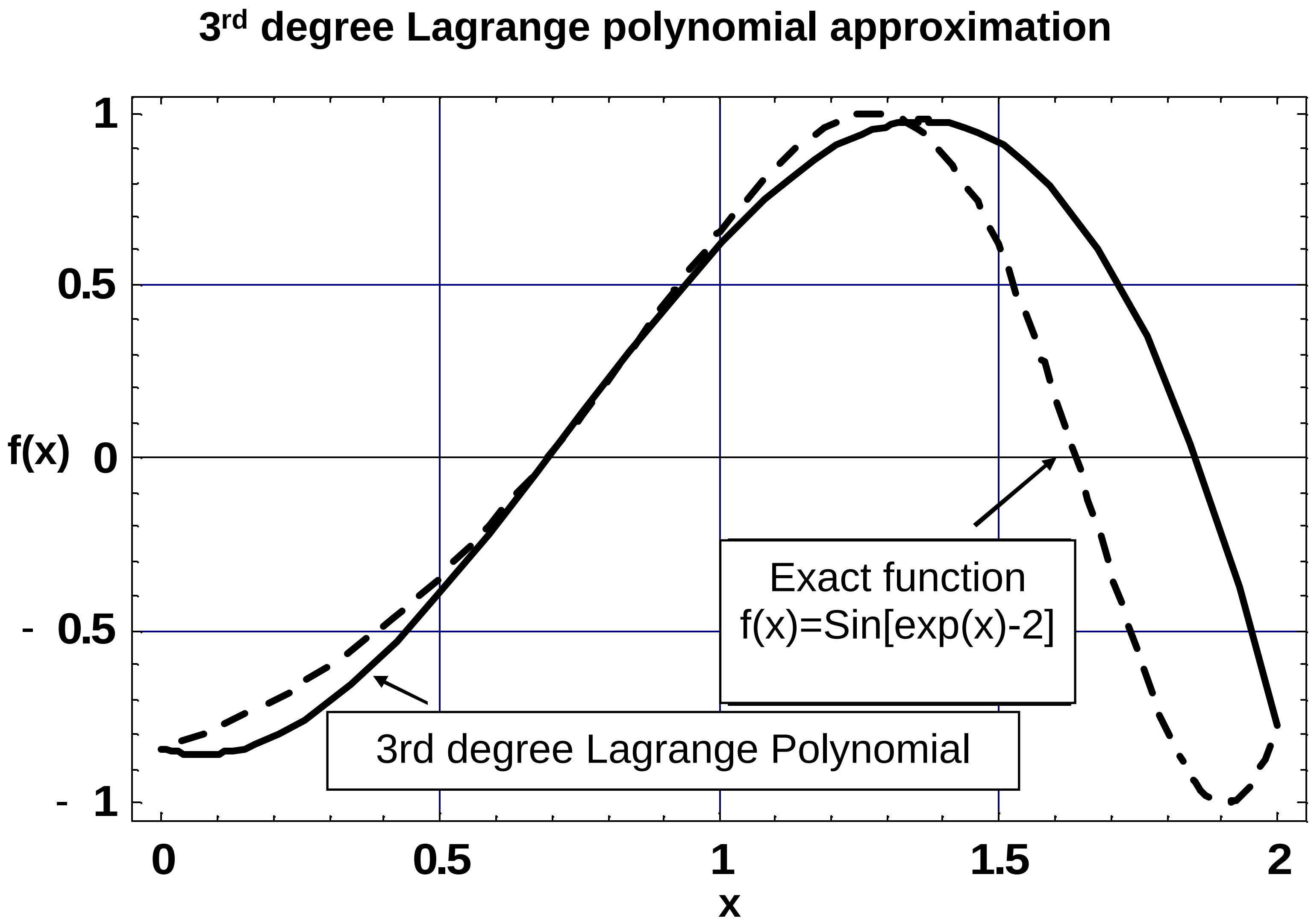
