

Find the roots of

$$x_1 x_2 x_3 x_4 = -4$$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 10$$

$$x_2 x_3^2 = -2$$

$$x_1^3 + x_2^2 + x_3^3 - x_4^2 = 2$$

**with the initial guess
(starting point)**

$$x_1 = 0.5$$

$$x_2 = -2.5$$

$$x_3 = 1.5$$

$$x_4 = 1.5$$

$$[f']^k \{\Delta x\}^k = -\{f\}^k \longrightarrow \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix}^k \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix}^k = - \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}^k$$

$$\{x\}^{k+1} = \{x\}^k + \{\Delta x\}^k$$

$$f_1 = x_1 x_2 x_3 x_4 + 4$$

$$f_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 10$$

$$f_3 = x_2 x_3^2 + 2$$

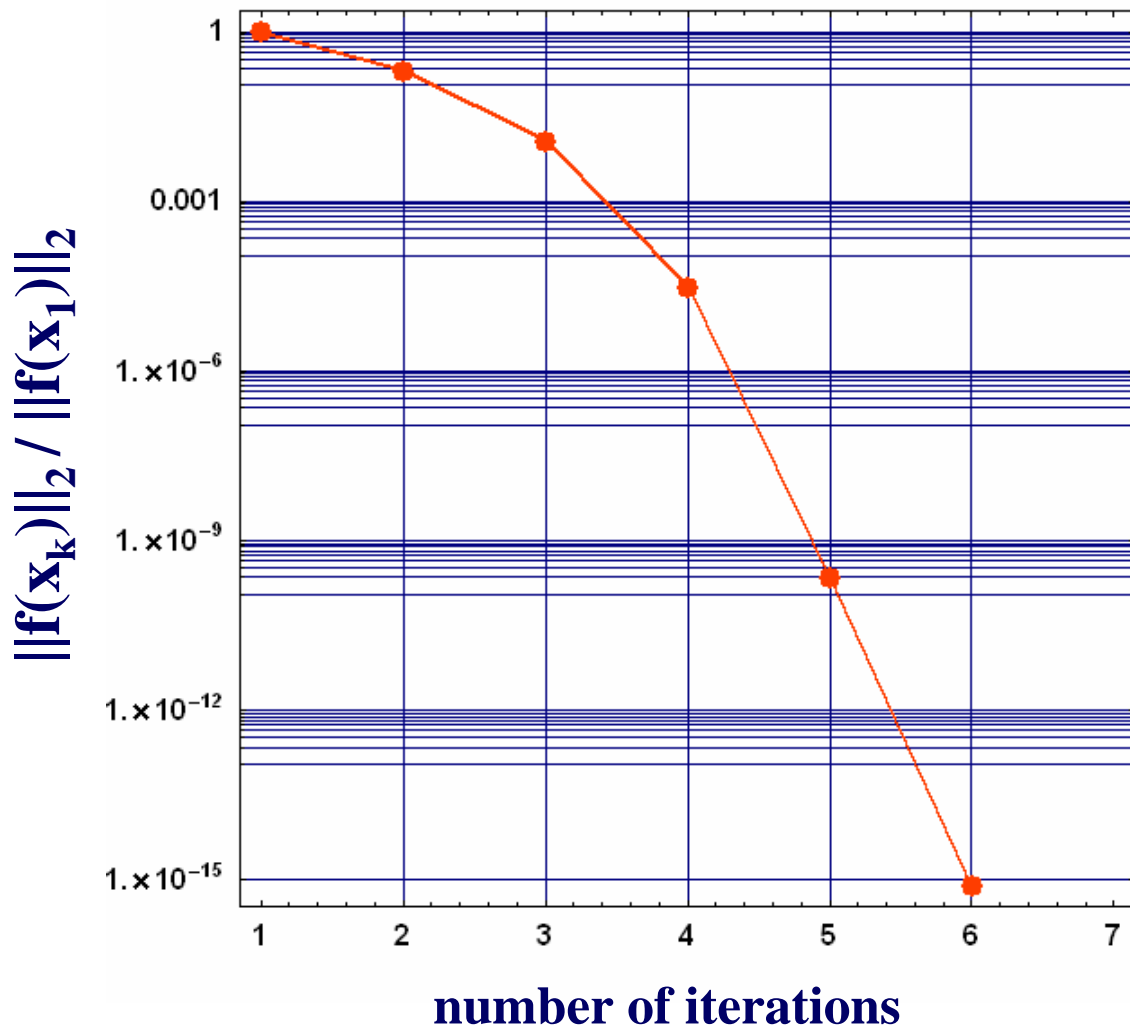
$$f_4 = x_1^3 + x_2^2 + x_3^3 - x_4^2 - 2$$

