Find the roots of

$$x_1 x_2 x_3 x_4 = -4$$
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 10$$
$$x_2 x_3^2 = -2$$
$$x_1^3 + x_2^2 + x_3^3 - x_4^2 = 2$$

with the initial guess (starting point)

$$x_1 = 0.5$$
$$x_2 = -2.5$$
$$x_3 = 1.5$$
$$x_4 = 1.5$$

$$[f']^k\{\Delta x\}^k=-\{f\}^k \longrightarrow \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix}^k \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix}^k = -\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}^k$$

$$\{x\}^{k+1}=\{x\}^k+\{\Delta x\}^k$$

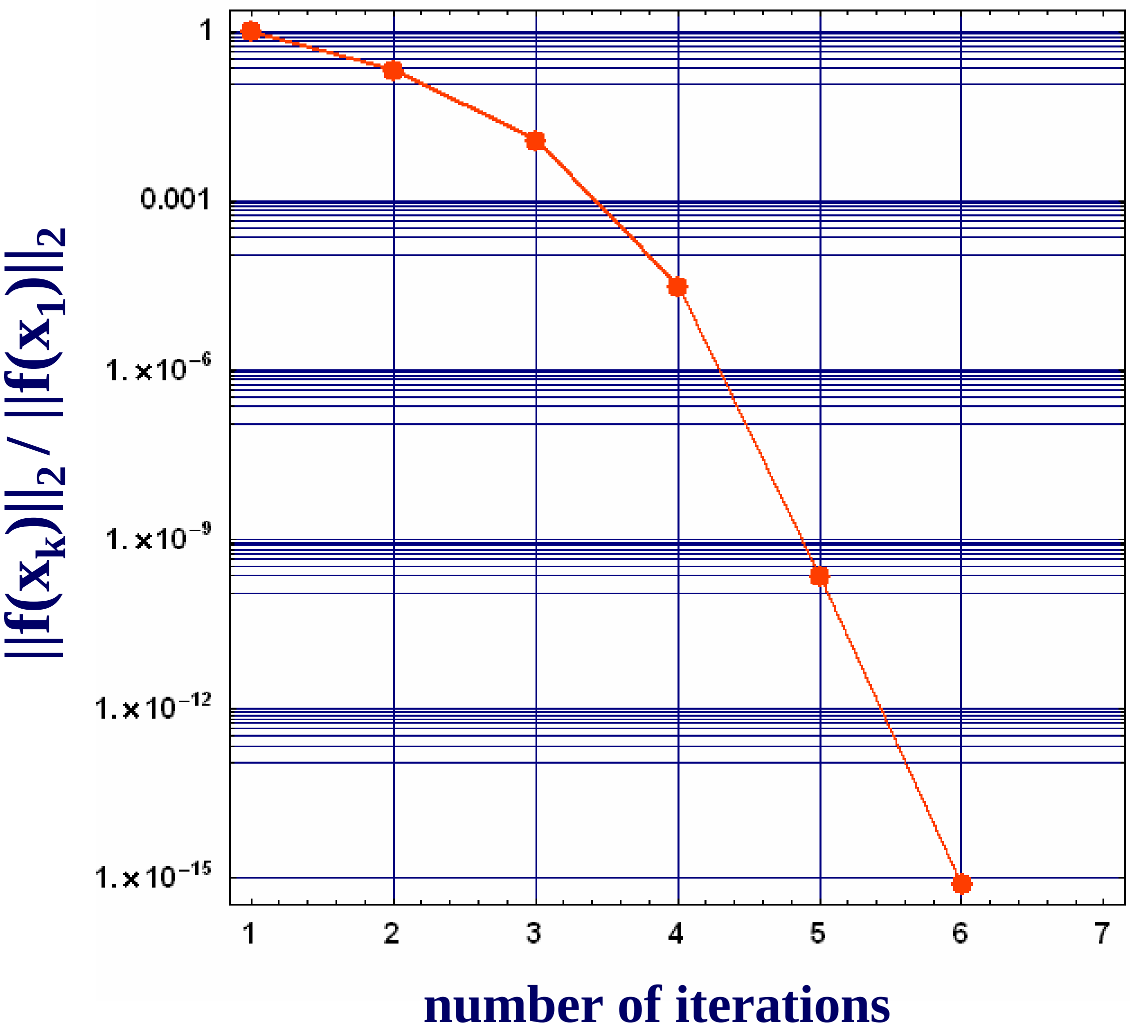
$$\boxed{\begin{aligned} f_1 &= x_1x_2x_3x_4+4 \\ f_2 &= x_1^2+x_2^2+x_3^2+x_4^2-10 \\ f_3 &= x_2x_3^2+2 \\ f_4 &= x_1^3+x_2^2+x_3^3-x_4^2-2 \end{aligned}}$$

