HW3

AERO 5830 - S. HOSDER

MATT PAHAYO

Question 1

Consider a simple two-spring system shown in the undeformed position in Figure 1a and after deformation in Figure 1b. The springs are linearly elastic with constants $K_1 = 8\ N/cm$ and $K_2 = 1N/cm$. The loads P_1 and P_2 are both constant and equal to 5 N. In terms of the original lengths of the springs $l_1 = l_2 = 10\ cm$ and the displacements x_1 and x_2, the total potential energy of the system is given as

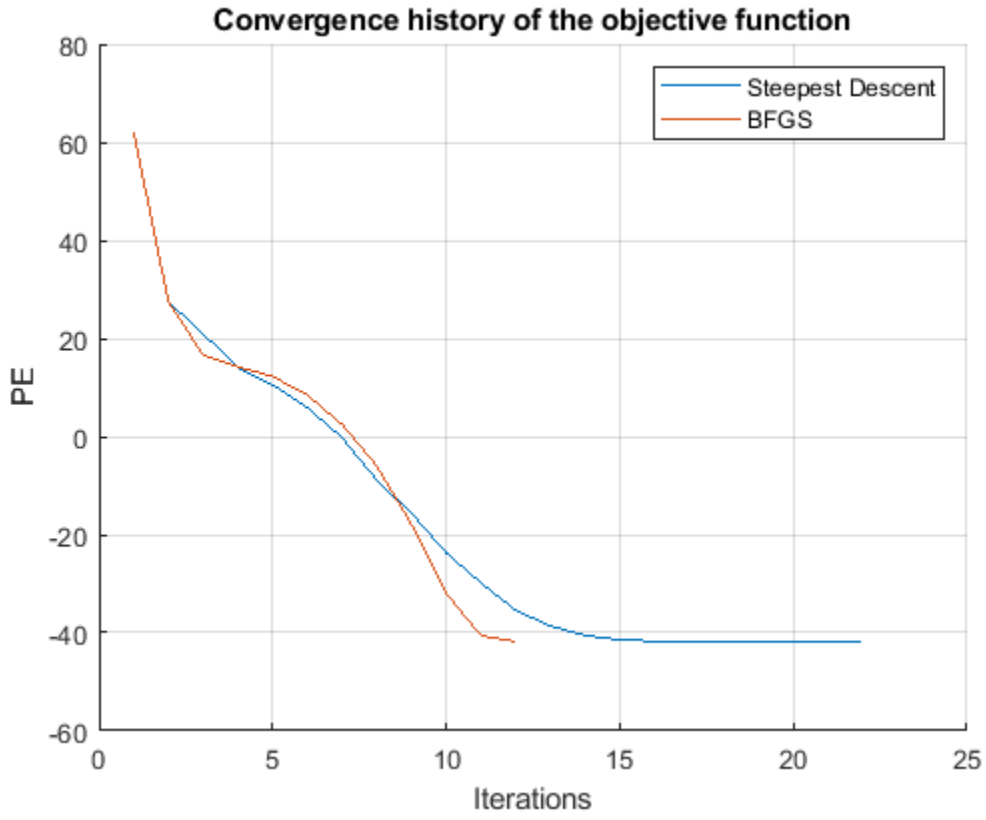
$$PE(x_1, x_2) = \frac{1}{2}K_1\left[\sqrt{x_1^2 + (l_1 - x_2)^2} - l_1\right]^2 + \frac{1}{2}K_2\left[\sqrt{x_1^2 + (l_2 + x_2)^2} - l_2\right]^2 - P_1x_1 - P_2x_2.$$

This equation can be solved for the equilibrium displacements as an unconstrained minimization problem where $\vec{x} = (x_1, x_2)$ is the independent design variable vector and $PE(x_1, x_2)$ is the objective function. Calculate the minimum objective function value and the corresponding

design variables by using **(i)** Steepest Descent, and **(ii)** the BFGS Variable Metric (Quasi-Newton) Methods. For each method, use the initial design point $\vec{x}^0 = \{-4, 5\}^T$ and limit the maximum number of iterations to 200. Use the convergence criterion $\left|PE(\vec{x}^k) - PE(\vec{x}^{k-1})\right| \leq 10^{-10}$ and make sure to satisfy this at 4 successive iterations. For each method, output the objective function and design variable vector values at the minimum along with the number of iterations to converge. Plot the convergence history of the objective function ($PE(\vec{x}^k)$ versus k, the iteration number) for each method. For one dimensional search at each iteration, you may use the routine you have developed in Homework 4 with $n = 15$ or 20 for golden section search. (Note that at the minimum, $PE = -41.8082$, $x_1 = 8.6321$, and $x_2 = 4.5319$.)

