25. Numerical Interpolation, Differentiation, and Integration

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25. Numerical Interpolation, Differentiation, and Integration

Numerical analysts have a tendency to accumulate a multiplicity of tools each designed for highly specialized operations and each requiring special knowledge to use properly. From the vast stock of formulas available we have culled the present selection. We hope that it will be useful. As with all such compendia, the reader may miss his favorites and find others whose utility he thinks is marginal.

We would have liked to give examples to illuminate the formulas, but this has not been feasible. Numerical analysis is partially a science and partially an art, and short of writing a textbook on the subject it has been impossible to indicate where and under what circumstances the various formulas are useful or accurate, or to elucidate the numerical difficulties to which one might be led by uncritical use. The formulas are therefore issued together with a caveat against their blind application.

Formulas

Notation: Abscissas: $x_0<x_1<\ldots$; functions: $f, g, \ldots$; values: $f(x_i)=f_i$, $f'(x_i)=f'_i$. $f', f^{(2)}, \ldots$ indicate 1st, 2d, . . . derivatives. If abscissas are equally spaced, $x_{i+1}-x_i=h$ and $f_p=f(x_0+ph)$ (p not necessarily integral). R, R_n indicate remainders.

25.1. Differences

Forward Differences

25.1.1

$$\Delta(f_n)=\Delta_n=\Delta_n^1=f_{n+1}-f_n$$

$$\Delta_n^2=\Delta_{n+1}^1-\Delta_n^1=f_{n+2}-2f_{n+1}+f_n$$

$$\Delta_n^3=\Delta_{n+1}^2-\Delta_n^2=f_{n+3}-3f_{n+2}+3f_{n+1}-f_n$$

$$\Delta_n^k=\Delta_{n+1}^{k-1}-\Delta_n^{k-1}=\sum_{j=0}^{k}(-1)^j\binom{k}{j}f_{n+k-j}$$

Central Differences

25.1.2

$$\delta(f_{n+\frac{1}{2}})=\delta_{n+\frac{1}{2}}=\delta_{n+\frac{1}{2}}^1=f_{n+1}-f_n$$

$$\delta_n^2=\delta_{n+\frac{1}{2}}^1-\delta_{n-\frac{1}{2}}^1=f_{n+1}-2f_n+f_{n-1}$$

$$\delta_{n+\frac{1}{2}}^3=\delta_{n+1}^2-\delta_n^2=f_{n+2}-3f_{n+1}+3f_n-f_{n-1}$$

$$\delta_n^{2k}=\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}f_{n+k-j}$$

$$\delta_{n+\frac{1}{2}}^{2k+1}=\sum_{j=0}^{2k+1}(-1)^j\binom{2k+1}{j}f_{n+k+1-j}$$

$$\delta_{\frac{1}{2}n}^k=\Delta_{\frac{1}{2}(n-k)}^k \text{ if } n \text{ and } k \text{ are of same parity.}$$

Forward Differences

$$\begin{array}{ccccc}
x_0 & f_0 & & & \\
 & & \Delta_0 & & \\
x_1 & f_1 & & \Delta_0^2 & \\
 & & \Delta_1 & & \Delta_0^3 \\
x_2 & f_2 & & \Delta_1^2 & \\
 & & \Delta_2 & & \\
x_3 & f_3 & & &
\end{array}$$

Central Differences

$$\begin{array}{ccccc}
x_{-1} & f_{-1} & & & \\
 & & \delta_{-\frac{1}{2}} & & \\
x_0 & f_0 & & \delta_0^2 & \\
 & & \delta_{\frac{1}{2}} & & \delta_{\frac{1}{2}}^3 \\
x_1 & f_1 & & \delta_1^2 & \\
 & & \delta_{3/2} & & \\
x_2 & f_2 & & &
\end{array}$$

Mean Differences

25.1.3 $$\mu(f_n)=\tfrac{1}{2}(f_{n+\frac{1}{2}}+f_{n-\frac{1}{2}})$$

Divided Differences

25.1.4 $$[x_0,x_1]=\frac{f_0-f_1}{x_0-x_1}=[x_1,x_0]$$

$$[x_0,x_1,x_2]=\frac{[x_0,x_1]-[x_1,x_2]}{x_0-x_2}$$

$$[x_0,x_1,\ldots,x_k]=\frac{[x_0,\ldots,x_{k-1}]-[x_1,\ldots,x_k]}{x_0-x_k}$$

Divided Differences in Terms of Functional Values

25.1.5 $$[x_0,x_1,\ldots,x_n]=\sum_{k=0}^{n}\frac{f_k}{\pi_n'(x_k)}$$

25.1.6 where $\pi_n(x)=(x-x_0)\ (x-x_1)\ \ldots\ (x-x_n)$ and $\pi_n'(x)$ is its derivative:

25.1.7

$$\pi'_n(x_k)=(x_k-x_0)\ \ldots\ (x_k-x_{k-1})(x_k-x_{k+1})\ \ldots\ (x_k-x_n)$$

Let D be a simply connected domain with a piecewise smooth boundary C and contain the points $z_0, \ldots, z_n$ in its interior. Let $f(z)$ be analytic in D and continuous in $D+C$. Then,

25.1.8 $$[z_0,z_1, \ldots,z_n]=\frac{1}{2\pi i}\int_C \frac{f(z)}{\prod_{k=0}^{n}(z-z_k)}\,dz$$

25.1.9 $$\Delta_0^n=h^nf^{(n)}(\xi) \qquad (x_0<\xi<x_n)$$

25.1.10

$$[x_0,x_1, \ldots,x_n]=\frac{\Delta_0^n}{n!h^n}=\frac{f^{(n)}(\xi)}{n!} \qquad (x_0<\xi<x_n)$$

25.1.11

$$[x_{-n},x_{-n+1}, \ldots,x_0, \ldots,x_n]=\frac{\delta_0^{2n}}{h^{2n}(2n)!}$$

Reciprocal Differences

25.1.12

$$\rho(x_0,x_1)=\frac{x_0-x_1}{f_0-f_1}$$

$$\rho_2(x_0,x_1,x_2)=\frac{x_0-x_2}{\rho(x_0,x_1)-\rho(x_1,x_2)}+f_1$$

$$\rho_3(x_0,x_1,x_2,x_3)=\frac{x_0-x_3}{\rho_2(x_0,x_1,x_2)-\rho_2(x_1,x_2,x_3)}+\rho(x_1,x_2)$$

$$\vdots$$

$$\rho_n(x_0,x_1, \ldots,x_n)=\frac{x_0-x_n}{\rho_{n-1}(x_0, \ldots,x_{n-1})-\rho_{n-1}(x_1, \ldots,x_n)}+\rho_{n-2}(x_1, \ldots,x_{n-1})$$

25.2. Interpolation

Lagrange Interpolation Formulas

25.2.1 $$f(x)=\sum_{i=0}^{n} l_i(x)f_i+R_n(x)$$

25.2.2

$$l_i(x)=\frac{\pi_n(x)}{(x-x_i)\pi'_n(x_i)}$$

$$=\frac{(x-x_0)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$$

Remainder in Lagrange Interpolation Formula

25.2.3

$$R_n(x)=\pi_n(x)\cdot[x_0,x_1, \ldots,x_n,x]$$

$$=\pi_n(x)\cdot\frac{f^{n+1}(\xi)}{(n+1)!} \qquad (x_0<\xi<x_n)$$

25.2.4

$$|R_n(x)|\le\frac{(x_n-x_0)^{n+1}}{(n+1)!}\max_{x_0\le x\le x_n}|f^{(n+1)}(x)|$$

25.2.5

$$R_n(z)=\frac{\pi_n(z)}{2\pi i}\int_C\frac{f(t)}{(t-z)(t-z_0)\ldots(t-z_n)}\,dt$$

The conditions of **25.1.8** are assumed here.

Lagrange Interpolation, Equally Spaced Abscissas

n Point Formula

25.2.6 $$f(x_0+ph)=\sum_k A_k^n(p)f_k+R_{n-1}$$

For n even, $\left(-\frac{1}{2}(n-2)\le k\le\frac{1}{2}n\right)$.

For n odd, $\left(-\frac{1}{2}(n-1)\le k\le\frac{1}{2}(n-1)\right)$.

25.2.7

$$A_k^n(p)=\frac{(-1)^{\frac{1}{2}n+k}}{\left(\frac{n-2}{2}+k\right)!(\frac{1}{2}n-k)!(p-k)}\prod_{t=1}^{n}(p+\tfrac{1}{2}n-t) \qquad n \text{ even.}$$

$$A_k^n(p)=\frac{(-1)^{\frac{1}{2}(n-1)+k}}{\left(\frac{n-1}{2}+k\right)!\left(\frac{n-1}{2}-k\right)!(p-k)}\prod_{t=0}^{n-1}\left(p+\frac{n-1}{2}-t\right), \qquad n \text{ odd.}$$

25.2.8

$$R_{n-1}=\frac{1}{n!}\prod_k(p-k)h^nf^{(n)}(\xi)$$

$$\approx\frac{1}{n!}\prod_k(p-k)\Delta_0^n \qquad (x_0<\xi<x_n)$$

k has the same range as in **25.2.6.**

Lagrange Two Point Interpolation Formula (Linear Interpolation)

25.2.9 $$f(x_0+ph)=(1-p)f_0+pf_1+R_1$$

25.2.10 $$R_1(p)\approx .125h^2f^{(2)}(\xi)\approx .125\Delta^2$$

Lagrange Three Point Interpolation Formula

25.2.11

$$f(x_0+ph)=A_{-1}f_{-1}+A_0f_0+A_1f_1+R_2$$
$$\approx\frac{p(p-1)}{2}f_{-1}+(1-p^2)f_0+\frac{p(p+1)}{2}f_1$$

25.2.12

$$R_2(p)\approx .065h^3f^{(3)}(\xi)\approx .065\Delta^3 \qquad (|p|\leq 1)$$

Lagrange Four Point Interpolation Formula

25.2.13

$$f(x_0+ph)=A_{-1}f_{-1}+A_0f_0+A_1f_1+A_2f_2+R_3$$
$$\approx\frac{-p(p-1)(p-2)}{6}f_{-1}+\frac{(p^2-1)(p-2)}{2}f_0$$
$$-\frac{p(p+1)(p-2)}{2}f_1+\frac{p(p^2-1)}{6}f_2$$

25.2.14 $\qquad R_3(p)\approx$

$$.024h^4f^{(4)}(\xi)\approx .024\Delta^4 \qquad (0<p<1)$$
$$.042h^4f^{(4)}(\xi)\approx .042\Delta^4 \qquad (-1<p<0,\ 1<p<2)$$
$$(x_{-1}<\xi<x_2)$$

Lagrange Five Point Interpolation Formula

25.2.15

$$f(x_0+ph)=\sum_{i=-2}^{2}A_if_i+R_4$$
$$\approx\frac{(p^2-1)p(p-2)}{24}f_{-2}-\frac{(p-1)p(p^2-4)}{6}f_{-1}$$
$$+\frac{(p^2-1)(p^2-4)}{4}f_0-\frac{(p+1)p(p^2-4)}{6}f_1$$
$$+\frac{(p^2-1)p(p+2)}{24}f_2$$

25.2.16 $\qquad R_4(p)\approx$

$$.012h^5f^{(5)}(\xi)\approx .012\Delta^5 \qquad (|p|<1)$$
$$.031h^5f^{(5)}(\xi)\approx .031\Delta^5 \qquad (1<|p|<2) \qquad (x_{-2}<\xi<x_2)$$

Lagrange Six Point Interpolation Formula

25.2.17

$$f(x_0+ph)=\sum_{i=-2}^{3}A_if_i+R_5$$
$$\approx\frac{-p(p^2-1)(p-2)(p-3)}{120}f_{-2}$$
$$+\frac{p(p-1)(p^2-4)(p-3)}{24}f_{-1}$$
$$-\frac{(p^2-1)(p^2-4)(p-3)}{12}f_0$$
$$+\frac{p(p+1)(p^2-4)(p-3)}{12}f_1-\frac{p(p^2-1)(p+2)(p-3)}{24}f_2$$
$$+\frac{p(p^2-1)(p^2-4)}{120}f_3$$

25.2.18 $\qquad R_5(p)\approx$

$$.0049h^6f^{(6)}(\xi)\approx .0049\Delta^6 \qquad (0<p<1)$$
$$.0071h^6f^{(6)}(\xi)\approx .0071\Delta^6 \qquad (-1<p<0,\ 1<p<2)$$
$$.024h^6f^{(6)}(\xi)\approx .024\Delta^6 \qquad (-2<p<-1,\ 2<p<3)$$
$$(x_{-2}<\xi<x_3)$$

Lagrange Seven Point Interpolation Formula

25.2.19

$$f(x_0+ph)=\sum_{i=-3}^{3}A_if_i+R_6$$

25.2.20

$$R_6(p)\approx\begin{cases}.0025h^7f^{(7)}(\xi)\approx .0025\Delta^7 & (|p|<1)\\ .0046h^7f^{(7)}(\xi)\approx .0046\Delta^7 & (1<|p|<2)\\ .019h^7f^{(7)}(\xi)\approx .019\Delta^7 & (2<|p|<3)\end{cases}$$
$$(x_{-3}<\xi<x_3)$$

Lagrange Eight Point Interpolation Formula

25.2.21

$$f(x_0+ph)=\sum_{i=-3}^{4}A_if_i+R_7$$

25.2.22

$$R_7(p)\approx\begin{cases}.0011h^8f^{(8)}(\xi)\approx .0011\Delta^8 & (0<p<1)\\ .0014h^8f^{(8)}(\xi)\approx .0014\Delta^8 & (-1<p<0)\ (1<p<2)\\ .0033h^8f^{(8)}(\xi)\approx .0033\Delta^8 & (-2<p<-1)\ (2<p<3)\\ .016h^8f^{(8)}(\xi)\approx .016\Delta^8 & (-3<p<-2)\ (3<p<4)\end{cases}$$
$$(x_{-3}<\xi<x_4)$$

Aitken's Iteration Method

Let $f(x|x_0, x_1, \ldots, x_k)$ denote the unique polynomial of k^{th} degree which coincides in value with $f(x)$ at $x_0, \ldots, x_k$.

25.2.23

$$f(x|x_0,x_1)=\frac{1}{x_1-x_0}\begin{vmatrix}f_0 & x_0-x\\ f_1 & x_1-x\end{vmatrix}$$

$$f(x|x_0,x_2)=\frac{1}{x_2-x_0}\begin{vmatrix}f_0 & x_0-x\\ f_2 & x_2-x\end{vmatrix}$$

$$f(x|x_0,x_1,x_2)=\frac{1}{x_2-x_1}\begin{vmatrix}f(x|x_0,x_1) & x_1-x\\ f(x|x_0,x_2) & x_2-x\end{vmatrix}$$

$$f(x|x_0,x_1,x_2,x_3)=\frac{1}{x_3-x_2}\begin{vmatrix}f(x|x_0,x_1,x_2) & x_2-x\\ f(x|x_0,x_1,x_3) & x_3-x\end{vmatrix}$$

Taylor Expansion

25.2.24

$$f(x)=f_0+(x-x_0)f_0'+\frac{(x-x_0)^2}{2!}f_0^{(2)}+\ldots+\frac{(x-x_0)^n}{n!}f_0^{(n)}+R_n$$

25.2.25

$$R_n=\int_{x_0}^{x}f^{(n+1)}(t)\,\frac{(x-t)^n}{n!}\,dt=\frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}(\xi)\qquad(x_0<\xi<x)$$

Newton's Divided Difference Interpolation Formula

25.2.26

$$f(x)=f_0+\sum_{k=1}^{n}\pi_{k-1}(x)\,[x_0,x_1,\ldots,x_k]+R_n$$

$$\begin{array}{llllll}
x_0 & f_0 & & & & \\
 & & [x_0,x_1] & & & \\
x_1 & f_1 & & [x_0,x_1,x_2] & & \\
 & & [x_1,x_2] & & [x_0,x_1,x_2,x_3] & \\
x_2 & f_2 & & [x_1,x_2,x_3] & & \\
 & & [x_2,x_3] & & & \\
x_3 & f_3 & & & &
\end{array}$$

25.2.27

$$R_n(x)=\pi_n(x)\,[x_0,\ldots,x_n,x]=\pi_n(x)\frac{f^{(n+1)}(\xi)}{(n+1)!}\qquad(x_0<\xi<x_n)$$

(For π_n see **25.1.6.**)

Newton's Forward Difference Formula

25.2.28

$$f(x_0+ph)=f_0+p\Delta_0+\binom{p}{2}\Delta_0^2+\ldots+\binom{p}{n}\Delta_0^n+R_n$$

$$\begin{array}{lllll}
x_0 & f_0 & & & \\
 & & \Delta_0 & & \\
x_1 & f_1 & & \Delta_0^2 & \\
 & & \Delta_1 & & \Delta_0^3 \\
x_2 & f_2 & & \Delta_1^2 & \\
 & & \Delta_2 & & \\
x_3 & f_3 & & &
\end{array}$$

25.2.29

$$R_n=h^{n+1}\binom{p}{n+1}f^{(n+1)}(\xi)\approx\binom{p}{n+1}\Delta_0^{n+1}\qquad(x_0<\xi<x_n)$$

Relation Between Newton and Lagrange Coefficients

25.2.30

$$\binom{p}{2}=A_{-1}^3(p)\qquad\binom{p}{3}=-A_{-1}^4(p)\qquad\binom{p}{4}=A_2^5(1-p)$$

$$\binom{p}{5}=A_3^6(2-p)$$

Everett's Formula

25.2.31

$$\begin{aligned}f(x_0+ph)&=(1-p)f_0+pf_1-\frac{p(p-1)(p-2)}{3!}\delta_0^2\\&\quad+\frac{(p+1)p(p-1)}{3!}\delta_1^2+\ldots-\binom{p+n-1}{2n+1}\delta_0^{2n}\\&\quad+\binom{p+n}{2n+1}\delta_1^{2n}+R_{2n}\\&=(1-p)f_0+pf_1+E_2\delta_0^2+F_2\delta_1^2+E_4\delta_0^4\\&\quad+F_4\delta_1^4+\ldots+R_{2n}\end{aligned}$$

$$\begin{array}{llll}
x_0 & f_0 & \delta_0^2 & \delta_0^4 \\
 & \delta_{\frac12} & \delta_{\frac12}^3 & \\
x_1 & f_1 & \delta_1^2 & \delta_1^4
\end{array}$$