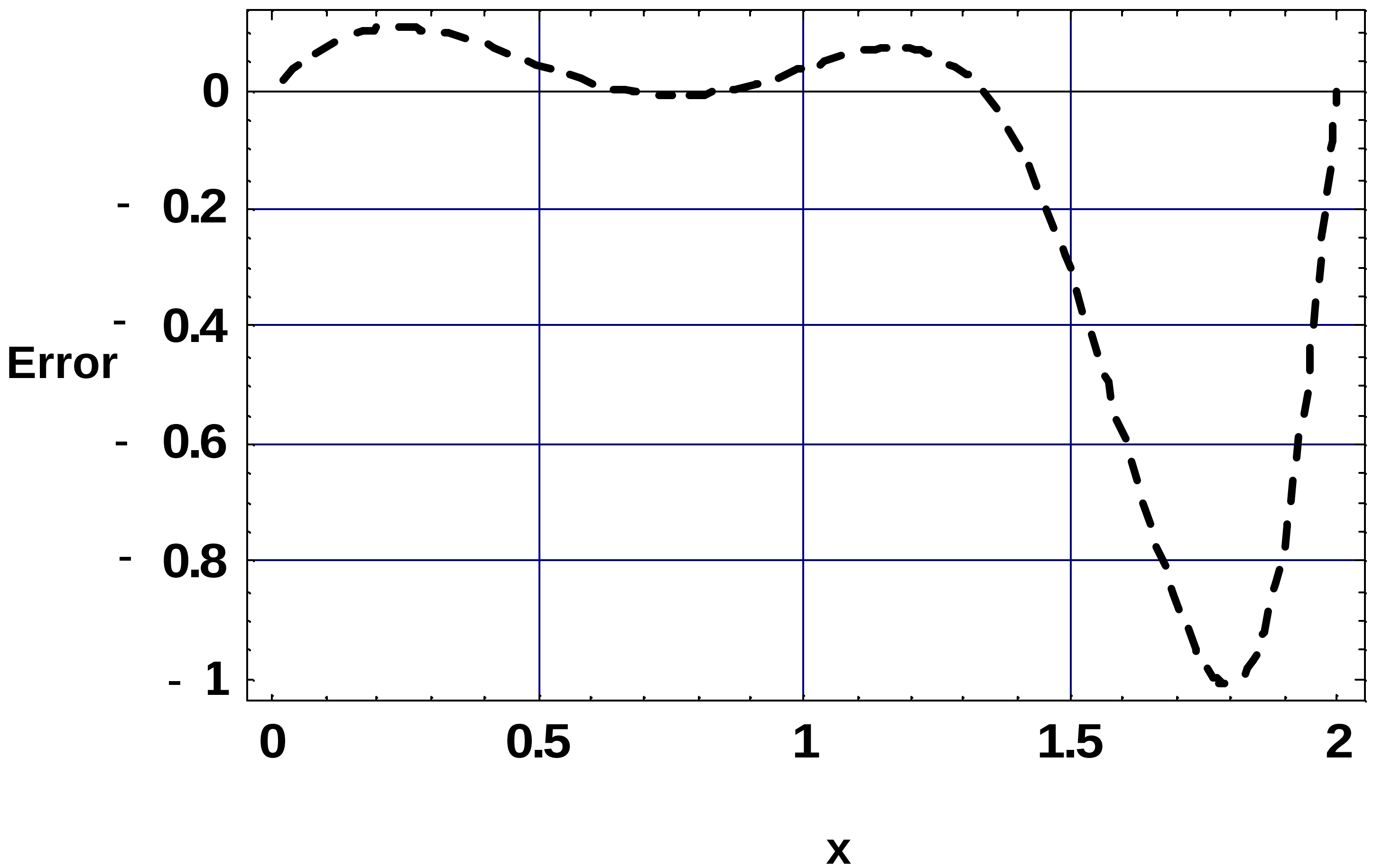
Question 1.

