22. Orthogonal Polynomials

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Contents

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22. Orthogonal Polynomials

Mathematical Properties

22.1. Definition of Orthogonal Polynomials

A system of polynomials $f_n(x)$, degree $[f_n(x)]=n$, is called orthogonal on the interval $a \leq x \leq b$, with respect to the weight function $w(x)$, if

22.1.1

$$\int_a^b w(x) f_n(x) f_m(x) dx = 0 \qquad (n \neq m;\, n, m = 0, 1, 2, \ldots)$$

The weight function $w(x)[w(x) \geq 0]$ determines the system $f_n(x)$ up to a constant factor in each polynomial. The specification of these factors is referred to as standardization. For suitably standardized orthogonal polynomials we set

22.1.2

$$\int_a^b w(x) f_n^2(x) dx = h_n,\; f_n(x) = k_n x^n + k_n' x^{n-1} + \ldots \qquad (n = 0, 1, 2, \ldots)$$

These polynomials satisfy a number of relationships of the same general form. The most important ones are:

Differential Equation

22.1.3 $$g_2(x) f_n'' + g_1(x) f_n' + a_n f_n = 0$$

where $g_2(x)$, $g_1(x)$ are independent of n and a_n a constant depending only on n.

Recurrence Relation

22.1.4 $$f_{n+1} = (a_n + x b_n) f_n - c_n f_{n-1}$$

where

22.1.5

$$b_n = \frac{k_{n+1}}{k_n}, \quad a_n = b_n \left(\frac{k_{n+1}'}{k_{n+1}} - \frac{k_n'}{k_n} \right), \quad c_n = \frac{k_{n+1} k_{n-1} h_n}{k_n^2 h_{n-1}}$$

Rodrigues' Formula

22.1.6 $$f_n = \frac{1}{e_n w(x)} \frac{d^n}{dx_n} \{ w(x) [g(x)]^n \}$$

where $g(x)$ is a polynomial in x independent of n. The system $\left\{ \frac{df_n}{dx} \right\}$ consists again of orthogonal polynomials.

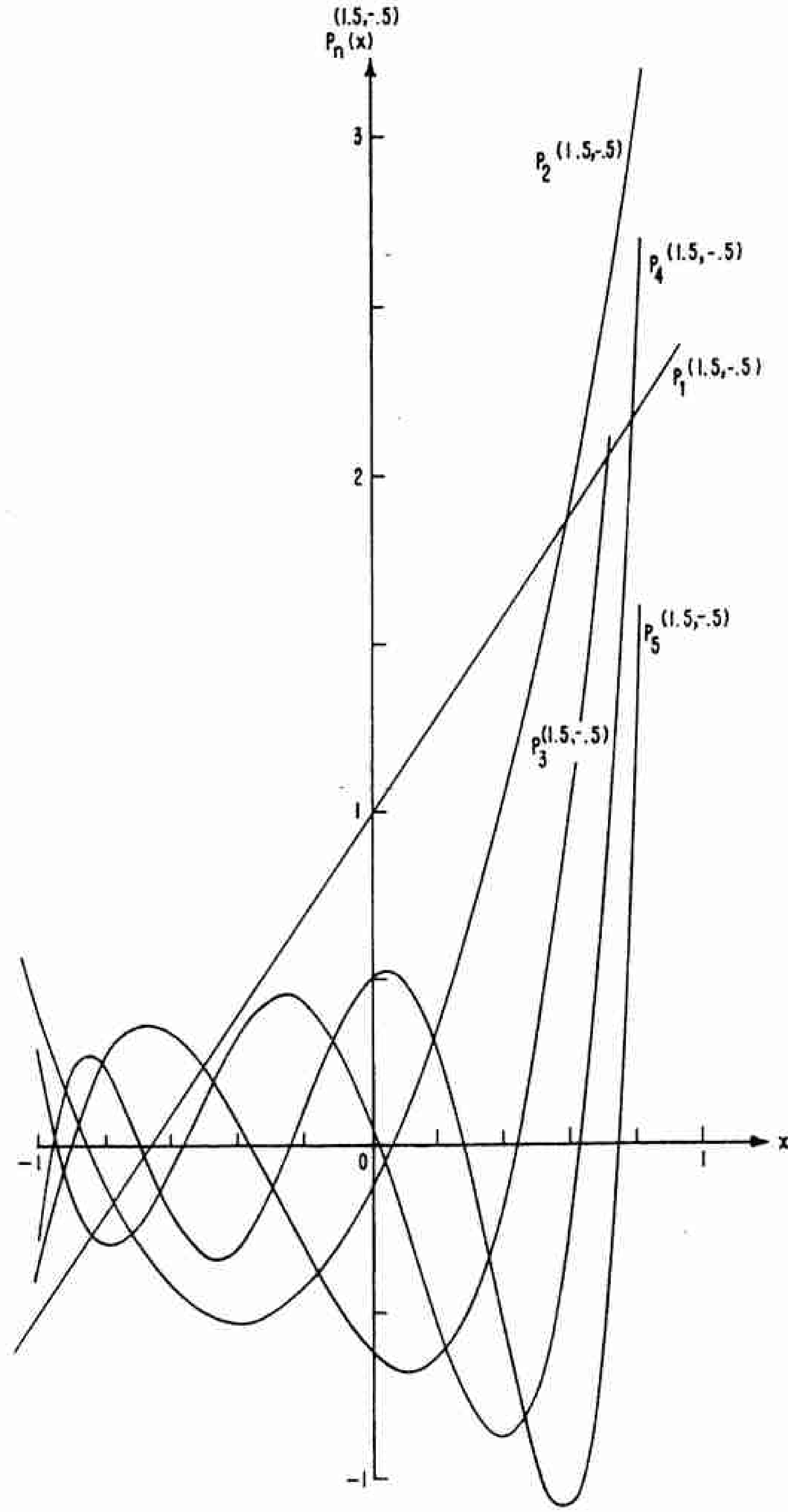

FIGURE 22.1. *Jacobi Polynomials* $P_n^{(\alpha,\beta)}(x)$, $\alpha=1.5$, $\beta=-.5$, $n=1(1)5$.

22.2. Orthogonality Relations

	$f_n(x)$	Name of Polynomial	a	b	$w(x)$	Standardization	h_n	Remarks
22.2.1	$P_n^{(\alpha,\beta)}(x)$	Jacobi	-1	1	$(1-x)^\alpha(1+x)^\beta$	$P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n}$	$\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}$	$\alpha>-1,\ \beta>-1$
22.2.2	$G_n(p, q, x)$	Jacobi	0	1	$(1-x)^{p-q}x^{q-1}$	$k_n=1$	$\frac{n!\Gamma(n+q)\Gamma(n+p)\Gamma(n+p-q+1)}{(2n+p)\Gamma^2(2n+p)}$	$p-q>-1,\ q>0$
22.2.3	$C_n^{(\alpha)}(x)$	Ultraspherical (Gegenbauer)	-1	1	$(1-x^2)^{\alpha-\frac{1}{2}}$	$C_n^{(\alpha)}(1)=\binom{n+2\alpha-1}{n}$ $(\alpha\neq 0)$	$\frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)[\Gamma(\alpha)]^2}$ $\alpha\neq 0$	$\alpha>-\frac{1}{2}$
						$C_n^{(0)}(1)=\frac{2}{n}$, $C_0^{(0)}(1)=1$	$\frac{2\pi}{n^2}$ $\alpha=0$	
22.2.4	$T_n(x)$	Chebyshev of the first kind	-1	1	$(1-x^2)^{-\frac{1}{2}}$	$T_n(1)=1$	$\begin{cases}\frac{\pi}{2} & n\neq 0\\ \pi & n=0\end{cases}$	
22.2.5	$U_n(x)$	Chebyshev of the second kind	-1	1	$(1-x^2)^{\frac{1}{2}}$	$U_n(1)=n+1$	$\frac{\pi}{2}$	∞
* 22.2.6	$C_n(x)$	Chebyshev of the first kind	-2	2	$\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}$	$C_n(2)=2$	$\begin{cases}4\pi & n\neq 0\\ 8\pi & n=0\end{cases}$	
* 22.2.7	$S_n(x)$	Chebyshev of the second kind	-2	2	$\left(1-\frac{x^2}{4}\right)^{\frac{1}{2}}$	$S_n(2)=n+1$	π	
22.2.8	$T_n^*(x)$	Shifted Chebyshev of the first kind	0	1	$(x-x^2)^{-\frac{1}{2}}$	$T_n^*(1)=1$	$\begin{cases}\frac{\pi}{2} & n\neq 0\\ \pi & n=0\end{cases}$	
22.2.9	$U_n^*(x)$	Shifted Chebyshev of the second kind	0	1	$(x-x^2)^{\frac{1}{2}}$	$U_n^*(1)=n+1$	$\frac{\pi}{8}$ *	
22.2.10	$P_n(x)$	Legendre (Spherical)	-1	1	1	$P_n(1)=1$	$\frac{2}{2n+1}$	
22.2.11	$P_n^*(x)$	Shifted Legendre	0	1	1		$\frac{1}{2n+1}$	

*See page II.

22.2. Orthogonality Relations—Continued

22.2.12	$L_n^{(\alpha)}(x)$	Generalized Laguerre	0	∞	$e^{-x}x^{\alpha}$	$k_n=\frac{(-1)^n}{n!}$	$\frac{\Gamma(\alpha+n+1)}{n!}$	$\alpha>-1$
22.2.13	$L_n(x)$	Laguerre	0	∞	e^{-x}	$k_n=\frac{(-1)^n}{n!}$	1	
* 22.2.14	$H_n(x)$	Hermite	$-\infty$	∞	e^{-x^2}	$e_n=(-1)^n$	$\sqrt{\pi}2^n n!$	
* 22.2.15	$He_n(x)$	Hermite	$-\infty$	∞	$e^{-\frac{x^2}{2}}$	$e_n=(-1)^n$	$\sqrt{2\pi}n!$	

*See page II.

22.3. Explicit Expressions

$$f_n(x)=d_n\sum_{m=0}^{N}c_m g_m(x)$$

	$f_n(x)$	N	d_n	c_m	$g_m(x)$	k_n	Remarks
22.3.1	$P_n^{(\alpha,\beta)}(x)$	n	$\frac{1}{2^n}$	$\binom{n+\alpha}{m}\binom{n+\beta}{n-m}$	$(x-1)^{n-m}(x+1)^m$	$\frac{1}{2^n}\binom{2n+\alpha+\beta}{n}$	$\alpha>-1,\ \beta>-1$
22.3.2	$P_n^{(\alpha,\beta)}(x)$	n	$\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)}$	$\binom{n}{m}\frac{\Gamma(\alpha+\beta+n+m+1)}{2^m\Gamma(\alpha+m+1)}$	$(x-1)^m$	$\frac{1}{2^n}\binom{2n+\alpha+\beta}{n}$	$\alpha>-1,\ \beta>-1$
22.3.3	$G_n(p,q,x)$	n	$\frac{\Gamma(q+n)}{\Gamma(p+2n)}$	$(-1)^m\binom{n}{m}\frac{\Gamma(p+2n-m)}{\Gamma(q+n-m)}$	x^{n-m}	1	$p-q>-1,\ q>0$
22.3.4	$C_n^{(\alpha)}(x)$	$\left[\frac{n}{2}\right]$	$\frac{1}{\Gamma(\alpha)}$	$(-1)^m\frac{\Gamma(\alpha+n-m)}{m!(n-2m)!}$	$(2x)^{n-2m}$	$\frac{2^n}{n!}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$	$\alpha>-\frac{1}{2},\ \alpha\neq 0$
22.3.5	$C_n^{(0)}(x)$	$\left[\frac{n}{2}\right]$	1	$(-1)^m\frac{(n-m-1)!}{m!(n-2m)!}$	$(2x)^{n-2m}$	$\frac{2^n}{n}\quad n\neq 0$	$n\neq 0,\ C_0^{(0)}(1)=1$
22.3.6	$T_n(x)$	$\left[\frac{n}{2}\right]$	$\frac{n}{2}$	$(-1)^m\frac{(n-m-1)!}{m!(n-2m)!}$	$(2x)^{n-2m}$	2^{n-1}	
22.3.7	$U_n(x)$	$\left[\frac{n}{2}\right]$	1	$(-1)^m\frac{(n-m)!}{m!(n-2m)!}$	$(2x)^{n-2m}$	2^n	
22.3.8	$P_n(x)$	$\left[\frac{n}{2}\right]$	$\frac{1}{2^n}$	$(-1)^m\binom{n}{m}\binom{2n-2m}{n}$	x^{n-2m}	$\frac{(2n)!}{2^n(n!)^2}$	
22.3.9	$L_n^{(\alpha)}(x)$	n	1	$(-1)^m\binom{n+\alpha}{n-m}\frac{1}{m!}$	x^m	$\frac{(-1)^n}{n!}$	$\alpha>-1$
22.3.10	$H_n(x)$	$\left[\frac{n}{2}\right]$	$n!$	$(-1)^m\frac{1}{m!(n-2m)!}$	$(2x)^{n-2m}$	2^n	see **22.11**
22.3.11	$He_n(x)$	$\left[\frac{n}{2}\right]$	$n!$	$(-1)^m\frac{1}{m!2^m(n-2m)!}$	x^{n-2m}	1	

