AE / ME 301 Solutions to Homework 2

Question 1
Part a :

Input [K] Matrix (in N / m) and right hand side vector {p} (in N)

In[35]:=
```
K = {{27.58, 7.004, -7.004, 0.0000, 0.0000}, {7.004, 29.57, -5.253, 0.0000, -24.32}, {-7.004, -5.253, 29.57, 0.0000, 0.0000},
    {0.0000, 0.0000, 0.0000, 27.58, -7.004}, {0.0000, -24.32, 0.0000, -7.004, 29.57}} * 10^6;
p = {0.0, 0.0, 0.0, 0.0, -45.0} * 10^3;
Print["K=", MatrixForm[K]]
Print["p=", MatrixForm[p]]
```

$$K=\begin{pmatrix} 2.758\times10^{7} & 7.004\times10^{6} & -7.004\times10^{6} & 0. & 0. \\ 7.004\times10^{6} & 2.957\times10^{7} & -5.253\times10^{6} & 0. & -2.432\times10^{7} \\ -7.004\times10^{6} & -5.253\times10^{6} & 2.957\times10^{7} & 0. & 0. \\ 0. & 0. & 0. & 2.758\times10^{7} & -7.004\times10^{6} \\ 0. & -2.432\times10^{7} & 0. & -7.004\times10^{6} & 2.957\times10^{7} \end{pmatrix}$$

$$p=\begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ -45\,000. \end{pmatrix}$$

Perform Gauss Elimination with Partial Pivoting :

In[41]:=
```
x = gausswpv[K, p];
Print["Solution vector x (in meters)=", MatrixForm[x]]
```

$$\text{Solution vector x (in meters)}=\begin{pmatrix} 0.00144044 \\ -0.00648249 \\ -0.000810405 \\ -0.00185182 \\ -0.00729199 \end{pmatrix}$$

Part b :

Input Hilbert Matrix and right hand side vector b for n = 5 :

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Input Hilbert Matrix and right hand side vector b for n = 5 :

```
hilbert = Table[1. / (i + j - 1.), {i, 1, 5}, { j, 1, 5}];
A = hilbert;
b = Table[1, {5}];
Print["Hilbert Matrix=", MatrixForm[A]]
Print["b=", MatrixForm[b]]
```

Hilbert Matrix= $\begin{pmatrix} 1. & 0.5 & 0.333333 & 0.25 & 0.2 \\ 0.5 & 0.333333 & 0.25 & 0.2 & 0.166667 \\ 0.333333 & 0.25 & 0.2 & 0.166667 & 0.142857 \\ 0.25 & 0.2 & 0.166667 & 0.142857 & 0.125 \\ 0.2 & 0.166667 & 0.142857 & 0.125 & 0.111111 \end{pmatrix}$

b= $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Perform Gauss Elimination with Partial Pivoting :

```
x = gausswpv[A, b];
Print["Solution vector x=", MatrixForm[x]]
```

Solution vector x= $\begin{pmatrix} 5. \\ -120. \\ 630. \\ -1120. \\ 630. \end{pmatrix}$

Input Hilbert Matrix and right hand side vector b for n = 10 :

```
    hilbert = Table[1. / (i + j - 1.), {i, 1, 10}, { j, 1, 10}];
A = hilbert;
b = Table[1, {10}];
Print["Hilbert Matrix="]
MatrixForm[A]
Print["b=", MatrixForm[b]]
```

Hilbert Matrix=

$$\begin{pmatrix} 1. & 0.5 & 0.333333 & 0.25 & 0.2 & 0.166667 & 0.142857 & 0.125 & 0.111111 & 0.1 \\ 0.5 & 0.333333 & 0.25 & 0.2 & 0.166667 & 0.142857 & 0.125 & 0.111111 & 0.1 & 0.0909091 \\ 0.333333 & 0.25 & 0.2 & 0.166667 & 0.142857 & 0.125 & 0.111111 & 0.1 & 0.0909091 & 0.0833333 \\ 0.25 & 0.2 & 0.166667 & 0.142857 & 0.125 & 0.111111 & 0.1 & 0.0909091 & 0.0833333 & 0.0769231 \\ 0.2 & 0.166667 & 0.142857 & 0.125 & 0.111111 & 0.1 & 0.0909091 & 0.0833333 & 0.0769231 & 0.0714286 \\ 0.166667 & 0.142857 & 0.125 & 0.111111 & 0.1 & 0.0909091 & 0.0833333 & 0.0769231 & 0.0714286 & 0.0666667 \\ 0.142857 & 0.125 & 0.111111 & 0.1 & 0.0909091 & 0.0833333 & 0.0769231 & 0.0714286 & 0.0666667 & 0.0625 \\ 0.125 & 0.111111 & 0.1 & 0.0909091 & 0.0833333 & 0.0769231 & 0.0714286 & 0.0666667 & 0.0625 & 0.0588235 \\ 0.111111 & 0.1 & 0.0909091 & 0.0833333 & 0.0769231 & 0.0714286 & 0.0666667 & 0.0625 & 0.0588235 & 0.0555556 \\ 0.1 & 0.0909091 & 0.0833333 & 0.0769231 & 0.0714286 & 0.0666667 & 0.0625 & 0.0588235 & 0.0555556 & 0.0526316 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Perform Gauss Elimination with Partial Pivoting:

```
x = gausswpv[A, b];
Print["Solution vector x=", MatrixForm[x]]
```

