MISSOURI S&T

Solution of Linear Set of Equations – 06

Dr. Serhat Hosder

Associate Professor of Aerospace Engineering

Mechanical and Aerospace Engineering

290B Toomey Hall

Missouri S&T

Rolla, MO 65409

Phone: 573-341-7239

E-mail: hosders@mst.edu

Outline

For Jacobi and Gauss-Seidel (indirect) methods:

1. Over/under-relaxation
2. Vector Norms and stopping criteria
3. Convergence of iterative methods

Matrix Form of Jacobi and Gauss-Seidel Iterations (1)

It is frequently convenient to examine (and code) the matrix form of these methods. Consider a different decomposition of A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

such that ***A=D-L-U*** where

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & 0 \\ \vdots & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ -a_{21} & \ddots & & \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{nn-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -a_{n-1n} \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

A = ***D*** - ***L*** - ***U***

Matrix Form of Jacobi and Gauss-Seidel Iterations (2)

The problem

$$Ax=b$$

$$(D - L - U)x=b$$

$$Dx=(L+U)x+b$$

$$D^{-1} = \begin{bmatrix} a_{11}^{-1} & 0 & .... & 0 \\ 0 & a_{22}^{-1} & & 0 \\ \vdots & & \ddots & \\ 0 & & & a_{nn}^{-1} \end{bmatrix}$$

If D^{-1} exists, then $x= D^{-1}(L+U)x + D^{-1}b$

Jacobi iteration: $x^{(k+1)} = D^{-1}(L+ U)x^{(k)} + D^{-1}b$

$$= x^{(k)} + D^{-1}r^{(k)}$$

$$\mathbf{\Delta x^{(k)}=D^{-1}r^{(k)}}$$

Where $r^{(k)} =b - A x^{(k)}$

Gauss-Seidel iteration: $(D - L)\, x^{(k+1)} = U x^{(k)} + b$

$$(D - L)\,(x^{(k+1)} - x^{(k)})= r^{(k)}$$

$$\mathbf{(D - L)\, \Delta x^{(k)}= r^{(k)}}$$

Over/Under-Relaxation

For the Jacobi and Gauss-Seidel methods, one can introduce a relaxation factor ω ($0 < \omega < 2$) in the algorithm

If $0 < \omega < 1$, the technique is called **Under-relaxation**.

If $1 < \omega < 2$, the technique is called **Over-relaxation**.

In many cases, over-relaxation accelerates the convergence and under-relaxation stabilizes the convergence of the algorithm.

Over/Under-Relaxation Applied to Jacobi Method

In Matrix Form:

$$\Delta x^k = \omega D^{-1} r^k$$

$$r^k = b - A x^k$$

$$x^{k+1} = x^k + \Delta x^k$$

In terms of individual components:

$$\Delta x_i^k = \omega \frac{r_i^k}{a_{ii}}$$

$$r_i^k = b_i - \sum_{j=1}^{n} a_{ij} x_j^k \qquad (i = 1,2,\ldots,n)$$

$$x_i^{k+1} = x_i^k + \Delta x_i^k$$

Over/Under Relaxation Applied to Gauss-Seidel Method

In Matrix Form:

$$(D-\omega L)\Delta x^{k} = \omega r^{k}$$

$$r^{k} = b - Ax^{k}$$

$$x^{k+1} = x^{k} + \Delta x^{k}$$

Over-Relaxation applied to Gauss-Seidel method is called Successive Over Relaxation (SOR)

In terms of individual components:

$$\Delta x_i^{k} = \omega \frac{\left[r_i^{k} - \sum_{j=1}^{i-1} a_{ij} \Delta x_j^{k} \right]}{a_{ii}} \qquad (i = 1,2,\ldots,n)$$

$$x_i^{k+1} = x_i^{k} + \Delta x_i^{k}$$

Solution strategy for the Matrix Form of the Gauss-Seidel Iteration with over/under-relaxation

$$(D-\omega L)\,\Delta x^k = \omega\, r^k$$

$$\begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ \omega a_{21} & a_{22} & 0 & \dots & 0 \\ \omega a_{31} & \omega a_{32} & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega a_{n1} & \omega a_{n2} & \omega a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{Bmatrix} \Delta x_1^k \\ \Delta x_2^k \\ \Delta x_3^k \\ \vdots \\ \Delta x_n^k \end{Bmatrix} = \begin{Bmatrix} \omega\, r_1^k \\ \omega\, r_2^k \\ \omega\, r_3^k \\ \vdots \\ \omega\, r_n^k \end{Bmatrix}$$

