Root Finding – 02

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Outline

- Error Definitions
- Rate of Convergence
- Rate of convergence for different methods
- Quadratic Rate of convergence for Newton's Method
- Work on an example problem

Error Definitions

- **Absolute Error:**
  - If a number "x_n" is an approximation to a number "p", then the absolute error is defined as:

$$\varepsilon_n = |x_n - p|$$

- **Relative Error**

$$(\varepsilon_n)_r = \frac{|x_n - p|}{|p|}$$

Rate of Convergence

- Let $x_0, x_1, x_2, \ldots x_n, \ldots$ is an infinite sequence that converges to "p" with $x_n \neq p$ for all n
- Absolute error for the n^{th} term: $\varepsilon_n = |x_n - p|$

  Absolute error for the $(n+1)^{th}$ term: $\varepsilon_{n+1} = |x_{n+1} - p|$

- If positive constants λ and α exists with

$$\lim_{n\to\infty} \frac{|x_{n+1} - p|}{|x_n - p|^\alpha} = \lambda \quad \text{or} \quad \varepsilon_{n+1} = \lambda\, \varepsilon_n^\alpha$$

- The sequence $\{x_n\}_{n=0}^{\infty}$ converges to "p" with order "α" and asymptotic error constant λ
- Here "α" is also called as **rate of convergence**.

Rate of Convergence (2)

- If $\alpha = 1$, $\{x_n\}_{n=0}^{\infty}$ is linearly convergent (**linear rate of convergence**)
- If $\alpha = 2$, $\{x_n\}_{n=0}^{\infty}$ is quadratically convergent (**quadratic rate of convergence**)
- **Example for the significance of the rate of convergence:**
  - Assume that you have ***method A*** with linear convergence and ***method B*** with quadratic convergence. Also assume that $\lambda=0.5$ and your initial absolute error is the same for both methods $\varepsilon_1=0.1$, then
  - **For method A**: n=14, $\varepsilon_{14} = 1.22x10^{-5}$
  - **For method B:** n=3, $\varepsilon_3 = 1.25x10^{-5}$

Rate of Convergence of different numerical methods for simple roots

Method	Convergence	α
Bisection	Linear	1
Newton	Quadratic	2
Secant	Super-linear	1.62

- Note that the rate of convergence of the above methods are observed for <u>simple roots</u>
- For simple roots, $f'(p) \neq 0$

Quadratic Rate of Convergence for Newton's Method

Let p a simple root of $f(x)=0$ => $f(p)=0$

Let x_n be the n^{th} approximation to the root , then the absolute error can be written as $\varepsilon_n = |x_n - p|$

Newton's Method: $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Expand $f(x)$ to Taylor series at $x = p$

$$f(x_n) = f(p) + f'(p)(x_n - p) + f''(p)\frac{(x_n - p)^2}{2!} + \dots.$$

Expand $f'(x)$ to Taylor series at $x = p$

$$f'(x_n) = f'(p) + f''(p)(x_n - p) + \dots.$$

Quadratic Rate of Convergence for Newton's Method (2)

Substitute $f(x_n)$ and $f'(x_n)$ in Newton's Formula:

$$x_{n+1} = x_n - \frac{\left(f(p) + f'(p)(x_n - p) + f''(p)\dfrac{(x_n - p)^2}{2!} + \dots \right)}{f'(p) + f''(p)(x_n - p) + \dots}$$

Define $\beta = \dfrac{f''(p)}{f'(p)}$ then

$$x_{n+1} = x_n - \frac{\left((x_n - p) + \dfrac{1}{2}\beta(x_n - p)^2 + \dots \right) \cancel{f'(p)}}{\left(1 + \beta(x_n - p) + \dots\right) \cancel{f'(p)}}$$

Re-write the above equation using the binomial expansion for the denominator and neglect the higher order terms

Note that binomial expansion: $(1+q)^m = 1 + mq + \dfrac{m(m-1)}{2!}q^2 + \dots$
(here q<<1.0)

Quadratic Rate of Convergence for Newton's Method (3)

to obtain

$$x_{n+1} = x_n - \left(\varepsilon_n + \frac{1}{2}\beta\varepsilon_n^2 \right)\left(1 - \beta\varepsilon_n\right)$$

subtract p from both sides and use the definition of the ***ε_n*** & ***ε_{n+1}***

$$\varepsilon_{n+1} = \varepsilon_n - \left(\varepsilon_n - \beta\varepsilon_n^2 + \frac{1}{2}\beta\varepsilon_n^2 - \frac{1}{2}\beta^2\varepsilon_n^3 \right)$$

finally;

