Root Finding – 01

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Introduction to Root Finding

- Also known as "solution of a non-linear function of a single variable"
- Standard form of the root finding problem

$$f(x) = 0$$

- We look for a value of "***x***" that satisfies the above equation
  - ***p=root of the equation (zero of the equation)***
- For example

$$f(x) = ax^3+bx^2+cx+d = 0 \quad or \quad tan(kx)=x$$

- In the latter case, we can re-write the equation in the standard form as

$$f(x) = tan(kx)-x = 0$$

Selected Numerical Methods

Bracketing methods

- Bisection Method (Interval halving) (5.2)
- Linear Interpolation (also named as Regula Falsi or False Position) (5.3)

Slope methods

- Secant Method (6.3)
- Newton's Method (6.2)
- Fixed Point Iteration (6.1)

Section in your text book

Bisection Method for a Single Root

- Let function $f(x)$ be a continuous function on the closed interval $[a,b]$ with $f(a)$ and $f(b)$ of opposite sign.

- By the Intermediate Value Theorem, there exists a number 'p' in [a,b] with $f(p)=0$.

- At each step, the method calls for repeated halving of subintervals of $[a,b]$ and locating the half containing the root p.

Bisection Method

Step1:

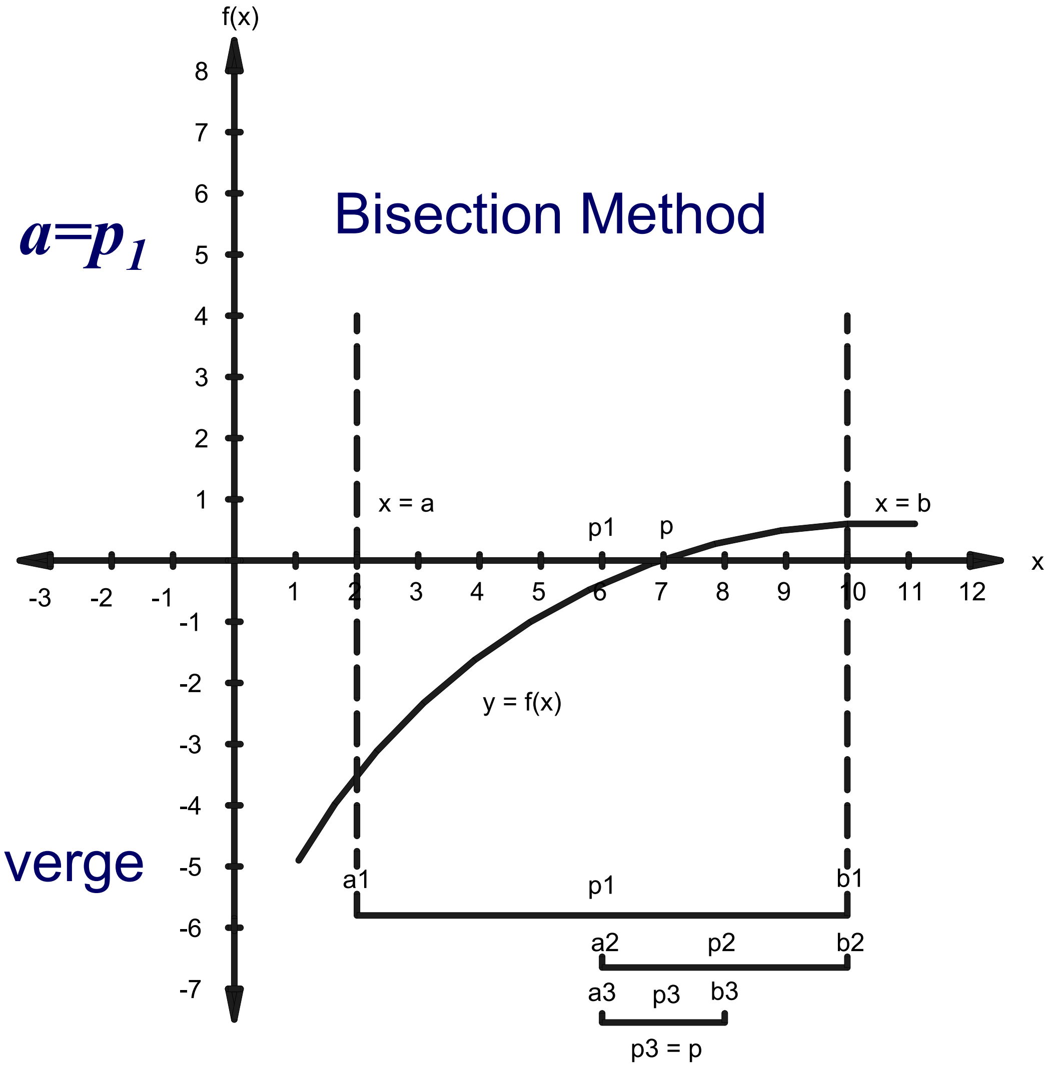
$p_1 = (a+b)/2$

if $f(p_1) \cdot f(b) < 0 \Rightarrow p \ \varepsilon \ (p_1, b)$ & $a = p_1$

else $p \ \varepsilon \ (a, p_1)$ & $b = p_1$

Step2:

Divide the new interval again and bracket the root



Advantages

- Simple to program
- Once you know [a,b], you will converge to a solution