$$[f']=\begin{bmatrix} x_2x_3x_4 & x_1x_3x_4 & x_1x_2x_4 & x_1x_2x_3 \\ 2x_1 & 2x_2 & 2x_3 & 2x_4 \\ 0 & x_3^2 & 2x_2x_3 & 0 \\ 3x_1^2 & 2x_2 & 3x_3^2 & -2x_4 \end{bmatrix}$$

$$\text{Initial Estimate} = \begin{pmatrix} 0.5 \\ -2.5 \\ 1.5 \\ 1.5 \end{pmatrix} \qquad \text{Solution} = \begin{pmatrix} 1. \\ -2. \\ 1. \\ 2. \end{pmatrix}$$

k	$\lVert f(x_k) \rVert_2$	$\lVert f(x_k) \rVert_2 / \lVert f(x_1) \rVert_2$
1	1.13886	1.
2	0.230074	0.20202
3	0.0133317	0.0117061
4	0.000034377	0.0000301854
5	2.42295×10^{-10}	2.12751×10^{-10}
6	9.15513×10^{-16}	8.03882×10^{-16}
7	0.	0.

Note that for this case I used the built-in LinearSolve[…] command to solve the linear set of equations at each iteration



Quadratic convergence observed for this initial estimate:

$$\text{Initial Estimate} = \begin{pmatrix} 0.5 \\ -2.5 \\ 1.5 \\ 1.5 \end{pmatrix}$$

$$\text{Initial Estimate} = \begin{pmatrix} 1. \\ 1. \\ 1. \\ 1. \end{pmatrix}$$

LinearSolve::nosol : Linear equation encountered which has no solution. More…

LinearSolve::nosol : Linear equation encountered which has no solution. More…

LinearSolve::nosol : Linear equation encountered which has no solution. More…

General::stop : Further output of LinearSolve::nosol will be suppressed during this calculation. More…

Out[95]= $Aborted

Convergence failed for the initial estimate $X^0=\{1.0,1.0,1.0,1.0\}^T$

Good initial estimates are important for the convergence of the Newton's Method for the solution of non-linear systems

