Solution of Linear Set of Equations – 03

Dr. Serhat Hosder

Associate Professor of Aerospace Engineering

Mechanical and Aerospace Engineering

290B Toomey Hall

Missouri S&T

Rolla, MO 65409

Phone: 573-341-7239

E-mail: hosders@mst.edu

Outline

Previously, we covered Gaussian elimination (Section 6.1 in your text book) and its computational work. We will continue with

1. Gauss-Jordan Method
2. Partial pivoting (Section 6.2)
3. LU Decomposition (Section 6.5)
4. Determinants (Section 6.4)

Gauss-Jordan Method

The Gauss-Jordan method is a modification of the Gauss elimination method. It converts the system of equations into diagonal form by putting zeros above the diagonal as well as below it on each elimination step. The solution is obtained by dividing the augmented right hand side vector by the diagonal entries.

Consider our previous example:

$$E_1 = 4x_1+2x_2+4x_3=20$$

$$E_2 = 2x_1+2x_2+3x_3=15$$

$$E_3 = -8x_1-2x_2+16x_3=36$$

The augmented matrix for the given set of equations is:

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 2 & 3 \\ -8 & -2 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \\ 36 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 4 & 2 & 4 & 20 \\ 2 & 2 & 3 & 15 \\ -8 & -2 & 16 & 36 \end{array}\right]$$

Gauss-Jordan Example

Step 1:

$$\left[\begin{array}{ccc|c} 4 & 2 & 4 & 20 \\ 2 & 2 & 3 & 15 \\ -8 & -2 & 16 & 36 \end{array}\right] \begin{array}{l} \\ E_2 - \left(2/4\right) E_1 \\ E_3 - \left(-8/4\right) E_1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 4 & 2 & 4 & 20 \\ 0 & 1 & 1 & 5 \\ 0 & 2 & 24 & 76 \end{array}\right]$$

Step 2:

$$\left[\begin{array}{ccc|c} 4 & 2 & 4 & 20 \\ 0 & 1 & 1 & 5 \\ 0 & 2 & 24 & 76 \end{array}\right] \begin{array}{l} E_1 - \left(2/1\right) E_2 \\ \\ E_3 - \left(2/1\right) E_2 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 4 & 0 & 2 & 10 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 22 & 66 \end{array}\right]$$

Step 3:

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 10 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 22 & 66 \end{array}\right] \begin{array}{l} E_1 - \left(2/22\right) E_3 \\ E_2 - \left(1/22\right) E_3 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 4 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 22 & 66 \end{array}\right]$$

Gauss-Jordan Concluded

Step 4:

Solve for the solution vector (i.e. divide the adjusted right hand side by the diagonal entries)

$$\left[\begin{array}{ccc|c} 4 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 22 & 66 \end{array}\right]$$

The solution to the system of equations is:

$$x_3 = 66/22 = 3$$

$$x_2 = 2/1 = 2$$

$$x_1 = 4/4 = 1$$

Pivoting

Let's apply the Gaussian elimination method to the problem:

$$x_1 + x_2 + x_3 = 1 \rightarrow (E_1)$$

$$x_1 + x_2 + 2x_3 = 2 \rightarrow (E_2)$$

$$x_1 + 2x_2 + 2x_3 = 1 \rightarrow (E_3)$$

The augmented matrix and the Gaussian elimination operations done on it are given by:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \end{array}\right] \Leftrightarrow \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ E_2 - E_1 \\ E_3 - E_1 \end{array}$$

Notice that the diagonal element in the pivot position becomes zero. Further application of Gaussian elimination will result in division by zero. Even if the pivot element is non-zero, a precision problem will appear if the element is very small compared to all the other elements. In such a case, pivoting is useful.

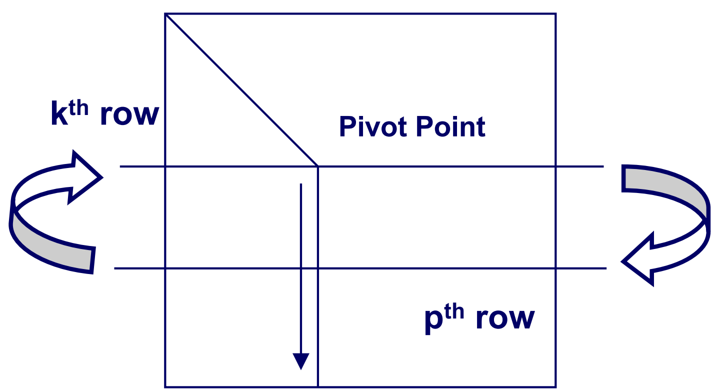
Partial Pivoting

On the k^{th} elimination step, the procedure of interchanging the k^{th} row with the row below the diagonal containing the largest element in absolute value in the pivot column is called Partial Pivoting.

i.e., find the integer p that satisfies

$$\left|a_{pk}^{(k)}\right| = \max_{k \le i \le n} \left|a_{ik}^{(k)}\right|$$

and interchange rows p and k. This improves the precision of the numerical computation and is frequently applied in solving engineering problems.