25.2.32

$$R_{2n}=h^{2n+2}\binom{p+n}{2n+2}f^{(2n+2)}(\xi)\approx\binom{p+n}{2n+2}\left[\frac{\Delta_{-n-1}^{2n+2}+\Delta_{-n}^{2n+2}}{2}\right]\qquad(x_{-n}<\xi<x_{n+1})$$

Relation Between Everett and Lagrange Coefficients

25.2.33

$$E_2=A_{-1}^4\qquad E_4=A_{-2}^6\qquad E_6=A_{-3}^8$$

$$F_2=A_2^4\qquad F_4=A_3^6\qquad F_6=A_4^8$$

Everett's Formula With Throwback (Modified Central Difference)

25.2.34

$$f(x_0+ph)=(1-p)f_0+pf_1+E_2\delta_{m,0}^2+F_2\delta_{m,1}^2+R$$

25.2.35 $\delta_m^2=\delta^2-.184\delta^4$

25.2.36 $R\approx.00045|\mu\delta_{\frac12}^4|+.00061|\delta_{\frac12}^5|$

25.2.37

$$f(x_0+ph)=(1-p)f_0+pf_1+E_2\delta_0^2+F_2\delta_1^2+E_4\delta_{m,0}^4+F_4\delta_{m,1}^4+R$$

25.2.38 $\delta_m^4=\delta^4-.207\delta^6+\ldots$

25.2.39 $R\approx.000032|\mu\delta_{\frac12}^6|+.000052|\delta_{\frac12}^7|$

25.2.40

$$f(x_0+ph)=(1-p)f_0+pf_1+E_2\delta_0^2+F_2\delta_1^2+E_4\delta_0^4+F_4\delta_1^4+E_6\delta_{m,0}^6+F_6\delta_{m,1}^6+R$$

25.2.41 $\delta_m^6=\delta^6-.218\delta^8+.049\delta^{10}+\ldots$

25.2.42 $R\approx.0000037|\mu\delta_{\frac12}^8|+\ldots$

Simultaneous Throwback

25.2.43

$$f(x_0+ph)=(1-p)f_0+pf_1+E_2\delta^2_{m,0}+F_2\delta^2_{m,1}+E_4\delta^4_{m,0}+F_4\delta^4_{m,1}+R$$

25.2.44 $\delta^2_m=\delta^2-.01312\delta^6+.0043\delta^8-.001\delta^{10}$

25.2.45 $\delta^4_m=\delta^4-.27827\delta^6+.0685\delta^8-.016\delta^{10}$

25.2.46 $R\approx.00000083|\mu\delta^6_{\frac{1}{2}}|+.00000094\delta^7$

Bessel's Formula With Throwback

25.2.47

$$f(x_0+ph)=(1-p)f_0+pf_1+B_2(\delta^2_{m,0}+\delta^2_{m,1})+B_3\delta^3_{\frac{1}{2}}+R,\ B_2=\frac{p(p-1)}{4},\ B_3=\frac{p(p-1)\,(p-\frac{1}{2})}{6}$$

25.2.48 $\delta^2_m=\delta^2-.184\delta^4$

25.2.49 $R\approx.00045|\mu\delta^4_{\frac{1}{2}}|+.00087|\delta^5_{\frac{1}{2}}|$

Thiele's Interpolation Formula

25.2.50

$$f(x)=f(x_1)+\cfrac{x-x_1}{\rho(x_1,x_2)+\cfrac{x-x_2}{\rho_2(x_1,x_2,x_3)-f(x_1)+\cfrac{x-x_3}{\begin{pmatrix}\rho_3(x_1,x_2,x_3,x_4)\\ -\rho(x_1,x_2)+\dots\end{pmatrix}}}}$$

(For reciprocal differences, ρ, see **25.1.12.**)

Trigonometric Interpolation

Gauss' Formula

25.2.51 $f(x)\approx\sum_{k=0}^{2n}f_k\zeta_k(x)=t_n(x)$

25.2.52

$$\zeta_k(x)=\frac{\sin\frac{1}{2}(x-x_0)\ldots\sin\frac{1}{2}(x-x_{k-1})}{\sin\frac{1}{2}(x_k-x_0)\ldots\sin\frac{1}{2}(x_k-x_{k-1})}\cdot\frac{\sin\frac{1}{2}(x-x_{k+1})\ldots\sin\frac{1}{2}(x-x_{2n})}{\sin\frac{1}{2}(x_k-x_{k+1})\ldots\sin\frac{1}{2}(x_k-x_{2n})}$$

$t_n(x)$ is a trigonometric polynomial of degree n such that $t_n(x_k)=f_k$ $(k=0,1,\ldots,2n)$

Harmonic Analysis

Equally spaced abscissas

$$x_0=0,\qquad x_1,\ldots,x_{m-1},x_m=2\pi$$

25.2.53

$$f(x)\approx\frac{1}{2}a_0+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)$$

25.2.54 $m=2n+1$

$$a_k=\frac{2}{2n+1}\sum_{r=0}^{2n}f_r\cos kx_r;\qquad b_k=\frac{2}{2n+1}\sum_{r=0}^{2n}f_r\sin kx_r\qquad(k=0,1,\ldots,n)$$

25.2.55 $m=2n$

$$a_k=\frac{1}{n}\sum_{r=0}^{2n-1}f_r\cos kx_r;\qquad b_k=\frac{1}{n}\sum_{r=0}^{2n-1}f_r\sin kx_r$$

$(k=0,1,\ldots,n)$ $\qquad(k=0,1,\ldots,n-1)$

b_n is arbitrary.

Subtabulation

Let $f(x)$ be tabulated initially in intervals of width h. It is desired to subtabulate $f(x)$ in intervals of width h/m. Let Δ and $\overline{\Delta}$ designate differences with respect to the original and the final intervals respectively. Thus $\overline{\Delta}_0=f\left(x_0+\frac{h}{m}\right)-f(x_0)$. Assuming that the original 5th order differences are zero,

25.2.56

$$\overline{\Delta}_0=\frac{1}{m}\Delta_0+\frac{1-m}{2m^2}\Delta^2_0+\frac{(1-m)(1-2m)}{6m^3}\Delta^3_0+\frac{(1-m)(1-2m)(1-3m)}{24m^4}\Delta^4_0$$

$$\overline{\Delta}^2_0=\frac{1}{m^2}\Delta^2_0+\frac{1-m}{m^3}\Delta^3_0+\frac{(1-m)(7-11m)}{12m^4}\Delta^4_0$$

$$\overline{\Delta}^3_0=\frac{1}{m^3}\Delta^3_0+\frac{3(1-m)}{2m^4}\Delta^4_0$$

$$\overline{\Delta}^4_0=\frac{1}{m^4}\Delta^4_0$$

From this information we may construct the final tabulation by addition. For $m=10$,

25.2.57

$$\overline{\Delta}_0=.1\Delta_0-.045\Delta^2_0+.0285\Delta^3_0-.02066\Delta^4_0$$
$$\overline{\Delta}^2_0=.01\Delta^2_0-.009\Delta^3_0+.007725\Delta^4_0$$
$$\overline{\Delta}^3_0=.001\Delta^3_0-.00135\Delta^4_0$$
$$\overline{\Delta}^4_0=.0001\Delta^4_0$$

Linear Inverse Interpolation

Find p, given $f_p(=f(x_0+ph))$.

Linear

25.2.58 $p\approx\dfrac{f_p-f_0}{f_1-f_0}$

Quadratic Inverse Interpolation

25.2.59

$$(f_1-2f_0+f_{-1})p^2+(f_1-f_{-1})p+2(f_0-f_p)\approx 0$$

Inverse Interpolation by Reversion of Series

25.2.60 Given $f(x_0+ph)=f_p=\sum_{k=0}^{\infty} a_k p^k$

25.2.61

$$p=\lambda+c_2\lambda^2+c_3\lambda^3+\ .\ .\ .,\ \lambda=(f_p-a_0)/a_1$$

25.2.62

$$c_2=-a_2/a_1$$

$$c_3=\frac{-a_3}{a_1}+2\left(\frac{a_2}{a_1}\right)^2$$

$$c_4=\frac{-a_4}{a_1}+\frac{5a_2a_3}{a_1^2}-\frac{5a_2^3}{a_1^3}$$

$$c_5=\frac{-a_5}{a_1}+\frac{6a_2a_4}{a_1^2}+\frac{3a_3^2}{a_1^2}-\frac{21a_2^2a_3}{a_1^3}+\frac{14a_2^4}{a_1^4}$$

Inversion of Newton's Forward Difference Formula

25.2.63

$$a_0=f_0$$

$$a_1=\Delta_0-\frac{\Delta_0^2}{2}+\frac{\Delta_0^3}{3}-\frac{\Delta_0^4}{4}+\ .\ .\ .$$

$$a_2=\frac{\Delta_0^2}{2}-\frac{\Delta_0^3}{2}+\frac{11\Delta_0^4}{24}+\ .\ .\ .$$

$$a_3=\frac{\Delta_0^3}{6}-\frac{\Delta_0^4}{4}+\ .\ .\ .$$

$$a_4=\frac{\Delta_0^4}{24}+\ .\ .\ .$$

(Used in conjunction with **25.2.62.**)

Inversion of Everett's Formula

25.2.64

$$a_0=f_0$$

$$a_1=\delta_{\frac{1}{2}}-\frac{\delta_0^2}{3}-\frac{\delta_1^2}{6}+\frac{\delta_0^4}{20}+\frac{\delta_1^4}{30}+\ .\ .\ .$$

$$a_2=\frac{\delta_0^2}{2}-\frac{\delta_0^4}{24}+\ .\ .\ .$$

$$a_3=\frac{-\delta_0^2+\delta_1^2}{6}-\frac{\delta_0^4+\delta_1^4}{24}+\ .\ .\ .$$

$$a_4=\frac{\delta_0^4}{24}+\ .\ .\ .$$

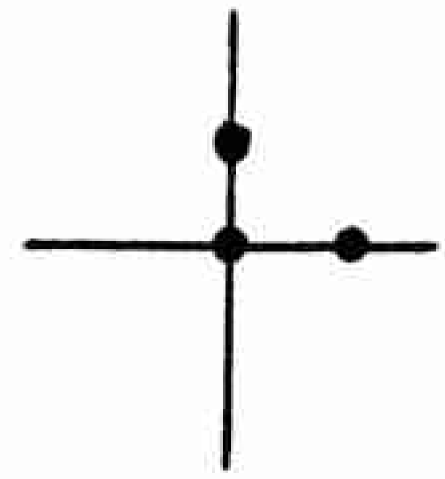
$$a_5=\frac{-\delta_0^4+\delta_1^4}{120}+\ .\ .\ .$$

(Used in conjunction with **25.2.62.**)

Bivariate Interpolation

Three Point Formula (Linear)

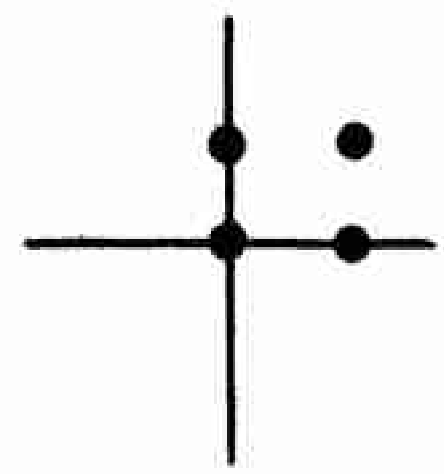
25.2.65



$$f(x_0+ph,y_0+qk)=(1-p-q)f_{0,0}+pf_{1,0}+qf_{0,1}+O(h^2)$$

Four Point Formula

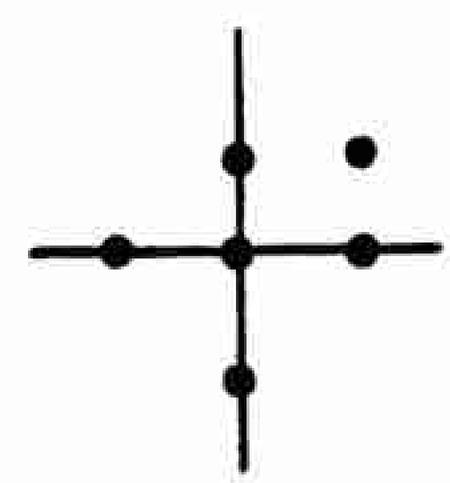
25.2.66



$$f(x_0+ph,y_0+qk)=(1-p)(1-q)f_{0,0}+p(1-q)f_{1,0}+q(1-p)f_{0,1}+pqf_{1,1}+O(h^2)$$

Six Point Formula

25.2.67



$$f(x_0+ph,y_0+qk)=\frac{q(q-1)}{2}f_{0,-1}+\frac{p(p-1)}{2}f_{-1,0}+(1+pq-p^2-q^2)f_{0,0}+\frac{p(p-2q+1)}{2}f_{1,0}+\frac{q(q-2p+1)}{2}f_{0,1}+pqf_{1,1}+O(h^3)$$

25.3. Differentiation

Lagrange's Formula

25.3.1 $$f'(x)=\sum_{k=0}^{n} l_k'(x)f_k+R_n'(x)$$

(See **25.2.1.**)

25.3.2 $$l_k'(x)=\sum_{\substack{j=0\\ j\neq k}}^{n}\frac{\pi_n(x)}{(x-x_k)(x-x_j)\pi_n'(x_k)}$$

25.3.3

$$R'_n(x)=\frac{f^{(n+1)}}{(n+1)!}(\xi)\pi'_n(x)+\frac{\pi_n(x)}{(n+1)!}\frac{d}{dx}f^{(n+1)}(\xi)$$

$$\xi=\xi(x)\quad(x_0<\xi<x_n)$$

Equally Spaced Abscissas

Three Points

25.3.4

$$f'_p=f'(x_0+ph)$$

$$=\frac{1}{h}\{(p-\tfrac{1}{2})f_{-1}-2pf_0+(p+\tfrac{1}{2})f_1\}+R'_2$$

Four Points

25.3.5

$$f'_p=f'(x_0+ph)=\frac{1}{h}\left\{-\frac{3p^2-6p+2}{6}f_{-1}+\frac{3p^2-4p-1}{2}f_0-\frac{3p^2-2p-2}{2}f_1+\frac{3p^2-1}{6}f_2\right\}+R'_3$$

Five Points

25.3.6

$$f'_p=f'(x_0+ph)=\frac{1}{h}\left\{\frac{2p^3-3p^2-p+1}{12}f_{-2}-\frac{4p^3-3p^2-8p+4}{6}f_{-1}+\frac{2p^3-5p}{2}f_0-\frac{4p^3+3p^2-8p-4}{6}f_1+\frac{2p^3+3p^2-p-1}{12}f_2\right\}+R'_4$$

For numerical values of differentiation coefficients see **Table 25.2.**

Markoff's Formulas

(Newton's Forward Difference Formula Differentiated)

25.3.7

$$f'(a_0+ph)=\frac{1}{h}\left[\Delta_0+\frac{2p-1}{2}\Delta_0^2+\frac{3p^2-6p+2}{6}\Delta_0^3+\ldots+\frac{d}{dp}\binom{p}{n}\Delta_0^n\right]+R'_n$$

25.3.8

$$R'_n=h^n f^{(n+1)}(\xi)\frac{d}{dp}\binom{p}{n+1}+h^{n+1}\binom{p}{n+1}\frac{d}{dx}f^{(n+1)}(\xi)$$

$$(a_0<\xi<a_n)$$

25.3.9 $$hf'_0=\Delta_0-\frac{1}{2}\Delta_0^2+\frac{1}{3}\Delta_0^3-\frac{1}{4}\Delta_0^4+\ldots$$

25.3.10 $$h^2f_0^{(2)}=\Delta_0^2-\Delta_0^3+\frac{11}{12}\Delta_0^4-\frac{5}{6}\Delta_0^5+\ldots$$

25.3.11

$$h^3f_0^{(3)}=\Delta_0^3-\frac{3}{2}\Delta_0^4+\frac{7}{4}\Delta_0^5-\frac{15}{8}\Delta_0^6+\ldots$$

25.3.12

$$h^4f_0^{(4)}=\Delta_0^4-2\Delta_0^5+\frac{17}{6}\Delta_0^6-\frac{7}{2}\Delta_0^7+\ldots$$

25.3.13

$$h^5f_0^{(5)}=\Delta_0^5-\frac{5}{2}\Delta_0^6+\frac{25}{6}\Delta_0^7-\frac{35}{6}\Delta_0^8+\ldots$$

Everett's Formula

25.3.14

$$hf'(x_0+ph)\approx-f_0+f_1-\frac{3p^2-6p+2}{6}\delta_0^2+\frac{3p^2-1}{6}\delta_1^2-\frac{5p^4-20p^3+15p^2+10p-6}{120}\delta_0^4+\frac{5p^4-15p^2+4}{120}\delta_1^4+\ldots-\left[\binom{p+n-1}{2n+1}\right]'\delta_0^{2n}+\left[\binom{p+n}{2n+1}\right]'\delta_1^{2n}$$

25.3.15

$$hf'_0\approx-f_0+f_1-\frac{1}{3}\delta_0^2-\frac{1}{6}\delta_1^2+\frac{1}{20}\delta_0^4+\frac{1}{30}\delta_1^4$$

Differences in Terms of Derivatives

25.3.16

$$\Delta_0\approx hf'_0+\frac{h^2}{2!}f_0^{(2)}+\frac{h^3}{3!}f_0^{(3)}+\frac{h^4}{4!}f_0^{(4)}+\frac{h^5}{5!}f_0^{(5)}$$