Solution vector x= $\begin{pmatrix} -9.99801 \\ 989.828 \\ -23756.3 \\ 240207. \\ -1.2611\times10^{6} \\ 3.78334\times10^{6} \\ -6.726\times10^{6} \\ 7.00059\times10^{6} \\ -3.93786\times10^{6} \\ 923700. \end{pmatrix}$

Question 2

Input Hilbert Matrix and right hand side vector b for n = 5:

```
hilbert = Table[1. / (i + j - 1.),  {i, 1, 5}, { j, 1, 5}];
A = hilbert;
b = Table[1., {5}];
Print["Hilbert Matrix=", MatrixForm[A]]
Print["b=", MatrixForm[b]]
```

Hilbert Matrix= $\begin{pmatrix} 1. & 0.5 & 0.333333 & 0.25 & 0.2 \\ 0.5 & 0.333333 & 0.25 & 0.2 & 0.166667 \\ 0.333333 & 0.25 & 0.2 & 0.166667 & 0.142857 \\ 0.25 & 0.2 & 0.166667 & 0.142857 & 0.125 \\ 0.2 & 0.166667 & 0.142857 & 0.125 & 0.111111 \end{pmatrix}$

b= $\begin{pmatrix} 1. \\ 1. \\ 1. \\ 1. \\ 1. \end{pmatrix}$

Perform LU Decomposition Using Crout' s Method:

```
    L = Flatten[LUdec[A][[1]], 1];
U = Flatten[LUdec[A][[2]], 1];

Print["L=", MatrixForm[L]]
Print["U=", MatrixForm[U]]
```

L= $\begin{pmatrix} 1. & 0. & 0. & 0. & 0. \\ 0.5 & 0.0833333 & 0. & 0. & 0. \\ 0.333333 & 0.0833333 & 0.00555556 & 0. & 0. \\ 0.25 & 0.075 & 0.00833333 & 0.000357143 & 0. \\ 0.2 & 0.0666667 & 0.00952381 & 0.000714286 & 0.0000226757 \end{pmatrix}$

U= $\begin{pmatrix} 1. & 0.5 & 0.333333 & 0.25 & 0.2 \\ 0. & 1. & 1. & 0.9 & 0.8 \\ 0. & 0. & 1. & 1.5 & 1.71429 \\ 0. & 0. & 0. & 1. & 2. \\ 0. & 0. & 0. & 0. & 1. \end{pmatrix}$

Perform Forward and Backward Substitutions to obtain the solution vector x

```
y = Forwsubs[L, b];
x = Backsubs[U, y];
Print["Solution vector x=", MatrixForm[x]]
```

Solution vector x= $\begin{pmatrix} 5. \\ -120. \\ 630. \\ -1120. \\ 630. \end{pmatrix}$

Input Hilbert Matrix and right hand side vector b for n = 10 :

```
hilbert = Table[1. / (i + j - 1.),  {i, 1, 10}, { j, 1, 10}];
A = hilbert;
b = Table[1., {10}];
Print["Hilbert Matrix="]
MatrixForm[A]
Print["b=", MatrixForm[b]]
```

