AE/ME 301 Solution to Homework #3

Question 1:

Part a)

For the linear problem [A]{x} = {b}, construct [A] and {b}
Hilbert Matrix for n=2 [A], the right hand side (RHS) vector {b}, and the initial vector {x0}:

```
hilbert = Table[1. / (i + j - 1.), {i, 1, 2}, {j, 1, 2}];
A = hilbert; Print["Hilbert Matrix (n=2) =", MatrixForm[A]];
b = Table[1., {i, 1, 2}]; Print["RHS vector b =", MatrixForm[b]];
x0 = Table[1., {2}]; Print["Initial guess x0=", MatrixForm[x0]];
```

Hilbert Matrix (n=2) $=\begin{pmatrix} 1. & 0.5 \\ 0.5 & 0.333333 \end{pmatrix}$

RHS vector b $=\begin{pmatrix} 1. \\ 1. \end{pmatrix}$

Initial guess x0$=\begin{pmatrix} 1. \\ 1. \end{pmatrix}$

Parametric Study for the Convergence of the Gauss-Seidel Scheme by changing the relaxation factor:

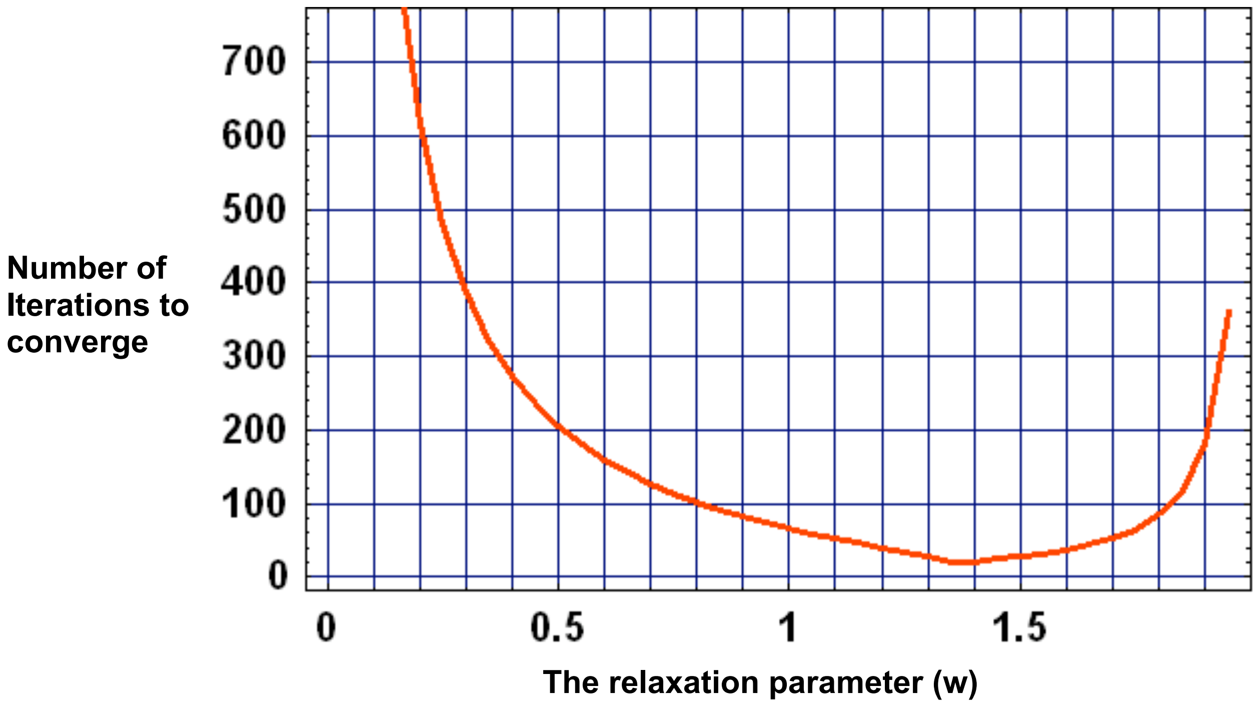
Call the Mathematica Routine (see Appendix A for the code) written for the Gauss-Seidel Scheme:

```
{w0, kmax} = gaussseidel[2, A, b, x0, 0.1, 5000, 8]
```

