Interpolation and Polynomial Approximation - Lecture 01

Dr. Serhat Hosder

Associate Professor of Aerospace Engineering

Mechanical and Aerospace Engineering

290B Toomey Hall

Missouri S&T

Rolla, MO 65409

Phone: 573-341-7239

E-mail: hosders@mst.edu

Outline

- Interpolation and polynomial approximation (Chapter 3 in the textbook)

- Finding a function that fits to a data given at discrete points is known "interpolation". You use the fitted function to approximate the values at the other points in your interval.

- Polynomial approximation will be useful in approximating integrals and derivatives

- We will cover
  - Lagrange Interpolation
  - Cubic Spline

- This lecture we will look at Lagrange Interpolation

Interpolation and Polynomial Approximation

- Assume that you have a data set given at $(n+1)$ discrete points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in an interval
- To find y value at any location in this interval, you need to approximate your data with a function. If this function is represented with a polynomial, then we call it a "polynomial approximation":
- An n^{th} degree polynomial:

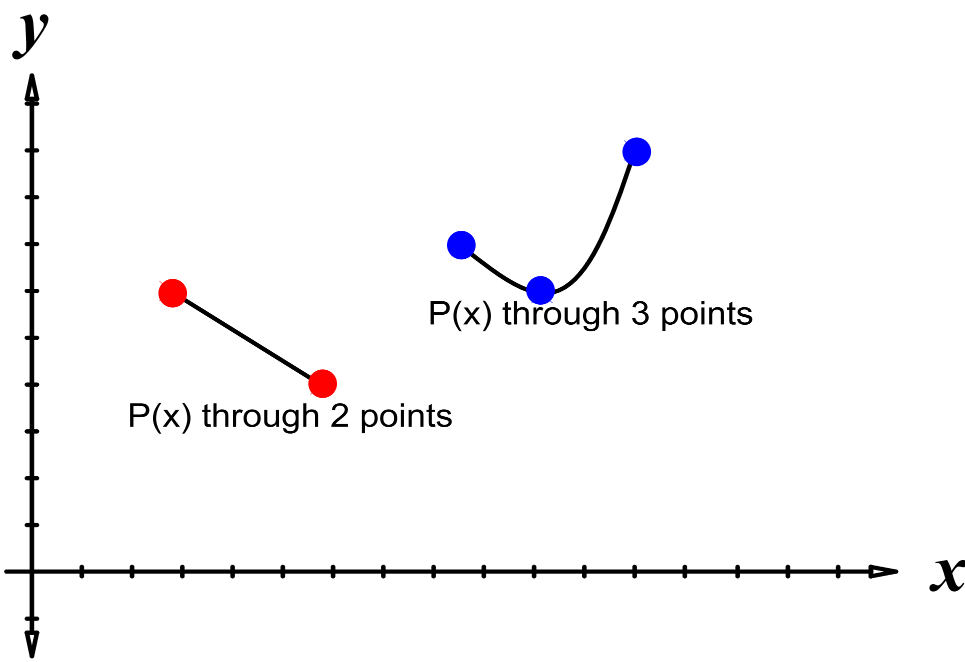
$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

- When you use $P_n(x)$ to approximate the y values at any location within the interval, you perform "interpolation"

- Polynomial approximation is also useful in approximating integrals and derivatives

Lagrange Interpolation

- Lagrange interpolation constructs the interpolating polynomial by using the data points where the function values are specified



Given 2 points, we can define a straight line (1st degree polynomial);

With 3 points, we can define a parabola (2nd degree polynomial)

Generalizing,

- **Given (n+1) points or data pairs, we can fit a polynomial of degree n through those points.**

Lagrange Interpolation for two given data points

- The equation of a line passing through 2 given points (x_0, y_0) and (x_1, y_1) by Linear Interpolation is

$$P_1(x) = \underbrace{\frac{(x-x_1)}{(x_0-x_1)}}_{} y_0 + \underbrace{\frac{(x-x_0)}{(x_1-x_0)}}_{} y_1$$

$$= \quad \mathrm{L}_{1,0} \quad y_0 + \quad \mathrm{L}_{1,1} \quad y_1$$

where $\mathbf{L}_{1,0}$ and $\mathbf{L}_{1,1}$ represent the Lagrange multipliers.