Hilbert Matrix=

$$\begin{pmatrix} 1. & 0.5 & 0.333333 & 0.25 & 0.2 & 0.166667 & 0.142857 & 0.125 & 0.111111 & 0.1 \\ 0.5 & 0.333333 & 0.25 & 0.2 & 0.166667 & 0.142857 & 0.125 & 0.111111 & 0.1 & 0.0909091 \\ 0.333333 & 0.25 & 0.2 & 0.166667 & 0.142857 & 0.125 & 0.111111 & 0.1 & 0.0909091 & 0.0833333 \\ 0.25 & 0.2 & 0.166667 & 0.142857 & 0.125 & 0.111111 & 0.1 & 0.0909091 & 0.0833333 & 0.0769231 \\ 0.2 & 0.166667 & 0.142857 & 0.125 & 0.111111 & 0.1 & 0.0909091 & 0.0833333 & 0.0769231 & 0.0714286 \\ 0.166667 & 0.142857 & 0.125 & 0.111111 & 0.1 & 0.0909091 & 0.0833333 & 0.0769231 & 0.0714286 & 0.0666667 \\ 0.142857 & 0.125 & 0.111111 & 0.1 & 0.0909091 & 0.0833333 & 0.0769231 & 0.0714286 & 0.0666667 & 0.0625 \\ 0.125 & 0.111111 & 0.1 & 0.0909091 & 0.0833333 & 0.0769231 & 0.0714286 & 0.0666667 & 0.0625 & 0.0588235 \\ 0.111111 & 0.1 & 0.0909091 & 0.0833333 & 0.0769231 & 0.0714286 & 0.0666667 & 0.0625 & 0.0588235 & 0.0555556 \\ 0.1 & 0.0909091 & 0.0833333 & 0.0769231 & 0.0714286 & 0.0666667 & 0.0625 & 0.0588235 & 0.0555556 & 0.0526316 \end{pmatrix}$$

$$b = \begin{pmatrix} 1. \\ 1. \\ 1. \\ 1. \\ 1. \\ 1. \\ 1. \\ 1. \\ 1. \\ 1. \end{pmatrix}$$

Perform LU Decomposition Using Crout' s Method :

```
L = Flatten[LUdec[A][[1]], 1];
U = Flatten[LUdec[A][[2]], 1];
```

```
Print["L=", MatrixForm[L]]
Print["U=", MatrixForm[U]]
```

L=

$$\begin{pmatrix}
1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0.5 & 0.0833333 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0.333333 & 0.0833333 & 0.00555556 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0.25 & 0.075 & 0.00833333 & 0.000357143 & 0. & 0. & 0. & 0. & 0. & 0. \\
0.2 & 0.0666667 & 0.00952381 & 0.000714286 & 0.0000226757 & 0. & 0. & 0. & 0. & 0. \\
0.166667 & 0.0595238 & 0.00992063 & 0.000992063 & 0.0000566893 & 1.43155\times 10^{-6} & 0. & 0. & 0. & 0. \\
0.142857 & 0.0535714 & 0.00992063 & 0.00119048 & 0.0000927644 & 4.29465\times 10^{-6} & 9.00975\times 10^{-8} & 0. & 0. & 0. \\
0.125 & 0.0486111 & 0.00972222 & 0.00132576 & 0.000126263 & 8.09376\times 10^{-6} & 3.15341\times 10^{-7} & 5.65997\times 10^{-9} & 0. & 0. \\
0.111111 & 0.0444444 & 0.00942761 & 0.00141414 & 0.0001554 & 0.0000123333 & 6.72728\times 10^{-7} & 2.26399\times 10^{-8} & 3.55135\times 10^{-10} & 0. \\
0.1 & 0.0409091 & 0.00909091 & 0.00146853 & 0.00017982 & 0.00001665 & 1.13523\times 10^{-6} & 5.39362\times 10^{-8} & 1.59811\times 10^{-9} & 2.22664\times 10^{-11}
\end{pmatrix}$$

$$U=\begin{pmatrix}
1. & 0.5 & 0.333333 & 0.25 & 0.2 & 0.166667 & 0.142857 & 0.125 & 0.111111 & 0.1 \\
0. & 1. & 1. & 0.9 & 0.8 & 0.714286 & 0.642857 & 0.583333 & 0.533333 & 0.490909 \\
0. & 0. & 1. & 1.5 & 1.71429 & 1.78571 & 1.78571 & 1.75 & 1.69697 & 1.63636 \\
0. & 0. & 0. & 1. & 2. & 2.77778 & 3.33333 & 3.71212 & 3.9596 & 4.11189 \\
0. & 0. & 0. & 0. & 1. & 2.5 & 4.09091 & 5.56818 & 6.85315 & 7.93007 \\
0. & 0. & 0. & 0. & 0. & 1. & 3. & 5.65385 & 8.61538 & 11.6308 \\
0. & 0. & 0. & 0. & 0. & 0. & 1. & 3.5 & 7.46667 & 12.6 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 4. & 9.52941 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 4.5 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1.
\end{pmatrix}$$