$$[f'] = \begin{bmatrix} x_2 x_3 x_4 & x_1 x_3 x_4 & x_1 x_2 x_4 & x_1 x_2 x_3 \\ 2x_1 & 2x_2 & 2x_3 & 2x_4 \\ 0 & x_3^2 & 2x_2 x_3 & 0 \\ 3x_1^2 & 2x_2 & 3x_3^2 & -2x_4 \end{bmatrix}$$

$$\text{Initial Estimate} = \begin{pmatrix} 0.5 \\ -2.5 \\ 1.5 \\ 1.5 \end{pmatrix} \quad \text{Solution} = \begin{pmatrix} 1. \\ -2. \\ 1. \\ 2. \end{pmatrix}$$

| k | $\ f(\mathbf{x}_k)\ _2$ | $\ f(\mathbf{x}_k)\ _2 / \ f(\mathbf{x}_1)\ _2$ |
|----------|---|---|
| 1 | 1.13886 | 1. |
| 2 | 0.230074 | 0.20202 |
| 3 | 0.0133317 | 0.0117061 |
| 4 | 0.000034377 | 0.0000301854 |
| 5 | 2.42295×10^{-10} | 2.12751×10^{-10} |
| 6 | 9.15513×10^{-16} | 8.03882×10^{-16} |
| 7 | 0. | 0. |

Note that for this case I used the built-in `LinearSolve[...]` command to solve the linear set of equations at each iteration



**Quadratic convergence
observed for this initial
estimate:**

$$\text{Initial Estimate} = \begin{pmatrix} 0.5 \\ -2.5 \\ 1.5 \\ 1.5 \end{pmatrix}$$

$$\text{Initial Estimate} = \begin{pmatrix} 1. \\ 1. \\ 1. \\ 1. \end{pmatrix}$$

```
LinearSolve ::nosol :
  Linear equation encountered which has no solution . More...

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  Linear equation encountered which has no solution . More...

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  Linear equation encountered which has no solution . More...

General ::stop : Further output of LinearSolve ::nosol will
  be suppressed during this calculation . More...
```