Results

	Steepest D	BFGS
x1	8.629644	8.631849
x2	4.531281	4.531588
iterations	23	13
F at {x}	-41.8082	-41.8082



Methodology

Get the search direction from the gradient. Equation 1 will give a unit vector and is for the Steepest Descent Method.

$$\overrightarrow{s^k} = \nabla \mathrm{F}(\overrightarrow{x^k}) / \left\| \nabla F(\overrightarrow{x^k}) \right\| \qquad (1)$$

Get alpha star by evaluating the function at the guess plus the search direction times alpha. Then find the minimum value of alpha by minimizing the alpha-function with a Golden Section algorithm and then fit a cubic function that is outputted by the Golden Section algorithm to get alpha star.

$$\frac{d}{d\alpha} F\left(\overrightarrow{x^k} + \alpha^k \overrightarrow{s^k}\right) = 0 \quad (2)$$

$$\overrightarrow{x^{k+1}} = \overrightarrow{x^k} + \alpha^k \overrightarrow{s^k} \qquad (3)$$

Everything on the right-hand side is known, so get the next iteration of x-vector. These steps are repeated until the stopping criteria is met (equation 4).

$$\left| F\left(\overrightarrow{x^{k+1}}\right) - F(\overrightarrow{x^k}) \right| < tol \qquad (4)$$

To use the BFGS algorithm all that needs to be changed is how the search direction is calculated. This is done by approximating the Hessian matrix at x^k and multiplying it by the negative of the gradient of the function at x^k.

$$\overrightarrow{s^k} = -H^k \nabla F(\overrightarrow{x^k}) \qquad (5)$$

To approximate the Hessian, let the first iteration be the identity matrix at the first iteration and to get the next define D.

At the end of the kth iteration define $\hat{H}^{k+1} = \hat{H}^k + D^k$

where symmetric update matrix D^k is given by

$$D^k = \frac{\sigma + \theta\tau}{\sigma^2}\vec{p}\vec{p}^T + \frac{\theta - 1}{\tau}\hat{H}^k\vec{y}\left(\hat{H}^k\vec{y}\right)^T - \frac{\theta}{\sigma}\left[\hat{H}^k\vec{y}\vec{p}^T + \vec{p}\left(\hat{H}^k\vec{y}\right)^T\right]$$

Where the change vectors are defined as:

$\vec{p} = \vec{x}^k - \vec{x}^{k-1}$ and $\vec{y} = \nabla F(\vec{x}^k) - \nabla F(\vec{x}^{k-1})$

and the scalers are defined as:

$\sigma = \vec{p}^T\vec{y}$ and $\tau = \vec{y}^T\hat{H}^k\vec{y}$

1. For $\theta = 0 \rightarrow$ DFP (Davidon - Fletcher - Powell) Method
2. For $\theta = 1 \rightarrow$ BFGS (Braydon - Fletcher - Goldfarb - Shanno) Method

Question 2

It can be shown that solution to a non-linear set of equations ($f_i(\vec{x}) = 0$, $i = 1, 2, ..., n$ and $\vec{x} = \{x_1, x_2, ..., x_n\}$) can be obtained by formulating the problem as a non-linear optimization problem with the objective function

$$F(\vec{x}) = f_1^2(\vec{x}) + f_2^2(\vec{x}) + ... + f_n^2(\vec{x}) \quad (1)$$

By implementing the above strategy, use Steepest Descent and the BFGS Variable Metric (Quasi-Newton) methods to find the solution vector to the following system of equations:

$$\begin{aligned} 4x_1 - x_2 + x_3 &= x_1 x_4 \\ -x_1 + 3x_2 - 2x_3 &= x_2 x_4 \\ x_1 - 2x_2 + 3x_3 &= x_3 x_4 \\ x_1^2 + x_2^2 + x_3^2 &= 1 \end{aligned}$$

For each method, obtain the solution using two starting vectors: $x^0 = \{1, 1, 1, 1\}^T$ and $x^0 = \{3, 3, 3, 3\}^T$ and use the same stopping criteria given in Question 1 with the objective function you have defined for this question. Comment on the the convergence of two methods.

Results

{x0} = [1;1;1;1]	Steepest Descent	BFGS
x1	0.001247359	-0.8195
x2	0.708618632	0.635988
x3	0.705587656	-0.89644
x4	1.005244657	5.389582
iterations	200	200
F at {x}	7.72E-06	11.82891

{x0} = [3;3;3;3]	Steepest Descent	BFGS
x1	0.301551553	-0.32746
x2	0.90455456	0.714646
x3	0.301409474	-0.65508
x4	2.000214261	7.862718
iterations	200	200
F at {x}	1.75E-08	20.76306

The Steepest Descent converges to a value for both guesses while BFGS does not seem to do so. The Descent converges to zero while the BFGS does not in the given number of iterations.

