Root Finding – 03

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Outline

- Define *root of multiplicity m* of a function
- Convergence rate of Newton's Method for a function that has multiple roots at the same point
- Learn *modified Newton's method* (Section 6.5 in text)
- Comparison of Newton's Method and Modified Newton's Method with an example
- Learn *Fixed Point Iteration* for solving root-finding problems and its convergence criteria (Section 6.1 in text)

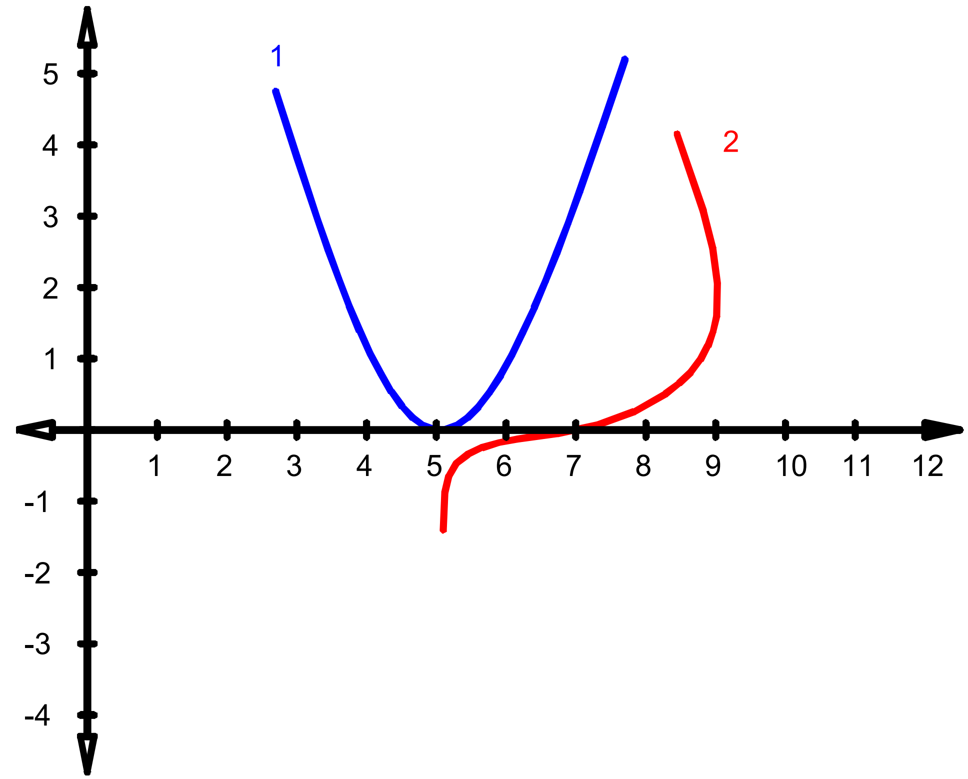
Multiple Roots

Definition:

- A root 'p' of a continuous function is said to be of multiplicity 'm' if

  $$f(x) = (x-p)^m q(x)$$

  where $q(x)$ is continuous and differentiable and $q(p) \neq 0$.
- A function has multiple roots, if the path traced by the function, is tangential to the x-axis.



Example: $f(x) = 2x^3-4x^2+2x = (x-1)^2\, 2x$

$p=1$ *is a root of multiplicity "2" of this function*

Rate of Convergence For Multiple Roots

- Note that for ***f(x) = (x-p)^{m}q(x)***

$$f'(p) = f''(p) = ... = f^{(m-1)}(p) = 0$$

- Remember for simple roots Newton's method has a quadratic rate of convergence

$$\varepsilon_{n+1} = \frac{1}{2}\beta_1\varepsilon_n^2$$

- The rate of convergence of Newton's method is reduced to first order in case of multiple roots
  - ***For m=2*** $$\varepsilon_{n+1} = \frac{1}{2}\varepsilon_n + \frac{1}{12}\beta_2\varepsilon_n^2$$

Approach for finding roots of multiplicity

- To solve $f(x) = (x\text{-}p)^m q(x)$ for multiple roots, we define a new function $\mu(x)$ given by

$$\grave{\imath}(x) = \frac{f(x)}{f'(x)}$$

  - $\mu(x)$ has the same roots as $f(x)$
  - $x{=}p$ should be a simple root for $\mu(x)$