25.3.17

$$\Delta_0^2\approx h^2f_0^{(2)}+h^3f_0^{(3)}+\frac{7}{12}h^4f_0^{(4)}+\frac{1}{4}h^5f_0^{(5)}$$

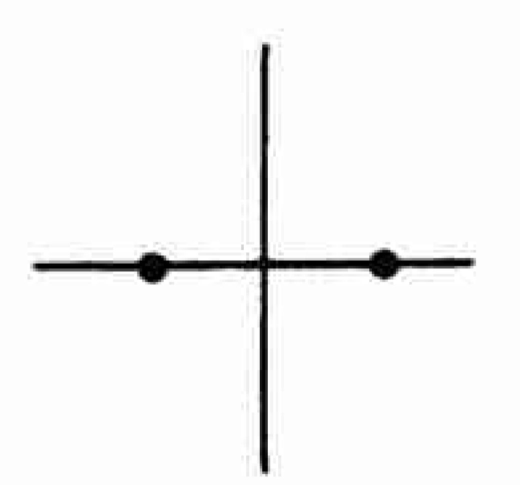
25.3.18 $$\Delta_0^3\approx h^3f_0^{(3)}+\frac{3}{2}h^4f_0^{(4)}+\frac{5}{4}f_0^{(5)}$$

25.3.19 $$\Delta_0^4\approx h^4f_0^{(4)}+2h^5f_0^{(5)}$$

25.3.20 $$\Delta_0^5\approx h^5f_0^{(5)}$$

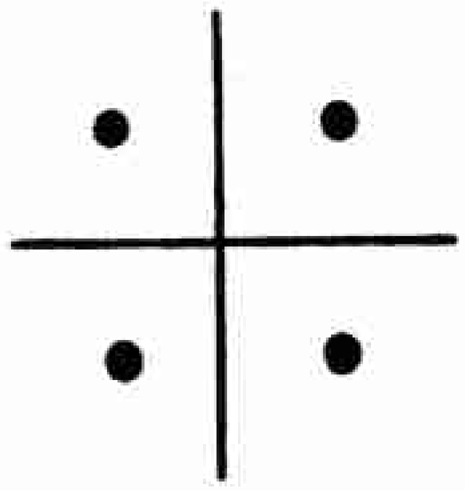
Partial Derivatives

25.3.21



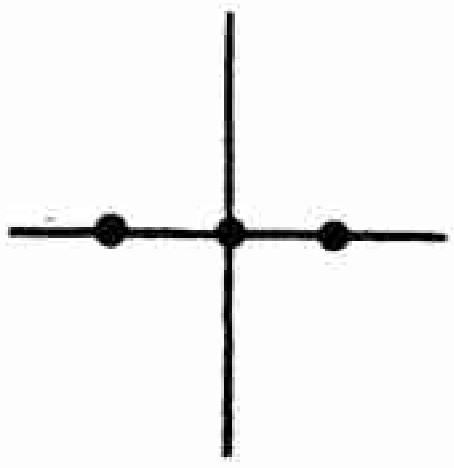
$$\frac{\partial f_{0,0}}{\partial x}=\frac{1}{2h}(f_{1,0}-f_{-1,0})+O(h^2)$$

25.3.22



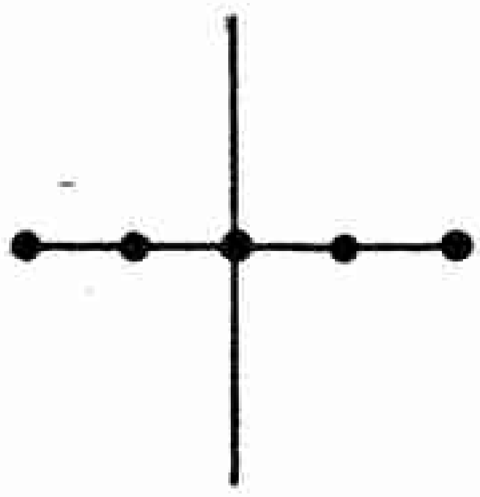
$$\frac{\partial f_{0,0}}{\partial x}=\frac{1}{4h}\left(f_{1,1}-f_{-1,1}+f_{1,-1}-f_{-1,-1}\right)+O(h^2)$$

25.3.23



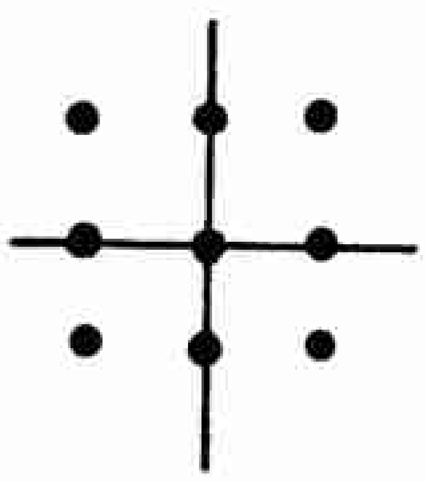
$$\frac{\partial^2 f_{0,0}}{\partial x^2}=\frac{1}{h^2}\left(f_{1,0}-2f_{0,0}+f_{-1,0}\right)+O(h^2)$$

25.3.24



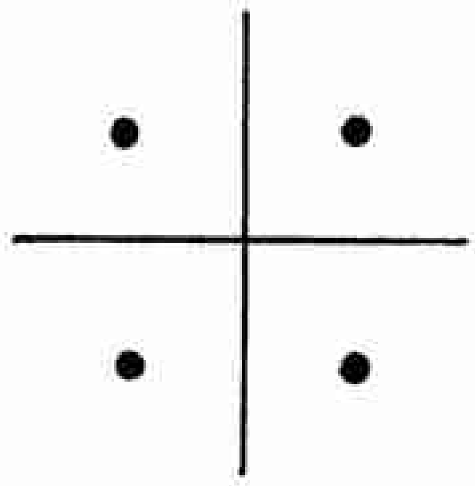
$$\frac{\partial^2 f_{0,0}}{\partial x^2}=\frac{1}{12h^2}\left(-f_{2,0}+16f_{1,0}-30f_{0,0}+16f_{-1,0}-f_{-2,0}\right)+O(h^4)$$

25.3.25



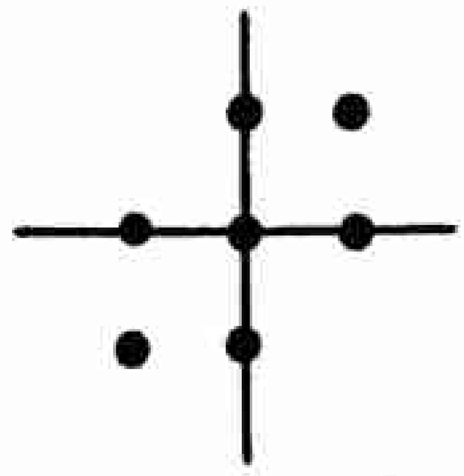
$$\frac{\partial^2 f_{0,0}}{\partial x^2}=\frac{1}{3h^2}\left(f_{1,1}-2f_{0,1}+f_{-1,1}+f_{1,0}-2f_{0,0}+f_{-1,0}+f_{1,-1}-2f_{0,-1}+f_{-1,-1}\right)+O(h^2)$$

25.3.26



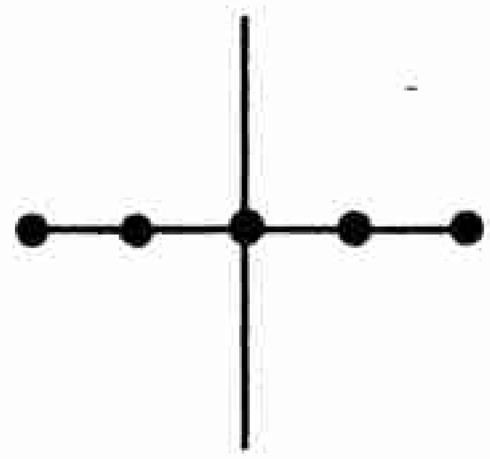
$$\frac{\partial^2 f_{0,0}}{\partial x\partial y}=\frac{1}{4h^2}\left(f_{1,1}-f_{1,-1}-f_{-1,1}+f_{-1,-1}\right)+O(h^2)$$

25.3.27



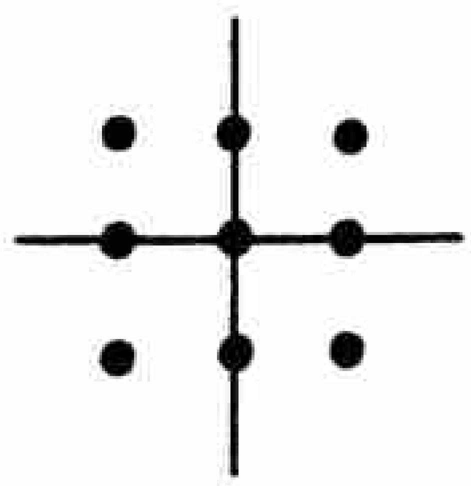
$$\frac{\partial^2 f_{0,0}}{\partial x\partial y}=\frac{-1}{2h^2}\left(f_{1,0}+f_{-1,0}+f_{0,1}+f_{0,-1}-2f_{0,0}-f_{1,1}-f_{-1,-1}\right)+O(h^2)$$

25.3.28



$$\frac{\partial^4 f_{0,0}}{\partial x^4}=\frac{1}{h^4}\left(f_{2,0}-4f_{1,0}+6f_{0,0}-4f_{-1,0}+f_{-2,0}\right)+O(h^2)$$

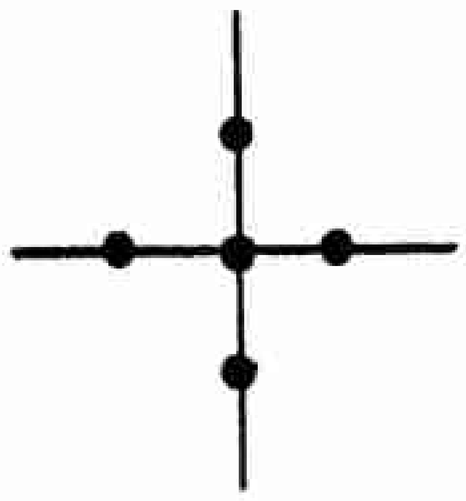
25.3.29



$$\frac{\partial^4 f_{0,0}}{\partial x^2\partial y^2}=\frac{1}{h^4}\left(f_{1,1}+f_{-1,1}+f_{1,-1}+f_{-1,-1}-2f_{1,0}-2f_{-1,0}-2f_{0,1}-2f_{0,-1}+4f_{0,0}\right)+O(h^2)$$

Laplacian

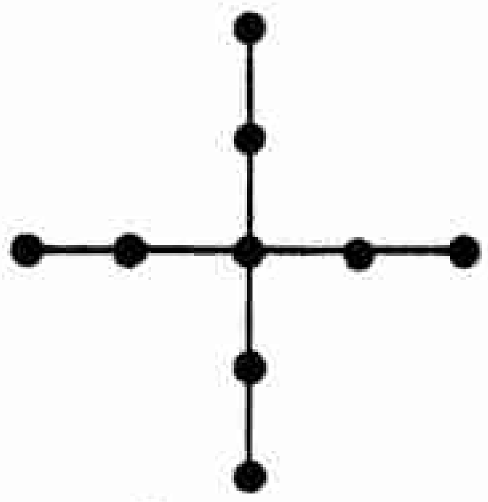
25.3.30



$$\nabla^2 u_{0,0}=\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)_{0,0}$$

$$=\frac{1}{h^2}\,(u_{1,0}+u_{0,1}+u_{-1,0}+u_{0,-1}-4u_{0,0})+O(h^2)$$

25.3.31

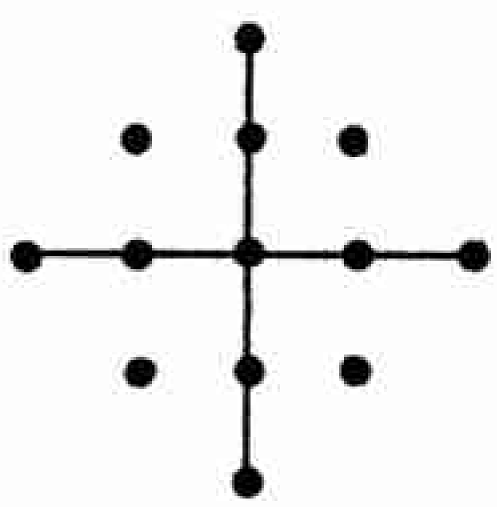


$$\nabla^2 u_{0,0}=\frac{1}{12h^2}\,[-60u_{0,0}+16(u_{1,0}+u_{0,1}+u_{-1,0}+u_{0,-1})$$

$$-(u_{2,0}+u_{0,2}+u_{-2,0}+u_{0,-2})]+O(h^4)$$

Biharmonic Operator

25.3.32



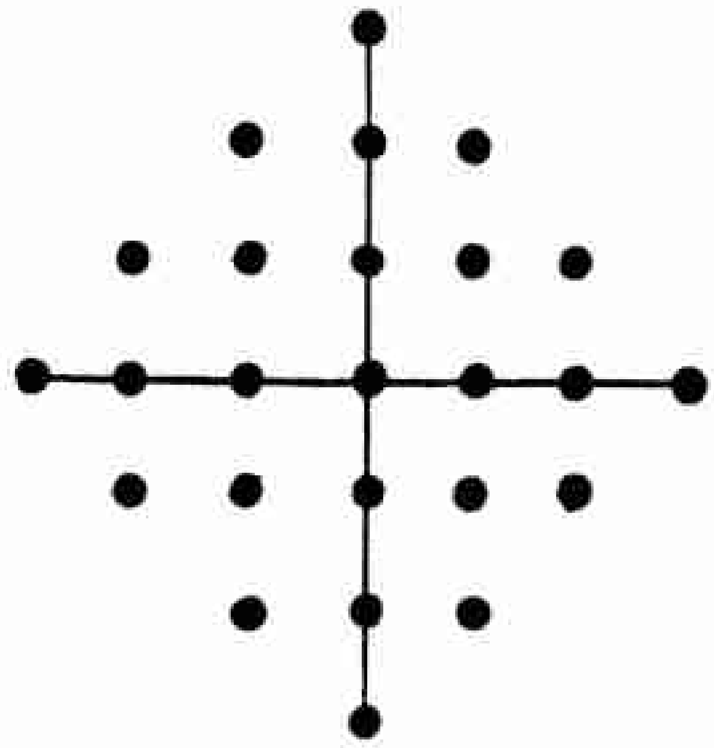
$$\nabla^4 u_{0,0}=\left(\frac{\partial^4 u}{\partial x^4}+2\,\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}\right)_{0,0}$$

$$=\frac{1}{h^4}[20u_{0,0}-8(u_{1,0}+u_{0,1}+u_{-1,0}+u_{0,-1})$$

$$+2(u_{1,1}+u_{1,-1}+u_{-1,1}+u_{-1,-1})$$

$$+(u_{0,2}+u_{2,0}+u_{-2,0}+u_{0,-2})]+O(h^2)$$

25.3.33



$$\nabla^4 u_{0,0}=\frac{1}{6h^4}\,[-(u_{0,3}+u_{0,-3}+u_{3,0}+u_{-3,0})$$

$$+14(u_{0,2}+u_{0,-2}+u_{2,0}+u_{-2,0})$$

$$-77(u_{0,1}+u_{0,-1}+u_{1,0}+u_{-1,0})$$

$$+184u_{0,0}+20(u_{1,1}+u_{1,-1}+u_{-1,1}+u_{-1,-1})$$

$$-(u_{1,2}+u_{2,1}+u_{1,-2}+u_{2,-1}+u_{-1,2}+u_{-2,1}$$

$$+u_{-1,-2}+u_{-2,-1})]+O(h^4)$$

25.4. Integration

Trapezoidal Rule

25.4.1

$$\int_{x_0}^{x_1} f(x)dx=\frac{h}{2}\,(f_0+f_1)-\frac{1}{2}\int_{x_0}^{x_1}(t-x_0)(x_1-t)f''(t)\,dt$$

$$=\frac{h}{2}\,(f_0+f_1)-\frac{h^3}{12}f''(\xi)\qquad (x_0<\xi<x_1)$$

Extended Trapezoidal Rule

25.4.2

$$\int_{x_0}^{x_m} f(x)dx=h\left[\frac{f_0}{2}+f_1+\ .\ .\ .\ +f_{m-1}+\frac{f_m}{2}\right]$$

$$-\frac{mh^3}{12}f''(\xi)$$

Error Term in Trapezoidal Formula for Periodic Functions

If $f(x)$ is periodic and has a continuous k^{th} derivative, and if the integral is taken over a period, then

25.4.3 $$|\text{Error}|\leq\frac{\text{constant}}{m^k}$$

Modified Trapezoidal Rule

25.4.4

$$\int_{x_0}^{x_m} f(x)dx=h\left[\frac{f_0}{2}+f_1+\ .\ .\ .\ +f_{m-1}+\frac{f_m}{2}\right]$$

$$+\frac{h}{24}\,[-f_{-1}+f_1+f_{m-1}-f_{m+1}]+\frac{11m}{720}\,h^5 f^{(4)}(\xi)$$

Simpson's Rule

25.4.5

$$\int_{x_0}^{x_2} f(x)dx=\frac{h}{3}[f_0+4f_1+f_2]$$
$$+\frac{1}{6}\int_{x_0}^{x_1}(x_0-t)^2(x_1-t)f^{(3)}(t)dt$$
$$+\frac{1}{6}\int_{x_1}^{x_2}(x_2-t)^2(x_1-t)f^{(3)}(t)dt$$
$$=\frac{h}{3}[f_0+4f_1+f_2]-\frac{h^5}{90}f^{(4)}(\xi)$$

Extended Simpson's Rule

25.4.6

$$\int_{x_0}^{x_{2n}} f(x)dx=\frac{h}{3}[f_0+4(f_1+f_3+\ldots+f_{2n-1})$$
$$+2(f_2+f_4+\ldots+f_{2n-2})+f_{2n}]-\frac{nh^5}{90}f^{(4)}(\xi)$$

Euler-Maclaurin Summation Formula

25.4.7

$$\int_{x_0}^{x_n} f(x)dx=h\left[\frac{f_0}{2}+f_1+f_2+\ldots+f_{n-1}+\frac{f_n}{2}\right]$$
$$-\frac{B_2}{2!}h^2(f'_n-f'_0)-\ldots-\frac{B_{2k}h^{2k}}{(2k)!}[f_n^{(2k-1)}-f_0^{(2k-1)}]+R_{2k}$$

$$R_{2k}=\frac{\theta n B_{2k+2}h^{2k+3}}{(2k+2)!}\max_{x_0\leq x\leq x_n}|f^{(2k+2)}(x)|, \qquad (-1\leq\theta\leq 1)$$

(For B_{2k}, Bernoulli numbers, see chapter **23.**)

If $f^{(2k+2)}(x)$ and $f^{(2k+4)}(x)$ do not change sign for $x_0<x<x_n$ then $|R_{2k}|$ is less than the first neglected term. If $f^{(2k+2)}(x)$ does not change sign for $x_0<x<x_n$, $|R_{2k}|$ is less than twice the first neglected term.