Methodology

To solve the system, the functions are first rearranged, so the functions equal zero. Then a function is defined where it is the L2norm of the other functions. The algorithms developed in question 1 are then used on this new function.

$$F(\vec{x}) = f_1^2(\vec{x}) + f_2^2(\vec{x}) + ... + f_n^2(\vec{x}) \quad (6)$$

```
classdef mOpt
    % mOpt is a multivariable optimizer

    methods (Static)
        function [Fq,q,iter,PE] = Steep(F,q0,tol)
            syms x1 x2 x3 x4 as
            % Steep is a function that utilizes the
            %   steepest descent method
            q = q0;
            q1(1) = q0(1);
            q2(1) = q0(2);
            q3(1) = q0(3);
            q4(1) = q0(4);
            i=2;
            q1(2) = 10^99;
            q2(2) = 10^99;
            q3(2) = 10^99;
            q4(2) = 10^99;
            gradF(x1,x2,x3,x4) = gradient(F,[x1,x2,x3,x4]);
            s = @(x1,x2,x3,x4) -gradF(x1,x2,x3,x4)/norm(gradF(x1,x2,x3,x4));
            while abs(F(q1(i),q2(i),q3(i),q4(i))-F(q1(i-1),q2(i-1),q3(i-
1),q4(i-1)))>tol
                if q1(2) == 10^99
                    q1(2) = q0(1);
                    q2(2) = q0(2);
                    q3(2) = q0(3);
                    q4(2) = q0(4);
                end
                search = double(s(q1(i),q2(i),q3(i),q4(i)));
                fas(as) =
F(q1(i)+search(1)*as,q2(i)+search(2)*as,q3(i)+search(3)*as,q4(i)+search(4)*as
);
                [xlow,w2,w1,xhigh] = onedOpt.Gold(0,2,20,0,-fas);
                [alpha,y] = cubicFit(xlow,w2,w1,xhigh,-fas);
                q = q + alpha(1)*search
                PE(i-1,1) = F(q1(i),q2(i),q3(i),q4(i));
                q1(i+1) = q(1);
                q2(i+1) = q(2);
                q3(i+1) = q(3);
                q4(i+1) = q(4);
                i = i + 1;
                if i == 201
                    break;
                end
            end
            Fq = F(q1(end),q2(end),q3(end),q4(end));
            iter = i - 1;
        end

        function [Fq,q,iter,PE] = BFGS(F,q0,theta,tol)
            %METHOD1 Summary of this method goes here
            %   Detailed explanation goes here
            syms x1 x2 x3 x4 as
            q = q0;
            q1(1) = q0(1);
            q2(1) = q0(2);
```

```
            q3(1) = q0(3);
            q4(1) = q0(4);
            i=2;
            q1(2) = 10^99;
            q2(2) = 10^99;
            q3(2) = 10^99;
            q4(2) = 10^99;
            gradF(x1,x2,x3,x4) = gradient(F,[x1,x2,x3,x4]);
            H = eye(length(q));
            D = 0;
            while abs(F(q1(i),q2(i),q3(i),q4(i))-F(q1(i-1),q2(i-1),q3(i-
1),q4(i-1)))>tol
                if q1(2) == 10^99
                    q1(2) = q0(1);
                    q2(2) = q0(2);
                    q3(2) = q0(3);
                    q4(2) = q0(4);
                end
                if i >2
                p = [q1(i)-q1(i-1);q2(i)-q2(i-1);q3(i)-q3(i-1);q4(i)-q4(i-
1)];
                y = double(gradF(q1(i),q2(i),q3(i),q4(i))-gradF(q1(i-1),q2(i-
1),q3(i-1),q4(i-1)));
                sigma = transpose(p)*y;
                tau = transpose(y)*H*y;
                D = (sigma+theta*tau)/sigma^2*p*transpose(p)...
                +(theta-1)/tau*H*y*transpose(H*y)-...
                theta/sigma*(H*y*transpose(p)+p*transpose(H*y));
                end
                H = H + D;
                search = double(-H*gradF(q1(i-1),q2(i-1),q4(i-1),q3(i-
1))/norm(-H*gradF(q1(i-1),q2(i-1),q3(i-1),q4(i-1))));
                fas(as) =
F(q1(i)+search(1)*as,q2(i)+search(2)*as,q3(i)+search(3)*as,q4(i)+search(4)*as
);
