Numerical Integration - Lecture 02

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Outline

- Numerical integration plays a key role in many engineering applications. The integration methods that we examine
  - Trapezoidal Rule
  - Simpson's 1/3 Rule

    Previous Lecture

  - Mid-point Rule
  - Romberg Integration

    This Lecture

  - Gauss Quadrature
  - Multiple Integrals

In this lecture we will also obtain the general form of closed Newton-Cotes formulas to approximate integrals.

Closed Newton-Cotes Formulas (1)

Let P_n be the n^{th} Lagrange interpolating polynomial

$$P_n(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x) \qquad \text{where} \qquad L_{n,k} = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$$

Recall that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{k=0}^{n} (x - x_k)$$

Integrating yields quadrature formulas and their error terms

$$\int_a^b f(x)dx = \int_a^b \sum_{k=0}^{n} f(x_k) L_{n,k}(x) dx + \int_a^b \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{k=0}^{n} (x - x_k) dx$$

$$= \sum_{k=0}^{n} f(x_k) \int_a^b L_{n,k}(x) dx + \frac{1}{(n+1)!} \int_a^b f^{(n+1)}(\xi(x)) \prod_{k=0}^{n} (x - x_k) dx$$

$$= \sum_{k=0}^{n} a_k f(x_k) \quad + \quad E(f) \qquad \text{where} \qquad a_k = \int_a^b L_{n,k}(x) dx$$

Quadrature formula + Error term

Closed Newton-Cotes Formulas (2)

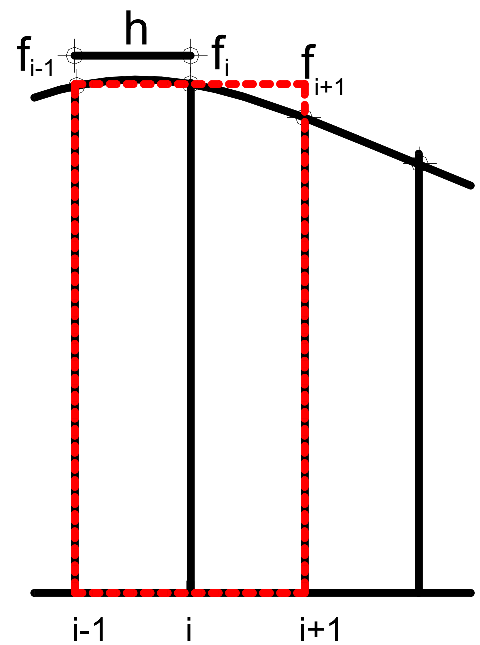
The Closed Newton-Cotes formulas are obtained by integrating Lagrange polynomials of varying degree, n.

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n} a_k f(x_k) \quad \text{where} \quad a_k = \int_a^b \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)} dx$$

n	Integration Rule
1	Trapezoidal Rule
2	Simpson's 1/3 Rule
3	Simpson's 3/8 Rule
etc.	

Mid-Point Rule

The Mid-Point Rule evaluates the function at the mid-point of the interval and multiplies by the width of the interval. Consider the figure on the right.



$$\int_{x_{i-1}}^{x_{i+1}} f(x)dx \approx f(x_i)2h$$

This is also referred to as an Open Newton-Cotes formula with n=0. (A polynomial of degree 0 is a constant). Note that the indexing scheme is different than the closed formulas in that only interior nodes are used in the Open formulas (see text for more information).

Error in Mid-Point Rule:

$$Local\ Error = \frac{h^3}{3} f''(\xi)$$

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Romberg Integration

Romberg Integration uses the composite trapezoidal rule and Richardson's extrapolation to evaluate an integral.

This method generates a table from the data pairs given as shown below:

Built using Trapezoidal rule **Romberg formula**

h_1=assumed	$I_{1,1}$		**Romberg formula**
$h_2 = h_1/2$	$I_{2,1}$	$I_{2,2}$	
$h_3 = h_2/2$	$I_{3,1}$	$I_{3,2}$	$I_{3,3}$
Error Order	h^2	h^4	h^6

The first column is generated using the composite trapezoidal rule

The remaining columns use simple extrapolation formula that do not require additional function evaluations.

Derivation of Romberg Integration

The general form of the Composite Trapezoidal Rule is

$$I = \frac{h}{2}\left[f_0 + f_n + 2\sum_{i=1}^{n-1} f_i \right] - \frac{h^2}{12}(b-a)f''(\xi) + \vartheta(h^4)$$

This is of the form:

$$I = \bar{I} + Ch^2 + Dh^4 + \vartheta(h^6) \qquad \rightarrow (1)$$

where $\bar{I} = \bar{I}(h)$ is the approximation from the Composite Trapezoidal Rule. To compute the first column, we simply refine the mesh starting with a value of h_1.

$$I = \bar{I}_{1,1} + Ch_1^2 + Dh_1^4 + \vartheta(h_1^6) \qquad \rightarrow (2)$$

Now define $h_2 = h_1/2$ and compute

$$I = \bar{I}_{2,1} + Ch_2^2 + Dh_2^4 + \vartheta(h_2^6) \qquad \rightarrow (3)$$

The first subscript indicates the mesh size. The second subscript=1 indicates that this column was computed with the Comp. Trap. Rule.

