Numerical Integration - Lecture 03

Dr. Serhat Hosder

Associate Professor of Aerospace Engineering

Mechanical and Aerospace Engineering

290B Toomey Hall

Missouri S&T

Rolla, MO 65409

Phone: 573-341-7239

E-mail: hosders@mst.edu

Outline

- Numerical integration plays a key role in many engineering applications. The integration methods that we examine
  - Trapezoidal Rule
  - Simpson’s 1/3 Rule
  - Mid-point Rule
  - Romberg Integration
  - Gauss Quadrature
  - Multiple Integrals

Generalization of Extrapolation used in Romberg Integration

h_1=assumed	$\bar{I}_{1,1}$		
$h_2 = h_1/2$	$\bar{I}_{2,1}$	$\bar{I}_{2,2} = \dfrac{4\bar{I}_{2,1} - \bar{I}_{1,1}}{3}$	
$h_3 = h_2/2$	$\bar{I}_{3,1}$	$\bar{I}_{3,2} = \dfrac{4\bar{I}_{3,1} - \bar{I}_{2,1}}{3}$	$\bar{I}_{3,3} = \dfrac{16\bar{I}_{3,2} - \bar{I}_{2,2}}{15}$
Error Order	h^2	h^4	h^6

The extrapolation formula generalizes to:

$$\bar{I}_{i,j} = \frac{4^{j-1}\,\bar{I}_{i,j-1} - \bar{I}_{i-1,j-1}}{4^{j-1} - 1} \quad \forall \begin{array}{l} i=2,3,\ldots..n \\ j=2,3,\ldots i \end{array}$$

Romberg Example (1)

Problem Set 4.5. **Problem 1 b.** ← **Page 211**

$$I = \int_0^1 f(x)dx \qquad f(x) = e^{-x} x^2$$

Exact value = $2 - \frac{5}{e}$ **=** 0.160603

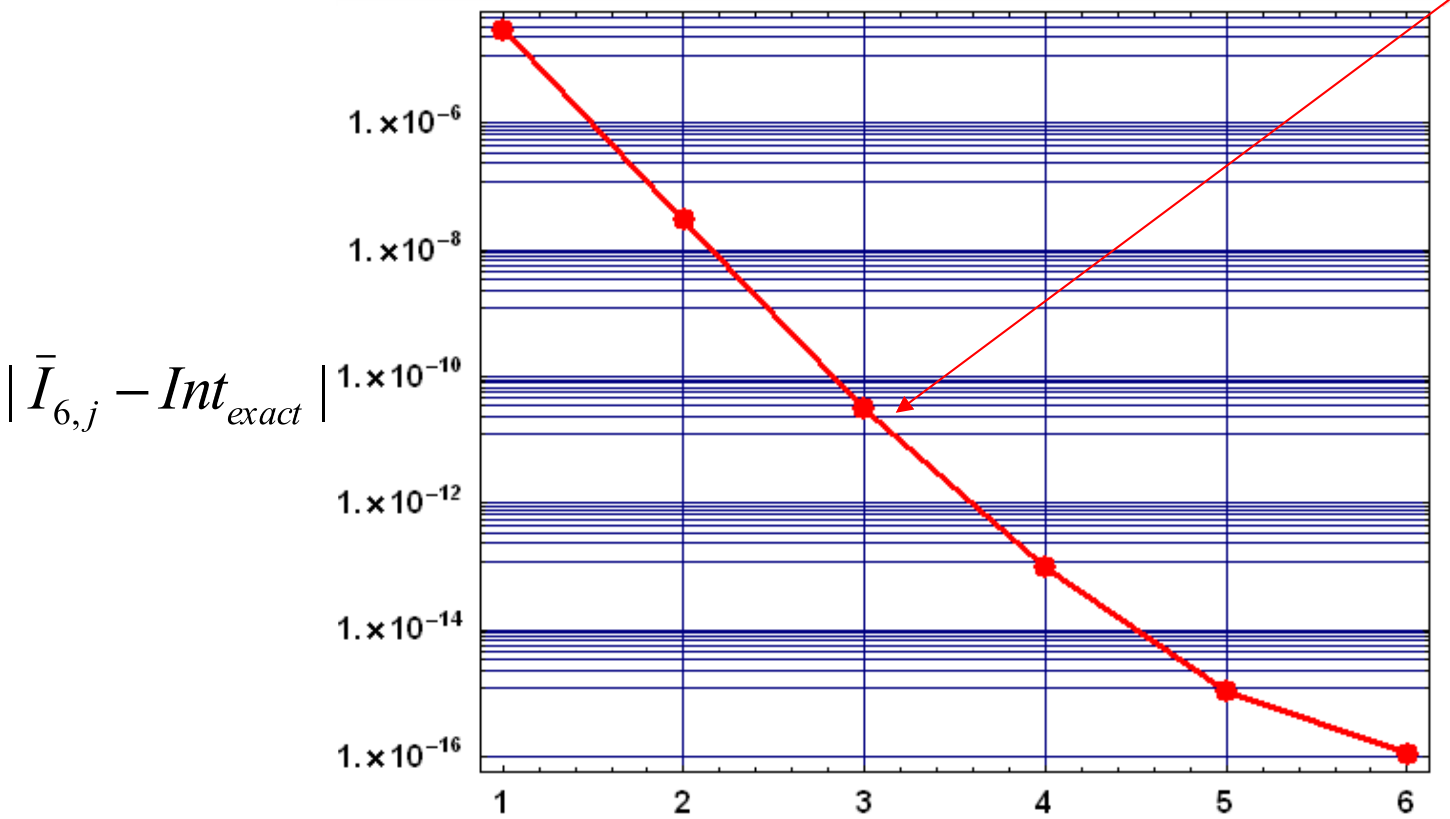
h_i (mesh size)	$\bar{I}_{i,1}$
1.	0.18394
0.5	0.167786
0.25	0.162488
0.125	0.16108
0.0625	0.160722
0.03125	0.160633