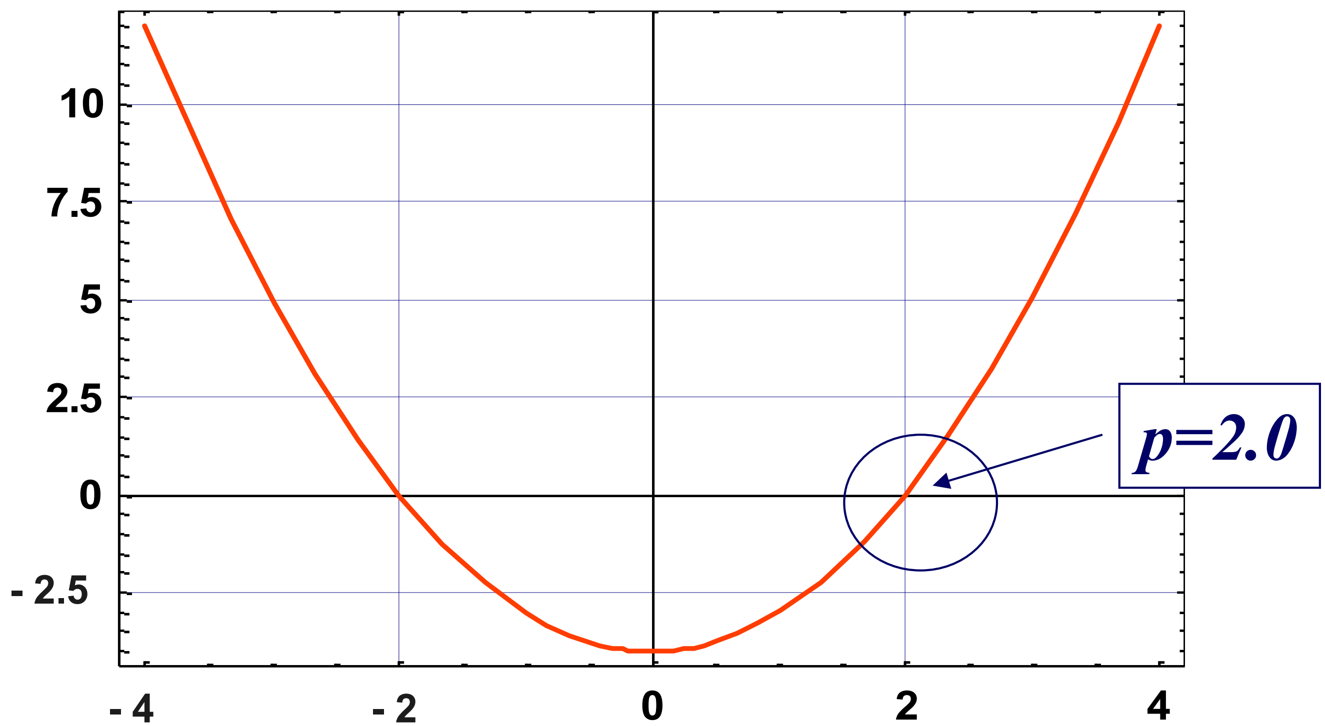
$$\boxed{\varepsilon_{n+1} = \frac{1}{2}\beta\varepsilon_n^2} \longrightarrow \varepsilon_{n+1} = \lambda\,\varepsilon_n^{\alpha}$$

where $\lambda = \dfrac{f''(p)}{2f'(p)}$ & $\alpha = 2$

So Newton's method converges quadratically for a simple root.

Root Finding Example

- Find the positive root of $x^2=4$ using bisection and Newton's methods.