An Improved Estimate

To generate the first term in the second column $\bar{I}_{2,2}$, we want to eliminate the constant C between equations (2) and (3).

Repeated from before

$$I = \bar{I}_{1,1} + Ch_1^2 + Dh_1^4 + \vartheta(h_1^6) \qquad \rightarrow (2)$$

$$I = \bar{I}_{2,1} + Ch_2^2 + Dh_2^4 + \vartheta(h_2^6) \qquad \rightarrow (3)$$

But since $h_2 = h_1/2$

$$I = \bar{I}_{2,1} + \frac{C}{4} h_1^2 + \frac{D}{16} h_1^4 + \vartheta(h_1^6) \quad \rightarrow (3a)$$

To do this, multiply (3a) by -4 and add to (2) to yield

$$-3I = \bar{I}_{1,1} - 4\bar{I}_{2,1} + \frac{3}{4} Dh_1^4 + \vartheta(h_1^6)$$

Solve for I

$$I = \frac{4\bar{I}_{2,1} - \bar{I}_{1,1}}{3} - \frac{D}{4} h_1^4 + \vartheta(h_1^6) = \bar{I}_{2,2} - 4Dh_2^4 + \vartheta(h_2^6) \quad \rightarrow (4)$$

The first term on the RHS is an improved approximation. The remainder is $\vartheta(h^4)$

Refine and Repeat

To get the third element in the first column, $\bar{I}_{3,1}$, we refine the mesh again, i.,e., $h_3 = h_2/2$ and compute it directly.

$$I = \bar{I}_{3,1} + Ch_3^2 + Dh_3^4 + \vartheta(h_3^6)$$

or

$$I = \bar{I}_{3,1} + \frac{C}{4}h_2^2 + \frac{D}{16}h_2^4 + \vartheta(h_2^6) \rightarrow (5)$$

From the previous slide:

$$I = \bar{I}_{2,1} + Ch_2^2 + Dh_2^4 + \vartheta(h_2^6) \rightarrow (3)$$

Eliminate C between (3) and (5) , (multiply 5) by -4 and add to (3))

$$-3I = \bar{I}_{2,1} - 4\bar{I}_{3,1} + \frac{3}{4}Dh_2^4 + \vartheta(h_2^6)$$

Solve for I

$$I = \frac{4\bar{I}_{3,1} - \bar{I}_{2,1}}{3} - \frac{D}{4}h_2^4 + \vartheta(h_2^6)$$

$$= \bar{I}_{3,2} - \frac{D}{4}h_2^4 + \vartheta(h_2^6) \quad \rightarrow (6)$$

where

$$\bar{I}_{3,2} \equiv \frac{4\bar{I}_{3,1} - \bar{I}_{2,1}}{3}$$

Extrapolate to Higher Order

To compute, $\bar{I}_{3,3}$, we need to eliminate D between equations (4) and (6). From the previous slides:

$$I = \bar{I}_{2,2} - 4Dh_2^4 + \vartheta(h_2^6) \quad \rightarrow (4) \qquad\qquad I = \bar{I}_{3,2} - \frac{D}{4}h_2^4 + \vartheta(h_2^6) \quad \rightarrow (6)$$

To eliminate D, multiply (6) by 16 and subtract from (4)

$$-15I = \bar{I}_{2,2} - 16\bar{I}_{3,2} + \vartheta(h_2^6)$$

Solve for I

$$I = \frac{16\bar{I}_{3,2} - \bar{I}_{2,2}}{15} + \vartheta(h_2^6)$$

$$I = \bar{I}_{3,3} + \vartheta(h_2^6)$$

This process can be continued by computing more terms (rows) in the first column and then expanding the Romberg table to the right (i.e. adding columns).

Generalization of the Extrapolation Formula

h_1=assumed	$\bar{I}_{1,1}$		
$h_2 = h_1/2$	$\bar{I}_{2,1}$	$\bar{I}_{2,2} = \dfrac{4\bar{I}_{2,1} - \bar{I}_{1,1}}{3}$	
$h_3 = h_2/2$	$\bar{I}_{3,1}$	$\bar{I}_{3,2} = \dfrac{4\bar{I}_{3,1} - \bar{I}_{2,1}}{3}$	$\bar{I}_{3,3} = \dfrac{16\bar{I}_{3,2} - \bar{I}_{2,2}}{15}$
Error Order	h^2	h^4	h^6

The extrapolation formula generalizes to:

$$\bar{I}_{i,j} = \frac{4^{j-1}\bar{I}_{i,j-1} - \bar{I}_{i-1,j-1}}{4^{j-1} - 1} \quad \forall \begin{array}{l} i=2,3,\ldots..n \\ j=2,3,\ldots i \end{array}$$

Summary

In this lecture, we have

- Developed a precise mathematical approach for obtaining the Newton-Cotes formulas
- Presented the Mid-Point rule as an example of the Open Newton-Cotes formulas
- Derived the first two extrapolation formula for Romberg Integration
- Generalized the extrapolation and constructed a Romberg table