- $$\mu(x) = \frac{f(x)}{f'(x)} = \frac{(x-p)^m q(x)}{m(x-p)^{m-1} q(x) + (x-p)^m q'(x)}$$

$$\mu(x) = (x-p)\frac{(x-p)^{m-1} q(x)}{(x-p)^{m-1}\left[mq(x) + (x-p)q'(x)\right]}$$

Approach for finding roots of multiplicity (2)

- We can write the final form as

$$\mu(x) = (x - p)\frac{q(x)}{mq(x) + (x - p)q'(x)}$$

- This function also has a root at $x=p$. However,

$$\frac{q(p)}{mq(p) + 0 \times q'(x)} = \frac{1}{m} \neq 0$$

- So p is a simple root of $\mu(x)$.
- We can apply Newton's Method to $\mu(x)$ to recover the quadratic convergence rate

Modified Newton's Method

- Apply Newton's Method to $\mu(x)$:

$$x_{n+1} = x_n - \frac{\mu(x_n)}{\mu'(x_n)}$$

using

$$\mu' = \frac{(f'^2 - ff'')}{f'^2} \quad where \quad \mu(x) = \frac{f(x)}{f'(x)}$$

$$\Rightarrow \frac{\mu}{\mu'} = \frac{f}{f'}\left(\frac{f'^2}{f'^2 - ff''}\right)$$

$$\Rightarrow \frac{\mu}{\mu'} = \left(\frac{ff'}{f'^2 - ff''}\right)$$

Modified Newton's Method

- Now this method is called ***modified Newton's method*** and the algorithm is given by

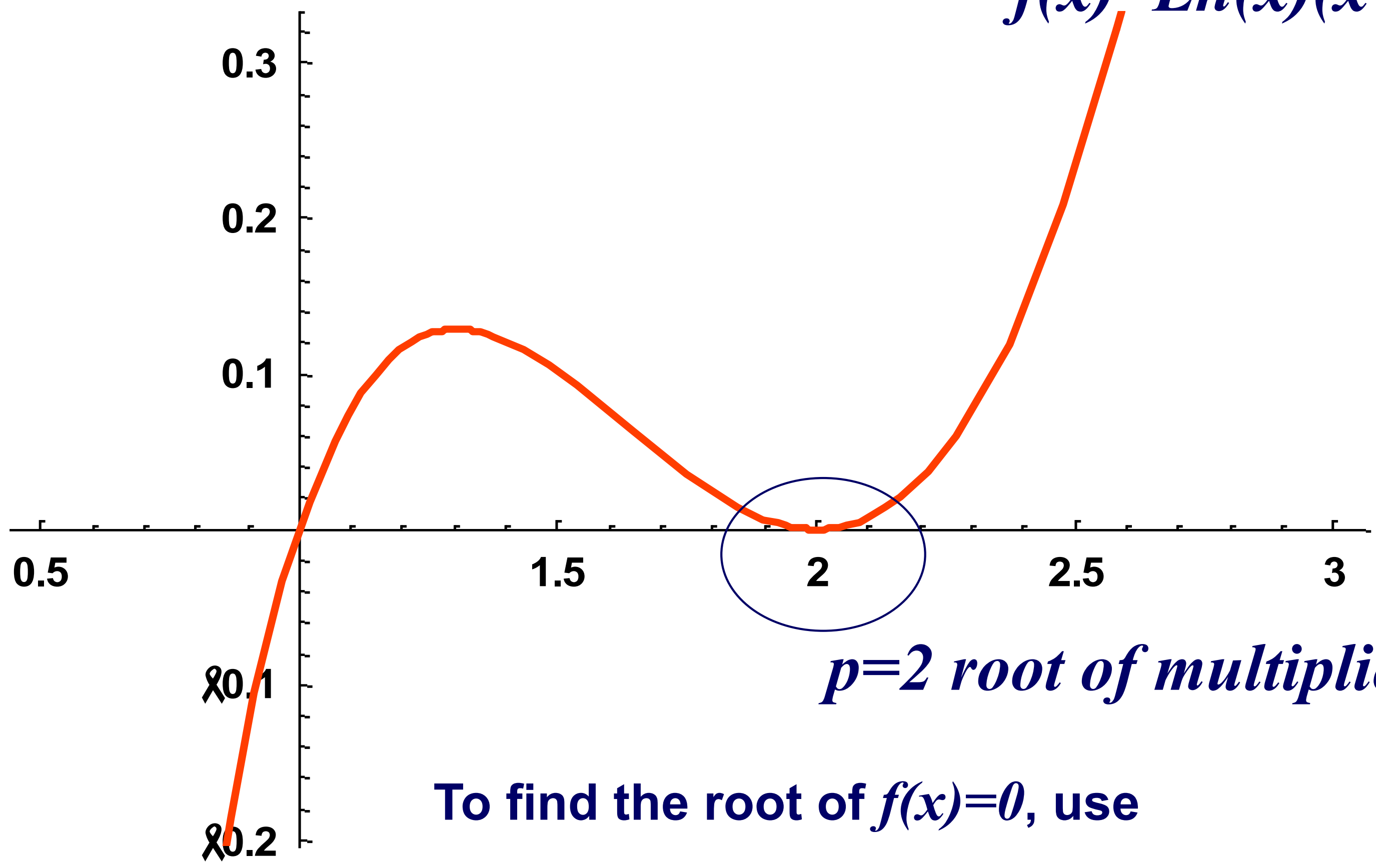
$$x_{n+1} = x_n - \frac{f(x_n).f'(x_n)}{\left[f'(x_n)\right]^2 - f(x_n).f''(x_n)}$$

