Basic Properties of Vectors and Matrices

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Outline

- Before the discussing the solution of linear set of equations, we will review the basic properties of matrices and vectors
  - commonly used in solving linear set of equations ***Ax=b:***

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\
. & . & . & \dots & . \\
. & . & . & \dots & . \\
. & . & . & \dots & . \\
a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn}
\end{bmatrix}
\begin{Bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{Bmatrix}
=
\begin{Bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{Bmatrix}
$$

- define triangular matrices which will be important in the study of direct methods for solving the linear problem ***Ax=b***.

Matrix Definition

Matrix:

A matrix is a rectangular array of numbers in which not only the value is important, but also its position.

A matrix A is said to be of order n x m, where n is the number of rows and m is the number of columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix} = [a_{ij}]$$

where $i = 1,2,3,\ldots n$
and $j = 1,2,3,\ldots m$

Matrix Arithmetic

Addition of two Matrices:

Two matrices can be added or subtracted if they are of the same size. i.e., they can be added if the number of rows and columns are the same.

It is symbolically represented as

$$c_{ij} = a_{ij} + b_{ij}$$
$$C = A + B$$

Multiplication of two Matrices:

Two matrices can be multiplied if the number of columns of the first matrix and the number of rows of the second matrix are equal.

i.e., if A is a matrix of order pxq, then it can be multiplied with B if the order of the matrix is qxr, to result in a matrix C whose order is $p{\times}r$.

Symbolically it is represented as $C = AB$,

$$C_{ij} = \sum_{k=1}^{q} a_{ik}\, b_{kj}$$

Matrix Multiplication

Example for Matrix Multiplication:

$$\begin{bmatrix} 3 & 1 & 0 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} x \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

The result of multiplying the 2 matrices will result in the value of the element

$$c_{3,2} = a_{3,1}b_{1,2} + a_{3,2}b_{2,2} + a_{3,3}b_{3,2} = 1+4+9 = 14$$

Properties of Matrix Multiplication

(i) Commutative Law:

In general, matrix multiplications are not commutative.

$$AB \neq BA$$

(ii) Distributive Law:

Distributive law holds for matrices, $(A + B)C = AC + BC$

$$\sum_{k=1}^{q}(a_{ik} + b_{ik})c_{kj} = \sum_{k=1}^{q} a_{ik}c_{kj} + \sum_{k=1}^{q} b_{ik}c_{kj}$$

Associative Law

(iii) Associative Law:

Associative Law holds for matrices.

i.e., $(AB)C = A(BC)$

This can be significant from the computational perspective, it implies that powers can be combined.

i.e., $A^3A^2 = A^5$

(iv) If $AC = BC$, then A need not be necessarily equal to B, $(A \neq B)$.

Example:

$$AC = BC$$

$$\begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$But, A \neq B$$

Basic properties and vectors

(v) If $AB = [0]_{null} \Rightarrow$ ***A is not equal to [0] and/or B not equal to [0]***

$$\begin{bmatrix} a_{11} & 0 \\ a_{12} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$But, A \neq 0 \;\; and \;\; B \neq 0$$

(vi) Multiplying a matrix with a scalar is equivalent to multiplying each element of the matrix by the scalar k.

$C = k[A] = [ka_{ij}] \;\; \forall\, i,j$

A Vector:

A vector is a matrix with only one row or column. They are called a Row Vector or Column Vector. The product of a row vector and a column vector is a scalar, called a "Scalar Product" or "Inner Product"

Example:

$$\begin{bmatrix} 4 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [-2]$$

Transpose and Trace of a Matrix

Transpose of a Matrix:

The transpose of a matrix is an interchange of rows and columns of a matrix. It is denoted as A^T and algebraically, it is written as

$[a_{ij}] \Rightarrow [a_{ji}]$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 9 & 5 \\ 4 & 3 & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 9 & 3 \\ 0 & 5 & 6 \end{bmatrix}$$

Note that $(A^T)^T = A$

You can also show that $C^T = B^T A^T$ *if* $C = AB$

Trace of a Matrix:

The trace of a square matrix is the sum of the diagonal elements.

For a square matrix, the number of rows = the number of columns

Inverse and Identity Matrix

Inverse of a Matrix:

For a square matrix, the inverse of a matrix is denoted as A^{-1}, where A and A^{-1} satisfy $AA^{-1} = I$ (Identity Matrix)

Identity Matrix:

Identity matrix of order 'n' is

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ 0 & . & . & . & . & . & . & 0 \\ 0 & . & . & . & . & . & . & 0 \\ 0 & . & . & . & . & 1 & . & 0 \\ 0 & . & . & . & . & 0 & 1 & 0 \\ 0 & . & . & . & . & 0 & 0 & 1 \end{bmatrix} = \delta_{ij} \ \forall \, i,j \qquad \text{where } \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Diagonal and Triangular Matrices

Diagonal Matrix:

Diagonal Matrix of order 'n' is a square matrix where all but the diagonal elements are zero.

$$\begin{bmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & . & \\ & & & 0 \\ 0 & 0 & . & \\ 0 & 0 & 0 & d_{nn} \end{bmatrix}$$

It is represented as ***[d_{ij}]*** where ***[d_{ij}] = 0, if*** $i \neq j$

Lower Triangular Matrix:

A matrix is lower triangular if all of the elements above the main diagonal ***[l_{ii}]*** are 0. This is represented as

$$[l_{ij}] = 0 \text{ if } i < j \text{ (or) } L = [l_{ij}] = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & \dots & \dots & \\ & & & 0 \\ l_{41} & \dots & \dots & \\ l_{51} & \dots & \dots & l_{nn} \end{bmatrix}$$

Upper Triangular Matrices

Upper Triangular Matrix:

A matrix is upper triangular if all of the elements below the main diagonal $[u_{ii}]$ are 0. This is represented as

$$[u_{ij}] = 0 \text{ if } i > j \text{ (or) } U = [u_{ij}] = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ 0 & 0 & \cdots & u_{mn} \\ 0 & 0 & \cdots & \\ 0 & 0 & 0 & u_{nn} \end{bmatrix}$$

Example:

Solve $x_1+x_2+x_3=3$

$2x_2-2x_3=0$

$4x_3=4$

In matrix form we have, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$

Solving by backward substitution, $x_3 = 1; x_2=1; x_1=1$

Summary

- We have reviewed basic properties of matrices and vectors, also defined upper triangular, lower triangular, and diagonal matrices
- Next, we will start to learn the techniques to solve Linear system of equations
  - Linear system of Equations can be broadly classified into two types:
  - Direct Methods
    - Direct methods typically convert the coefficient matrix into triangular form and obtain the solution by backward or forward substitution. We will look at methods such as Gaussian elimination, LU Decomposition, pivoting, and tridiagonal systems.
  - Iterative Methods
    - Iterative methods start with an initial guess and implement an algorithm to converge to the solution vector x. Examples include Jacobi and Gauss-Seidel iteration with over/under-relaxation