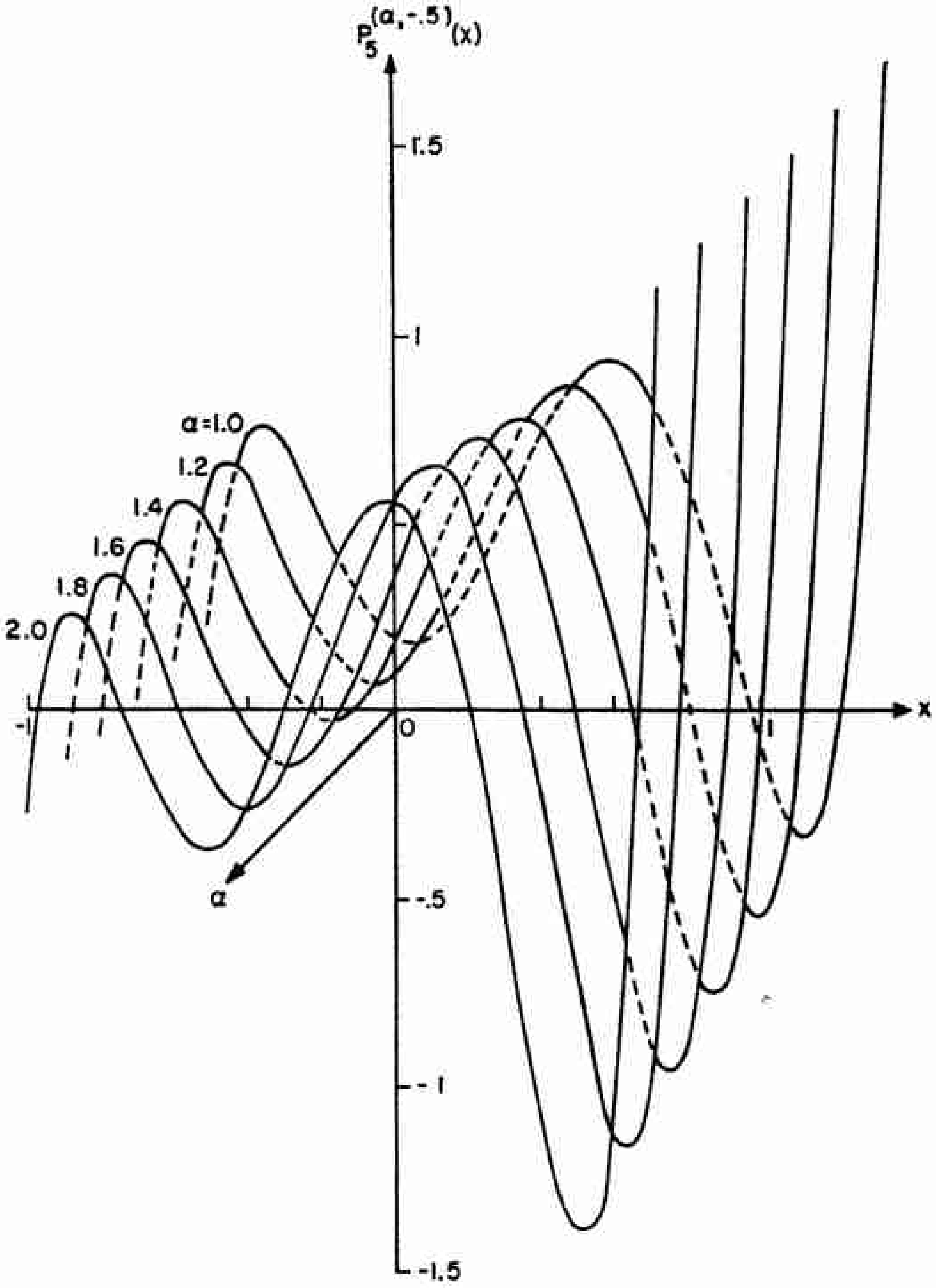

FIGURE 22.2. *Jacobi Polynomials* $P_n^{(\alpha,\beta)}(x)$, $\alpha=1(.2)2,\ \beta=-.5,\ n=5.$

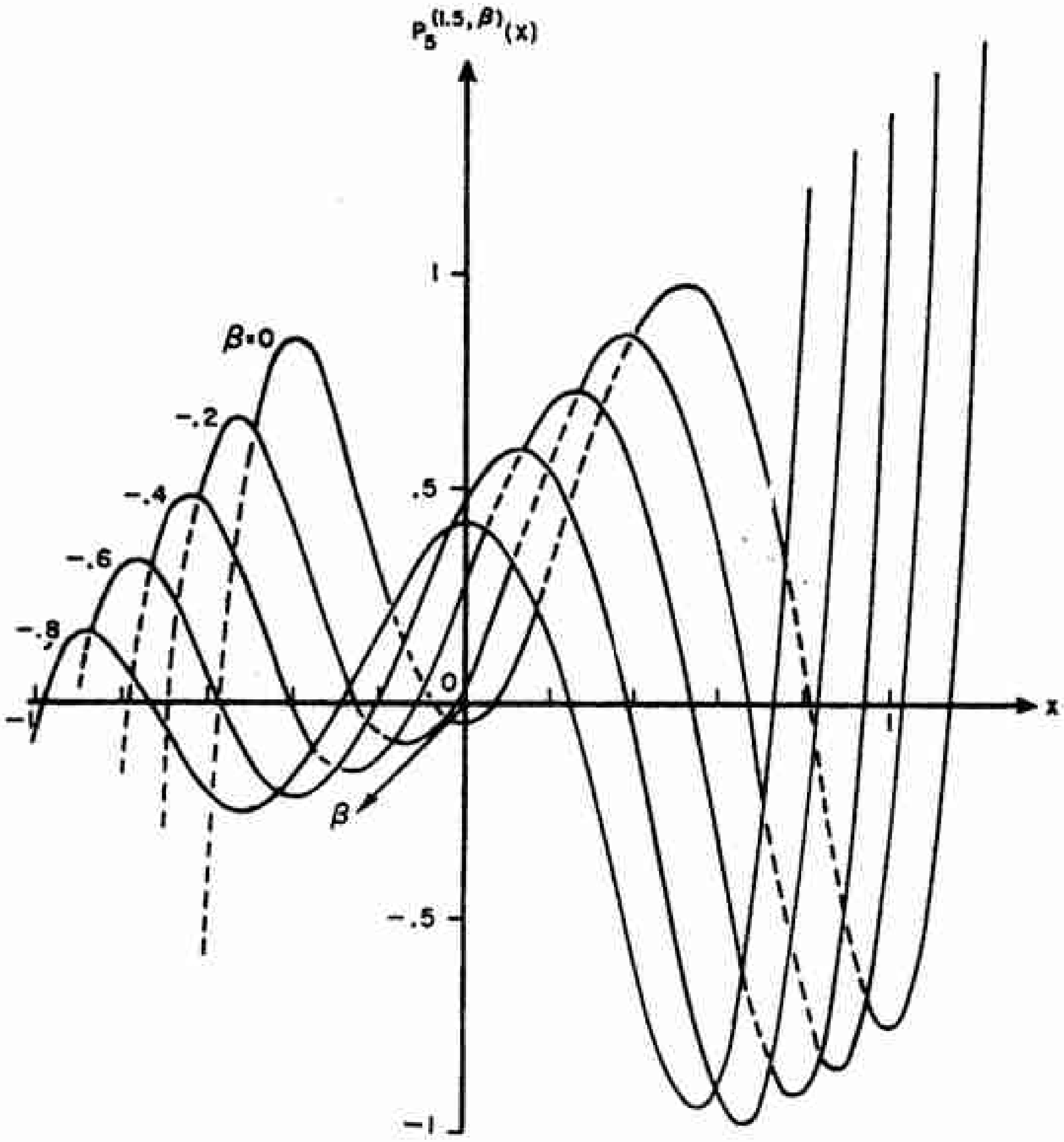

FIGURE 22.3. *Jacobi Polynomials* $P_n^{(\alpha,\beta)}(x)$, $\alpha=1.5,\ \beta=-.8(.2)0,\ n=5.$

Explicit Expressions Involving Trigonometric Functions

$$f_n(\cos\theta)=\sum_{m=0}^{n} a_m \cos\,(n-2m)\theta$$

	$f_n(\cos\theta)$	a_m	Remarks
22.3.12	$C_n^{(\alpha)}(\cos\theta)$	$\dfrac{\Gamma(\alpha+m)\Gamma(\alpha+n-m)}{m!(n-m)![\Gamma(\alpha)]^2}$	$\alpha\neq 0$
22.3.13	$P_n(\cos\theta)$	$\dfrac{1}{4^n}\binom{2m}{m}\binom{2n-2m}{n-m}$	

22.3.14 $$C_n^{(0)}(\cos\theta)=\frac{2}{n}\cos n\theta$$

22.3.15 $$T_n(\cos\theta)=\cos n\theta$$

22.3.16 $$U_n(\cos\theta)=\frac{\sin\,(n+1)\theta}{\sin\theta}$$

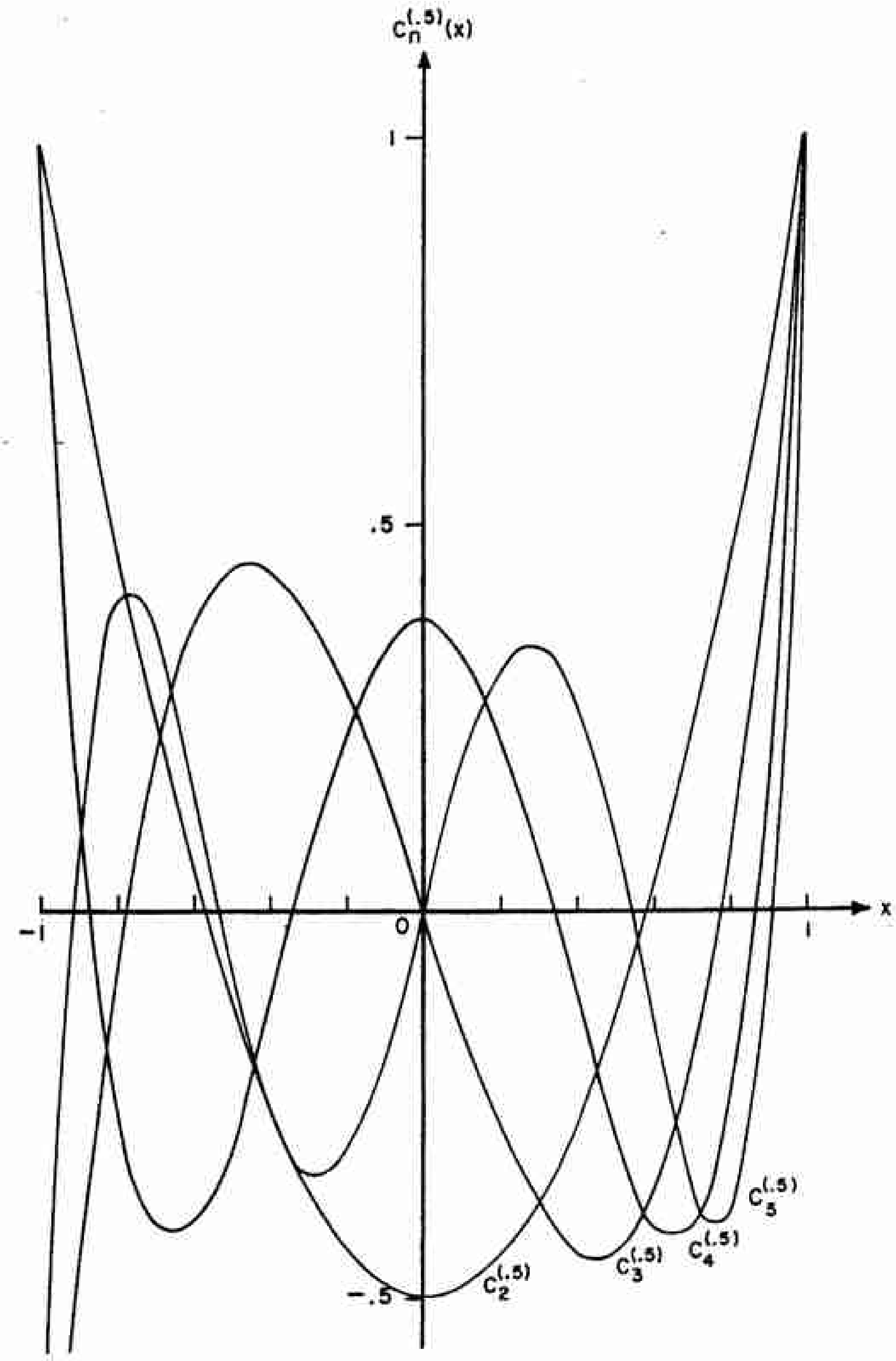

FIGURE 22.4. *Gegenbauer (Ultraspherical) Polynomials* $C_n^{(\alpha)}(x)$, $\alpha=.5,\ n=2(1)5.$

22.4. Special Values

	$f_n(x)$	$f_n(-x)$	$f_n(1)$	$f_n(0)$	$f_0(x)$	$f_1(x)$
22.4.1	$P_n^{(\alpha,\beta)}(x)$	$(-1)^n P_n^{(\beta,\alpha)}(x)$	$\binom{n+\alpha}{n}$ *		1	$\frac{1}{2}[\alpha-\beta+(\alpha+\beta+2)x]$
22.4.2	$C_n^{(\alpha)}(x)$ $\alpha\neq 0$	$(-1)^n C_n^{(\alpha)}(x)$	$\binom{n+2\alpha-1}{n}$	$\begin{cases} 0, & n=2m+1 \\ (-1)^{n/2}\frac{\Gamma(\alpha+n/2)}{\Gamma(\alpha)(n/2)!}, & n=2m \end{cases}$	1	$2\alpha x$
22.4.3	$C_n^{(0)}(x)$	$(-1)^n C_n^{(0)}(x)$	$\frac{2}{n}$, $n\neq 0$	$\begin{cases} \frac{(-1)^m}{m}, & n=2m\neq 0 \\ 0, & n=2m+1 \end{cases}$	1	$2x$
22.4.4	$T_n(x)$	$(-1)^n T_n(x)$	1	$\begin{cases} (-1)^m, & n=2m \\ 0, & n=2m+1 \end{cases}$	1	x
22.4.5	$U_n(x)$	$(-1)^n U_n(x)$	$n+1$	$\begin{cases} (-1)^m, & n=2m \\ 0, & n=2m+1 \end{cases}$	1	$2x$
22.4.6	$P_n(x)$	$(-1)^n P_n(x)$	1	$\begin{cases} \frac{(-1)^m}{4^m}\binom{2m}{m}, & n=2m^* \\ 0, & n=2m+1 \end{cases}$	1	x
22.4.7	$L_n^{(\alpha)}(x)$			$\binom{n+\alpha}{n}$	1	$-x+\alpha+1$
22.4.8	$H_n(x)$	$(-1)^n H_n(x)$		$\begin{cases} (-1)^m\frac{(2m)!}{m!}, & n=2m \\ 0, & n=2m+1 \end{cases}$	1	$2x$