I have calculated the integral with the trapezoidal rule using 6 different mesh sizes (increasing the number of panels by a factor of 2 each time)

Romberg Example (2)

Romberg Table:

$1.83939720586\times10^{-1}$	0.	0.	0.	0.	0.
$1.67786192757\times10^{-1}$	$1.62401683481\times10^{-1}$	0.	0.	0.	0.
$1.62488405093\times10^{-1}$	$1.60722475872\times10^{-1}$	$1.60610528698\times10^{-1}$	0.	0.	0.
$1.6107989608\times10^{-1}$	$1.60610393075\times10^{-1}$	$1.60602920889\times10^{-1}$	$1.6060280013\times10^{-1}$	0.	0.
$1.60722427236\times10^{-1}$	$1.60603270955\times10^{-1}$	$1.60602796147\times10^{-1}$	$1.60602794167\times10^{-1}$	$1.60602794144\times10^{-1}$	0.
$1.60632724789\times10^{-1}$	$1.60602823973\times10^{-1}$	$1.60602794174\times10^{-1}$	$1.60602794143\times10^{-1}$	$1.60602794143\times10^{-1}$	$1.60602794143\times10^{-1}$



j

Gauss Quadrature – Basic Principle

The Gauss Quadrature Rule for finding an integral numerically is given by the function

$$\int_a^b f(x)dx = \sum_{i=1}^{n} c_i f(x_i)$$

where c_i are the weights of the function $f(x)$ at x_i

and x_i are the zeros(roots) of the n^{th} degree Legendre polynomial

We have 2n parameters for evaluating the integral

$c_1, c_2, \ldots\ c_n$ and $x_1, x_2, x_3, \ldots x_n$.

Approach: Choose the parameters to exactly integrate the largest class of polynomials possible. With 2n parameters (n points), the class of polynomials of degree 2n-1 can be integrated exactly.

Gauss Quadrature Derivation for n=2

Determine the weights and zero's for n=2 on the interval [-1,1]

$$\int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2)$$

Since n=2, the quadrature formula should be exact for polynomials of degree (2n-1) = 3 or less, i.e., it should be exact for f(x) = x^m, for m = 0,1,2,3,…(2n-1)

$$\left.\begin{aligned}
m = 0 &\rightarrow \int_{-1}^{1} 1dx = c_1 + c_2 = 2 \\
m = 1 &\rightarrow \int_{-1}^{1} xdx = c_1 x_1 + c_2 x_2 = 0 \\
m = 2 &\rightarrow \int_{-1}^{1} x^2 dx = c_1 x_1^2 + c_2 x_2^2 = \tfrac{2}{3} \\
m = 3 &\rightarrow \int_{-1}^{1} x^3 dx = c_1 x_1^3 + c_2 x_2^3 = 0
\end{aligned}\right\}$$

4 equations in
4 unknowns

Gauss Quadrature n=2

We have 4 equations and 4 unknowns from which the solution can be obtained. The solution to this set of equations is

$c_1 = c_2 = 1;\ x_1 = \frac{1}{\sqrt{3}};\ x_2 = -\frac{1}{\sqrt{3}}$

$$\frac{1}{\sqrt{3}} = 0.5773502692$$

Example with n = 2

$$\int_{-1}^{1} \mathrm{Cos}[x]\, dx = 1.68294$$

(Exact Value)

$$\approx c_1 f\left(\frac{1}{\sqrt{3}}\right) + c_2 f\left(-\frac{1}{\sqrt{3}}\right)$$

$$= (1)\,(0.837912) + (1)(0.837912)$$

$$= 1.67582$$

More on Gauss Quadrature

We can extend this approach using the orthogonality property of the Legendre's polynomials on the interval [-1,1]. These polynomials are given by

$$L_0(x) = 1 \qquad\qquad L_1(x) = x$$

The subsequent polynomials are developed by recursion.

