3801ICT Numerical Algorithms Milestone 2

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Trimester 1 - 2019

Question 1

The deflection of a boat mast can be modelled as

$$\frac{d^2y}{dz^2} = \frac{f}{2EI}(L-z)^2$$

Where f = wind force, E = modulus of elasticity, L = mast length, and I = moment of inertia. Calculate the deflection if y = 0 and dy/dz = 0 at z = 0 using f = 60, L = 30, E = 1.25*10^8 and I = 0.05

The analytical solution was for this second order differential equation was found here https://www.wolframalpha.com/input/?i=y%27%27+%3D+(60%2F(2+*+0.05+*+(1.25*10%5E8)))+*+(30+-+x)%5E2++,y(0)%3D0,+y%27(0)%3D0

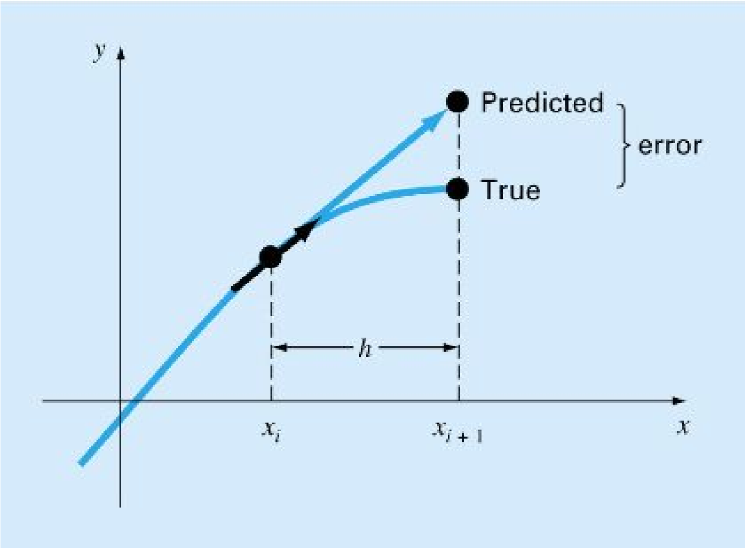
$$y = x^2 * (4.0 * 10^{-7} * x^2 - 0.000048x + 0.00216$$

This will be used to determine the effectiveness of the multiple numerical approximation methods used.

Euler's method

Linear extrapolation method based off the derivative at a specific point. Error of $O(h^2)$, is accurate only if h is very small.

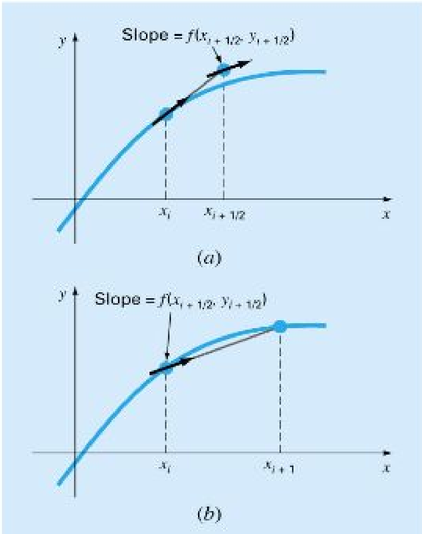
$$y_{i+1} = y_i + f(x_i, y_i)h$$



Midpoint method

Extends Euler's method to predict a value of y at the midpoint of the interval. Error of $O(h^4)$

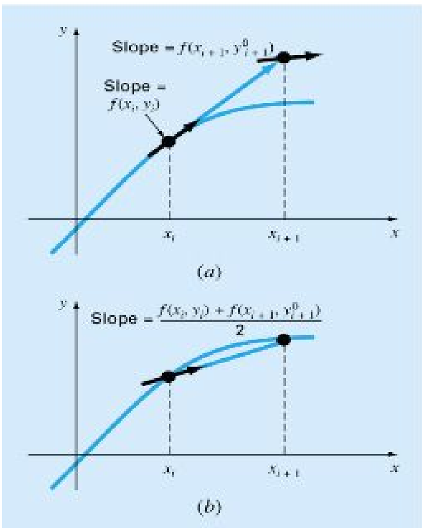
$$y_{i+1} = y_i + f(x_{i+1/2},\ y_{i+1/2})h$$



Huen's method

Extends Euler's method by determining two derivatives for the interval - at the start and end point. These are then averaged which provide a better estimate for the target value compared to just Euler's. This is called having a predictor and a corrector. Error $O(h^4)$

$$Predictor:\ y^0_{i+1} = y_i + f(x_i,\ y_i)h$$
$$Corrector:\ y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y^0_{i+1})}{2}h$$



RK4

RK4 is the most commonly used method, there are higher orders but generally RK4 is sufficient. Runge-Kutta methods achieve the accuracy of a taylor series approach without requiring the calculation of higher derivatives.