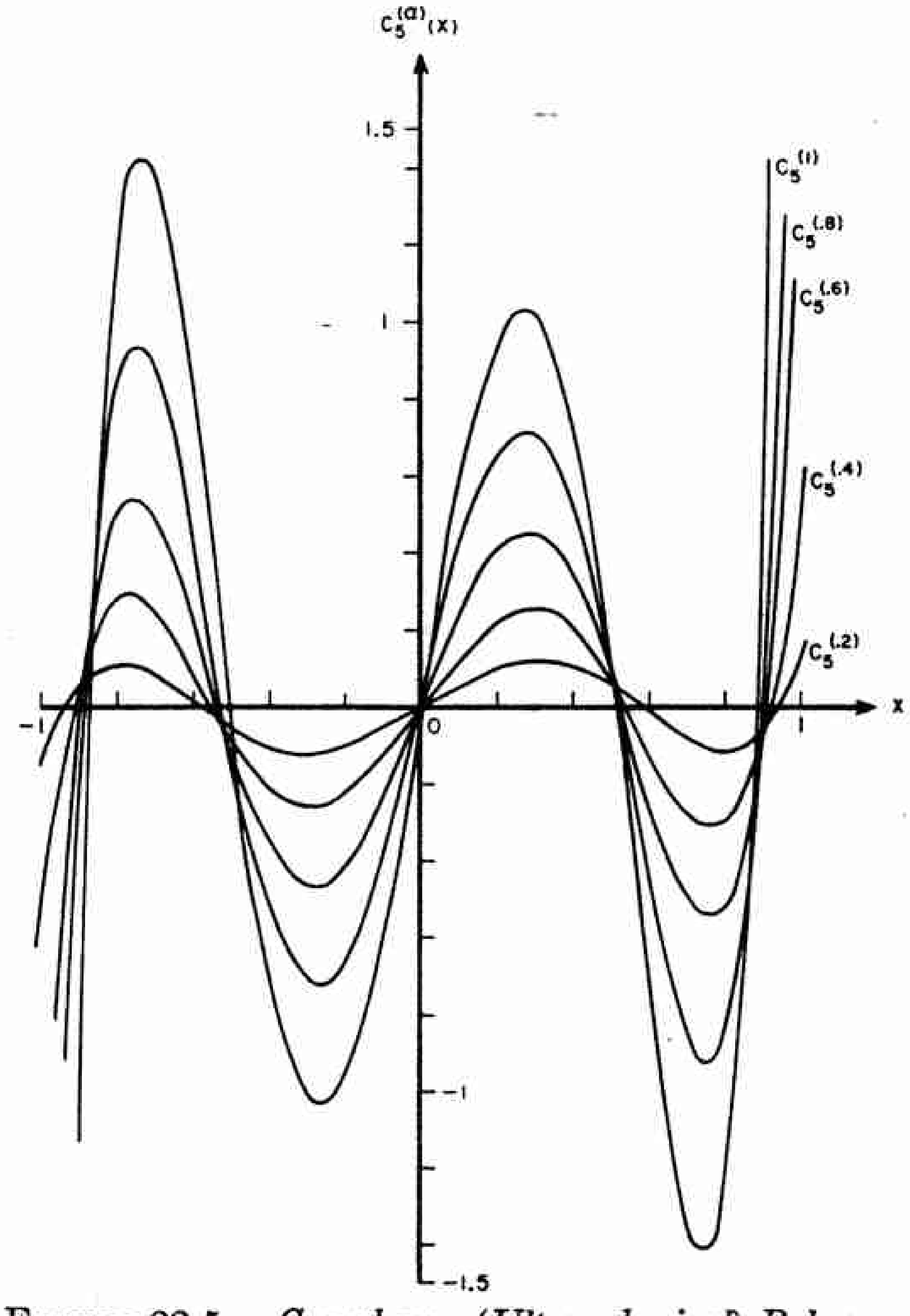

FIGURE 22.5. *Gegenbauer (Ultraspherical) Polynomials* $C_n^{(\alpha)}(x)$, $\alpha=.2(.2)1$, $n=5$.

22.5. Interrelations

Interrelations Between Orthogonal Polynomials of the Same Family

Jacobi Polynomials

22.5.1

$$P_n^{(\alpha,\beta)}(x)=\frac{\Gamma(2n+\alpha+\beta+1)}{n!\,\Gamma(n+\alpha+\beta+1)}G_n\left(\alpha+\beta+1,\beta+1,\frac{x+1}{2}\right)$$

22.5.2

$$G_n(p,q,x)=\frac{n!\,\Gamma(n+p)}{\Gamma(2n+p)}P_n^{(p-q,q-1)}(2x-1)$$

(see [22.21]).

22.5.3

$$F_n(p,q,x)=(-1)^n n!\,\frac{\Gamma(q)}{\Gamma(q+n)}P_n^{(p-q,q-1)}(2x-1)$$

(see [22.13]).

Ultraspherical Polynomials

22.5.4 $$C_n^{(0)}(x)=\lim_{\alpha\to 0}\frac{1}{\alpha}C_n^{(\alpha)}(x)$$

Chebyshev Polynomials

22.5.5 $$T_n(x)=\tfrac{1}{2}C_n(2x)=T_n^*\left(\frac{1+x}{2}\right)$$

22.5.6 $$T_n(x)=U_n(x)-xU_{n-1}(x)$$

*See page II.

22.5.7 $$T_n(x)=xU_{n-1}(x)-U_{n-2}(x)$$

22.5.8 $$T_n(x)=\tfrac{1}{2}\left[U_n(x)-U_{n-2}(x)\right]$$

22.5.9 $$U_n(x)=S_n(2x)=U_n^*\left(\frac{1+x}{2}\right)$$

22.5.10 $$U_{n-1}(x)=\frac{1}{1-x^2}\left[xT_n(x)-T_{n+1}(x)\right]$$

22.5.11 $$C_n(x)=2T_n\left(\frac{x}{2}\right)=2T_n^*\left(\frac{x+2}{4}\right)$$

22.5.12 $$C_n(x)=S_n(x)-S_{n-2}(x)$$

22.5.13 $$S_n(x)=U_n\left(\frac{x}{2}\right)=U_n^*\left(\frac{x+2}{4}\right)$$

22.5.14 $$T_n^*(x)=T_n(2x-1)=\tfrac{1}{2}\,C_n(4x-2)$$

(see [22.22]).

22.5.15 $$U_n^*(x)=S_n(4x-2)=U_n(2x-1)$$

(see [22.22]).

Generalized Laguerre Polynomials

22.5.16 $$L_n^{(0)}(x)=L_n(x)$$

22.5.17 $$L_n^{(m)}(x)=(-1)^m\frac{d^m}{dx^m}\left[L_{n+m}(x)\right]$$

Hermite Polynomials

22.5.18 $$He_n(x)=2^{-n/2}H_n\left(\frac{x}{\sqrt{2}}\right)$$

(see [22.20]).

22.5.19 $$H_n(x)=2^{n/2}He_n(x\sqrt{2})$$

(see [22.13], [22.20]).

Interrelations Between Orthogonal Polynomials of Different Families

Jacobi Polynomials

22.5.20
$$P_n^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x)=\frac{\Gamma(2\alpha)\Gamma(\alpha+n+\frac{1}{2})}{\Gamma(2\alpha+n)\Gamma(\alpha+\frac{1}{2})}\,C_n^{(\alpha)}(x)$$

22.5.21
$$P_n^{(\alpha,\frac{1}{2})}(x)=\frac{(\frac{1}{2})_{n+1}}{\sqrt{\frac{x+1}{2}}\,(\alpha+\frac{1}{2})_{n+1}}\,C_{2n+1}^{(\alpha+\frac{1}{2})}\left(\sqrt{\frac{x+1}{2}}\right)$$

22.5.22 $$P_n^{(\alpha,-\frac{1}{2})}(x)=\frac{(\frac{1}{2})_n}{(\alpha+\frac{1}{2})_n}\,C_{2n}^{(\alpha+\frac{1}{2})}\left(\sqrt{\frac{x+1}{2}}\right)$$

22.5.23 $$P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)=\frac{1}{4^n}\binom{2n}{n}T_n(x)$$

22.5.24 $$P_n^{(0,0)}(x)=P_n(x)$$

Ultraspherical Polynomials

22.5.25
$$C_{2n}^{(\alpha)}(x)=\frac{\Gamma(\alpha+n)n!2^{2n}}{\Gamma(\alpha)(2n)!}\,P_n^{(\alpha-\frac{1}{2},-\frac{1}{2})}(2x^2-1)$$
$$(\alpha\neq 0)$$

22.5.26
$$C_{2n+1}^{(\alpha)}(x)=\frac{\Gamma(\alpha+n+1)n!2^{2n+1}}{\Gamma(\alpha)(2n+1)!}\,xP_n^{(\alpha-\frac{1}{2},\frac{1}{2})}(2x^2-1)$$
$$(\alpha\neq 0)$$

22.5.27
$$C_n^{(\alpha)}(x)=\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(2\alpha+n)}{\Gamma(2\alpha)\Gamma(\alpha+n+\frac{1}{2})}\,P_n^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x)$$
$$(\alpha\neq 0)$$

22.5.28
$$C_n^{(0)}(x)=\frac{2}{n}T_n(x)=2\,\frac{(n-1)!}{\Gamma(n+\frac{1}{2})}\sqrt{\pi}P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)$$ *

Chebyshev Polynomials

22.5.29 $$T_{2n+1}(x)=\frac{n!\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}\,xP_n^{(-\frac{1}{2},\frac{1}{2})}(2x^2-1)$$

22.5.30 $$U_{2n}(x)=\frac{n!\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}\,P_n^{(\frac{1}{2},-\frac{1}{2})}(2x^2-1)$$

22.5.31 $$T_n(x)=\frac{n!\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}\,P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)$$

22.5.32 $$U_n(x)=\frac{(n+1)!\sqrt{\pi}}{2\Gamma(n+\frac{3}{2})}\,P_n^{(\frac{1}{2},\frac{1}{2})}(x)$$

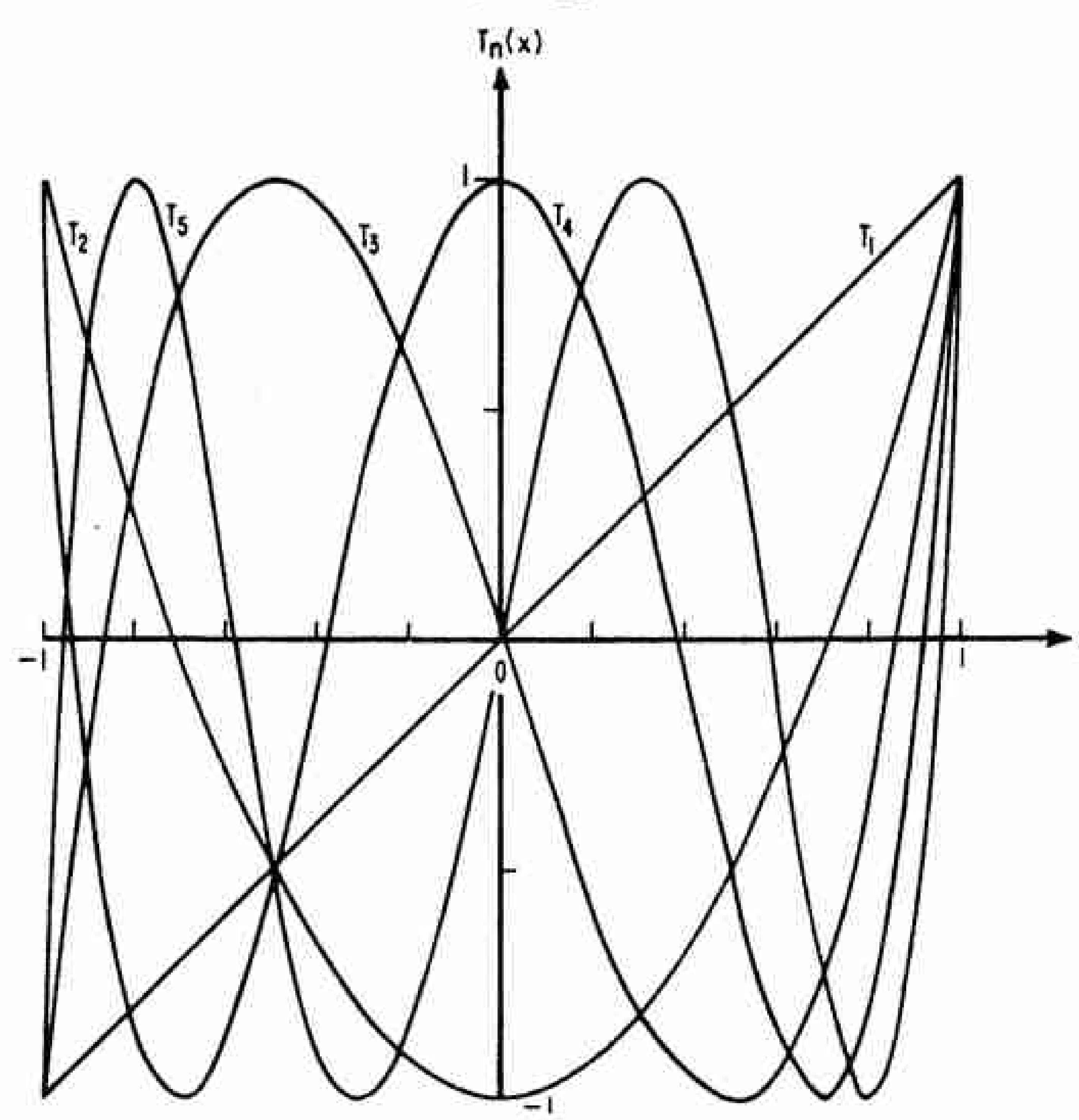

FIGURE 22.6. *Chebyshev Polynomials* $T_n(x)$, $n=1(1)5$.