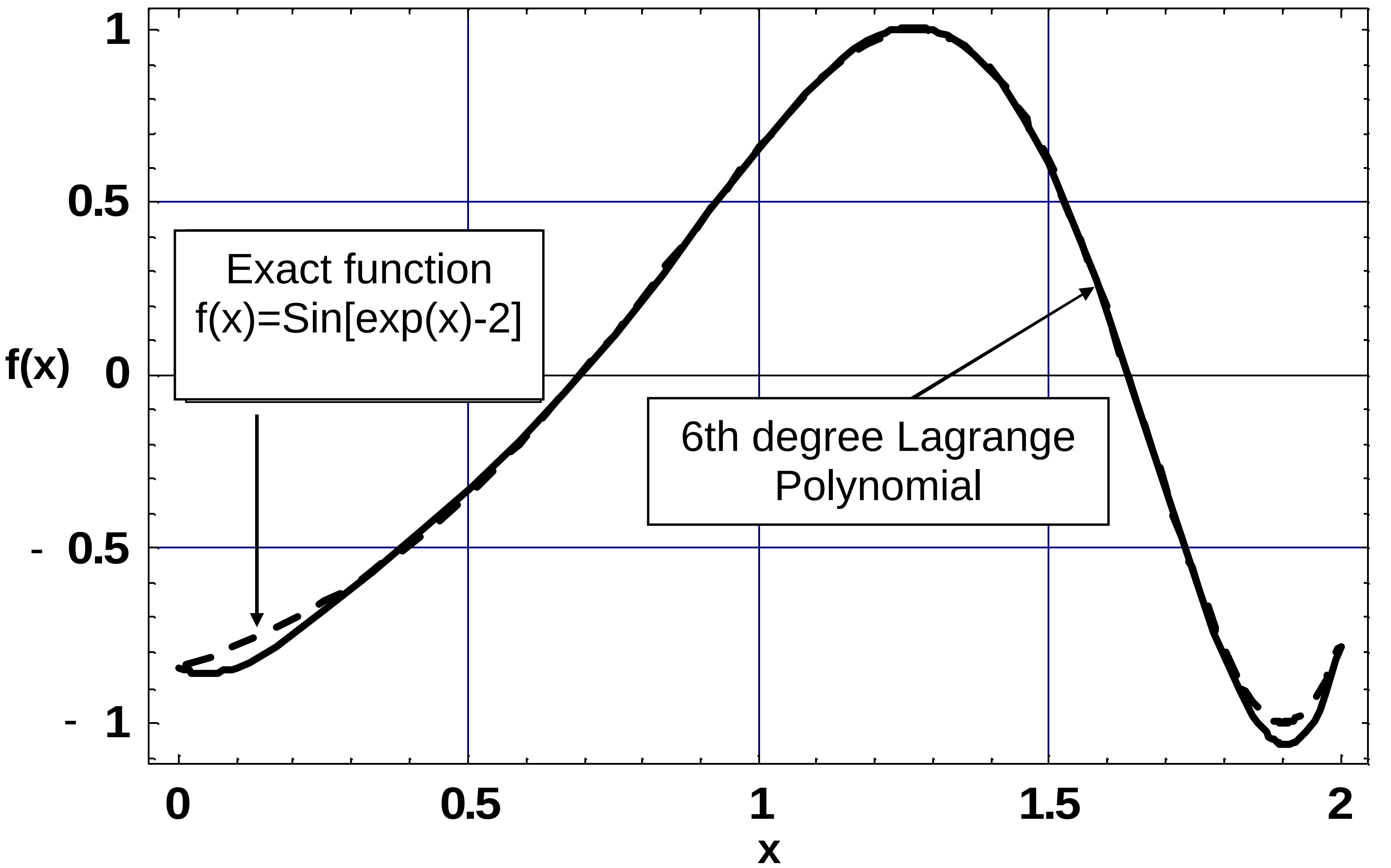


3rd degree Lagrange polynomial approximation
1
0.5
0
- 0.5
- 1
f(x)
0
0.5
1
1.5
2
x
Exact function
f(x)=Sin[exp(x)-2]
3rd degree Lagrange Polynomial

