Solution of Ordinary Differential Equations (Initial Value Problems) - Lecture 05

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Outline

In this lecture we will

- review the most commonly used form of 4 stage Runge-Kutta scheme
- work on two example problems using Euler explicit and 4 stage Runge-Kutta scheme
  - Problem with a high frequency content
  - Problem with discontinuity

Four-Stage Runge-Kutta

The 4-Stage Runge-Kutta methods are the most popular. This involves comparing terms to fourth order and results in 11 equations in 13 unknowns. This leads to a 2-parameter family of schemes that yield 4th order accuracy.

$$
\boxed{
\begin{aligned}
w_{i+1} &= w_i + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] \\
k_1 &= \Delta t \, f(t_i, w_i) \\
k_2 &= \Delta t \, f\left(t_i + \frac{\Delta t}{2}, w_i + \frac{k_1}{2}\right) \\
k_3 &= \Delta t \, f\left(t_i + \frac{\Delta t}{2}, w_i + \frac{k_2}{2}\right) \\
k_4 &= \Delta t \, f\left(t_i + \Delta t, w_i + k_3\right)
\end{aligned}
}
$$

The most common choice of parameters is shown on the right. Note that intermediate function evaluations are required.

Runge-Kutta (RK) Example 1

This example has a lot of high frequency content

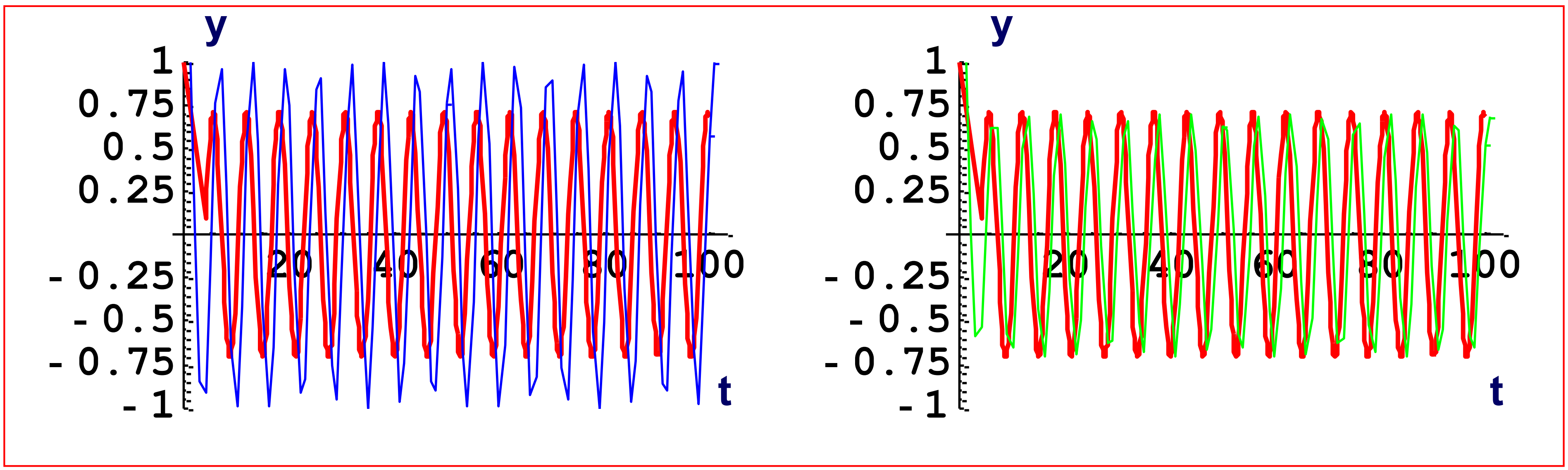
$$\frac{dy}{dt} = -y - \sin(t) \qquad y(0) = 1$$

Use Euler Explicit and 4 stage Runge-Kutta to numerically solve ***y*** between ***t=0*** and ***t=100***

The exact solution is

$$\left\{\left\{y[t] \to \frac{1}{2} e^{-t} \left(1 + e^{t} \mathrm{Cos}[t] - e^{t} \mathrm{Sin}[t]\right)\right\}\right\}$$

$h = \Delta t = 1.0$ — Euler Explicit — Exact — 4 Stage Runge-Kutta



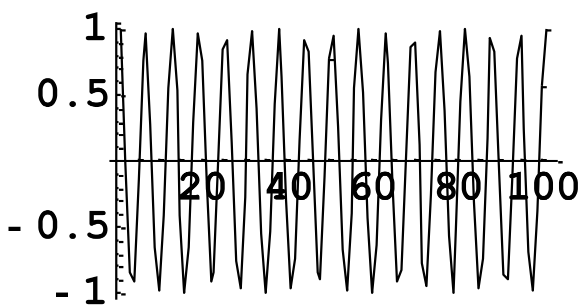
Note the significant improvement with the 4-Stage RK method.

