Gradient-Based Non-Linear Optimization – Lecture 03

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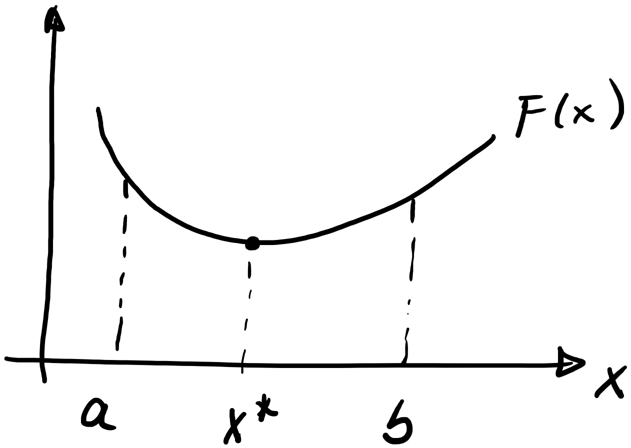
1-D, Unconstraint, Optimization

$$\frac{dF}{dx} = \boxed{g(x) = 0}$$



F(x) = Objective Function of a single variable "x".

Minimize F(x) on [a, b]

to find $x^* =$ minimum

Steps for 1-D minimization

① Determine the bounds on the minimum of F(x) (i.e., [a, b])

initial interval

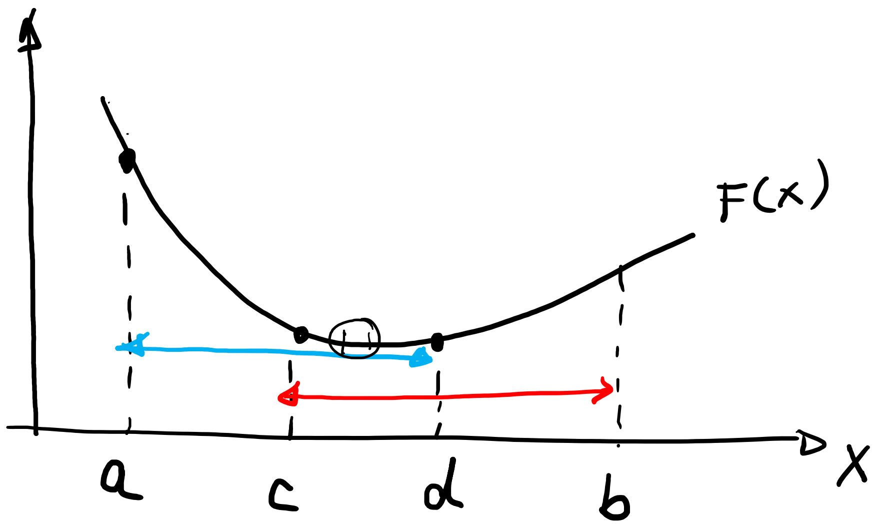
② Use an algorithm to find the minimum within [a, b]

↳ Solve $\frac{dF}{dx} = g(x) = 0$ (root finding)

↳ Golden Section Algorithm ⇒ used to reduce the size of the interval where the minimum exists

↳ Optional step: Fit a cubic polynomial to the points of last interval and predict the min. from poly. fit.