Lagrange Formula

25.4.8

$$\int_a^b f(x)dx=\sum_{i=0}^{n}(L_i^{(n)}(b)-L_i^{(n)}(a))f_i+R_n$$

(See **25.2.1.**)

25.4.9

$$L_i^{(n)}(x)=\frac{1}{\pi'_n(x_i)}\int_{x_0}^{x}\frac{\pi_n(t)}{t-x_i}dt=\int_{x_0}^{x}l_i(t)dt$$

25.4.10 $$R_n=\frac{1}{(n+1)!}\int_a^b\pi_n(x)f^{(n+1)}(\xi(x))dx$$

Equally Spaced Abscissas

25.4.11

$$\int_{x_0}^{x_k} f(x)dx=\frac{1}{h^n}\sum_{i=0}^{n}f_i\frac{(-1)^{n-i}}{i!(n-i)!}\int_{x_0}^{x_k}\frac{\pi_n(x)}{x-x_i}dx+R_n$$

25.4.12 $$\int_{x_m}^{x_{m+1}} f(x)dx=h\sum_{i=-\left[\frac{n-1}{2}\right]}^{\left[\frac{n}{2}\right]}A_i(m)f_i+R_n \qquad *$$

(See **Table 25.3** for $A_i(m)$.)

Newton-Cotes Formulas (Closed Type)

(For Trapezoidal and Simpson's Rules see **25.4.1–25.4.6.**)

25.4.13 **(Simpson's $\frac{3}{8}$ rule)**

$$\int_{x_0}^{x_3} f(x)dx=\frac{3h}{8}(f_0+3f_1+3f_2+f_3)-\frac{3f^{(4)}(\xi)h^5}{80}$$

25.4.14 **(Bode's rule)**

$$\int_{x_0}^{x_4} f(x)dx=\frac{2h}{45}(7f_0+32f_1+12f_2$$
$$+32f_3+7f_4)-\frac{8f^{(6)}(\xi)h^7}{945}$$

25.4.15

$$\int_{x_0}^{x_5} f(x)dx=\frac{5h}{288}(19f_0+75f_1+50f_2+50f_3$$
$$+75f_4+19f_5)-\frac{275f^{(6)}(\xi)h^7}{12096}$$

25.4.16

$$\int_{x_0}^{x_6} f(x)dx=\frac{h}{140}(41f_0+216f_1+27f_2+272f_3$$
$$+27f_4+216f_5+41f_6)-\frac{9f^{(8)}(\xi)h^9}{1400}$$

25.4.17

$$\int_{x_0}^{x_7} f(x)dx=\frac{7h}{17280}(751f_0+3577f_1+1323f_2$$
$$+2989f_3+2989f_4+1323f_5+3577f_6$$
$$+751f_7)-\frac{8183f^{(8)}(\xi)h^9}{518400}$$

25.4.18

$$\int_{x_0}^{x_8} f(x)dx=\frac{4h}{14175}(989f_0+5888f_1-928f_2$$
$$+10496f_3-4540f_4+10496f_5-928f_6+5888f_7$$
$$+989f_8)-\frac{2368}{467775}f^{(10)}(\xi)h^{11}$$

25.4.19

$$\int_{x_0}^{x_9} f(x)dx=\frac{9h}{89600}\{2857(f_0+f_9)$$
$$+15741(f_1+f_8)+1080(f_2+f_7)+19344(f_3+f_6)$$
$$+5778(f_4+f_5)\}-\frac{173}{14620}f^{(10)}(\xi)h^{11}$$

*See page II.

25.4.20

$$\int_{x_0}^{x_{10}} f(x)dx=\frac{5h}{299376}\{16067(f_0+f_{10})+106300(f_1+f_9)-48525(f_2+f_8)+272400(f_3+f_7)-260550(f_4+f_6)+427368f_5\}-\frac{1346350}{326918592}f^{(12)}(\xi)h^{13}$$

Newton-Cotes Formulas (Open Type)

25.4.21

$$\int_{x_0}^{x_3} f(x)dx=\frac{3h}{2}(f_1+f_2)+\frac{f^{(2)}(\xi)h^3}{4}$$

25.4.22

$$\int_{x_0}^{x_4} f(x)dx=\frac{4h}{3}(2f_1-f_2+2f_3)+\frac{28f^{(4)}(\xi)h^5}{90}$$

25.4.23

$$\int_{x_0}^{x_5} f(x)dx=\frac{5h}{24}(11f_1+f_2+f_3+11f_4)+\frac{95f^{(4)}(\xi)h^5}{144}$$

25.4.24

$$\int_{x_0}^{x_6} f(x)dx=\frac{6h}{20}(11f_1-14f_2+26f_3-14f_4+11f_5)+\frac{41f^{(6)}(\xi)h^7}{140}$$

25.4.25

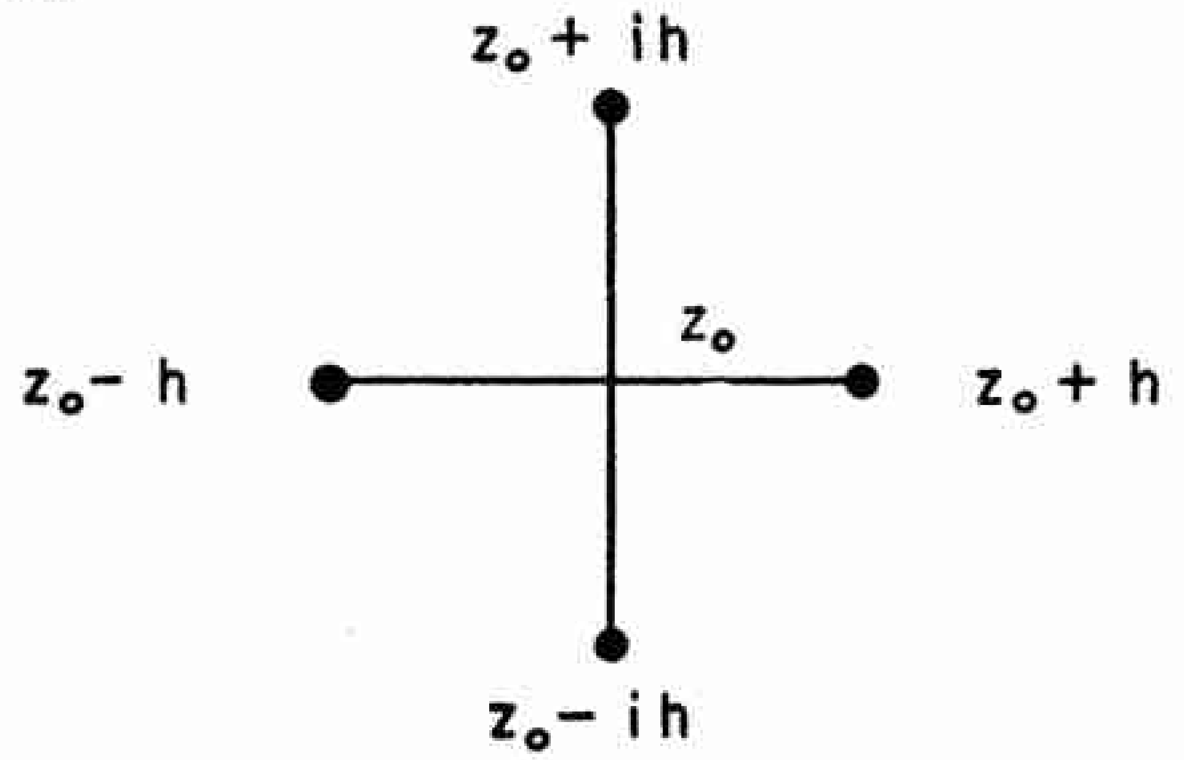
$$\int_{x_0}^{x_7} f(x)dx=\frac{7h}{1440}(611f_1-453f_2+562f_3+562f_4-453f_5+611f_6)+\frac{5257}{8640}f^{(6)}(\xi)h^7$$

25.4.26

$$\int_{x_0}^{x_8} f(x)dx=\frac{8h}{945}(460f_1-954f_2+2196f_3-2459f_4+2196f_5-954f_6+460f_7)+\frac{3956}{14175}f^{(8)}(\xi)h^9$$

Five Point Rule for Analytic Functions

25.4.27



$$\int_{z_0-h}^{z_0+h} f(z)dz=\frac{h}{15}\{24f(z_0)+4[f(z_0+h)+f(z_0-h)]-[f(z_0+ih)+f(z_0-ih)]\}+R$$

$|R|\leq\frac{|h|^7}{1890}\operatorname*{Max}_{z\in S}|f^{(6)}(z)|$, S designates the square with vertices $z_0+i^kh(k=0,1,2,3)$; h can be complex.

Chebyshev's Equal Weight Integration Formula

25.4.28 $$\int_{-1}^{1} f(x)dx=\frac{2}{n}\sum_{i=1}^{n} f(x_i)+R_n$$

Abscissas: x_i is the i^{th} zero of the polynomial part of

$$x^n \exp\left[\frac{-n}{2\cdot 3x^2}-\frac{n}{4\cdot 5x^3}-\frac{n}{6\cdot 7x^4}-\ldots\right]$$

(See **Table 25.5** for x_i.)

For $n=8$ and $n\geq 10$ some of the zeros are complex.
Remainder:

$$R_n=\int_{-1}^{+1}\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\xi)dx-\frac{2}{n(n+1)!}\sum_{i=1}^{n} x_i^{n+1}f^{(n+1)}(\xi_i)$$

where $\xi=\xi(x)$ satisfies $0\leq\xi\leq x$ and $0\leq\xi_i\leq x_i$

$$(i=1,\ldots,n)$$

Integration Formulas of Gaussian Type

(For Orthogonal Polynomials see chapter **22**)

Gauss' Formula

25.4.29 $$\int_{-1}^{1} f(x)dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials: Legendre polynomials $P_n(x)$, $P_n(1)=1$

Abscissas: x_i is the i^{th} zero of $P_n(x)$

Weights: $w_i=2/(1-x_i^2)[P_n'(x_i)]^2$ *
(See **Table 25.4** for x_i and w_i.)

$$R_n=\frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3}f^{(2n)}(\xi)\qquad(-1<\xi<1)$$

Gauss' Formula, Arbitrary Interval

25.4.30 $$\int_a^b f(y)dy=\frac{b-a}{2}\sum_{i=1}^{n} w_i f(y_i)+R_n$$

$$y_i=\left(\frac{b-a}{2}\right)x_i+\left(\frac{b+a}{2}\right)$$

*See page II.

Related orthogonal polynomials: $P_n(x)$, $P_n(1)=1$
Abscissas: x_i is the i^{th} zero of $P_n(x)$
* Weights: $w_i=2/(1-x_i^2)\,[P_n'(x_i)]^2$

$$R_n=\frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3}f^{(2n)}(\xi) \quad *$$

Radau's Integration Formula

25.4.31

$$\int_{-1}^{1} f(x)dx=\frac{2}{n^2}f_{-1}+\sum_{i=1}^{n-1} w_i f(x_i)+R_n$$

Related polynomials:

$$\frac{P_{n-1}(x)+P_n(x)}{x+1}$$

Abscissas: x_i is the i^{th} zero of

$$\frac{P_{n-1}(x)+P_n(x)}{x+1}$$

Weights:

$$w_i=\frac{1}{n^2}\frac{1-x_i}{[P_{n-1}(x_i)]^2}=\frac{1}{1-x_i}\frac{1}{[P'_{n-1}(x_i)]^2}$$

Remainder:

$$R_n=\frac{2^{2n-1}\cdot n}{[(2n-1)!]^3}[(n-1)!]^4 f^{(2n-1)}(\xi) \qquad (-1<\xi<1)$$

Lobatto's Integration Formula

25.4.32

$$\int_{-1}^{1} f(x)dx=\frac{2}{n(n-1)}[f(1)+f(-1)] +\sum_{i=2}^{n-1} w_i f(x_i)+R_n$$

Related polynomials: $P'_{n-1}(x)$

Abscissas: x_i is the $(i-1)^{\text{st}}$ zero of $P'_{n-1}(x)$

Weights:

$$w_i=\frac{2}{n(n-1)[P_{n-1}(x_i)]^2} \qquad (x_i\neq\pm1)$$

(See **Table 25.6** for x_i and w_i.)

Remainder:

$$R_n=\frac{-n(n-1)^3 2^{2n-1}[(n-2)!]^4}{(2n-1)[(2n-2)!]^3}f^{(2n-2)}(\xi) \qquad (-1<\xi<1)$$

25.4.33

$$\int_0^1 x^k f(x)dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials:

$$q_n(x)=\sqrt{k+2n+1}\,P_n^{(k,0)}(1-2x)$$

(For the Jacobi polynomials $P_n^{(k,0)}$ see chapter **22**.)

Abscissas:

x_i is the i^{th} zero of $q_n(x)$

Weights:

$$w_i=\left\{\sum_{j=0}^{n-1}[q_j(x_i)]^2\right\}^{-1} \quad *$$

(See **Table 25.8** for x_i and w_i.)

Remainder:

$$R_n=\frac{f^{(2n)}(\xi)}{(k+2n+1)(2n)!}\left[\frac{n!(k+n)!}{(k+2n)!}\right]^2 \qquad (0<\xi<1)$$

25.4.34

$$\int_0^1 f(x)\sqrt{1-x}\,dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{1-x}}P_{2n+1}(\sqrt{1-x}),\ P_{2n+1}(1)=1$$

Abscissas: $x_i=1-\xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n+1}(x)$.
Weights: $w_i=2\xi_i^2 w_i^{(2n+1)}$ where $w_i^{(2n+1)}$ are the Gaussian weights of order $2n+1$.
Remainder:

$$R_n=\frac{2^{4n+3}[(2n+1)!]^4}{(2n)!(4n+3)[(4n+2)!]^2}f^{(2n)}(\xi) \qquad (0<\xi<1)$$

25.4.35

$$\int_a^b f(y)\sqrt{b-y}\,dy=(b-a)^{3/2}\sum_{i=1}^{n} w_i f(y_i)$$

$$y_i=a+(b-a)x_i$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{1-x}}P_{2n+1}(\sqrt{1-x}),\ P_{2n+1}(1)=1$$

Abscissas: $x_i=1-\xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n+1}(x)$.
Weights: $w_i=2\xi_i^2 w_i^{(2n+1)}$ where $w_i^{(2n+1)}$ are the Gaussian weights of order $2n+1$.

*See page II.

25.4.36 $$\int_0^1 \frac{f(x)}{\sqrt{1-x}}\,dx=\sum_{i=1}^n w_i f(x_i)+R_n$$

Related orthogonal polynomials:

$$P_{2n}(\sqrt{1-x}),\ P_{2n}(1)=1$$

Abscissas: $x_i=1-\xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n}(x)$.
Weights: $w_i=2w_i^{(2n)}$, $w_i^{(2n)}$ are the Gaussian weights of order $2n$.
Remainder:

$$R_n=\frac{2^{4n+1}}{4n+1}\frac{[(2n)!]^3}{[(4n)!]^2}f^{(2n)}(\xi)\qquad(0<\xi<1)$$

25.4.37 $$\int_a^b \frac{f(y)}{\sqrt{b-y}}\,dy=\sqrt{b-a}\sum_{i=1}^n w_i f(y_i)+R_n$$

$$y_i=a+(b-a)x_i$$

Related orthogonal polynomials:

$$P_{2n}(\sqrt{1-x}),P_{2n}(1)=1$$

Abscissas:
$x_i=1-\xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n}(x)$.

Weights: $w_i=2w_i^{(2n)}$, $w_i^{(2n)}$ are the Gaussian weights of order $2n$.