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*See page II.

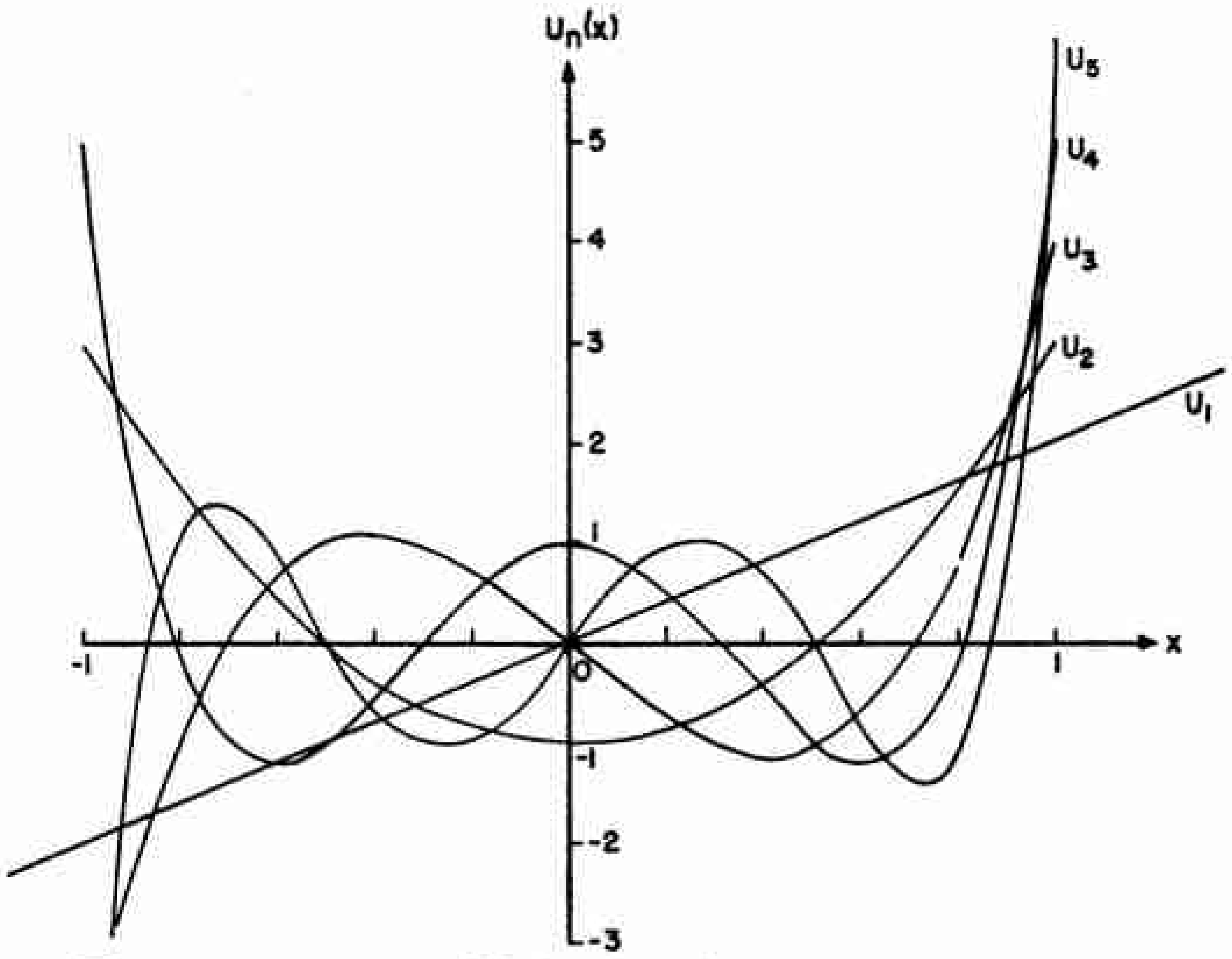

FIGURE 22.7. *Chebyshev Polynomials* $U_n(x)$, $n=1(1)5$.

22.5.33 $$T_n(x)=\frac{n}{2}\,C_n^{(0)}(x)$$

22.5.34 $$U_n(x)=C_n^{(1)}(x)$$

Legendre Polynomials

22.5.35 $$P_n(x)=P_n^{(0,0)}(x)$$

22.5.36 $$P_n(x)=C_n^{(1/2)}(x)$$

22.5.37

$$\frac{d^m}{dx^m}[P_n(x)]=1\cdot 3\ldots(2m-1)C_{n-m}^{(m+\frac{1}{2})}(x) \qquad (m\leq n)$$

Generalized Laguerre Polynomials

22.5.38 $$L_n^{(-1/2)}(x)=\frac{(-1)^n}{n!2^{2n}}H_{2n}(\sqrt{x})$$

22.5.39 $$L_n^{(1/2)}(x)=\frac{(-1)^n}{n!2^{2n+1}\sqrt{x}}H_{2n+1}(\sqrt{x})$$

Hermite Polynomials

22.5.40 $$H_{2m}(x)=(-1)^m2^{2m}m!L_m^{(-1/2)}(x^2)$$

22.5.41 $$H_{2m+1}(x)=(-1)^m2^{2m+1}m!xL_m^{(1/2)}(x^2)$$

Orthogonal Polynomials as Hypergeometric Functions (see chapter 15)

$$f_n(x)=dF(a,\,b;\,c;\,g(x))$$

For each of the listed polynomials there are numerous other representations in terms of hypergeometric functions.

	$f_n(x)$	d	a	b	c	$g(x)$
22.5.42	$P_n^{(\alpha,\beta)}(x)$	$\binom{n+\alpha}{n}$	$-n$	$n+\alpha+\beta+1$	$\alpha+1$	$\frac{1-x}{2}$
22.5.43	$P_n^{(\alpha,\beta)}(x)$	$\binom{2n+\alpha+\beta}{n}\left(\frac{x-1}{2}\right)^n$	$-n$	$-n-\alpha$	$-2n-\alpha-\beta$	$\frac{2}{1-x}$
22.5.44	$P_n^{(\alpha,\beta)}(x)$	$\binom{n+\alpha}{n}\left(\frac{1+x}{2}\right)^n$	$-n$	$-n-\beta$	$\alpha+1$	$\frac{x-1}{x+1}$
22.5.45	$P_n^{(\alpha,\beta)}(x)$	$\binom{n+\beta}{n}\left(\frac{x-1}{2}\right)^n$	$-n$	$-n-\alpha$	$\beta+1$	$\frac{x+1}{x-1}$
22.5.46	$C_n^{(\alpha)}(x)$	$\frac{\Gamma(n+2\alpha)}{n!\Gamma(2\alpha)}$	$-n$	$n+2\alpha$	$\alpha+\frac{1}{2}$	$\frac{1-x}{2}$
22.5.47	$T_n(x)$	1	$-n$	n	$\frac{1}{2}$	$\frac{1-x}{2}$
22.5.48	$U_n(x)$	$n+1$	$-n$	$n+2$ *	$\frac{3}{2}$	$\frac{1-x}{2}$
22.5.49	$P_n(x)$	1	$-n$	$n+1$	1	$\frac{1-x}{2}$
22.5.50	$P_n(x)$	$\binom{2n}{n}\left(\frac{x-1}{2}\right)^n$	$-n$	$-n$	$-2n$	$\frac{2}{1-x}$
22.5.51	$P_n(x)$	$\binom{2n}{n}\left(\frac{x}{2}\right)^n$	$-\frac{n}{2}$	$\frac{1-n}{2}$	$\frac{1}{2}-n$	$\frac{1}{x^2}$
22.5.52	$P_{2n}(x)$	$(-1)^n\frac{(2n)!}{2^{2n}(n!)^2}$	$-n$	$n+\frac{1}{2}$	$\frac{1}{2}$	x^2
22.5.53	$P_{2n+1}(x)$	$(-1)^n\frac{(2n+1)!}{2^{2n}(n!)^2}x$	$-n$	$n+\frac{3}{2}$	$\frac{3}{2}$	x^2

Orthogonal Polynomials as Confluent Hypergeometric Functions (see chapter 13)

22.5.54 $L_n^{(\alpha)}(x)=\binom{n+\alpha}{n}M(-n,\alpha+1,x)$

Orthogonal Polynomials as Parabolic Cylinder Functions (see chapter 19)

22.5.55 $H_n(x)=2^nU\left(\frac{1}{2}-\frac{1}{2}n,\frac{3}{2},x^2\right)$

22.5.56 $H_{2m}(x)=(-1)^m\frac{(2m)!}{m!}M\left(-m,\frac{1}{2},x^2\right)$

22.5.57

* $H_{2m+1}(x)=(-1)^m\frac{(2m+1)!}{m!}2xM\left(-m,\frac{3}{2},x^2\right)$

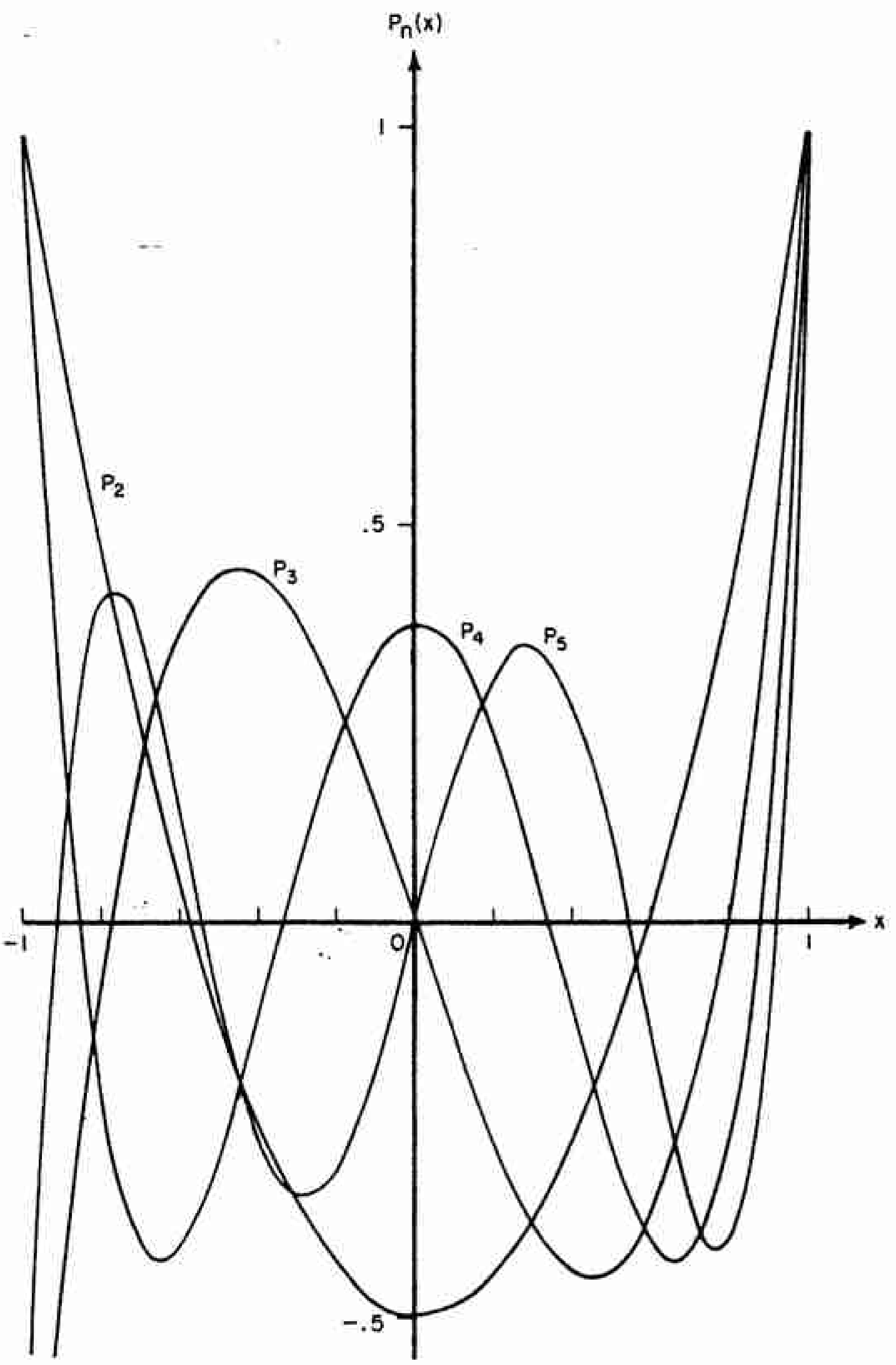

FIGURE 22.8. *Legendre Polynomials* $P_n(x)$, $n=2(1)5$.