Perform Forward and Backward Substitutions to obtain the solution vector x

```
y = Forwsubs[L, b];
x = Backsubs[U, y];
Print["Solution vector x=", MatrixForm[x]]
```

Solution vector x= $\begin{pmatrix} -9.99852 \\ 989.869 \\ -23757.2 \\ 240214. \\ -1.26113 \times 10^{6} \\ 3.78343 \times 10^{6} \\ -6.72614 \times 10^{6} \\ 7.00072 \times 10^{6} \\ -3.93792 \times 10^{6} \\ 923714. \end{pmatrix}$

Question 3

Consider a system of "n" linear equation

$$[A]\{x\} = \{b\}$$

PART A)

Number of operations for LU decomposition :

(1) First count the # of operations (column operations) to obtain $[L]$ matrix where $[A]=[L][U]$

We know that for $j=1$:

$$l_{i1} = a_{i1} \quad (1,2,\ldots,n) \Rightarrow \text{No operations}$$

And

for $j = 2, 3, \ldots n$:

$$\left\{ l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \quad \text{for } i = j, j+1, \ldots n \right.$$

$\hookrightarrow$ We use this formula for each value of j $(2,3,\ldots n)$. So at a particular j value (or column operation in obtaining L) we repeat this formula. First look at the number of operations that will come from the summation (Σ).

Assume $j=3$ then

$$\sum_{k=1}^{j-1} l_{ik} u_{kj} = \sum_{k=1}^{2} l_{ik} u_{kj} = l_{i1} u_{13} + l_{i2} u_{23}$$

$\Rightarrow$ which gives 2 multiplications and 1 addition.

So in general this summation will give me
$(j-1)$ multiplications and $(j-2)$ additions.
for a fixed value of "i" and "j".

I have to perform this summation for each value of "i" at a fixed "j". Since my "i" index changes between $i = j, j+1, \dots, n$, I have to repeat the summation $[n-j+1]$ times. Then for a fixed "j" (column operation in obtaining L), I will have

$$(j-1) \times (n-j+1) \Rightarrow \text{multiplications} \qquad (1)$$
$$(j-2) \times (n-j+1) \Rightarrow \text{additions.} \qquad (2)$$

for Since $l_{ij} = a_{ij} - \left[\sum_{k=1}^{j-1} l_{ik} u_{kj} \right]$, For a fixed value of "$j$" and "$i$", I will have $\underline{\underline{1}}$ subtraction.

For a fixed value of "j", this will give me

$$(n-j+1) \Rightarrow \text{subtractions} \qquad (3)$$

since I have to perform this for each value of "i".

Expressions (1), (2) and (3) give the # of operations for a fixed value of "j". In order to find the total # of multiplications at this step, I have to evaluate Expression (1) with each value of j $(j = 2, 3, \dots n)$ and sum up the resulting terms:

Total # of multiplications $= (2-1)(n-1) + (3-1)(n-2)$

$+ (4-1)(n-3) + \dots + (j-1)(n-j+1)$ (4)

$$= \sum_{j=2}^{n} (j-1)(n-j+1) \quad (5)$$

In a similar way we can show that

Total # of additions $= \sum_{j=2}^{n} (j-2)(n-j+1)$ (6)

And

Total # of subtractions $= \sum_{j=2}^{n} (n-j+1)$ (7)

I can combine total # of additions & subtractions:

Total # of adds/subtr $= \sum_{j=2}^{n} (j-2)(n-j+1) + \sum_{j=2}^{n} (n-j+1)$

$$= \sum_{j=2}^{n} (j-1)(n-j+1) \quad (8)$$

Since total # of divisions = 0

Total # multip./divisions $= \sum_{j=2}^{n} (j-1)(n-j+1)$

$$= \sum_{j=2}^{n} \left[(n+2)j - j^2 - (n+1)\right]$$

$$= \sum_{j=2}^{n} (n+2)j - \sum_{j=2}^{n} j^2 - \sum_{j=2}^{n} (n+1)$$

$$= (n+2)\sum_{j=2}^{n} j - \sum_{j=2}^{n} j^2 - (n+1)\sum_{j=2}^{n} 1.$$

Remember $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ and $\sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1$

