MATLAB

2. EQUATIONS.

MATRIX

$$[\mathbf{A}]_{m\times n} = \left.\overbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}}^{n\ columns}\right\} m\ rows$$

MATRIX MULTIPLICATION

$$[\mathbf{A}]_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

EQUAL

$$[\mathbf{B}]_{n\times p} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1p} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{np} \end{pmatrix}$$

MATRIX MULTIPLICATION

$$[\mathbf{AB}]_{m\times p} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boxed{a_{i1} \quad a_{i2} \quad a_{i3} \quad \cdots \quad a_{in}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2j} & \cdots & b_{2p} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3j} & \cdots & b_{3p} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix}$$

$$[\mathbf{AB}]_{(i,j)} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

EXAMPLE

$$[\mathbf{A}]_{4\times 3} = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 4 & 2 \\ 2 & 1 & 4 \\ 0 & 1 & 3 \end{pmatrix} \qquad [\mathbf{B}]_{3\times 4} = \begin{pmatrix} 2 & 0 & 4 & 1 \\ 0 & 3 & 1 & 0 \\ 1 & 2 & 1 & 4 \end{pmatrix}$$

$$[\mathbf{AB}]_{4\times 4} = \begin{pmatrix} 9 & 13 & 14 & 22 \\ 4 & 16 & 10 & 9 \\ 8 & 11 & 13 & 18 \\ 3 & 9 & 4 & 12 \end{pmatrix} \qquad [\mathbf{BA}]_{3\times 3} = \begin{pmatrix} 12 & 7 & 29 \\ 5 & 13 & 10 \\ 6 & 14 & 25 \end{pmatrix}$$

IDENTITY MATRIX

$$[\mathbf{I}]_{n\times n} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$[\mathbf{I}]_{m\times m}[\mathbf{A}]_{m\times n} = [\mathbf{A}]_{m\times n}[\mathbf{I}]_{n\times n} = [\mathbf{A}]_{m\times n}$$

INVERTIBILITY

A SQUARE MATRIX $[\mathbf{A}]_{n\times n}$ IS **INVERTIBLE** IF THERE EXISTS A MATRIX $[\mathbf{A}^{-1}]_{n\times n}$ SUCH THAT

$$[\mathbf{A}\mathbf{A}^{-1}]_{n\times n} = [\mathbf{A}^{-1}\mathbf{A}]_{n\times n} = [\mathbf{I}]_{n\times n}$$

SYSTEM OF LINEAR EQUATIONS:

$$[\mathbf{A}]_{n\times n}[\mathbf{B}]_{n\times 1} = [\mathbf{C}]_{n\times 1}$$

SOLUTION:

$$[\mathbf{B}]_{n\times 1} = [\mathbf{A}^{-1}\mathbf{C}]_{n\times 1}$$

SYSTEMS OF LINEAR EQUATIONS

A system of n linear equations in n unknowns may be written in matrix form as

$$[\mathbf{A}]_{n\times n}[\mathbf{B}]_{n\times 1} = [\mathbf{C}]_{n\times 1} \qquad (*)$$

I If $[\mathbf{C}]_{n\times 1} = [\mathbf{0}]_{n\times 1}$, and $\det[\mathbf{A}]_{n\times n} \neq 0$, no non-zero solution to $(*)$ exists.

II If $[\mathbf{C}]_{n\times 1} = [\mathbf{0}]_{n\times 1}$, and $\det[\mathbf{A}]_{n\times n} = 0$, at least one family of non-zero solutions to $(*)$ exists.

III If $\det[\mathbf{A}]_{n\times n} = 0$, there are values of $[\mathbf{C}]_{n\times 1} \neq [\mathbf{0}]_{n\times 1}$, for which no solution to $(*)$ exists.

IV If $[\mathbf{C}]_{n\times 1} \neq [\mathbf{0}]_{n\times 1}$, and $\det[\mathbf{A}]_{n\times n} \neq 0$, exactly one solution to $(*)$ exists.

```
>> A=[8 2;1 3];
>> C=[5;2];
>> B=A\C;
>> B

B =

    0.5000
    0.5000

>> D=C\A;
>> D

D =

    1.4483    0.5517
```

$$8x + 2y = 5, \quad x + 3y = 2.$$

$$5x = 8, \quad 5y = 2, \quad 2x = 1, \quad 2y = 3.$$

EIGENVECTORS AND EIGENVALUES

A number λ is an **EIGENVALUE** of $\left[\mathbf{A}\right]_{n\times n}$ if there exists a column vector $\left[\mathbf{v}\right]_{n\times 1} \neq \left[\mathbf{0}\right]_{n\times 1}$ such that

$$\left[\mathbf{A}\right]_{n\times n}\left[\mathbf{v}\right]_{n\times 1} = \lambda\left[\mathbf{v}\right]_{n\times 1}$$

Such a vector $\left[\mathbf{v}\right]_{n\times 1}$ is called an **EIGENVECTOR** of $\left[\mathbf{A}\right]_{n\times 1}$ corresponding to the eigenvalue λ.