22.5.58

$$H_n(x)=2^{n/2}e^{x^2/2}D_n(\sqrt{2}x)=2^{n/2}e^{x^2/2}U\left(-n-\frac{1}{2},\sqrt{2}x\right)$$

22.5.59 $He_n(x)=e^{x^2/4}D_n(x)=e^{x^2/4}U\left(-n-\frac{1}{2},x\right)$

Orthogonal Polynomials as Legendre Functions (see chapter 8)

22.5.60

$C_n^{(\alpha)}(x)=$

$$\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(2\alpha+n)}{n!\Gamma(2\alpha)}\left[\frac{1}{4}(x^2-1)\right]^{\frac{1}{4}-\frac{\alpha}{2}}P_{n+\alpha-\frac{1}{2}}^{(\frac{1}{2}-\alpha)}(x)$$

$(\alpha\neq 0)$

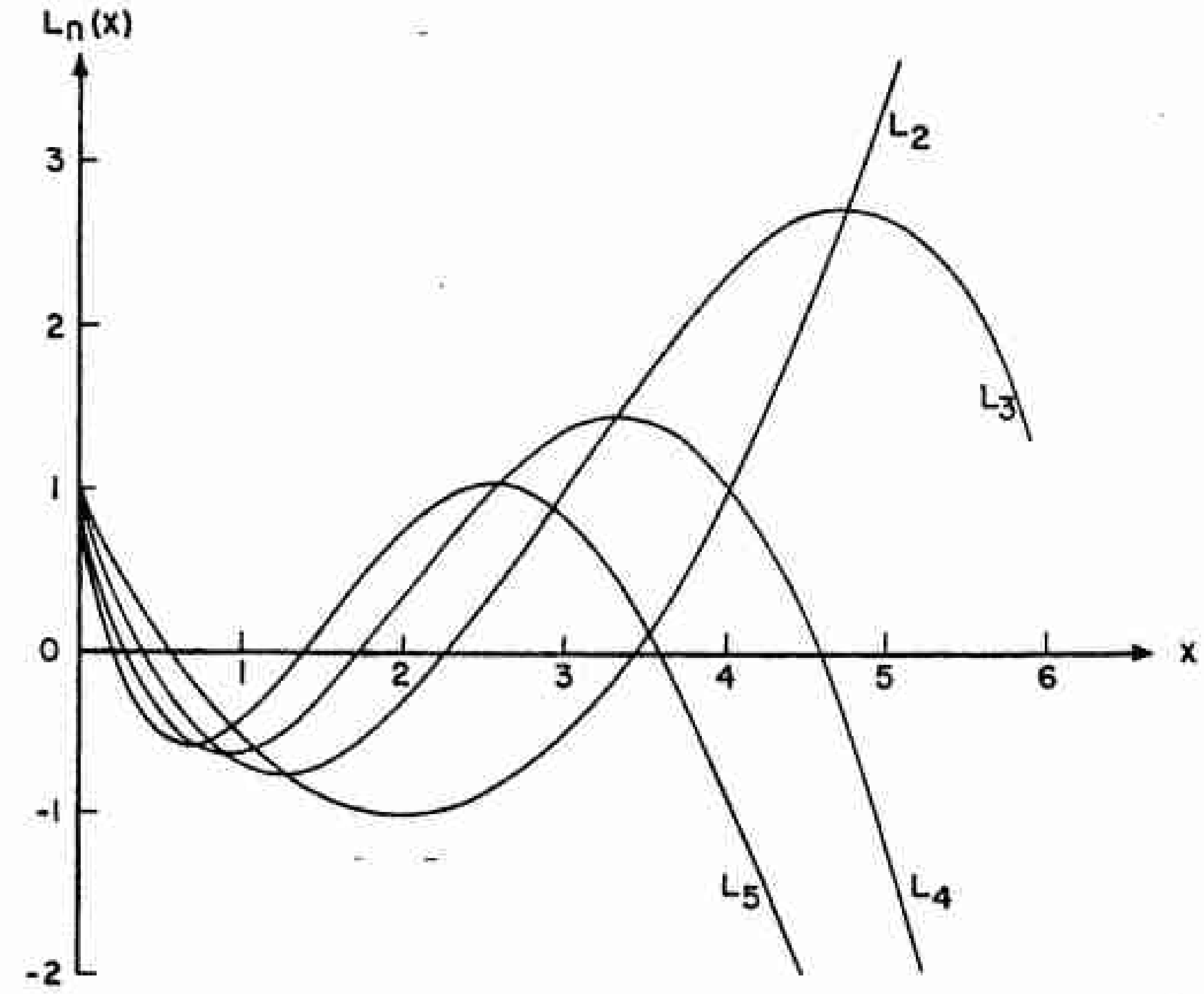

FIGURE 22.9. *Laguerre Polynomials* $L_n(x)$, $n=2(1)5$.

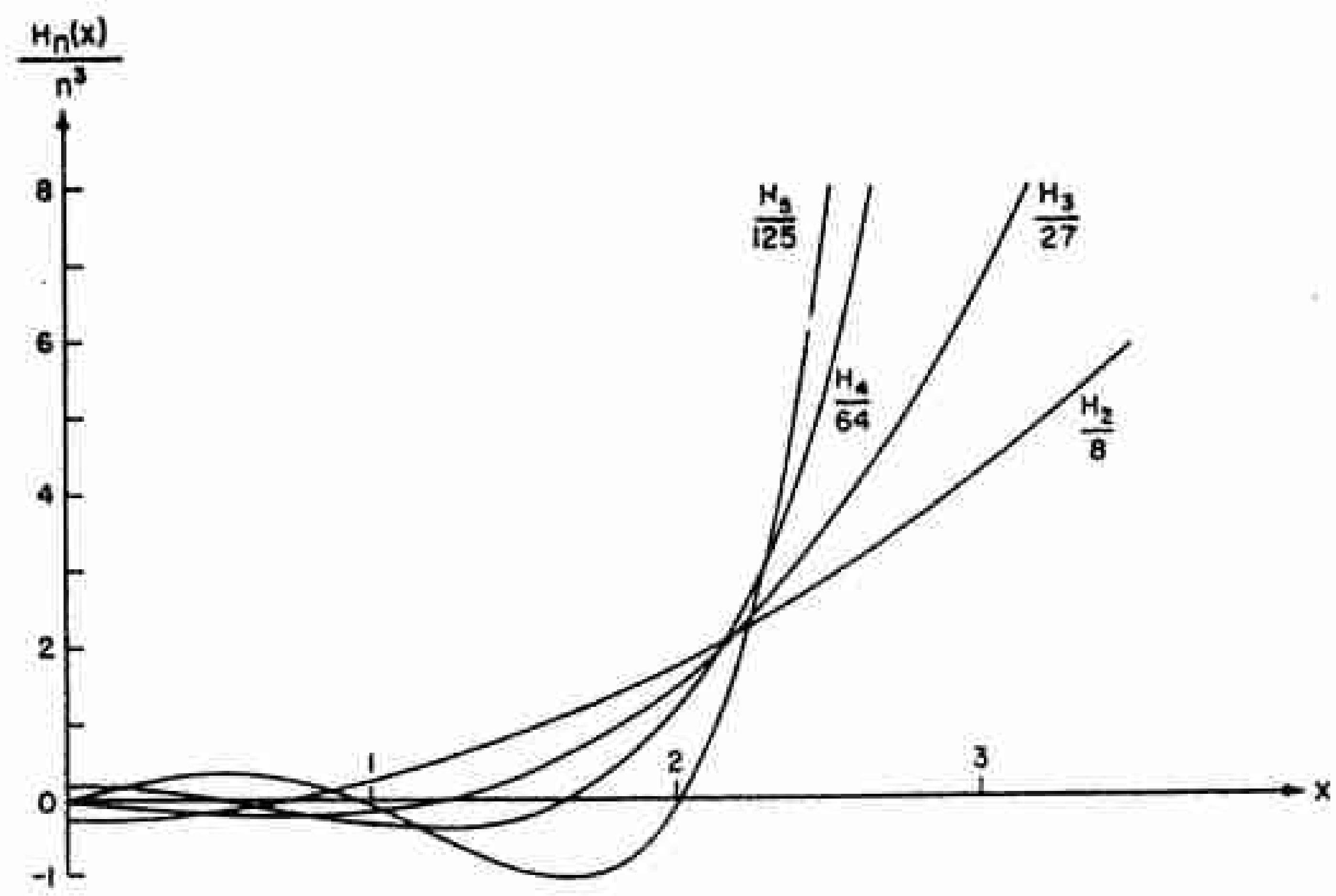

FIGURE 22.10. *Hermite Polynomials* $\frac{H_n(x)}{n^3}$, $n=2(1)5$.

*See page II.

22.6. Differential Equations

$$g_2(x)y''+g_1(x)y'+g_0(x)y=0$$

	y	$g_2(x)$	$g_1(x)$	$g_0(x)$
22.6.1	$P_n^{(\alpha,\beta)}(x)$	$1-x^2$	$\beta-\alpha-(\alpha+\beta+2)x$	$n(n+\alpha+\beta+1)$
22.6.2	$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)$	$1-x^2$	$\alpha-\beta+(\alpha+\beta-2)x$	$(n+1)(n+\alpha+\beta)$
22.6.3	$(1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}}P_n^{(\alpha,\beta)}(x)$	1	0	$\frac{1}{4}\frac{1-\alpha^2}{(1-x)^2}+\frac{1}{4}\frac{1-\beta^2}{(1+x)^2}+\frac{2n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)}{2(1-x^2)}$
22.6.4	$\left(\sin\frac{x}{2}\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{x}{2}\right)^{\beta+\frac{1}{2}}P_n^{(\alpha,\beta)}(\cos x)$	1	0	$\frac{1-4\alpha^2}{16\sin^2\frac{x}{2}}+\frac{1-4\beta^2}{16\cos^2\frac{x}{2}}+\left(n+\frac{\alpha+\beta+1}{2}\right)^2$
22.6.5	$C_n^{(\alpha)}(x)$	$1-x^2$	$-(2\alpha+1)x$	$n(n+2\alpha)$
22.6.6	$(1-x^2)^{\alpha-\frac{1}{2}}C_n^{(\alpha)}(x)$	$1-x^2$	$(2\alpha-3)x$	$(n+1)(n+2\alpha-1)$
22.6.7	$(1-x^2)^{\frac{\alpha}{2}+\frac{1}{4}}C_n^{(\alpha)}(x)$	1	0	$\frac{(n+\alpha)^2}{1-x^2}+\frac{2+4\alpha-4\alpha^2+x^2}{4(1-x^2)^2}$
22.6.8	$(\sin x)^\alpha C_n^{(\alpha)}(\cos x)$	1	0	$(n+\alpha)^2+\frac{\alpha(1-\alpha)}{\sin^2 x}$
22.6.9	$T_n(x)$	$1-x^2$	$-x$	n^2
22.6.10	$T_n(\cos x)$	1	0	n^2
22.6.11	$\frac{1}{\sqrt{1-x^2}}T_n(x)$; $U_{n-1}(x)$ *	$1-x^2$	$-3x$	n^2-1
22.6.12	$U_n(x)$	$1-x^2$	$-3x$	$n(n+2)$
22.6.13	$P_n(x)$	$1-x^2$	$-2x$	$n(n+1)$
22.6.14	$\sqrt{1-x^2}P_n(x)$	1	0	$\frac{n(n+1)}{1-x^2}+\frac{1}{(1-x^2)^2}$
22.6.15	$L_n^{(\alpha)}(x)$	x	$\alpha+1-x$	n
22.6.16	$e^{-x}x^{\alpha/2}L_n^{(\alpha)}(x)$ *	x	$x+1$	$n+\frac{\alpha}{2}+1-\frac{\alpha^2}{4x}$
22.6.17	$e^{-x/2}x^{(\alpha+1)/2}L_n^{(\alpha)}(x)$	1	0	$\frac{2n+\alpha+1}{2x}+\frac{1-\alpha^2}{4x^2}-\frac{1}{4}$
22.6.18	$e^{-x^2/2}x^{\alpha+\frac{1}{2}}L_n^{(\alpha)}(x^2)$	1	0	$4n+2\alpha+2-x^2+\frac{1-4\alpha^2}{4x^2}$
22.6.19	$H_n(x)$	1	$-2x$	$2n$
22.6.20	$e^{-\frac{x^2}{2}}H_n(x)$	1	0	$2n+1-x^2$
22.6.21	$He_n(x)$	1	$-x$	n

*See page II.