$\sum_{j=1}^{n} j^2 = \frac{(2n+1)(n+1)n}{6}$ and $\sum_{j=1}^{n} j^2 = \frac{(2n+1)(n+1)n}{6} - 1$

Then total # of mult/div:

$$= (n+2)\left[\frac{n(n+1)}{2} - 1\right] - \left[\frac{(2n+1)(n+1)n}{6} - 1\right] - (n+1)(n-1)$$

$$\boxed{\text{Total \# of mult/div} = \frac{n^3}{6} - \frac{n}{6}} \quad (9)$$

Since Expressions (5) and (8) are the same

$$\boxed{\text{Total \# of adds/subt} = \frac{n^3}{6} - \frac{n}{6}} \quad (10)$$

(2) Number of Operations to obtain [U] matrix:

We have seen that for $i=1$:

$$u_{1j} = \frac{a_{1j}}{l_{11}} \quad (j=2, 3, \ldots n) \qquad (11)$$

This step will give us $\Rightarrow$ $(n-1)$ divisions.

For $i = 2, 3, \ldots (n-1)$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}}{l_{ii}} \quad \text{for } j = i+1, i+2, \ldots n \qquad (12)$$

In a similar way to our analysis for finding the # of operations in obtaining [L] matrix we can show that

Summation in (12) requires $(i-1)\times[n-i]$ multiplications & $(i-2)\times[n-i]$ additions. $\left.\right\}$ For fixed "i" & $j=i+1,\dots,n$.

For the subtractions & divisions $\Rightarrow$ $u_{ij} = \dfrac{a_{ij} - \text{Summation}}{l_{ii}}$ (13)

So this step will require $[n-i]$ subtractions and $[n-i]$ divisions for fixed "i" & $j=i+1, i+2, \dots n$.

Then the total # of operation will be obtained by:

$$\text{Total \# of mult} = \sum_{i=2}^{n-1} (i-1)(n-i) \qquad (14)$$

$$\text{Total \# of div} = (n-1) + \sum_{i=2}^{n-1} (n-i) \qquad (15)$$

$$\text{Total \# of additions} = \sum_{i=2}^{n-1} (i-2)(n-i) \qquad (16)$$

$$\text{Total \# of subtractions} = \sum_{i=2}^{n-1} (n-i) \qquad (17)$$

or

$$\text{Total \# of mult/div} = \sum_{i=2}^{n-1} (i-1)(n-i) + \sum_{i=2}^{n-1} (n-i) + (n-1)$$

$$= (n-1) + \sum_{i=2}^{n-1} i(n-i) \qquad (18)$$

$$= (n-1) + n\sum_{i=2}^{n-1} i - \sum_{i=2}^{n-1} i^2$$

$$= (n-1) + n\left[\frac{n(n-1)}{2} - 1\right] - \left[\frac{(2n-1)(n-1)n}{6} - 1\right]$$

Note that here we used

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} \quad \text{and} \quad \sum_{i=1}^{n-1} i^2 = \frac{(2n-1)(n-1)n}{6}$$

Total # of mult/divisions

$$= n-1 + \frac{n^3}{2} - \frac{n^2}{2} - n - \frac{n^3}{3} + \frac{n^2}{2} - \frac{n}{6} + 1$$

$$\boxed{\text{Total \# of mult/divisions} = \frac{n^3}{6} - \frac{n}{6}} \quad (19)$$

Using (16) and (17), total # of adds/subts:

$$= \sum_{i=2}^{n-1} (i-2)(n-i) + \sum_{i=2}^{n-1} (n-i) = \sum_{i=2}^{n-1} (i-1)(n-i)$$

$$= \sum_{i=2}^{n-1} [(n+1)i - i^2 - n] = (n+1)\sum_{i=2}^{n-1} i - \sum_{i=2}^{n-1} i^2 - n\sum_{i=2}^{n-1} 1$$

$$= n+1\left[\frac{n(n-1)}{2} - 1\right] - \left[\frac{(2n-1)(n-1)n}{6} - 1\right] - n(n-2)$$

$$\boxed{\text{Total \# of adds/subt} = \frac{n^3}{6} - \frac{n^2}{2} + n/3} \quad (20)$$

Now to find the overall total # of operations, we have to add the results obtaining [U] and [L]. Using expressions (9), (10), (19) and (20):

Then for LU decomposition using Crout's method ($A = LU$)

Total # of mult/div $= \frac{n^3}{6} - \frac{n}{6} + \frac{n^3}{6} - \frac{n}{6}$

$$= \boxed{\frac{n^3}{3} - \frac{n}{3}} \quad (21)$$

Total # of adds/subt $= \frac{n^3}{6} - \frac{n}{6} + \frac{n^3}{6} - \frac{n^2}{2} + \frac{n}{3}$

$$= \boxed{\frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}} \quad (22)$$

PART B.)