A number λ is an **EIGENVALUE** of $\left[\mathbf{A}\right]_{n\times n}$ precisely if it is a root of the n^{th}–order polynomial equation

$$\det\left[\mathbf{A} - \lambda\mathbf{I}\right]_{n\times n} = 0 \qquad (**)$$

Equation $(**)$ is called the characteristic equation of the matrix $\left[\mathbf{A}\right]_{n\times n}$.

SYMMETRIC 3x3 MATRIX

$$\det\left[\mathbf{A}-\lambda\mathbf{1}\right]=\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{12} & a_{22}-\lambda & a_{23} \\ a_{13} & a_{23} & a_{33}-\lambda \end{vmatrix}=-\lambda^3+\mathrm{I}_1\lambda^2-\mathrm{I}_2\lambda+\mathrm{I}_3$$

$$\mathrm{I}_1=a_{11}+a_{22}+a_{33}$$

$$\mathrm{I}_2=a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{12}^2-a_{13}^2-a_{23}^2$$

$$\mathrm{I}_3=\det\left[\mathbf{A}\right]=a_{11}a_{22}a_{33}+2a_{12}a_{23}a_{13}-a_{11}a_{23}^2-a_{22}a_{13}^2--a_{33}a_{12}^2$$

I All three roots of the polynomial are real.

II If $\lambda_1 \neq \lambda_2$ are distinct eigenvalues of $\left[\mathbf{A}\right]$ and $\left[\mathbf{v_1}\right]$, $\left[\mathbf{v_2}\right]$ are eigenvectors corresponding to λ_1, λ_2 respectively, then

$$\left[(\mathbf{v_1})^{\mathrm{T}}(\mathbf{v_2})\right]=0 \qquad (***)$$

III If the roots of the polynomial are λ_1, λ_1, and $\lambda_2 \neq \lambda_1$ [I.E., if λ_1 is a double root,] and if $\left[\mathbf{v_2}\right]$ is an eigenvector corresponding to λ_2, then any vector $\left[\mathbf{v_1}\right]$ satisfying $(***)$ is an eigenvector corresponding to λ_1. If λ_1 is a triple root, every vector is an eigenvector corresponding to λ_1.

```
>> V=[3 5 -2;5 8 1;-2 1 6];
>> V

V =

     3     5    -2
     5     8     1
    -2     1     6

>> [C,D]=eig(V);
>> C

C =

   -0.8113   -0.2412   -0.5325
    0.4953    0.2004   -0.8453
   -0.3106    0.9496    0.0431

>> D

D =

   -0.8180         0         0
         0    6.7191         0
         0         0   11.0988

>> e1=C(:,1);
>> e1

e1 =

   -0.8113
    0.4953
   -0.3106

>> e2=C(:,2);
>> e3=C(:,3);
>> e1'*e2

ans =

     0

>> e1'*e3

ans =

   1.5613e-17
```

```
>> A=[2.36 0 -.48;0 2 0;-.48 0 2.64];
>> A

A =

    2.3600         0   -0.4800
         0    2.0000         0
   -0.4800         0    2.6400

>> [E,F]=eig(A);
>> E

E =

    0.5555   -0.5757   -0.6000
   -0.7196   -0.6944   -0.0000
    0.4166   -0.4318    0.8000

>> F

F =

    2.0000         0         0
         0    2.0000         0
         0         0    3.0000
```

```
>> B=[3 -.8 -.6;.8 1.72 -.96;.6 -.96 2.28];
>> B

B =

    3.0000   -0.8000   -0.6000
    0.8000    1.7200   -0.9600
    0.6000   -0.9600    2.2800

>> [G,H]=eig(B);
>> G

G =

    0.7071   -0.7071         0
    0.5657   -0.5657   -0.6000
    0.4243   -0.4243    0.8000

>> H

H =

    2.0000         0         0
         0    2.0000         0
         0         0    3.0000
```

```
>> R=[.8 .48 .36;0 .6 -.8;-.6 .64 .48];
>> R

R =

    0.8000    0.4800    0.3600
         0    0.6000   -0.8000
   -0.6000    0.6400    0.4800

>> S=R*R';
>> S

S =

    1.0000         0   -0.0000
         0    1.0000         0
   -0.0000         0    1.0000

>> % R IS A ROTATION MATRIX.
>> [P,Q]=eig(R);
>> P

P =

  -0.8018             0.1572 - 0.3922i   0.1572 + 0.3922i
  -0.5345             0.1048 + 0.5883i   0.1048 - 0.5883i
   0.2673             0.6814             0.6814

>> Q

Q =

   1.0000                  0                  0
        0             0.4400 + 0.8980i        0
        0                  0             0.4400 - 0.8980i
```