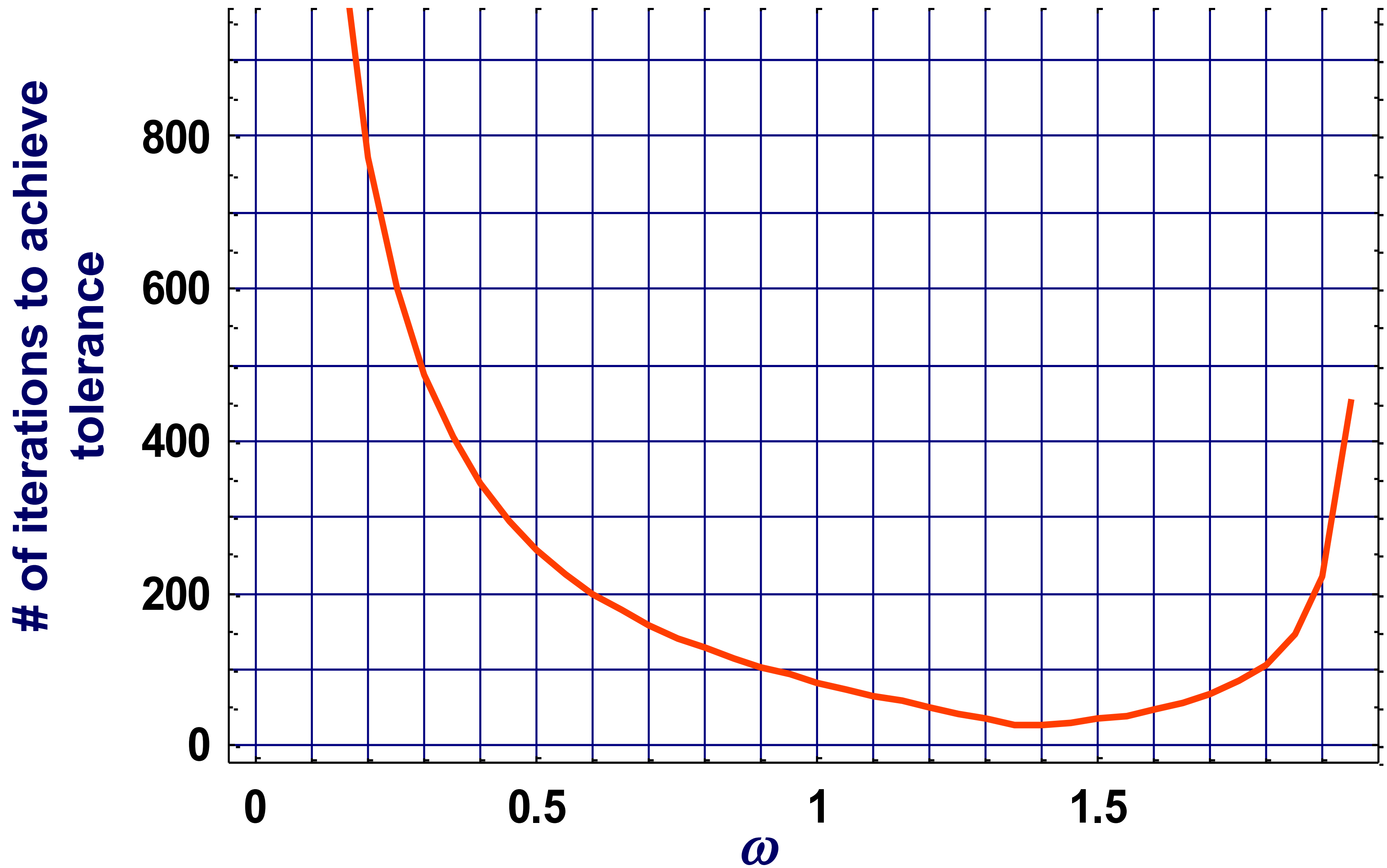
Note that ***(D-ωL)*** is a lower triangular matrix. Therefore we can use forward substitution to find $\Delta \boldsymbol{x}^{\boldsymbol{k}}_{\boldsymbol{i}}$

$$\Delta x_1^k = \omega \frac{r_1^k}{a_{11}}$$

$$\Delta x_i^k = \omega \frac{r_i^k - \sum_{j=1}^{i-1} a_{ij} \Delta x_j^k}{a_{ii}} \quad (i = 2,3,\dots.,n)$$