Multiple Right Hand Sides

A system of equations with multiple right hand sides represented as
$Ax_1=b_1; Ax_2=b_2; Ax_3=b_3;\ldots$ or equivalently as $AX=B$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_2 \cdots \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_m \right] = \left[\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}_1 \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}_2 \cdots \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}_m \right]$$

The LHS can be converted to upper triangular form and all RHS vectors adjusted during one elimination phase. Then, ***m*** backward substitutions can be performed for each solution vector

$$\left[\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{matrix} \right| \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}_1 \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}_2 \cdots \left. \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}_m \right]$$

Motivation for Lower/Upper (LU) Decomposition

Many problems in engineering, occur with dependence. i.e.,

$Ax_1=b_1;$ $\quad Ax_2=b_2(x_1);$ $\quad Ax_3=b_3(x_2);$ $\quad \ldots$

The Gaussian elimination procedure for problems with multiple right hand sides that we examined is not particularly efficient for this class of problems since all of the the right hand side vectors need to be known in advance.

LU decomposition methods (note that they are not unique) have been proposed to deal with this issue. In reality, LU decompositions are minor variations on Gaussian elimination in which the decomposition matrices are stored for future use.

LU Decomposition Method

If the matrix A is non-singular matrix, then the rows of A can be ordered such that an LU decomposition exists.

$$A = LU$$

The matrix A is written as the product of L (Lower triangular matrix) and U (Upper triangular matrix). Consider the problem, $Ax=b$.

$$\Rightarrow \quad LUx \quad = b$$

$$\Rightarrow \quad L(Ux) = b$$

$$\Rightarrow \quad L\,x^{*} \quad = b \text{ (where } x^{*} = Ux)$$

i.e., this decomposes $Ax=b$ into 2 problems:

$L\,x^{*} = b$ (Forward substitution) and

$Ux = x^{*}$ (Backward substitution)

By storing the LU decomposition, one can solve $Ax=b$ efficiently with multiple right hand side vectors, including dependent ones.

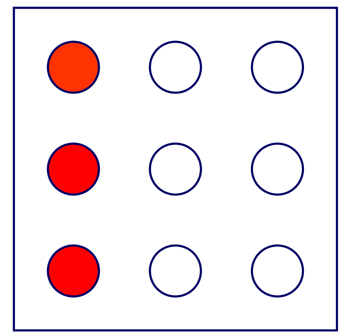
Crout's (LU) Decomposition

Crout's method is one of many possible $\boldsymbol{LU}$ decompositions in which 1's are put on the diagonal of the $\boldsymbol{U}$ matrix. We can determine the $\boldsymbol{L}$ and $\boldsymbol{U}$ elements by the definition $\boldsymbol{LU}=\boldsymbol{A}$.

#1: Solve for 1st column of $\boldsymbol{L}$:

Multiply the rows of $\boldsymbol{L}$ with the 1st column of $\boldsymbol{U}$:

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



$\Rightarrow \boldsymbol{l_{11}} = \boldsymbol{a_{11}}$ (Row 1 x Column 1)

$\Rightarrow \boldsymbol{l_{21}} = \boldsymbol{a_{21}}$ (Row 2 x Column 1)

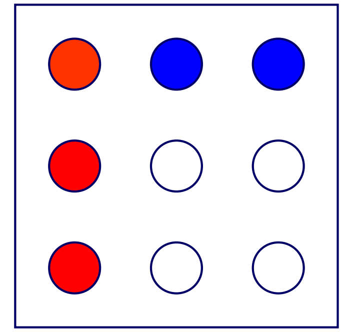
$\Rightarrow \boldsymbol{l_{31}} = \boldsymbol{a_{31}}$ (Row 3 x Column 1)

We now have all the elements of the first column of the L matrix.

