Solution of Linear Set of Equations – 01

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Outline

- We will start learning the techniques to solve linear system of equations

- The methods used for the solution of linear system of equations can be broadly classified into two types:

- Direct Methods:
  - Direct methods typically convert the coefficient matrix into triangular form and obtain the solution by backward or forward substitution.

- Iterative Methods
  - Iterative methods start with an initial guess and implement an algorithm to converge to the solution vector. Examples include Jacobi and Gauss-Seidel iteration with over/under-relaxation

System of Equations in Matrix Form

Any linear system of equations can be written in matrix form.

Example:

$x_1+x_2+x_3 = 3$

$2x_1-x_2+x_3=2$ Set of 3 equations to be solved

$-x_1+x_2+5x_3=5$

These equations can be written in matrix form as $Ax = b$ where the matrix A contains the coefficients of the unknowns, x is the unknown column vector and b is the column vector of known constants. The equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 1 & 5 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

Direct Methods for Solving $Ax=b$

- By a direct method for solving a linear system of equations, we mean a method that after a finite number of steps (arithmetic operations, row/column interchanges) gives the exact solution in the absence of rounding errors.

- For systems in which A is full (i.e., most of the elements are non-zero), direct elimination methods are almost always the most efficient.

Direct Methods To be Covered

The methods and topics that we will discuss during the next few lectures are:

- Cramer’s Rule
- Upper Triangular Systems
- Gauss Elimination method
- Gauss-Jordan method
- Pivoting
- LU Decomposition
- Tri-diagonal equations

Cramer's Rule

If the system of equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$Ax = b \longrightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

We denote the determinant of the matrix A as $|A|$ and if it is not equal to zero, the solution to the system of equations is given by

$$x_1 = \frac{|A_1|}{|A|} \quad and \quad x_2 = \frac{|A_2|}{|A|}$$

where $|A_1|$ is obtained by replacing the first column in A with the RHS vector b and $|A_2|$ is obtained by replacing the second column in A with the vector b.

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}; |A_1| = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}; |A_2| = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}; x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}; x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Cramer's Rule Example

Consider the coupled linear equations, $\boldsymbol{Ax=b}$

$x_1 + x_2 = 3$ *or* $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$x_1 + 2x_2 = 5$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = (a_{11}a_{22} - a_{12}a_{21}) = 2 - 1 = 1$$

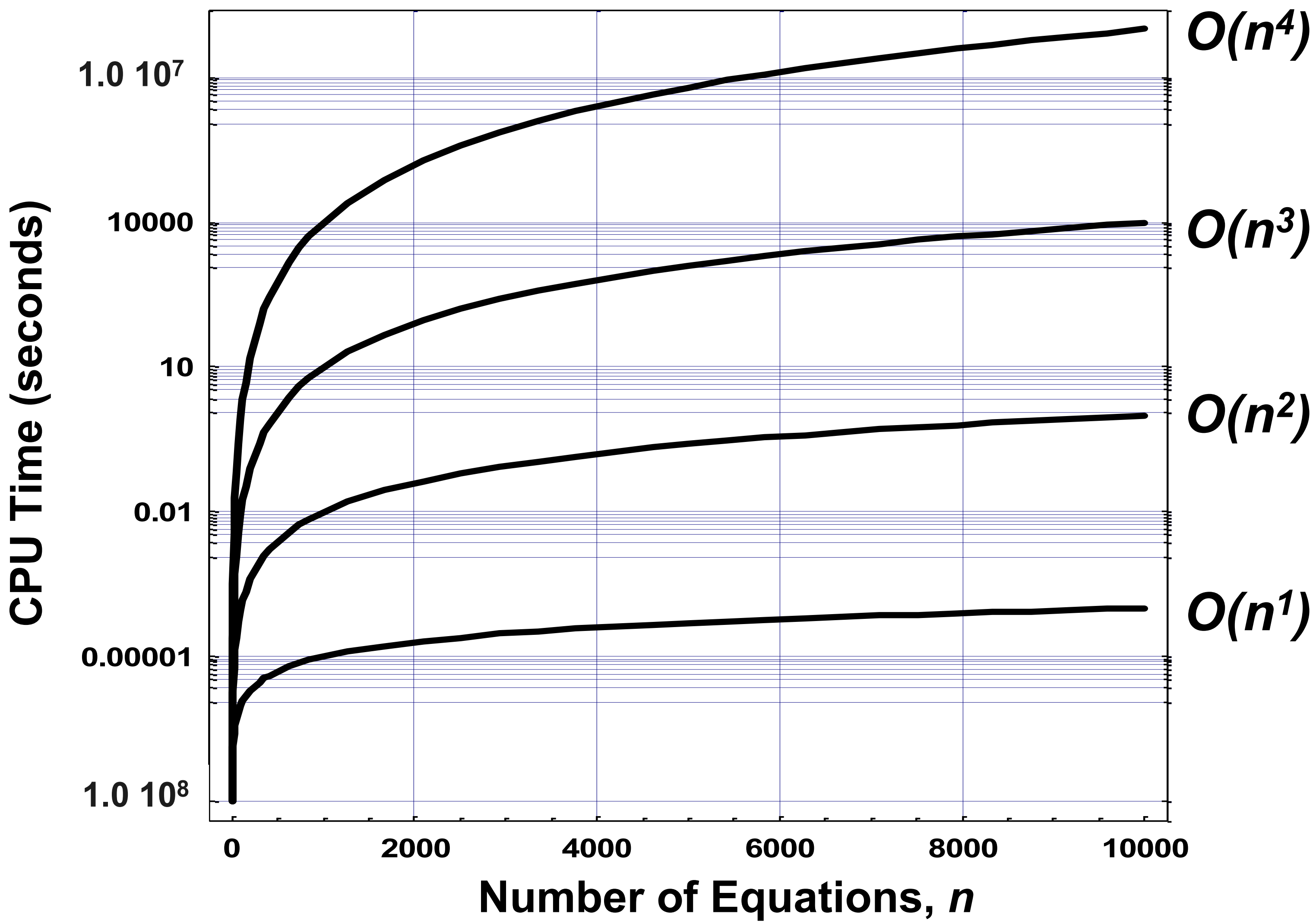
$$|A_1| = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = (b_1 a_{22} - a_{12} b_2) = 6 - 5 = 1$$