22.7. Recurrence Relations

Recurrence Relations With Respect to the Degree n

$$a_{1n}f_{n+1}(x)=(a_{2n}+a_{3n}x)f_n(x)-a_{4n}f_{n-1}(x)$$

	f_n	a_{1n}	a_{2n}	a_{3n}	a_{4n}
22.7.1	$P_n^{(\alpha,\beta)}(x)$	$2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)$	$(2n+\alpha+\beta+1)(\alpha^2-\beta^2)$	$(2n+\alpha+\beta)_3$	$2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)$
22.7.2	$G_n(p,q,x)$	$(2n+p-2)_4(2n+p-1)$	$-[2n(n+p)+q(p-1)](2n+p-2)_3$	$(2n+p-2)_4(2n+p-1)$	$n(n+q-1)(n+p-1)(n+p-q)(2n+p+1)$
22.7.3	$C_n^{(\alpha)}(x)$	$n+1$	0	$2(n+\alpha)$	$n+2\alpha-1$
22.7.4	$T_n(x)$	1	0	2	1
22.7.5	$U_n(x)$	1	0	2	1
22.7.6	$S_n(x)$	1	0	1	1
22.7.7	$C_n(x)$	1	0	1	1
22.7.8	$T_n^*(x)$	1	-2	4	1
22.7.9	$U_n^*(x)$	1	-2	4	1
22.7.10	$P_n(x)$	$n+1$	0	$2n+1$	n
22.7.11	$P_n^*(x)$	$n+1$	$-2n-1$	$4n+2$	n
22.7.12	$L_n^{(\alpha)}(x)$	$n+1$	$2n+\alpha+1$	-1	$n+\alpha$
22.7.13	$H_n(x)$	1	0	2	$2n$
22.7.14	$He_n(x)$	1	0	1	n

Miscellaneous Recurrence Relations

Jacobi Polynomials

22.7.15

$$\left(n+\frac{\alpha}{2}+\frac{\beta}{2}+1\right)(1-x)P_n^{(\alpha+1,\beta)}(x)=(n+\alpha+1)P_n^{(\alpha,\beta)}(x)-(n+1)P_{n+1}^{(\alpha,\beta)}(x)$$

22.7.16

$$\left(n+\frac{\alpha}{2}+\frac{\beta}{2}+1\right)(1+x)P_n^{(\alpha,\beta+1)}(x)=(n+\beta+1)P_n^{(\alpha,\beta)}(x)+(n+1)P_{n+1}^{(\alpha,\beta)}(x)$$

22.7.17

$$(1-x)P_n^{(\alpha+1,\beta)}(x)+(1+x)P_n^{(\alpha,\beta+1)}(x)=2P_n^{(\alpha,\beta)}(x)$$

22.7.18

$$(2n+\alpha+\beta)P_n^{(\alpha-1,\beta)}(x)=(n+\alpha+\beta)P_n^{(\alpha,\beta)}(x)-(n+\beta)P_{n-1}^{(\alpha,\beta)}(x)$$

22.7.19

$$(2n+\alpha+\beta)P_n^{(\alpha,\beta-1)}(x)=(n+\alpha+\beta)P_n^{(\alpha,\beta)}(x)+(n+\alpha)P_{n-1}^{(\alpha,\beta)}(x)$$

22.7.20 $$P_n^{(\alpha,\beta-1)}(x)-P_n^{(\alpha-1,\beta)}(x)=P_{n-1}^{(\alpha,\beta)}(x)$$

Ultraspherical Polynomials

22.7.21

$$2\alpha(1-x^2)C_{n-1}^{(\alpha+1)}(x)=(2\alpha+n-1)C_{n-1}^{(\alpha)}(x)-nxC_n^{(\alpha)}(x)$$

22.7.22

$$=(n+2\alpha)xC_n^{(\alpha)}(x)-(n+1)C_{n+1}^{(\alpha)}(x)$$

22.7.23 $$(n+\alpha)C_{n+1}^{(\alpha-1)}(x)=(\alpha-1)[C_{n+1}^{(\alpha)}(x)-C_{n-1}^{(\alpha)}(x)]$$

Chebyshev Polynomials

22.7.24

$$2T_m(x)T_n(x)=T_{n+m}(x)+T_{n-m}(x)\qquad(n\geq m)$$ *

22.7.25

$$2(x^2-1)U_{m-1}(x)U_{n-1}(x)=T_{n+m}(x)-T_{n-m}(x)\qquad(n\geq m)$$

22.7.26

$$2T_m(x)U_{n-1}(x)=U_{n+m-1}(x)+U_{n-m-1}(x)\qquad(n>m)$$

22.7.27

$$2T_n(x)U_{m-1}(x)=U_{n+m-1}(x)-U_{n-m-1}(x)\qquad(n>m)$$

22.7.28 $$2T_n(x)U_{n-1}(x)=U_{2n-1}(x)$$

*See page II.

Generalized Laguerre Polynomials

22.7.29

$$L_n^{(\alpha+1)}(x)=\frac{1}{x}[(x-n)L_n^{(\alpha)}(x)+(\alpha+n)L_{n-1}^{(\alpha)}(x)]$$

22.7.30 $L_n^{(\alpha-1)}(x)=L_n^{(\alpha)}(x)-L_{n-1}^{(\alpha)}(x)$

22.7.31

$$L_n^{(\alpha+1)}(x)=\frac{1}{x}[(n+\alpha+1)L_n^{(\alpha)}(x)-(n+1)L_{n+1}^{(\alpha)}(x)]$$

22.7.32

$$L_n^{(\alpha-1)}(x)=\frac{1}{n+\alpha}[(n+1)L_{n+1}^{(\alpha)}(x)-(n+1-x)L_n^{(\alpha)}(x)]$$

22.8. Differential Relations

$$g_2(x)\frac{d}{dx}f_n(x)=g_1(x)f_n(x)+g_0(x)f_{n-1}(x)$$

	f_n	g_2	g_1	g_0
22.8.1	$P_n^{(\alpha,\beta)}(x)$	$(2n+\alpha+\beta)(1-x^2)$	$n[\alpha-\beta-(2n+\alpha+\beta)x]$	$2(n+\alpha)(n+\beta)$
22.8.2	$C_n^{(\alpha)}(x)$	$1-x^2$	$-nx$	$n+2\alpha-1$
22.8.3	$T_n(x)$	$1-x^2$	$-nx$	n
22.8.4	$U_n(x)$	$1-x^2$	$-nx$	$n+1$
22.8.5	$P_n(x)$	$1-x^2$	$-nx$	n
22.8.6	$L_n^{(\alpha)}(x)$	x	n	$-(n+\alpha)$
22.8.7	$H_n(x)$	1	0	$2n$
22.8.8	$He_n(x)$	1	0	n

22.9. Generating Functions

$$g(x,z)=\sum_{n=0}^{\infty}a_nf_n(x)z^n \qquad R=\sqrt{1-2xz+z^2}$$

	$f_n(x)$	a_n	$g(x,z)$	Remarks
22.9.1	$P_n^{(\alpha,\beta)}(x)$	$2^{-\alpha-\beta}$	$R^{-1}(1-z+R)^{-\alpha}(1+z+R)^{-\beta}$	$\|z\|<1$
22.9.2	$C_n^{(\alpha)}(x)$	$\frac{2^{\frac{1}{2}-\alpha}\Gamma(\alpha+\frac{1}{2}+n)\Gamma(2\alpha)}{\Gamma(\alpha+\frac{1}{2})\Gamma(2\alpha+n)}$	$R^{-1}(1-xz+R)^{\frac{1}{2}-\alpha}$	$\|z\|<1, \alpha\neq 0$
22.9.3	$C_n^{(\alpha)}(x)$	1	$R^{-2\alpha}$	$\|z\|<1, \alpha\neq 0$
22.9.4	$C_n^{(0)}(x)$	1	$-\ln R^2$	$\|z\|<1$
22.9.5	$C_n^{(\alpha)}(x)$	$\frac{\Gamma(2\alpha)}{\Gamma(\alpha+\frac{1}{2})\Gamma(2\alpha+n)}$	$e^{z\cos\theta}\left(\frac{z}{2}\sin\theta\right)^{\frac{1}{2}-\alpha}J_{\alpha-\frac{1}{2}}(z\sin\theta)$	$x=\cos\theta$
22.9.6	$T_n(x)$	2	$\left(\frac{1-z^2}{R^2}+1\right)$	$-1<x<1$ $\|z\|<1$
22.9.7	$T_n(x)$	$\frac{\sqrt{2}}{4^n}\binom{2n}{n}$	$R^{-1}(1-xz+R)^{1/2}$	$-1<x<1$ $\|z\|<1$
22.9.8	$T_n(x)$	$\frac{1}{n}$	$1-\frac{1}{2}\ln R^2$	$a_0=1$ $-1<x<1$ $\|z\|<1$
22.9.9	$T_n(x)$	1	$\frac{1-xz}{R^2}$	$-1<x<1$ $\|z\|<1$
22.9.10	$U_n(x)$	1	R^{-2}	$-1<x<1$ $\|z\|<1$
22.9.11	$U_n(x)$	$\frac{\sqrt{2}}{4^{n+1}}\binom{2n+2}{n+1}$	$\frac{1}{R}(1-xz+R)^{-1/2}$ *	$-1<x<1$ $\|z\|<1$

*See page II.

22.9. Generating Functions—Continued

$$g(x,z)=\sum_{n=0}^{\infty} a_n f_n(x) z^n \qquad R=\sqrt{1-2xz+z^2}$$

	$f_n(x)$	a_n	$g(x,z)$	Remarks
22.9.12	$P_n(x)$	1	R^{-1}	$-1<x<1$ $\|z\|<1$
22.9.13	$P_n(x)$	$\frac{1}{n!}$	$e^{z\cos\theta}J_0(z\sin\theta)$	$x=\cos\theta$
22.9.14	$S_n(x)$	1	$(1-xz+z^2)^{-1}$	$-2<x<2$ $\|z\|<1$
22.9.15	$L_n^{(\alpha)}(x)$	1	$(1-z)^{-\alpha-1}\exp\left(\frac{xz}{z-1}\right)$	$\|z\|<1$
22.9.16	$L_n^{(\alpha)}(x)$	$\frac{1}{\Gamma(n+\alpha+1)}$	$(xz)^{-\frac{1}{2}\alpha}e^zJ_\alpha[2(xz)^{1/2}]$	
22.9.17	$H_n(x)$	$\frac{1}{n!}$	e^{2xz-z^2}	
22.9.18	$H_{2n}(x)$	$\frac{(-1)^n}{(2n)!}$	$e^z\cos(2x\sqrt{z})$ *	
22.9.19	$H_{2n+1}(x)$	$\frac{(-1)^n}{(2n+1)!}$	$z^{-1/2}e^z\sin(2x\sqrt{z})$ *	

22.10. Integral Representations

Contour Integral Representations

$f_n(x)=\frac{g_0(x)}{2\pi i}\int_C [g_1(z,x)]^n g_2(z,x)dz$ where C is a closed contour taken around $z=a$ in the positive sense

	$f_n(x)$	$g_0(x)$	$g_1(z,x)$	$g_2(z,x)$	a	Remarks
22.10.1	$P_n^{(\alpha,\beta)}(x)$	$\frac{1}{(1-x)^\alpha(1+x)^\beta}$	$\frac{z^2-1}{2(z-x)}$	$\frac{(1-z)^\alpha(1+z)^\beta}{z-x}$	x	±1 outside C
22.10.2	$C_n^{(\alpha)}(x)$	1	$1/z$	$(1-2xz+z^2)^{-\alpha}z^{-1}$	0	Both zeros of $1-2xz+z^2$ outside C, $\alpha>0$
22.10.3	$T_n(x)$	1/2	$1/z$	$\frac{1-z^2}{z(1-2xz+z^2)}$	0	Both zeros of $1-2xz+z^2$ outside C
22.10.4	$U_n(x)$	1	$1/z$	$\frac{1}{z(1-2xz+z^2)}$	0	Both zeros of $1-2xz+z^2$ outside C
22.10.5	$P_n(x)$	1	$1/z$	$\frac{1}{z}(1-2xz+z^2)^{-1/2}$	0	Both zeros of $1-2xz+z^2$ outside C
22.10.6	$P_n(x)$	$\frac{1}{2^n}$	$\frac{z^2-1}{z-x}$	$\frac{1}{z-x}$	x	
22.10.7	$L_n^{(\alpha)}(x)$	$e^x x^{-\alpha}$	$\frac{z}{z-x}$	$\frac{z^\alpha}{z-x}e^{-z}$	x	Zero outside C
22.10.8	$L_n^{(\alpha)}(x)$	1	$1+\frac{x}{z}$	$e^{-z}\left(1+\frac{z}{x}\right)^\alpha 1/z$	0	$z=-x$ outside C
22.10.9	$H_n(x)$	$n!$	$1/z$	$\frac{e^{2zx-z^2}}{z}$	0	

Miscellaneous Integral Representations

22.10.10 $$C_n^{(\alpha)}(x)=\frac{2^{(1-2\alpha)}\Gamma(n+2\alpha)}{n![\Gamma(\alpha)]^2}\int_0^\pi [x+\sqrt{x^2-1}\cos\phi]^n(\sin\phi)^{2\alpha-1}d\phi \qquad (\alpha>0)$$

22.10.11 $$C_n^{(\alpha)}(\cos\theta)=\frac{2^{1-\alpha}\Gamma(n+2\alpha)}{n![\Gamma(\alpha)]^2}(\sin\theta)^{1-2\alpha}\int_0^\theta \frac{\cos(n+\alpha)\phi}{(\cos\phi-\cos\theta)^{1-\alpha}}d\phi \qquad (\alpha>0)$$

*See page II.

22.10.12 $$P_n(\cos\theta)=\frac{1}{\pi}\int_0^{\pi}(\cos\theta+i\sin\theta\cos\phi)^n d\phi$$

22.10.13 $$P_n(\cos\theta)=\frac{\sqrt{2}}{\pi}\int_{\theta}^{\pi}\frac{\sin(n+\frac{1}{2})\phi\, d\phi}{(\cos\theta-\cos\phi)^{\frac{1}{2}}}$$

22.10.14 $$L_n^{(\alpha)}(x)=\frac{e^x x^{-\frac{\alpha}{2}}}{n!}\int_0^{\infty}e^{-t}t^{n+\frac{\alpha}{2}}J_\alpha(2\sqrt{tx})dt$$

22.10.15

$$H_n(x)=e^{x^2}\frac{2^{n+1}}{\sqrt{\pi}}\int_0^{\infty}e^{-t^2}t^n\cos\left(2xt-\frac{n}{2}\pi\right)dt$$

22.11. Rodrigues' Formula

$$f_n(x)=\frac{1}{a_n\rho(x)}\frac{d^n}{dx^n}\{\rho(x)(g(x))^n\}$$

The polynomials given in the following table are the only orthogonal polynomials which satisfy this formula.

