Solution of Ordinary Differential Equations (Initial Value Problems) - Lecture 01

Dr. Serhat Hosder

Associate Professor of Aerospace Engineering

Mechanical and Aerospace Engineering

290B Toomey Hall

Missouri S&T

Rolla, MO 65409

Phone: 573-341-7239

E-mail: hosders@mst.edu

Outline

In this lecture we will

- Briefly introduce a first-order Ordinary Differential Equation (ODE) in the form of an Initial Value Problem (IVP)

- Show Extension to an n^{th} order ODE
  - Conversion of an n^{th} order ODE to a system of 1^{st} Order ODE's

- Learn Euler Explicit method
  - Derivation, Truncation Error
  - Demonstrate the application on an example IVP

Numerical methods for the solution of IVP's are given in Chapter 5 of your textbook

A 1st Order Scalar ODE

Our discussion will include the numerical solution of ODE's in the form:

$$\frac{dy}{dt} = f(t, y) \text{ for } a \leq t \leq b \qquad (1)$$

subject to the initial condition

$$y(a) = y_0$$

This is an ***Initial Value Problem***. Second- and higher-order ODE's can be converted to a system of coupled first order ODE's of the form shown above.

A 2nd Order Scalar ODE

Consider the second order ODE:

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt}) \qquad \text{on} \qquad a \le t \le b \qquad (2)$$

subject to the initial conditions

$$y(a) = \alpha_1 \qquad \frac{dy}{dt}(a) = \alpha_2$$

We would like to re-write this as a system of two 1st order ODE's.

Let

$$z_1(t) \equiv y(t) \qquad z_2(t) \equiv \frac{dy}{dt}(t)$$

Then, by definition,

$$\frac{dz_1}{dt} = \frac{dy}{dt} = z_2$$

$$z_1(a) = \alpha_1$$

System of two coupled 1st order ODE's

From Eqn. (2) we have

$$\frac{dz_2}{dt} = \frac{d^2y}{dt^2} = f(t, z_1, z_2)$$

$$z_2(a) = \alpha_2$$

A System of 1st Order ODE's

A general first order system of coupled ODE's can be written as:

$$\frac{dy_1}{dt} = f_1(t, y_1, y_2, y_3, \ldots y_n) \quad ; \quad y_1(a) = \alpha_1$$

$$\frac{dy_2}{dt} = f_2(t, y_1, y_2, y_3, \ldots y_n) \quad ; \quad y_2(a) = \alpha_2$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\frac{dy_n}{dt} = f_n(t, y_1, y_2, y_3, \ldots y_n) \quad ; \quad y_n(a) = \alpha_n$$

In matrix notation, this system is simply

$$\frac{d\vec{y}}{dt} = \vec{f}(t, \vec{y}) \quad ; \quad \vec{y}(a) = \vec{\alpha}$$

So an nth Order ODE can be written as a system of "n" 1st Order ODE's in this form.

Ordinary Differential Equations

In this class, we will focus the remainder of our attention on the numerical solution of the scalar, 1st Order ODE in the form of an Initial Value Problem as given by Equation (1). The methods below directly apply to this problem:

1. Euler's explicit method
2. Adam-Bashforth method (Open, Explicit)
3. Euler's implicit method
4. Adam's Moulton's method (Closed, Implicit)
5. Runge-Kutta method

Note that the extension of these methods to a system of equations (or an n^{th} Order ODE) is straightforward.

Euler Explicit Method

Consider the initial value problem

$$\frac{dy}{dt} = f(t, y) \quad ; \quad y(a) = y_0 \qquad a \le t \le b$$

Define a mesh such that $t_i = a + ih \quad \forall \quad i = 0,1,2,\cdots,N$ where $h = (b-a)/N$

Expand $y(t_{i+1}) \equiv y_{i+1}$ in a Taylor series about t_i

$$y_{i+1} = y_i + (t_{i+1} - t_i)y_i' + \frac{(t_{i+1} - t_i)^2}{2} y''(\xi_i)$$

But $h = t_{i+1} - t_i$ and $y_i' = f_i$. Substituting into the above yields

$$y_{i+1} = y_i + hf_i + \frac{h^2}{2} y''\left(\xi_i\right) \quad ; \quad f_i \equiv f(t_i, y(t_i))$$

The Euler Explicit method neglects the error term to yield the algorithm

$$\boxed{w_{i+1} = w_i + hf_i \quad \forall\, i = 0,1,\cdots N-1 \quad ; \quad w_0 = y(a)}$$

where w_i is the numerical approximation to the exact solution y_i.