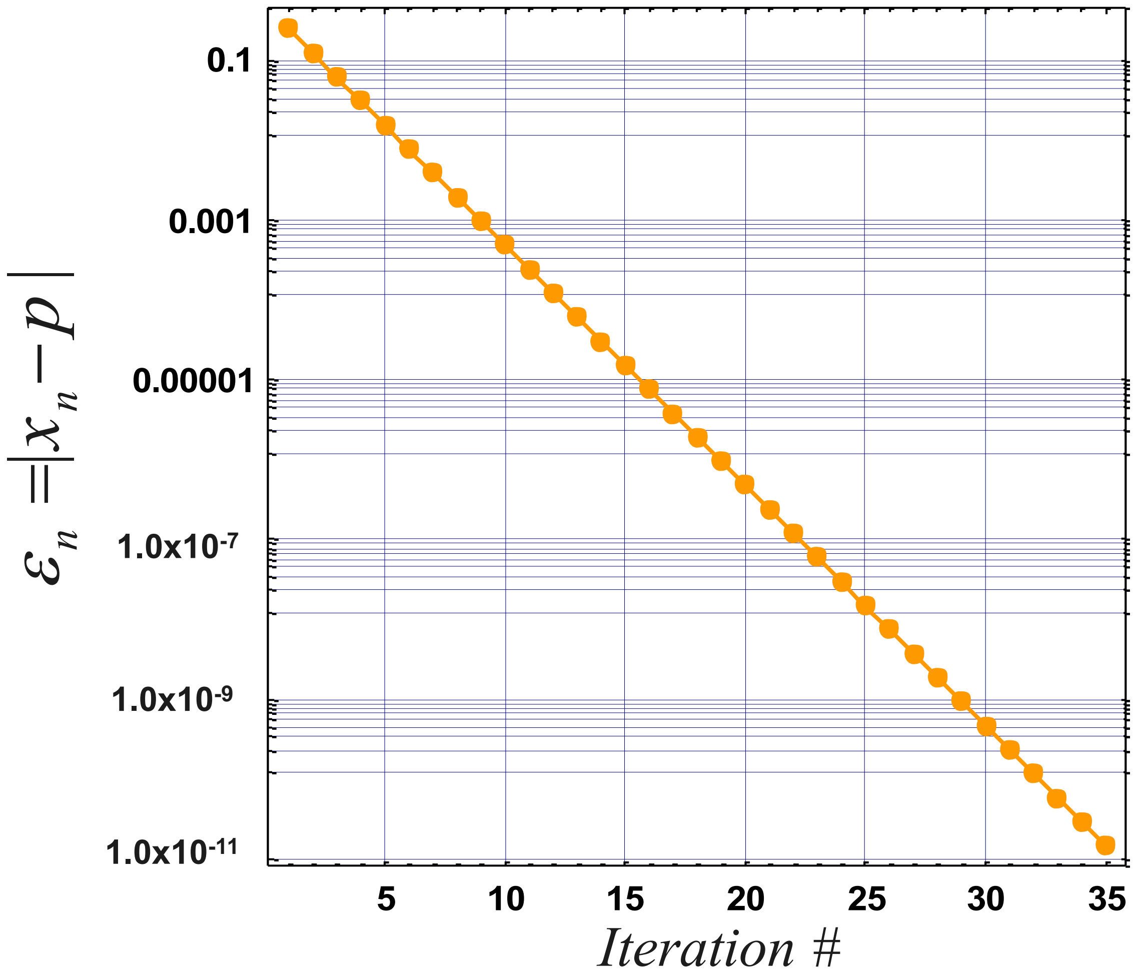
- Standard form: $f(x)= x^2-4 = 0$
- Use the stopping criteria of $|f(x_n)| \leq 10^{-10}$

Bisection Method

- ***Initial interval* = [1.0 , 2.5]**
- $p_n=(a_n+b_n)/2$ ***& Sign check***

a	b	x_n	$\|f(x_n)\|$
1.	2.5	1.75	1.
1.75	2.5	2.125	0.55
1.75	2.125	1.9375	0.2625
1.9375	2.125	2.03125	0.134375
1.9375	2.03125	1.984375	0.06640625
1.984375	2.03125	2.0078125	0.0333984375
1.984375	2.0078125	1.99609375	0.016650390625
1.99609375	2.0078125	2.001953125	0.00833740234375
⋮	⋮		
1.99999999627	2.00000000745	2.00000000186	$7.94728597005\times10^{-9}$
1.99999999627	2.00000000186	1.99999999907	$3.97364298503\times10^{-9}$
1.99999999907	2.00000000186	2.00000000047	$1.98682149251\times10^{-9}$
1.99999999907	2.00000000047	1.99999999977	$9.93410746257\times10^{-10}$
1.99999999977	2.00000000047	2.00000000012	$4.96705373128\times10^{-10}$
1.99999999977	2.00000000012	1.99999999994	$2.48352686564\times10^{-10}$
1.99999999994	2.00000000012	2.00000000003	$1.24176343282\times10^{-10}$
1.99999999994	2.00000000003	1.99999999999	$6.2088171641\times10^{-11}$