Golden Section Search Algorithm for 1-D optimization



At a given iteration

interval bounds $\Rightarrow [a, b]$

$$d - a = b - c$$

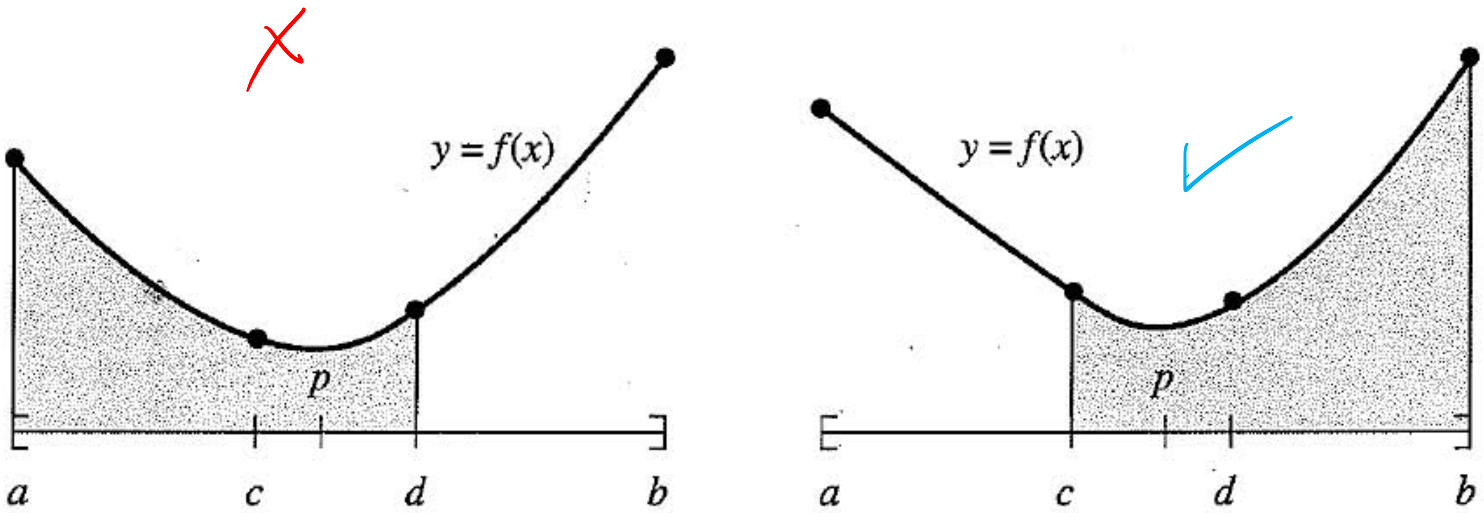
$$\frac{d-a}{b-a} = r$$

$$\frac{b-c}{b-a} = r$$

r = fixed ratio

Objective of the Golden Search Algorithm: To reduce the interval size at each iteration with a single function evaluation while selecting the interval points at fixed ratios relative to the interval size at that iteration.

$$\boxed{\frac{d_0 - a_0}{b_0 - a_0} = r \qquad \frac{b_0 - c_0}{b_0 - a_0} = r}$$



If $f(c) \leqq f(d)$, then squeeze from the right and use $[a, d]$.

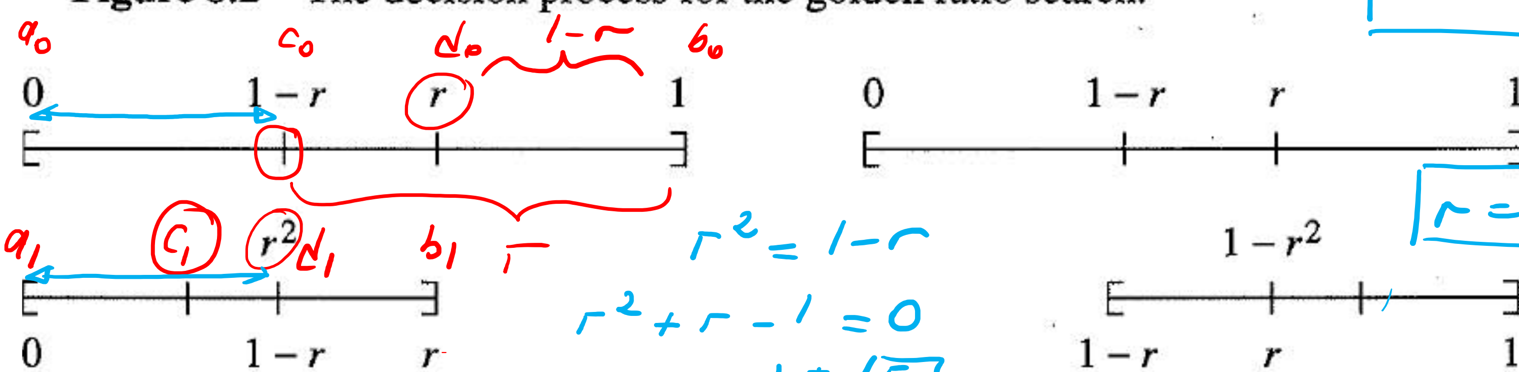
If $f(d) < f(c)$, then squeeze from the left and use $[c, b]$.

