Solution of Ordinary Differential Equations (Initial Value Problems) - Lecture 02

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Outline

- Multi-Step Methods for the solution of an Initial Value Problems

- Adams-Bashforth Methods (Open or explicit multi-step methods)

Multi-Step Methods

Single step methods use data at a single point to advance the solution to the next step. The Euler Explicit method is an example of such a method.

Methods that use data at more than one mesh point to determine the next approximation to the solution of the ODE are called multi-step methods.

Multi-step methods require starting values that should be obtained by self-starting methods of equivalent order.

Not surprisingly, multi-step methods can be derived by integrating polynomials that pass through specified data points.

We will begin our discussion of multi-step methods by examining the Adams-Bashforth family of explicit multi-step methods.

Adams-Bashfort Method (Open or Explicit Formulas)

Again, we consider the initial value problem

$$\frac{dy}{dt} = f(t,y) \quad ; \quad y(a) = y_0 \qquad\qquad a \le t \le b$$

Expand $y(t_{i+1}) \equiv y_{i+1}$ in a Taylor series about t_i

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \cdots$$ But $$y' = f; \quad y'' = f'; \quad y''' = f''; \quad etc.$$

i.e.,

$$y_{i+1} = y_i + hf_i + \frac{h^2}{2} f_i' + \frac{h^3}{6} f_i'' + \cdots$$

Replace f_i' by a first order backward difference + it's error term

$$y_{i+1} = y_i + h\left\{ f_i + \frac{h}{2}\left[\frac{f_i - f_{i-1}}{h} + \frac{h}{2} f_i'' + \vartheta(h^2) \right] + \frac{h^2}{6} f_i'' + \cdots \right\}$$

Adams-Bashfort Method (Open or Explicit Formulas)

Grouping terms yields:

$$y_{i+1} = y_i + h\left[\frac{3}{2}f_i - \frac{1}{2}f_{i-1}\right] + \frac{5}{12}h^3 f_i'' + \vartheta(h^4)$$

Thus the Adams-Bashforth Two-Step Explicit Method is:

$$w_{i+1} = w_i + h\left[\frac{3}{2}f_i - \frac{1}{2}f_{i-1}\right] \quad \forall\, i = 1,2,\cdots N-1$$

with $w_0 = y_0, \quad w_1 = \alpha_1$

Note that the method is not self-starting. That is to say that it requires some method to generate w_1 to get it started. The local truncation error is $\vartheta(h^2)$.

The higher order Adams-Bashforth formulas can be derived in a similar fashion. They can be generalized using backward difference polynomials.

Higher-Order Adams Bashforth Methods

Adams-Bashforth three step method (i=2,3,…N-1)

$$w_{i+1} = w_i + \frac{h}{12}\left[23 f_i - 16 f_{i-1} + 5 f_{i-2}\right] \quad ;\left[\mathrm{TE} = \frac{3}{8} y^{iv}(\mu_i) h^3\right]$$

Adams-Bashforth four step method (i=3,4,…N-1)

$$w_{i+1} = w_i + \frac{h}{24}\left[55 f_i - 59 f_{i-1} + 37 f_{i-2} - 9 f_{i-3}\right] \quad ;\left[\mathrm{TE} = \frac{251}{720} y^{v}(\mu_i) h^4\right]$$

Adams-Bashforth five step method (i=4,5,…N-1)

$$w_{i+1} = w_i + \frac{h}{720}\left[1901 f_i - 2774 f_{i-1} + 2616 f_{i-2} - 1274 f_{i-3} + 251 f_{i-4}\right]$$

$$;\left[\mathrm{TE} = \frac{95}{288} y^{vi}(\mu_i) h^5\right]$$

Starting values for each of these methods must be obtained. (Runge-Kutta is a common approach for starting these methods).

