Solution of Linear Set of Equations – 04

Dr. Serhat Hosder

Associate Professor of Aerospace Engineering

Mechanical and Aerospace Engineering

290B Toomey Hall

Missouri S&T

Rolla, MO 65409

Phone: 573-341-7239

E-mail: hosders@mst.edu

Outline

In the previous lectures, we discussed direct methods for solving linear problems in which A was full. We will now consider the special case in which A is tri-diagonal.

Tri-diagonal equations is discussed in Section 6.6 in the textbook

We will learn Thomas algorithm for the solution of tri-diagonal system of linear equations.

Tri-diagonal Matrices

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & a_4 & b_4 & c_4 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & a_5 & b_5 & c_5 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & a_6 & b_6 & c_6 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_n & b_n \end{bmatrix}$$

A matrix in the form shown is tri-diagonal. These matrices are of special interest due to their application in engineering problems.

The elements along the main diagonal are labeled b_i , the elements one below the main diagonal are labeled a_i and the elements one above the main diagonal are labeled c_i.

The **Thomas algorithm** can be used if the coefficient matrix is in tri-diagonal form.

Solution to Tri-Diagonal Systems (1)

The system of equations in matrix form is $\boldsymbol{Ax = d}$.

In **Thomas Algorithm,** the coefficient matrix A in tri-diagonal form is decomposed into $\boldsymbol{LU}$ form: $\boldsymbol{A=LU}$

$$
\begin{bmatrix}
b_1 & c_1 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\
a_2 & b_2 & c_2 & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\
0 & a_3 & b_3 & c_3 & 0 & 0 & 0 & \dots & \dots & 0 \\
0 & 0 & a_4 & b_4 & c_4 & 0 & 0 & \dots & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
\dots & \dots & \dots & 0 & a_i & b_i & c_i & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{n-1} & b_{n-1} & c_{n-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_n & b_n
\end{bmatrix}
=
\begin{bmatrix}
\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\
\alpha_2 & \beta_2 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\
0 & \alpha_3 & \beta_3 & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\
0 & 0 & \alpha_4 & \beta_4 & 0 & 0 & 0 & \dots & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
\dots & \dots & \dots & 0 & \alpha_i & \beta_i & 0 & \dots & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{n-1} & \beta_{n-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_n & \beta_n
\end{bmatrix}
\begin{bmatrix}
1 & \gamma_1 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\
0 & 1 & \gamma_2 & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\
0 & 0 & 1 & \gamma_3 & 0 & 0 & 0 & \dots & \dots & 0 \\
0 & 0 & 0 & 1 & \gamma_4 & 0 & 0 & \dots & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
\dots & \dots & \dots & \dots & 0 & 1 & \gamma_i & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \gamma_{n-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$\boldsymbol{A}$ **=** $\boldsymbol{L}$ $\boldsymbol{U}$

The i^{th} equation of the system is:

$$\boldsymbol{a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i} \qquad \textbf{(1)}$$

Solution to Tri-Diagonal Systems (2)

By the LU decomposition method,

$Ax = d$ (But $A = LU$)

$(LU)x = d \qquad \Rightarrow L(Ux) = d \qquad \Rightarrow L\,x^* = d$ **where** $U\,x = x^*$

Recall that the i^{th} equation of the system is

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i \qquad (1)$$

From LU decomposition,

$$x_i + \gamma_i x_{i+1} = x_i^* \qquad (i.e.\ U\,x = x^*) \qquad (2)$$

$$x_{i-1} + \gamma_{i-1} x_i = x^*_{i-1} \qquad (3)$$

Substitute (3) into (1)

$$a_i (x^*_{i-1} - \gamma_{i-1} x_i) + b_i x_i + c_i x_{i+1} = d_i$$

$$(b_i - a_i \gamma_{i-1})\, x_i + c_i x_{i+1} = d_i - a_i x^*_{i-1}$$

Compare eqns (2) and (4)

$$x_i + \left(\frac{c_i}{b_i - a_i \gamma_{i-1}} \right) x_{i+1} = \left(\frac{d_i - a_i x^*_{i-1}}{b_i - a_i \gamma_{i-1}} \right) \qquad (4)$$

Solution to Tri-Diagonal Systems (3)

Comparing equations *(2)* and *(4)*,

$$\gamma_i = \left(\frac{c_i}{b_i - a_i \gamma_{i-1}} \right) and \;\; x_i^* = \left(\frac{d_i - a_i x_{i-1}^*}{b_i - a_i \gamma_{i-1}} \right) \qquad \rightarrow (5)$$

Substituting ***i = 1 and noting that*** $a_1=0$, we have

$$\gamma_1 = \frac{c_1}{b_1} \;\; and \;\; x_1^* = \frac{d_1}{b_1}$$

Starting with $i=1$, equation *(5)* is solved for $i = 2, 3, 4, \ldots n$

Then solve *(2)* for x_i, i.e., $x_i = x^*_i - \gamma_i x_{i+1}$

Starting with $x_n = x^*_n$ (since $\gamma_n = c_n = 0$) and for $i = n-1, n-2, \ldots, 1$

The operation count for Thomas algorithm varies linearly as a function of ***n – the number of unknown variables.***