Disadvantages

- Convergence is slow
- Algorithm complex for multiple roots

Stopping Criteria

For the bisection method, we can use

$$\frac{b_n - a_n}{2} \leq Tolerance$$

Other stopping criteria that can also be applied to any of the iterative methods used in root finding:

1: $$|x_n - x_{n-1}| \leq Tolerance_1$$

2: $$\frac{|x_n - x_{n-1}|}{|x_n|} \leq Tolerance_2$$

3: $$|f(x_n)| \leq Tolerance_3$$

Where x_n is the approximation to the root (p) in n^{th} iteration

4: Always set a limit for the number of iterations (i.e. N_{max})

Newton’s Method

- Newton’s Method is one of the most powerful and well-known methods for solving a root-finding problem. This method is also referred to as Newton-Raphson’s Method.

- Assumptions
  - The function $f(x)$ and $f'(x)$ are continuous in the given interval [a,b]
  - The analytic expression for the function is known. It requires a single initial guess, but converges faster than most other methods.

- Newton’s method can be established analytically or graphically.

Derivation of Newton's Method

Using a Taylor's series expansion,

$$f(x_{n+1}) = f(x_n) + f'(x_n)(x_{n+1} - x_n) + f''(x_n)\frac{(x_{n+1} - x_n)^2}{2!} + f'''(x_n)\frac{(x_{n+1} - x_n)^3}{3!} + \ldots$$

and neglecting higher order terms, we have

$$f(x_{n+1}) = f(x_n) + f'(x_n)(x_{n+1} - x_n)$$

If $\boldsymbol{x_{n+1}}$ is the root of the given function, $\boldsymbol{f(x_{n+1}) = 0}$.

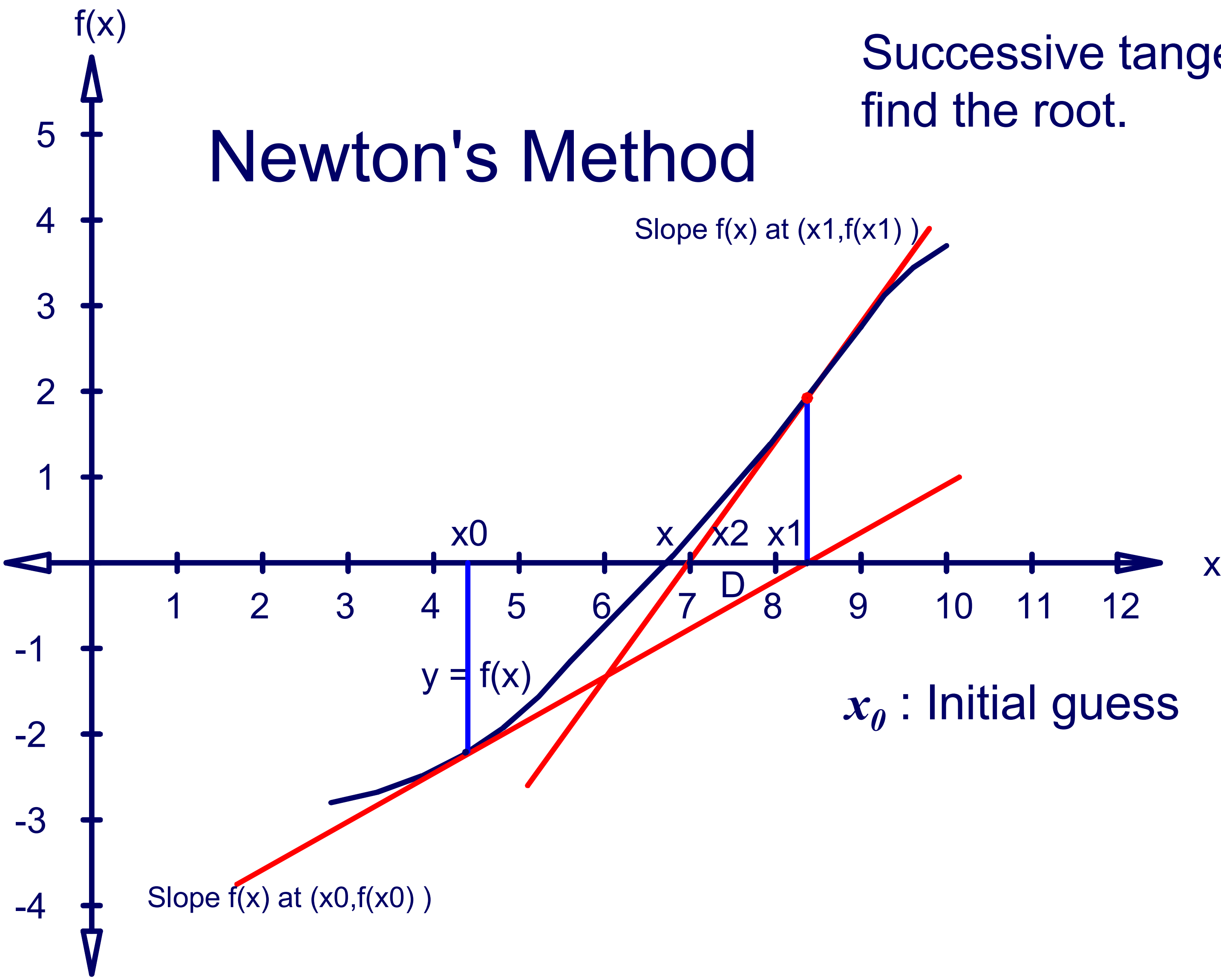
$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$$

