times \boldsymbol{u})\boldsymbol{v} - \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v})$$

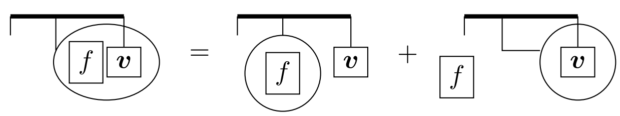
Use Leibnitz rule and anti-symmetry. Follow the rule in 1.4.3; stick the branch of what you differentiate immediately right of the derivative.



1.5.7 $\operatorname{curl}(fv) = \operatorname{grad} f \times v + f \operatorname{curl} v$

$$\operatorname{curl}(f\boldsymbol{v}) = \operatorname{grad} f \times \boldsymbol{v} + f \operatorname{curl} \boldsymbol{v}$$
$$\nabla \times (f\boldsymbol{v}) = \nabla f \times \boldsymbol{v} + f \nabla \times \boldsymbol{v}$$

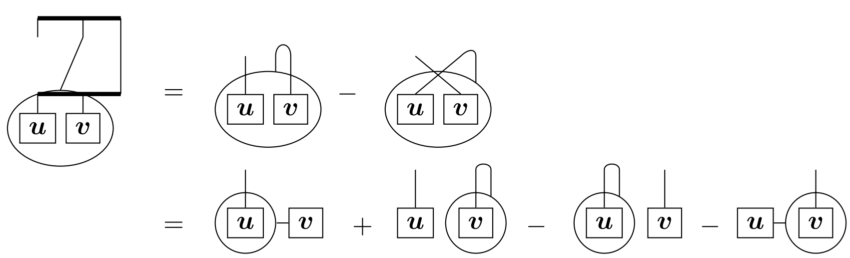
Expand by Leibnitz rule.



1.5.8 $\operatorname{curl}(u \times v) = (v \cdot \operatorname{grad})u + u \operatorname{div} v - v \operatorname{div} u - (u \cdot \operatorname{grad})v$

$$\operatorname{curl}(\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{v} \cdot \operatorname{grad})\boldsymbol{u} + \boldsymbol{u}\ \operatorname{div} \boldsymbol{v} - \boldsymbol{v}\ \operatorname{div} \boldsymbol{u} - (\boldsymbol{u} \cdot \operatorname{grad})\boldsymbol{v}$$
$$\nabla \times (\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} + \boldsymbol{u}(\nabla \cdot \boldsymbol{v}) - \boldsymbol{v}(\nabla \cdot \boldsymbol{u}) - (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}$$

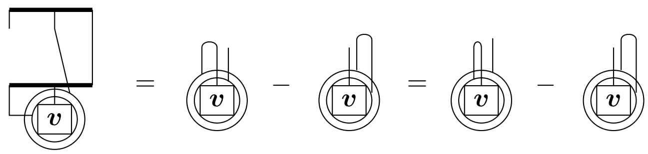
Contraction of Levi-Civita symbols and Leibnitz rule.



1.5.9 $\operatorname{curl} \operatorname{curl} = \operatorname{grad} \operatorname{div} - \triangle$

$$\operatorname{curl} \operatorname{curl} \boldsymbol{v} = \operatorname{grad} \operatorname{div} \boldsymbol{v} - \triangle \boldsymbol{v}$$
$$\nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - (\nabla \cdot \nabla)\boldsymbol{v}$$

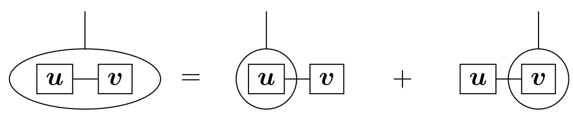
Levi-Civita contraction and symmetry of derivatives.



1.5.10 $\mathrm{grad}(u \cdot v) = v \times \mathrm{curl}\, u + (v \cdot \mathrm{grad})u + u \times \mathrm{curl}\, v + (u \cdot \mathrm{grad})v$

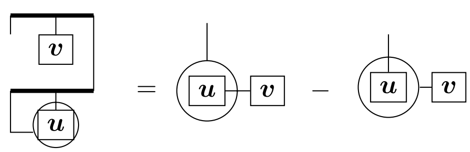
$$\mathrm{grad}(\boldsymbol{u} \cdot \boldsymbol{v}) = \boldsymbol{v} \times \mathrm{curl}\, \boldsymbol{u} + (\boldsymbol{v} \cdot \mathrm{grad})\boldsymbol{u} + \boldsymbol{u} \times \mathrm{curl}\, \boldsymbol{v} + (\boldsymbol{u} \cdot \mathrm{grad})\boldsymbol{v}$$
$$\nabla(\boldsymbol{u} \cdot \boldsymbol{v}) = \boldsymbol{v} \times (\nabla \times \boldsymbol{u}) + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} + \boldsymbol{u} \times (\nabla \times \boldsymbol{v}) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}$$

First expand by Leibniz rule.