Adams-Bashforth 2 Step Example

This is our last example that we did with the Euler explicit method

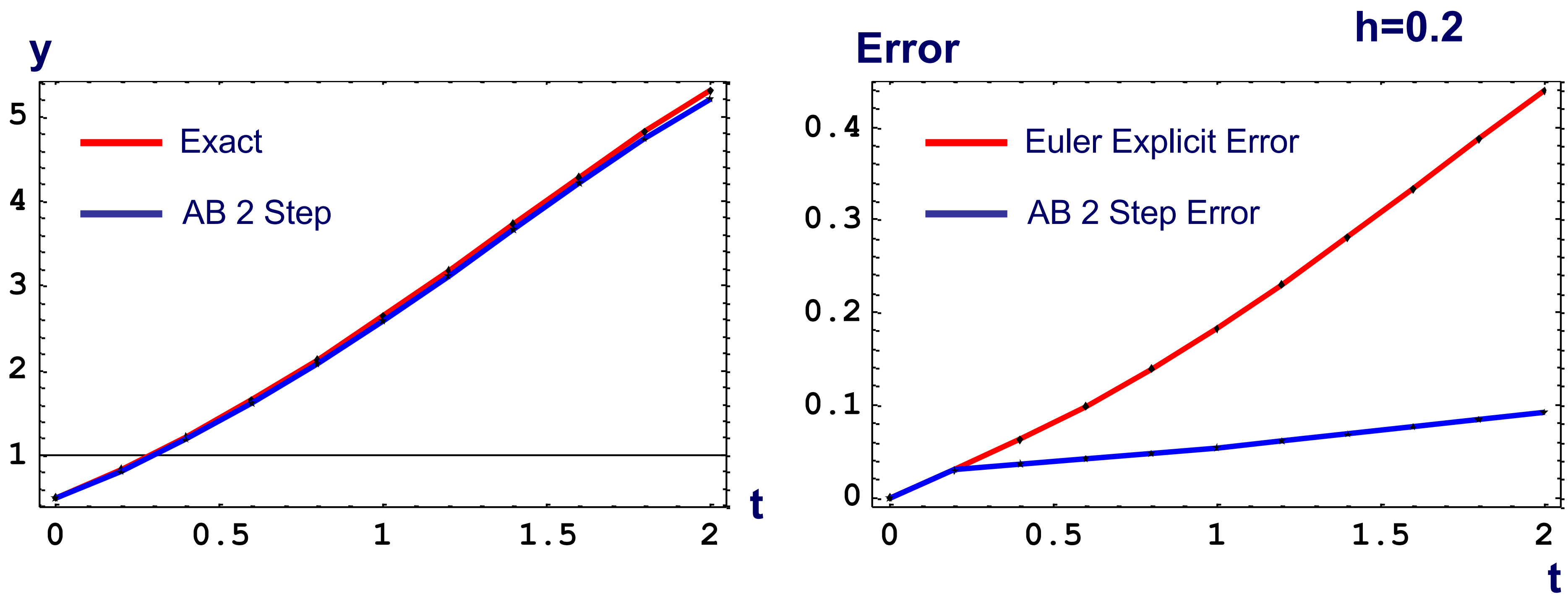
$$\frac{dy}{dt} = y - t^2 + 1 \qquad y(0) = 0.5 \qquad 0 \le t \le 2$$

The exact solution is: $y(t) = (1+t)^2 - e^t / 2$ **h=0.2**

i	Time	Exact Solution	Euler Explicit	Euler Exp. Error	Adams-Bash. 2-Step	AB 2Step Error
0	0.	0.5	0.5	0.	0.5	0.
1	0.2	0.829299	0.8	-0.0292986	0.8	-0.0292986
2	0.4	1.21409	1.152	-0.0620877	1.178	-0.0360877
3	0.6	1.64894	1.5504	-0.0985406	1.6074	-0.0415406
4	0.8	2.12723	1.98848	-0.13875	2.07982	-0.0474095
5	1.	2.64086	2.45818	-0.182683	2.58703	-0.0538331
6	1.2	3.17994	2.94981	-0.23013	3.11915	-0.0607897
7	1.4	3.7324	3.45177	-0.280627	3.66419	-0.0682053
8	1.6	4.28348	3.95013	-0.333356	4.20754	-0.0759458
9	1.8	4.81518	4.42815	-0.387023	4.73138	-0.0837964
10	2.	5.30547	4.86578	-0.439687	5.21404	-0.0914319

Euler explicit method was used to start the AB 2-step method.
Note the much smaller error of AB 2 step as time advances.

Adams-Bashforth 2 Step Example



Note that the errors are the same after the first step since I used the Euler explicit method to start the AB 2-Step method. However, the errors are much smaller with the AB 2-Step method as time advances.

Summary

In this lecture we have

- Learnt Adams-Bashforth (explicit) family of schemes obtained using backward differences
- Demonstrated the application of 2-step Adams-Bashforth method on an example problem.