Self Study 2

ALG-CPH-F23: Algorithms and Data Structures (SW2)

Instructions. This exam consists of **four questions** and you have **four hours** to solve them. Note that each question begins on a new page. The maximum possible score is **100 points**, but bear in mind that these points are only advisory, and will be weighted following the exam. Please provide your answers digitally, either in a separate document, directly on this exam PDF, or both.
Remember to upload your digital answers as a single PDF file. It will not be possible to hand in anything physically for this exam. Throughout the exam, make sure to clearly indicate which question you are answering.

- Before starting with the exam, please make sure you read and understand these guidelines.
- For each question in the exam, make sure you read the text carefully, and understand what the problem is before solving it. Terms in bold are of particular importance.
- During the exam, you are allowed to use **almost all** aides, including notes, slides, and your internet access / online search to the best of your abilities. However, you are **not** allowed to use tools which rely on AI (e.g. no use of ChatGPT), or that **directly solve** the problem you are working on. For instance, using a Master Theorem solver is not allowed.
- You are **not** allowed to communicate with others during the exam, beyond asking for administrative help from the administrative staff.
- It is highly recommended to have a look at the **entire** exam before starting, so you get an idea of the time each question might take.
- You may answer in English, Danish, Norwegian, Swedish, or in any combination of these.

Question 1. 20 Pts

Identifying asymptotic notation. (Note: lg means logarithm in base 2)

(1.1) [5 Pts] Mark **ALL** the correct answers. $\lg n + 2n^2 + 3\lg n + \sqrt{n^2}$ is:

☐ **a)** $\Theta(\lg n)$ ☐ **b)** $\Theta(n)$ ☐ **c)** $\Theta(\sqrt{n})$ ☐ **d)** $\Theta(n^2)$ ☐ **e)** $\Theta(2^n)$

(1.2) [5 Pts] Mark **ALL** the correct answers. $n \lg n^4 + n + 4000!$ is:

☐ **a)** $O(\lg n)$ ☐ **b)** $O(n)$ ☐ **c)** $O(n \lg n)$ ☐ **d)** $O(n^2)$ ☐ **e)** $O(2^n)$

(1.3) [5 Pts] Mark **ALL** the correct answers. $\lg n + 42^2 + n^2$ is

☐ **a)** $\Omega(\lg n)$ ☐ **b)** $\Omega(n)$ ☐ **c)** $\Omega(n \lg n)$ ☐ **d)** $\Omega(n^2)$ ☐ **e)** $\Omega(2^n)$

(1.4) [5 Pts] Mark **ALL** the correct answers. Consider the following recurrence

$$T(n) = \begin{cases} 1 & \text{if n}\leq 1 \\ \text{T(n/2) + 40} & \text{if n} > 1 \end{cases}$$

☐ **a)** The master method can be used to solve this recurrence.

☐ **b)** The substitution method can be used to solve this recurrence.

☐ **c)** Using the first case of the master theorem, we can prove that $T(n) = \Theta(\lg n)$.

☐ **d)** Using the second case of the master theorem, we can prove that $T(n) = \Theta(\lg n)$.

☐ **e)** Using the third case of the master theorem, we can prove that $T(n) = \Theta(\lg n)$.

Solution 1.

(1.1) $= n^2 = \Theta(n^2)$. Correct answer: (d).

(1.2) $= nlgn$. Correct answers: (c, d, e).

(1.3) $= n^2$ Correct answers: (a, b, c, d).

(1.4) Master method, a=1, b=2, f(n)=40=1. Case 2, as f(n)=1 is in $\Theta(1)$ Correct answers: (a, b, d).

Question 2. 36 Pts

Linear Probing: $h(k, i) = (h'(k) + i) \mod m$

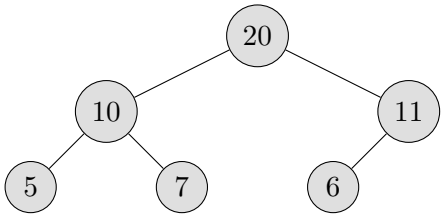
Quadratic Probing: $h(k, i) = (h'(k) + c_1 i + c_2 i^2) \mod m$

Double Hashing: $h(k, i) = (h_1(k) + i h_2(k)) \mod m$

(2.1) [9 Pts] Consider inserting the keys 20; 20; 28; 23; 5; 6; into a hash table of length $m = 8$, represented by the 8 boxes by each probing function. Write down the result of inserting these keys with the following hash functions, bearing in mind the probe definitions listed above. Assume indexation starting at 0, and enter the numbers into the corresponding boxes.

