Numerical Integration - Lecture 04

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Outline

- Numerical integration plays a key role in many engineering applications. The integration methods that we examine
  - Trapezoidal Rule
  - Simpson's 1/3 Rule
  - Mid-point Rule
  - Romberg Integration
  - Gauss Quadrature
  - Multiple Integrals ⟶ **(This lecture)**

  - Multiple integrals are very common in engineering applications. The approach that we will use is based on integrating a function over a single variable holding the remaining variables constant. This is repeated for each independent variable.

Multiple Integrals

The evaluation of multiple integrals is a straightforward extension of the methods that we have already discussed. As an example, consider the double integral

$$\iint_R f(x,y)\,dA$$

over the region $\mathrm{R} = \left\{(x,y) \middle| a \le x \le b, c \le y \le d\right\}$

The integral over the 2-D space

$$I = \iint_R f(x,y)\,dA$$

Approach: Break it up into a sequence of 1D problems

$$= \int_{\mathrm{a}}^{\mathrm{b}} \left[\int_c^d f(x,y)dy\right]dx$$

The term in brackets is a function of x only.

$$= \int_{\mathrm{a}}^{\mathrm{b}} \left[g(x)\right]dx$$

The integration rules we developed can be applied here.

Multiple Integrals – Simpson's Rule

For a multiple integral, Simpson's integration rule can be used:

$$I = \int_a^b g(x)dx = \frac{h_1}{3}\left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right] \qquad h_1 = \frac{b-a}{2}$$

By definition

$$g(a) = \int_c^d f(a,y)dy = \frac{h_2}{3}\left[f(a,c) + 4f\left(a,\frac{c+d}{2}\right) + f(a,d)\right] \qquad h_2 = \frac{d-c}{2}$$

Likewise,

$$g\left(\frac{a+b}{2}\right) = \int_c^d f(\frac{a+b}{2},y)dy = \frac{h_2}{3}\left[f\left(\frac{a+b}{2},c\right) + 4f\left(\frac{a+b}{2},\frac{c+d}{2}\right) + f\left(\frac{a+b}{2},d\right)\right]$$

Finally,

$$g(b) = \int_c^d f(b,y)dy = \frac{h_2}{3}\left[f(b,c) + 4f\left(b,\frac{c+d}{2}\right) + f(b,d)\right]$$

This approach can be extended to any of the methods we discussed earlier and to any number of dimensions

Example for Simpson's Integration Rule

Example:

$$I = \int_0^1 \left(\int_{-1}^{1} xy^2 dy \right) dx = \frac{1}{6}\left[g(0) + 4g(0.5) + g(1) \right]$$

$$g(x) = \frac{1}{3}\left[\mathrm{f(x,\text{-}1) + 4f(x,0) + f(x,1)} \right]$$

$$g(0) = \frac{1}{3}[0 + 0 + 0] = 0$$

$$g(½) = \frac{1}{3}\left[\frac{1}{2} + 0 + \frac{1}{2} \right] = \frac{1}{3}$$

$$g(1) = \frac{1}{3}\left[1 + 0 + 1 \right] = \frac{2}{3}$$

Example for Simpson's Integration Rule

From before,

$$I = \int_0^1 \left(\int_{-1}^{1} xy^2 dy \right) dx = \frac{1}{6}\left[g(0) + 4g(0.5) + g(1) \right]$$

$$= \frac{1}{6}\left[0 + 4\left(\frac{1}{3}\right) + \frac{2}{3} \right] = \frac{1}{3}$$

The exact value is

$$\int_0^1 \int_{-1}^{1} xy^2 dy dx = \int_0^1 \left. \frac{xy^3}{3} \right|_{-1}^{1} dx = \int_0^1 \frac{2x}{3} dx = \left. \frac{x^2}{3} \right|_0^1 = \frac{1}{3}$$

Different Methods

Let's work the same problem with Trapezoidal Rule in both directions (x and y) using a single panel:

$$I = \int_0^1 \left(\int_{-1}^{1} xy^2 dy \right) dx = (1)\frac{g(0)+g(1)}{2}$$

$$g(x) \approx (2)\ \frac{f(x,-1)+f(x,+1)}{2}$$

$$g(0) \approx (2)\ \frac{0+0}{2} = 0; \quad g(1) \approx (2)\ \frac{1+1}{2} = 2$$

$$I = \int_0^1 \left(\int_{-1}^{1} xy^2 dy \right) dx \approx (1)\frac{0+2}{2} = 1$$

The truncation error is $\vartheta\left(h^p\right)$ in each direction.