- In $\mathbf{L}_{1,0}$: The first subscript, '**1**', represents the degree of the polynomial and the second subscript, '**0**', represents the point that the data passes through.
- Generalizing, $n+1$ data pairs will lead to a polynomial of degree n having $n+1$ terms.

$$P_n(x) = \sum_{k=0}^{n} L_{n,k} y_k$$

- Where $L_{n,k}$ is the Lagrange multiplier and n is the degree of the polynomial

Lagrange Interpolation - Generalized

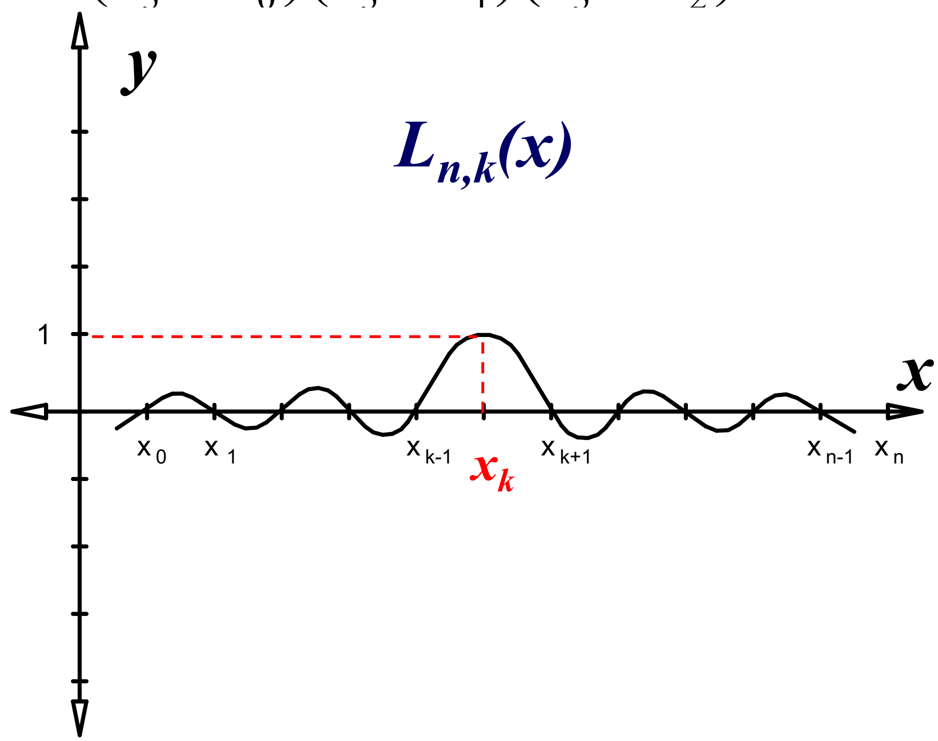
If the data pairs we have are (x_0,f_0), (x_1,f_1), (x_2,f_2) and (x_3,f_3) then the polynomial is given by:

$$P_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f_3$$

In the general case, the n^{th} Lagrange polynomial for the k^{th} point is:

$$\boxed{L_{n,k}(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}}$$



Lagrange Interpolation - Theorem

If $x_0, x_1, x_2, x_3, \ldots, x_n$ are $n+1$ distinct numbers and f is a function given at those numbers, then there exists a unique polynomial of at most degree n, with the property that

$f(x_k) = P_n(x_k)$ *for* $k = 0, 1, 2, 3, \ldots n$

The polynomial is

$$P_n(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x)$$

where $L_{n,k}$ are the Lagrange interpolating polynomials of degree n given by

$$L_{n,k} = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$$

Lagrange Interpolation - Example 1

Given the data pairs:

x	2	2.5	4
$f(x)$	.5	.4	.25

$f(x)=1/x$

Write the Lagrange polynomial of degree two that passes through the data points. Solution:

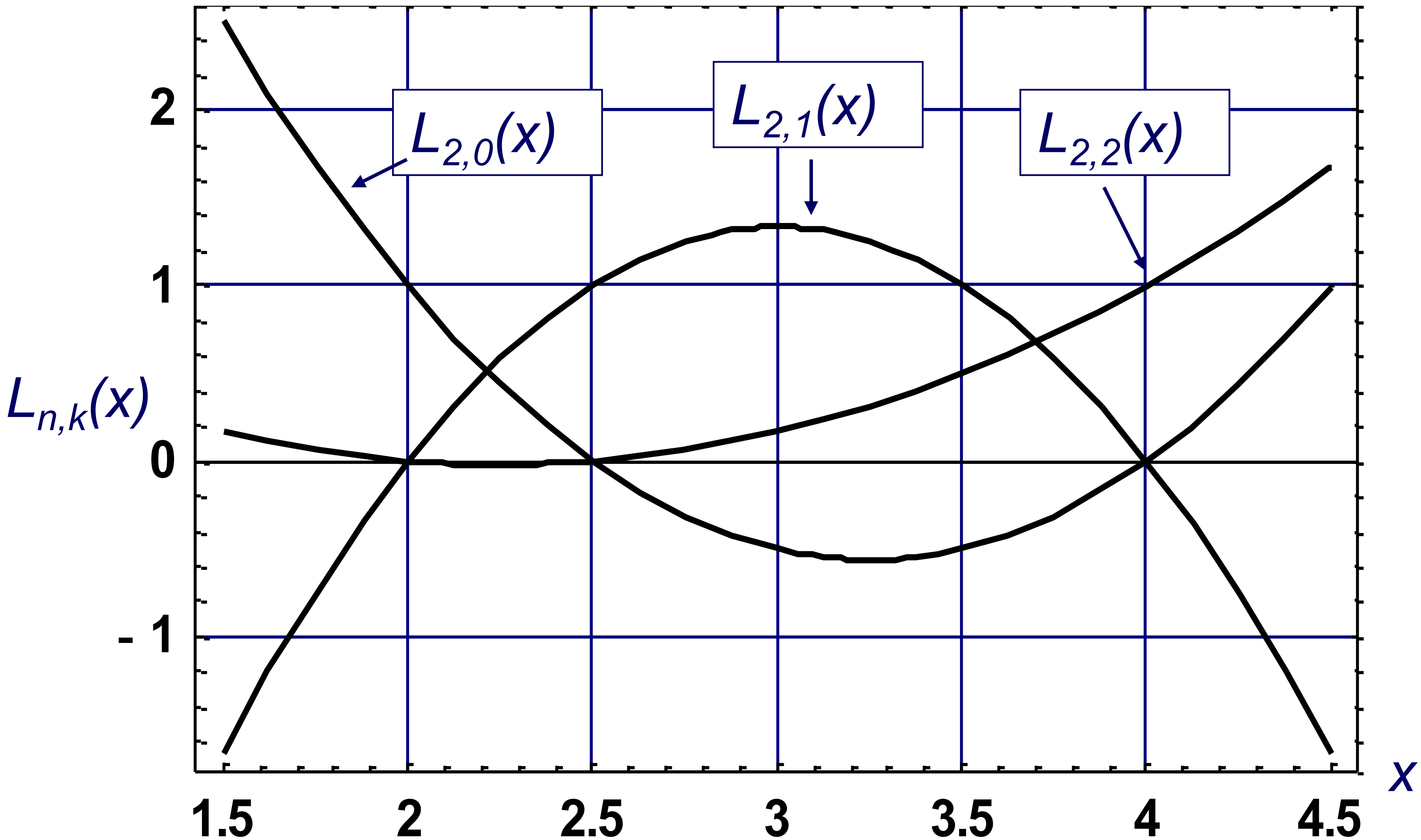
$$P_2(x) = \sum_{k=0}^{2} f(x_k)\,L_{n,k}(x) = f(x_0)\,L_{2,0}(x) + f(x_1)\,L_{2,1}(x) + f(x_2)\,L_{2,2}(x)$$

$$L_{2,0} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-2.5)(x-4)}{(2-2.5)(2-4)} = (x-6.5)x+10$$

$$L_{2,1} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-2)(x-4)}{(2.5-2)(2.5-4)} = -\frac{4}{3}[(x-6)x+8]$$

$$L_{2,2} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-2)(x-2.5)}{(4-2)(4-2.5)} = \frac{1}{3}[(x-4.5)x+5]$$

Lagrange Interpolation - Example 1



Given the data pairs:

x	2	2.5	4
$f(x)$	.5	.4	.25

Lagrange Interpolation - Example 1

From the definition:

$$P_2(x) = f(x_0)\,L_{2,0}(x) + f(x_1)\,L_{2,1}(x) + f(x_2)\,L_{2,2}(x)$$

$$= 0.5\,\{(x-6.5)x+10\}$$

$$+0.4\left\{-\frac{4}{3}[(x-6)x+8]\right\}$$

$$+0.25\left\{\frac{1}{3}[(x-4.5)x+5]\right\}$$

Simplifying:

$$\boxed{P_2(x) = 0.05\,x^2 - .425x + 1.15}$$

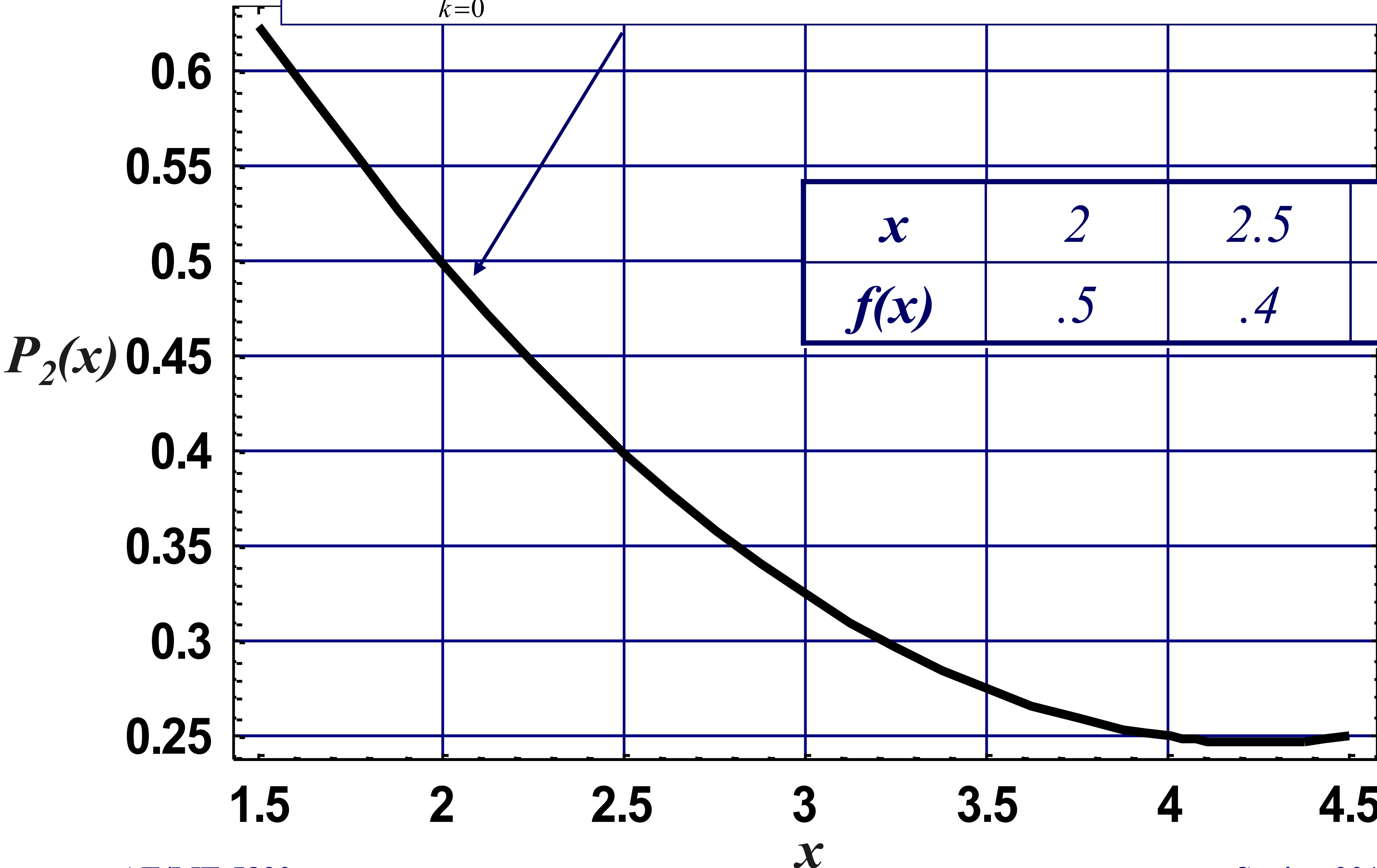
The table on the right shows that the polynomial passes through the data and interpolates in between