Crout's Method for a 3x3 system continued

#2: Solve for 1st row of U :

Multiply 1st row of L matrix with the 2nd & 3rd columns of U



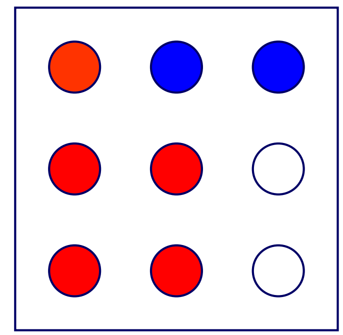
$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$l_{11}u_{12} = a_{12} \Rightarrow u_{12}=a_{12}/l_{11}$$

$$l_{11}u_{13} = a_{13} \Rightarrow u_{13}=a_{13}/l_{11}$$

#3: Solve for 2nd column of L:

Multiply 2nd and 3rd rows of L with 2nd column of U



$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$l_{21}u_{12}+l_{22}=a_{22}$$

$$\Rightarrow l_{22} = a_{22}-l_{21}u_{12}$$

$$l_{31}u_{12}+l_{32}=a_{32}$$

$$\Rightarrow l_{32} = a_{32}-l_{31}u_{12}$$

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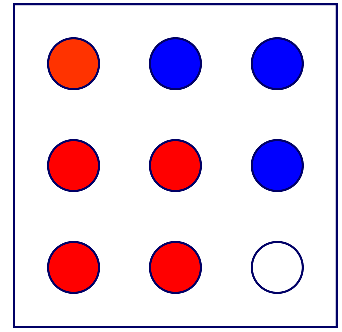
Crout's Method for a 3x3 system concluded

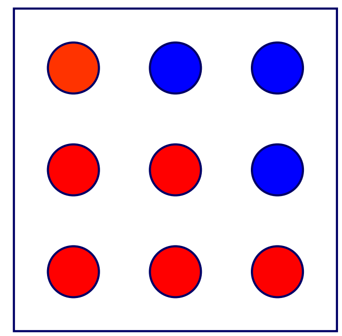
#4: Solve for 2nd row of U :

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



$l_{21}u_{13}+l_{22}u_{23}=a_{23}$

$\Rightarrow u_{23}=(a_{23}-l_{21}u_{13})/l_{22}$



#5: Solve for 3rd column of L:

Multiply the 3rd row of L with the 3rd column of U

$l_{33}=[a_{33}-(l_{31}u_{13}+l_{32}u_{23})]$

Generalization of Crout's LU Decomposition (1)

In Crout's method, the diagonal entries of the U matrix contain 1's. In this approach, the individual elements can be obtained from:

Fix the j^{th} column and compute

For j=1 : $$l_{i1} = a_{i1} \quad (i=1,2,..n)$$

For j=2,3,…,n : $$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \qquad \forall\, i = j, j+1, \ldots n$$

Fix the i^{th} row and compute

For i=1 : $$u_{1j} = \frac{a_{1j}}{l_{11}} \quad (j=2,3,..n)$$

For i=2,3,…,n-1 : $$u_{ij} = \left[a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \right] \Big/ l_{ii} \qquad \forall\, j = i+1, i+2, \ldots n$$

Generalization of Crout's LU Decomposition (2)

Obtain the intermediate variable x^* by forward substitution

$x_1^* = b_1/l_{11}$ followed by

$$x_i^* = \left[b_i - \sum_{k=1}^{i-1} l_{ik} x_k^* \right] / l_{ii} \qquad \forall\, i = 2,3,\ldots n$$

Then solve the linear system by backward substitution

$x_n = x^*_n$ followed by

$$x_i = x_i^* - \sum_{k=i+1}^{n} u_{ik} x_k \qquad \forall\, i = n-1, n-2, \ldots 1$$

Determinants

The determinant of a triangular matrix is the product of its diagonal entries.

$$\det(L) = \prod_{i=1}^{n} l_{ii} \qquad \det(U) = \prod_{i=1}^{n} u_{ii}$$

Properties of determinant of a matrix:

1. If a matrix A' is obtained from A by one row interchange, then

$$det(A') = -det(A)$$

If there are r row interchanges, then $det(A) = (-1)^r det(A')$

2. If A is of order nxn and k is a scalar, then

$$det(kA) = k^n det(A)$$

3. $det(A^T) = det(A)$

4. $det(A.B) = det(A).det(B)$

5. Determinant by Crout's method for A: If with A' is obtained from A by r row interchanges and $A' = LU$ *then*

$$det(A') = det(LU) = det(L).det(U) = det(L)$$

$$\det(A) = (-1)^r \prod_{i=1}^{n} l_{ii}$$

Summary

- We described Gauss-Jordan elimination and worked an example
- We have seen partial pivoting which is used to improve numerical precision and to prevent division by zero in Gauss elimination.
- We discussed problems with multiple RHS vectors. The conversion of the matrix to triangular form only needs to be done once at a cost of $O(n^3)$ operations.
- Derived the LU decomposition using Crout's method for a $3x3$ system and extended it to an nxn system
- Discussed computing determinants of a matrix from its LU decomposition