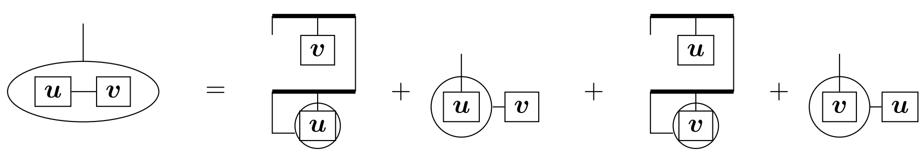


(1)

We cannot write this operation in the normal way, so let's consider how Levi-Civita contraction can help. The first term of right-hand side comes from the following contraction.



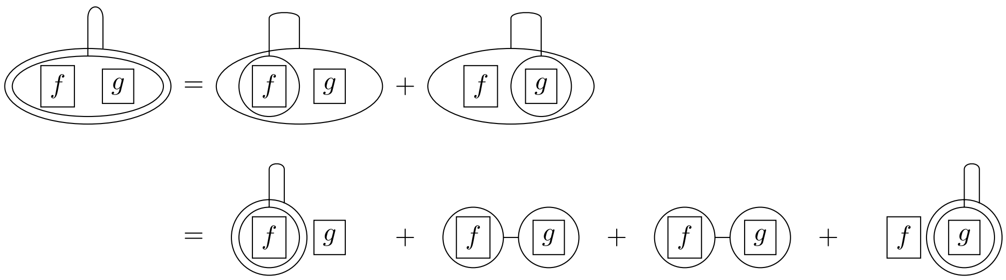
As well in the second term, we can represent it using the following notation.



1.5.11 $\triangle(fg) = (\triangle f)g + 2\ \mathrm{grad}\, f \cdot \mathrm{grad}\, g + f(\triangle g)$

$$\triangle(fg) = (\triangle f)g + 2\ \mathrm{grad}\, f \cdot \mathrm{grad}\, g + f(\triangle g)$$
$$\nabla \cdot \nabla(fg) = (\nabla \cdot \nabla f)g + 2(\nabla f) \cdot (\nabla g) + f(\nabla \cdot \nabla g)$$

Leibnitz rule twice.

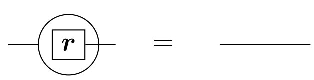


1.6 Derivative formula on a position vector

When you differentiate a position vector with a nabla, you can contract as the follows:

$$\frac{\partial}{\partial r_i} r_j \boldsymbol{e}_j = \delta_{ij} \boldsymbol{e}_j$$

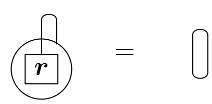
Displaying this in graphical notation, the position vector and the circle disappear, and both ends are connected.



1.6.1 $\operatorname{div} r = 3$

$$\operatorname{div} \boldsymbol{r} = \nabla \cdot \boldsymbol{r} = \delta_{ii} = 3$$

R.H.S. is actually dimension of the space; $\operatorname{div} \boldsymbol{r} = 4$ in 4-dim, n in n-dim.

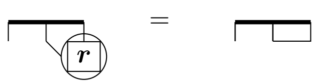


The ring of R.H.S. represents connection of both ends of Kronecker delta, i.e. δ_{ii}. In 3-dim space summed up through $i = 1, 2, 3$.

1.6.2 $\operatorname{curl} r = 0$

$$\operatorname{curl} \boldsymbol{r} = \nabla \times \boldsymbol{r} = 0$$

Contract the derivative of position.



R.H.S. represents $\epsilon_{ijk}\delta_{jk} = \epsilon_{ijj}$ so that zero.

2 Levi-Civita symbols

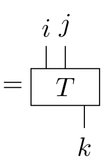
2.1 What you need to know

This section requires an understanding of the following concepts:

- Upper/lower indices
- Levi-Civita symbol
- Einstein notation
- Kronecker delta
- Determinant and inverse matrix

2.2 Display of tensors

In Penrose graphical notation, tensors are drawn as a box with branches corresponding to the rank of the tensor.

$T^{ij}_{k} =$ 

The direction of these branches indicates upper/lower indices.

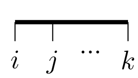
2.2.1 Kronecker delta

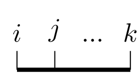
Kronecker delta is displayed as a branch that connects two open ends.

$\delta^{i}_{j} =$ 

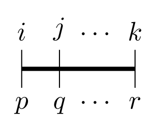
2.2.2 Levi-Civita symbol

Horizontal thick line represents Levi-Civita symbol.

$\epsilon_{ij\cdots k} =$ 

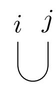
, $\epsilon^{ij\cdots k} =$ 

Penrose's paper represents the product of two Levi-Civita symbols with the same rank as shown in the following figure. We will also be using this notation.

$\epsilon^{ij\cdots k}_{pq\cdots r} =$ 

2.2.3 Metric tensor

Bending Kronecker delta, you can represent metric tensor.