Figure 8.2 The decision process for the golden ratio search.

$$\boxed{r = \frac{-1+\sqrt{5}}{2}}$$

$k=0$

$k=1$



$$\boxed{r = 0.61803}$$

$$r^2 = 1 - r$$

$$r^2 + r - 1 = 0$$

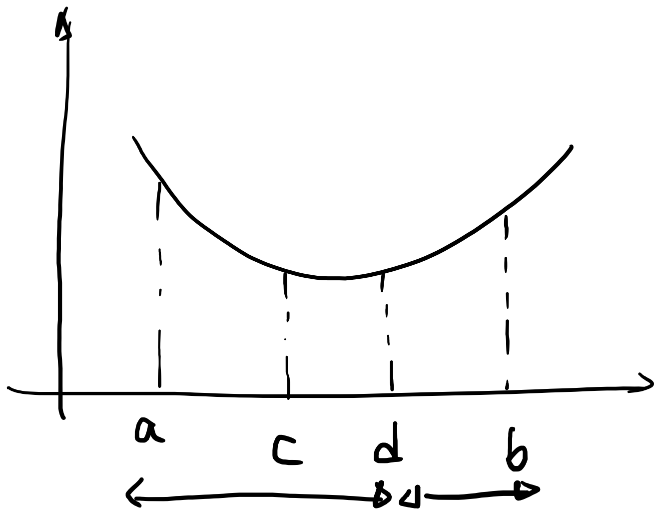
$$r_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$$

$$\frac{d_1 - a_1}{b_1 - a_1} = r$$

Squeeze from the right and the new interval is $[0, r]$.

Squeeze from the left and the new interval is $[1 - r, 1]$.

Figure 8.3 The intervals involved in the golden ratio search.



$$r = 0.61803$$

$$\frac{d-a}{b-d} = \frac{r}{1-r} = \underbrace{1.61803}_{\text{Golden Section Ratio}}$$

$$\frac{c-a}{b-a} = (1-r) \longrightarrow c = a + (b-a).(1-r)$$

$$\boxed{c = (1-r)\,b + r\,a} \quad \text{Eqn 1.}$$

$$\frac{d-a}{b-a} = r \longrightarrow d = a + r\,(b-a)$$

$$\boxed{d = (1-r)\,a + r\,b} \quad \text{Eqn 2.}$$

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General Algorithm For Golden Search Method

k = iteration number

At $k=0$ → Pick $[a_0, b_0]$ initial interval

→ Calculate c_0 using Eqn 1

Calculate d_0 " Eqn 2.

At the k^{th} step: Compare $f(c_{k-1})$ with $f(d_{k-1})$

If $f(c_{k-1}) \leq f(d_{k-1})$

$a_k = a_{k-1}$

$b_k = d_{k-1}$

$d_k = c_{k-1}$

$c_k = (1-r)b_k + r\,a_k$ (Eqn 1)

If $f(c_{k-1}) > f(d_{k-1})$

$a_k = c_{k-1}$

$b_k = b_{k-1}$

$c_k = d_{k-1}$

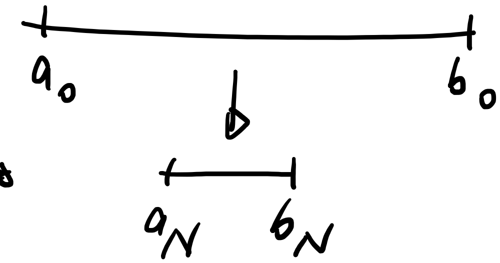
$d_k = (1-r)\,a_k + r\,b_k$ (Eqn 2)

Repeat this procedure until you reduce the interval size to a desired length.