                [xlow,w2,w1,xhigh] = onedOpt.Gold(0,2,20,0,-fas);
                [alpha,y] = cubicFit(xlow,w2,w1,xhigh,-fas);
                q = q + alpha(1)*search;
                PE(i-1,1) = F(q1(i),q2(i),q3(i),q4(i));
                q1(i+1) = q(1);
                q2(i+1) = q(2);
                q3(i+1) = q(3);
                q4(i+1) = q(4);
                i = i + 1;
                if i == 201
                    break;
                end
            end
            Fq = F(q1(end),q2(end),q3(end),q4(end));
            iter = i - 1;
        end

    end
end
```

```
classdef onedOpt
    % onedOpt is a class of 1-d optimization functions

    methods (Static)
        function [xlow,x2,x1,xhigh] = Gold(xlow,xhigh,n,es,f)
            % Gold is the Golden section algorithm for 1-d opt.
            % if using a set convergence target set es to the according
            % value and set iter to any value else set it to zero for a
targeted amount of iterations
            R = (sqrt(5)-1)/2;
            if es>0
               n = log10(es)/log10(R);
            end
            iter = 1;
            d = R*(xhigh-xlow);
            x1 = xlow + d;
            x2 = xhigh - d;
            f1 = double(f(x1));
            f2 = double(f(x2));
            if f1>f2
                xopt = x1;
                fx = f1;
            else
                xopt = x2;
                fx = f2;
            end

            while iter<n
                d = R*d;
                if f1>f2
                    xlow = x2;
                    x2 = x1;
                    x1 = xlow + d;
                    f2 = f1;
                    f1 = double(f(x1));
                else
                    xhigh = x1;
                    x1 = x2;
                    x2 = xhigh - d;
                    f1 = f2;
                    f2 = double(f(x2));
                end
                if f1>f2
                    fx = f1;
                else
                    fx = f2;
                end
                iter = iter + 1;
            end
        end
    end
end
```

```
function [x,y] = cubicFit(xlow,x2,x1,xhigh,f)
%cubicFit fits a cubic function into to the specified points
%   takes values from Gold
q1 = x1^3*(x2-xlow)-x2^3*(x1-xlow)+xlow^3*(x1-x2);
q2 = xhigh^3*(x2-xlow)-x2^3*(xhigh-xlow)+xlow^3*(xhigh-x2);
q3 = (x1-x2)*(x2-xlow)*(x1-xlow);
q4 = (xhigh-x2)*(x2-xlow)*(xhigh-xlow);
q5 = double(f(x1))*(x2-x1)-double(f(x2))*(x1-xlow)+double(f(x1-x2));
q6 = double(f(xhigh))*(x2-x1)-double(f(x2))*(xhigh-xlow)+double(f(xhigh-x2));

a3 = (q3*q6-q4*q5)/(q2*q3-q1*q4);
a2 = (q5-a3*q1)/q3;
a1 = (double(f(x2)-f(xlow)))/(x2-xlow)-a3*(x2^3-xlow^3)/(x2-xlow)-...
    a2*(xlow+x2);
del = a2^2-3*a1*a3;

x(1) = double(-a2+sqrt(del))/3/a3;
x(2) = double(-a2-sqrt(del))/3/a3;
y(1) = double(f(x(1)));
y(2) = double(f(x(2)));
end


% main.m
clc
clear all
close all

format longg

syms x1 x2 x3 x4 as
q1 = -4;
q2 = 5;
q3 = 0;
q4 = 0;
q0 = [q1;q2;q3;q4];
tol =10^-10;
k1 = 8;
k2 = 1;
P1 = 5;
P2 = 5 ;
l1 =10;
l2 = 10;
theta = 1;

F = @(x1,x2,x3,x4) 1/2*k1*(sqrt(x1^2+(l1-x2)^2)-
l1)^2+k2/2*(sqrt(x1^2+(l2+x2)^2)-l2)^2-P1*x1-P2*x2;

[Fq,q,iter,PE] = mOpt.BFGS(F,q0,theta,10^-6)
[Fs,qs,iters,PEs] = mOpt.Steep(F,q0,10^-6)

hold on
plot(1:length(PEs),PEs)
plot(1:length(PE),PE)
xlabel('Iterations')
```

```
legend('Steepest Descent','BFGS')
ylabel('PE')
title('Convergence history of the objective function')
grid on

x0 = [1;1;1;1]
F = @(x1,x2,x3,x4) (4*x1-x2+x3-x1*x4)^2+ ...
    (-x1+3*x2-2*x3-x2*x4)^2 + (x1-2*x3+3*x3-x3*x4)^2 + ...
    (x1^2+x2^2+x3^2-1)^2
[Fq,q,iter,PE] = mOpt.BFGS(F,x0,theta,tol)
[Fq,q,iter,PE] = mOpt.Steep(F,x0,tol)
```