x	$P_2(x)$	$f(x)$
2	.5	.5
2.5	.4	.4
3	.325	.333
4	.25	.25

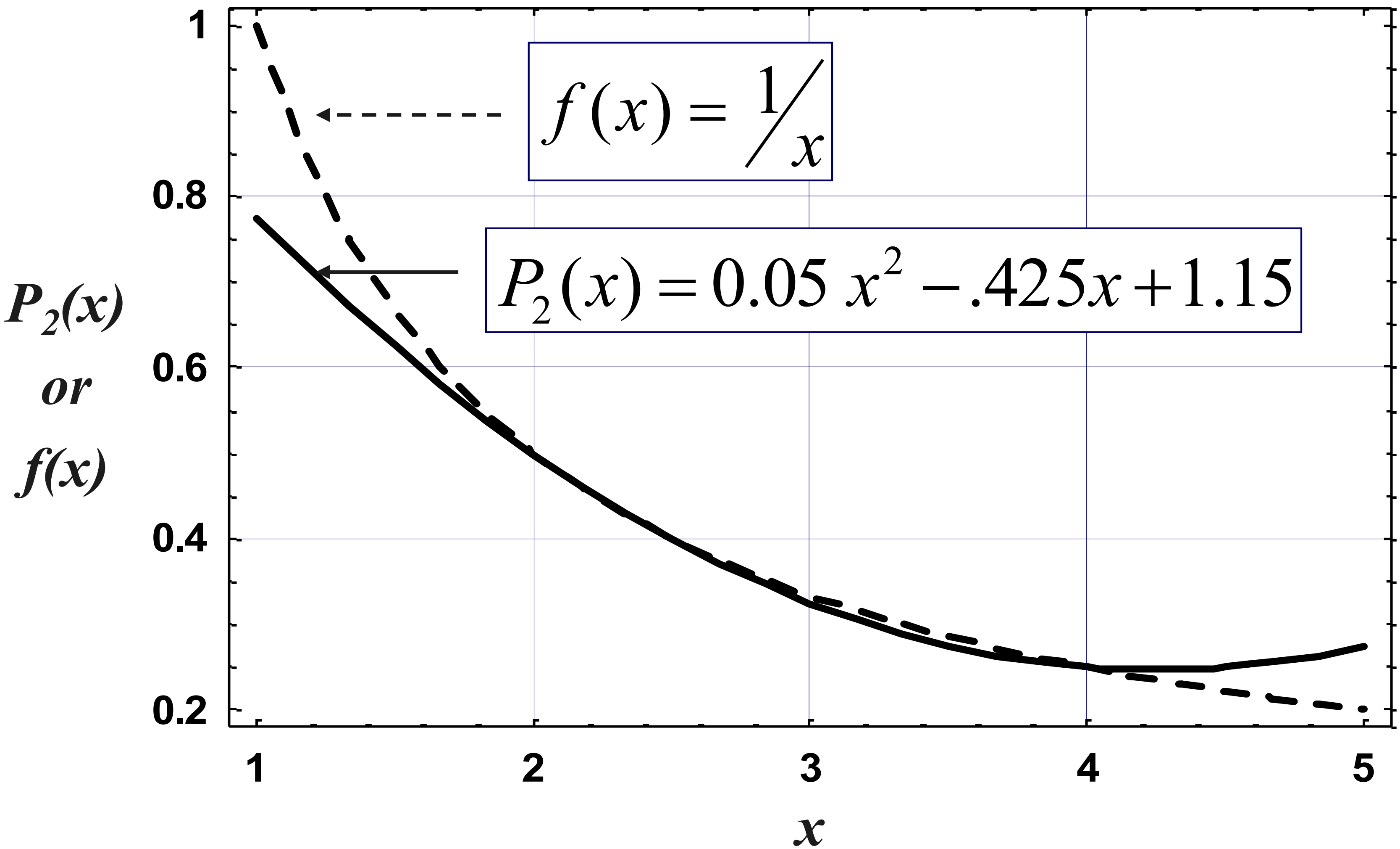
Lagrange Interpolating Polynomial $P_2(x)$

$$P_2(x) = \sum_{k=0}^{2} f(x_k)\,L_{n,k}(x) = f(x_0)\,L_{2,0}(x) + f(x_1)\,L_{2,1}(x) + f(x_2)\,L_{2,2}(x)$$

x	2	2.5	4
$f(x)$	.5	.4	.25



Graphical Comparison of $P_2(x)$ and $f(x)$



Lagrange Interpolation - Example 2

Given the data pairs:

x	*0*	*0.25*	*0.5*	*1.0*
f(x)	*1.0*	*1.21632*	*1.35701*	*1.38177*

Write the Lagrange polynomial of degree three that passes through the data points. Solution:

$$P_3(x) = \sum_{k=0}^{3} f(x_k)\, L_{n,k}(x) = f(x_0)\, L_{3,0}(x) + f(x_1)\, L_{3,1}(x) + f(x_2)\, L_{3,2}(x) + f(x_3)\, L_{3,3}(x)$$

$$P_3(x) = \frac{\mathrm{f0}\,(\mathrm{x}-\mathrm{x1})\,(\mathrm{x}-\mathrm{x2})\,(\mathrm{x}-\mathrm{x3})}{(\mathrm{x0}-\mathrm{x1})\,(\mathrm{x0}-\mathrm{x2})\,(\mathrm{x0}-\mathrm{x3})} + \frac{\mathrm{f1}\,(\mathrm{x}-\mathrm{x0})\,(\mathrm{x}-\mathrm{x2})\,(\mathrm{x}-\mathrm{x3})}{(-\mathrm{x0}+\mathrm{x1})\,(\mathrm{x1}-\mathrm{x2})\,(\mathrm{x1}-\mathrm{x3})} +$$