	Index: 0	1	2	3	4	5	6	7
a) Linear Probing with $h'(k) = k$:	☐	☐	☐	☐	☐	☐	☐	☐
b) Quadratic Probing with $h'(k) = k, c_1 = 2, c_2 = 3$:	☐	☐	☐	☐	☐	☐	☐	☐
c) Double Hashing with $h_1(k) = k, h_2(k) = (2k)$:	☐	☐	☐	☐	☐	☐	☐	☐

(2.2) [5 Pts] Consider the following tree, T, and mark **ALL** the correct answers below:



☐ **a)** T satisfies the Min-Heap property

☐ **b)** The height of T is 2

☐ **c)** T satisfies the binary search tree property

☐ **d)** T corresponds to a binary tree interpretation of the array $[20, 10, 10, 5, 6, 7]$

☐ **e)** If we insert 20 as a right child of the node with the value 11, T satisfies the Min-Heap Property

(2.3) [16 Pts] Consider the following recursive algorithm, BEST-CAT. The input is an array $A[1 \ldots A.length]$ of pictures, some of which include one or more cats. When the recursion bottoms out, the procedure calls a cat quality-detection procedure, CAT-QUALITY, which returns a value showing the quality of that cat. Assume that CAT-QUALITY uses constant time to process a single input. At every call, we want to check that we don't have any dogs in the pictures, hence line 7-8 calls such a procedure for each element in A. Assume that the procedure VERIFY-NO-DOGS uses constant time to process a single input. Note that this procedure does not make any changes to A.

```
BEST-CAT(A)
1  if length(A) == 1
2        return CAT-QUALITY(A[1])
3  else
4        newsize = length(A)/2
5        a = BEST-CAT(A[0 : newsize])
6        b = BEST-CAT(A[newsize : 2 * newsize])
7        for i = 0 to length(A)
8              VERIFY-NO-DOGS(A[i])
9        return max(a, b, c)
```

Assume that the input size at the first call is $n = 2^k$ for some $k \in \mathbb{N}$. Further assume that the notation A[i:j] returns a sub-array of A from index i (inclusive) to index j (exclusive).

(a) [2 Pts] The worst case running time of lines 1-2 is: $O(\ )$

(b) [2 Pts] The worst case running time of lines 7-8 is: $O(\ )$

(c) [4 Pts] Write the recurrence $T(n)$ describing the running time of BEST-CAT.

(d) [8 Pts] Use the Master Theorem to prove the running time of this recurrence, showing all the steps you use to reach your conclusion.

(2.4) [6 Pts] In this assignment, you will work with a Singly Linked List. Assume that your list has an attribute L.head which points at the head of the list (*nil* if L is empty). Further assume that each element in the list has the attributes x.next (*nil* if x is the tail).

(a) [4 Pts] Give a procedure which takes as input a Singly Linked List and converts it to a Doubly Linked List. The procedure **must** run in linear time, i.e., $\Theta(n)$. Name the procedure SINGLY-TO-DOUBLY-LL. Write pseudocode and provide a brief description in words how your procedure works.

(b) [2 Pts] Analyse your version of SINGLY-TO-DOUBLY-LL. Write down both the runtime per line in your pseudocode, and the worst-case running time in big-O notation.

Solution 2.

(2.1) Double-hashing has an error in the exercise and should be disregarded.

(2.2) The tree does not satisfy the Min-Heap property, as all children are smaller than their parents. The height of the tree is 3 since its longest path from the root to a leaf has 3 edges. The tree does not correspond to the binary tree interpretation of the array.

Correct answers: (b).

