Solution of Ordinary Differential Equations (Initial Value Problems) - Lecture 03

Dr. Serhat Hosder

Associate Professor of Aerospace Engineering

Mechanical and Aerospace Engineering

290B Toomey Hall

Missouri S&T

Rolla, MO 65409

Phone: 573-341-7239

E-mail: hosders@mst.edu

Outline

In this lecture we will

- introduce a general framework for deriving all of the Adams family of Multi-Step methods via polynomial approximation

- discuss different approaches for solving implicit equations
  - Predictor-Corrector
  - Linearization
- compare Euler Explicit and Implicit methods by example

A More General Formulation for the Derivation of Adams Formulas

Back to the Initial Value Problem

$$\frac{dy}{dt} = f(t, y) \quad ; \quad y(a) = y_0 \qquad a \le t \le b$$

Rearranging and integrating between t_i and t_{i+1} yields

$$\int_{t_i}^{t_{i+1}} dy = y_{i+1} - y_i = \int_{t_i}^{t_{i+1}} f(t, y(t))dt$$

Note that this expression is exact. As we have done before, we can approximate the function f with a polynomial.

$$f(t, y(t)) \approx P_n(t, y(t))$$

Where $\boldsymbol{P_n}$ is an $\boldsymbol{n^{th}}$ degree Polynomial that passes through specified points (e.g. $\{t_i, f_i\}$, …).

Example Derivation

Consider the Lagrange Polynomial of degree zero that passes thru i

$$f(t,y(t)) \approx P_0(t,y(t)) = f(t_i,y(t_i)) \equiv f_i$$

Note that given t_i and $y(t_i)$, f_i is a known number (a constant)

$$\int_{t_i}^{t_{i+1}} P_0(t,y(t))dt = \int_{t_i}^{t_{i+1}} f_i dt = f_i \int_{t_i}^{t_{i+1}} dt = f_i h$$

Thus, our approximation to the solution to the ODE will be

$$w_{i+1} - w_i = \int_{t_i}^{t_{i+1}} P_0(t,w(t))dt = f_i h$$

Euler Explicit Approximation

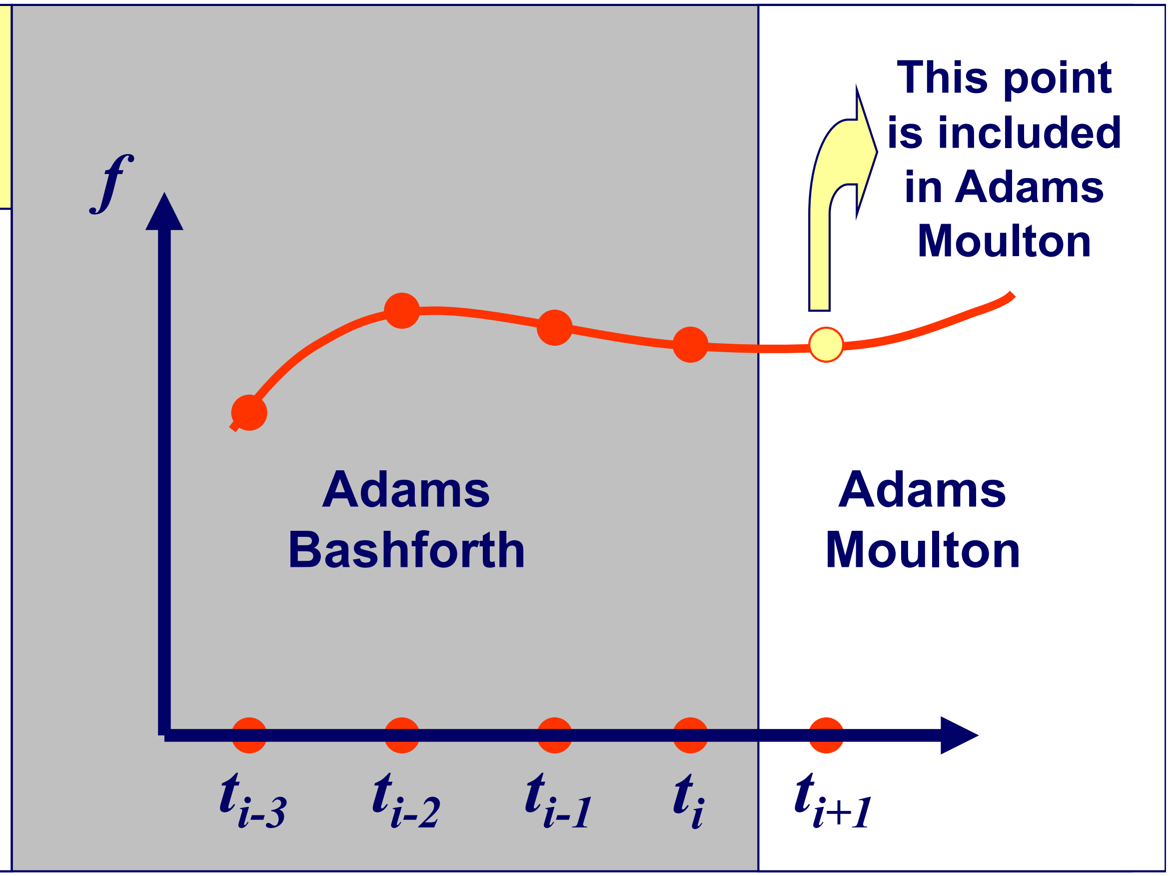
$$y_{i+1} - y_i = \int_{t_i}^{t_{i+1}} f(t,y(t))dt$$