In delta form $\Delta_n x = -\dfrac{f(x_n)}{f'(x_n)}$ and $x_{n+1} = x_n + \Delta_n x$

Newton’s Method Graphically

Successive tangents are used to find the root.



x_0 : Initial guess

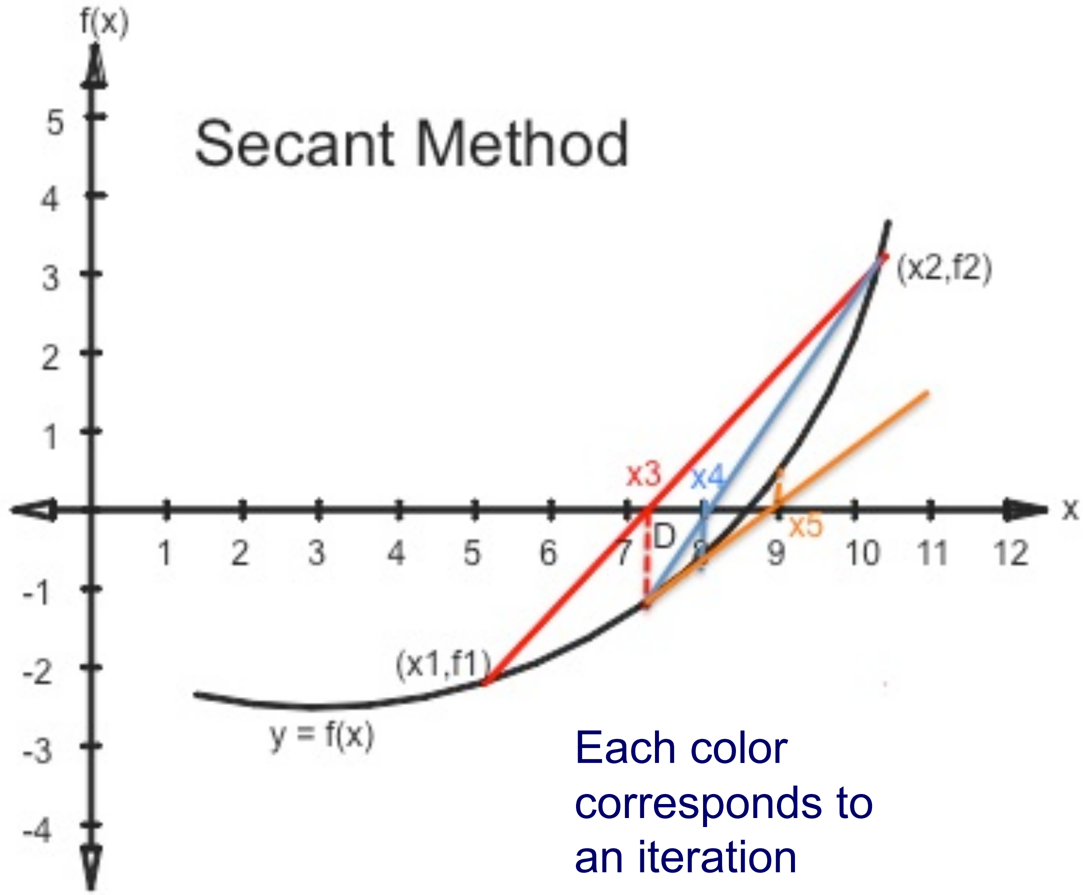
Secant Method

- Similar to *Regula-Falsi* Method, with the only difference that it does not bracket the root.

- A common method used when the analytical form of the function is not available

$$x_{n+1} = x_n - \frac{f(x_n)}{\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$

