Solution of Linear Set of Equations – 02

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Outline

Previously, we discussed Cramer's rule, computational work, and triangular systems. In this lecture, we will discuss methods to convert a linear problem into triangular form.

- Gaussian elimination for a 3x3
- Generalize Gaussian elimination (for programming)
- Computational work of Gaussian elimination

Gauss Elimination Method

Gauss elimination method works by

(i) performing simple arithmetic operations on the rows to result in an upper triangular matrix, and

(ii) using the method of backward-substitution to solve the upper triangular system

As an example, consider the following equations:

$$E_1 = 4x_1+2x_2+4x_3=20$$

$$E_2 = 2x_1+2x_2+3x_3=15$$

$$E_3 = -8x_1-2x_2+16x_3=36$$

Writing the equations in the matrix form $\boldsymbol{Ax=b}$ yields

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 2 & 3 \\ -8 & -2 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \\ 36 \end{bmatrix}$$

Gauss Elimination Example on a *3x3 Ax=b* system (1)

For conciseness, we combine the matrix A and b into a single matrix called the ***Augmented Matrix***. For the system of equations under consideration, the ***Augmented Matrix*** is:

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 2 & 3 \\ -8 & -2 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \\ 36 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 4 & 2 & 4 & 20 \\ 2 & 2 & 3 & 15 \\ -8 & -2 & 16 & 36 \end{array}\right]$$

***Conversion to Upper triangular form*:**

Operations that keep the solution invariant:

1. Multiplying both sides of the equation with a constant λ_i (Replace E_i with $\lambda_i E_i$)
2. Multiplying an Equation E_j with λ_j , then replacing E_i with $E_i + \lambda_j E_j$
3. Re-numbering (re-ordering) the equations: $E_i \longleftrightarrow E_j$

Gauss Elimination Example on a *3x3* *Ax=b* system (2)

Elimination Step 1:

$$\left[\begin{array}{ccc|c} 4 & 2 & 4 & 20 \\ 2 & 2 & 3 & 15 \\ -8 & -2 & 16 & 36 \end{array}\right] \begin{array}{l} \\ E_2 - \left(\tfrac{2}{4}\right)E_1 \\ E_3 - \left(\tfrac{-8}{4}\right)E_1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 4 & 2 & 4 & 20 \\ 0 & 1 & 1 & 5 \\ 0 & 2 & 24 & 76 \end{array}\right]$$

Elimination Step 2:

$$\left[\begin{array}{ccc|c} 4 & 2 & 4 & 20 \\ 0 & 1 & 1 & 5 \\ 0 & 2 & 24 & 76 \end{array}\right] \begin{array}{l} \\ \\ E_3 - \left(\tfrac{2}{1}\right)E_2 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 4 & 2 & 4 & 20 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 22 & 66 \end{array}\right]$$

Gauss Elimination Example on a *3x3* *Ax=b* system (3)

This ***augmented matrix*** is now in upper triangular matrix form and the right hand side has been adjusted.

$$\begin{bmatrix} 4 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 22 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 5 \\ 66 \end{bmatrix}$$

This system can now be solved by the method of ***backward-substitution***.

$$x_3 = 66/22 = 3$$

$$x_2 = (5 - 1x_3)/1 = (5 - 3)/1 = 2$$

$$x_1 = (20 - 2x_2 - 4x_3)/4 = (20 - 4 - 12)/4 = 1$$

Thus, the solution to the given set of equations is:

$$\boldsymbol{x_1 = 1\ ;\ x_2 = 2;\ x_3 = 3}$$

Gauss Elimination for a $n \times n$ system

Consider the system of equations:

$$
\begin{aligned}
a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + \ldots. + a_{1n}^{(1)}x_n &= b_1^{(1)} \\
a_{21}^{(1)}x_1 + a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \ldots. + a_{2n}^{(1)}x_n &= b_2^{(1)} \\
a_{31}^{(1)}x_1 + a_{32}^{(1)}x_2 + a_{33}^{(1)}x_3 + \ldots. + a_{3n}^{(1)}x_n &= b_3^{(1)} \\
\vdots \qquad \vdots \qquad\qquad\qquad\qquad & \quad \vdots \\
a_{n1}^{(1)}x_1 + a_{n2}^{(1)}x_2 + a_{n3}^{(1)}x_3 + \ldots. + a_{nn}^{(1)}x_n &= b_n^{(1)}
\end{aligned}
$$