Exact

Note that this approximation is the Euler Explicit Scheme. To get the 2-Step AB method, pass P_1 through {i-1,i}, etc..

Adams-Bashforth and Adams-Moulton

	Data Points Used	
P_n	Adams Bashforth	Adams Moulton
P_0	$\{i\}$	$\{i+1\}$
P_1	$\{i-1, i\}$	$\{i, i+1\}$
P_2	$\{i-2, i-1, i\}$	$\{i-1, i, i+1\}$
	Explicit	Implicit



Strictly speaking, the Adams methods start with P_1. The methods for P_0 are the Euler Explicit and Euler Implicit Methods.

Euler Implicit

Consider the Lagrange Polynomial of degree zero through i+1

$$f(t,y(t)) \approx P_0(t,y(t)) = f(t_{i+1},y(t_{i+1})) \equiv f_{i+1}$$

Thus, the Euler Implicit algorithm is:

$w_{i+1}-w_i = \int_{t_i}^{t_{i+1}} P_0(t,w(t))dt$ $= f(t_{i+1},w(t_{i+1}))h$ $w_{i+1}-w_i = f_{i+1}h$ **Euler Implicit**	$w_{i+1}-w_i = \int_{t_i}^{t_{i+1}} P_0(t,w(t))dt$ $= f(t_i,w(t_i))h$ $w_{i+1}-w_i = f_i h$ **Euler Explicit**

Note however that $f_{i+1} = f(t_{i+1},w_{i+1})$. The unknown w_{i+1} appears **implicitly** in the function and cannot be solved for directly unless f is a linear function in w. We can predict and correct or we can linearize.

Predictor and Corrector

We can solve this implicit equation by starting with a predicted value, w^p, from an explicit method (e.g. from the Euler Explicit method):

$$w^c{}_{i+1} = w_i + hf(t_{i+1}, w^p_{i+1})$$

Feed the corrected value back in as the predicted value until convergence

Iteration Loop for fixed "i"

One can then use the above to obtain a corrected value, w^c. The process can be repeated by using the corrected value from the previous step as the predicted value on the next step keeping i fixed.

When two successive values are sufficiently close (to a user-defined tolerance), then the iterative process is stopped on this "i+1st" step and one moves on to the "i+2nd" step.

Newton Linearization

We can also solve this implicit equation by linearization. We shall consider a "Newton" linearization. The (non-linear) Euler Implicit algorithm is

$$w_{i+1} = w_i + hf(t_{i+1}, w_{i+1}) = w_i + hf_{i+1} \quad \Rightarrow \boxed{\Delta w_i = hf_{i+1}}$$

Linearize f with respect to w by expanding it in a Taylor series

$$f_{i+1} = f_i + \left(\frac{\partial f}{\partial w}\right)_i \Delta w_i + \ldots \qquad \text{where} \quad \Delta w_i = w_{i+1} - w_i$$

Substitute

$$\Delta w_i = h\left[f_i + \left(\frac{\partial f}{\partial w}\right)_i \Delta w_i\right]$$

Rearrange

$$\frac{\Delta w_i}{h} - \left(\frac{\partial f}{\partial w}\right)_i \Delta w_i = f_i$$

Operator Form

$$\Rightarrow \left[\frac{1}{h} - \left(\frac{\partial f}{\partial w}\right)_i\right]\Delta w_i = f_i$$