$g^{ij} =$  , $\qquad g_{ij} =$ 

2.3 Determinant and inverse matrix

Levi-Civita symbols are often used for determinants and inverse matrices. From now on, we have the lower indices to represent rows, and upper indices to represent columns;

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow M_1^2 = b.$$

2.3.1 Determinant

According to the definition, $n \times n$ matrix M satisfies

$$\det M = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) M_1^{\sigma_1} \cdots M_n^{\sigma_n},$$

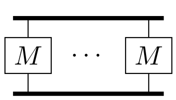
with symmetry group $\mathfrak{S}_n$. Note that the sign of permutation is equivalent to that of Levi-Civita symbol, i.e. $\operatorname{sgn}(\sigma) = \epsilon^{1\cdots n}_{\sigma_1 \cdots \sigma_n}$. Therefore

$$\det M = \epsilon^{1\cdots n}_{\sigma_1 \cdots \sigma_n} M_1^{\sigma_1} \cdots M_n^{\sigma_n},$$

but it is difficult to manipulate the lower indices in this notation, then we use the following representation:

$$\det M = \frac{1}{n!} \epsilon^{1\cdots n}_{\sigma_1 \cdots \sigma_n} \epsilon^{\tau_1 \cdots \tau_n}_{1 \cdots n} M^{\sigma_1}_{\tau_1} \cdots M^{\sigma_n}_{\tau_n} \tag{2}$$

Drawing this in graphical notation, we get

$\det M = \dfrac{1}{\dim M!}$  .

2.3.2 Cofactor expansion

Since we get the representation of determinant in graphical notation, we anticipate that cofactor expansion can be interpreted in an intrinsic way.

$$\det M = \sum_j (-1)^{i+j} M_i^j \det \begin{pmatrix} M_1^1 & \cdots & M_1^{j-1} & M_1^{j+1} & \cdots & M_1^n \\ \vdots & & \vdots & \vdots & & \vdots \\ M_{i-1}^1 & \cdots & M_{i-1}^{j-1} & M_{i-1}^{j+1} & \cdots & M_{i-1}^n \\ M_{i+1}^1 & \cdots & M_{i+1}^{j-1} & M_{i+1}^{j+1} & \cdots & M_{i+1}^n \\ \vdots & & \vdots & \vdots & & \vdots \\ M_n^1 & \cdots & M_n^{j-1} & M_n^{j+1} & \cdots & M_n^n \end{pmatrix} \tag{3}$$

Here the index i is fixed, but clouds are cleared when you run i, too. Using (2),

$$\det M = \frac{1}{n}\sum_{i,j}(-1)^{i+j}M_i^j\frac{1}{(n-1)!}\epsilon^{1\cdots i-1,i+1\cdots n}_{\sigma_1\cdots\sigma_{i-1}\sigma_{i+1}\cdots\sigma_n}\epsilon^{\tau_1\cdots\tau_{j-1}\tau_{j+1}\cdots\tau_n}_{1\cdots j-1,j+1\cdots n}M^{\sigma_1}_{\tau_1}\cdots M^{\sigma_n}_{\tau_n}.$$

σ_k, τ_k runs $\{1,\cdots,i-1,i+1,\cdots,n\},\{1,\cdots j-1,j+1,\cdots n\}$, respectively. Remark that LEvi-Civita part is

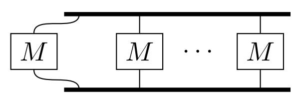
$$\begin{aligned}
&\epsilon^{\sigma_1\cdots\sigma_{i-1}\sigma_{i+1}\cdots\sigma_n}_{1\cdots i-1,i+1\cdots n}\epsilon^{1\cdots j-1,j+1\cdots n}_{\tau_1\cdots\tau_{j-1}\tau_{j+1}\cdots\tau_n}\\
&\quad=\epsilon^{\sigma_1\cdots\sigma_{i-1},i,\sigma_{i+1}\cdots\sigma_n}_{1\cdots i-1,i,i+1\cdots n}\epsilon^{1\cdots j-1,j,j+1\cdots n}_{\tau_1\cdots\tau_{j-1},j,\tau_{j+1}\cdots\tau_n}\\
&\quad=\delta^i_{\sigma_i}\delta^{\tau_j}_j\epsilon^{\sigma_1\cdots\sigma_{i-1}\sigma_i\sigma_{i+1}\cdots\sigma_n}_{1\cdots i-1,i,i+1\cdots n}\epsilon^{1\cdots j-1,j,j+1\cdots n}_{\tau_1\cdots\tau_{j-1}\tau_j\tau_{j+1}\cdots\tau_n}\\
&\quad=(-1)^{i+j}\delta^i_{\sigma_i}\delta^{\tau_j}_j\epsilon^{\sigma_i\sigma_1\cdots\sigma_{i-1}\sigma_{i+1}\cdots\sigma_n}_{1\cdots i-1,i,i+1\cdots n}\epsilon^{1\cdots j-1,j,j+1\cdots n}_{\tau_j\tau_1\cdots\tau_{j-1}\tau_{j+1}\cdots\tau_n}
\end{aligned}$$

thus

$$\det M = \frac{1}{n}M^{\tau_j}_{\sigma_i}\frac{1}{(n-1)!}\epsilon^{\sigma_i\sigma_1\cdots\sigma_{i-1}\sigma_{i+1}\cdots\sigma_n}_{1\cdots i-1,i,i+1\cdots n}\epsilon^{1\cdots j-1,j,j+1\cdots n}_{\tau_j\tau_1\cdots\tau_{j-1}\tau_{j+1}\cdots\tau_n}M^{\sigma_1}_{\tau_1}\cdots M^{\sigma_n}_{\tau_n}.$$