- Note that this method requires more work since you need to calculate $d^2 f(x)/dx^2$

- Let us now see the application of this method on an example root finding problem

Root Finding Example For Multiple Roots

$f(x)=Ln(x)(x-2)^2$

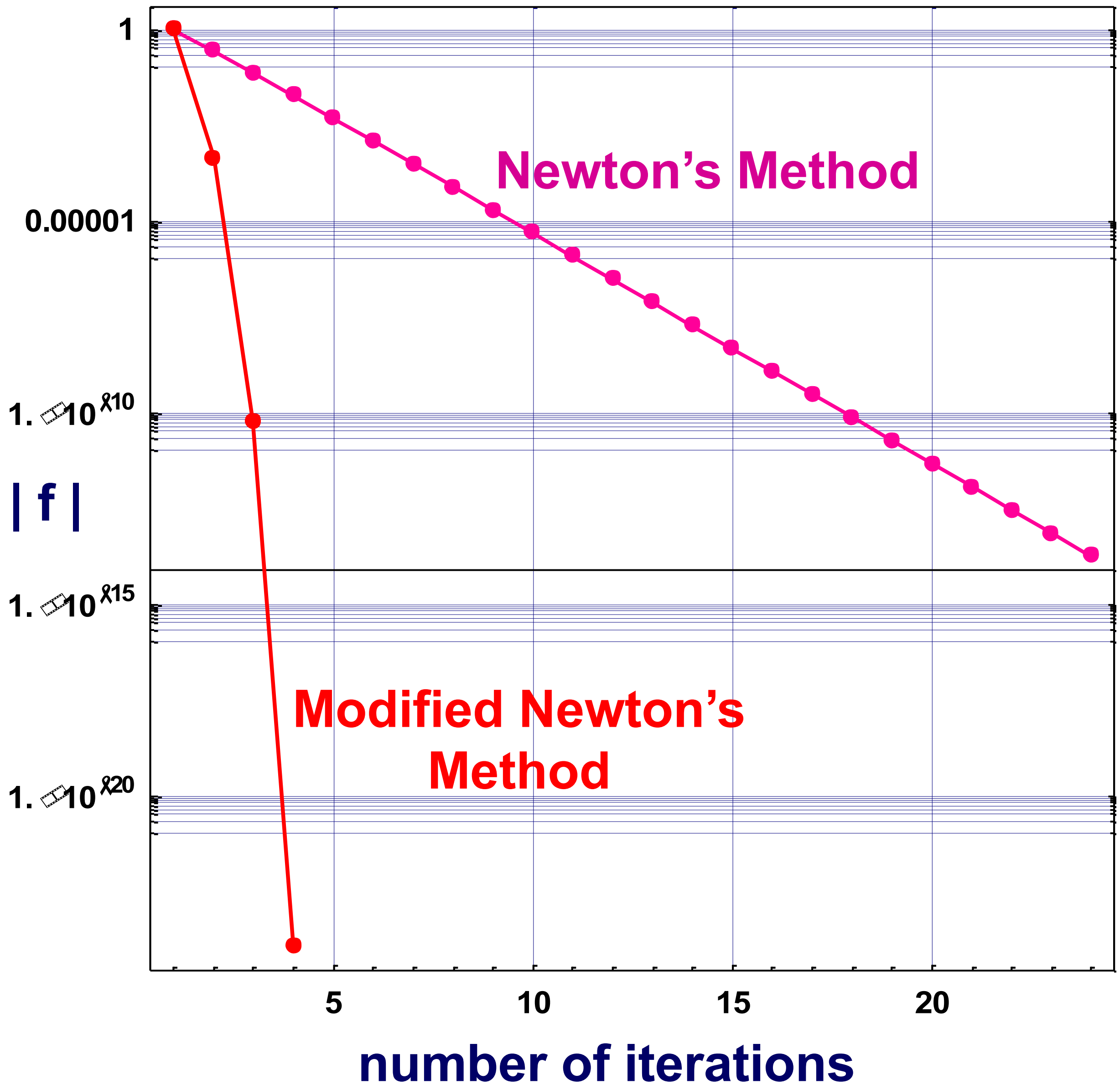


p=2 root of multiplicity 2

To find the root of $f(x)=0$, use

1. **Newton's Method** ***(x_0=3.0)***
2. **Modified Newton's Method** ***(x_0=3.0)***

Root Finding Example For Multiple Roots (2)



- Newton's method converge linearly to a root of multiplicity
- Modified Newton's Method exhibits quadratic convergence

Fixed Point Iteration (FPI)

- A number p is a ***fixed point*** for a given function $g(x)$ if $g(p)=p$
- Under some constraints, the algorithm $x_{n+1} = g(x_n)$ for $n = 0, 1, 2, \ldots$ will converge to a fixed point
- For a root finding problem in the form $f(x)=0$ we can define function $g(x)$ in a number of ways as

  $g(x) = x - f(x)$ *or* $g(x) = x + f(x)$ *or* $g(x) = x + c\, f(x)$ *where c is a constant*
- There can be MANY possible choices for $g(x)$. Consider:

1. $f(x) = x^2 - 2x - 3 = 0$

   $g_1(x) = (2x+3)^{1/2}$ *or* $g_2(x) = 3/(x-2)$ *or* $g_3(x) = (x^2-3)/2$.
2. $f(x) = x^2 - 5 = 0$