25.4.38 $$\int_{-1}^{+1} \frac{f(x)}{\sqrt{1-x^2}}\,dx=\sum_{i=1}^n w_i f(x_i)+R_n$$

Related orthogonal polynomials: Chebyshev Polynomials of First Kind

$$T_n(x),T_n(1)=\frac{1}{2^{n-1}}$$

Abscissas:

$$x_i=\cos\frac{(2i-1)\pi}{2n}$$

Weights:

$$w_i=\frac{\pi}{n}$$

Remainder:

$$R_n=\frac{\pi}{(2n)!2^{2n-1}}f^{(2n)}(\xi)\quad(-1<\xi<1)$$

25.4.39

$$\int_a^b \frac{f(y)dy}{\sqrt{(y-a)(b-y)}}=\sum_{i=1}^n w_i f(y_i)+R_n$$

$$y_i=\frac{b+a}{2}+\frac{b-a}{2}x_i$$

Related orthogonal polynomials:

$$T_n(x),T_n(1)=\frac{1}{2^{n-1}}$$

Abscissas:

$$x_i=\cos\frac{(2i-1)\pi}{2n}$$

Weights:

$$w_i=\frac{\pi}{n}$$

25.4.40

$$\int_{-1}^{+1} f(x)\sqrt{1-x^2}dx=\sum_{i=1}^n w_i f(x_i)+R_n$$

Related orthogonal polynomials: Chebyshev Polynomials of Second Kind

$$U_n(x)=\frac{\sin[(n+1)\arccos x]}{\sin(\arccos x)}$$ *

Abscissas:

$$x_i=\cos\frac{i}{n+1}\pi$$ *

Weights:

$$w_i=\frac{\pi}{n+1}\sin^2\frac{i}{n+1}\pi$$ *

Remainder:

$$R_n=\frac{\pi}{(2n)!2^{2n+1}}f^{(2n)}(\xi)\qquad(-1<\xi<1)$$

25.4.41

$$\int_a^b \sqrt{(y-a)(b-y)}f(y)dy=\left(\frac{b-a}{2}\right)^2\sum_{i=1}^n w_i f(y_i)+R_n$$

$$y_i=\frac{b+a}{2}+\frac{b-a}{2}x_i$$

Related orthogonal polynomials:

$$U_n(x)=\frac{\sin[(n+1)\arccos x]}{\sin(\arccos x)}$$ *

Abscissas:

$$x_i=\cos\frac{i}{n+1}\pi$$ *

Weights:

$$w_i=\frac{\pi}{n+1}\sin^2\frac{i}{n+1}\pi$$ *

25.4.42 $$\int_0^1 f(x)\sqrt{\frac{x}{1-x}}\,dx=\sum_{i=1}^n w_i f(x_i)+R_n$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{x}}T_{2n+1}(\sqrt{x})$$

Abscissas:

$$x_i=\cos^2\frac{2i-1}{2n+1}\cdot\frac{\pi}{2}$$

Weights:

$$w_i=\frac{2\pi}{2n+1}x_i$$

Remainder:

$$R_n=\frac{\pi}{(2n)!2^{4n+1}}f^{(2n)}(\xi)\qquad(0<\xi<1)$$

25.4.43

$$\int_a^b f(x)\sqrt{\frac{x-a}{b-x}}\,dx=(b-a)\sum_{i=1}^n w_i f(y_i)+R_n$$

$$y_i=a+(b-a)x_i$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{x}}\,T_{2n+1}(\sqrt{x})$$

Abscissas:

$$x_i=\cos^2\frac{2i-1}{2n+1}\cdot\frac{\pi}{2}$$

Weights:

$$w_i=\frac{2\pi}{2n+1}\,x_i$$

25.4.44 $$\int_0^1 \ln x\, f(x)dx=\sum_{i=1}^n w_i f(x_i)+R_n$$

Related orthogonal polynomials: polynomials orthogonal with respect to the weight function $-\ln x$
Abscissas: See **Table 25.7**
Weights: See **Table 25.7**

25.4.45

$$\int_0^\infty e^{-x}f(x)dx=\sum_{i=1}^n w_i f(x_i)+R_n$$

Related orthogonal polynomials: Laguerre polynomials $L_n(x)$.
Abscissas: x_i is the i^{th} zero of $L_n(x)$
Weights:

* $$w_i=\frac{x_i}{(n+1)^2[L_{n+1}(x_i)]^2}$$

(See **Table 25.9** for x_i and w_i.)
Remainder:

$$R_n=\frac{(n!)^2}{(2n)!}f^{(2n)}(\xi)\qquad(0<\xi<\infty)$$

25.4.46

$$\int_{-\infty}^\infty e^{-x^2}f(x)dx=\sum_{i=1}^n w_i f(x_i)+R_n$$

Related orthogonal polynomials: Hermite polynomials $H_n(x)$.
Abscissas: x_i is the i^{th} zero of $H_n(x)$
Weights:

$$\frac{2^{n-1}n!\sqrt{\pi}}{n^2[H_{n-1}(x_i)]^2}$$

(See **Table 25.10** for x_i and w_i.)

Remainder:

$$R_n=\frac{n!\sqrt{\pi}}{2^n(2n)!}f^{(2n)}(\xi)\qquad(-\infty<\xi<\infty)$$

Filon's Integration Formula [3]

25.4.47

$$\int_{x_0}^{x_{2n}} f(x)\cos tx\,dx=h\Big[\alpha(th)(f_{2n}\sin tx_{2n}-f_0\sin tx_0)+\beta(th)\cdot C_{2n}+\gamma(th)\cdot C_{2n-1}+\frac{2}{45}th^4S'_{2n-1}\Big]-R_n$$

25.4.48

$$C_{2n}=\sum_{i=0}^n f_{2i}\cos(tx_{2i})-\tfrac{1}{2}[f_{2n}\cos tx_{2n}+f_0\cos tx_0]$$

25.4.49

$$C_{2n-1}=\sum_{i=1}^n f_{2i-1}\cos tx_{2i-1}$$

25.4.50

$$S'_{2n-1}=\sum_{i=1}^n f^{(3)}_{2i-1}\sin tx_{2i-1}$$

25.4.51

$$R_n=\frac{1}{90}nh^5f^{(4)}(\xi)+O(th^7)$$

25.4.52

$$\alpha(\theta)=\frac{1}{\theta}+\frac{\sin 2\theta}{2\theta^2}-\frac{2\sin^2\theta}{\theta^3}$$

$$\beta(\theta)=2\left(\frac{1+\cos^2\theta}{\theta^2}-\frac{\sin 2\theta}{\theta^3}\right)$$

$$\gamma(\theta)=4\left(\frac{\sin\theta}{\theta^3}-\frac{\cos\theta}{\theta^2}\right)$$

For small θ we have

25.4.53

$$\alpha=\frac{2\theta^3}{45}-\frac{2\theta^5}{315}+\frac{2\theta^7}{4725}-\dots$$

$$\beta=\frac{2}{3}+\frac{2\theta^2}{15}-\frac{4\theta^4}{105}+\frac{2\theta^6}{567}-\dots$$

$$\gamma=\frac{4}{3}-\frac{2\theta^2}{15}+\frac{\theta^4}{210}-\frac{\theta^6}{11340}+\dots$$

25.4.54

$$\int_{x_0}^{x_{2n}} f(x)\sin tx\,dx=h\Big[\alpha(th)(f_0\cos tx_0-f_{2n}\cos tx_{2n})+\beta S_{2n}+\gamma S_{2n-1}+\frac{2}{45}th^4C'_{2n-1}\Big]-R_n$$

25.4.55

$$S_{2n}=\sum_{i=0}^n f_{2i}\sin(tx_{2i})-\frac{1}{2}[f_{2n}\sin(tx_{2n})+f_0\sin(tx_0)]$$

[3] For certain difficulties associated with this formula, see the article by J. W. Tukey, p. 400, "On Numerical Approximation," Ed. R. E. Langer, Madison, 1959.

*See page II.

25.4.56 $$S_{2n-1}=\sum_{i=1}^{n} f_{2i-1}\sin(tx_{2i-1})$$

25.4.57 $$C'_{2n-1}=\sum_{i=1}^{n} f^{(3)}_{2i-1}\cos(tx_{2i-1})$$

(See **Table 25.11** for α, β, γ.)

Iterated Integrals

25.4.58

$$\int_0^x dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_3} dt_2 \int_0^{t_2} f(t_1)dt_1 = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}f(t)dt$$

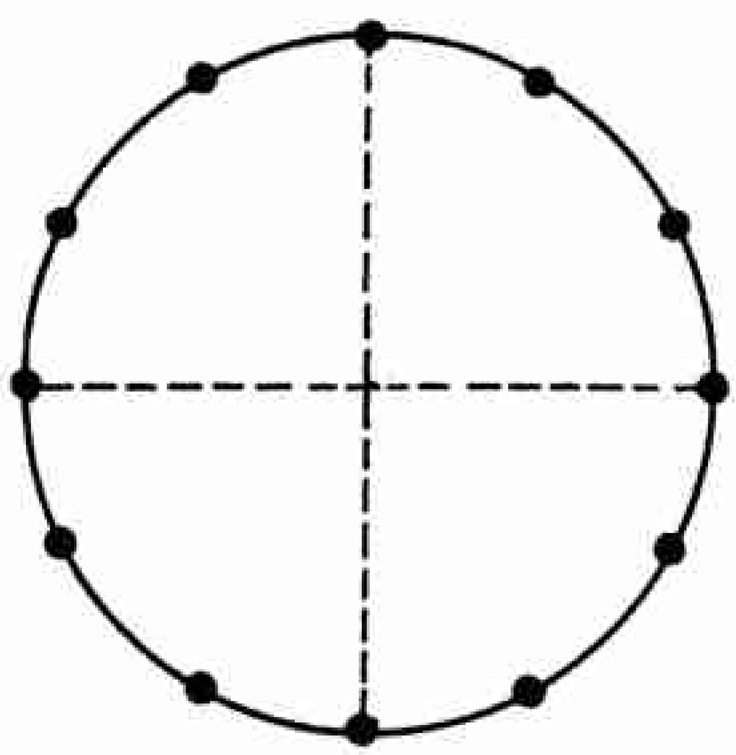
25.4.59

$$\int_a^x dt_n \int_a^{t_n} dt_{n-1} \ldots \int_a^{t_3} dt_2 \int_a^{t_2} f(t_1)dt_1 = \frac{(x-a)^n}{(n-1)!}\int_0^1 t^{n-1}f(x-(x-a)t)dt$$

Multidimensional Integration

Circumference of Circle Γ: $x^2+y^2=h^2$.

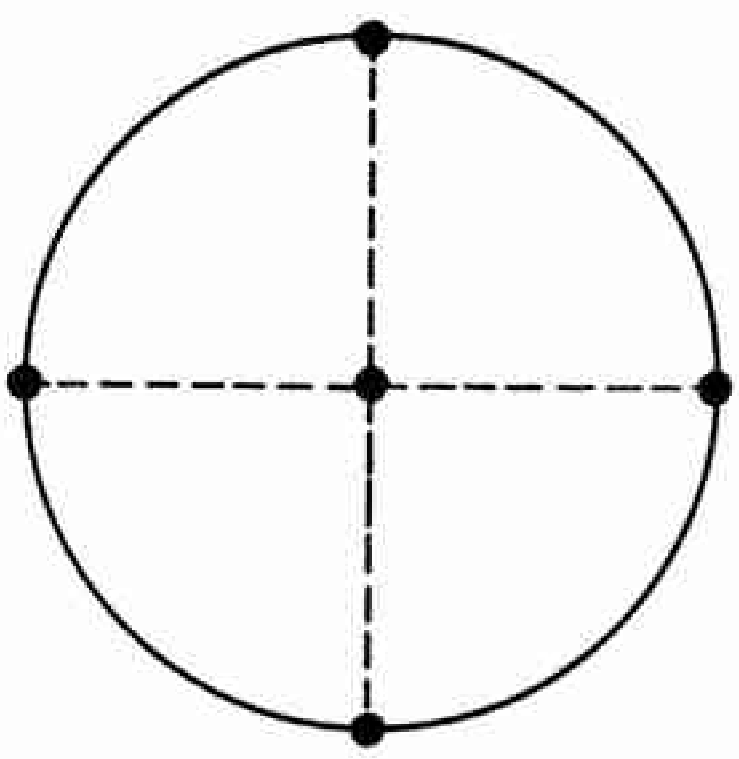
25.4.60



$$\frac{1}{2\pi h}\int_\Gamma f(x,y)ds=\frac{1}{2m}\sum_{n=1}^{2m} f\left(h\cos\frac{\pi n}{m}, h\sin\frac{\pi n}{m}\right)+O(h^{2m-2})$$

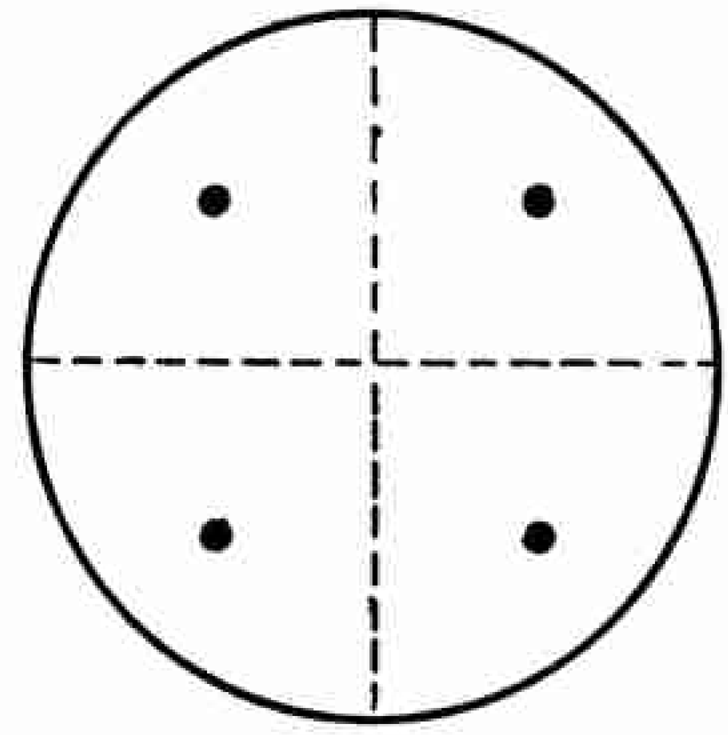
Circle C: $x^2+y^2\leq h^2$.

25.4.61

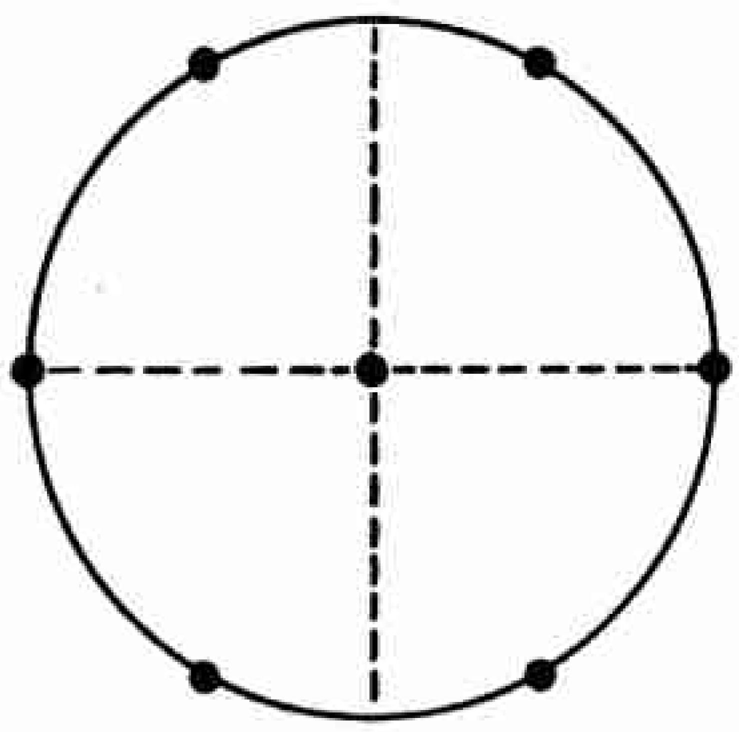


$$\frac{1}{\pi h^2}\iint_C f(x,y)dxdy=\sum_{i=1}^{n} w_i f(x_i,y_i)+R$$

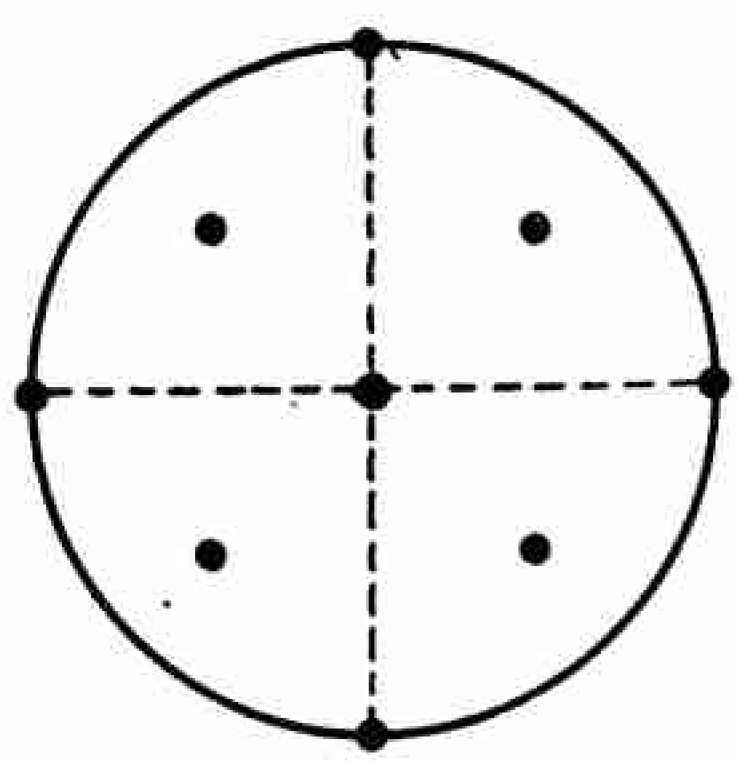
(x_i, y_i)	w_i	
$(0,0)$	1/2	$R=O(h^4)$
$(\pm h, 0)$, $(0, \pm h)$	1/8	