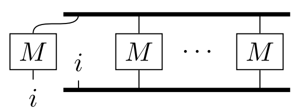
Comparing with graphical notation of determinant, $M^{\tau_j}_{\sigma_i}$ corresponds to the first M, and remaining $n-1$ M's composes a cofactor.

$$\det M = \frac{1}{\dim M!}$$



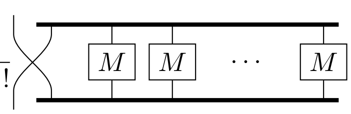
If you do not run an index as well as (3), the branch of the first M is cut off to leave the same indices:

$$\det M = \frac{1}{(\dim M - 1)!}$$



This has to be the product of matrix M and its adjugate $\tilde{M}$. Thus the adjugate matrix is,

$$\tilde{M} = \frac{1}{(\dim M - 1)!}$$

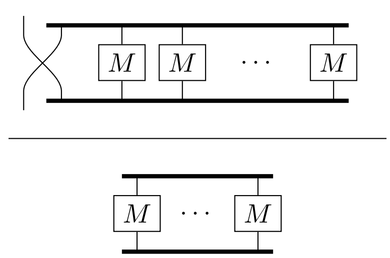


. (4)

2.3.3 Inverse matrix

According to definition, $M^{-1} = \tilde{M}/M$, i.e.

$M^{-1} = \dim M$ $M^{-1} = \dim M$



.

2.4 Contraction of Levi-Civita symbols

Contraction of Levi-Civita symbols is often defined with a determinant of Kronecker delta:

$$\epsilon^{ij\cdots k}\epsilon_{pq\cdots r} = \begin{vmatrix} \delta^i_p & \delta^i_q & \cdots & \delta^i_r \\ \delta^j_p & \delta^j_q & \cdots & \delta^j_r \\ \vdots & \vdots & \ddots & \ddots \\ \delta^k_p & \delta^k_q & \cdots & \delta^k_r \end{vmatrix}$$

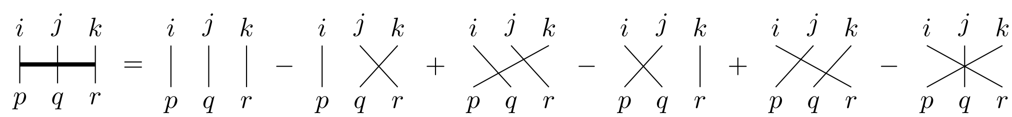
Since it can be challenging to express expansions in general dimensions using graphical notation, we will discuss contraction formulas in three and four dimensions, which are frequently used in relativity theory. However, it is possible to calculate contractions in the same way in other dimensions.

2.4.1 Contraction in 3-dim.

Writing down with Kronecker delta,

$$\begin{aligned} \epsilon^{ijk}\epsilon_{pqr} &= \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ \delta^k_p & \delta^k_q & \delta^k_r \end{vmatrix} \\ &= \delta^i_p\delta^j_q\delta^k_r - \delta^i_p\delta^j_r\delta^k_q + \delta^i_q\delta^j_r\delta^k_p - \delta^i_q\delta^j_p\delta^k_r + \delta^i_r\delta^j_p\delta^k_q - \delta^i_r\delta^j_q\delta^k_p. \end{aligned}$$

Draw all the connections between the upper and lower ends, and indicate the sign corresponding to the permutation.

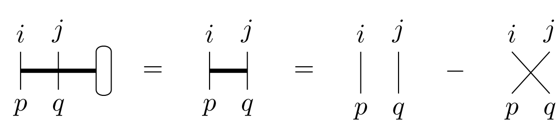


Graphical notation of Levi-Civita symbol plays a significant role when some of the branches are contracted.

When an pair of upper and lower ends is connected, as well as in 1.3.1, the rank of the symbol decreases. It is imortant to note that all three numbers in i, j, k are different, as in p, q, r.

$$\begin{aligned} \epsilon^{ijk}\epsilon_{pqk} &= \begin{vmatrix} \delta^i_p & \delta^i_q & 0 \\ \delta^j_p & \delta^j_q & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \delta^i_p & \delta^i_q \\ \delta^j_p & \delta^j_q \end{vmatrix} \\ &= \delta^i_p\delta^j_q - \delta^i_q\delta^j_p \end{aligned}$$

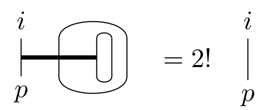
In graphical notation, erase the contracted branches.



Two pairs contractions needs a factor 2! corresponding to the permutations of the contracted two indices is introduced.

$$\epsilon^{ijk}\epsilon_{pjk} = 2!\begin{vmatrix} \delta^i_p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\delta^i_p$$

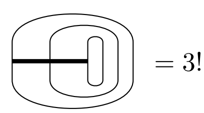
This is as well in graphical notation.



The reason for the factor of 2! in the case of two pairs of contractions is intuitive. Consider the example where two pairs of indices (j, q) and (k, r) are contracted in a set of three indices $(i, j, k), (p, q, r)$. In Einstein notation, you have to sum up the following two cases: the inner connected line is δ^j_q, the outer δ^k_r; and the inner connected line is δ^k_r, the outer δ^j_q. Thus you need to count each case, resulting in a factor of 2!.