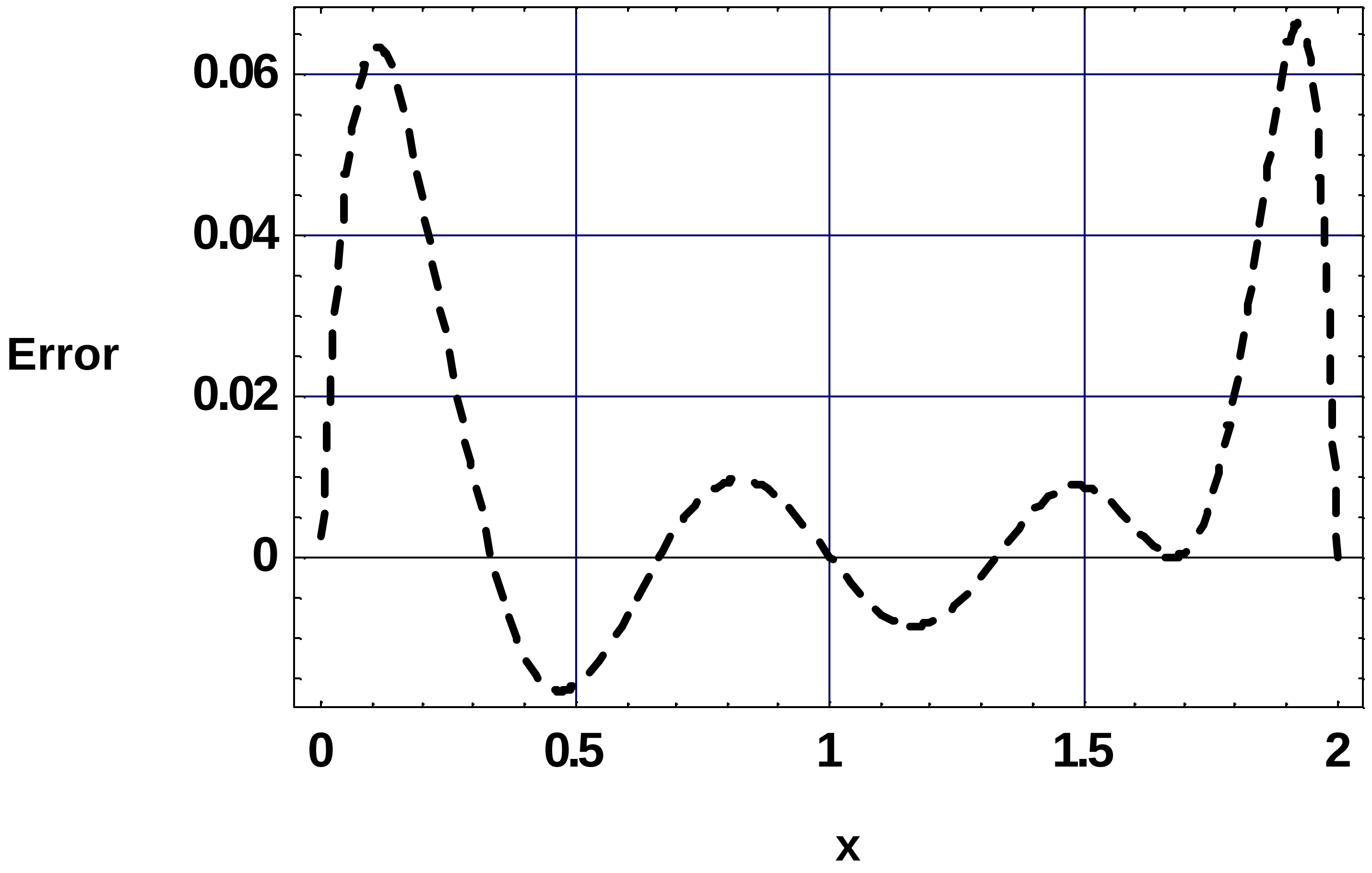
Error distribution for the 3rd degree Lagrange polynomial approximation :



6^{th} degree Lagrange polynomial approximation:



Error distribution for the 6th degree Lagrange polynomial approximation :



Error values for the 3rd and the 6th degree Lagrange polynomial approximation :

Polynomial Degree	x=0.1	x=0.9	x=1.5	x=1.9
3	0.0769269	0.00868774	-0.302344	-0.779357
6	0.0627347	0.00724272	0.00859439	0.0620758