For $Ax = b$, using LU decomposition:

$A = LU$ so

$L\underbrace{Ux}_{y} = b \Rightarrow Ly = b$ (Forward Subst.)

$Ux = y$ (Backword subst.)

Forword substitution.

$$y_1 = \frac{b_1}{l_{11}} \rightarrow 1 \text{ divisions} \quad (23)$$

$$y_i = \frac{b_i - \sum_{k=1}^{i-1} l_{ik} y_k}{l_{ii}} \quad i = 2, 3, \ldots, n. \quad (24)$$

From the summation we'll have $(i-1)$ multiplications and $(i-2)$ additions for a fixed value of "i".

Expression (24) will also give 1 subtraction and 1 division for a fixed value of "i".

For $i = 2, 3, \cdots n$

Total # of multip. $= \sum_{i=2}^{n} (i-1)$

come from (23)

Total # of div $= (n-1) + 1 = n$

Total # of subtractions $= (n-1)$

Total # of additions $= \sum_{i=2}^{n} (i-2)$

Then for forward substitution

Total # of mult./div $= n + \sum_{i=2}^{n} (i-1)$

$$= n + \sum_{i=2}^{n} i - \sum_{i=2}^{n} 1 = n + \left[\frac{n(n+1)}{2} - 1 \right] - (n-1)$$

$$\boxed{= \frac{n^2}{2} + \frac{n}{2}} \quad (24)$$

Total # of adds/subt $= (n-1) + \sum_{i=2}^{n} (i-2)$

$$= (n-1) + \sum_{i=2}^{n} i - 2 \sum_{i=2}^{n} 1.$$

$$= (n-1) + \left[\frac{n(n+1)}{2} - 1 \right] - 2(n-1)$$

$$\boxed{= \frac{n^2}{2} - \frac{n}{2}} \quad (25)$$

Backword Substitution :

We know that $u_{ii} = 1$ for $(i = 1, 2, \dots n)$

$x_n = y_n \rightarrow$ no operations

$$x_i = y_i - \sum_{k=i+1}^{n} u_{ik} x_k \quad i = n-1, n-2, \dots 1$$

# of multiplications $= \sum_{i=1}^{n-1} (n-i)$

$$= n\sum_{i=1}^{n-1} 1 - \sum_{i=1}^{n-1} i = n(n-1) + \frac{n(n-1)}{2}$$

$$= \frac{n^2}{2} - \frac{n}{2}$$

# of divisions $= 0$

# of multiplications / divisions $= \boxed{\frac{n^2}{2} - \frac{n}{2}}$

# of adds $= \sum_{i=1}^{n-1} (n-i-1)$

# of subts $= (n-1)$

# of adds/subts $= n-1 + \sum_{i=1}^{n-1} (n-i-1)$

$$= n-1 + (n-1)\sum_{i=1}^{n-1} 1 - \sum_{i=1}^{n-1} i$$

$$= n-1 + (n-1)^2 - \frac{n(n-1)}{2} = n-1 + n^2 - 2n + 1 - \frac{n^2}{2} + \frac{n}{2}$$

$$= \boxed{\frac{n^2}{2} - \frac{n}{2}}$$

So total # of operations = $\underbrace{(\text{total \# of op.})}_{\text{obtained in part a}}{}_{LU}$

$+ (\text{total \# of op.})_{\text{Forward S.}} + (\text{total \# of op.})_{\text{Back. S.}}$

Total # of mult./div $= \frac{n^3}{3} - \frac{n}{3} + \frac{n^2}{2} + \frac{n}{2} + \frac{n^2}{2} - \frac{n}{2}$

$= \boxed{\frac{n^3}{3} + n^2 - \frac{n}{3}}$ (Same as the one obtained for Gauss Elimination)

Total # of adds/subts $= \frac{n^3}{3} - \cancel{\frac{n^2}{2}} + \frac{n}{6} + \cancel{\frac{n^2}{2}} - \frac{n}{2} + \frac{n^2}{2} - \frac{n}{2}$

$= \boxed{\frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}}$ (Same as the one obtained for Gauss Elimination)