$$y_{i+1} = y_i + \varphi(x_i, y_i, h)h$$

$$\varphi = a_1k_1 + a_2k_2 + \cdots + a_nk_n \quad \textit{Increment function}$$

$$a's = \text{constants}$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1h, y_i + q_{11}k_1h) \quad \textit{p's and q's are constants}$$

$$k_3 = f(x_i + p_2h, y_i + q_{21}k_1h + q_{22}k_2h)$$

$$\vdots$$

$$k_n = f(x_i + p_{n-1}h, y_i + q_{n-1}k_1h + q_{n-1,2}k_2h + \cdots + q_{n-1,n-1}k_{n-1}h)$$

RK4 is a weight sum of 4 estimates if change.

$$y_{i+1} = y_i + (k_1 + 2k_2 + 2k_3 + k_4)h / 6$$

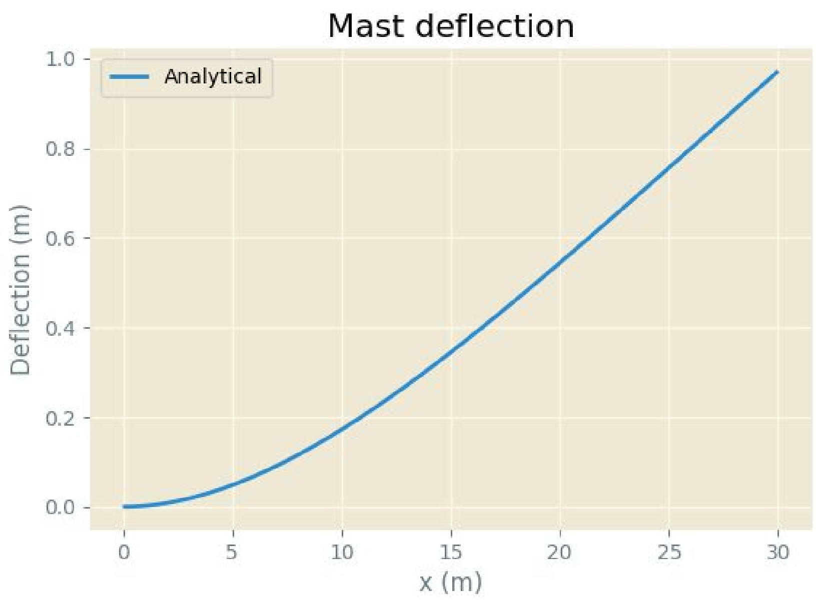
$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h/2, y_i + k_1h/2)$$