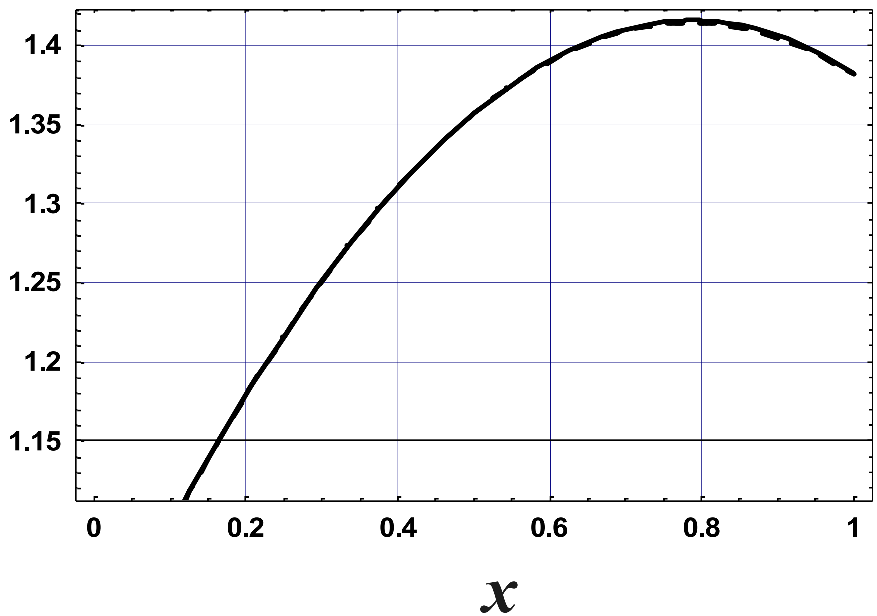
$$\frac{\mathrm{f2}\,(\mathrm{x}-\mathrm{x0})\,(\mathrm{x}-\mathrm{x1})\,(\mathrm{x}-\mathrm{x3})}{(-\mathrm{x0}+\mathrm{x2})\,(-\mathrm{x1}+\mathrm{x2})\,(\mathrm{x2}-\mathrm{x3})} + \frac{\mathrm{f3}\,(\mathrm{x}-\mathrm{x0})\,(\mathrm{x}-\mathrm{x1})\,(\mathrm{x}-\mathrm{x2})}{(-\mathrm{x0}+\mathrm{x3})\,(-\mathrm{x1}+\mathrm{x3})\,(-\mathrm{x2}+\mathrm{x3})}$$

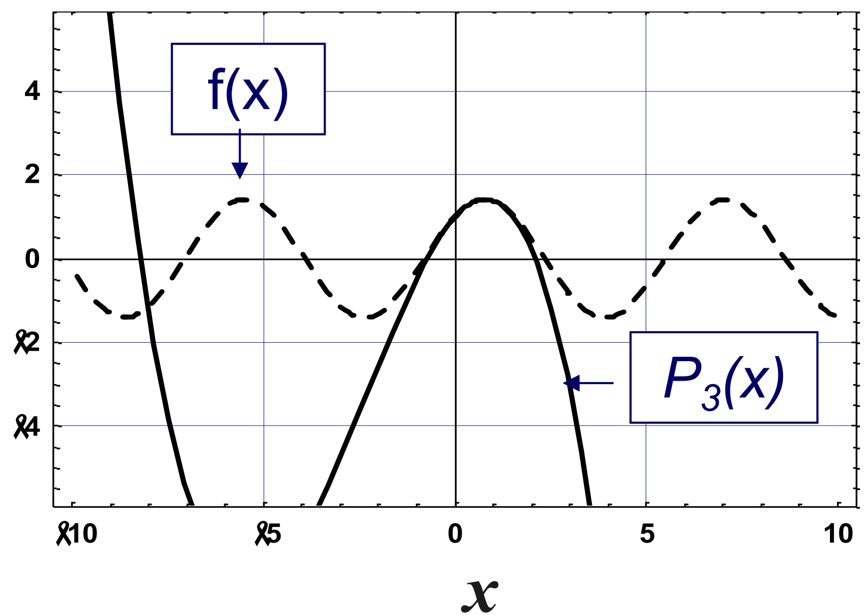
Graphical Comparison of $P_3(x)$ and $f(x)$

After some algebra, we can obtain:

$$P_3(x) = -0.079318\,x^3 - 0.545509\,x^2 + 1.0066x + 1$$

The actual function is: $f(x) = \cos x + \sin x$

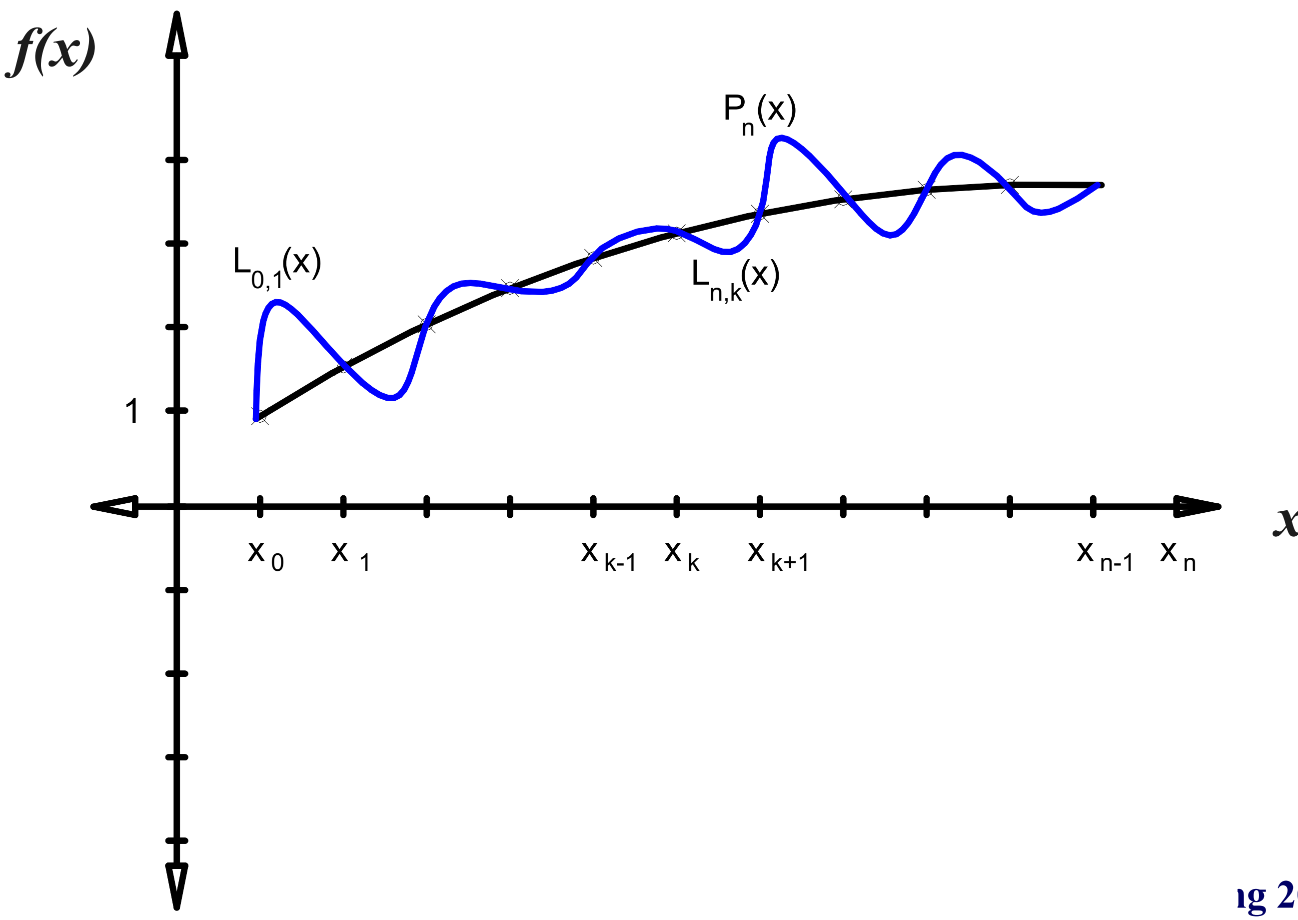




Visually, the approximation is nearly identical to the actual function in the interpolation range and far from it outside of this range (see plot on right).

Disadvantage of using high degree Lagrange Polynomials

Although the Lagrange polynomial passes through the given data pairs, for high degree polynomial interpolation, oscillations frequently arise particularly near the end points. Thus, low degree polynomials are normally employed.



Lagrange Interpolation – Example 3

Approximate the function:

$$f(x) = \frac{1}{1+25x^2}$$

on the interval {-1,+1} using equally spaced data. Plot the function and the 4th and 10th order Lagrange polynomials on this interval.

Solution: From a simple code, we can obtain:

the fourth degree Lagrange Polynomial:

$$1. + 0.\,x - 4.27719\,x^2 + 0.\,x^3 + 3.31565\,x^4$$

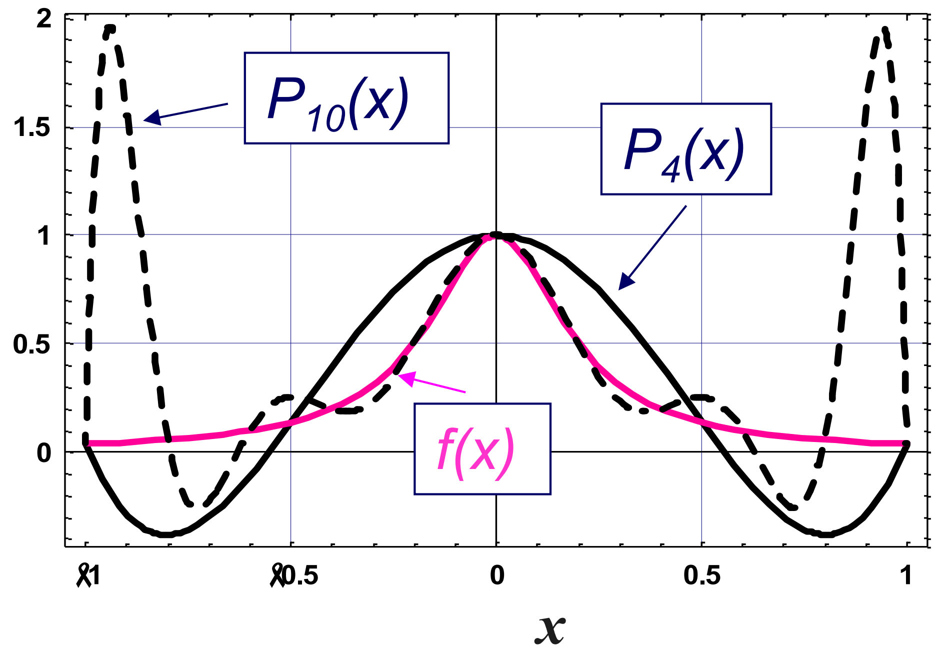
the tenth degree Lagrange Polynomial:

$$1. - 4.44089\times 10^{-16}\,x - 16.8552\,x^2 - 7.4607\times 10^{-14}\,x^3 + 123.36\,x^4 +$$
$$1.77636\times 10^{-13}\,x^5 - 381.434\,x^6 + 3.97904\times 10^{-13}\,x^7 + 494.91\,x^8 - 2.06057\times 10^{-13}\,x^9 - 220.942\,x^{10}$$

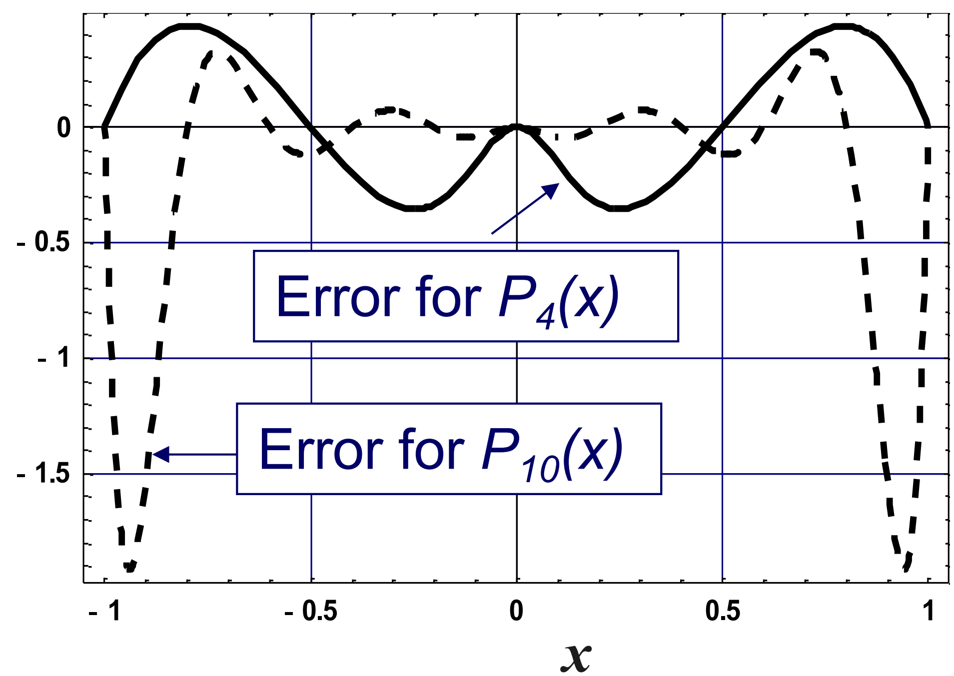
Lagrange Interpolation – High Degree

The function that we are approximating is: $f(x) = \dfrac{1}{1+25x^2}$

A comparison of the function and polynomial approximations



Error - Interpolation



Note that although the 10th degree polynomial matches the function at eleven data points, the approximation is unacceptable. Because of this, high degree polynomial approximation is not commonly used. The large wiggles near the endpoints are also typical of high degree polynomial approximations.

Summary

In today's lecture

- We have developed the Lagrange polynomials by passing polynomials through the data points, $(x, f(x))$.
- We looked at simple examples (linear, quadratic) and extended the method to Lagrange Polynomials of degree n.
- We worked on two examples to demonstrate the application of the Lagrange polynomials and interpolation
- We have shown that using high degree polynomials may cause oscillatory behavior especially near the end points of the function estimated