Numerical Solution to a PDE

1D Time-dependent Heat Equation

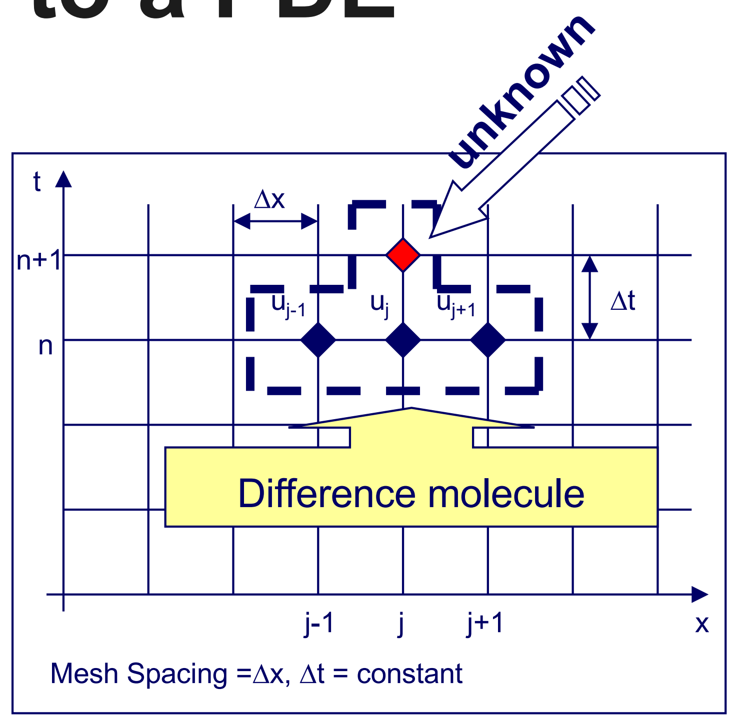
$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Conditions:

$u(x,0) = u_0(x)$ – Initial Condition

$u(0,t) = u_L(t)$ – Boundary Condition

$u(L,t) = u_R(t)$ – Boundary Condition



Let's use a 2nd order accurate central difference approximation for the spatial derivative and the Euler explicit time integration method.

unknown

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

This is a finite difference equation with a molecule as shown above.

Values at time 'n' are known. We seek values at time 'n+1'.

Operator Notation

Define $\delta u_j \equiv u_{j+1/2} - u_{j-1/2}$ and $\Delta u^n \equiv u^{n+1} - u^n$ and $\lambda \equiv \alpha \dfrac{\Delta t}{\Delta x^2}$

Then

$$\delta^2 u_j = \delta(\delta u_j)$$

$$= \delta(u_{j+1/2} - u_{j-1/2}) = \delta u_{j+1/2} - \delta u_{j-1/2}$$

$$= (u_{j+1} - u_j) - (u_j - u_{j-1}) = u_{j+1} - 2u_j + u_{j-1}$$

Returning to our algorithm: $\dfrac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$

In operator notation, this is : $\Delta u_j^n = \lambda \delta^2 u_j^n$

followed by the update step $u_j^{n+1} = u_j^n + \Delta u_j^n$

Euler explicit in operator form

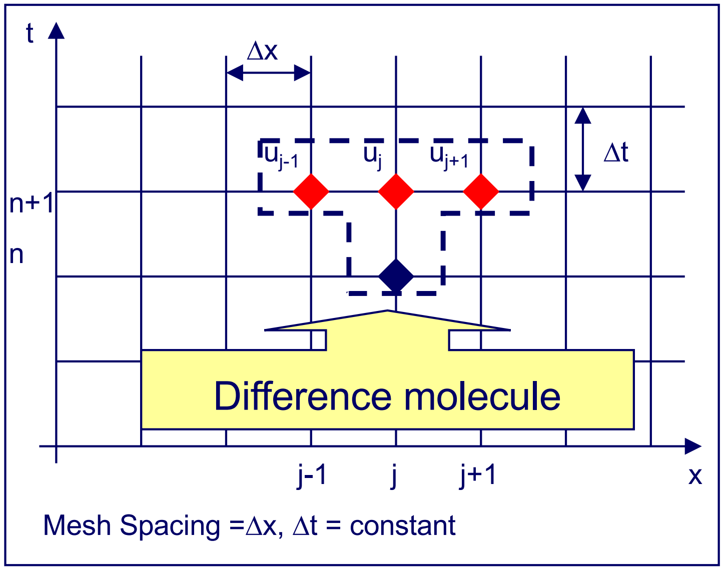
Euler Implicit for 1-D Heat Equation (1)

For our PDE example

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

The Euler implicit method is

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+!}}{\Delta x^2}$$



Note that the only difference is that the RHS is evaluated at ‘n+1’.
(Recall that in the Euler explicit method it is evaluate at ‘n’.)

$$\frac{\Delta^n u_j}{\Delta t} = \alpha \frac{\delta^2 u_j^{\,n+1}}{\Delta x^2}$$

$$\Delta u_j^n = \lambda \delta^2 \left(u_j^{\,n} + \Delta u_j^n \right)$$

Note the change in the difference molecule above. This algorithm results in a coupled (tri-diagonal) set of equations that we can easily solve.

Euler Implicit for 1-D Heat Equation (2)

From the previous slide, we have

$$\Delta u_j^n = \lambda\delta^2\left(u_j^{\ n} + \Delta u_j^n\right)$$

$$(1-\lambda\delta^2)\Delta u_j^n = \lambda\delta^2 u_j^{\ n}$$

In finite difference form, this is:

$$-\lambda\Delta u_{j-1}^n + (1+2\lambda)\Delta u_j^n - \lambda\Delta u_{j+1}^n = \lambda\delta^2 u_j^{\ n}$$

The linear system in matrix form is thus

$$\begin{pmatrix} b & c & & & \\ a & b & c & & \\ & \ddots & \ddots & \ddots & \\ & & a & b & c \\ & & & a & b \end{pmatrix}\begin{pmatrix} \\ \Delta u_{j-1}^n \\ \Delta u_j^n \\ \Delta u_{j+1}^n \\ \\ \end{pmatrix} = \begin{pmatrix} \\ \lambda\delta^2 u_{j-1}^n \\ \lambda\delta^2 u_j^n \\ \lambda\delta^2 u_{j+1}^n \\ \\ \end{pmatrix}$$

At the interior nodes

$$a = -\lambda$$

$$b = 1 + 2\lambda$$

$$c = -\lambda$$