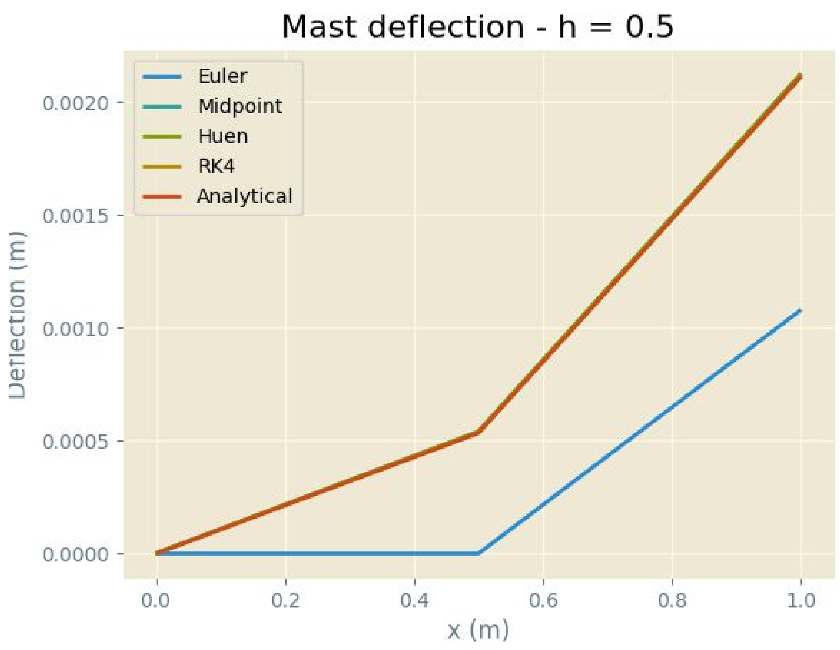
$$k_3 = f(x_i + h/2, y_i + k_2h/2)$$

$$k_4 = f(x_i + h, y_i + k_3h)$$

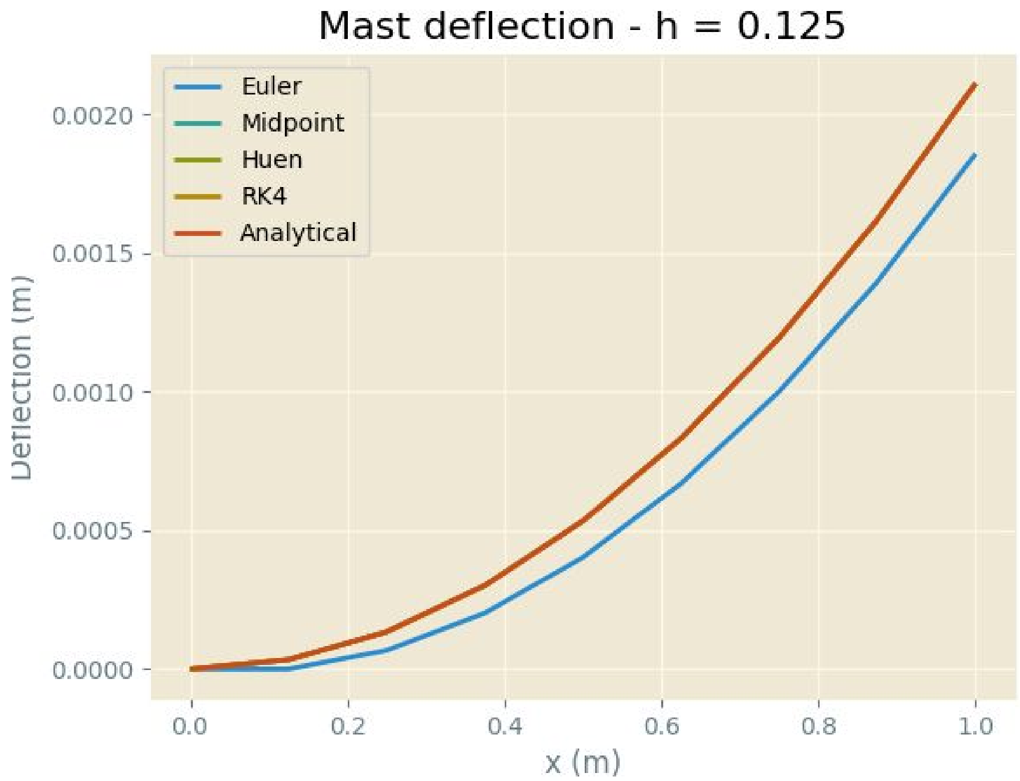
Analysis



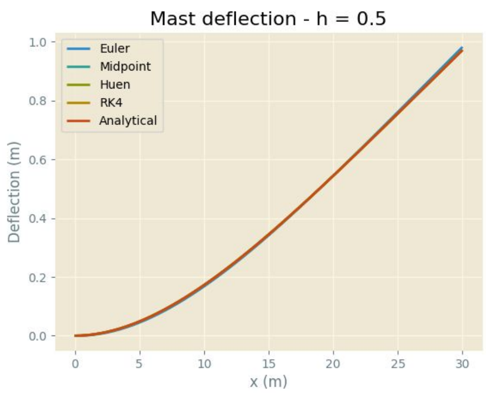
The above graph shows the deflection of the 30m mast using the analytical function - 0 error.

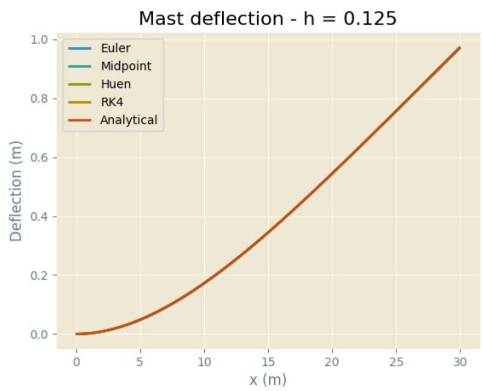


Using a h value of 0.5 between the interval of 0 and 1 it is clear that Euler's method is by far the least accurate, whereas the other methods are closely aligned with the analytical. The interval of 0 to 1 is used for demonstration purposes as it is a good region to illustrate the differences in the numerical methods.



Even decreasing the h (step size) to a quarter the Euler method is still highly inaccurate. As expected the other methods. The next two following graphs will show the region 0-30m the errors are not as graphically obvious as the previous graphs but visualises the approximations over the whole region.