(2.3) (a) O(1)

(b) O(n)

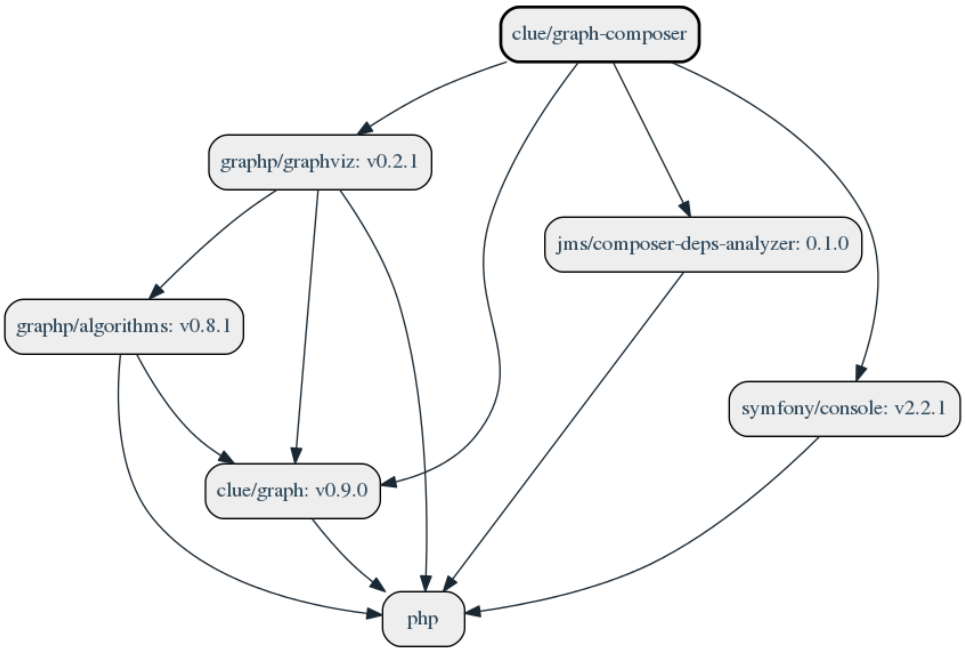
(c)

$$T(n) = \begin{cases} c_1 & \text{if } n \leq 1, \\ 2T(n/2) + T(n) & \text{if } n > 1. \end{cases}$$

(d) This should evaluate to $\Theta(nlgn)$.

(2.4) (a) The solution must take into account an update of L.head, x.next, x.prev, and correctly use the properties of a doubly linked list. E.g. assuming that the input is an array, or that the length of L is known, or that L can be indexed as an array, is not a correct interpretation of the assignment. A single loop should suffice, starting at L.head, terminating when x.next=nil, and swapping each x.next with x.prev.

(b) The following analysis should therefore also be relatively straight-forward in showing runtime $= \Theta(n)$




```
DFS(G)
1  for each vertex u ∈ G.V
2      u.color = WHITE
3      u.π = NIL
4  time = 0
5  for each vertex u ∈ G.V
6      if u.color == WHITE
7          DFS-VISIT(G, u)
```

```
TOPOLOGICAL-SORT(G)
1  call DFS(G) to compute finishing times v.f for each vertex v
2  as each vertex is finished, insert it onto the front of a linked list
3  return the linked list of vertices
```

```
DFS-VISIT(G, u)
1  time = time + 1
2  u.d = time
3  u.color = GRAY
4  for each v ∈ G.Adj[u]
5      if v.color == WHITE
6          v.π = u
7          DFS-VISIT(G, v)
8  u.color = BLACK
9  time = time + 1
10 u.f = time
```

Question 3. 15 Pts

Consider the dependency graph for various software packages, G, as well as the pseudocode for Topological-Sort and DFS (all shown above).

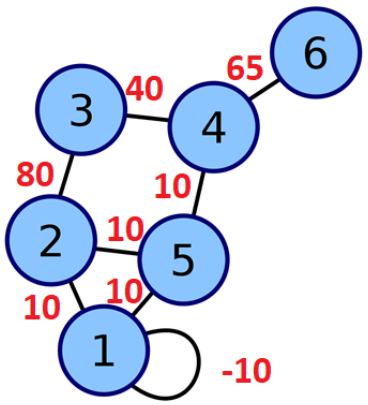
(3.1) [10 Pts] In order to compile this graph, we need to first perform a topological sort, in order to get a compilation order. Choose a starting vertex, and manually go through the steps of Topological-Sort on G. Provide **both** the resulting linked-list, as well as the **start** and **finish** times for each vertex during the DFS.