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Out[95]= $Aborted
```

Convergence failed for the initial estimate $X^0=\{1.0,1.0,1.0,1.0\}^T$

Good initial estimates are important for the convergence of the Newton's Method for the solution of non-linear systems

Used the initial estimate $\mathbf{X}^0=\{0.5,-2.5,1.5,1.5\}^T$

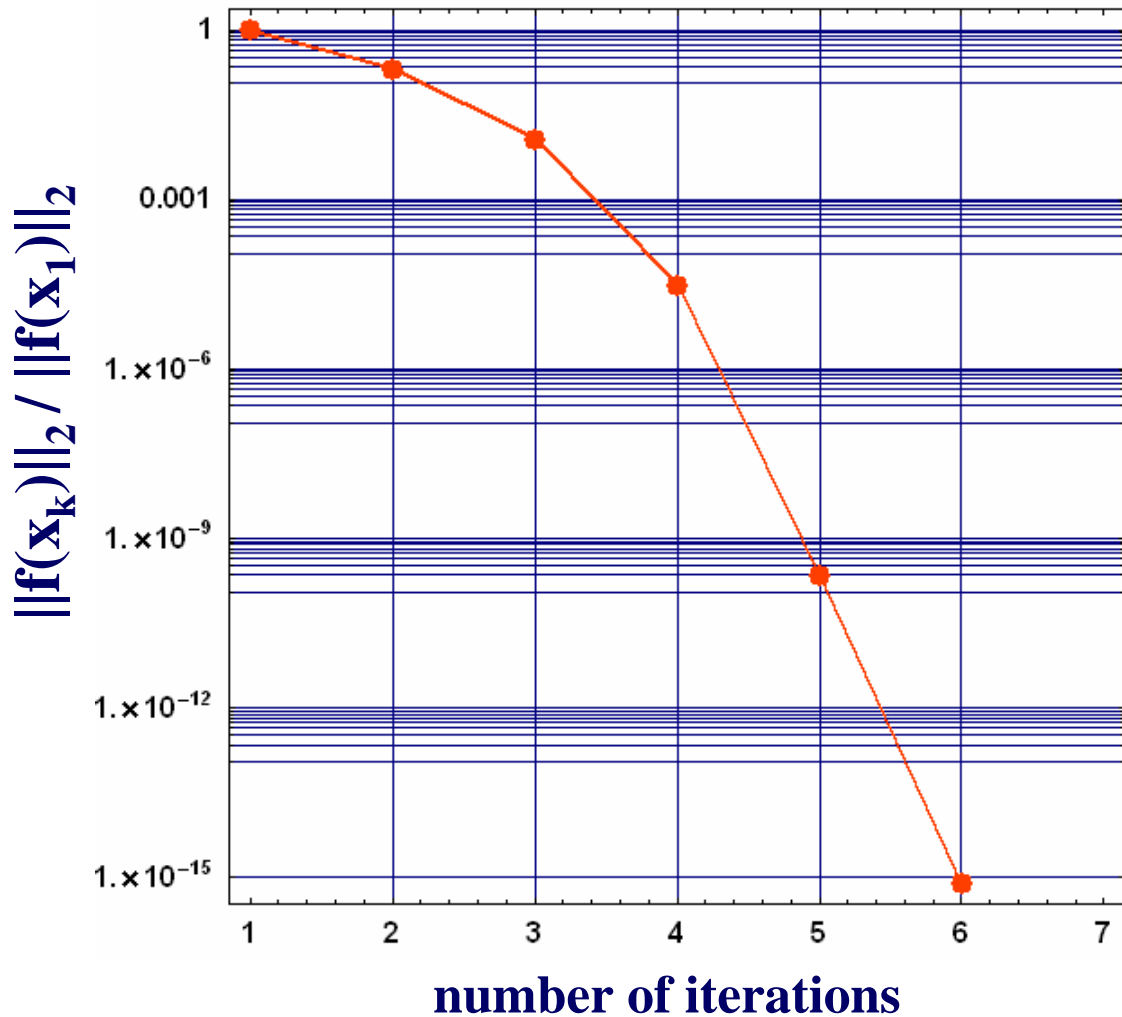
| k | $\ \mathbf{f}(\mathbf{x}_k)\ _2$ | $\ \mathbf{f}(\mathbf{x}_k)\ _2 / \ \mathbf{f}(\mathbf{x}_1)\ _2$ |
|-----|---|---|
| 1 | 1. | 1.13886 |
| 2 | 6.49495 | 7.39687 |
| 3 | 64.0895 | 72.9893 |
| 4 | $3.087617976789494 \times 10^{658}$ | $3.516379249160662 \times 10^{658}$ |
| 5 | $3.115413502965002 \times 10^{64666}$ | $3.54803459389495 \times 10^{64666}$ |
| 6 | $1.254358281939305 \times 10^{6401926}$ | $1.42854442057969 \times 10^{6401926}$ |
| 7 | Overflow[] | Overflow[] |
| 8 | Overflow[] | Overflow[] |
| 9 | Indeterminate | Indeterminate |
| 10 | Indeterminate | Indeterminate |

When the linear problem at each iteration is solved with Gauss-Seidel with a relaxation parameter of 1.1 (over-relaxation), the method diverges. We can try under-relaxation ($w < 1.0$).

Used the initial estimate $X^0=\{0.5,-2.5,1.5,1.5\}^T$

| k | $\ f(x_k)\ _2$ | $\ f(x_k)\ _2 / \ f(x_1)\ _2$ |
|----------|----------------------------------|---|
| 1 | 1. | 1.13886 |
| 2 | 0.20202 | 0.230074 |
| 3 | 0.0117061 | 0.0133317 |
| 4 | 0.0000301854 | 0.000034377 |
| 5 | 2.12751×10^{-10} | 2.42295×10^{-10} |
| 6 | 8.03882×10^{-16} | 9.15513×10^{-16} |
| 7 | 0. | 0. |

- **When the linear problem at each iteration is solved with Gauss-Seidel with a relaxation parameter of 0.5 (under-relaxation), the method converges.**
- **Note that the precision goal to solve the linear problem on each step was 10^{-20} The precision goal to solve the non-linear problem was 10^{-16}**



Quadratic convergence observed when the linear problem at each iteration is solved with Gauss-Seidel with a relaxation parameter of 0.5 (under-relaxation),