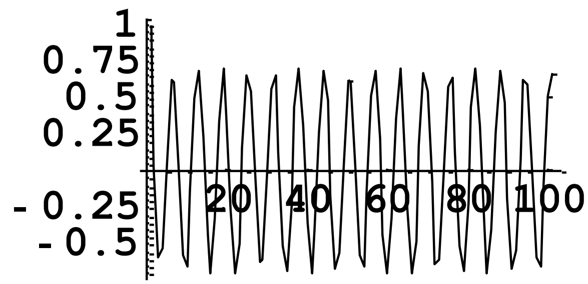
RK Example 1 – Error for Δt=1.0

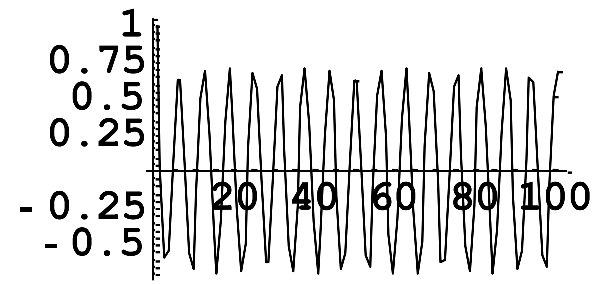
The Step Size for this calculation was $h = \Delta t = 1.0$