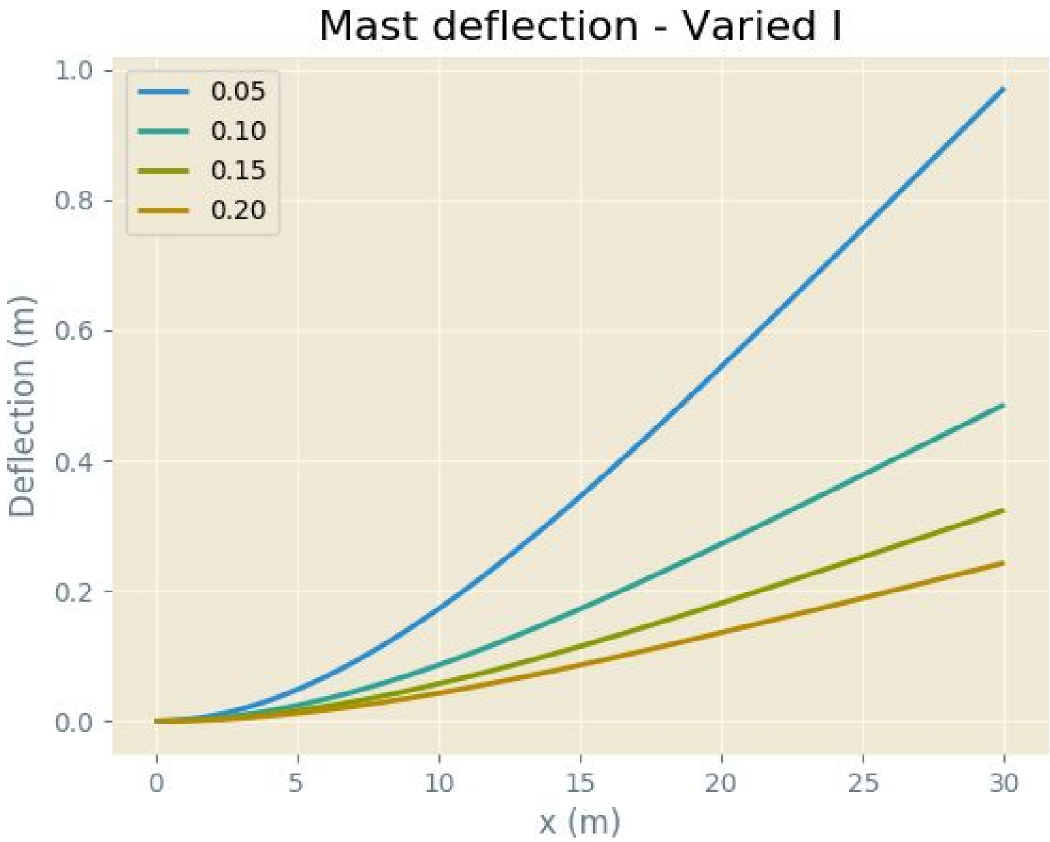




These graphs prove that even with Euler’s method an accurate approximation can be made with a small step size however it is better to just us RK4. There are two main reasons for this, rk4 requires less iterations to get an equivalent value with Euler’s and the fact that once the step size decreases rounding issues present themself.

Experimentation

As an experiment multiple values of I were used - 'moment of inertia' the RK4 method was used to approximate the equation value as it has a high accuracy which was shown previously.



This graph shows exactly what you would expect which is that as I decreases the deflection increases.

Question 2

Given the system of linear equations comprising of 3 equations with 3 unknowns use Gauss Elimination, Gauss-Jordan and LU decomposition to solve for the unknowns. The image below shows the 3 linear equations and augmented matrix.

$$\boxed{\begin{aligned} 0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\ 0.3x_1 - 0.2x_2 + 10x_3 &= 71.4 \\ 3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \end{aligned}}$$

$$\underset{A}{\begin{bmatrix} 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \\ 3 & -0.1 & -0.2 \end{bmatrix}} \times \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{C}{\begin{bmatrix} -19.3 \\ 71.4 \\ 7.85 \end{bmatrix}}$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \\ 3 & -0.1 & -0.2 & 7.85 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} \frac{1}{10} & 7 & \frac{-3}{10} & \frac{-193}{10} \\ \frac{3}{10} & \frac{-1}{5} & 10 & \frac{357}{5} \\ 3 & \frac{-1}{10} & \frac{-1}{5} & \frac{157}{20} \end{array}\right]$$

Gauss Elimination

Manipulate the equations so that the values under the pivot (red) are 0, this creates an upper triangular matrix form. Then the process of back substitution can be performed, find the value of X3, use that to find X2, use that to find X1. This process can be used to find N unknown variables in N linear equations.

Gauss Elimination

$$\left[\begin{array}{ccc|c} \frac{1}{10} & 7 & \frac{-3}{10} & \frac{-193}{10} \\ \frac{3}{10} & \frac{-1}{5} & 10 & \frac{357}{5} \\ 3 & \frac{-1}{10} & \frac{-1}{5} & \frac{157}{20} \end{array}\right]$$

$R_2 - 3R_1$

$$\left[\begin{array}{ccc|c} \frac{1}{10} & 7 & \frac{-3}{10} & \frac{-193}{10} \\ 0 & \frac{-106}{5} & \frac{109}{10} & \frac{1293}{10} \\ 3 & \frac{-1}{10} & \frac{-1}{5} & \frac{157}{20} \end{array}\right]$$

$R_3 - 30R$

$$\left[\begin{array}{ccc|c} \frac{1}{10} & 7 & \frac{-3}{10} & \frac{-193}{10} \\ 0 & \frac{-106}{5} & \frac{109}{10} & \frac{1293}{10} \\ 0 & \frac{-2101}{10} & \frac{44}{5} & \frac{11737}{20} \end{array}\right]$$