SINGLE NON-LINEAR EQUATION

NEWTON'S METHOD

PROBLEM:– SOLVE THE NON-LINEAR EQUATION

$$f(x) = 0. \tag{E}$$

IDEA:– SUPPOSE $\bar{x}$ IS A SOLUTION OF (E), AND THAT x_1 IS A GUESSED VALUE FOR $\bar{x}$. BY TAYLOR'S THEOREM,

$$0 = f(\bar{x}) = f(x_1) + f'(x_1)(\bar{x} - x_1) + O(\bar{x} - x_1)^2$$

IF THE INITIAL GUESS IS "GOOD ENOUGH", $(\bar{x} - x_1)^2 \approx 0$, AND WE GET AN ESTIMATE FOR $\bar{x}$ OF THE FORM

$$\bar{x} \approx x_1 - \frac{f(x_1)}{f'(x_1)} = x_2.$$

WE TAKE x_2 AS OUR NEW GUESS, AND EXPECT IT TO BE A BETTER ESTIMATE FOR THE ROOT THAN x_1.
BY REPEATING THIS PROCESS, WE MAY HOPE TO FORCE THE UPDATED GUESS EVER CLOSER TO THE ROOT.

$\boxed{\textbf{NEWTON ALGORITHM}}$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{N}$$

$\boxed{\text{OBSERVATIONS}}$

(1) PROCESS STOPS IF $f(x_k) = 0$.

(2) IF $f'(x_k) = 0$, $f(x_k) \neq 0$, THEN $x_{k+1} = \pm\infty$:– **NOT** AN IMPROVED GUESS.

(3) IF $|f'(x_k)| \ll 1$, $f(x_k) \neq 0$, THEN NUMERICAL INSTABILITY :– **NOT** AN IMPROVED GUESS.

(4) THERE ARE THEOREMS TO THE EFFECT THAT, IF A ROOT EXISTS, AND THE INITIAL GUESS IS ‘CLOSE ENOUGH” TO IT, THE NEWTON PROCESS CONVERGES TO THE ROOT. SADLY, THESE DON’T OFFER AN UP-FRONT TEST FOR “CLOSE ENOUGHNESS”.

NEWTON ALGORITHM

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \qquad (N)$$

EXAMPLE

CONSIDER THE FUNCTION $f(x) = x^2\cos(x) + \sin(x) - 1$. THIS FUNCTION HAS A ZERO AT $\bar{x} = \frac{\pi}{2} \approx 1.5708$.

STARTING FROM INITIAL GUESS $x_1 = 2$, THE SUCCESSIVE ITERATES ARE :–

$$x_1 = 2, \quad x_2 = 1.6930, \quad x_3 = 1.5874, \quad x_4 = 1.5712, \quad x_5 = 1.5708.$$

PROCESS CONVERGES IN 4 ITERATIONS.

NEWTON CODE

```
format long e;
Maxits=500;
Tol=1.0e-8;
X=2;

for j=1:Maxits;
    x=X;
    F=cos(x)*x^2+sin(x)-1;
    G=(2*x+1)*cos(x)-sin(x)*x^2;

    deltaX=-F/G;
    X=X+deltaX;
    x=X;

    Fcheck=abs(cos(x)*x^2+sin(x)-1);

    if Fcheck<Tol
        break
    end
end

if j<Maxits

    fprintf('x=%-12e \n',X);

    fprintf('Iterations=%-5g\n\n',j);

else

    fprintf('we did not converge!\n ');
end
```