Euler Explicit | **4 Stage Runge-Kutta** | **Exact**

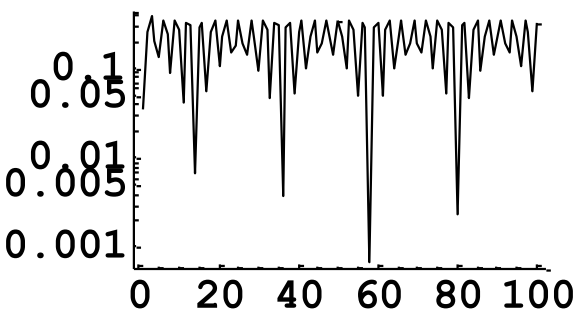
Solution

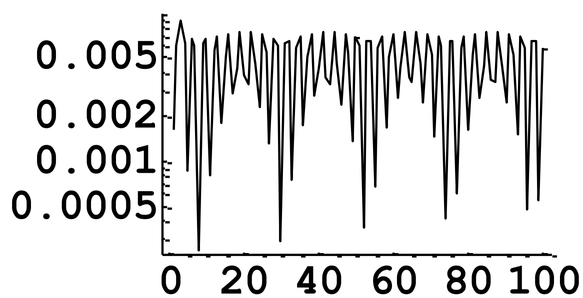






Error





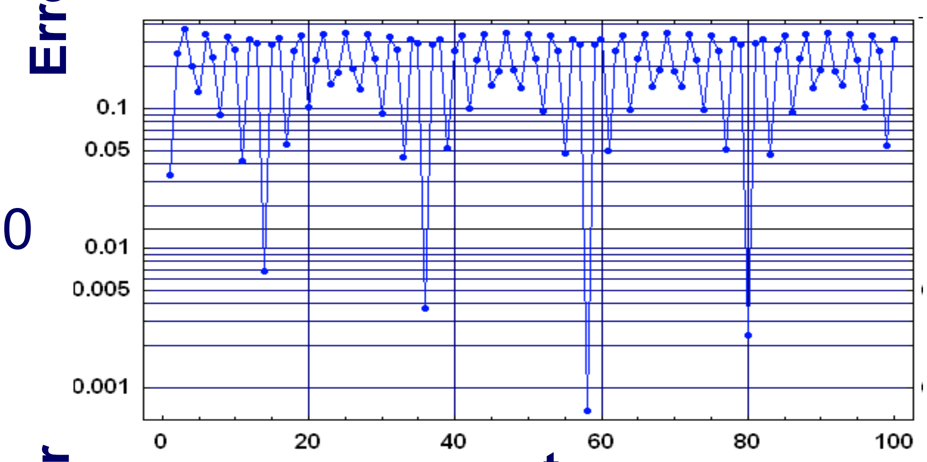
Error=|Sol_{exact}-$Sol_{numerical}$|

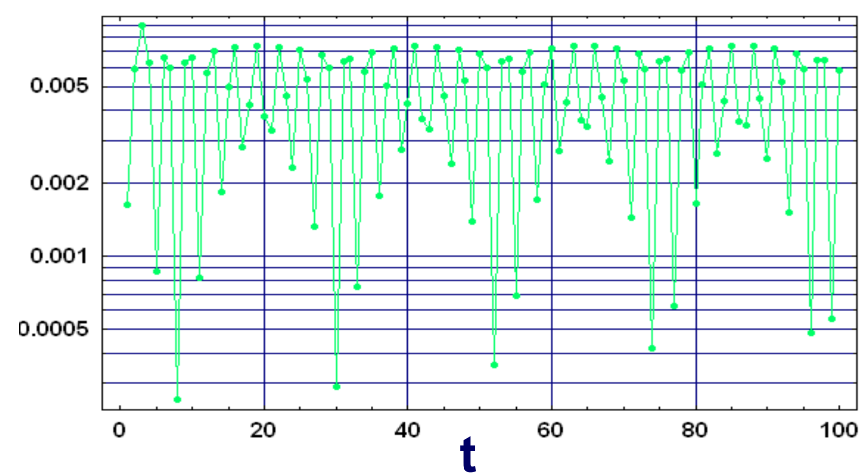
Note the magnitude of the errors for this step size.

RK Example 1 – Error for Δt=1.0 & 0.1

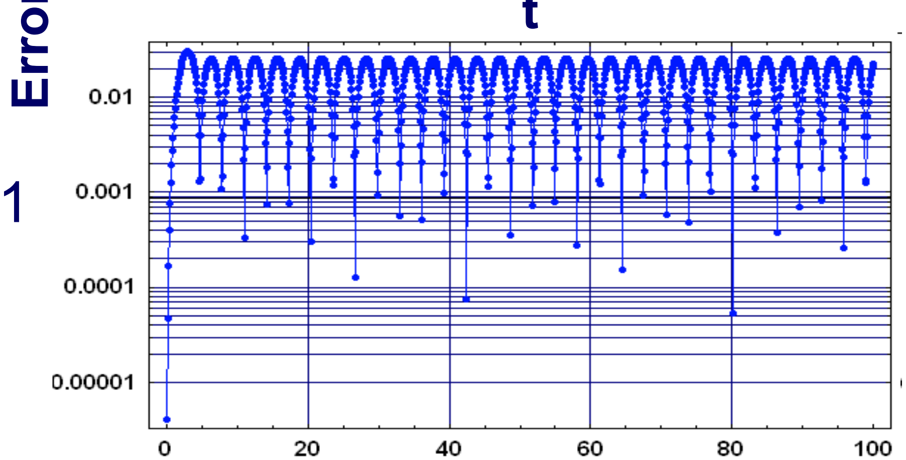
Euler Explicit | **4 Stage Runge-Kutta**