Used the initial estimate $X^0=\{0.5,-2.5,1.5,1.5\}^T$

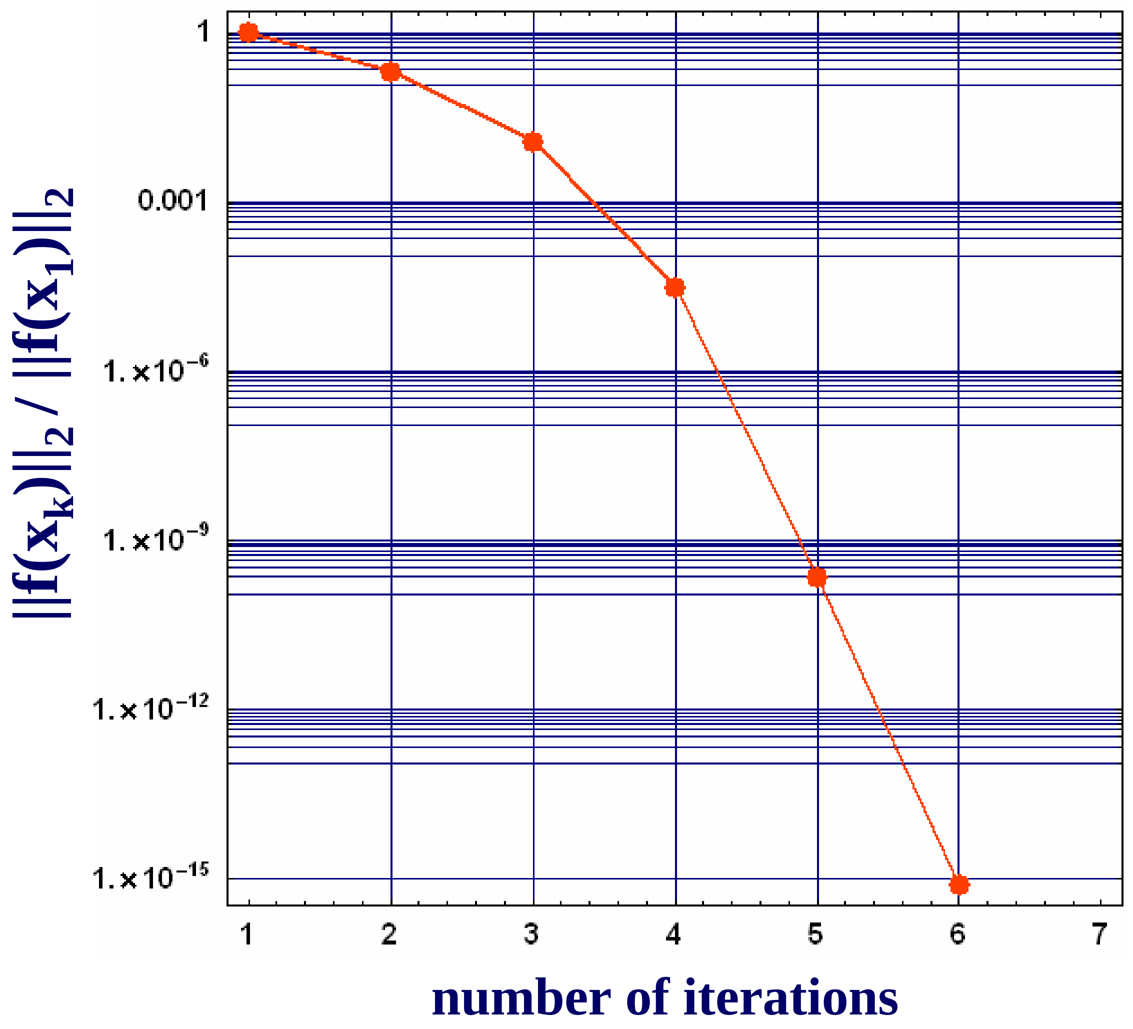
k	$\|\|f(x_k)\|\|_2$	$\|\|f(x_k)\|\|_2 / \|\|f(x_1)\|\|_2$
1	1.	1.13886
2	6.49495	7.39687
3	64.0895	72.9893
4	$3.087617976789494 \times 10^{658}$	$3.516379249160662 \times 10^{658}$
5	$3.115413502965002 \times 10^{64666}$	$3.54803459389495 \times 10^{64666}$
6	$1.254358281939305 \times 10^{6401926}$	$1.42854442057969 \times 10^{6401926}$
7	Overflow[]	Overflow[]
8	Overflow[]	Overflow[]
9	Indeterminate	Indeterminate
10	Indeterminate	Indeterminate

When the linear problem at each iteration is solved with Gauss-Seidel with a relaxation parameter of 1.1 (over-relaxation), the method diverges. We can try under-relaxation (w<1.0).

Used the initial estimate $X^0=\{0.5,-2.5,1.5,1.5\}^T$

k	$\|\|f(x_k)\|\|_2$	$\|\|f(x_k)\|\|_2 / \|\|f(x_1)\|\|_2$
1	1.	1.13886
2	0.20202	0.230074
3	0.0117061	0.0133317
4	0.0000301854	0.000034377
5	2.12751×10^{-10}	2.42295×10^{-10}
6	8.03882×10^{-16}	9.15513×10^{-16}
7	0.	0.

- **When the linear problem at each iteration is solved with Gauss-Seidel with a relaxation parameter of 0.5 (under-relaxation), the method converges.**
- **Note that the precision goal to solve the linear problem on each step was 10^{-20} The precision goal to solve the non-linear problem was 10^{-16}**



Quadratic convergence observed when the linear problem at each iteration is solved with Gauss-Seidel with a relaxation parameter of 0.5 (under-relaxation),