Convergence Criterion for Golden Section Algorithm

Initial interval size $\longrightarrow$ $b_0 - a_0 \Rightarrow$

At the N^{th} iteration $\longrightarrow$ $b_N - a_N \Rightarrow$



$$\frac{b_N - a_N}{b_0 - a_0} = \boxed{\varepsilon = r^N}$$

$\boxed{r = 0.61803}$

ε = Relative Tolerance

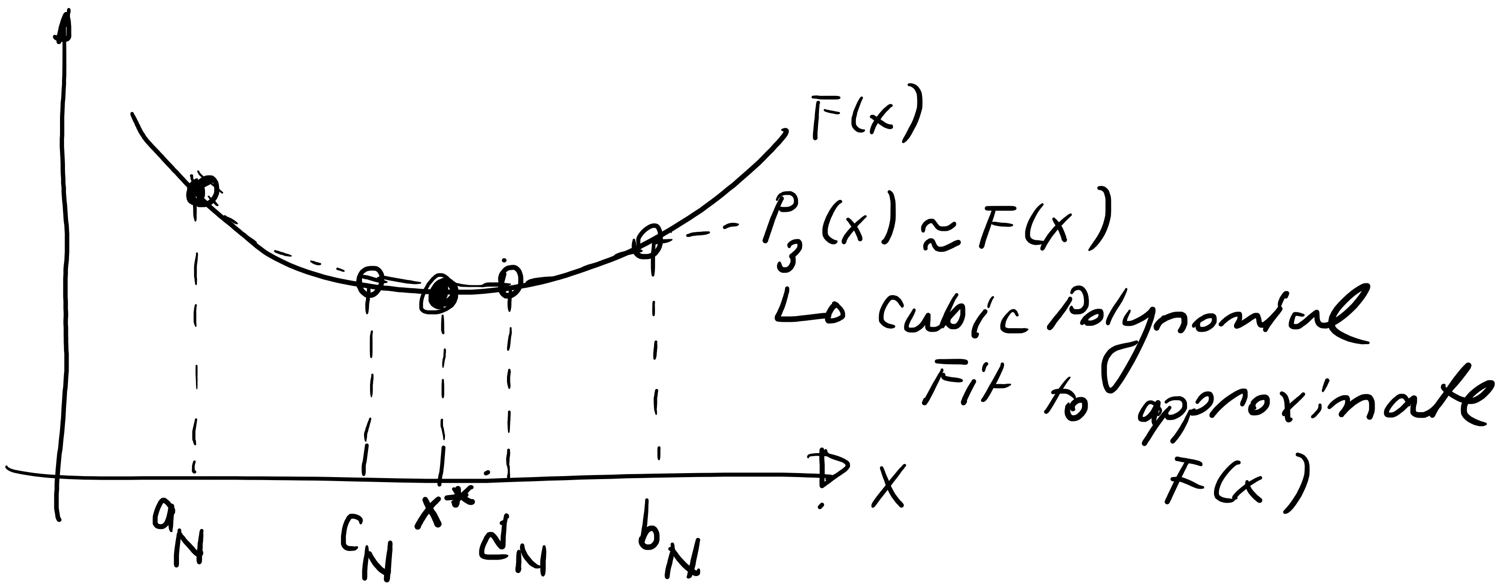
$$r^N = \varepsilon \rightarrow \log(r^N) = \log(\varepsilon)$$

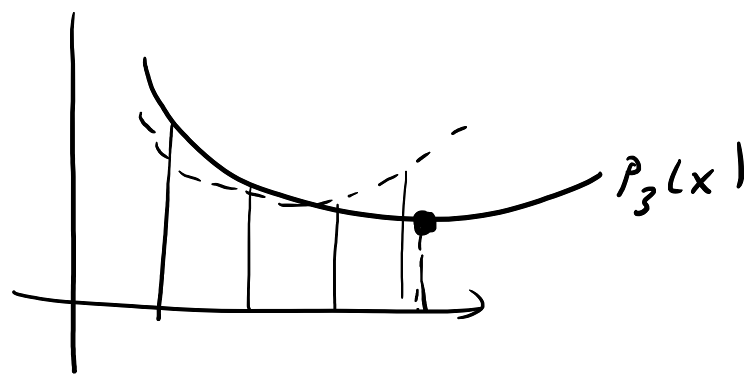
$$N \cdot \log(r) = \log(\varepsilon)$$

$$\boxed{N = \frac{\log(\varepsilon)}{\log(r)}}$$

Example: $\boxed{\varepsilon = 0.01}$

$N = 9.6 \Rightarrow 10$ iterations required.