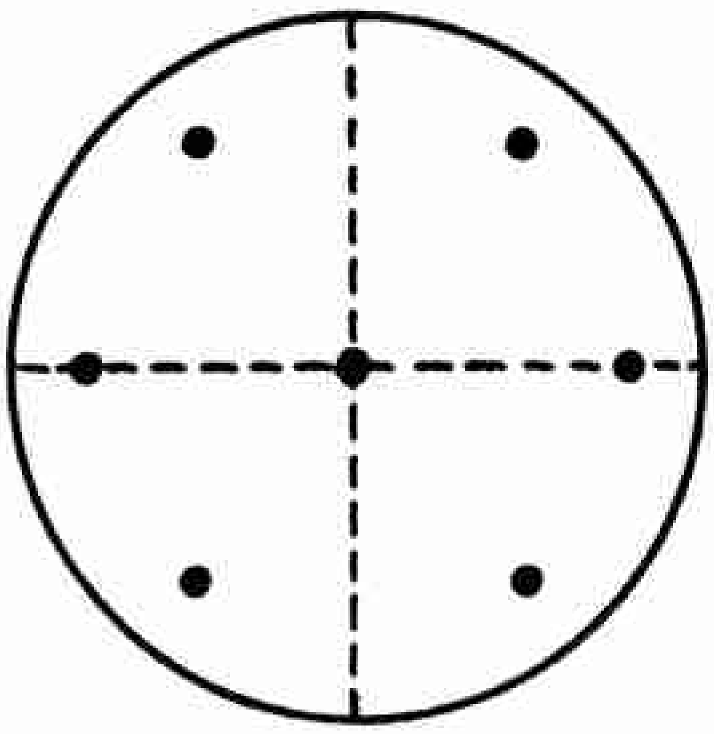
(x_i, y_i)	w_i	
$\left(\pm\frac{h}{2}, \pm\frac{h}{2}\right)$	1/4	$R=O(h^4)$



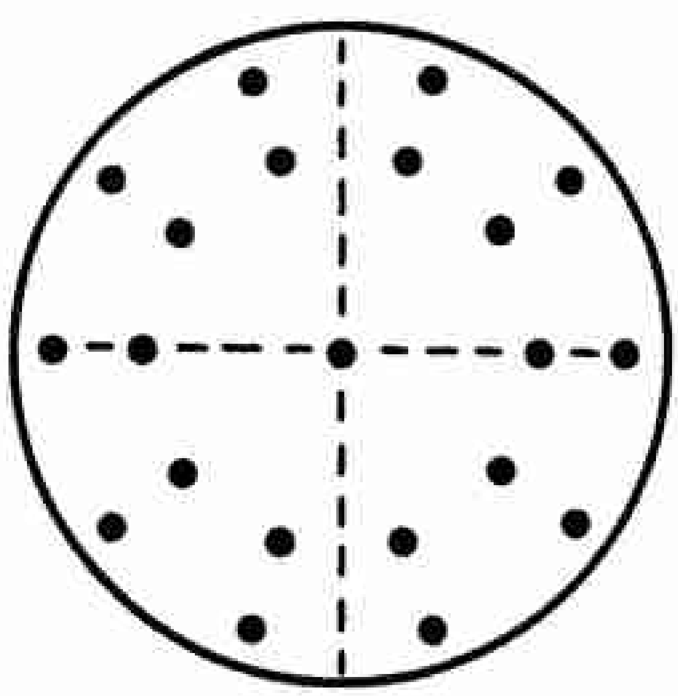
(x_i, y_i)	w_i	
$(0,0)$	1/2	
$(\pm h, 0)$	1/12	$R=O(h^4)$
$\left(\pm\frac{h}{2}, \pm\frac{h}{2}\sqrt{3}\right)$	1/12	



(x_i, y_i)	w_i	
$(0,0)$	$1/6$	
$(\pm h, 0)$	$1/24$	$R=O(h^6)$
$(0, \pm h)$	$1/24$	
$\left(\pm\frac{h}{2}, \pm\frac{h}{2}\right)$	$1/6$	



(x_i, y_i)	w_i	
$(0,0)$	$1/4$	
$\left(\pm\sqrt{\frac{2}{3}}\,h, 0\right)$	$1/8$	$R=O(h^6)$
$\left(\pm\sqrt{\frac{1}{6}}\,h, \pm\frac{h}{2}\sqrt{2}\right)$	$1/8$	



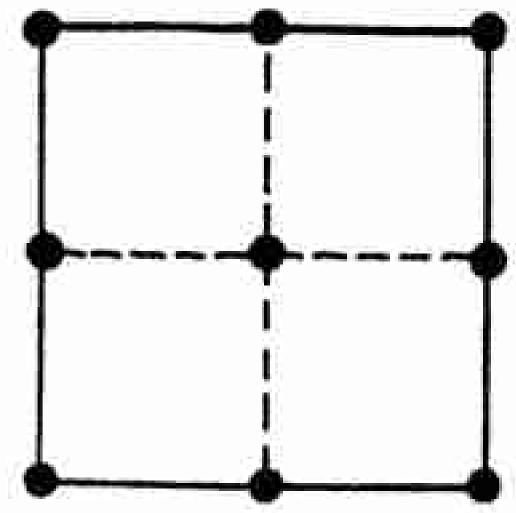
(x_i, y_i)	w_i
$(0,0)$	$1/9$
$\left(\sqrt{\frac{6-\sqrt{6}}{10}}\,h\cos\frac{2\pi k}{10}, \sqrt{\frac{6-\sqrt{6}}{10}}\,h\sin\frac{2\pi k}{10}\right)$	$\frac{16+\sqrt{6}}{360}$
$(k=1, \ldots, 10)$	
$\left(\sqrt{\frac{6+\sqrt{6}}{10}}\,h\cos\frac{2\pi k}{10}, \sqrt{\frac{6+\sqrt{6}}{10}}\,h\sin\frac{2\pi k}{10}\right)$	$\frac{16-\sqrt{6}}{360}$

$R=O(h^{10})$

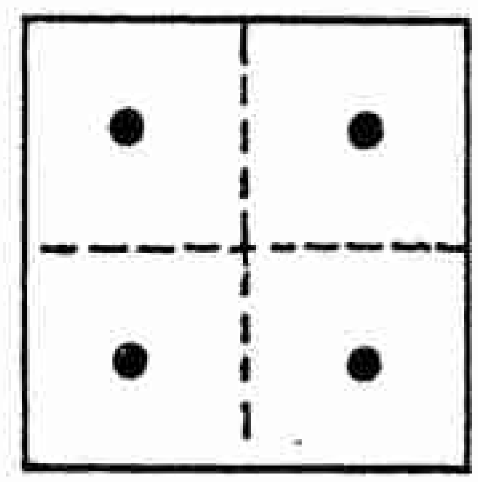
Square[4] S: $|x|\le h, |y|\le h$

25.4.62

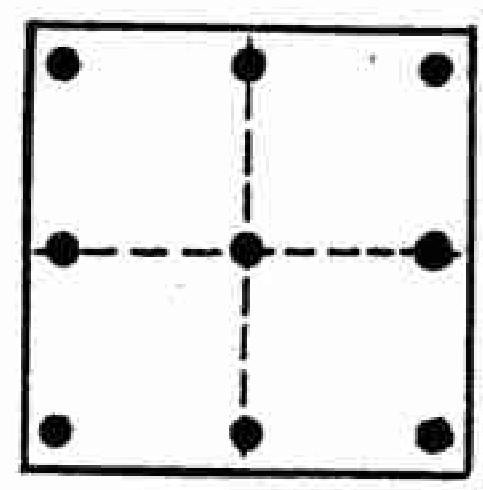
$$\frac{1}{4h^2}\iint_S f(x,y)\,dx\,dy=\sum_{i=1}^{n} w_i f(x_i,y_i)+R$$



(x_i, y_i)	w_i	
$(0,0)$	$4/9$	
$(\pm h, \pm h)$	$1/36$	$R=O(h^4)$
$(\pm h, 0)$	$1/9$	
$(0, \pm h)$	$1/9$	



(x_i, y_i)	w_i	
$\left(\pm h\sqrt{\frac{1}{3}}, \pm h\sqrt{\frac{1}{3}}\right)$	$1/4$	$R=O(h^4)$



(x_i, y_i)	w_i
$(0,0)$	$16/81$

[4] For regions, such as the square, cube, cylinder, etc., which are the Cartesian products of lower dimensional regions, one may always develop integration rules by "multiplying together" the lower dimensional rules. Thus if

$$\int_0^1 f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

is a one dimensional rule, then

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy \approx \sum_{i,j=1}^{n} w_i w_j f(x_i, x_j)$$

becomes a two dimensional rule. Such rules are not necessarily the most "economical".

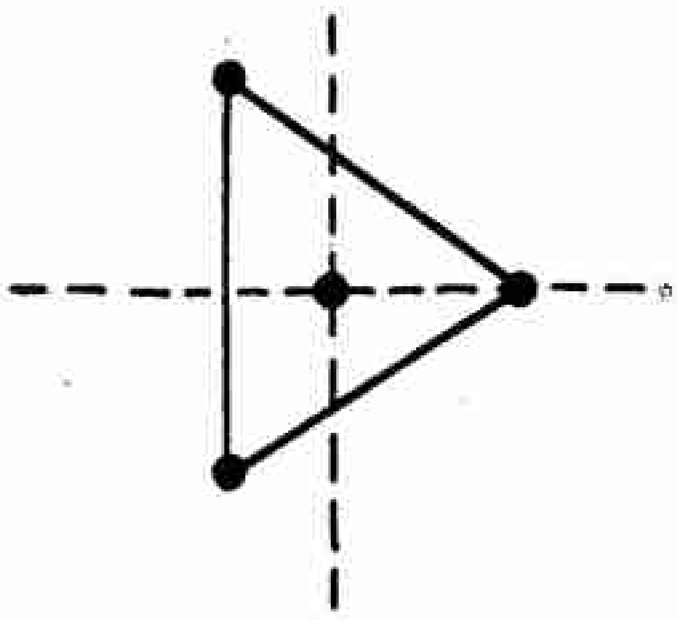
(x_i, y_i)	w_i	
$\left(\pm\sqrt{\frac{3}{5}}\,h, \pm\sqrt{\frac{3}{5}}\,h\right)$	25/324	
$\left(0, \pm\sqrt{\frac{3}{5}}\,h\right)$	10/81	$R=O(h^6)$
$\left(\pm\sqrt{\frac{3}{5}}\,h, 0\right)$	10/81	

Equilateral Triangle T

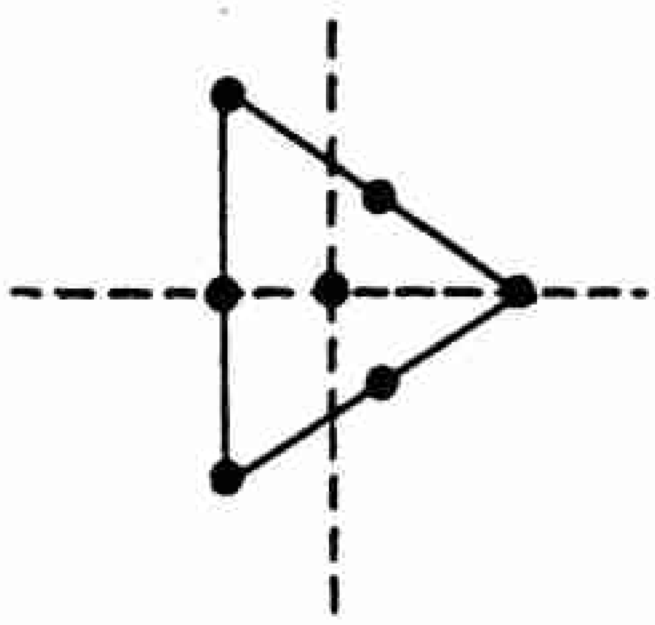
Radius of Circumscribed Circle$=h$

25.4.63

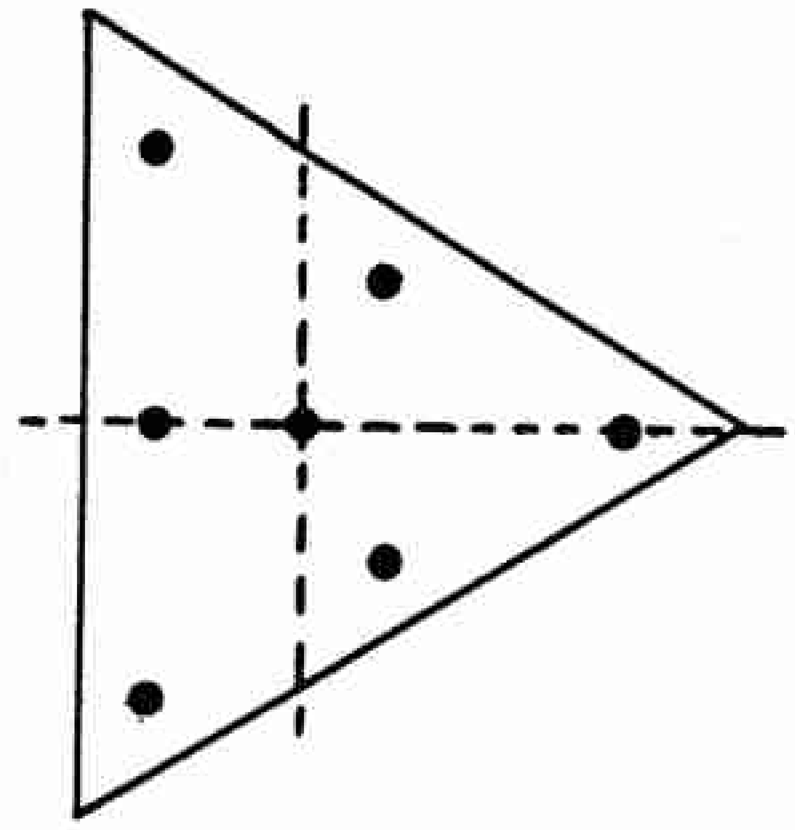
$$\frac{1}{\frac{3}{4}\sqrt{3}h^2}\iint_T f(x,y)\,dx\,dy=\sum_{i=1}^{n} w_i f(x_i,y_i)+R$$



(x_i, y_i)	w_i	
$(0,0)$	3/4	
$(h,0)$	1/12	$R=O(h^3)$
$\left(-\frac{h}{2}, \pm\frac{h}{2}\sqrt{3}\right)$	1/12	



(x_i, y_i)	w_i	
$(0,0)$	27/60	
$(h,0)$	3/60	
$\left(-\frac{h}{2}, \pm\frac{h}{2}\sqrt{3}\right)$	3/60	$R=O(h^4)$
$\left(-\frac{h}{2}, 0\right)$	8/60	
$\left(\frac{h}{4}, \pm\frac{h}{4}\sqrt{3}\right)$	8/60	



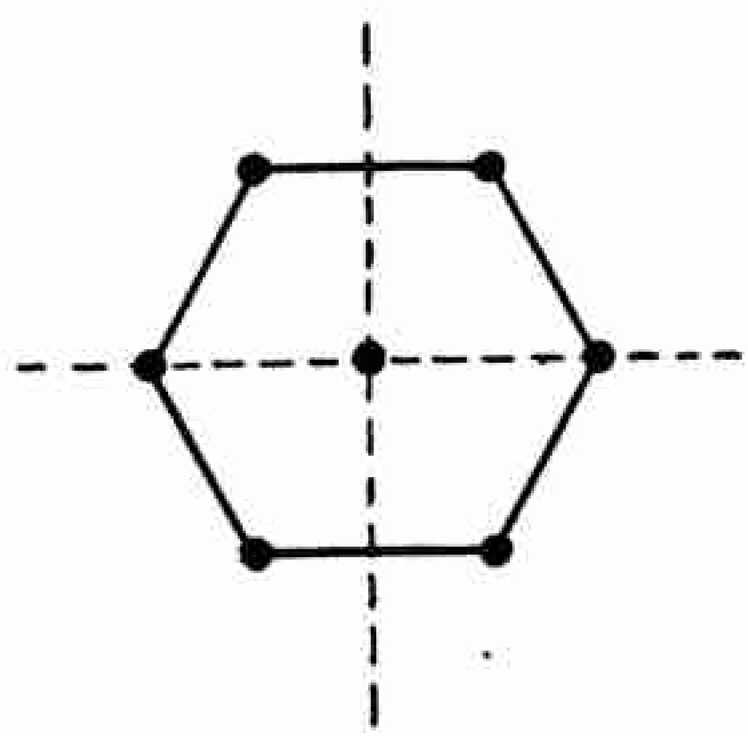
(x_i, y_i)	w_i	
$(0,0)$	270/1200	
$\left(\left(\frac{\sqrt{15}+1}{7}\right)h, 0\right)$	$\frac{155-\sqrt{15}}{1200}$	
$\left(\left(\frac{-\sqrt{15}+1}{14}\right)h, \pm\left(\frac{\sqrt{15}+1}{14}\right)\sqrt{3}h\right)$		$R=O(h^6)$
$\left(\left(-\frac{\sqrt{15}-1}{7}\right)h, 0\right)$	$\frac{155+\sqrt{15}}{1200}$	
$\left(\left(\frac{\sqrt{15}-1}{14}\right)h, \pm\left(\frac{\sqrt{15}-1}{14}\right)\sqrt{3}h\right)$		

Regular Hexagon H

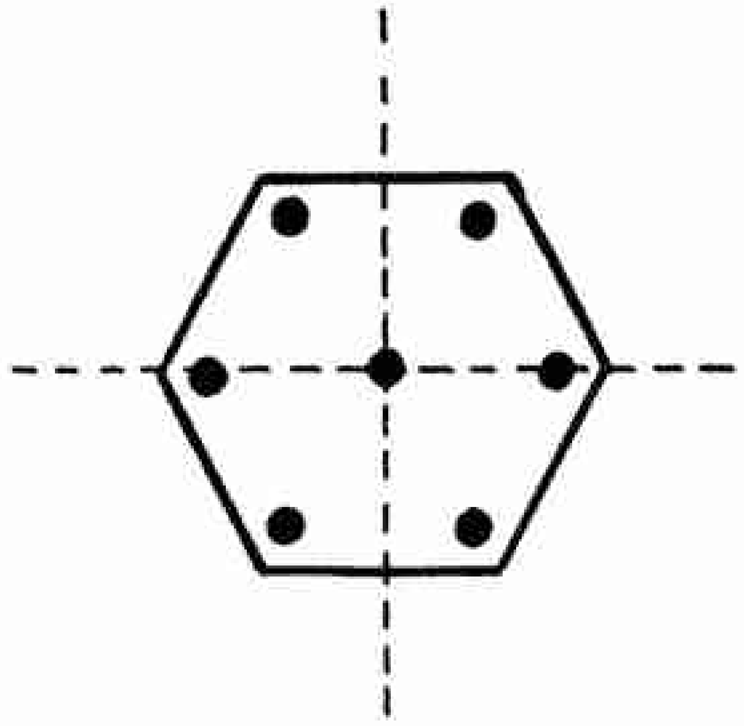
Radius of Circumscribed Circle$=h$

25.4.64

$$\frac{1}{\frac{3}{2}\sqrt{3}h^2}\iint_H f(x,y)\,dx\,dy=\sum_{i=1}^{n} w_i f(x_i, y_i)+R$$