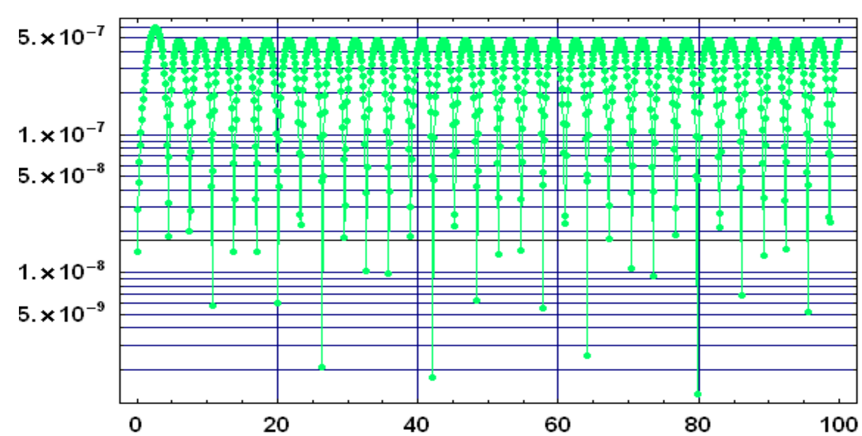
Δt=1.0





Δt=0.1





Effect of decreasing the step size 1 order of magnitude

The error of Euler Explicit went down **one** order of magnitude

The error of 4 stage Runge-Kutta went down **four** orders of magnitude

Runge-Kutta (RK) Example 2

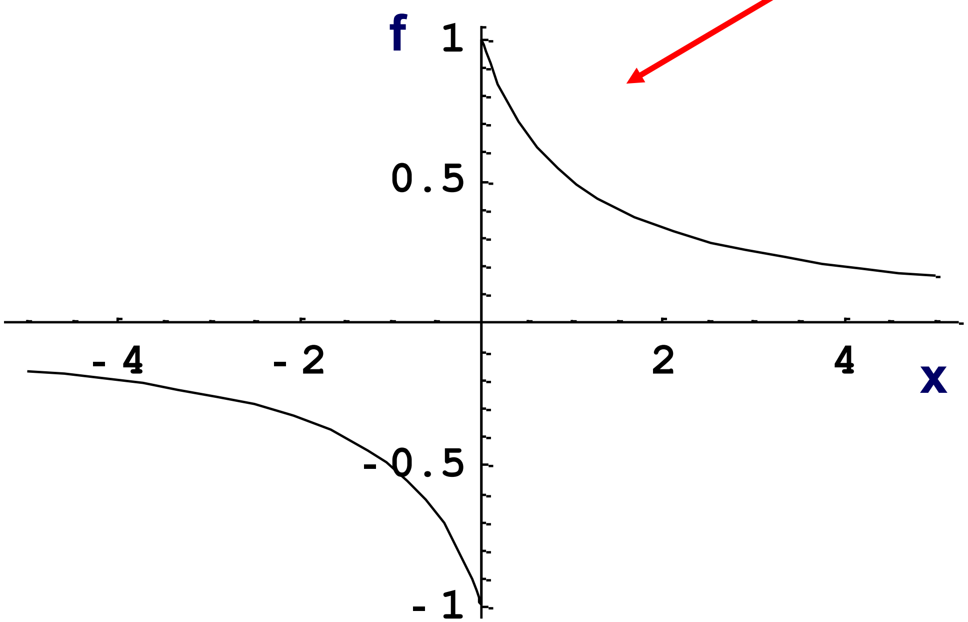
This example has a discontinuity in the forcing function.

$$\frac{dy}{dx} = f(x) = \begin{cases} 1/(x-1) & x<0 \\ 1/(x+1) & x>0 \end{cases}$$