When contracting all three indices, you have to set a factor of 3!.

$$\epsilon^{ijk}\epsilon_{ijk} = 3!\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6$$



You can understand intuitively by considering permutation of the three pairs: $(i, p), (j, q), (k, r)$.

2.4.2 Contraction in 4-dim.

We will use the Euclid metric for the rest of this section. Note that in Minkowski metric, the metric tensor

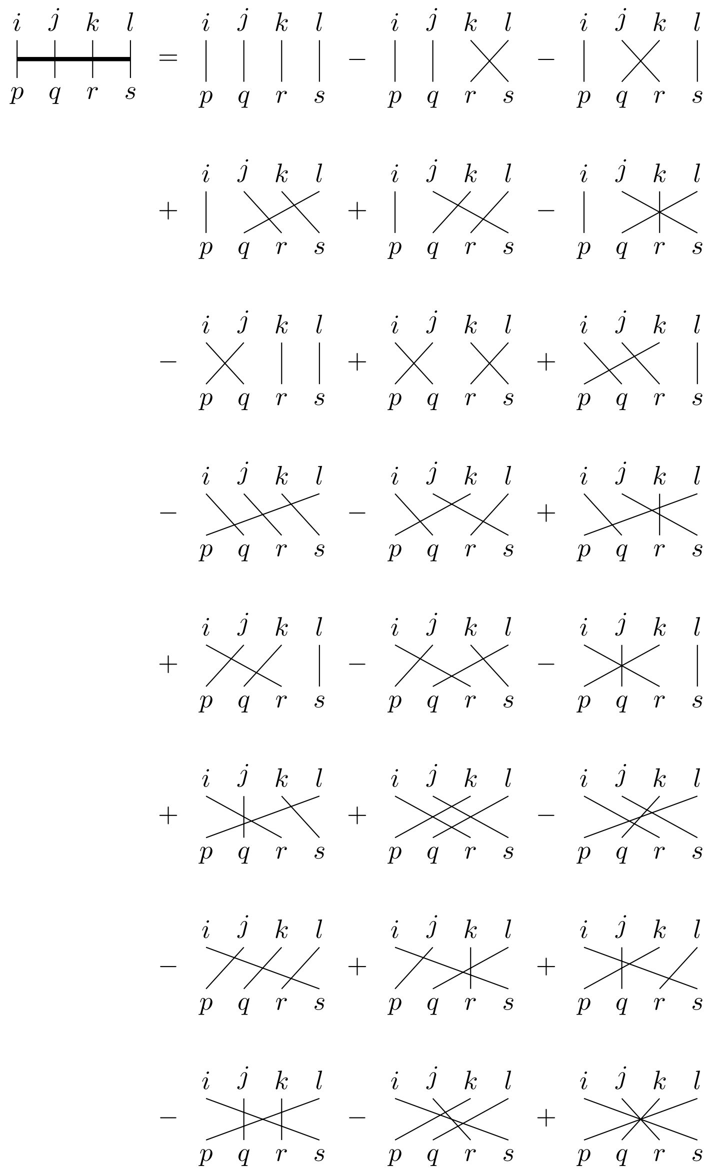
$$g_{ij} = \begin{cases} 1 & i = j = 0 \\ -1 & i = j \neq 0 \\ 0 & i \neq j \end{cases}$$

is multiplied to all the cotangent indices, resulting in negative values for all results in this section.

Computing the troublesome expansion with Kronecker delta, we get the following terms:

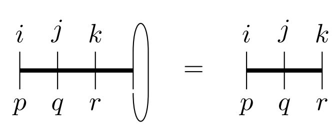
$$
\begin{aligned}
\epsilon^{ijkl}\epsilon_{pqrs} =& \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r & \delta^i_s \\ \delta^j_p & \delta^j_q & \delta^j_r & \delta^j_s \\ \delta^k_p & \delta^k_q & \delta^k_r & \delta^k_s \\ \delta^l_p & \delta^l_q & \delta^l_r & \delta^l_s \end{vmatrix} \\
=& \delta^i_p\delta^j_q\delta^k_r\delta^l_s - \delta^i_p\delta^j_q\delta^k_s\delta^l_r - \delta^i_p\delta^j_r\delta^k_q\delta^l_s + \delta^i_p\delta^j_r\delta^k_s\delta^l_q + \delta^i_p\delta^j_s\delta^k_q\delta^l_r - \delta^i_p\delta^j_s\delta^k_r\delta^l_q \\
& - \delta^i_q\delta^j_p\delta^k_r\delta^l_s + \delta^i_q\delta^j_p\delta^k_s\delta^l_r + \delta^i_q\delta^j_r\delta^k_p\delta^l_s - \delta^i_q\delta^j_r\delta^k_s\delta^l_p - \delta^i_q\delta^j_s\delta^k_p\delta^l_r + \delta^i_q\delta^j_s\delta^k_r\delta^l_p \\
& + \delta^i_r\delta^j_p\delta^k_q\delta^l_s - \delta^i_r\delta^j_p\delta^k_s\delta^l_q - \delta^i_r\delta^j_q\delta^k_p\delta^l_s + \delta^i_r\delta^j_q\delta^k_s\delta^l_p + \delta^i_r\delta^j_s\delta^k_p\delta^l_q - \delta^i_r\delta^j_s\delta^k_q\delta^l_p \\
& - \delta^i_s\delta^j_p\delta^k_q\delta^l_r + \delta^i_s\delta^j_p\delta^k_r\delta^l_q + \delta^i_s\delta^j_q\delta^k_p\delta^l_r - \delta^i_s\delta^j_q\delta^k_r\delta^l_p - \delta^i_s\delta^j_r\delta^k_p\delta^l_q + \delta^i_s\delta^j_r\delta^k_q\delta^l_p
\end{aligned}
$$

Displaying this expansion in graphical notation is also difficult.