Convergence of Bisection Method



Linear rate of convergence observed for the bisection method

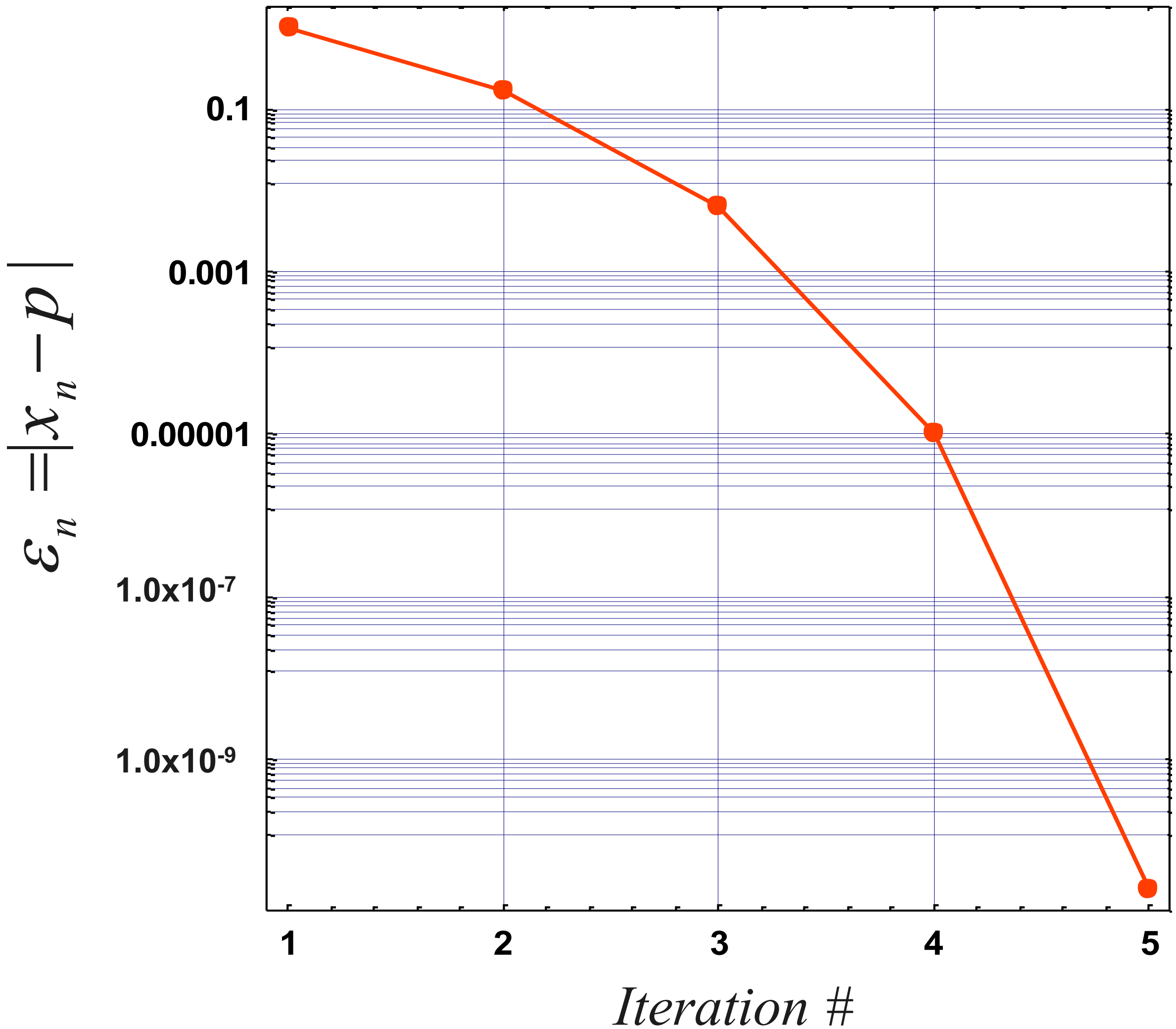
Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad or \quad \Delta_n x = -\frac{f(x_n)}{f'(x_n)} \quad and \quad x_{n+1} = x_n + \Delta_n x$$