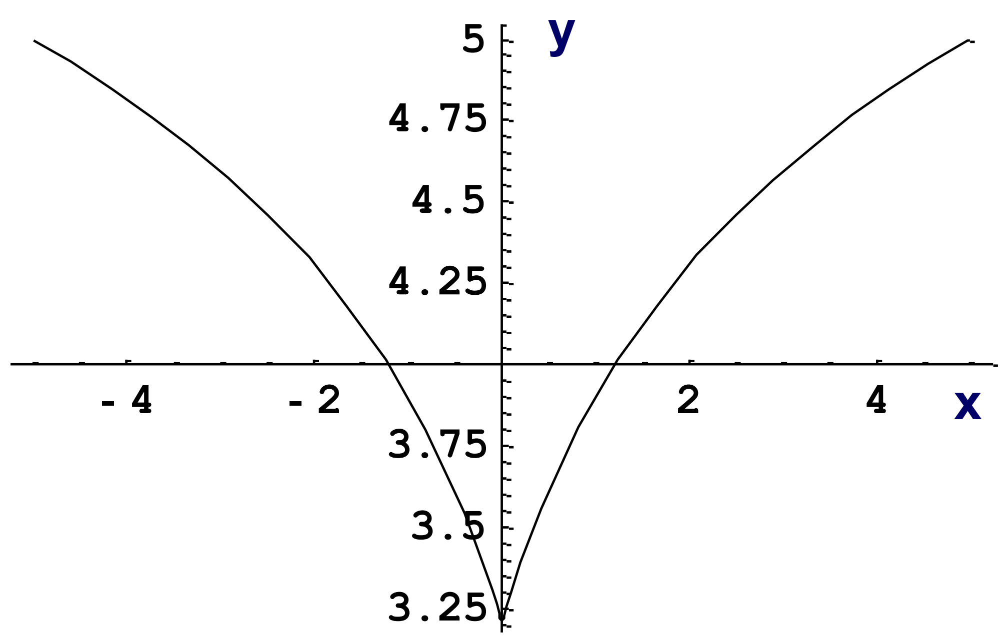
$$subject\ to\ y(-5)=5$$

Note the f is discontinuous in x

The forcing function

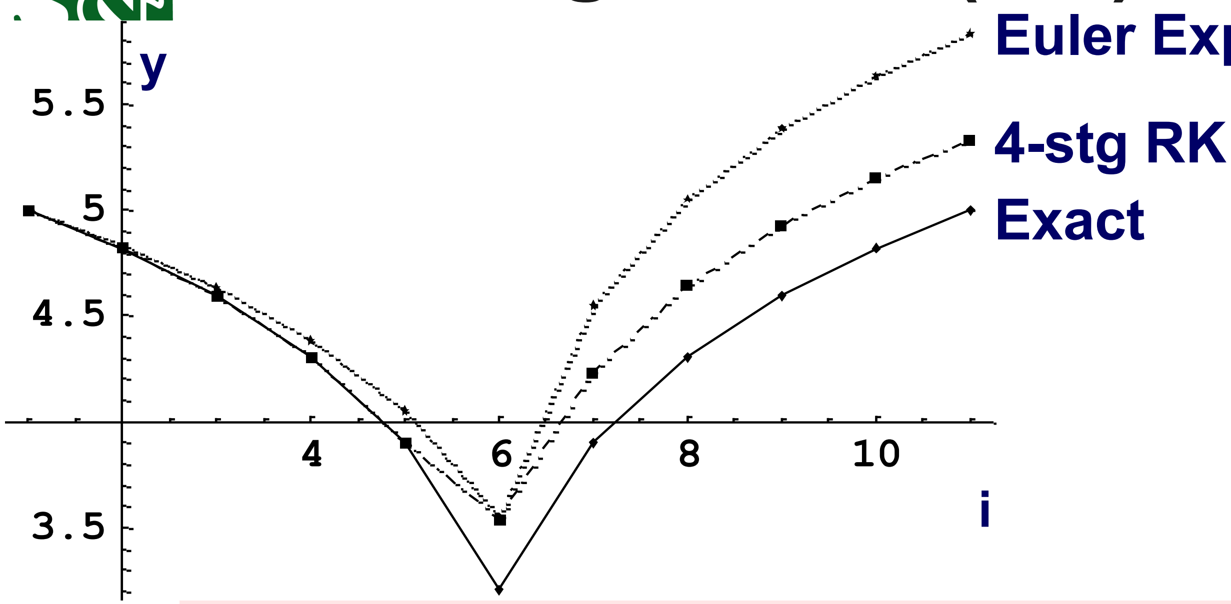


The solution



The issue here is what happens after you integrate through the discontinuity.

Runge-Kutta (RK) Example 2

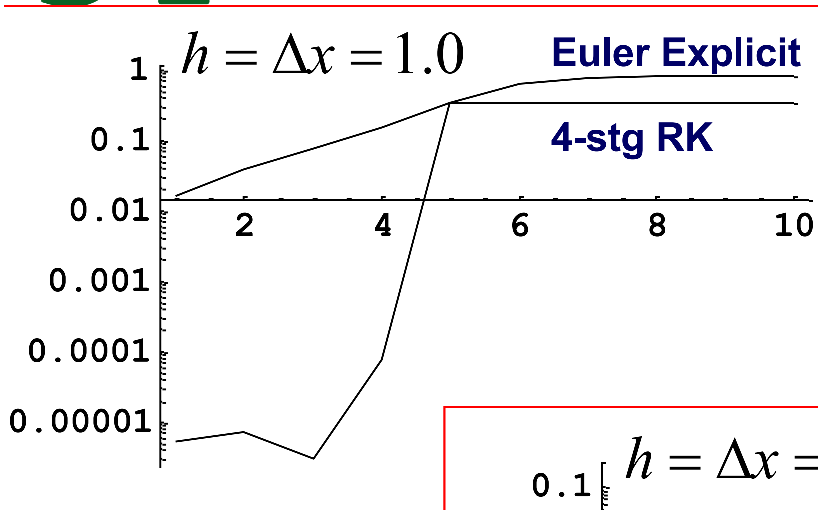


Here are solutions on a coarse grid with dx=1.