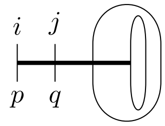
Contraction decreases the rank.

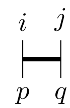
$$\epsilon^{ijkl}\epsilon_{pqrl} = \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r & 0 \\ \delta^j_p & \delta^j_q & \delta^j_r & 0 \\ \delta^k_p & \delta^k_q & \delta^k_r & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ \delta^k_p & \delta^k_q & \delta^k_r \end{vmatrix};$$



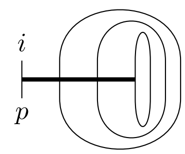
When multiple indices are contracted, set factors $2!, 3!, 4!$.

$$\epsilon^{ijkl}\epsilon_{pqkl} = 2!\begin{vmatrix} \delta^i_p & \delta^i_q & 0 & 0 \\ \delta^j_p & \delta^j_q & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2!\begin{vmatrix} \delta^i_p & \delta^i_q \\ \delta^j_p & \delta^j_q \end{vmatrix};$$



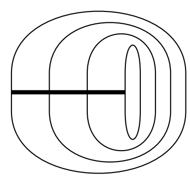
$= 2!$


$$\epsilon^{ijkl}\epsilon_{pjkl} = 3!\begin{vmatrix} \delta^i_p & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3!\delta^i_p;$$



$= 3!$


$$\epsilon^{ijkl}\epsilon_{ijkl} = 4!;$$


$= 4!$

3 Differential form analysis

3.1 What you need to know

In this section, an understanding of the following concepts is required:

- differential form
- Levi-Civita symbol
- wedge product
- exterior product
- interior product

All the indices follow Einstein notation. Derivative symbol ∂_i differentiates only the immediate following quantity, i.e., $\partial_i A^j B^k = \partial_i(A^j)B^k$.

We write n-ord. symmetric group as $\mathfrak{S}_n$. The sign of $P \in \mathfrak{S}_n$ is $\mathrm{sgn}(P)$. A k-ranked Levi-Civita symbol is defined as follows:

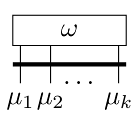
$$\epsilon^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_k} \equiv \det \begin{pmatrix} \delta^{\mu_1}_{\nu_1} & \cdots & \delta^{\mu_1}_{\nu_k} \\ \vdots & \ddots & \vdots \\ \delta^{\mu_k}_{\nu_1} & \cdots & \delta^{\mu_k}_{\nu_k} \end{pmatrix} = \sum_{P \in \mathfrak{S}_n} \mathrm{sgn}(P) \delta^{P(\mu_1)}_{\nu_1} \cdots \delta^{P(\mu_k)}_{\nu_k}$$

3.2 Differential forms and operations

3.2.1 k-form

$$\omega \equiv \omega_{\mu_1\cdots\mu_k} \, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k} = \omega_{\mu_1\cdots\mu_k} \epsilon^{\mu_1\cdots\mu_k}_{\mu'_1\cdots\mu'_k} \, \mathrm{d}x^{\mu'_1} \otimes \cdots \otimes \mathrm{d}x^{\mu'_k} \, .$$

Using representation of Levi-Civita symbol, a k-form can be represented as a box with k branches:

$\omega_{\mu_1\cdots\mu_n} =$ 

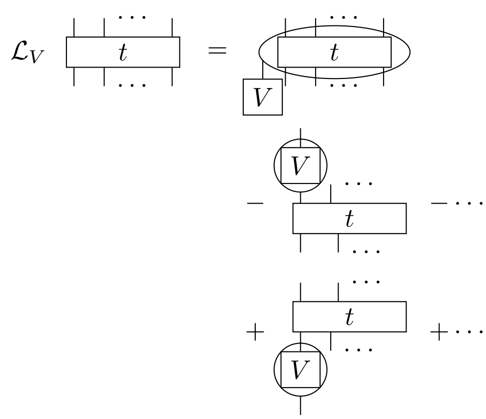
The symbol of the Levi-Civita tensor is drawn as a thick horizontal line. The rectangle labelled ω is the factor $\omega_{\mu'_1\cdots\mu'_k}$. That is, the entire picture represents $\omega_{\mu'_1\cdots\mu'_k} \epsilon^{\mu'_1\cdots\mu'_k}_{\mu_1\cdots\mu_k}$, thus a k-form.

In graphical notation, it is difficult to display the basis, so we do not write $\mathrm{d}x$ and other symbols explicitly. The branches with empty upper ends represent the basis for the tangent space ∂_μ, and the branches with empty lower ends represents the basis for the cotangent space $\mathrm{d}x^\mu$.