$R_3 - 10R_2$

$$\left[\begin{array}{ccc|c} \frac{1}{10} & 7 & \frac{-3}{10} & \frac{-193}{10} \\ 0 & \frac{-106}{5} & \frac{109}{10} & \frac{1293}{10} \\ 0 & 0 & \frac{-12006}{121} & \frac{-50703}{73} \end{array}\right]$$

$$\frac{1}{10}x_1 + 7x_2 - \frac{3}{10}x_3 = \frac{-193}{10}$$

$$\frac{-106}{5}x_2 + \frac{109}{10}x_3 = \frac{1293}{10}$$

$$\frac{-12006}{121}x_3 = \frac{-50703}{73}$$

Back Substitution

$$x_3 = \frac{\frac{-50703}{73}}{\frac{-12006}{121}} = 7$$

$$x_2 = \frac{\frac{1293}{10} - \frac{109}{10}x_3}{\frac{-106}{5}} = \frac{-5}{2} = -2.5$$

$$x_1 = \frac{\frac{-193}{10} + \frac{3}{10}x_3 - 7x_2}{\frac{1}{10}} = 3$$

$$X = \begin{bmatrix} 3 \\ -2.5 \\ 7 \end{bmatrix}$$

Check

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$3(3) - 0.1(-2.5) - 0.2(7) = 7.85$$

$$7.85 = 7.85$$

Gauss Jordan

Similar to Gauss elimination, but make values above and below the pivot 0, and the values on the diagonal 1. This process removes the need for back substitution. The pivot is normalized (made 1) and then the values above and below are eliminated.

Gauss - Jordan

$$\left[\begin{array}{ccc|c} \frac{1}{10} & 7 & \frac{-3}{10} & \frac{-193}{10} \\ \frac{3}{10} & \frac{-1}{5} & 10 & \frac{357}{5} \\ 3 & \frac{-1}{10} & \frac{-1}{5} & \frac{157}{20} \end{array}\right]$$

$R_1 \times 10$

$$\left[\begin{array}{ccc|c} 1 & 70 & -3 & -193 \\ \frac{3}{10} & \frac{-1}{5} & 10 & \frac{357}{5} \\ 3 & \frac{-1}{10} & \frac{-1}{5} & \frac{157}{20} \end{array}\right]$$

$R_2 - \left(\frac{3}{10}\right) R_1$

$$\left[\begin{array}{ccc|c} 1 & 70 & -3 & -193 \\ 0 & \frac{-106}{5} & \frac{109}{10} & \frac{1293}{10} \\ 3 & \frac{-1}{10} & \frac{-1}{5} & \frac{157}{20} \end{array}\right]$$

$R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 70 & -3 & -193 \\ 0 & \frac{-106}{5} & \frac{109}{10} & \frac{1293}{10} \\ 0 & \frac{-2101}{10} & \frac{44}{5} & \frac{11737}{20} \end{array}\right]$$

$\frac{1}{21} R_2$

$$\left[\begin{array}{ccc|c} 1 & 70 & -3 & -193 \\ 0 & 1 & \frac{-109}{212} & \frac{-1293}{212} \\ 0 & \frac{-2101}{10} & \frac{44}{5} & \frac{11737}{20} \end{array}\right]$$

$R_1 - 70R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3497}{106} & \frac{21288}{91} \\ 0 & 1 & \frac{-109}{212} & \frac{-1293}{212} \\ 0 & \frac{-2101}{10} & \frac{44}{5} & \frac{11737}{20} \end{array}\right]$$

$R_3 - (-210)R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3497}{106} & \frac{21288}{91} \\ 0 & 1 & \frac{-109}{212} & \frac{-1293}{212} \\ 0 & 0 & \frac{-12006}{121} & \frac{-50703}{73} \end{array}\right]$$

$R_3 \times \left(\frac{1}{99}\right)$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3497}{106} & \frac{21288}{91} \\ 0 & 1 & \frac{-109}{212} & \frac{-1293}{212} \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$R_1 - 33R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & \frac{-109}{212} & \frac{-1293}{212} \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$R_2 - \left(\frac{-18}{35}\right) R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & \frac{-5}{2} \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$x_3 = 7$

$x_2 = \frac{-5}{2} = 2.5$

$x_1 = 3$

$$X = \begin{bmatrix} 3 \\ -2.5 \\ 7 \end{bmatrix}$$

LU Decomposition

Use the values obtained through Gauss Elimination and solve for Z in LZ = C, then substitute Z into UX = Z to find X.

LU Decomposition

$$AX = C$$
$$LUX = C$$
$$L^{-1}LUX = L^{-1}C$$
$$IUX = L^{-1}C$$
$$UX = L^{-1}C$$
$$L^{-1}C = Z$$
$$LZ = C \quad \text{—(1)}$$
$$UX = Z \quad \text{—(2)}$$

Using values obtained in Gauss Elimination

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 30 & 10 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{10} & 7 & \frac{-3}{10} \\ 0 & \frac{-106}{5} & \frac{109}{10} \\ 0 & 0 & \frac{-12006}{121} \end{bmatrix}$$

$$LZ = C$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 30 & 10 & 1 \end{bmatrix} \times \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -19.3 \\ 71.4 \\ 7.85 \end{bmatrix}$$

$$z_1 = -19.3$$
$$3z_1 + z_2 = 71.4$$
$$z_2 = 129.3$$
$$30z_1 + 10z_2 + z_3 = 7.85$$
$$z_3 = -706.15$$

$$Z = \begin{bmatrix} -19.3 \\ 129.3 \\ -706.15 \end{bmatrix}$$

$$UX = Z$$

$$\begin{bmatrix} \frac{1}{10} & 7 & \frac{-3}{10} \\ 0 & \frac{-106}{5} & \frac{109}{10} \\ 0 & 0 & \frac{-12006}{121} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -19.3 \\ 129.3 \\ -706.15 \end{bmatrix}$$