(3.2) [5 Pts] Imagine a new case, in which there is a second graph, H, containing a distinct set of packages and dependencies, which is completely independent from G with a single exception: The node *php* in graph G has an outgoing arrow to a node in H, which **only** has outgoing arrows. If we now want to compile H, we want to take advantage of the fact that we already have a linked-list showing the compilation order for G. Describe in words how you can get a linked list with the compilation order of H independently from G. Which linked-list of packages should be compiled first, the one for G or the one for H?

Solution 3.

(3.1) A correct answer will contain a correct topological sort given as a linked list, and the corresponding start/finish times for the DFS.

(3.2) Simply doing a topological sort on H without G would give a new linked list for compiling H. There is a node in H which depends on php, hence G must be compiled first. Then the topological sort of H can be followed.




```
BELLMAN-FORD(G, w, s)
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  for i = 1 to |G.V| − 1
3      for each edge (u, v) ∈ G.E
4          RELAX(u, v, w)
RELAX(u, v, w)
1  if v.d > u.d + w(u, v)
2      v.d = u.d + w(u, v)
3      v.π = u
```

Figure 1: A graph G, and pseudocode for the procedures BELLMAN-FORD and RELAX

Question 4. 29 Pts

You are working with the implementation of route guidance for electric vehicles. Consider the above graph $G = (V, E)$, which represents possible routes within a small country. Each vertex V represents a city, and each edge E represents a road. The weights given by the weight function, $w : E \longrightarrow \mathbb{R}$, representing the **battery consumption in percentage points** for driving from a city to the next. When it is possible to charge a vehicle's battery, this is currently indicated by the presence of a negative cycle in the graph (i.e. from vertex 1 to 1). Assume that your vehicle starts off with 100% battery power.

(4.1) [6 Pts] Under normal circumstances (i.e. finding the shortest route without any other restrictions), the single-source shortest paths algorithms covered in the course do not work in this graph. Explain briefly why *(i) Bellman-Ford*, *(ii) Dijkstra's Algorithm* and *(iii) Single-Source Shortest Path in DAG* would not work.

(4.2) [3 Pts] Examine the graph. Following the restrictions above (i.e. battery must remain ≥ 0), is there a possible route from 6 to 2? How about from 6 to 3? Write down these routes.

(4.3) [20 Pts] Describe how you can solve the problem of route guidance for electric vehicles in the graph G, with the battery usage outlined above. As a hint, possible solutions can involve modification of **Bellman-Ford**, **Relax**, and/or **the graph** G. Explain how your proposed solution works, and describe the steps it would take to find a possible path from 6 to 3 with your solution. Take care that your solution does not allow the battery go below 0, and does not get stuck in an infinite charging loop. If you modify existing pseudocode, provide your answer in pseudocode as well as an explanation in words. If you modify the graph G, draw the new graph, and explain in words how your solution works.

Solution 4.

(4.1) (i) no negative cycles allowed, (ii) assumes non-negative weights, (iii), G is not a DAG

(4.2) Yes, e.g. 6 - 4 - 5 - 2 and 6 - 4 - 5 - 1 (repeat) - 2 - 3.

(4.3) Many possibilities exist, as covered in detail in the lecture and the corresponding case-work. One relatively straight-forward solution could be to modify the data structure such that (1) is a special node with some property denoting this. Then, `RELAX` can be modified to disallow visiting this node once the battery has been re-charged up to 100. This would allow, e.g. BF, to find a route going down to 1, using max. 10 recharge-cycles, and then be forced to leave again. Note that this would also require BF to be modified to use more than G.V-1 iterations.

Another option is to first try to attain a straight-forward route from u to v, and if this is not possible, force the route from u to 1, and then rerun BF for a route from 1 to v.

Thank you for your hard work and active participation in this course!
Remember to upload your answers to the exam as a single PDF file.

Best of luck!
- Johannes