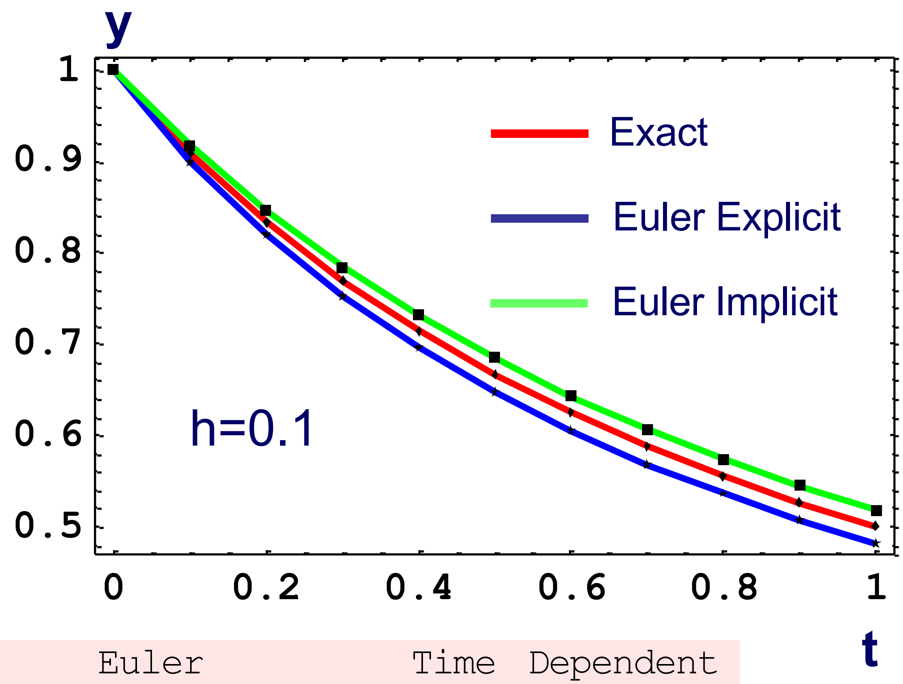
Linear in Δw

Euler Implicit Example

Here is an IVP example:

$$\frac{dy}{dt} = f(y) = -y^2 \qquad y(0) = 1$$

The algorithm is: $\left[\frac{1}{h} + 2w_i\right]\Delta w_i = f_i$



Exact	Euler Explicit	Time Dependent Error	Euler Implicit	Time Dependent Error
1.	1.	0.	1.	0.
0.909091	0.9	- 0.00909091	0.916667	0.00757576
0.833333	0.819	- 0.0143333	0.845657	0.0123239
0.769231	0.751924	- 0.0173069	0.784489	0.0152583
0.714286	0.695385	- 0.0189008	0.731293	0.0170074
0.666667	0.647029	- 0.0196377	0.684638	0.0179712
0.625	0.605164	- 0.0198357	0.64341	0.0184102
0.588235	0.568542	- 0.0196934	0.606732	0.018497
0.555556	0.536218	- 0.0193376	0.573904	0.018348
0.526316	0.507465	- 0.0188508	0.544358	0.0180425
0.5	0.481713	- 0.0182871	0.517635	0.0176351

Adams-Moulton Method

Some of the Adam-Moulton formula are:

Two-Step Method: (polynomial through i-1,i, i+1}

$$w_{i+1} = w_i + \frac{h}{12}\left[5f_{i+1} + 8f_i - f_{i-1}\right] \quad ;\left[\mathrm{TE} = -\frac{1}{24}y^{iv}(\mu_i)h^3\right]$$

$$where\ \mu \in \left(t_{i\text{-}1}, t_{i+1}\right)$$

Three-Step Method: {polynomial through i-2,i-1,i,i+1}

$$w_{i+1} = w_i + \frac{h}{24}\left[9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2}\right] \quad ;\left[\mathrm{TE} = -\frac{19}{720}y^{v}(\mu_i)h^4\right]$$

$$where\ \mu \in \left(t_{i\text{-}2}, t_{i+1}\right)$$

Outline

In this lecture we have

- Discussed the Adams family of Multi-Step methods for solving ODE's.
- Introduced a general framework for deriving all of the Adams formulas via polynomial approximation
- Discussed different approaches for solving implicit equations
  - Predictor-Corrector
  - Linearization
- Compared Euler Explicit and Implicit methods by example