$$x_3 = \frac{\frac{14123}{20}}{\frac{12006}{121}}$$
$$= 7$$
$$\frac{-106}{5}x_2 + \frac{109}{10}z_3 = 129.3$$
$$x_2 = \frac{-5}{2} = -2.5$$
$$\frac{1}{10}x_1 + 7x_2 + \frac{-3}{10}x_3 = -19.3$$
$$x_1 = 3$$

$$X = \begin{bmatrix} 3 \\ -2.5 \\ 7 \end{bmatrix}$$

$AX = C$ ✓

Question 3

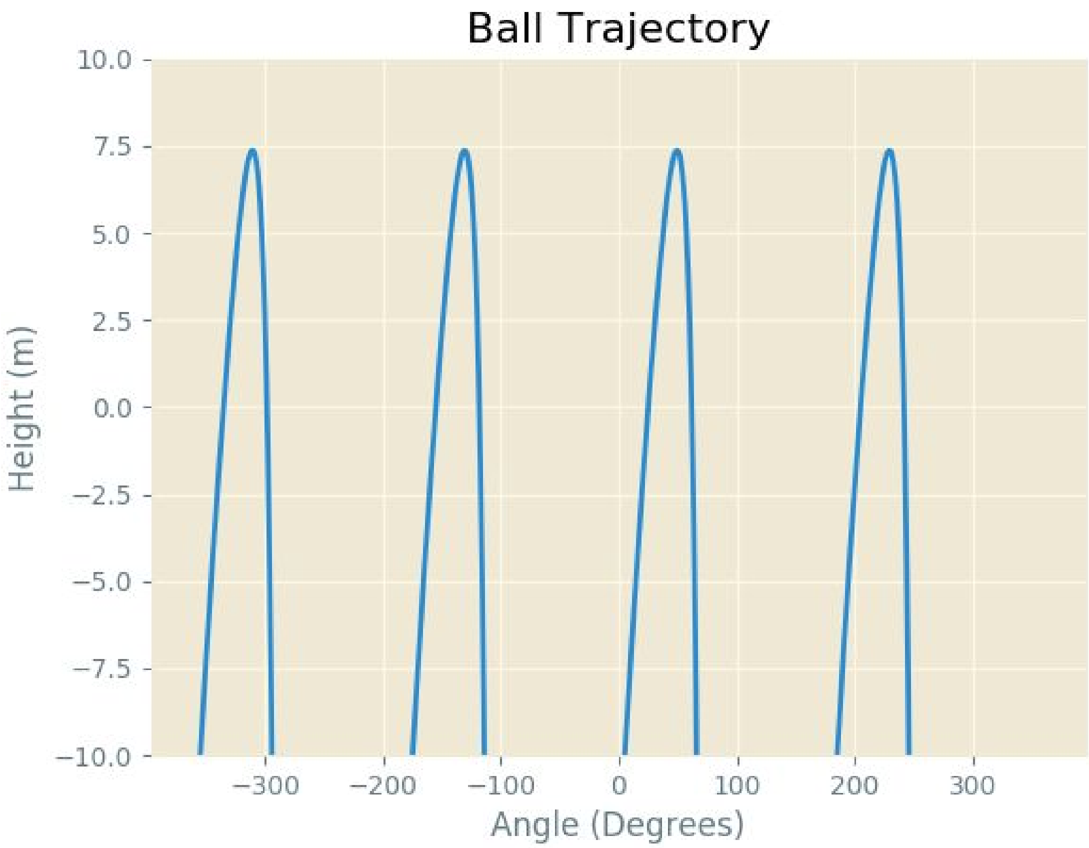
The trajectory of a thrown ball is defined by the (x, y) coordinates that can be modelled as:

$$y = (tan\theta_0)x - \frac{g}{2v_0^2cos^2\theta_0}x^2 + y_0$$

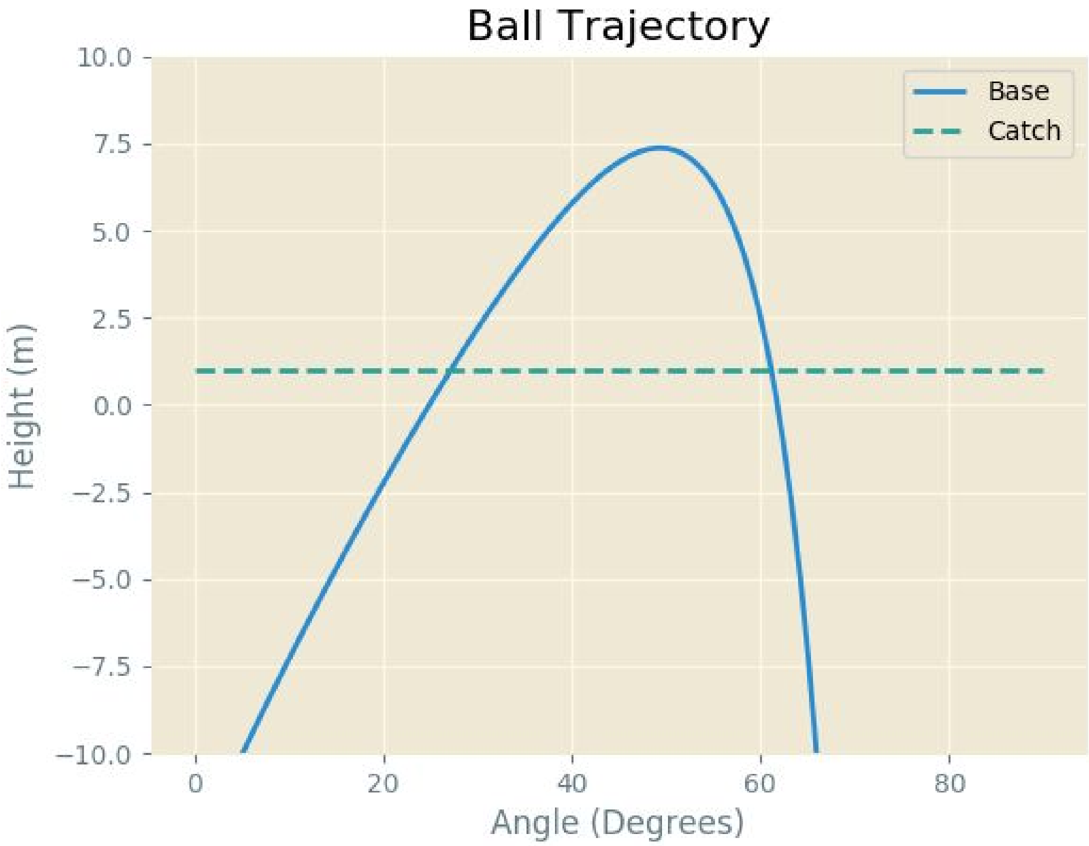
The task is to find the initial angle θ_0 , if the initial velocity v_0 = 20m/s and the distance to the catcher x is 35m. The ball leaves the thrower's hand at y_0 = 2m and the catcher is to catch the ball at 1m. 9.81m/s^2 is also used for g.

Substituting the values.

$$y = (tan\theta_0)35 - \frac{9.81}{2*20^2*cos^2\theta_0}35^2 + 2$$



The graph shows that the function is cyclic in nature as expected and has a period of 180 degrees. It is evident that there are two initial angles in which the ball can be caught from the same location - a low and high angle, this is intuitive. All further analysis will occur in the interval 0-90 degrees.



Same graph as before but in the interval 0-90 degrees with a line at 1m - the elevation in which the ball is to be caught from, as stated before there is two points in which these lines intersect. These points can be approximated using numerical methods and 'verified' graphically.

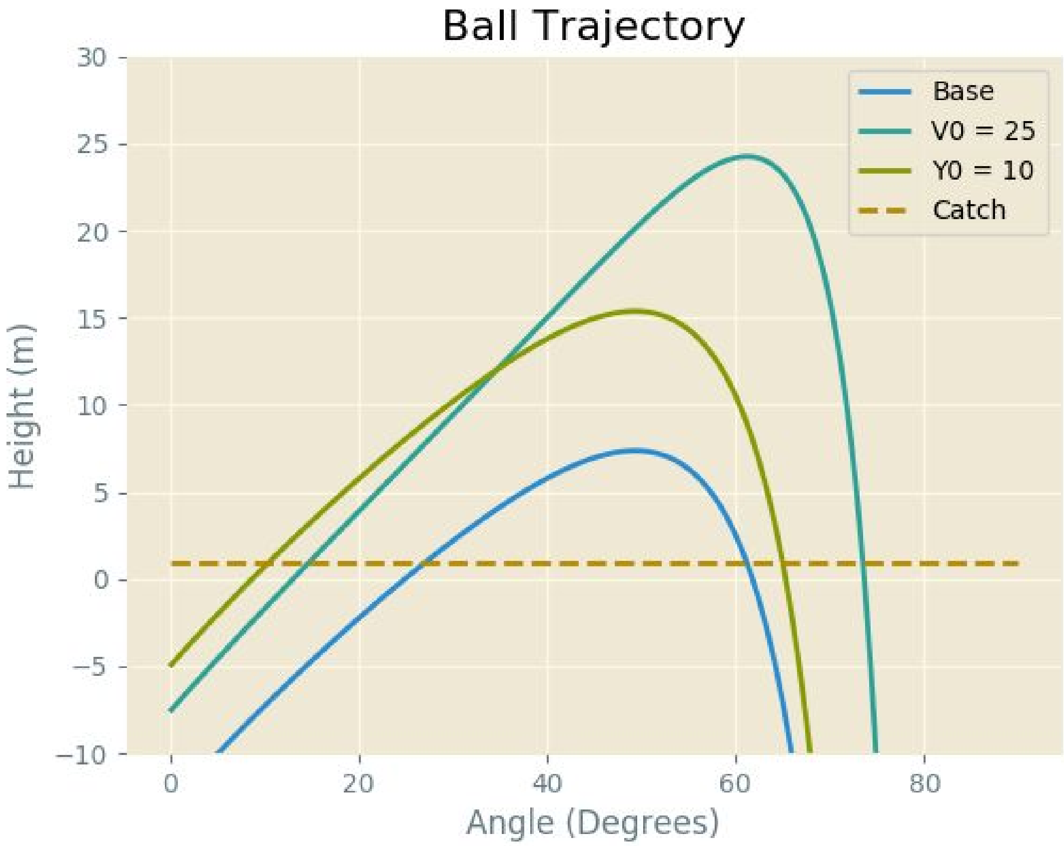
The method used to find the points of intersection was based off the bisection search method which finds roots of functions, this was modified to find a specific value as shown below. **NOTE:** because this function is cyclic the values in the domain of 0-90 degrees are the only ones required, to find the others 180 degrees can simply be added or subtracted.

```
Function bisectionSearch(target, left, right)
       Precision ← 0.0001

       If f(left) < f(right)
              Increasing ← true
       Else
              Increasing ← false

       While (right - left) >= precision
              Mid ← (left + right) / 2

              If f(mid) == target
                     Return mid

              If increasing
                     If f(mid) < target
                            Left ← mid
                     Else
                            Right ← mid

              Else
                     If f(mid) > target
                            Left ← mid
                     Else
                            Right ← mid

       Return mid
```