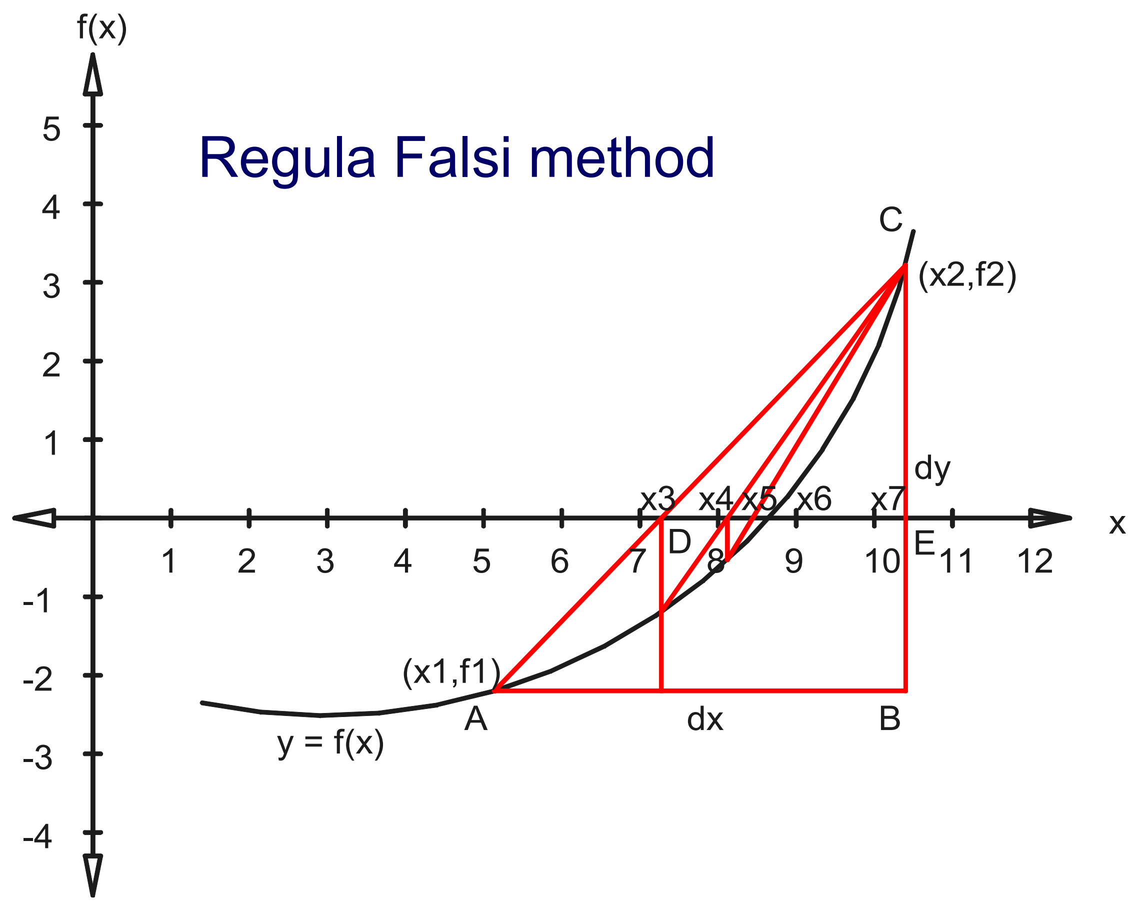
- If it converges, it converges faster but is not as robust



Each color corresponds to an iteration

Method of Linear Interpolation
(also called Regula Falsi or False Position)

- $f(x)$ continuous on $[x_1, x_2]$
- $f(x_1) = f_1$ & $f(x_2) = f_2$
- The line joining the end points of the interval is used iteratively to find the root.
- The method *brackets* the root every iteration.