F(x)
P3(x) ≈ F(x)
↳ cubic Polynomial Fit to approximate F(x)
X
aN
cN
x*
dN
bN


P3(x)

Cubic polynomial fit to the given four data points:

$a, f(a)$ $c, f(c)$ $d, f(d)$ $b, f(b)$

$$(x_1, f_1) \quad (x_2, f_2) \quad (x_3, f_3) \quad (x_4, f_4)$$

Define:

$$q_1 = x_3^3(x_2 - x_1) - x_2^3(x_3 - x_1) + x_1^3(x_3 - x_2)$$

$$q_2 = x_4^3(x_2 - x_1) - x_2^3(x_4 - x_1) + x_1^3(x_4 - x_2)$$

$$q_3 = (x_3 - x_2)(x_2 - x_1)(x_3 - x_1)$$

$$q_4 = (x_4 - x_2)(x_2 - x_1)(x_4 - x_1)$$

$$q_5 = f_3(x_2 - x_1) - f_2(x_3 - x_1) + f_1(x_3 - x_2)$$

$$q_6 = f_4(x_2 - x_1) - f_2(x_4 - x_1) + f_1(x_4 - x_2)$$

Cubic polynomial approximation to the function:

$$\boxed{\tilde{f}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3}$$

where

$$a_3 = \frac{q_3 q_6 - q_4 q_5}{q_2 q_3 - q_1 q_4}$$

$$a_2 = \frac{q_5 - a_3 q_1}{q_3}$$

$$a_1 = \frac{f_2 - f_1}{x_2 - x_1} - a_3 \frac{x_2^3 - x_1^3}{x_2 - x_1} - a_2 (x_1 + x_2)$$

$$a_0 = f_1 - a_1 x_1 - a_2 x_1^2 - a_3 x_1^3$$

→ minimum point has to be a stationary point

→ $\frac{d\tilde{f}(x)}{dx} = 0$ (Necessary condition)

Approximation to the minimum function point using the cubic polynomial fit:

$$\frac{d\tilde{f}(x)}{dx} = a_1 + 2a_2x + 3a_3x^2$$

$$\frac{d\tilde{f}}{dx} = 0 \rightarrow \boxed{a_1 + 2a_2x + 3a_3x^2 = 0}$$

$$\Delta = a_2^2 - 3a_1a_3$$

$$\hat{x}_1 = \frac{-a_2 + \sqrt{\Delta}}{3a_3} \qquad \hat{x}_2 = \frac{-a_2 - \sqrt{\Delta}}{3a_3}$$

$\Delta > 0$ $\quad \hat{x}_1$ and $\hat{x}_2$ are real and unique roots

$\Delta = 0$ $\quad \hat{x}_1$ and $\hat{x}_2$ are real and $\hat{x}_1 = \hat{x}_2$

$\Delta < 0$ $\quad \hat{x}_1$ and $\hat{x}_2$ are complex conjugate roots

$$\frac{d^2\tilde{f}(x)}{dx^2} = 2a_2 + 6a_3x$$

$$\left.\frac{d^2\tilde{f}(x)}{dx^2}\right|_{x=\hat{x}_1} = 2\sqrt{\Delta} \quad \left(\text{For } \Delta > 0 \Rightarrow \frac{d^2\tilde{f}}{dx^2} > 0\right)$$

$$\left.\frac{d^2\tilde{f}(x)}{dx^2}\right|_{x=\hat{x}_2} = -2\sqrt{\Delta} \quad \left(\text{For } \Delta > 0 \Rightarrow \frac{d^2\tilde{f}}{dx^2} < 0\right)$$

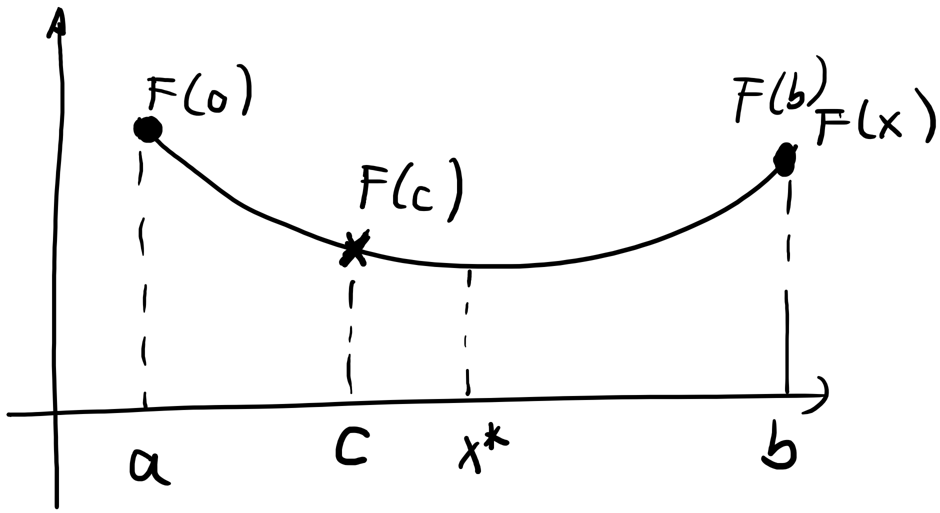
$\Delta > 0$ $\boxed{\hat{x}_1 \rightarrow \text{minimum}}$ and $\hat{x}_2 \rightarrow$ maximum