Original System

$$
\begin{aligned}
a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + \ldots. + a_{1n}^{(1)}x_n &= b_1^{(1)} \\
0 \quad + a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 + \ldots. + a_{2n}^{(2)}x_n &= b_2^{(2)} \\
0 \quad + a_{32}^{(2)}x_2 + a_{33}^{(2)}x_3 + \ldots. + a_{3n}^{(2)}x_n &= b_3^{(2)} \\
\vdots \qquad \vdots \qquad\qquad\qquad\qquad & \quad \vdots \\
0 \quad + a_{n2}^{(2)}x_2 + a_{n3}^{(2)}x_3 + \ldots. + a_{nn}^{(2)}x_n &= b_n^{(2)}
\end{aligned}
$$

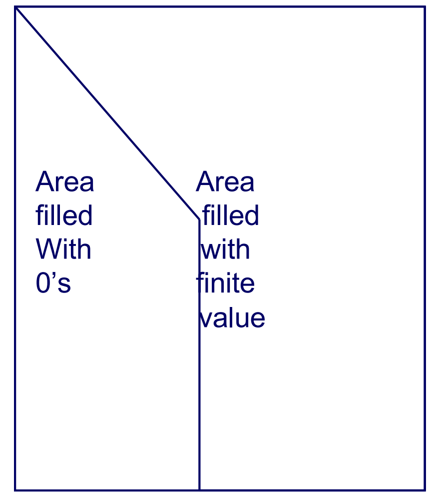
System after 1 elimination step

The procedure for simplifying the system of the equations is to convert the coefficients below the diagonal to zero one column at a time.

The k^{th} elimination step

After the $(k\text{-}1)^{th}$ elimination step, we have :

$$
\begin{array}{l}
a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{1n}^{(1)}x_n = b_1^{(1)} \\
0 \quad + a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{2n}^{(2)}x_n = b_2^{(2)} \\
0 \quad + \quad 0 \quad + a_{33}^{(3)}x_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{3n}^{(3)}x_n = b_3^{(3)} \\
\ldots \qquad\qquad \ldots \quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \ldots \quad = \ldots \\
0 \quad + \quad 0 \quad + \quad 0 \quad + \ldots\ldots 0 \ldots\ldots + a_{kk}^{(k)}x_k + \ldots\ldots\ldots\ldots + a_{kn}^{(k)}x_n = b_k^{(k)} \\
\ldots \qquad\qquad \ldots \quad + \ldots\ldots 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \ldots \quad = \ldots \\
0 \quad + \quad 0 \quad + \quad 0 \quad + \ldots\ldots 0 \ldots\ldots + a_{nk}^{(k)}x_k + \ldots\ldots\ldots\ldots + a_{nn}^{(k)}x_n = b_n^{(k)}
\end{array}
$$



We have to eliminate $a_{ik}^{(k)}$ $\forall\ i = k+1, n$. We define a multiplier, m_{ik} given by

$$m_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} \forall\ i = k+1, \ldots, n \longrightarrow$$ ***This operation involves (n-k) divisions.***

Then multiply *row k* by m_{ik} and subtract from ***row i*** for ***i = k+1 to n***. This yields the new coefficients

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - m_{ik}a_{kj}^{(k)} \quad \forall\ i, j = k+1, n \longrightarrow \left\{ \begin{array}{l} (n\text{-}k)^2 \textit{ multiplications} \\ (n\text{-}k)^2 \textit{ subtractions} \end{array} \right.$$

$$b_i^{(k+1)} = b_i^{(k)} - m_{ik}b_k^{(k)} \quad \forall\ i = k+1, n \longrightarrow \left\{ \begin{array}{l} (n\text{-}k) \textit{ multiplications} \\ (n\text{-}k) \textit{ subtractions} \end{array} \right.$$