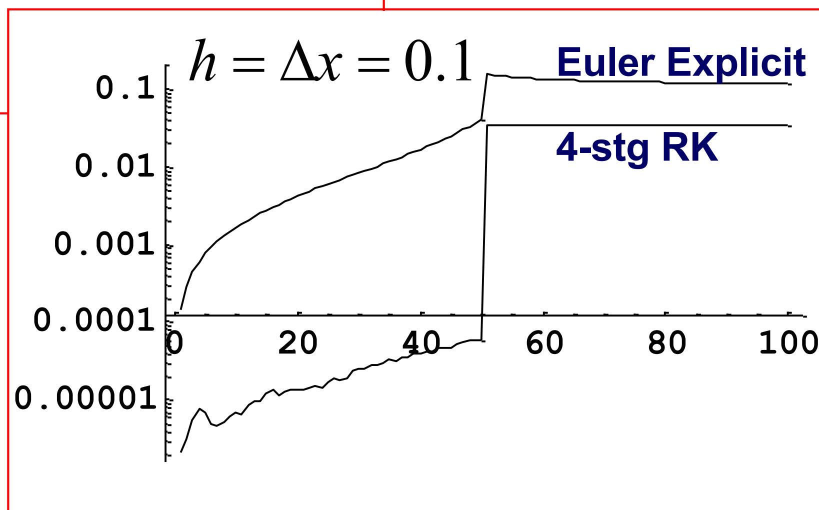
The table shows that the RK error is much less than the Euler explicit error until the discontinuity

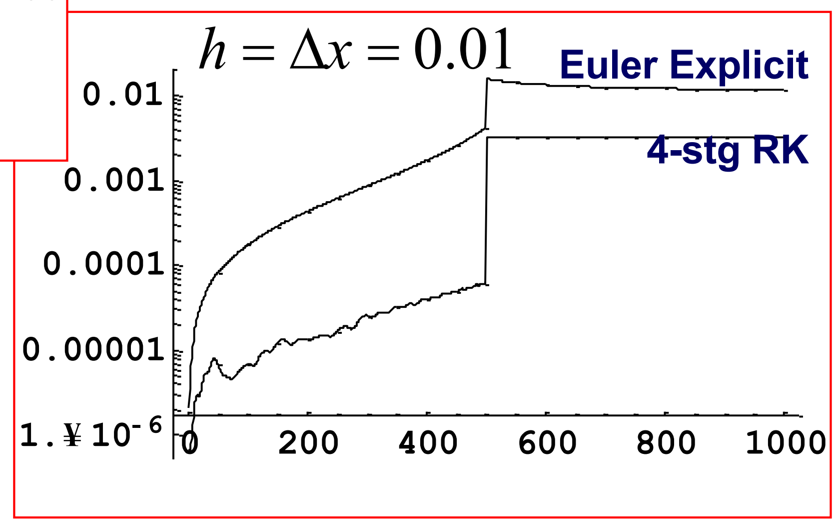
x_i	Exact	Euler Explicit	Error	4-Stage Runge-Kutta	Error
-5	5.	5.	0.	5.	0.
-4	4.81767	4.83333	0.0156618	4.81768	5.21643 ¥ 10^{-6}
-3	4.59452	4.63333	0.038812	4.59453	7.24917 ¥ 10^{-6}
-2	4.30683	4.38333	0.0765061	4.30683	2.94171 ¥ 10^{-6}
-1	3.90135	4.05	0.148651	3.90127	-0.00007413
0	3.20818	3.55	0.341819	3.54016	0.331983
1	3.90132	4.55	0.648679	4.23461	0.333287
2	4.30678	5.05	0.743216	4.64016	0.33338
3	4.59446	5.38333	0.788869	4.92786	0.333398
4	4.81761	5.63333	0.815727	5.15101	0.333404
5	4.99993	5.83333	0.833407	5.33333	0.333407

Runge-Kutta (RK) Example 2



Note that the RK error is much smaller than the Euler Explicit error on any mesh until the value of x=0 is crossed. At that point, the errors "jump" and approach the same order of magnitude (more or less).





Taylor series expansions are not valid across a discontinuity. This impacts the accuracy of the method and is an important topic in compressible fluid dynamics.

Summary

In this lecture we have

- reviewed the most commonly used form of 4-stage Runge-Kutta scheme
- examined the 4-stage RK performance relative to the Euler explicit method on two examples
  - A problem with high frequency content
  - A problem with discontinuity