Using this algorithm on the above function resulted in the following values.

	Point 1	**Point 2**
X value - angle (degrees)	27.20365524	61.15974426
Y - height (m)	1.000012784	1.000077865

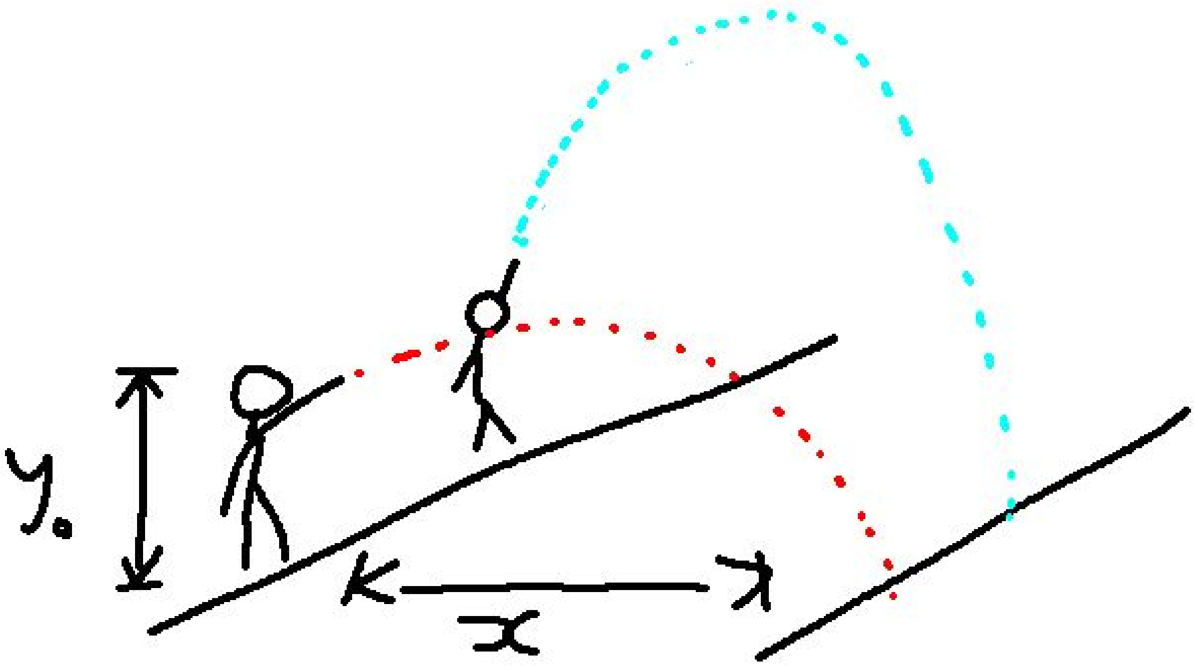
This is a numerical method hence the Y values are not exactly 1.0 but are accurate to of a high degree. These values can be confirmed graphically against the previous graph.



Further analysis of the problem involved exploration of the values of the variables V0 and Y0. The initial velocity was changed from 20m/s to 25m/s and initial height from 2 metres to 10 metres - a 5x increase. The results are illustrated above and points of intersection with y=1 below.

	Point 1		Point 2	
Function	**X value - angle (degrees)**	**Y - height (m)**	**X value - angle (degrees)**	**Y - height (m)**
Base	27.20365524	1.000012784	61.15974426	1.000077865
V0 = 25	14.86717224	1.000004499	73.49632263	0.999587106
Y0 = 10	10.59547424	1.000004143	64.98374939	1.000031675

From the results it is evident that increasing the initial height mostly contributes to having a shallower angle for Point 1, whereas increasing the initial velocity increases the angle for Point 2 - it becomes negatively skewed. This is intuitive as to compensate for the initial velocity the angle can be increased where most of the energy goes into the y axis and not the x axis. This is visualised in the figure below.



y_0
x