The table showing the solution vector (x), relaxation parameter (w), and the number of iterations to converge:

x1	x2	w	iter#
-1.9999999669783262	5.999999943206736	0.1	1298
-1.9999999683457186	5.999999945983924	0.2	616
-1.9999999691225123	5.999999947770701	0.3	388
-1.9999999703219886	5.999999950296077	0.4	274
-1.9999999730440396	5.999999955364466	0.5	206
-1.9999999744492993	5.999999958243207	0.6	160
-1.9999999761606808	5.99999996163628	0.7	127
-1.9999999779102042	5.9999999651026314	0.8	102
-1.99999997846251	5.999999966740367	0.9	82
-1.9999999810799576	5.999999971619936	1.	66
-1.9999999813688163	5.999999973134799	1.1	52
-1.9999999856513406	5.999999980510824	1.2	40
-1.9999999927410328	5.99999999128924	1.3	28
-1.999999998757993	5.99999998958416	1.4	21
-2.0000000211586	6.000000036403118	1.5	27
-2.0000000144608654	6.000000028482961	1.6	37
-2.00000001352447	6.000000021554384	1.7	54
-1.9999999851749612	5.999999978002207	1.8	86
-1.9999999837651952	5.999999975370578	1.9	181

The graph showing the number of iterations to converge vs. the relaxation parameter (w):



Part b)

For the linear problem [A]{x} = {b}, construct [A] and {b}
Hilbert Matrix for n=3 [A], the right hand side (RHS) vector {b}, and the initial vector {x0}:

```
hilbert = Table[1. / (i + j - 1.),  {i, 1, 3}, {j, 1, 3}];
A = hilbert; Print["Hilbert Matrix (n=3) =", MatrixForm[A]];
b = Table[1., {i, 1, 3}]; Print["RHS vector b =", MatrixForm[b]];
x0 = Table[1., {3}]; Print["Initial guess x0=", MatrixForm[x0]];
```

Hilbert Matrix (n=3) = $\begin{pmatrix} 1. & 0.5 & 0.333333 \\ 0.5 & 0.333333 & 0.25 \\ 0.333333 & 0.25 & 0.2 \end{pmatrix}$

RHS vector b = $\begin{pmatrix} 1. \\ 1. \\ 1. \end{pmatrix}$

Initial guess x0= $\begin{pmatrix} 1. \\ 1. \\ 1. \end{pmatrix}$

Parametric Study for the Convergence of the Gauss-Seidel Scheme by changing the relaxation factor:

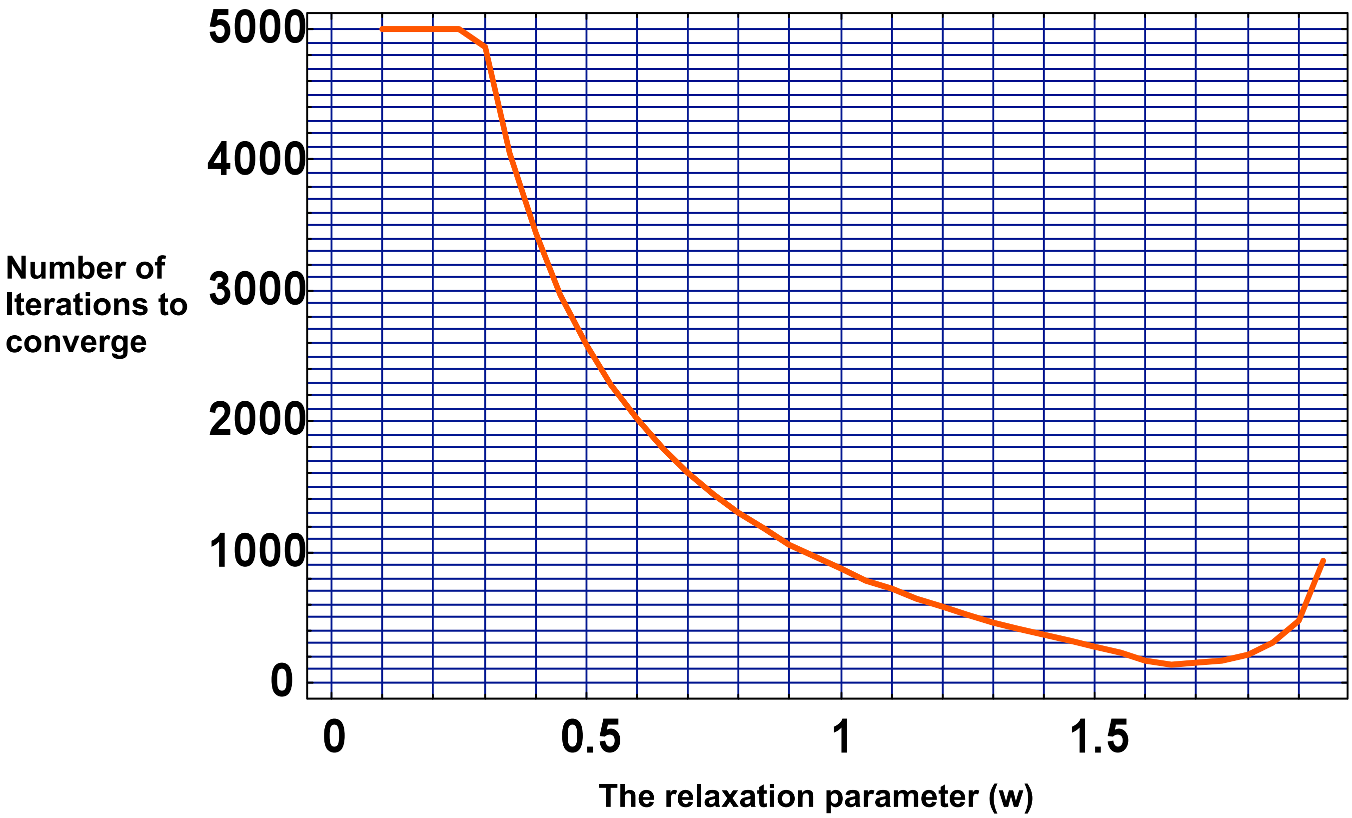
Call the Mathematica Routine (see Appendix for the code) written for the Gauss-Seidel Scheme:

```
{w0, kmax} = gausseidel[3, A, b, x0, 0.1, 5000, 8]
```

The table showing the solution vector (x), relaxation parameter (w), and the number of iterations to converge:

x1	x2	x3	w	iter #
2.9680162257924523	-23.827341738059598	29.836001493412205	0.1	5000
2.999877695188248	-23.99934206734267	29.999376132336785	0.2	5000
2.9999996236745274	-23.999997983455742	29.9999980915138	0.3	4869
2.9999996304097576	-23.999998028192792	29.999998137867532	0.4	3442
2.999999639368635	-23.999998085481337	29.99999819638738	0.5	2586
2.999999649674085	-23.99999815064818	29.999998262660213	0.6	2015
2.999999662371268	-23.999998229212434	29.999998341868135	0.7	1607
2.9999996787987366	-23.99999832807182	29.999998440407573	0.8	1301
2.9999996953554438	-23.999998428377122	29.99999854063969	0.9	1062
2.9999997137665875	-23.99999853916755	29.999998651015122	1.	870
2.9999997348976017	-23.999998664813056	29.99999877553301	1.1	712
2.9999997584545812	-23.99999880370473	29.99999891263418	1.2	579
2.9999997881411	-23.99999897373962	29.999999078333577	1.3	465
2.9999998182542873	-23.99999914668102	29.999999246903467	1.4	364
2.9999998527229796	-23.99999934192883	29.9999994357915	1.5	271
2.9999998994757884	-23.999999598788563	29.999999680041494	1.6	173
2.999999994208918	-24.000000101859232	30.000000147167874	1.7	149
2.9999998796877843	-23.999999319117933	29.99999935583052	1.8	221
2.9999999441742715	-23.999999581533466	29.999999560681083	1.9	467

The graph showing the number of iterations to converge vs. the relaxation parameter (w):



Question 2: Part b)

The following operations were done in Mathematica environment. See Appendix B for the actual Mathematica routine written for solving non-linear system of equations using Newton's Method.