	$f_n(x)$	a_n	$\rho(x)$	$g(x)$
22.11.1	$P_n^{(\alpha,\beta)}(x)$	$(-1)^n 2^n n!$	$(1-x)^\alpha(1+x)^\beta$	$1-x^2$
22.11.2	$C_n^{(\alpha)}(x)$	$(-1)^n 2^n n!\,\frac{\Gamma(2\alpha)\Gamma(\alpha+n+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})\Gamma(n+2\alpha)}$	$(1-x^2)^{\alpha-\frac{1}{2}}$	$1-x^2$
22.11.3	$T_n(x)$	$(-1)^n 2^n\,\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}$ *	$(1-x^2)^{-\frac{1}{2}}$	$1-x^2$
22.11.4	$U_n(x)$	$(-1)^n 2^{n+1}\,\frac{\Gamma(n+\frac{3}{2})}{(n+1)\sqrt{\pi}}$	$(1-x^2)^{\frac{1}{2}}$	$1-x^2$
22.11.5	$P_n(x)$	$(-1)^n 2^n n!$	1	$1-x^2$
22.11.6	$L_n^{(\alpha)}(x)$	$n!$	$e^{-x}x^\alpha$	x
22.11.7	$H_n(x)$	$(-1)^n$	e^{-x^2}	1
22.11.8	$He_n(x)$	$(-1)^n$	$e^{-x^2/2}$	1

22.12. Sum Formulas

Christoffel-Darboux Formula

22.12.1

$$\sum_{m=0}^{n}\frac{1}{h_m}f_m(x)f_m(y)=\frac{k_n}{k_{n+1}h_n}\,\frac{f_{n+1}(x)f_n(y)-f_n(x)f_{n+1}(y)}{x-y}$$

Miscellaneous Sum Formulas (Only a Limited Selection Is Given Here.)

22.12.2 $$\sum_{m=0}^{n}T_{2m}(x)=\frac{1}{2}[1+U_{2n}(x)]$$

22.12.3 $$\sum_{m=0}^{n-1}T_{2m+1}(x)=\frac{1}{2}U_{2n-1}(x)$$

22.12.4 $$\sum_{m=0}^{n}U_{2m}(x)=\frac{1-T_{2n+2}(x)}{2(1-x^2)}$$

22.12.5 $$\sum_{m=0}^{n-1}U_{2m+1}(x)=\frac{x-T_{2n+1}(x)}{2(1-x^2)}$$

22.12.6 $$\sum_{m=0}^{n}L_m^{(\alpha)}(x)L_{n-m}^{(\beta)}(y)=L_n^{(\alpha+\beta+1)}(x+y)$$

22.12.7 $$\sum_{m=0}^{n}\binom{n+\alpha}{m}\mu^{n-m}(1-\mu)^m L_{n-m}^{(\alpha)}(x)=L_n^{(\alpha)}(\mu x)$$

22.12.8

$$H_n(x+y)=\frac{1}{2^{n/2}}\sum_{k=0}^{n}\binom{n}{k}H_k(\sqrt{2}x)H_{n-k}(\sqrt{2}y)$$

22.13. Integrals Involving Orthogonal Polynomials

22.13.1

$$2n\int_0^x(1-y)^\alpha(1+y)^\beta P_n^{(\alpha,\beta)}(y)dy$$
$$=P_{n-1}^{(\alpha+1,\beta+1)}(0)-(1-x)^{\alpha+1}(1+x)^{\beta+1}P_{n-1}^{(\alpha+1,\beta+1)}(x)$$

22.13.2

$$\frac{n(2\alpha+n)}{2\alpha}\int_0^x(1-y^2)^{\alpha-\frac{1}{2}}C_n^{(\alpha)}(y)dy$$
$$=C_{n-1}^{(\alpha+1)}(0)-(1-x^2)^{\alpha+\frac{1}{2}}C_{n-1}^{(\alpha+1)}(x)$$

22.13.3 $$\int_{-1}^{1}\frac{T_n(y)dy}{(y-x)\sqrt{1-y^2}}=\pi U_{n-1}(x)$$

22.13.4 $$\int_{-1}^{1}\frac{\sqrt{1-y^2}U_{n-1}(y)dy}{(y-x)}=-\pi T_n(x)$$ *

22.13.5 $$\int_{-1}^{1}(1-x)^{-1/2}P_n(x)dx=\frac{2^{3/2}}{2n+1}$$ *

22.13.6 $$\int_0^{\pi}P_{2n}(\cos\theta)d\theta=\frac{\pi}{16^n}\binom{2n}{n}^2$$

22.13.7 $$\int_0^{\pi}P_{2n+1}(\cos\theta)\cos\theta\, d\theta=\frac{\pi}{4^{2n+1}}\binom{2n}{n}\binom{2n+2}{n+1}$$

*See page II.

22.13.8

$$\int_0^1 x^\lambda P_{2n}(x)dx=\frac{(-1)^n\Gamma\left(n-\frac{\lambda}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\lambda}{2}\right)}{2\Gamma\left(-\frac{\lambda}{2}\right)\Gamma\left(n+\frac{3}{2}+\frac{\lambda}{2}\right)}\quad(\lambda>-1)$$

22.13.9

$$\int_0^1 x^\lambda P_{2n+1}(x)dx=\frac{(-1)^n\Gamma\left(n+\frac{1}{2}-\frac{\lambda}{2}\right)\Gamma\left(1+\frac{\lambda}{2}\right)}{2\Gamma\left(n+2+\frac{\lambda}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\lambda}{2}\right)}\quad(\lambda>-2)$$

22.13.10

$$\int_{-1}^x\frac{P_n(t)dt}{\sqrt{x-t}}=\frac{1}{(n+\frac{1}{2})\sqrt{1+x}}[T_n(x)+T_{n+1}(x)]$$

22.13.11

$$\int_x^1\frac{P_n(t)dt}{\sqrt{t-x}}=\frac{1}{(n+\frac{1}{2})\sqrt{1-x}}[T_n(x)-T_{n+1}(x)]$$

22.13.12 $$\int_x^\infty e^{-t}L_n^{(\alpha)}(t)dt=e^{-x}[L_n^{(\alpha)}(x)-L_{n-1}^{(\alpha)}(x)]$$

22.13.13

$$\Gamma(\alpha+\beta+n+1)\int_0^x(x-t)^{\beta-1}t^\alpha L_n^{(\alpha)}(t)dt=\Gamma(\alpha+n+1)\Gamma(\beta)x^{\alpha+\beta}L_n^{(\alpha+\beta)}(x)\quad(\mathscr{R}\alpha>-1,\ \mathscr{R}\beta>0)$$

22.13.14

$$\int_0^x L_m(t)L_n(x-t)dt=\int_0^x L_{m+n}(t)dt=L_{m+n}(x)-L_{m+n+1}(x)$$

22.13.15 $$\int_0^x e^{-t^2}H_n(t)dt=H_{n-1}(0)-e^{-x^2}H_{n-1}(x)$$

22.13.16 $$\int_0^x H_n(t)dt=\frac{1}{2(n+1)}[H_{n+1}(x)-H_{n+1}(0)]$$

22.13.17 $$\int_{-\infty}^\infty e^{-t^2}H_{2m}(tx)dt=\sqrt{\pi}\,\frac{(2m)!}{m!}(x^2-1)^m$$

22.13.18

$$\int_{-\infty}^\infty e^{-t^2}tH_{2m+1}(tx)dt=\sqrt{\pi}\,\frac{(2m+1)!}{m!}x(x^2-1)^m$$

22.13.19 $$\int_{-\infty}^\infty e^{-t^2}t^nH_n(xt)dt=\sqrt{\pi}n!P_n(x)$$

22.13.20

$$\int_0^\infty e^{-t^2}[H_n(t)]^2\cos(xt)dt=\sqrt{\pi}2^{n-1}n!e^{-\frac{1}{4}x^2}L_n\left(\frac{x^2}{2}\right)$$

22.14. Inequalities

22.14.1

$$|P_n^{(\alpha,\beta)}(x)|\leq\begin{cases}\binom{n+q}{n}\approx n^q, & \text{if } q=\max(\alpha,\beta)\geq-1/2\quad(\alpha>-1,\beta>-1)\\ |P_n^{(\alpha,\beta)}(x')|\approx\sqrt{\frac{1}{n}}, & \text{if } q<-\frac{1}{2}\end{cases}$$

x' maximum point nearest to $\frac{\beta-\alpha}{\alpha+\beta+1}$

22.14.2

$$|C_n^{(\alpha)}(x)|\leq\begin{cases}\binom{n+2\alpha-1}{n} & (\alpha>0)\\ |C_n^{(\alpha)}(x')| & \left(-\frac{1}{2}<\alpha<0\right)\end{cases}$$

$x'=0$ if $n=2m$; $x'=$maximum point nearest zero if $n=2m+1$

22.14.3

$$|C_n^{(\alpha)}(\cos\theta)|<2^{1-\alpha}\frac{n^{\alpha-1}}{(\sin\theta)^\alpha\Gamma(\alpha)}\quad(0<\alpha<1,0<\theta<\pi)$$

22.14.4 $$|T_n(x)|\leq1\quad(-1\leq x\leq1)$$

22.14.5 $$\left|\frac{dT_n(x)}{dx}\right|\leq n^2\quad(-1\leq x\leq1)$$

22.14.6 $$|U_n(x)|\leq n+1\quad(-1\leq x\leq1)$$

22.14.7 $$|P_n(x)|\leq1\quad(-1\leq x\leq1)$$

22.14.8 $$\left|\frac{dP_n(x)}{dx}\right|\leq\frac{1}{2}n(n+1)\quad(-1\leq x\leq1)$$

22.14.9 $$|P_n(x)|\leq\sqrt{\frac{2}{\pi n}}\frac{1}{\sqrt[4]{1-x^2}}\quad(-1<x\leq1)^*$$

22.14.10

$$P_n^2(x)-P_{n-1}(x)P_{n+1}(x)<\frac{2n+1}{3n(n+1)}\quad(-1\leq x\leq1)$$

22.14.11

$$P_n^2(x)-P_{n-1}(x)P_{n+1}(x)\geq\frac{1-P_n^2(x)}{(2n-1)(n+1)}\quad(-1\leq x\leq1)$$

22.14.12 $$|L_n(x)|\leq e^{x/2}\quad(x\geq0)$$

22.14.13 $$|L_n^{(\alpha)}(x)|\leq\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}e^{x/2}\quad(\alpha\geq0,\ x\geq0)$$

22.14.14

$$|L_n^{(\alpha)}(x)|\leq\left[2-\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}\right]e^{x/2}\quad(-1<\alpha<0,\ x\geq0)$$

*See page II.

22.14.15 $|H_{2m}(x)|\leq e^{x^2/2}2^{2m}m!\left[2-\frac{1}{2^{2m}}\binom{2m}{m}\right]$

22.14.16 $|H_{2m+1}(x)|\leq xe^{x^2/2}\frac{(2m+2)!}{(m+1)!}\quad(x\geq 0)$

22.14.17 $|H_n(x)|<e^{x^2/2}k2^{n/2}\sqrt{n!}\qquad k\approx 1.086435$

22.15. Limit Relations

22.15.1

$$\lim_{n\to\infty}\left[\frac{1}{n^\alpha}P_n^{(\alpha,\beta)}\left(\cos\frac{x}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n^\alpha}P_n^{(\alpha,\beta)}\left(1-\frac{x^2}{2n^2}\right)=\left(\frac{2}{x}\right)^\alpha J_\alpha(x)$$

22.15.2 $\lim_{n\to\infty}\left[\frac{1}{n^\alpha}L_n^{(\alpha)}\left(\frac{x}{n}\right)\right]=x^{-\alpha/2}J_\alpha(2\sqrt{x})$

22.15.3 $\lim_{n\to\infty}\left[\frac{(-1)^n\sqrt{n}}{4^n n!}H_{2n}\left(\frac{x}{2\sqrt{n}}\right)\right]=\frac{1}{\sqrt{\pi}}\cos x$

22.15.4 $\lim_{n\to\infty}\left[\frac{(-1)^n}{4^n n!}H_{2n+1}\left(\frac{x}{2\sqrt{n}}\right)\right]=\frac{2}{\sqrt{\pi}}\sin x$

22.15.5 $\lim_{\beta\to\infty}P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right)=L_n^{(\alpha)}(x)$

22.15.6 $\lim_{\alpha\to\infty}\frac{1}{\alpha^{n/2}}C_n^{(\alpha)}\left(\frac{x}{\sqrt{\alpha}}\right)=\frac{1}{n!}H_n(x)$

For asymptotic expansions, see [22.5] and [22.17].

22.16. Zeros

For tables of the zeros and associated weight factors necessary for the Gaussian-type quadrature formulas see chapter **25.** All the zeros of the orthogonal polynomials are real, simple and located in the interior of the interval of orthogonality.