Linear Interpolation

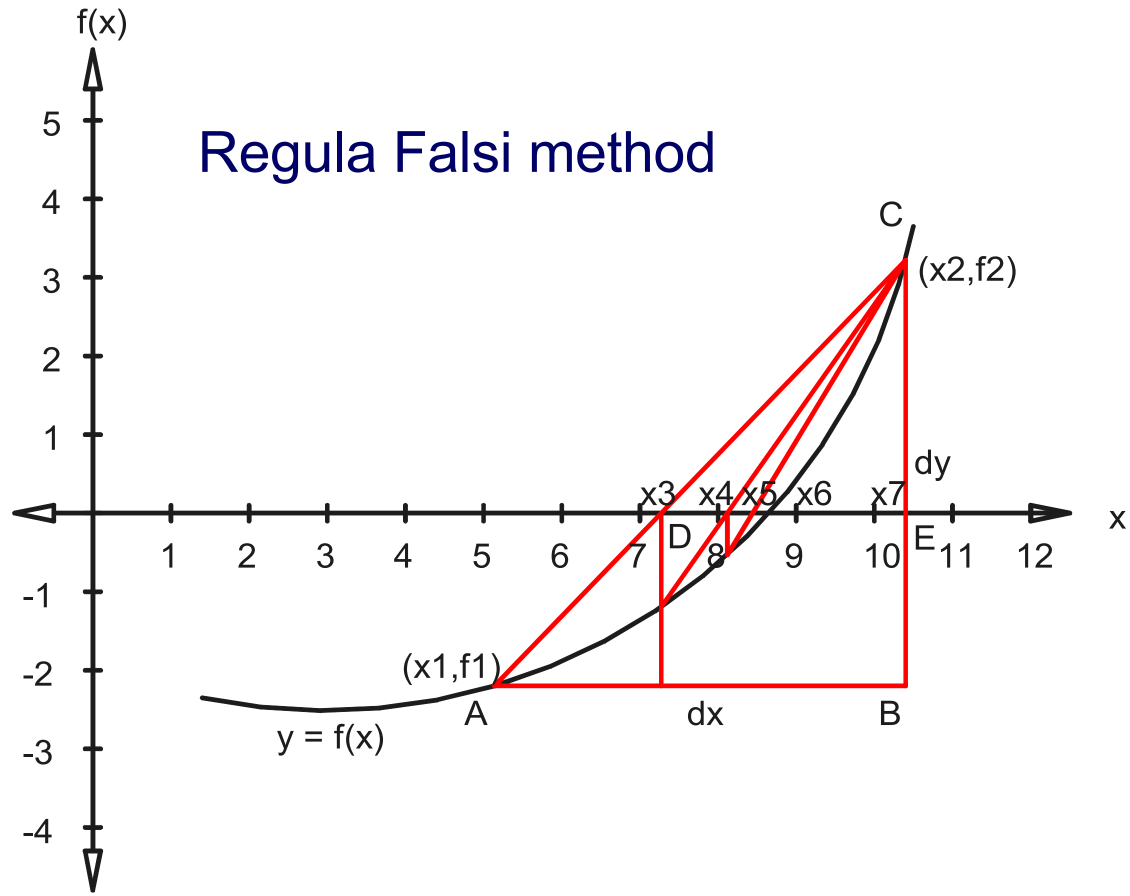
ΔABC similar to ΔDEC,

$$\frac{x_2 - x_3}{x_2 - x_1} = \frac{f_2 - f_3}{f_2 - f_1}$$

Replacing $f_3 = 0$, and generalizing, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$

Slope of the secant line joining the end points.



The location of the root is checked every iteration using the sign of $f(x_n) . f(x_{n+1})$

Summary

1. Bisection Method : Bracketing method that divides the initial interval into two halves and does a sign check to find out which half the root lies in. This method is iteratively repeated to a given tolerance.

2. *Regula Falsi* Method : Bracketing method that uses the slope of the secant line to obtain the next estimate.

3. Secant Method : A slope method that is equivalent to *Regula Falsi* Method except that it does NOT bracket the root.

4. Newton’s Method : Slope method that uses the derivative (tangent line) of the function to obtain the next estimate. It is the fastest of the basic root-finding methods discussed but is not as robust. We will discuss this in more detail in the following classes. Note that Newton’s method only requires one starting value unlike the other 3 methods listed above.