Numerical Differentiation - List of Some Finite Difference Approximations

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Table 3.1 Difference approximations using more than three points

Derivative	Finite-difference representation
$\left.\dfrac{\partial^3 u}{\partial x^3}\right)_{i,j} =$	$\dfrac{u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}}{2h^3} + O(h^2)$
$\left.\dfrac{\partial^4 u}{\partial x^4}\right)_{i,j} =$	$\dfrac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4} + O(h^2)$
$\left.\dfrac{\partial^2 u}{\partial x^2}\right)_{i,j} =$	$\dfrac{-u_{i+3,j} + 4u_{i+2,j} - 5u_{i+1,j} + 2u_{i,j}}{h^2} + O(h^2)$
$\left.\dfrac{\partial^3 u}{\partial x^3}\right)_{i,j} =$	$\dfrac{-3u_{i+4,j} + 14u_{i+3,j} - 24u_{i+2,j} + 18u_{i+1,j} - 5u_{i,j}}{2h^3} + O(h^2)$
$\left.\dfrac{\partial^2 u}{\partial x^2}\right)_{i,j} =$	$\dfrac{2u_{i,j} - 5u_{i-1,j} + 4u_{i-2,j} - u_{i-3,j}}{h^2} + O(h^2)$
$\left.\dfrac{\partial^3 u}{\partial x^3}\right)_{i,j} =$	$\dfrac{5u_{i,j} - 18u_{i-1,j} + 24u_{i-2,j} - 14u_{i-3,j} + 3u_{i-4,j}}{2h^3} + O(h^2)$
$\left.\dfrac{\partial u}{\partial x}\right)_{i,j} =$	$\dfrac{-u_{i+2,j} + 8u_{i+1,j} - 8u_{i-1,j} + u_{i-2,j}}{12h} + O(h^4)$
$\left.\dfrac{\partial^2 u}{\partial x^2}\right)_{i,j} =$	$\dfrac{-u_{i+2,j} + 16u_{i+1,j} - 30u_{i,j} + 16u_{i-1,j} - u_{i-2,j}}{12h^2} + O(h^4)$

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Table 3.2 Difference approximations for mixed partial derivatives

Derivative	Finite-difference representation
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{\Delta x}\left(\dfrac{u_{i+1,j}-u_{i+1,j-1}}{\Delta y}-\dfrac{u_{i,j}-u_{i,j-1}}{\Delta y}\right)+O(\Delta x,\Delta y)$
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{\Delta x}\left(\dfrac{u_{i,j+1}-u_{i,j}}{\Delta y}-\dfrac{u_{i-1,j+1}-u_{i-1,j}}{\Delta y}\right)+O(\Delta x,\Delta y)$
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{\Delta x}\left(\dfrac{u_{i,j}-u_{i,j-1}}{\Delta y}-\dfrac{u_{i-1,j}-u_{i-1,j-1}}{\Delta y}\right)+O(\Delta x,\Delta y)$
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{\Delta x}\left(\dfrac{u_{i+1,j+1}-u_{i+1,j}}{\Delta y}-\dfrac{u_{i,j+1}-u_{i,j}}{\Delta y}\right)+O(\Delta x,\Delta y)$
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{\Delta x}\left(\dfrac{u_{i+1,j+1}-u_{i+1,j-1}}{2\,\Delta y}-\dfrac{u_{i,j+1}-u_{i,j-1}}{2\,\Delta y}\right)+O[\Delta x,(\Delta y)^2]$
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{\Delta x}\left(\dfrac{u_{i,j+1}-u_{i,j-1}}{2\,\Delta y}-\dfrac{u_{i-1,j+1}-u_{i-1,j-1}}{2\,\Delta y}\right)+O[\Delta x,(\Delta y)^2]$
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{2\,\Delta x}\left(\dfrac{u_{i+1,j+1}-u_{i+1,j-1}}{2\,\Delta y}-\dfrac{u_{i-1,j+1}-u_{i-1,j-1}}{2\,\Delta y}\right)+O[(\Delta x)^2,(\Delta y)^2]$
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{2\,\Delta x}\left(\dfrac{u_{i+1,j+1}-u_{i+1,j}}{\Delta y}-\dfrac{u_{i-1,j+1}-u_{i-1,j}}{\Delta y}\right)+O[(\Delta x)^2,\Delta y]$
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{2\,\Delta x}\left(\dfrac{u_{i+1,j}-u_{i+1,j-1}}{\Delta y}-\dfrac{u_{i-1,j}-u_{i-1,j-1}}{\Delta y}\right)+O[(\Delta x)^2,\Delta y]$

Table 3.3 Some useful results from polynomial fitting

Polynomial degree	Wall value of function or derivative
1	$\left.\frac{\partial T}{\partial y}\right)_{i,j} = \frac{T_{i,j+1} - T_{i,j}}{h} + O(h)$
1	$T_{i,j} = T_{i,j+1} - h\left.\frac{\partial T}{\partial y}\right)_{i,j} + O(h^2)$
2	$\left.\frac{\partial T}{\partial y}\right)_{i,j} = \frac{1}{2h}(-3T_{i,j} + 4T_{i,j+1} - T_{i,j+2}) + O(h^2)$
2	$T_{i,j} = \frac{1}{3}\left[4T_{i,j+1} - T_{i,j+2} - 2h\left.\frac{\partial T}{\partial y}\right)_{i,j}\right] + O(h^3)$
3	$\left.\frac{\partial T}{\partial y}\right)_{i,j} = \frac{1}{6h}(-11T_{i,j} + 18T_{i,j+1} - 9T_{i,j+2} + 2T_{i,j+3}) + O(h^3)$
3	$T_{i,j} = \frac{1}{11}\left[18T_{i,j+1} - 9T_{i,j+2} + 2T_{i,j+3} - 6h\left.\frac{\partial T}{\partial y}\right)_{i,j}\right] + O(h^4)$
4	$\left.\frac{\partial T}{\partial y}\right)_{i,j} = \frac{1}{12h}(-25T_{i,j} + 48T_{i,j+1} - 36T_{i,j+2} + 16T_{i,j+3} - 3T_{i,j+4}) + O(h^4)$
4	$T_{i,j} = \frac{1}{25}\left[48T_{i,j+1} - 36T_{i,j+2} + 16T_{i,j+3} - 3T_{i,j+4} - 12h\left.\frac{\partial T}{\partial y}\right)_{i,j}\right] + O(h^5)$