To improve results, we may refine h or increase p

Different Methods

Let's work the same problem with Trapezoidal Rule in x direction and Simpson's 1/3 rule in y direction

$$I = \int_0^1 \left(\int_{-1}^{1} xy^2 dy \right) dx = (1)\frac{g(0) + g(1)}{2}$$

Use Simpson's Rule in y direction

$$g(x) = \left(\frac{1}{3}\right)\left[f(x,-1) + 4f(x,0) + f(x,+1)\right]$$

$$g(0) = \left(\frac{1}{3}\right)\left[0 + 4(0) + 0\right] = 0$$

$$g(1) = \left(\frac{1}{3}\right)\left[1 + 4(0) + 1\right] = \frac{2}{3}$$

$$\longrightarrow \quad I = \int_0^1 \left(\int_{-1}^{1} xy^2 dy \right) dx = (1)\frac{0 + 2/3}{2} = \frac{1}{3}$$

To get the exact result, all it takes is a method in x that is exact for $P_1(x)$ and a method in y that is exact for $P_2(y)$.

What if the integration limits are functions of an independent variable?

What if the integral is in the form:

$$\mathrm{I} = \int_{\mathrm{a}}^{\mathrm{b}} \left[\int_{c(x)}^{d(x)} f(x,y)dy \right] dx$$

$$= \int_{\mathrm{a}}^{\mathrm{b}} \left[g(x) \right] dx \qquad where\ g(x) = \int_{c(x)}^{d(x)} f(x,y)dy$$

$$Again\ define\ h_1 = \frac{b-a}{2} \quad then$$

Using Simpson's Rule:

$$I \approx \frac{h_1}{3}\left[g(a) + 4g\left(\frac{b+a}{2}\right) + g(b) \right]$$

$$g(a) = \int_{c(a)}^{d(a)} f(a,y)dy \quad and \quad h_{21} = \frac{d(a)-c(a)}{2}$$

$$g(a) = \frac{h_{21}}{3}\left[f\left(a,c(a)\right) + 4f\left(a,\frac{c(a)+d(a)}{2}\right) + f\left(a,d(a)\right) \right]$$

What if the integration limits are functions of an independent variable ?

$$g\left(\frac{a+b}{2}\right)=\int_{c(\frac{a+b}{2})}^{d(\frac{a+b}{2})} f(\frac{a+b}{2},y)dy \quad and \quad h_{22}=\frac{d(\frac{a+b}{2})-c(\frac{a+b}{2})}{2}$$

$$g(\frac{a+b}{2})=\frac{h_{22}}{3}\left[f\left(\frac{a+b}{2},c(\frac{a+b}{2})\right)+4f\left(\frac{a+b}{2},\frac{c(\frac{a+b}{2})+d(\frac{a+b}{2})}{2}\right)+f\left(\frac{a+b}{2},d(\frac{a+b}{2})\right)\right]$$

And finally for *g(b)*:

$$g(b)=\int_{c(b)}^{d(b)} f(b,y)dy \quad and \quad h_{23}=\frac{d(b)-c(b)}{2}$$

$$g(b)=\frac{h_{23}}{3}\left[f\left(b,c(b)\right)+4f\left(b,\frac{c(b)+d(b)}{2}\right)+f\left(b,d(b)\right)\right]$$

Gauss Quadrature – Basic Principle

The Gauss Quadrature Rule for finding an integral numerically is given by the function

$$\int_a^b f(x)dx = \sum_{i=1}^{n} c_i f(x_i)$$

where c_i *are the weights of the function* $f(x_i)$ *at* x_i

and x_i *are the zeros(roots) of the* n^{th} *Legendre polynomial*

We have 2n parameters for evaluating the integral

$c_1, c_2, \ldots. c_n$ and $x_1, x_2, x_3, \ldots x_n$.

Approach: Choose the parameters to exactly integrate the largest class of polynomials possible. With 2n parameters, the class of polynomials of degree 2n-1 can be integrated exactly.

Zeros (Roots) and Weights (Coefficients) for Gauss-Legendre Quadrature on [-1,1]

n	Zeros	Weights
2	-0.57735	1.
	0.57735	1.
3	-0.774597	0.555556
	0.	0.888889
	0.774597	0.555556
4	-0.861136	0.347855
	-0.339981	0.652145
	0.339981	0.652145
	0.861136	0.347855
5	-0.90618	0.236927
	-0.538469	0.478629
	0.	0.568889
	0.538469	0.478629
	0.90618	0.236927

This is Table 4.11 in the text on page 225.