OUTPUT

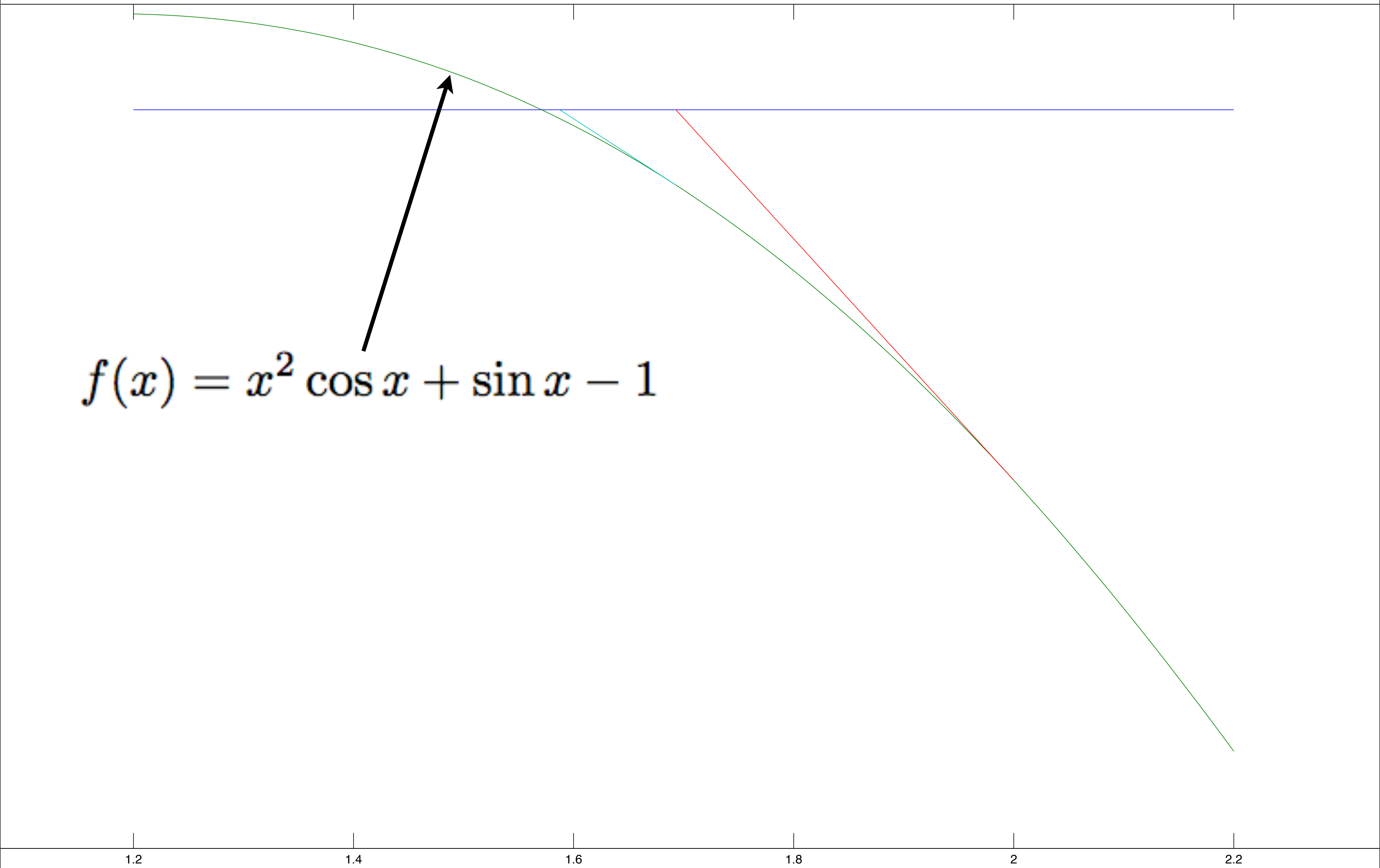
```
x=1.570796e+00
Iterations=5

>> x

x =

     1.570796326794971e+00
```

GEOMETRICAL INTERPRETATION OF THE NEWTON ALGORITHM:- NEW GUESS= VALUE OF x AT WHICH THE TANGENT TO THE GRAPH AT THE OLD GUESS CUTS THE x-AXIS.