   $g_1(x) = x-(x^2-5)/6$ *or* $g_2(x) = x-(x^2-5)/3$ *or*

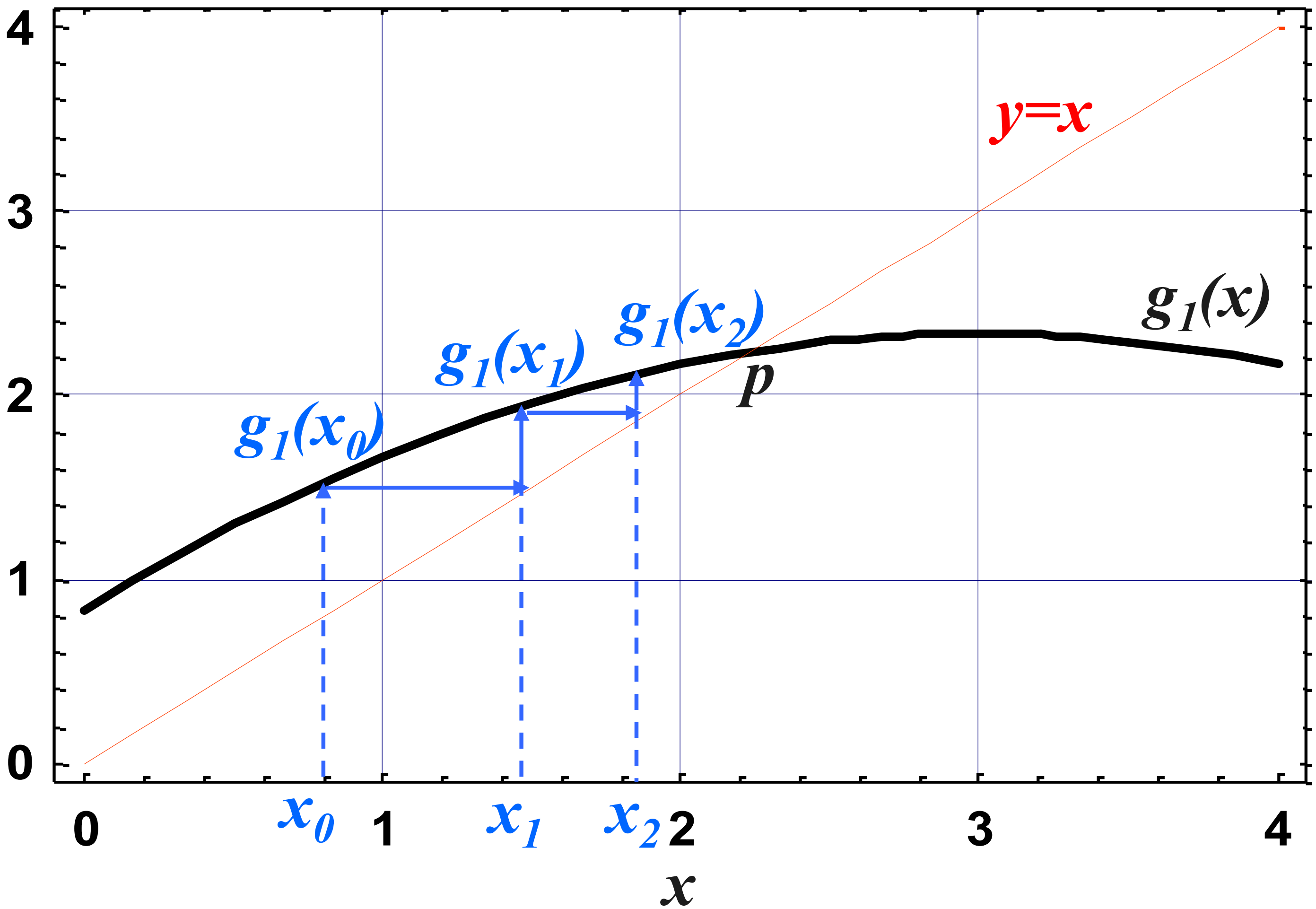
   $g_3(x) = x+(x^2-5)/3$

Let us work on this example

Monotonic convergence for FPI

f(x) = x² – 5 = 0 *and consider* $g_1(x) = x-(x^2-5)/6$

Note that c=-1/6 in g(x)= x + c f(x)



Requirement for monotonic convergence:

$$\boxed{0 \le g'(x) \le 1}$$

(in region of interest)

For our problem:

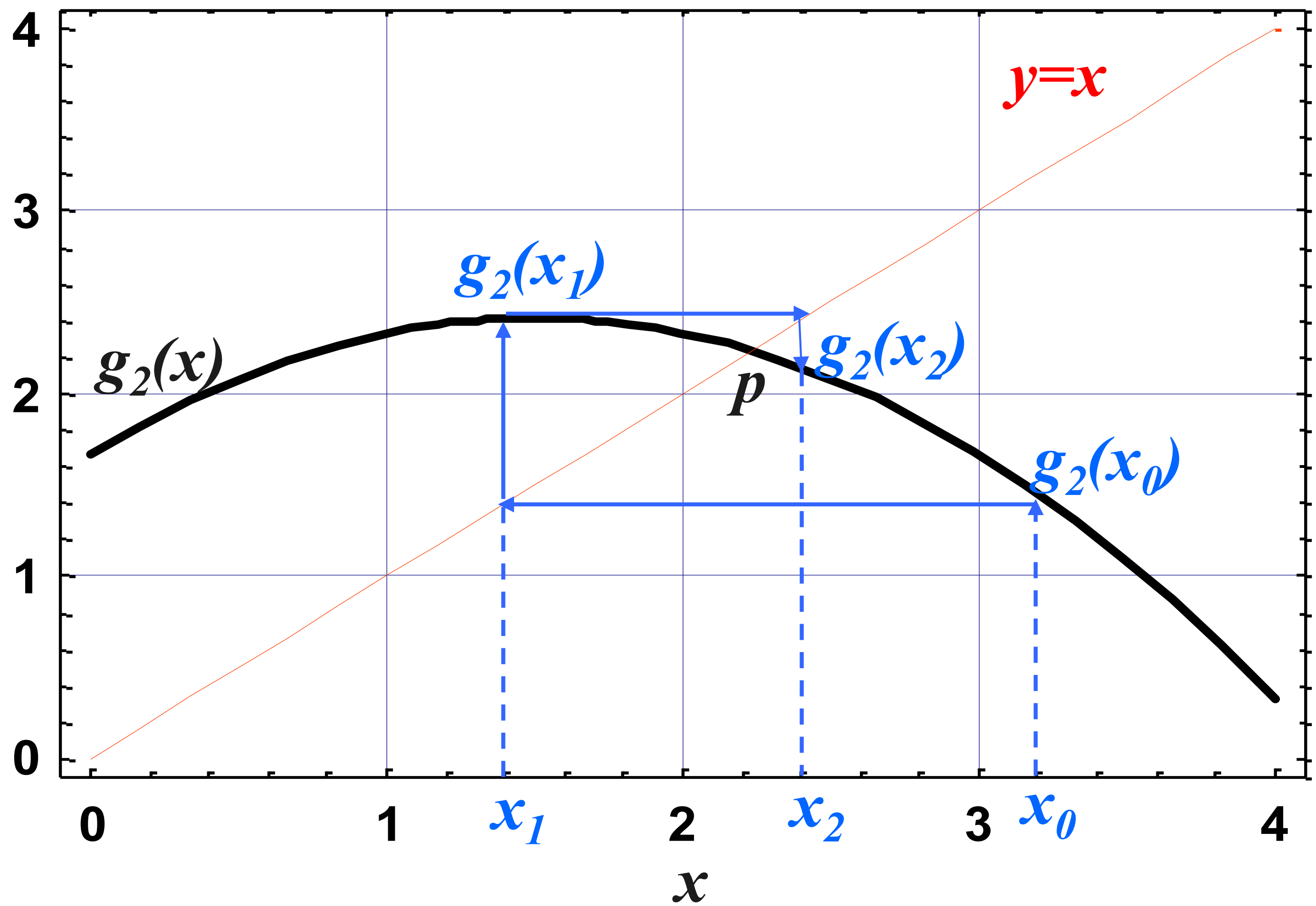
$$g_1'(x) = 1 - \frac{x}{3}$$

$$g_1'(p) = 0.254644$$

Oscillatory convergence for FPI

f(x) = x² – 5 = 0 *and consider* $g_2(x) = x-(x^2-5)/3$

Note that c=-1/3 in g(x)= x + c f(x)



Oscillatory convergence observed when:

$$\boxed{-1 \le g'(x) \le 0}$$

For our problem:

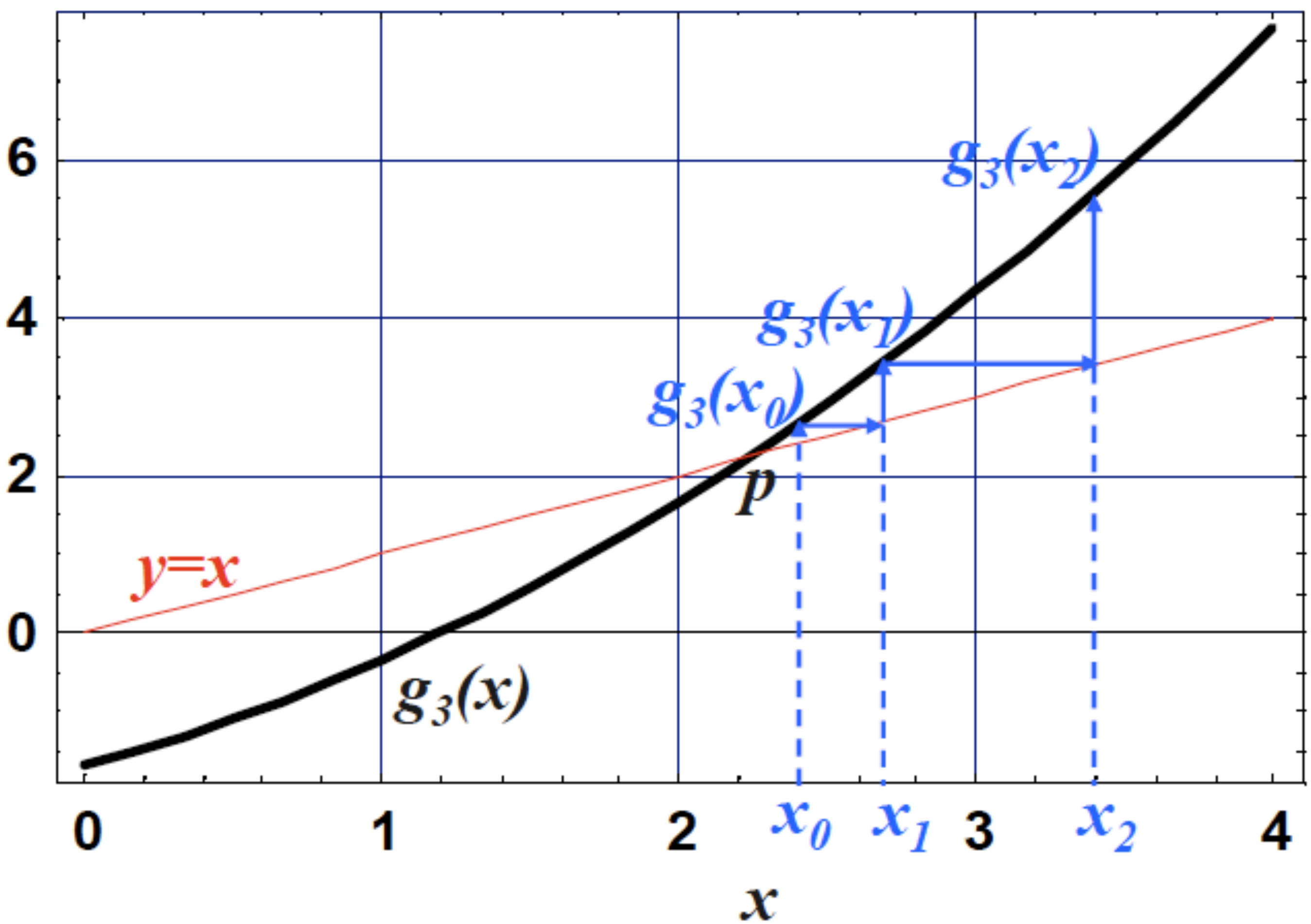
$$g_2'(x) = 1 - \frac{2x}{3}$$

$$g_2'(p) = -0.490712$$

Divergence for FPI

f(x) = x² – 5 = 0 *and consider* $g_3(x) = x+(x^2-5)/3$

Note that c=1/3 in g(x)= x + c f(x)



Divergence observed when:

$$\boxed{|g'(x)| > 1}$$

For our problem:

$$g_3'(x) = 1 + \frac{2x}{3}$$

$$g_3'(p) = 2.49071$$

Example on choosing *g(x)* function for a FPI

To find the positive root of $f(x) = x^2 - 5 = 0$ with a fixed point iteration, first I define

$$g(x) = x + cf(x) = x + c\left(x^2 - 5\right)$$

Then write the expression for its first derivative

$$g'(x) = 1 + cf'(x) = 1 + 2cx$$

Using the requirement for monotonic convergence $0 \le g'(x) \le 1$ in the interval [2,3]:

$$For\ x = 3.0 \Rightarrow \quad 0 \le 1 + 6c \le 1 \quad \Rightarrow -\frac{1}{6} \le c \le 0$$

$$For\ x = 2.0 \Rightarrow \quad 0 \le 1 + 4c \le 1 \quad \Rightarrow -\frac{1}{4} \le c \le 0$$

I choose $c = -1/6$

$$g(x) = x - \frac{1}{6}\left(x^2 - 5\right)$$

Summary

– We have defined what we mean by *root of multiplicity m* of a function and learned its effect on the rate of convergence

– Defined *modified Newton's method*

– Introduced *Fixed Point Iteration (FPI)* scheme for solving root finding problems and showed the converge criteria for *FPI*