Explicit and Asymptotic Formulas and Inequalities

Notations:

$x_m^{(n)}$ mth zero of $f_n(x)\ (x_1^{(n)}<x_2^{(n)}<\ldots<x_n^{(n)})$

$\theta_m^{(n)}=\arccos x_{n-m+1}^{(n)}\ (0<\theta_1^{(n)}<\theta_2^{(n)}<\ldots<\theta_n^{(n)}<\pi)$

$j_{\alpha,m}$, mth positive zero of the Bessel function $J_\alpha(x)$

$0<j_{\alpha,1}<j_{\alpha,2}<\ldots$

	$f_n(x)$	Relation
22.16.1	$P_n^{(\alpha,\beta)}(\cos\theta)$	$\lim_{n\to\infty} n\theta_m^{(n)}=j_{\alpha,m}\qquad(\alpha>-1,\ \beta>-1)$
22.16.2	$C_n^{(\alpha)}(x)$	$x_m^{(n)}=1-\frac{j^2_{\alpha-\frac{1}{2},m}}{2n^2}\left[1-\frac{2\alpha}{n}+O\left(\frac{1}{n^2}\right)\right]$
22.16.3	$C_n^{(\alpha)}(\cos\theta)$	$\frac{(m+\alpha-1)\pi}{n+\alpha}\leq\theta_m^{(n)}\leq\frac{m\pi}{n+\alpha}\qquad(0\leq\alpha\leq 1)$
22.16.4	$T_n(x)$	$x_m^{(n)}=\cos\frac{2m-1}{2n}\pi$
22.16.5	$U_n(x)$	$x_m^{(n)}=\cos\frac{m}{n+1}\pi$
22.16.6	$P_n(\cos\theta)$	$\begin{cases}\frac{2m-1}{2n+1}\pi\leq\theta_m^{(n)}\leq\frac{2m}{2n+1}\pi\\ \theta_m^{(n)}=\frac{4m-1}{4n+2}\pi+\frac{1}{8n^2}\cot\frac{4m-1}{4n+2}\pi+O(n^{-3})\end{cases}$
22.16.7	$P_n(x)$	$\begin{cases}x_m^{(n)}=1-\frac{j^2_{0,m}}{2n^2}\left[1-\frac{1}{n}+O(n^{-2})\right]\\ x_m^{(n)}=1-\frac{4\xi_m^{(n)}}{2n+1+\xi_m^{(n)}};\ \xi_m^{(n)}=\frac{j^2_{0,m}}{4n+2}\left[1+\frac{j^2_{0,m}-2}{12(2n+1)^2}\right]+O\left(\frac{1}{n^5}\right)\end{cases}$
22.16.8	$L_n^{(\alpha)}(x)$	$\left.\begin{cases}x_m^{(n)}>\frac{j^2_{\alpha,m}}{4k_n}\\ x_m^{(n)}<\frac{k_m}{k_n}\left(2k_m+\sqrt{4k_m^2+\frac{1}{4}-\alpha^2}\right)\\ x_m^{(n)}=\frac{j^2_{\alpha,m}}{4k_n}\left(1+\frac{2(\alpha^2-1)+j^2_{\alpha,m}}{48k_n^2}\right)+O(n^{-5})\end{cases}\right\}\ k_r=r+\frac{\alpha+1}{2}$

For error estimates see [22.6].

22.17. Orthogonal Polynomials of a Discrete Variable

In this section some polynomials $f_n(x)$ are listed which are orthogonal with respect to the scalar product

22.17.1 $$(f_n, f_m)=\sum_i w^*(x_i)f_n(x_i)f_m(x_i).$$

The x_i are the integers in the interval $a \le x_i \le b$ and $w^*(x_i)$ is a positive function such that $\sum_i w^*(x_i)$ is finite. The constant factor which is still free in each polynomial when only the orthogonality condition is given is defined here by the explicit representation (which corresponds to the Rodrigues' formula)

22.17.2 $$f_n(x)=\frac{1}{r_n w^*(x)}\Delta^n[w^*(x)g(x, n)]$$

where $g(x, n)=g(x)g(x-1)\ .\ .\ .\ g(x-n+1)$ and $g(x)$ is a polynomial in x independent of n.

Name	a	b	$w^*(x)$	r_n	$g(x, n)$	Remarks
Chebyshev	0	$N-1$	1	$1/n!$	$\binom{x}{n}\binom{x-N}{n}$	
Krawtchouk	0	N	$p^x q^{N-x}\binom{N}{x}$	$(-1)^n n!$	$\frac{q^n x!}{(x-n)!}$	$p, q>0;\ p+q=1$
Charlier	0	∞	$\frac{e^{-a}a^x}{x!}$	$(-1)^n\sqrt{a^n n!}$	$\frac{x!}{(x-n)!}$	$a>0$
Meixner	0	∞	$\frac{c^x\Gamma(b+x)}{\Gamma(b)x!}$	c^n	$\frac{x!}{(x-n)!}$	$b>0,\ 0<c<1$
Hahn	0	∞	$\frac{\Gamma(b)\Gamma(c+x)\Gamma(d+x)}{x!\Gamma(b+x)\Gamma(c)\Gamma(d)}$	$n!$	$\frac{x!\Gamma(b+x)}{(x-n)!\Gamma(b+x-n)}$	

For a more complete list of the properties of these polynomials see [22.5] and [22.17].

Numerical Methods

22.18. Use and Extension of the Tables

Evaluation of an orthogonal polynomial for which the coefficients are given numerically.

Example 1. Evaluate $L_6(1.5)$ and its first and second derivative using **Table 22.10** and the Horner scheme.

1	-36	450	-2400	5400	-4320	720
$x=1.5$	1.5	-51.75	597.375	-2703.9375	4044.09375	-413.859375
1	-34.5	398.25	-1802.625	2696.0625	-275.90625	306.140625
1.5	1.5	-49.5	523.125	-1919.25	1165.21875	$L_6=\frac{306.140625}{720}=.42519\ 53$
1	-33.0	348.75	-1279.500	776.8125	889.3125	
1.5	1.5	-47.25	452.250	-1240.875		$L_6'=\frac{889.3125}{720}=1.23515\ 625$
1	-31.5	301.50	-827.250	-464.0625		$L_6''=2\frac{[-464.0625]}{720}=-1.28906\ 25$

Evaluation of an orthogonal polynomial using the explicit representation when the coefficients are not given numerically.

If an isolated value of the orthogonal polynomial $f_n(x)$ is to be computed, use the proper explicit expression rewritten in the form

$$f_n(x)=d_n(x)a_0(x)$$

and generate $a_0(x)$ recursively, where

$$a_{m-1}(x)=1-\frac{b_m}{c_m}f(x)a_m(x) \qquad (m=n,\ n-1,\ \ldots,\ 2,\ 1,\ a_n(x)=1).$$

The $d_n(x)$, b_m, c_m, $f(x)$ for the polynomials of this chapter are listed in the following table:

$f_n(x)$	$d_n(x)$	b_m	c_m	$f(x)$
$P_n^{(\alpha,\beta)}$	$\binom{n+\alpha}{n}$	$(n-m+1)(\alpha+\beta+n+m)$	$2m(\alpha+m)$	$1-x$
$C_{2n}^{(\alpha)}$	$(-1)^n\frac{(\alpha)_n}{n!}$	$2(n-m+1)(\alpha+n+m-1)$	$m(2m-1)$	x^2
$C_{2n+1}^{(\alpha)}$	$(-1)^n\frac{(\alpha)_{n+1}}{n!}2x$	$2(n-m+1)(\alpha+n+m)$	$m(2m+1)$	x^2
T_{2n}	$(-1)^n$	$2(n-m+1)(n+m-1)$	$m(2m-1)$	x^2
T_{2n+1}	$(-1)^n(2n+1)x$	$2(n-m+1)(n+m)$	$m(2m+1)$	x^2
U_{2n}	$(-1)^n$	$2(n-m+1)(n+m)$	$m(2m-1)$	x^2
U_{2n+1}	$(-1)^n2(n+1)x$	$2(n-m+1)(n+m+1)$	$m(2m+1)$	x^2
P_{2n}	$\frac{(-1)^n}{4^n}\binom{2n}{n}$	$(n-m+1)(2n+2m-1)$	$m(2m-1)$	x^2
P_{2n+1}	$\frac{(-1)^n}{4^n}\binom{2n+1}{n}(n+1)x$	$(n-m+1)(2n+2m+1)$	$m(2m+1)$	x^2
$L_n^{(\alpha)}$	$\binom{n+\alpha}{n}$	$n-m+1$	$m(\alpha+m)$	x
H_{2n}	$(-1)^n\frac{(2n)!}{n!}$	$2(n-m+1)$	$m(2m-1)$	x^2
H_{2n+1}	$(-1)^n\frac{(2n+1)!}{n!}2x$	$2(n-m+1)$	$m(2m+1)$	x^2

Example 2. Compute $P_8^{(1/2,3/2)}(2)$. Here $d_8=\binom{8.5}{8}=3.33847$, $f(2)=-1$.

m	8	7	6	5	4	3	2	1	0
a_m	1	1. 132353	1. 366667	1. 841026	3. 008392	6. 849651	26. 44156	223. 1091	6545. 533
b_m	18	34	48	60	70	78	84	88	90
c_m	136	105	78	55	36	21	10	3	0

$P_8^{(1/2,3/2)}(2)=d_8a_0(2)=(3.33847)(6545.533)=21852.07$

Evaluation of orthogonal polynomials by means of their recurrence relations

Example 3. Compute $C_n^{(\frac{1}{4})}(2.5)$ for $n=2,3,4,5,6$.

From **Table 22.2** $C_0^{(\frac{1}{4})}=1$, $C_1^{(\frac{1}{4})}=1.25$ and from **22.7** the recurrence relation is

$$C_{n+1}^{(\frac{1}{4})}(2.5)=[5(n+\tfrac{1}{4})C_n^{(\frac{1}{4})}(2.5)-(n-\tfrac{1}{2})C_{n-1}^{(\frac{1}{4})}(2.5)]\frac{1}{n+1}.$$

n	2	3	4	5	6
$C_n^{(\frac{1}{4})}(2.5)$	3. 65625	13. 08594	50. 87648	207. 0649	867. 7516

Check: Compute $C_6^{(\frac{1}{4})}(2.5)$ by the method of **Example 2.**

Change of Interval of Orthogonality

In some applications it is more convenient to use polynomials orthogonal on the interval [0, 1]. One can obtain the new polynomials from the ones given in this chapter by the substitution $x=2\bar{x}-1$. The coefficients of the new polynomial can be computed from the old by the following recursive scheme, provided the standardization is not changed. If

$$f_n(x)=\sum_{m=0}^{n} a_m x^m, \quad f_n^*(x)=f_n(2x-1)=\sum_{m=0}^{n} a_m^* x^m$$

then the a_m^* are given recursively by the a_m through the relations

$a_m^{(j)}=2a_m^{(j-1)}-a_{m+1}^{(j)};\ m=n-1,\ n-2,\ \ldots,\ j;\ j=0,\ 1,\ 2,\ \ldots,\ n$

$a_m^{(-1)}=a_m/2,\ m=0,\ 1,\ 2,\ \ldots,\ n$

$a_n^{(j)}=2^j a_n,\ j=0,\ 1,\ 2,\ \ldots,\ n$ and $a_m^{(m)}=a_m^*;\ m=0,\ 1,\ 2,\ \ldots,\ n.$

Example 4. Given $T_5(x)=5x-20x^3+16x^5$, find $T_5^*(x)$.

$j \backslash m$	5	4	3	2	1	0
−1	$8=a_5^{(-1)}$	0	$-10=a_3^{(-1)}$	0	$2.5=a_1^{(-1)}$	0
0	16	−16	−4	4	1	$-1=a_0^*$
1	32	−64	56	−48	$50=a_1^*$	
2	64	−192	304	$-400=a_2^*$		
3	128	−512	$1120=a_3^*$			
4	256	$-1280=a_4^*$				
5	$512=a_5^*$					

Hence, $T_5^*(x)=512x^5-1280x^4+1120x^3-400x^2+50x-1$.

22.19. Least Square Approximations

Problem: Given a function $f(x)$ (analytically or in form of a table) in a domain D (which may be a continuous interval or a set of discrete points).[2] Approximate $f(x)$ by a polynomial $F_n(x)$ of given degree n such that a weighted sum of the squares of the errors in D is least.

Solution: Let $w(x)\geq 0$ be the weight function chosen according to the relative importance of the errors in different parts of D. Let $f_m(x)$ be orthogonal polynomials in D relative to $w(x)$, i.e. $(f_m, f_n)=0$ for $m\neq n$, where

$$(f,g)=\begin{cases}\displaystyle\int_D w(x)f(x)g(x)dx & \text{if } D \text{ is a continuous interval}\\[2ex] \displaystyle\sum_{m=1}^{N} w(x_m)f(x_m)g(x_m) & \text{if } D \text{ is a set of } N \text{ discrete points } x_m.\end{cases}$$

Then

$$F_n(x)=\sum_{m=0}^{n} a_m f_m(x)$$

where

* $$a_m=(f,f_m)/(f_m,f_m).$$

[2] $f(x)$ has to be square integrable, see e.g. [22.17].

*See page II.

D a Continuous Interval

Example 5. Find a least square polynomial of degree 5 for $f(x)=\dfrac{1}{1+x}$, in the interval $2\leq x\leq 5$, using the weight function

$$w(x)=\frac{1}{\sqrt{(x-2)(5-x)}}$$

which stresses the importance of the errors at the ends of the interval.

Reduction to interval $[-1,1]$, $t=\dfrac{2x-7}{3}$

$$w(x(t))=\frac{2}{3}\frac{1}{\sqrt{1-t^2}}$$

From **22.2**, $f_m(t)=T_m(t)$ and

$$a_m=\frac{4}{3\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\frac{1}{t+3}T_m(t)dt \qquad (m\neq 0)$$

$$a_0=\frac{2}{3\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\frac{dt}{t+3}$$

Example 7. Economize $f(x)=1+x/2+x^2/3+x^3/4+x^4/5+x^5/6$ with $R=.05$.

From **Table 22.3**

$$f(x)=\frac{1}{120}[149T_0(x)+32T_2(x)+3T_4(x)]+\frac{1}{96}[76T_1(x)+11T_3(x)+T_5(x)]$$

so

$$\bar{f}(x)=\frac{1}{120}[149T_0(x)+32T_2(x)]+\frac{1}{96}[76T_1(x)+11T_3(x)]$$

since

$$|\bar{f}(x)-f(x)|\leq\frac{1}{40}+\frac{1}{96}<.05$$

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