6th degree Lagrange Polynomial:

$$f(x) = -0.841471 - 0.836756\,x + 10.4029\,x^2 - 24.5576\,x^3 + 30.0517\,x^4 - 16.9888\,x^5 + 3.42808\,x^6$$

3rd degree Lagrange Polynomial:

$$f(x) = -0.841471 - 0.505227\,x + 3.66638\,x^2 - 1.69915\,x^3$$

Mathematica Program to calculate the Lagrange Polynomial of degree n

Lagrange Interpolation:

```
In[24]:= Clear[x, x1, x2, f1, f2, f3, fexact]

In[25]:= L[i_, xdata_, x_] := Module[{product, LagrangePol = 1},
           Do[If[k > i || k < i, product = (x - xdata[[k]]) / (xdata[[i]] - xdata[[k]]);
              LagrangePol = LagrangePol * product];
            , {k, 1, Length[xdata]}];
           LagrangePol]

In[26]:= xdata = {x0, x1, x2, x3, x4, x5, x6}
         ydata = {f0, f1, f2, f3, f4, f5, f6}

Out[26]= {x0, x1, x2, x3, x4, x5, x6}

Out[27]= {f0, f1, f2, f3, f4, f5, f6}

In[28]:= LagPoly[f_, n_] := Module[{polynomial}, polynomial = Sum[L[i, xdata, x] * f[[i]], {i, n}];
            polynomial];
```