$\Delta = 0$ $\hat{x}_1$ and $\hat{x}_2$ neither a minimum nor maximum

$\Delta < 0$ no real roots

Finding the bounds on the minimum of $F(x)$

(or picking the initial interval which includes the min).



For $a < c < b$

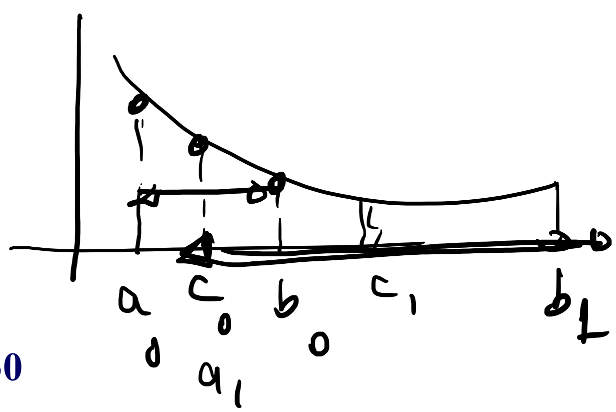
If $F(c) < F(b)$ and

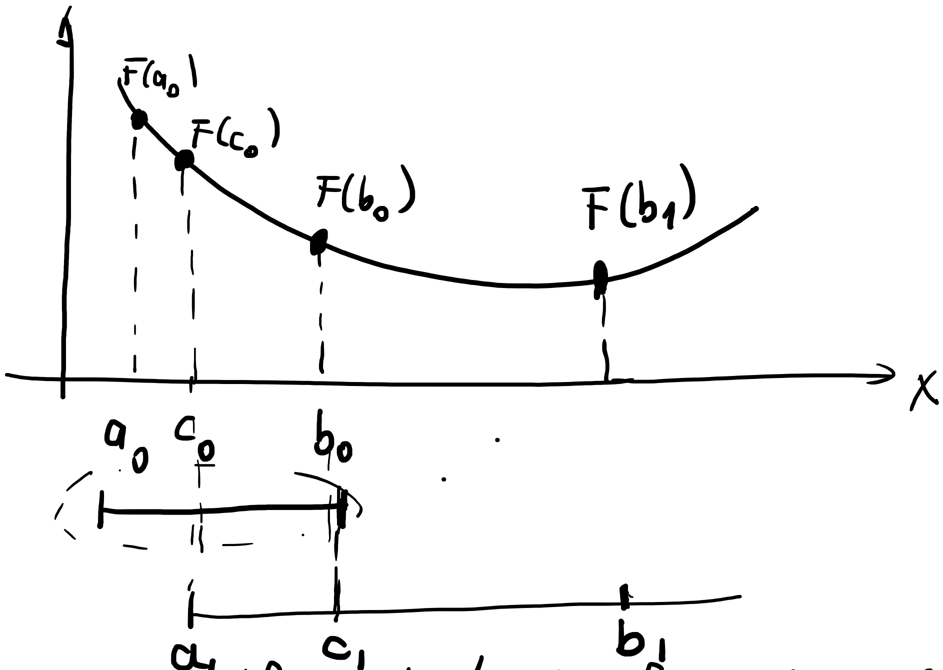
$F(c) < F(a)$

Then the interval

$[a, b]$ includes the

minimum x^*.





→ start with $[a_0, b_0]$

→ $c_0 = (1-r)b_0 + r\,a_0$

→ Check if $F(c_0) < F(b_0)$

and if $F(c_0) < F(a_0)$

If true → $[a_0, b_0]$

If not true, for the next interval check

$$a = c_0$$

$$c = b_0$$

$$b = (1+\tau)c - \tau a \quad \text{where} \quad \tau = 1+r$$

Repeat the function check, update if necessary.