$f(x) = x^2 \cos x + \sin x - 1$
1.2
1.4
1.6
1.8
2
2.2

SYSTEM OF NON-LINEAR EQUATIONS

NEWTON-RAPHSON METHOD

PROBLEM:– SOLVE THE NON-LINEAR SYSTEM OF EQUATIONS:–

$$\vec{f}(\vec{x}) = \begin{pmatrix} f_1(x_1, x_2 \ldots x_n) \\ f_2(x_1, x_2 \ldots x_n) \\ \vdots \\ f_n(x_1, x_2 \ldots x_n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \vec{0} \qquad (S)$$

IDEA:– SUPPOSE $\vec{\bar{x}}$ IS A SOLUTION OF (S), AND THAT $\vec{x}^{(1)}$ IS A GUESSED VALUE FOR $\vec{\bar{x}}$. BY THE MULTIVARIABLE VERSION OF TAYLOR'S THEOREM,

$$\vec{0} = \vec{f}(\vec{\bar{x}}) = \vec{f}(\vec{x}^{(1)}) + [\mathbf{J}(\vec{x}^{(1)})](\vec{\bar{x}} - \vec{x}^{(1)}) + O|\vec{\bar{x}} - \vec{x}^{(1)}|^2.$$

HERE $[\mathbf{J}(\vec{x}^{(1)})]$ IS THE JACOBIAN MATRIX OF THE SYSTEM AT $\vec{x}^{(1)}$,THAT IS, THE MATRIX WHOSE (i, j) ENTRY IS

$$[\mathbf{J}(\vec{x}^{(1)})](i, j) = \frac{\partial f_i}{\partial x_j}(x_1^{(1)}, x_2^{(1)} \ldots x_n^{(1)}).$$

AS IN THE 1×1 CASE ABOVE, WE WRITE

$$\vec{0} \approx \vec{f}(\vec{x}^{(1)}) + [\mathbf{J}(\vec{x}^{(1)})](\vec{\bar{x}} - \vec{x}^{(1)})$$

WHICH YIELDS THE NEW, WE HOPE IMPROVED, GUESS

$$\vec{\bar{x}} \approx \vec{x}^{(1)} - [\mathbf{J}^{-\mathbf{1}}(\vec{x}^{(1)})]\vec{f}(\vec{x}^{(1)}) = \vec{x}^{(2)}$$

WE TAKE $\vec{x}^{(2)}$ AS OUR NEW GUESS, AND EXPECT IT TO BE A BETTER ESTIMATE FOR THE ROOT THAN $\vec{x}^{(1)}$.
BY REPEATING THIS PROCESS, WE MAY HOPE TO FORCE THE UPDATED GUESS EVER CLOSER TO THE ROOT.

NEWTON-RAPHSON ALGORITHM

$$\vec{x}^{(k+1)} = \vec{x}^{(k)} - [\mathbf{J^{-1}}(\vec{x}^{(k)})]\vec{f}(\vec{x}^{(k)}) \qquad (NR)$$

$\boxed{\text{OBSERVATIONS}}$

(1) PROCESS STOPS IF $\vec{f}(\vec{x}^{(k)}) = \vec{0}$.

(2) IF $\det[\mathbf{J}(\vec{x}^{(k)})] = 0$, PROCESS BREAKS DOWN.

(3) IF $|\det[\mathbf{J}(\vec{x}^{(k)})]| \ll 1$, $\vec{f}(\vec{x}^{(k)}) \neq \vec{0}$, THEN NUMERICAL INSTABILITY :– **NOT** AN IMPROVED GUESS.

(4) THERE ARE THEOREMS TO THE EFFECT THAT, IF A ROOT EXISTS, AND THE INITIAL GUESS IS 'CLOSE ENOUGH" TO IT, THE NEWTON PROCESS CONVERGES TO THE ROOT. SADLY, THESE DON'T OFFER AN UP-FRONT TEST FOR "CLOSE ENOUGHNESS".

EXAMPLE: TRANSCENDENTAL COMPLEX EQUATION.