The roots are the zeros of the n^{th} degree Legendre polynomial

An arbitrary interval [a,b] is mapped onto [-1,1] via a linear transformation

$$x = \frac{b+a}{2} + \frac{b-a}{2} t \quad where \quad dx = \frac{(b-a)}{2} dt$$

$$\int_a^b f(x)dx = \frac{(b-a)}{2} \int_{-1}^{1} f[x(t)]dt = \frac{(b-a)}{2} \sum_{i=1}^{n} c_i f(x(t_i))$$

Gauss Quadrature Example for Single Integral

Evaluate the integral

$$\int_0^{\pi/2} \sin x \, dx = 1$$ using Gauss Quadrature and Trapezoidal Rule

$$b = \tfrac{\pi}{2};\ \ a = 0; \qquad x = \tfrac{\pi}{4} + \tfrac{\pi}{4} t = (1 + t)\tfrac{\pi}{4}$$

$$With\ \mathrm{n} = 2,\ \mathrm{c}_1 = c_2 = 1;\ t_1 = -0.577;\ t_2 = 0.577$$

$$I_{QUAD} \ = \ \tfrac{\pi}{4}\left[\sin\left(\tfrac{\pi}{4}(1 - 0.577)\right) + \sin\left(\tfrac{\pi}{4}(1 + 0.577)\right)\right]$$

$$I_{QUAD} = 0.9984716$$

The Single Panel Trapezoidal Rule gives $I_{TRAP} = 0.7584$
(also requires two function evaluations)

Gauss Quadrature Example for Multiple Integrals

$$\boxed{\mathrm{I} = \int_{1.4}^{2.0}\left[\int_{1.0}^{1.5} \ln(x+2y)dy\right]dx}$$

Integrate using a 3-point Gauss quadrature in both directions

$$R = \{(x,y) | 1.4 \le x \le 2.0, 1.0 \le y \le 1.5\}$$

$$\textit{To apply Gauss Quadrature, we have to make the transformation:}$$

$$\hat{R} = \{(u,v) | -1 \le u \le 1, -1 \le v \le 1\}$$

$$x = \frac{b+a}{2} + \frac{b-a}{2}u \rightarrow x = 1.7 + 0.3u \text{ and } dx = 0.3du$$

$$y = \frac{d+c}{2} + \frac{d-c}{2}v \rightarrow y = 1.25 + 0.25v \text{ and } dy = 0.25dv$$

$$I = 0.075\int_{-1}^{1}\int_{-1}^{1} \ln\left[(1.7+0.3u) + 2(1.25+0.25v)\right]dvdu$$

$$I = 0.075\int_{-1}^{1}\int_{-1}^{1} \underbrace{\ln\left[4.2+0.3u+0.5v\right]}_{f(u,v)}dvdu = 0.075\int_{-1}^{1}\int_{-1}^{1} f(u,v)dvdu$$

Gauss Quadrature Example for Multiple Integrals

$$g(u) = \int_{-1}^{1} f(u,v)dv \quad then \quad I = 0.075 \int_{-1}^{1} g(u)du$$

Apply a 3-point Gauss Quadrature in both directions

$$\hat{I} = 0.075\left[c_1 g(u_1) + c_2 g(u_2) + c_3 g(u_3)\right] \quad where$$

$$g(u_1) = \int_{-1}^{1} f(u_1,v)dv \approx c_1 f(u_1,v_1) + c_2 f(u_1,v_2) + c_3 f(u_1,v_3)$$

$$g(u_2) = \int_{-1}^{1} f(u_2,v)dv \approx c_1 f(u_2,v_1) + c_2 f(u_2,v_2) + c_3 f(u_2,v_3)$$

$$g(u_3) = \int_{-1}^{1} f(u_3,v)dv \approx c_1 f(u_3,v_1) + c_2 f(u_3,v_2) + c_3 f(u_3,v_3)$$

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Gauss Quadrature Example for Multiple Integrals

$$\hat{I} = 0.075\left[c_1 g(u_1) + c_2 g(u_2) + c_3 g(u_3)\right]$$

Using the expression for $g(u_1)$, $g(u_2)$, and $g(u_3)$ from previous slide

$$\hat{I} = 0.075\sum_{i=1}^{3}\sum_{j=1}^{3} c_i c_j f\left(u_i, v_j\right)$$

$$where \quad f\left(u_i, v_j\right) = \ln\left(4.2 + 0.3u_i + 0.5v_j\right)$$

For a 3-point Gauss quadrature:

$$u_1 = v_1 = -0.7745966692 \quad and \quad c_1 = 0.5555555555$$

$$u_2 = v_2 = 0.0 \quad and \quad c_2 = 0.8888888888$$

$$u_3 = v_3 = 0.7745966692 \quad and \quad c_3 = 0.5555555555$$

Final Result:

$$\hat{I} = 0.4295545314 \qquad and \qquad Error = 4.8\times10^{-9}$$

Summary

In this lecture we have

- Learnt to evaluate **Multiple Integrals** by the extension of one-dimensional quadrature rules.
    - Integration holding all variables constant but one was accomplished by using earlier methods
    - Repeated for each independent variable
    - Showed the approach when the integration limits are functions of one of the independent variables
    - Worked on an example to show how Gauss Quadrature can be applied to multiple integrals