3.2.2 Lie derivative

$$
\begin{aligned}
&\mathcal{L}_V (t^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_k} \partial_{\mu_1} \otimes \cdots \partial_{\mu_l} \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_k})\\
&= V^\lambda \partial_\lambda t^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_k} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_l} \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_k}\\
&\quad - t^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_k} \partial_{\mu_1} V^\lambda \partial_\lambda \otimes \cdots \otimes \partial_{\mu_l} \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_k}\\
&\quad - \cdots\\
&\quad - t^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_k} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_l} V^\lambda \partial_\lambda \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_k}\\
&\quad + t^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_k} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_l} \otimes \partial_\lambda V^{\nu_1}\, \mathrm{d}x^\lambda \otimes \cdots \otimes \mathrm{d}x^{\nu_k}\\
&\quad + \cdots\\
&\quad + t^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_k} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_l} \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \partial_\lambda V^{\nu_k}\, \mathrm{d}x^\lambda
\end{aligned}
$$

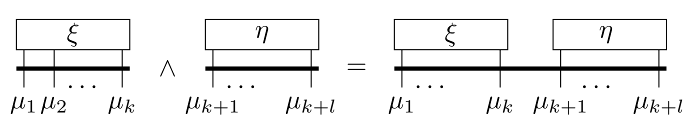
In graphical notation, this is drawn as the following figure.



3.2.3 Wedge product

$$
\begin{aligned}
&(\xi_{\mu_1\cdots\mu_k}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}) \wedge (\eta_{\mu_{k+1}\cdots\mu_{k+l}}\, \mathrm{d}x^{\mu_{k+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_{k+l}})\\
&= \xi_{\mu_1\cdots\mu_k} \eta_{\mu_{k+1}\cdots\mu_{k+l}}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{k+l}}\\
&= \xi_{\mu_1\cdots\mu_k} \eta_{\mu_{k+1}\cdots\mu_{k+l}}\, \epsilon^{\mu_1\cdots\mu_k\mu_{k+1}\cdots\mu_{k+l}}_{\mu'_1\cdots\mu'_k\mu'_{k+1}\cdots\mu_{k+l}}\, \mathrm{d}x^{\mu'_1} \otimes \cdots \otimes \mathrm{d}x^{\mu'_{k+l}}
\end{aligned}
$$

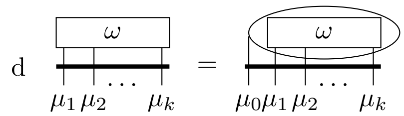
In graphical notation, connect each thick line that pierces the differential form.



3.2.4 Exterior product

$$
\begin{aligned}
\mathrm{d}\,(\omega_{\mu_1\cdots\mu_k}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}) &= \partial_{\mu_0} \omega_{\mu_1\cdots\mu_k}\, \mathrm{d}x^{\mu_0} \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}\\
&= \partial_{\mu_0} \omega_{\mu_1\cdots\mu_k}\, \epsilon^{\mu_0\mu_1\cdots\mu_k}_{\mu'_0\mu'_1\cdots\mu'_k}\, \mathrm{d}x^{\mu'_0} \otimes \cdots \otimes \mathrm{d}x^{\mu'_k}
\end{aligned}
$$

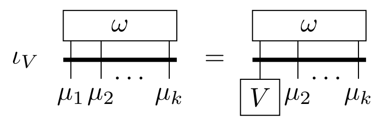
Surround the rectangle of factor by a circle and connect a branch into the beginning of the thick line.



3.2.5 Interior product

$$\iota_V(\omega_{\mu_1\cdots\mu_k}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_k}) = \omega_{\mu_1\cdots\mu_k}\epsilon^{\mu_1\cdots\mu_k}_{\mu'_1\cdots\mu'_k}V^{\mu'_1}\,\mathrm{d}x^{\mu'_2}\otimes\cdots\otimes\mathrm{d}x^{\mu'_k}$$

Put V at the first branch.



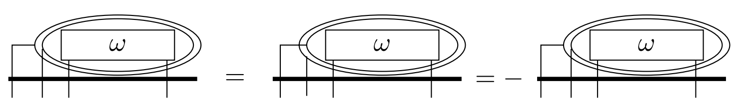
3.3 Formulae with differential forms

All the following formulae are proved in the same way as Einstein notation.

3.3.1 Poincaré lemma

$$\mathrm{d}^2 = 0$$

Use symmetry of the second derivatives and anti-symmetry of Levi-Civita symbol.



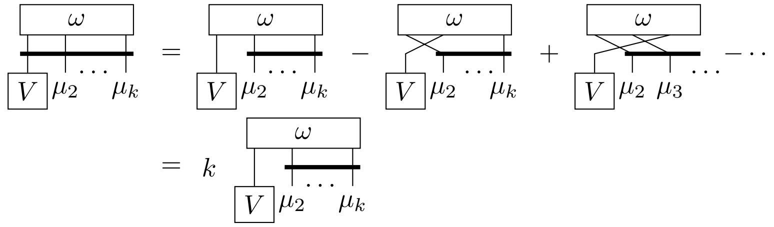
L.H.S. and R.H.S. differs only the sign thus the value must be zero.