$$\sin(x+iy) = x+iy$$

$$\cosh y = \frac{x}{\sin x} \qquad (1)$$

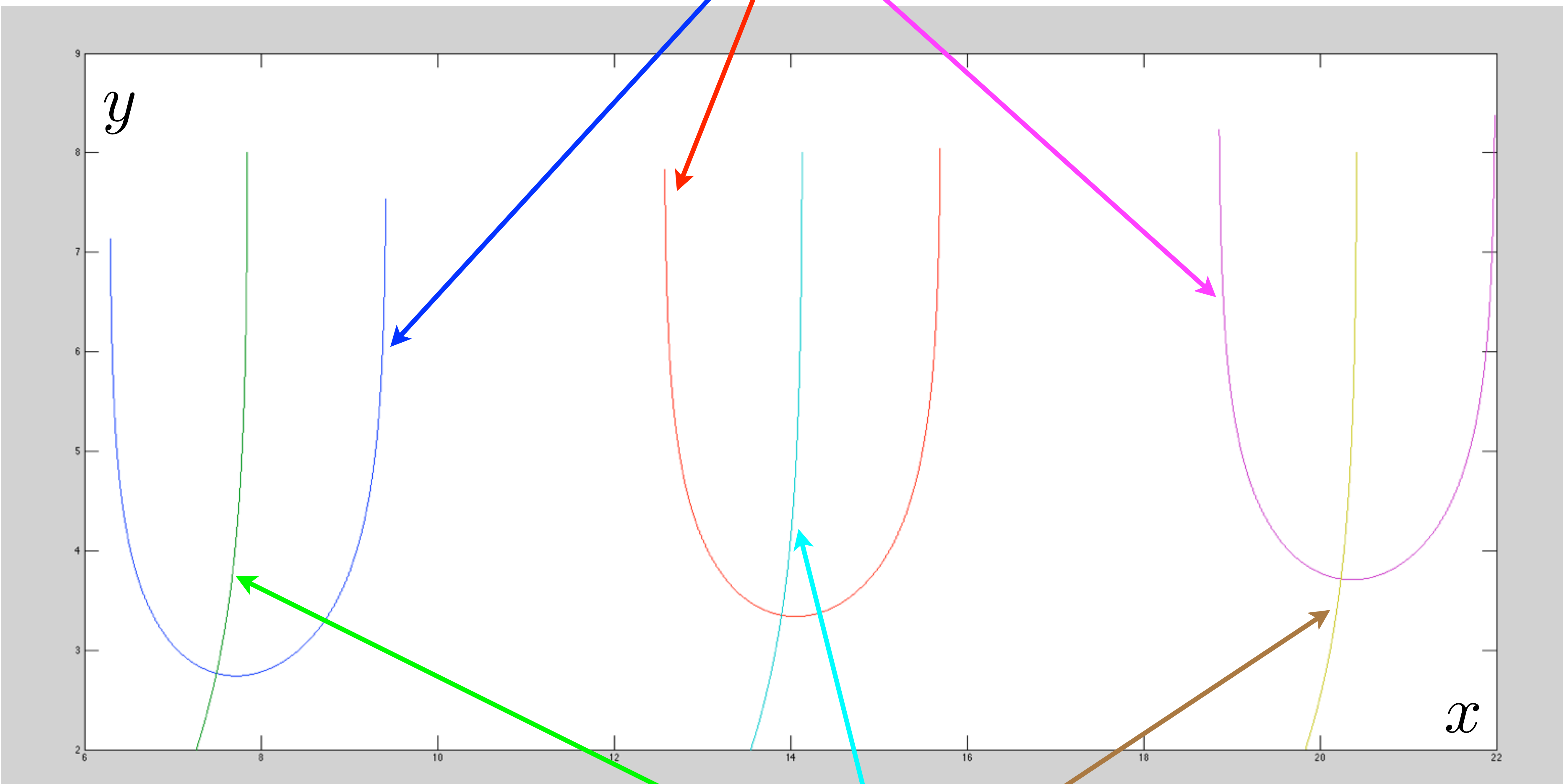
$$\cos x = \frac{y}{\sinh y} \qquad (2)$$

FOR $p = 1, 2 \ldots$ EXACTLY ONE ROOT WITH $x > 0$, $y > 0$ EXISTS IN THE RANGE $[2p\pi, (2p+1)\pi]$. G.H. HARDY FOUND AN ESTIMATE FOR THIS ROOT:-