$$|A_2| = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = (a_{11} b_2 - b_1 a_{21}) = 5 - 3 = 2$$

This yields $x_1 = \dfrac{|A_1|}{|A|} = \dfrac{1}{1} = 1$ $x_2 = \dfrac{|A_2|}{|A|} = \dfrac{2}{1} = 2$

Impact of Algorithm

Assumes CPU sustains 100 Mflops:



Impact of Algorithm (2)

Assume 100 Mflops and *n*=10,000

Algorithm Behavior	Estimated CPU Time (seconds)	
$O(n^4)$	10^8	~3 years
$O(n^3)$	10000	~3 hours
$O(n^2)$	1	
$O(n^1)$	0.0001	

Note that for Cramer's Algorithm, the operation count behaves as $O(n^4)$

Formal Solution of *Ax=b*

The formal solution of $Ax = b$ is given by $x = A^{-1}b$

For example;

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

In this case, the solution is given by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/8 & 1/4 & -1/8 \\ 11/16 & -3/8 & -1/16 \\ -1/16 & 1/8 & 3/16 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

However, when solving a linear system of equations, most numerical methods do not directly form A^{-1}. Instead, they convert the system to Upper triangular from and then back-substitute to obtain the solution.

Upper Triangular Form $Ux=b$

Suppose we could get a system in Upper Triangular form

$3x_1 + 2x_2 + x_3 = 7 \quad \rightarrow (1)$

$2x_2 - 2x_3 = -2 \quad \rightarrow (2)$

$4x_3 = 8 \quad \rightarrow (3)$

These equations can be represented as

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 8 \end{bmatrix}$$

Solve by backward-substitution

$$x_3 = 8/4 = 2$$

$$x_2 = (-2 - (-2)x_3)/2 = (-2 + 4)/2 = 1$$

$$x_1 = (7 - 2x_2 - 1x_3)/3 = (7 - 2 - 2)/3 = 1$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

General Solution for an Upper Triangular Matrix

For an nxn Upper Triangular Matrix

$$\begin{bmatrix} u_{11} & u_{12} & \dots\dots\dots & u_{1n} \\ 0 & u_{22} & \dots\dots\dots & u_{2n} \\ . & \dots & \dots\dots\dots & \dots \\ 0 & \dots & u_{ii}\dots\dots & u_{in} \\ . & \dots & \dots\dots\dots & . \\ . & \dots & \dots\dots\dots & . \\ 0 & 0 & 0\dots\dots\dots & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_i \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_i \\ . \\ . \\ b_n \end{bmatrix}$$

The solution to this problem can be obtained in two simple steps:

#1:

$$x_n = {b_n}/{u_{nn}}$$

#2:

$$x_i = \frac{\left[b_i - \sum_{k=i+1}^{n} u_{ik} x_k \right]}{u_{ii}} \forall i = n-1, n-2, \dots, 1$$

Operation Count

The efficiency of a method to find the solution of a system of linear equations is determined by the number of operations required to arrive at the solution.

Row No.	Addition, Subtraction	Multiplication	Division
n	0	0	1
$n-1$	1	1	1
$n-2$	2	2	1
. .	. .	. .	. .
1	$n-1$	$n-1$	1
Total	$n(n-1)/2$	$n(n-1)/2$	n

For large n, the computational work associated with backward substitution behaves as **O(n^2)**

Useful Relations for Operation Count

For obtaining the totals on the previous slide, I used the first relation below. We will find the second relation useful in examining Gaussian Elimination

$$\sum_{j=1}^{m} j = \frac{m(m+1)}{2}$$

$$\sum_{j=1}^{m} j^2 = \frac{m(m+1)(2m+1)}{6}$$

For example, to get the total multiplications in backward substitution, we have

$$\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2} = \frac{1}{2}n^2 - \frac{1}{2}n$$

Summary

- Defined what we mean by direct methods for solving $Ax=b$
- We have seen Cramer's rule, a direct method that uses determinants for obtaining the solution vector x. The operation count behaves as $O(n^4)$ hence this is a very inefficient method for large values of n.
- We graphically examined the behavior of methods with different operation counts for large values of n to understand the importance of minimizing the number of floating point operations.
- Finally, we discussed solving triangular systems by back-substitution. This is a key component to Gaussian Elimination. We found that the operation count for solving a triangular system of equations by back substitution is proportional to n^2