3.3.2 Another representation of interior product

$$\iota_V\omega = k\omega_{\mu_1\cdots\mu_k}\epsilon^{\mu_2\cdots\mu_k}_{\mu'_2\cdots\mu'_k}V^{\mu'_1}\,\mathrm{d}x^{\mu'_2}\otimes\cdots\otimes\mathrm{d}x^{\mu'_k}$$

You can display interior product ι_V. Note that the indices of Levi-Civita symbol starts from μ_2, μ'_2.

First put the branch of ω to V off the thick line.

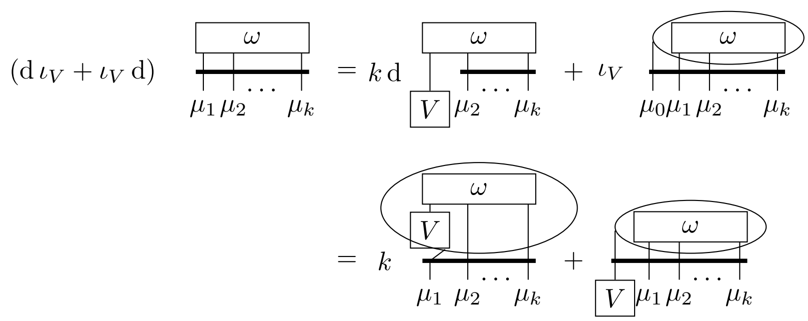


Remember $\omega_{\mu_1 \cdots \mu_k}$ is anti-symmetric for its indices. The positive terms on R.H.S. of the first line can be transformed into the first term by even permutations, and negative terms by odd permutations, so that all the terms are equivalent. There are k terms, thus we obtain the second line.

3.3.3 Cartan formula

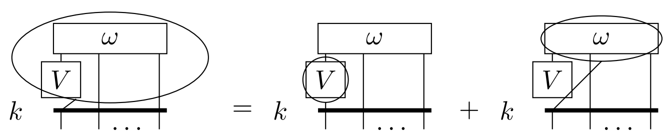
$$(\mathrm{d}\iota_V + \iota_V\,\mathrm{d})\omega = \mathcal{L}_V\omega$$

First draw down the L.H.S. We use another representation of interior product (cf. 3.3.2).

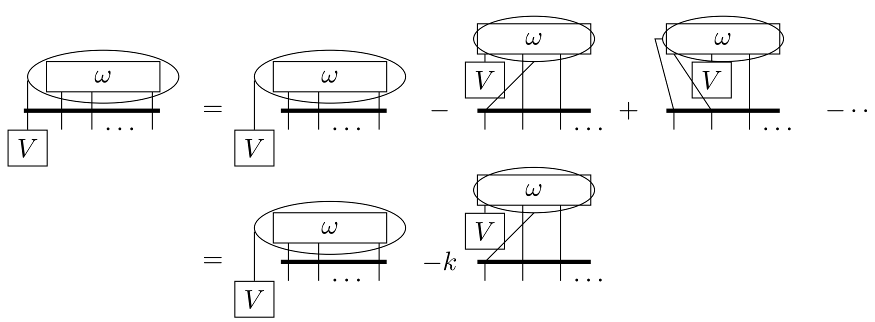


(5)

Apply Leibniz rule to the first term.

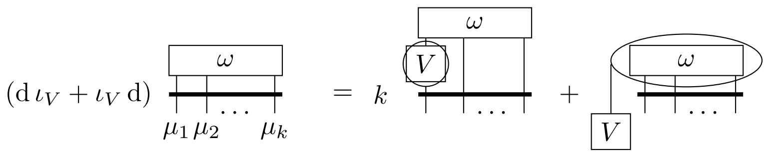


Put V off the thick line for the second term in (5).



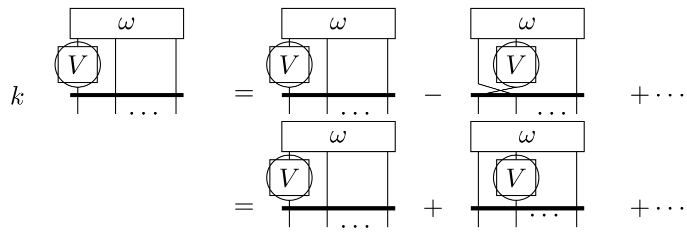
Focus on the second and the following terms in the first line. The positive terms can be transformed by even permutation, and negative terms by odd permutation, into the second term; all terms except the first are equivalent, thus the second line.

The questioned formula (5) is, therefore, equivalent to the following figure:



(6)

The first term in R.H.S. is expanded according to anti-symmetry for indices of $\omega_{\mu_1\cdots\mu_k}$.



Returning to the former figure, you will find Lie derivative of k-form.

References

[1] Roger Penrose, *Applications of Negative Dimensional Tensors*, Academic Press, 1971.

[2] E. Landau, E. M. Lifschitz, *The Classical Theory of Fields* Butterworth-Heinemann, 1980.

[3] M. Nakahara *Geometry, Topology and Physics, Second Edition* CRC Press, 2003