$$x \approx (2p+0.5)\pi, \quad y \approx \ln(4p+1)\pi.$$

Asymptotes: $2p\pi, \ (2p+1)\pi, \ p = 1, 2 \ldots$

$$\cosh y = \frac{x}{\sin x}$$



Asymptotes: $(2p+0.5)\pi, \ p = 1, 2 \ldots$

$$\cos x = \frac{y}{\sinh y}$$

```
format long e;
J=zeros(2,2);
Re=zeros(10,1);
Im=zeros(10,1);
Maxits=500;
Tol=1.0e-8;
for p=1:10;
X=[(2*p+1)*pi;log((4*p+1)*pi)];

for j=1:Maxits;
    x=X(1);
    y=X(2);
    F1=cosh(y)*sin(x)-x;
    F2=sinh(y)*cos(x)-y;
    J(1,1)=cosh(y)*cos(x)-1;
    J(1,2)=sinh(y)*sin(x);
    J(2,1)=-sinh(y)*sin(x);
    J(2,2)=cosh(y)*cos(x)-1;
    deltaX=J\[-F1;-F2];
    X=X+deltaX;
    x=X(1);
    y=X(2);
    F1check=abs(cosh(y)*sin(x)-x);
    F2check=abs(sinh(y)*cos(x)-y);
    Fmax=max([F1check,F2check] );
    if Fmax<Tol
        break
    end
end


if j<Maxits
    fprintf('p=%-6.2f \n',p);
    fprintf('Re(z)=%-12e \n',X(1));
    fprintf('Im(z)=%-12e \n',X(2));
    fprintf('Iterations=%-5g\n\n',j);

else
    fprintf('p=%-6.2f degrees\n',p);
    fprintf('we did not converge!\n ');
end
Re(p)=X(1);
Im(p)=X(2);
end
```

Jacobian storage

maximum number of iterations

initial guess

update X

update x, y

convergence check

p=1.00
Re(z)=7.497676e+00
Im(z)=2.768678e+00
Iterations=10

p=2.00
Re(z)=1.389996e+01
Im(z)=3.352210e+00
Iterations=9

p=3.00
Re(z)=2.023852e+01
Im(z)=3.716768e+00
Iterations=8

p=4.00
Re(z)=2.655455e+01
Im(z)=3.983142e+00
Iterations=8

p=5.00
Re(z)=3.285974e+01
Im(z)=4.193251e+00
Iterations=8

p=6.00
Re(z)=3.915882e+01
Im(z)=4.366795e+00
Iterations=8

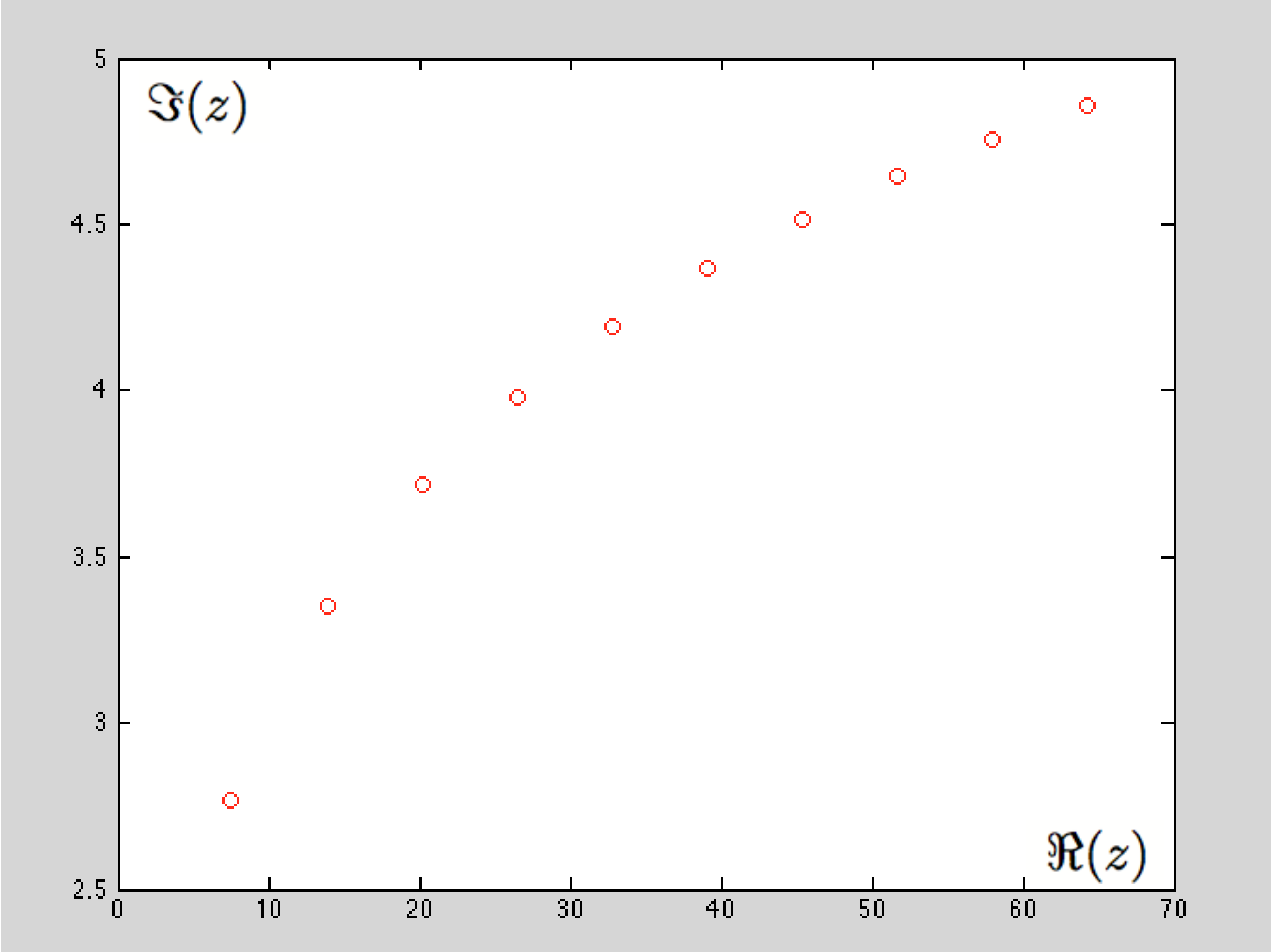
p=7.00
Re(z)=4.545407e+01
Im(z)=4.514640e+00
Iterations=8