Relaxation parameter

$Ax=b$, where $A=(2x2)$ Hilbert Matrix and $b=[1, 1]^T$



Vector Norms

- Let f be a vector with n components:

$$f = \begin{bmatrix} f_1 & f_2 & \dots & f_n \end{bmatrix}^T$$

then

L_∞ *Norm*:

$$\|f\|_\infty = \max |f_i|_{1 \le i \le n}$$

L_1 *Norm*:

$$\|f\|_1 = |f_1| + |f_2| + \dots + |f_n|$$

L_2 *Norm*:

$$\|f\|_2 = \sqrt{f_1^2 + f_2^2 + \dots + f_n^2} = \left[\sum_{i=1}^{n} f_i^2\right]^{1/2}$$

Convergence criteria for Jacobi and Gauss-Seidel Methods

- A measure of convergence can be given as:

$$\frac{\left\| r^k \right\|_p}{\left\| r^0 \right\|_p} \le Tolerance \quad where \quad r^k = b - Ax^k$$

10^{-10}

$$p = 1, 2, or\, \infty$$

- Another possible stopping criterion:

$$\frac{\left\| x^{k+1} - x^k \right\|_p}{\left\| x^k \right\|_p} \le Tolerance$$

More on convergence of the iterative methods (1)

- **Spectral Radius**: Largest eigenvalue of a given matrix in absolute magnitude

$$\rho(A) = \max|\lambda_i| \quad \textit{where } \lambda_i \textit{ are the eigenvalues of } A \textit{ matrix}$$

$$\left(Ax = \lambda x \quad \Rightarrow \left[A - \lambda I\right]x = 0 \;\Rightarrow |A - \lambda I| = 0\right)$$

- We can write an iterative method in the form:

$$x^{k+1} = Tx^k + c$$

where $\boldsymbol{T}$ is an $(\boldsymbol{nxn})$ matrix, $\boldsymbol{x}$ is the variable vector $(\boldsymbol{nx1})$, and $\boldsymbol{c}$ is a $(\boldsymbol{nx1})$ vector

- Then **necessary condition** for convergence to the unique solution of $\boldsymbol{Ax=b}$ *(or $\boldsymbol{x=Tx+c}$)* for any $\boldsymbol{x^0}$:

$$\boxed{\rho(T) < 1}$$

- Remember the **sufficient condition** for convergence was $\boldsymbol{A}$ to be diagonally dominant

More on convergence of the iterative methods (2)

- **For Jacobi Iteration necessary condition for convergence:**

$$x^{k+1} = D^{-1}(L+U)x^k + D^{-1}b \quad \Rightarrow T = D^{-1}(L+U)$$

$$\boxed{\rho(T) = \rho\left[D^{-1}(L+U)\right] < 1}$$

- **For Gauss-Seidel Iteration necessary condition for convergence:**

$$x^{k+1} = (D-L)^{-1}Ux^k + (D-L)^{-1}b \quad \Rightarrow T = (D-L)^{-1}U$$

$$\boxed{\rho(T) = \rho\left[(D-L)^{-1}U\right] < 1}$$

- **Also note that**

$$\left\|x^k - x\right\| \approx \left[\rho(T)\right]^k \left\|x^0 - x\right\| \longrightarrow \begin{array}{lll} \rho << 1 & \Rightarrow & \textit{fast convergence} \\ \rho \approx 1 & \Rightarrow & \textit{slow convergence} \\ \rho > 1 & \Rightarrow & \textit{divergence} \end{array}$$

Summary

In this lecture

- We have seen
  - over/under-relaxation methods applied to Jacobi and Gauss Seidel iterations
  - relaxation parameter
- We have defined different vector norms and the convergence criteria for the iterative methods
- We have defined spectral radius and its importance in the convergence of iterative methods