Error of the Euler Explicit Scheme

In solving IVP's, it is useful to define the local truncation error.

The local truncation error at a specified step is defined to be the amount by which the exact solution to the differential equation fails to satisfy the difference approximation

The local truncation error of Euler's Explicit method is (see the text for proof, section 5.3 – very simple),

$$\tau_{i+1}(h) = \frac{h}{2} y''(\xi_i) \qquad t_i < \xi_i < t_{i+1}$$

Thus, we say that the local error is $\boldsymbol{O(h)}$.

Note that this error is a "local error", since it measures the accuracy of the method at the specified step assuming that the method was exact at the previous step.

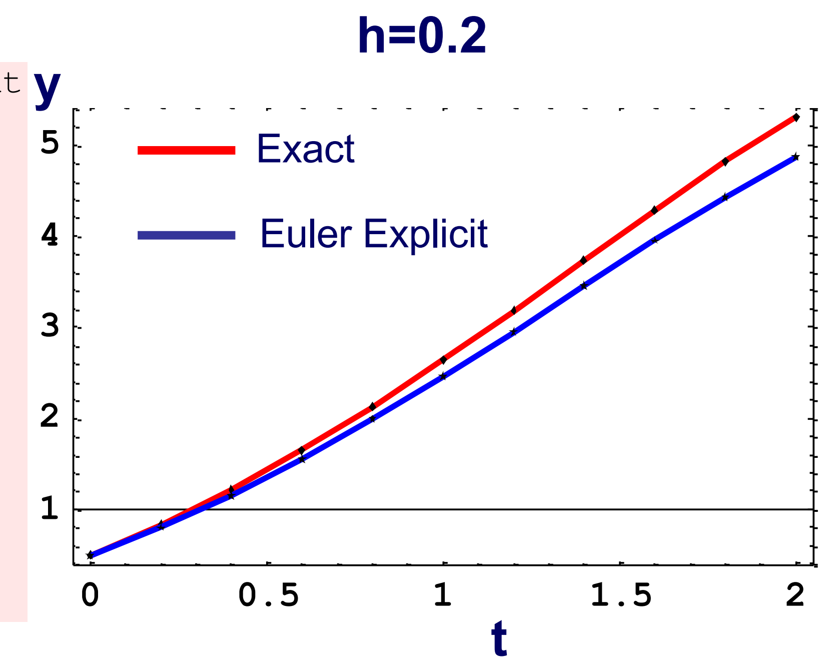
Euler Explicit Example

Here is an IVP example from your text.

$$\frac{dy}{dt} = y - t^2 + 1 \qquad y(0) = 0.5 \qquad 0 \le t \le 2$$

The exact solution is: $y(t) = (1+t)^2 - e^t/2$

i	Time	Exact Solution	Euler Explicit	Time Dependent Error
0	0.	0.5	0.5	0.
1	0.2	0.829299	0.8	-0.0292986
2	0.4	1.21409	1.152	-0.0620877
3	0.6	1.64894	1.5504	-0.0985406
4	0.8	2.12723	1.98848	-0.13875
5	1.	2.64086	2.45818	-0.182683
6	1.2	3.17994	2.94981	-0.23013
7	1.4	3.7324	3.45177	-0.280627
8	1.6	4.28348	3.95013	-0.333356
9	1.8	4.81518	4.42815	-0.387023
10	2.	5.30547	4.86578	-0.439687



Note the accumulation of the error. In this case, the error is growing linearly. This error is expected to grow in no worse than a linear manner.

Summary

In this lecture we have

- Discussed the form of the Ordinary Differential equation (IVP) that we will examine with different numerical methods
- Demonstrated by example that a second order ODE can be converted to a coupled system of two first order ODE’s
- Derived the Euler Explicit method based on a Taylor series expansion and showed the local error term for the method
- Worked on a simple example using Euler Explicit method