p=8.00
Re(z)=5.174677e+01
Im(z)=4.643428e+00
Iterations=8

p=9.00
Re(z)=5.803766e+01
Im(z)=4.757515e+00
Iterations=8

p=10.00
Re(z)=6.432723e+01
Im(z)=4.859917e+00
Iterations=8

FIRST 10 ROOTS



FIRST 10 ROOTS

1	7.497676277776809+2.768678282988873i
2	13.89995971397646+3.352209884853505i
3	20.23851770783222+3.716767679759906i
4	26.55454726549142+3.983141640339900i
5	32.85974100506986+4.193251470431218i
6	39.15881652006499+4.366795117670618i
7	45.45407146435511+4.514640449481303i
8	51.74676830282178+4.643427957051896i
9	58.03766205909427+4.757515118081621i
10	64.32723371328557+4.859916647897096i

LXVI. *Roots of Sin z=z.*

By A. P. Hillman and H. E. Salzer *
(Mathematical Tables Project, National Bureau of Standards, U.S.A.).

[Received March 15, 1943.]

The following table gives the first ten non-zero roots of $\sin z=z$ in the first quadrant to six decimal places. Obviously the roots are symmetrically situated in the four quadrants.

Equating real and imaginary parts of $\sin(x+iy)=x+iy$, one obtains

$$\sin x \cosh y=x \qquad (1)$$

and

$$\cos x \sinh y=y. \qquad (2)$$

Equivalent to (1) and (2) are equations

$$x=\coth y\,(\sinh^2 y-y^2)^{\frac{1}{2}}\equiv x_1(y), \qquad (3)$$

$$x=\operatorname{arc}\cos(y/\sinh y)\equiv x_2(y). \qquad (4)$$

Roots of sin z=z where z=x+iy.

n.	x.	y.
1.........	7·497676	2·768678
2.........	13·899960	3·352210
3.........	20·238518	3·716768
4.........	26·554547	3·983142
5.........	32·859741	4·193251
6.........	39·158817	4·366795
7.........	45·454071	4·514640
8.........	51·746768	4·643428
9.........	58·037662	4·757515
10.........	64·327234	4·859917

The problem of solving $\sin z=z$ is therefore reduced to the finding of the real zeros of the real function $x_1(y)-x_2(y)$. The imaginary part of the n-th root was obtained as follows:—Three values of x_1-x_2 (at interval in y not more than 0·01) were calculated (for the appropriate branch of x_2) in the neighbourhood of G. H. Hardy's approximation

$$y=\log(4n+1)\pi\ \dagger.$$

Quadratic inverse interpolation was then used to find the tabulated y. $x_2(y)$ was taken as the tabulated x. Finally

$$\Delta y=(x_1-x_2)\Big/\frac{d(x_1-x_2)}{dy} \quad \text{and} \quad \Delta x=\Delta y\frac{dx_2}{dy}$$

were calculated as first approximations to the errors in y and x respectively. Δy and Δx were always less than 10^{-7}.