(x_i, y_i)	w_i	
$(0,0)$	21/36	
$\left(\pm\frac{h}{2}, \pm\frac{h}{2}\sqrt{3}\right)$	5/72	$R=O(h^4)$
$(\pm h, 0)$	5/72	

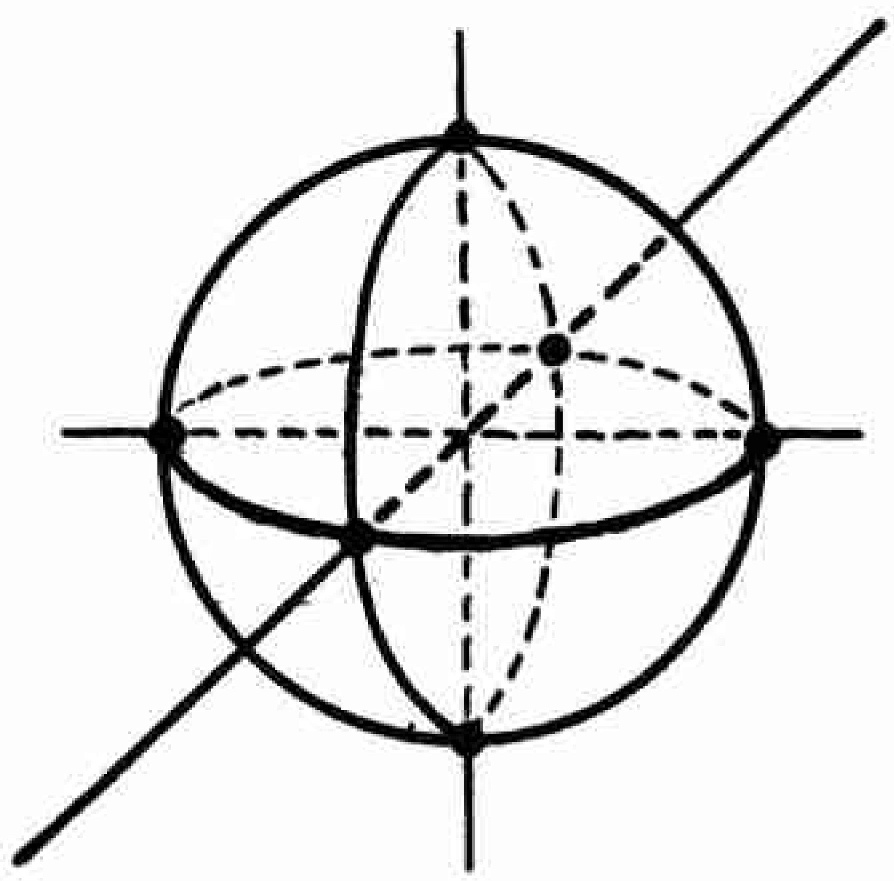


(x_i,y_i)	w_i	
$(0,0)$	258/1008	
$\left(\pm\frac{h}{10}\sqrt{14},\pm\frac{h}{10}\sqrt{42}\right)$	125/1008	$R=O(h^6)$
$\left(\pm h\frac{\sqrt{14}}{5},0\right)$	125/1008	

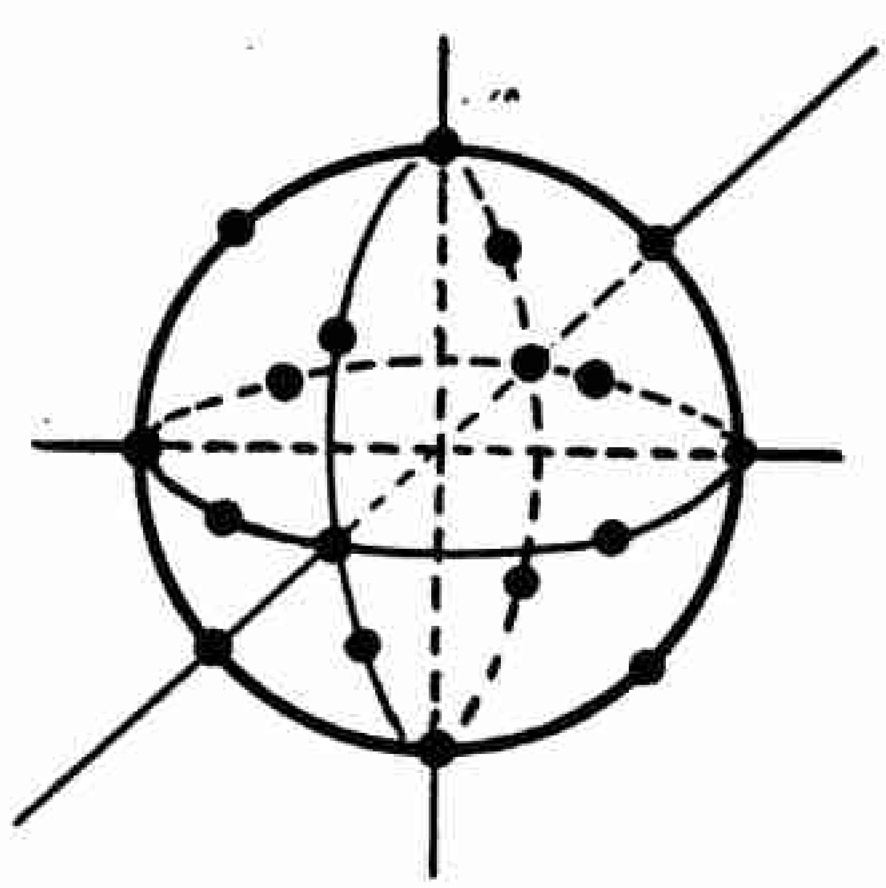
Surface of Sphere Σ: $x^2+y^2+z^2=h^2$

25.4.65

$$\frac{1}{4\pi h^2}\int\int_{\Sigma}f(x,y,z)d\sigma=\sum_{i=1}^{n}w_if(x_i,y_i,z_i)+R$$



(x_i,y_i,z_i)	w_i	
$(\pm h,0,0)$	1/6	
$(0,\pm h,0)$	1/6	$R=O(h^4)$
$(0,0,\pm h)$	1/6	



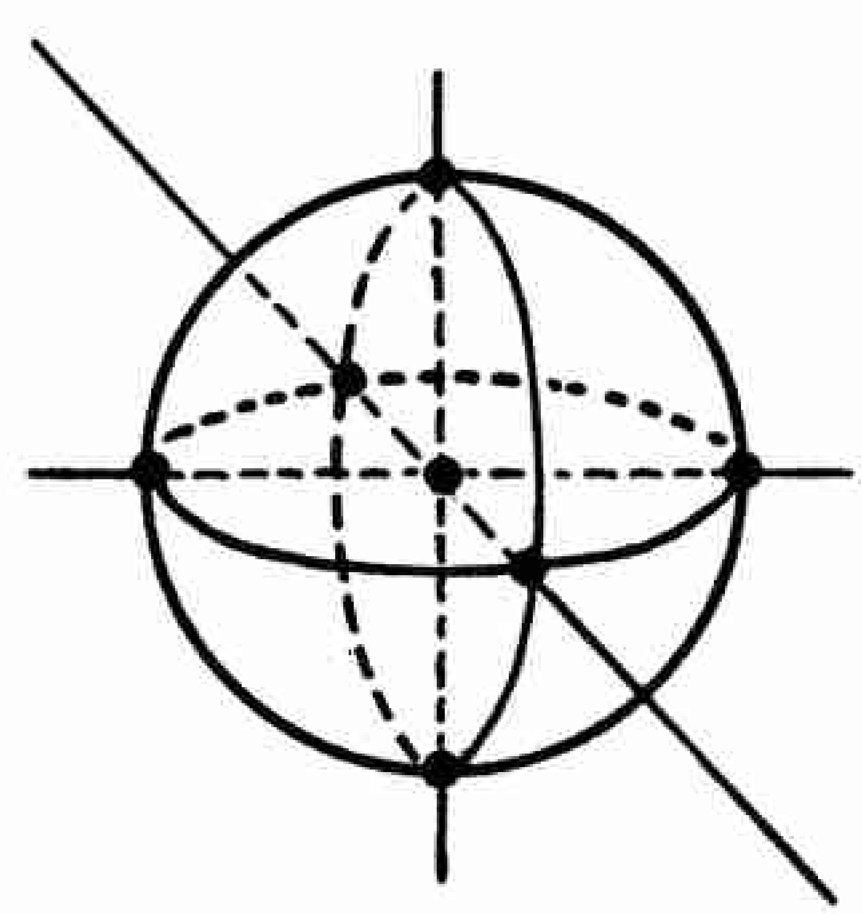
(x_i,y_i,z_i)	w_i	
$\left(\pm\sqrt{\frac{1}{2}}h,\pm\sqrt{\frac{1}{2}}h,0\right)$		
$\left(\pm\sqrt{\frac{1}{2}}h,0,\pm\sqrt{\frac{1}{2}}h\right)$	1/15	
$\left(0,\pm\sqrt{\frac{1}{2}}h,\pm\sqrt{\frac{1}{2}}h\right)$		
		$R=O(h^6)$
$(\pm h,0,0)$		
$(0,\pm h,0)$	1/30	
$(0,0,\pm h)$		

(x_i,y_i,z_i)	w_i	
$\left(\pm\sqrt{\frac{1}{3}}h,\pm\sqrt{\frac{1}{3}}h,\pm\sqrt{\frac{1}{3}}h\right)$	27/840	
$\left(\pm\sqrt{\frac{1}{2}}h,\pm\sqrt{\frac{1}{2}}h,0\right)$		
$\left(\pm\sqrt{\frac{1}{2}}h,0,\pm\sqrt{\frac{1}{2}}h\right)$	32/840	$R=O(h^8)$
$\left(0,\pm\sqrt{\frac{1}{2}}h,\pm\sqrt{\frac{1}{2}}h\right)$		
$(\pm h,0,0)$		
$(0,\pm h,0)$	40/840	
$(0,0,\pm h)$		

Sphere S: $x^2+y^2+z^2\le h^2$

25.4.66

$$\frac{1}{\frac{4}{3}\pi h^3}\int\int\int_{S}f(x,y,z)dxdydz=\sum_{i=1}^{n}w_if(x_i,y_i,z_i)+R$$



(x_i,y_i,z_i)	w_i
$(0,0,0)$	$2/5$
$(\pm h,0,0)$	$1/10$
$(0,\pm h,0)$	$1/10$
$(0,0,\pm h)$	$1/10$

$R=O(h^4)$

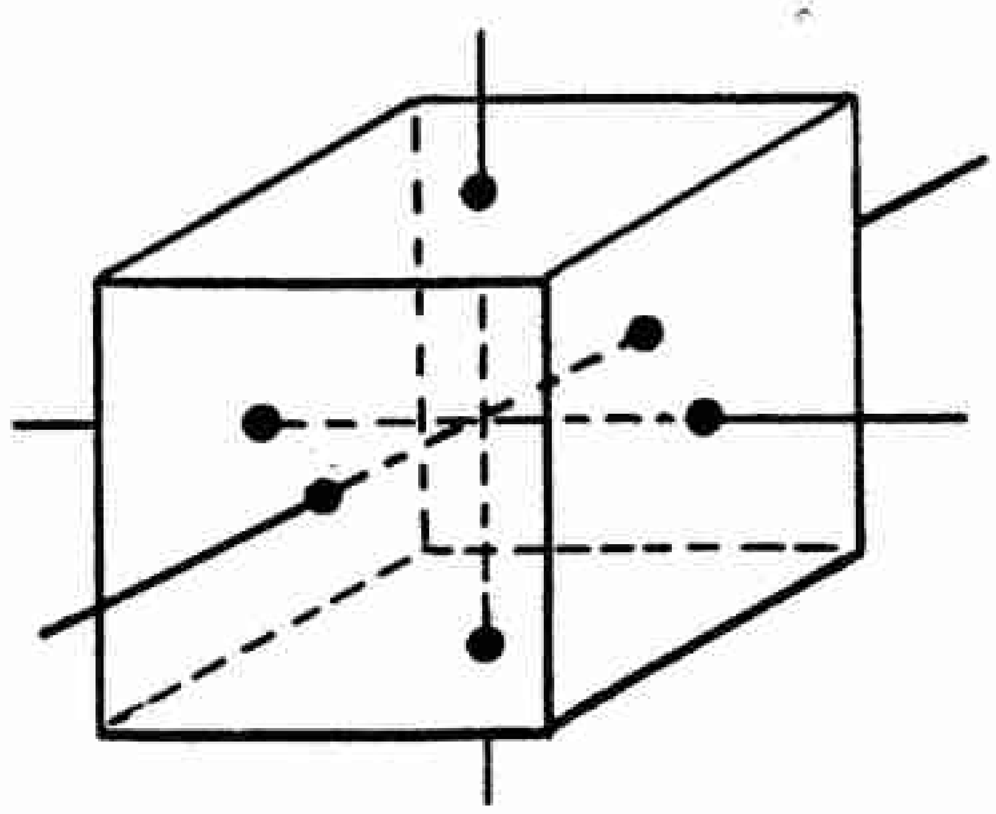
Cube[5] C: $|x|\leq h$

$|y|\leq h$

$|z|\leq h$

25.4.67

$$\frac{1}{8h^3}\iiint_C f(x,y,z)dxdydz=\sum_{i=1}^{n} w_i f(x_i,y_i,z_i)+R$$



(x_i,y_i,z_i)	w_i
$(\pm h,0,0)$	$1/6$
$(0,\pm h,0)$	$1/6$
$(0,0,\pm h)$	$1/6$

$R=O(h^4)$

25.4.68

$$\frac{1}{8h^3}\iiint_C f(x,y,z)dxdydz$$

$$=\frac{1}{360}\left[-496f_m+128\sum f_r+8\sum f_f+5\sum f_v\right]+O(h^6)$$

25.4.69

$$=\frac{1}{450}\left[91\sum f_f-40\sum f_e+16\sum f_d\right]+O(h^6)$$

where $f_m=f(0,0,0)$.

[5] See footnote to **25.4.62**.

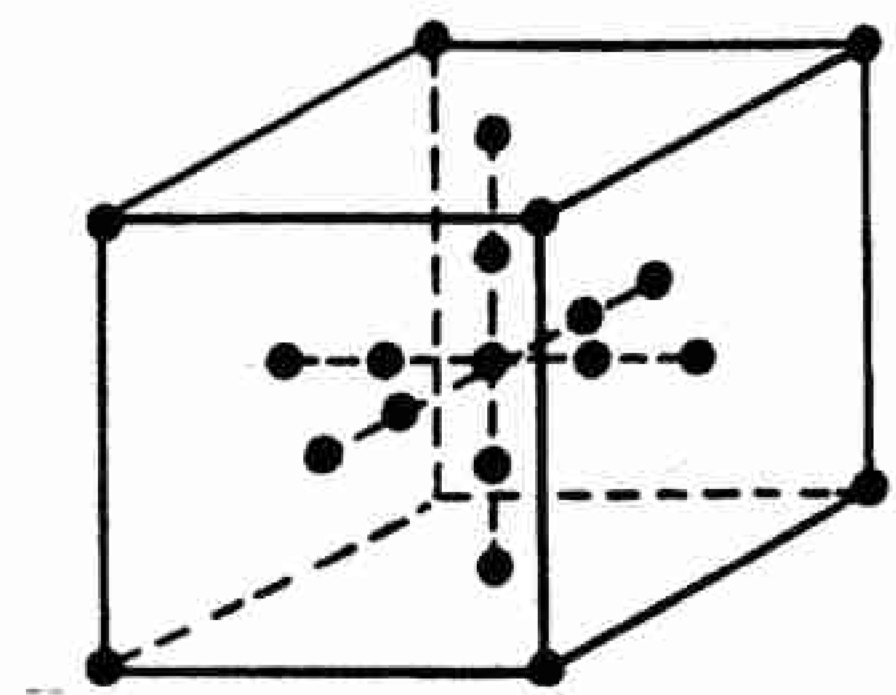
$\sum f_r$=sum of values of f at the 6 points midway from the center of C to the 6 faces.

$\sum f_f$=sum of values of f at the 6 centers of the faces of C.

$\sum f_v$=sum of values of f at the 8 vertices of C.

$\sum f_e$=sum of values of f at the 12 midpoints of edges of C.

$\sum f_d$=sum of values of f at the 4 points on the diagonals of each face at a distance of $\frac{1}{2}\sqrt{5}h$ from the center of the face.



Tetrahedron: $\mathscr{T}$

25.4.70

$$\frac{1}{V}\iiint_{\mathscr{T}} f(x,y,z)dxdydz=\frac{1}{40}\sum f_v+\frac{9}{40}\sum f_f$$

$$+\text{terms of 4}^{\text{th}}\text{ order}$$

$$=\frac{32}{60}f_m+\frac{1}{60}\sum f_v+\frac{4}{60}\sum f_e$$

$$+\text{terms of 4}^{\text{th}}\text{ order}$$

where

V: Volume of $\mathscr{T}$

$\sum f_v$: Sum of values of the function at the vertices of $\mathscr{T}$.

$\sum f_e$: Sum of values of the function at midpoints of the edges of $\mathscr{T}$.

$\sum f_f$: Sum of values of the function at the center of gravity of the faces of $\mathscr{T}$.

f_m: Value of function at center of gravity of $\mathscr{T}$.