$$(n+1)L_{n+1}(x) - (2n+1)xL_n(x)+nL_{n-1}(x) = 0$$

(for n=1, 2, 3, …)

With n=1, one can obtain $L_2(x) = \dfrac{3x^2 - 1}{2}$

The roots can be obtained by solving $L_2(x)=0$. This yields,

$$x = \pm\frac{1}{\sqrt{3}}$$

Legendre Polynomials

The first 11 Legendre Polynomials are:

n	Legendre Polynomial
0	1
1	x
2	$-\frac{1}{2}+\frac{3x^2}{2}$
3	$-\frac{3x}{2}+\frac{5x^3}{2}$
4	$\frac{3}{8}-\frac{15x^2}{4}+\frac{35x^4}{8}$
5	$\frac{15x}{8}-\frac{35x^3}{4}+\frac{63x^5}{8}$
6	$-\frac{5}{16}+\frac{105x^2}{16}-\frac{315x^4}{16}+\frac{231x^6}{16}$
7	$-\frac{35x}{16}+\frac{315x^3}{16}-\frac{693x^5}{16}+\frac{429x^7}{16}$
8	$\frac{35}{128}-\frac{315x^2}{32}+\frac{3465x^4}{64}-\frac{3003x^6}{32}+\frac{6435x^8}{128}$
9	$\frac{315x}{128}-\frac{1155x^3}{32}+\frac{9009x^5}{64}-\frac{6435x^7}{32}+\frac{12155x^9}{128}$
10	$-\frac{63}{256}+\frac{3465x^2}{256}-\frac{15015x^4}{128}+\frac{45045x^6}{128}-\frac{109395x^8}{256}+\frac{46189x^{10}}{256}$
11	$-\frac{693x}{256}+\frac{15015x^3}{256}-\frac{45045x^5}{128}+\frac{109395x^7}{128}-\frac{230945x^9}{256}+\frac{88179x^{11}}{256}$

Zeros (Roots) and Weights (Coefficients) for Gauss-Legendre Quadrature on [-1,1]

n	Zeros	Weights
2	-0.57735	1.
	0.57735	1.
3	-0.774597	0.555556
	0.	0.888889
	0.774597	0.555556
4	-0.861136	0.347855
	-0.339981	0.652145
	0.339981	0.652145
	0.861136	0.347855
5	-0.90618	0.236927
	-0.538469	0.478629
	0.	0.568889
	0.538469	0.478629
	0.90618	0.236927

This is Table 22.1 in the text on page 646.

The roots are the zeros of the n^{th} degree Legendre polynomial

MISSOURI S&T

Gauss Quadrature Theorem

Theorem:

If P is any polynomial of degree 2n-1 or less, then

$$\int_{-1}^{1} P(x)dx = \sum_{i=1}^{n} c_i P(x_i)$$

$$where\ \mathrm{c_i} = \int_{-1}^{1} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i - x_j)}.dx$$

$$and\ x_1, x_2, x_3, \ldots x_n\ are\ the\ zeros\ of\ the\ n^{th}\ Legendre\ polynomial$$

The interval [a,b] is mapped onto [-1,1] via a linear transformation

$$\boxed{\begin{aligned} &x = \frac{b+a}{2} + \frac{b-a}{2}t \quad where \quad dx = \frac{(b-a)}{2}dt \\ &\int_a^b f(x)dx = \frac{(b-a)}{2}\int_{-1}^{1} f[x(t)]dt = \frac{(b-a)}{2}\sum_{i=1}^{n} c_i f(x(t_i)) \end{aligned}}$$

Gauss Quadrature Example

Evaluate the integral

$$\int_0^{\pi/2} \sin x \, dx = 1$$ using Gauss Quadrature and Trapezoidal Rule

$$b = \tfrac{\pi}{2}; \;\; a = 0; \qquad x = \tfrac{\pi}{4} + \tfrac{\pi}{4} t = (1+t)\tfrac{\pi}{4}$$

$$With\ \mathrm{n} = 2,\ \mathrm{c}_1 = c_2 = 1;\ t_1 = -0.577;\ t_2 = 0.577$$

$$I_{QUAD} = \tfrac{\pi}{4}\left[\sin\left(\tfrac{\pi}{4}(1-0.577)\right) + \sin\left(\tfrac{\pi}{4}(1+0.577)\right)\right]$$

$$I_{QUAD} = 0.9984716$$

The Single Panel Trapezoidal Rule gives $I_{TRAP} = 0.7584$
(also requires two function evaluations)

Summary

In this lecture we have

- Worked on an example problem using Romberg Integration and constructed the elements of the extrapolation table
- Introduced Gaussian Quadrature
  - Requires explicit knowledge of the function
  - Points are chosen to maximize the accuracy
  - n-point Gauss quadrature integrates polynomials of degree 2n-1 or less *exactly*