Initial point x_1=3.0

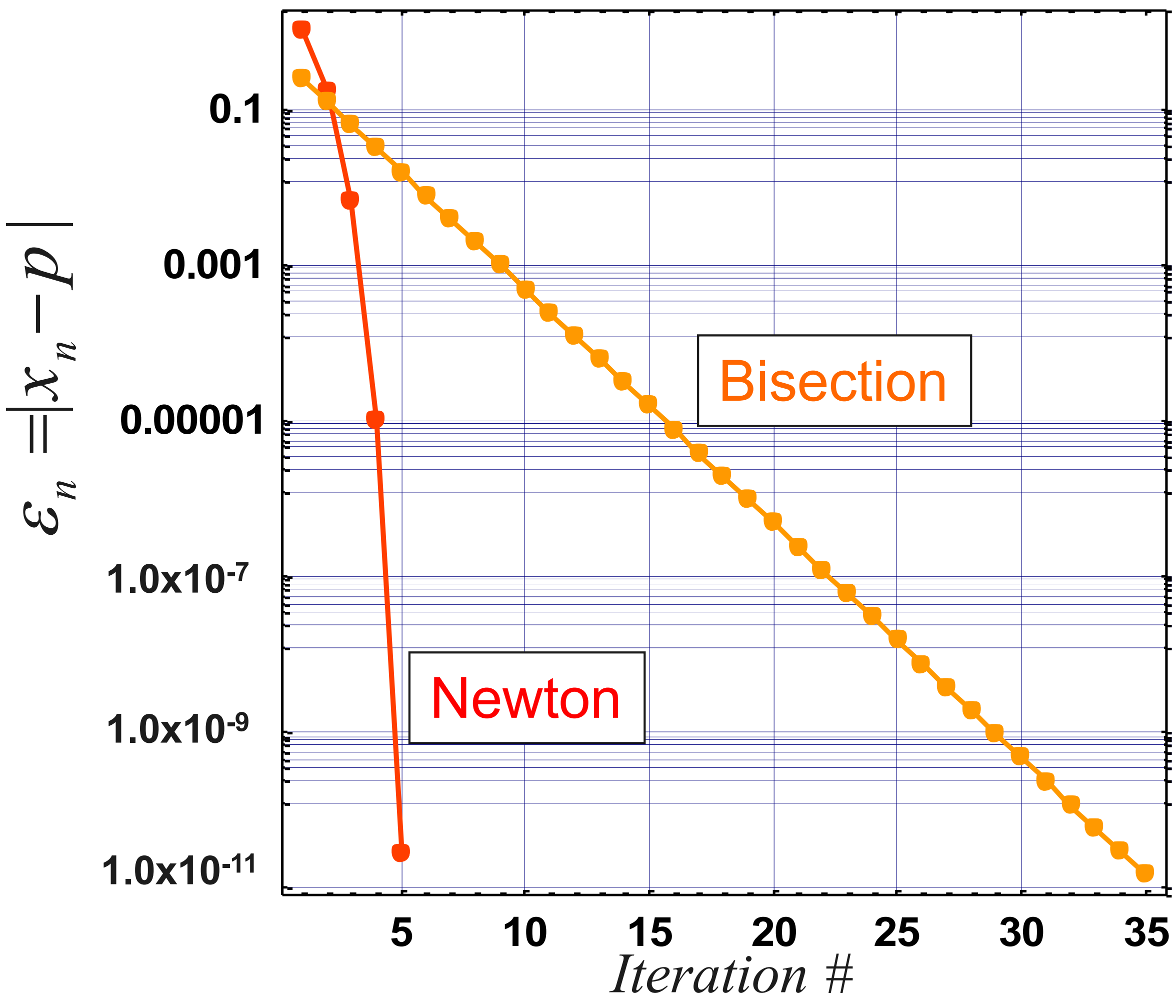
Iter #	x_n	$\lvert f(x_n) \rvert$	$\Delta_n x$	x_{n+1}
1	3.	5.	-0.833333333333	2.16666666667
2	2.16666666667	0.694444444444	-0.160256410256	2.00641025641
3	2.00641025641	0.0256821170283	-0.00640001638404	2.00001024003
4	2.00001024003	0.0000409602097164	-0.0000102400000002	2.00000000003
5	2.00000000003	$1.04856567873 \times 10^{-10}$	$-2.62141419679 \times 10^{-11}$	2.

Convergence of Newton's Method



Quadratic rate of convergence observed for the Newton's method

Comparison of Convergence Rates for Newton's and Bisection Methods



Newton's Method is much more efficient than bisection method in terms of convergence

Summary

– We defined the absolute error, relative error and the rate of convergence
– Discussed the importance of convergence rate
– Derived the rate of convergence for the Newton's Method
– Worked on a root-finding example using the Newton's and Bisection methods and compared the convergence of each algorithm