25.5. Ordinary Differential Equations[6]

First Order: $y'=f(x, y)$

Point Slope Formula

25.5.1 $$y_{n+1}=y_n+hy'_n+O(h^2)$$

25.5.2 $$y_{n+1}=y_{n-1}+2hy'_n+O(h^3)$$

Trapezoidal Formula

25.5.3 $$y_{n+1}=y_n+\frac{h}{2}(y'_{n+1}+y'_n)+O(h^3)$$

Adams' Extrapolation Formula

25.5.4

$$y_{n+1}=y_n+\frac{h}{24}(55y'_n-59y'_{n-1}+37y'_{n-2}-9y'_{n-3})+O(h^5)$$

Adams' Interpolation Formula

25.5.5

$$y_{n+1}=y_n+\frac{h}{24}(9y'_{n+1}+19y'_n-5y'_{n-1}+y'_{n-2})+O(h^5)$$

Runge-Kutta Methods

Second Order

25.5.6

$$y_{n+1}=y_n+\frac{1}{2}(k_1+k_2)+O(h^3)$$
$$k_1=hf(x_n,y_n), k_2=hf(x_n+h,y_n+k_1)$$

25.5.7

$$y_{n+1}=y_n+k_2+O(h^3)$$
$$k_1=hf(x_n,y_n), k_2=hf\left(x_n+\frac{1}{2}h,y_n+\frac{1}{2}k_1\right)$$

Third Order

25.5.8

$$y_{n+1}=y_n+\frac{1}{6}k_1+\frac{2}{3}k_2+\frac{1}{6}k_3+O(h^4)$$
$$k_1=hf(x_n,y_n), k_2=hf\left(x_n+\frac{1}{2}h,y_n+\frac{1}{2}k_1\right)$$
$$k_3=hf(x_n+h,y_n-k_1+2k_2)$$

25.5.9

$$y_{n+1}=y_n+\frac{1}{4}k_1+\frac{3}{4}k_3+O(h^4)$$
$$k_1=hf(x_n,y_n), k_2=hf\left(x_n+\frac{1}{3}h,y_n+\frac{1}{3}k_1\right)$$
$$k_3=hf\left(x_n+\frac{2}{3}h,y_n+\frac{2}{3}k_2\right)$$

Fourth Order

25.5.10

$$y_{n+1}=y_n+\frac{1}{6}k_1+\frac{1}{3}k_2+\frac{1}{3}k_3+\frac{1}{6}k_4+O(h^5)$$
$$k_1=hf(x_n,y_n), k_2=hf\left(x_n+\frac{1}{2}h,y_n+\frac{1}{2}k_1\right)$$
$$k_3=hf\left(x_n+\frac{1}{2}h,y_n+\frac{1}{2}k_2\right), k_4=hf(x_n+h,y_n+k_3)$$

25.5.11

$$y_{n+1}=y_n+\frac{1}{8}k_1+\frac{3}{8}k_2+\frac{3}{8}k_3+\frac{1}{8}k_4+O(h^5)$$
$$k_1=hf(x_n,y_n), k_2=hf\left(x_n+\frac{1}{3}h,y_n+\frac{1}{3}k_1\right)$$
$$k_3=hf\left(x_n+\frac{2}{3}h,y_n-\frac{1}{3}k_1+k_2\right),$$
$$k_4=hf(x_n+h,y_n+k_1-k_2+k_3)$$

Gill's Method

25.5.12

$$y_{n+1}=y_n+\frac{1}{6}\left(k_1+2\left(1-\sqrt{\frac{1}{2}}\right)k_2\right.$$
$$\left.+2\left(1+\sqrt{\frac{1}{2}}\right)k_3+k_4\right)+O(h^5)$$
$$k_1=hf(x_n, y_n)$$
$$k_2=hf\left(x_n+\frac{1}{2}h, y_n+\frac{1}{2}k_1\right)$$
$$k_3=hf\left(x_n+\frac{1}{2}h, y_n+\left(-\frac{1}{2}+\sqrt{\frac{1}{2}}\right)k_1\right.$$
$$\left.+\left(1-\sqrt{\frac{1}{2}}\right)k_2\right)$$
$$k_4=hf\left(x_n+h, y_n-\sqrt{\frac{1}{2}}k_2+\left(1+\sqrt{\frac{1}{2}}\right)k_3\right)$$

Predictor-Corrector Methods

Milne's Methods

25.5.13

$$\text{P:}\quad y_{n+1}=y_{n-3}+\frac{4h}{3}(2y'_n-y'_{n-1}+2y'_{n-2})+O(h^5)$$

$$\text{C:}\quad y_{n+1}=y_{n-1}+\frac{h}{3}(y'_{n-1}+4y'_n+y'_{n+1})+O(h^5)$$

[6] The reader is cautioned against possible instabilities especially in formulas **25.5.2** and **25.5.13**. See, e.g. [25.11], [25.12].

25.5.14

$$\text{P:}\quad y_{n+1}=y_{n-5}+\frac{3h}{10}(11y'_n-14y'_{n-1}+26y'_{n-2}-14y'_{n-3}+11y'_{n-4})+O(h^7)$$

$$\text{C:}\quad y_{n+1}=y_{n-3}+\frac{2h}{45}(7y'_{n+1}+32y'_n+12y'_{n-1}+32y'_{n-2}+7y'_{n-3})+O(h^7)$$

Formulas Using Higher Derivatives

25.5.15

$$\text{P:}\quad y_{n+1}=y_{n-2}+3(y_n-y_{n-1})+h^2(y''_n-y''_{n-1})+O(h^5)$$

$$\text{C:}\quad y_{n+1}=y_n+\frac{h}{2}(y'_{n+1}+y'_n)-\frac{h^2}{12}(y''_{n+1}-y''_n)+O(h^5)$$

25.5.16

$$\text{P:}\quad y_{n+1}=y_{n-2}+3(y_n-y_{n-1})+\frac{h^3}{2}(y'''_n+y'''_{n-1})+O(h^7)$$

$$\text{C:}\quad y_{n+1}=y_n+\frac{h}{2}(y'_{n+1}+y'_n)-\frac{h^2}{10}(y''_{n+1}-y''_n)+\frac{h^3}{120}(y'''_{n+1}+y'''_n)+O(h^7)$$

Systems of Differential Equations

First Order: $y'=f(x,y,z),\ z'=g(x,y,z).$

Second Order Runge-Kutta

25.5.17

$$y_{n+1}=y_n+\frac{1}{2}(k_1+k_2)+O(h^3),\qquad z_{n+1}=z_n+\frac{1}{2}(l_1+l_2)+O(h^3)$$

$$k_1=hf(x_n,y_n,z_n),\qquad l_1=hg(x_n,y_n,z_n)$$

$$k_2=hf(x_n+h,y_n+k_1,z_n+l_1),\qquad l_2=hg(x_n+h,y_n+k_1,z_n+l_1)$$

Fourth Order Runge-Kutta

25.5.18

$$y_{n+1}=y_n+\frac{1}{6}(k_1+2k_2+2k_3+k_4)+O(h^5),\qquad z_{n+1}=z_n+\frac{1}{6}(l_1+2l_2+2l_3+l_4)+O(h^5)$$

$$k_1=hf(x_n,y_n,z_n)\qquad l_1=hg(x_n,y_n,z_n)$$

$$k_2=hf\left(x_n+\frac{1}{2}h,y_n+\frac{1}{2}k_1,z_n+\frac{1}{2}l_1\right)\qquad l_2=hg\left(x_n+\frac{h}{2},y_n+\frac{k_1}{2},z_n+\frac{l_1}{2}\right)$$

$$k_3=hf\left(x_n+\frac{1}{2}h,y_n+\frac{1}{2}k_2,z_n+\frac{1}{2}l_2\right)\qquad l_3=hg\left(x_n+\frac{h}{2},y_n+\frac{k_2}{2},z_n+\frac{l_2}{2}\right)$$

$$k_4=hf(x_n+h,y_n+k_3,z_n+l_3)\qquad l_4=hg(x_n+h,y_n+k_3,z_n+l_3)$$

Second Order: $y''=f(x,y,y')$

Milne's Method

25.5.19

$$\text{P:}\quad y'_{n+1}=y'_{n-3}+\frac{4h}{3}(2y''_{n-2}-y''_{n-1}+2y''_n)+O(h^5)$$

$$\text{C:}\quad y'_{n+1}=y'_{n-1}+\frac{h}{3}(y''_{n-1}+4y''_n+y''_{n+1})+O(h^5)$$

Runge-Kutta Method

25.5.20

$$y_{n+1}=y_n+h\left[y'_n+\frac{1}{6}(k_1+k_2+k_3)\right]+O(h^5)$$

$$y'_{n+1}=y'_n+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$k_1=hf(x_n,y_n,y'_n)$$

$$k_2=hf\left(x_n+\frac{1}{2}h,y_n+\frac{h}{2}y'_n+\frac{h}{8}k_1,y'_n+\frac{k_1}{2}\right)$$

$$k_3=hf\left(x_n+\frac{1}{2}h,y_n+\frac{h}{2}y'_n+\frac{h}{8}k_1,y'_n+\frac{k_2}{2}\right)$$ *

$$k_4=hf\left(x_n+h,y_n+hy'_n+\frac{h}{2}k_3,y'_n+k_3\right)$$

Second Order: $y''=f(x,y)$

Milne's Method

25.5.21

$$\text{P:}\quad y_{n+1}=y_n+y_{n-2}-y_{n-3}+\frac{h^2}{4}(5y''_n+2y''_{n-1}+5y''_{n-2})+O(h^6)$$

$$\text{C:}\quad y_n=2y_{n-1}-y_{n-2}+\frac{h^2}{12}(y''_n+10y''_{n-1}+y''_{n-2})+O(h^6)$$

Runge-Kutta Method

25.5.22 $$y_{n+1}=y_n+h\left(y'_n+\frac{1}{6}(k_1+2k_2)\right)+O(h^4)$$

$$y'_{n+1}=y'_n+\frac{1}{6}k_1+\frac{2}{3}k_2+\frac{1}{6}k_3$$

$$k_1=hf(x_n,y_n)$$

$$k_2=hf\left(x_n+\frac{h}{2},y_n+\frac{h}{2}y'_n+\frac{h}{8}k_1\right)$$

$$k_3=hf\left(x_n+h,y_n+hy'_n+\frac{h}{2}k_2\right).$$

References

Texts

(For textbooks on numerical analysis, see texts in chapter 3)

[25.1] J. Balbrecht and L. Collatz, Zur numerischen Auswertung mehrdimensionaler Integrale, Z. Angew. Math. Mech. **38**, 1–15 (1958).

[25.2] Berthod-Zaborowski, Le calcul des intégrales de la forme: $\int_0^1 f(x) \log x \, dx$.
H. Mineur, Techniques de calcul numérique, pp. 555–556 (Librairie Polytechnique Ch. Béranger, Paris, France, 1952).

[25.3] W. G. Bickley, Formulae for numerical integration, Math. Gaz. **23**, 352 (1939).

[25.4] W. G. Bickley, Formulae for numerical differentiation, Math. Gaz. **25**, 19–27 (1941).

[25.5] W. G. Bickley, Finite difference formulae for the square lattice, Quart. J. Mech. Appl. Math., **1**, 35–42 (1948).

[25.6] G. Birkhoff and D. Young, Numerical quadrature of analytic and harmonic functions, J. Math. Phys. **29**, 217–221 (1950).

[25.7] L. Fox, The use and construction of mathematical tables, Mathematical Tables vol. I, National Physical Laboratory (Her Majesty's Stationery Office, London, England, 1956).

[25.8] S. Gill, Process for the step-by-step integration of differential equations in an automatic digital computing machine, Proc. Cambridge Philos. Soc. **47**, 96–108 (1951).

[25.9] P. C. Hammer and A. H. Stroud, Numerical evaluation of multiple integrals II, Math. Tables Aids Comp. **12**, 272–280 (1958).

[25.10] P. C. Hammer and A. W. Wymore, Numerical evaluation of multiple integrals I, Math. Tables Aids Comp. **11**, 59–67 (1957).

[25.11] P. Henrici, Discrete variable methods in ordinary differential equations (John Wiley & Sons, Inc., New York, N. Y., 1961).

[25.12] F. B. Hildebrand, Introduction to numerical analysis (McGraw-Hill Book Co., Inc., New York, N.Y., 1956).

[25.13] Z. Kopal, Numerical analysis (John Wiley & Sons, Inc., New York, N.Y., 1955).

[25.14] A. A. Markoff, Differenzenrechnung (B. G. Teubner, Leipzig, Germany, 1896).

[25.15] S. E. Mikeladze, Quadrature formulas for a regular function, Soobšč. Akad. Nauk Gruzin. SSR. **17**, 289–296 (1956).

[25.16] W. E. Milne, A note on the numerical integration of differential equations, J. Research NBS **43**, 537–542 (1949) RP2046.

[25.17] D. J. Panov, Formelsammlung zur numerischen Behandlung partieller Differentialgleichungen nach dem Differenzenverfahren (Akad. Verlag, Berlin, Germany, 1955).

[25.18] R. Radau, Études sur les formules d'approximation qui servent à calculer la valeur d'une intégrale définie, J. Math. Pures Appl. (3) **6**, 283–336 (1880).

[25.19] R. D. Richtmeyer, Difference methods for initial-value problems (Interscience Publishers, New York, N.Y., 1957).

[25.20] M. Sadowsky, A formula for approximate computation of a triple integral, Amer. Math. Monthly **47**, 539–543 (1940).

[25.21] H. E. Salzer, A new formula for inverse interpolation, Bull. Amer. Math. Soc. **50**, 513–516 (1944).

[25.22] H. E. Salzer, Formulas for complex Cartesian interpolation of higher degree, J. Math. Phys. **28**, 200–203 (1949).

[25.23] H. E. Salzer, Formulas for numerical integration of first and second order differential equations in the complex plane, J. Math. Phys. **29**, 207–216 (1950).

[25.24] H. E. Salzer, Formulas for numerical differentiation in the complex plane, J. Math. Phys. **31**, 155–169 (1952).

[25.25] A. Sard, Integral representations of remainders, Duke Math. J. **15**, 333–345 (1948).

[25.26] A. Sard, Remainders: functions of several variables, Acta Math. **84**, 319–346 (1951).

[25.27] G. Schulz, Formelsammlung zur praktischen Mathematik (DeGruyter and Co., Berlin, Germany, 1945).

[25.28] A. H. Stroud, A bibliography on approximate integration, Math. Comp. **15**, 52–80 (1961).

[25.29] G. J. Tranter, Integral transforms in mathematical physics (John Wiley & Sons, Inc., New York, N.Y., 1951).

[25.30] G. W. Tyler, Numerical integration with several variables, Canad. J. Math. **5**, 393–412 (1953).

Tables

[25.31] L. J. Comrie, Chambers' six-figure mathematical tables, vol. 2 (W. R. Chambers, Ltd., London, England, 1949).

[25.32] P. Davis and P. Rabinowitz, Abscissas and weights for Gaussian quadratures of high order, J. Research NBS **56**, 35–37 (1956) RP2645.

[25.33] P. Davis and P. Rabinowitz, Additional abscissas and weights for Gaussian quadratures of high order: Values for $n=64$, 80, and 96, J. Research NBS **60**, 613–614 (1958) RP2875.

[25.34] E. W. Dijkstra and A. van Wijngaarden, Table of Everett's interpolation coefficients (Elcelsior's Photo-offset, The Hague, Holland, 1955).

[25.35] H. Fishman, Numerical integration constants, Math. Tables Aids Comp. **11**, 1–9 (1957).

[25.36] H. J. Gawlik, Zeros of Legendre polynomials of orders 2–64 and weight coefficients of Gauss quadrature formulae, A.R.D.E. Memo (B) 77/58, Fort Halstead, Kent, England (1958).

[25.37] Gt. Britain H.M. Nautical Almanac Office, Interpolation and allied tables (Her Majesty's Stationery Office, London, England, 1956).

[25.38] I. M. Longman, Tables for the rapid and accurate numerical evaluation of certain infinite integrals involving Bessel functions, Math. Tables Aids Comp. **11**, 166–180 (1957).

[25.39] A. N. Lowan, N. Davids, and A. Levenson, Table of the zeros of the Legendre polynomials of order 1–16 and the weight coefficients for Gauss' mechanical quadrature formula, Bull. Amer. Math. Soc. **48**, 739–743 (1942).

[25.40] National Bureau of Standards, Tables of Lagrangian interpolation coefficients (Columbia Univ. Press, New York, N.Y., 1944).

[25.41] National Bureau of Standards, Collected Short Tables of the Computation Laboratory, Tables of functions and of zeros of functions, Applied Math. Series 37 (U.S. Government Printing Office, Washington, D.C., 1954).

[25.42] P. Rabinowitz, Abscissas and weights for Lobatto quadrature of high order, Math. Tables Aids Comp. **69**, 47–52 (1960).

[25.43] P. Rabinowitz and G. Weiss, Tables of abscissas and weights for numerical evaluation of integrals of the form

$$\int_0^\infty e^{-x}x^n f(x)\,dx,$$

Math. Tables Aids Comp. **68**, 285–294 (1959).

[25.44] H. E. Salzer, Tables for facilitating the use of Chebyshev's quadrature formula, J. Math. Phys. **26**, 191–194 (1947).

[25.45] H. E. Salzer and R. Zucker, Table of the zeros and weight factors of the first fifteen Laguerre polynomials, Bull. Amer. Math. Soc. **55**, 1004–1012 (1949).

[25.46] H. E. Salzer, R. Zucker, and R. Capuano, Table of the zeros and weight factors of the first twenty Hermite polynomials, J. Research NBS **48**, 111–116 (1952) RP2294.

[25.47] H. E. Salzer, Table of coefficients for obtaining the first derivative without differences, NBS Applied Math. Series 2 (U.S. Government Printing Office, Washington, D.C., 1948).

[25.48] H. E. Salzer, Coefficients for facilitating trigonometric interpolation, J. Math. Phys. **27**, 274–278 (1949).

[25.49] H. E. Salzer and P. T. Roberson, Table of coefficients for obtaining the second derivative without differences, Convair-Astronautics, San Diego, Calif. (1957).

[25.50] H. E. Salzer, Tables of osculatory interpolation coefficients, NBS Applied Math. Series 56 (U.S. Government Printing Office, Washington, D.C., 1958).