Define the Jacobien for a general (nxn) system :

atlantis01.aoe.vt.edu) In[160]:=

```
Jacobien[F_, X_, n_] := Table[D[F[[i]], X[[j]]], {i, 1, n}, {j, 1, n}];
```

Define the non - linear functions f1, f2, f3, and, f4 :

```
f1[x_] := 4 * x[[1]] - x[[2]] + x[[3]] - x[[1]] * x[[4]]
f2[x_] := -x[[1]] + 3 * x[[2]] - 2 * x[[3]] - x[[2]] * x[[4]]
f3[x_] := x[[1]] - 2 * x[[2]] + 3 * x[[3]] - x[[3]] * x[[4]]
f4[x_] := x[[1]] ^ 2 + x[[2]] ^ 2 + x[[3]] ^ 2 - 1.0
```

Define X an F vectors :

atlantis01.aoe.vt.edu) In[165]:=

```
X = {x1, x2, x3, x4};
Print["X=", MatrixForm[X]];
F = {f1[X], f2[X], f3[X], f4[X]};
Print["F=", MatrixForm[F]];
```

$$X=\begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix}$$

$$F=\begin{pmatrix} 4\,x1 - x2 + x3 - x1\,x4 \\ -x1 + 3\,x2 - 2\,x3 - x2\,x4 \\ x1 - 2\,x2 + 3\,x3 - x3\,x4 \\ -1. + x1^2 + x2^2 + x3^2 \end{pmatrix}$$

1st starting point :

atlantis01.aoe.vt.edu) In[174]:=

```
X0 = {1., 1., 1., 1.}; Print["X0=", MatrixForm[X0]];
```

$$X0=\begin{pmatrix} 1. \\ 1. \\ 1. \\ 1. \end{pmatrix}$$

atlantis01.aoe.vt.edu) In[175]:=

```
sol = nonlinearnewton[X0, 100, 4, -16];
Print["Solution Vector=", MatrixForm[sol], "   ", "F=", MatrixForm[F /. Thread[X → sol]]];
```

L2Norm Residual=7.41236×10^{-17} iteration number to converge=99

$$\text{Solution Vector=}\begin{pmatrix} 1.77396\times10^{-17} \\ 0.707107 \\ 0.707107 \\ 1. \end{pmatrix} \quad F=\begin{pmatrix} 9.32827\times10^{-17} \\ -1.11022\times10^{-16} \\ -1.11022\times10^{-16} \\ -2.22045\times10^{-16} \end{pmatrix}$$

2 nd starting point :

atlantis01.aoe.vt.edu) In[176]:=

```
X0 = {3., 3., 3., 3.}; Print["X0=", MatrixForm[X0]];
```

$$X0=\begin{pmatrix} 3. \\ 3. \\ 3. \\ 3. \end{pmatrix}$$

atlantis01.aoe.vt.edu) In[177]:=

```
sol = nonlinearnewton[X0, 100, 4, -16];
Print["Solution Vector=", MatrixForm[sol], "   ", "F=", MatrixForm[F /. Thread[X → sol]]];
```

L2Norm Residual=1.99402×10^{-18} iteration number to converge=8

$$\text{Solution Vector=}\begin{pmatrix} 0.816497 \\ 0.408248 \\ -0.408248 \\ 3. \end{pmatrix} \quad F=\begin{pmatrix} 0. \\ 0. \\ 0. \\ -5.55112 \times 10^{-17} \end{pmatrix}$$

3 rd starting point :

atlantis01.aoe.vt.edu) In[178]:=

```
X0 = {6., 6., 6., 6.}; Print["X0=", MatrixForm[X0]];
```

$$X0=\begin{pmatrix} 6. \\ 6. \\ 6. \\ 6. \end{pmatrix}$$

atlantis01.aoe.vt.edu) In[179]:=

```
sol = nonlinearnewton[X0, 100, 4, -16];
Print["Solution Vector=", MatrixForm[sol], "   ", "F=", MatrixForm[F /. Thread[X → sol]]];
```

L2Norm Residual=9.56769×10^{-19} iteration number to converge=11

$$\text{Solution Vector=}\begin{pmatrix} 0.57735 \\ -0.57735 \\ 0.57735 \\ 6. \end{pmatrix} \quad F=\begin{pmatrix} 0. \\ 0. \\ 0. \\ -1.11022 \times 10^{-16} \end{pmatrix}$$

As can be seen from the above results, the initial starting vector is important also for the solution of non-linear system of equations. Different starting vectors may lead to converging to a different solution vector (point).