After last elimination step

After ***n-1*** elimination steps, we have an upper triangular system

$$
\begin{array}{l}
a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{1n}^{(1)}x_n = b_1^{(1)} \\
0 \quad + a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{2n}^{(2)}x_n = b_2^{(2)} \\
0 \quad + \quad 0 \quad + a_{33}^{(3)}x_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{3n}^{(3)}x_n = b_3^{(3)} \\
\ldots \\
0 \quad + \quad 0 \quad + \quad 0 \quad + \ldots\ldots\ldots\ldots\ldots + a_{kk}^{(k)}x_k + \ldots\ldots\ldots\ldots + a_{kn}^{(k)}x_n = b_k^{(k)} \\
\ldots \\
0 \quad + \quad 0 \quad + \quad 0 \quad + \ldots\ldots\ldots\ldots\ldots + \quad 0 \quad + \ldots\ldots\ldots\ldots + a_{nn}^{(n)}x_n = b_n^{(n)}
\end{array}
$$

By backward substitution, we have

$$x_n = \frac{b_n^{(n)}}{a_{nn}^{(n)}}$$

$$x_i = \frac{\left[b_i^{(i)} - \sum_{k=i+1}^{n} a_{ik}^{(i)} x_k \right]}{a_{ii}^{(i)}} \quad \forall\, i = n-1, n-2, \ldots, 1$$

What's the operation count?

First, consider the conversion of *A* to upper triangular form.

On the k^{th} elimination step it takes:

$n-k$	divisions	$\longrightarrow$	forming the m_{ik}
$(n-k)^2$	multiplications	$\longrightarrow$	computing a^{k+1}
$(n-k)^2$	subtractions	$\longrightarrow$	computing a^{k+1}

We have to do $n-1$ elimination steps to get A in upper triangular form. This yields

$$\sum_{k=1}^{n-1}(n-k) = \sum_{k=1}^{n-1} n - \sum_{k=1}^{n-1} k = n(n-1) - \frac{n(n-1)}{2}$$

$$= \frac{1}{2}\left(n^2 - n\right) \quad \text{divisions}$$

Operation Count (2)

$$\sum_{k=1}^{n-1}(n-k)^2 = \sum_{k=1}^{n-1}\left(n^2 - 2nk + k^2\right)$$

$$= \sum_{k=1}^{n-1} n^2 - \sum_{k=1}^{n-1} 2nk + \sum_{k=1}^{n-1} k^2$$

$$= n^2(n-1) - \frac{2n^2(n-1)}{2} + \frac{(2n-1)(n-1)n}{6}$$

$$= \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}$$ multiplications & subtractions

Now consider the updating of the RHS (*b*) vector:

On the k^{th} elimination step it takes:

0	divisions
(n-k)	multiplications
(n-k)	subtractions

Operation Count (3)

We have to do ***n-1*** elimination steps:

$$\sum_{k=1}^{n-1}(n-k)=\sum_{k=1}^{n-1}n-\sum_{k=1}^{n-1}k=n(n-1)-\frac{n(n-1)}{2}$$

$$=\frac{1}{2}\left(n^2-n\right)$$ Multiplications & subtractions

Finally consider the backward substitution (from our last lecture):

$$\frac{1}{2}\left(n^2-n\right)$$ Multiplications & additions

n divisions

Gauss Elimination Work

	Additions and subtractions	*Multiplications*	*Divisions*
Conversion to upper triangular form	$\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}$	$\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}$	$\frac{n^2}{2}-\frac{n}{2}$
Adjusting the right hand side	$\frac{n^2}{2}-\frac{n}{2}$	$\frac{n^2}{2}-\frac{n}{2}$	0
Backward Substitution	$\frac{n^2}{2}-\frac{n}{2}$	$\frac{n^2}{2}-\frac{n}{2}$	n
Total	$\frac{n^3}{3}+\frac{n^2}{2}-\frac{5n}{6}$	$\frac{n^3}{3}+\frac{n^2}{2}-\frac{5n}{6}$	$\frac{n^2}{2}+\frac{n}{2}$
Behavior for large n	$O\left(\frac{n^3}{3}\right)$	$O\left(\frac{n^3}{3}\right)$	$O\left(\frac{n^2}{2}\right)$

Summary

- Worked through Gaussian Elimination for a system of 3 equations
- Generalized the method to a system of n unknowns
- Counted the operations required for
  - adjusting the coefficient matrix
  - adjusting the right hand side
  - backward substitution