Sample output is given for the polynomial of degree n=6

In[29]:= `P = LagPoly[ydata, 7]`

Out[29]=

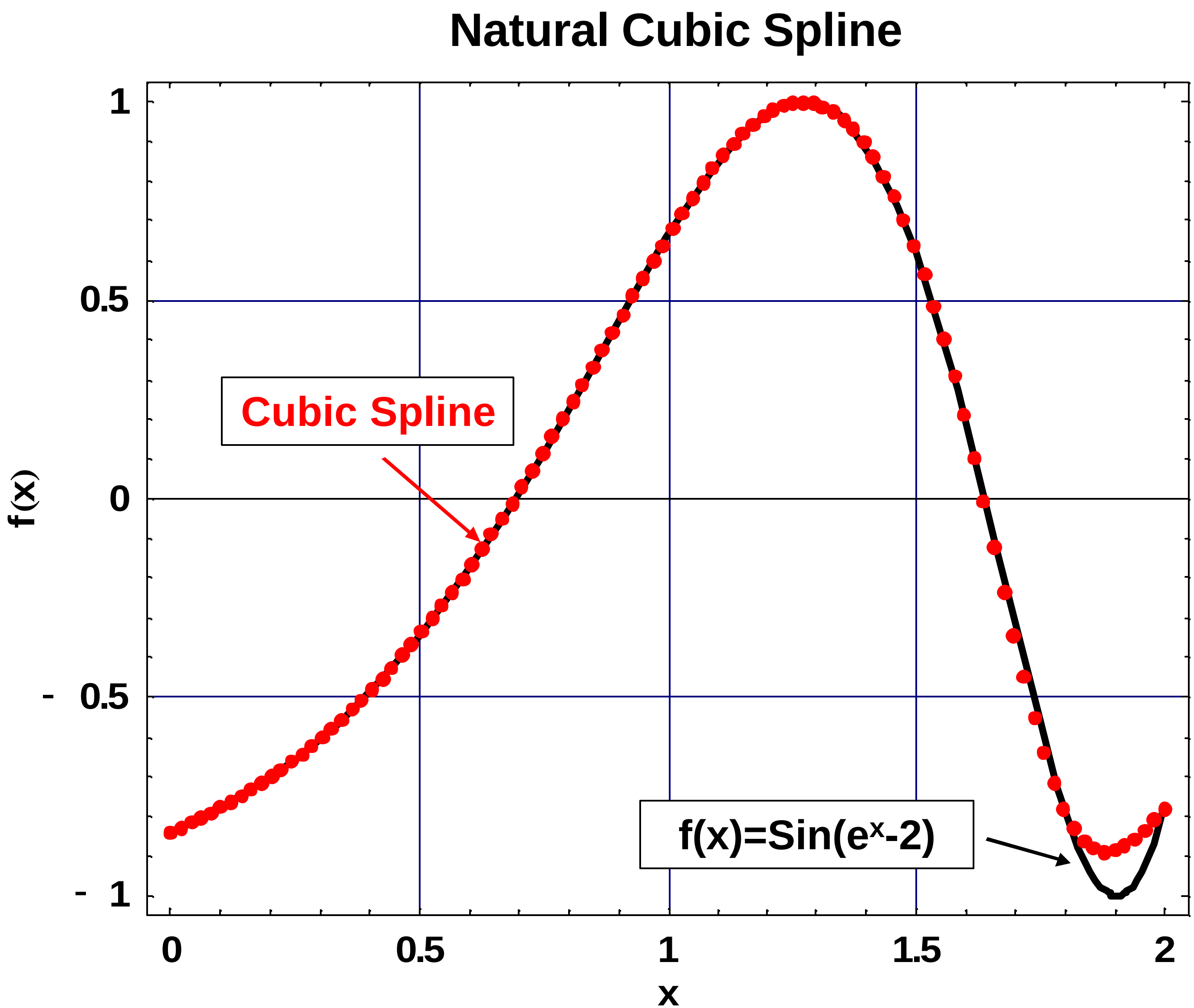
$$\frac{f0\,(x-x1)\,(x-x2)\,(x-x3)\,(x-x4)\,(x-x5)\,(x-x6)}{(x0-x1)\,(x0-x2)\,(x0-x3)\,(x0-x4)\,(x0-x5)\,(x0-x6)}+$$
$$\frac{f1\,(x-x0)\,(x-x2)\,(x-x3)\,(x-x4)\,(x-x5)\,(x-x6)}{(-x0+x1)\,(x1-x2)\,(x1-x3)\,(x1-x4)\,(x1-x5)\,(x1-x6)}+$$
$$\frac{f2\,(x-x0)\,(x-x1)\,(x-x3)\,(x-x4)\,(x-x5)\,(x-x6)}{(-x0+x2)\,(-x1+x2)\,(x2-x3)\,(x2-x4)\,(x2-x5)\,(x2-x6)}+$$
$$\frac{f3\,(x-x0)\,(x-x1)\,(x-x2)\,(x-x4)\,(x-x5)\,(x-x6)}{(-x0+x3)\,(-x1+x3)\,(-x2+x3)\,(x3-x4)\,(x3-x5)\,(x3-x6)}+$$
$$\frac{f4\,(x-x0)\,(x-x1)\,(x-x2)\,(x-x3)\,(x-x5)\,(x-x6)}{(-x0+x4)\,(-x1+x4)\,(-x2+x4)\,(-x3+x4)\,(x4-x5)\,(x4-x6)}+$$
$$\frac{f5\,(x-x0)\,(x-x1)\,(x-x2)\,(x-x3)\,(x-x4)\,(x-x6)}{(-x0+x5)\,(-x1+x5)\,(-x2+x5)\,(-x3+x5)\,(-x4+x5)\,(x5-x6)}+$$
$$\frac{f6\,(x-x0)\,(x-x1)\,(x-x2)\,(x-x3)\,(x-x4)\,(x-x5)}{(-x0+x6)\,(-x1+x6)\,(-x2+x6)\,(-x3+x6)\,(-x4+x6)\,(-x5+x6)}$$

Equally spaced 7 data points and corresponding function values

```
In[31]:= n = 7;
         xdata2 = Table[0 + (i - 1) * 2. / (n - 1), {i, n}]
         ydata2 = Table[fexact[xdata2[[i]]], {i, n}]

Out[32]= {0, 0.333333, 0.666667, 1., 1.33333, 1.66667, 2.}

Out[33]= {-0.841471, -0.568258, -0.0522422, 0.658092, 0.975267, -0.152302, -0.779664}

In[34]:= P /. Thread[xdata → xdata2];
         P6x = % /. Thread[ydata → ydata2];
         Print["f(x)=", Expand[%]]
```

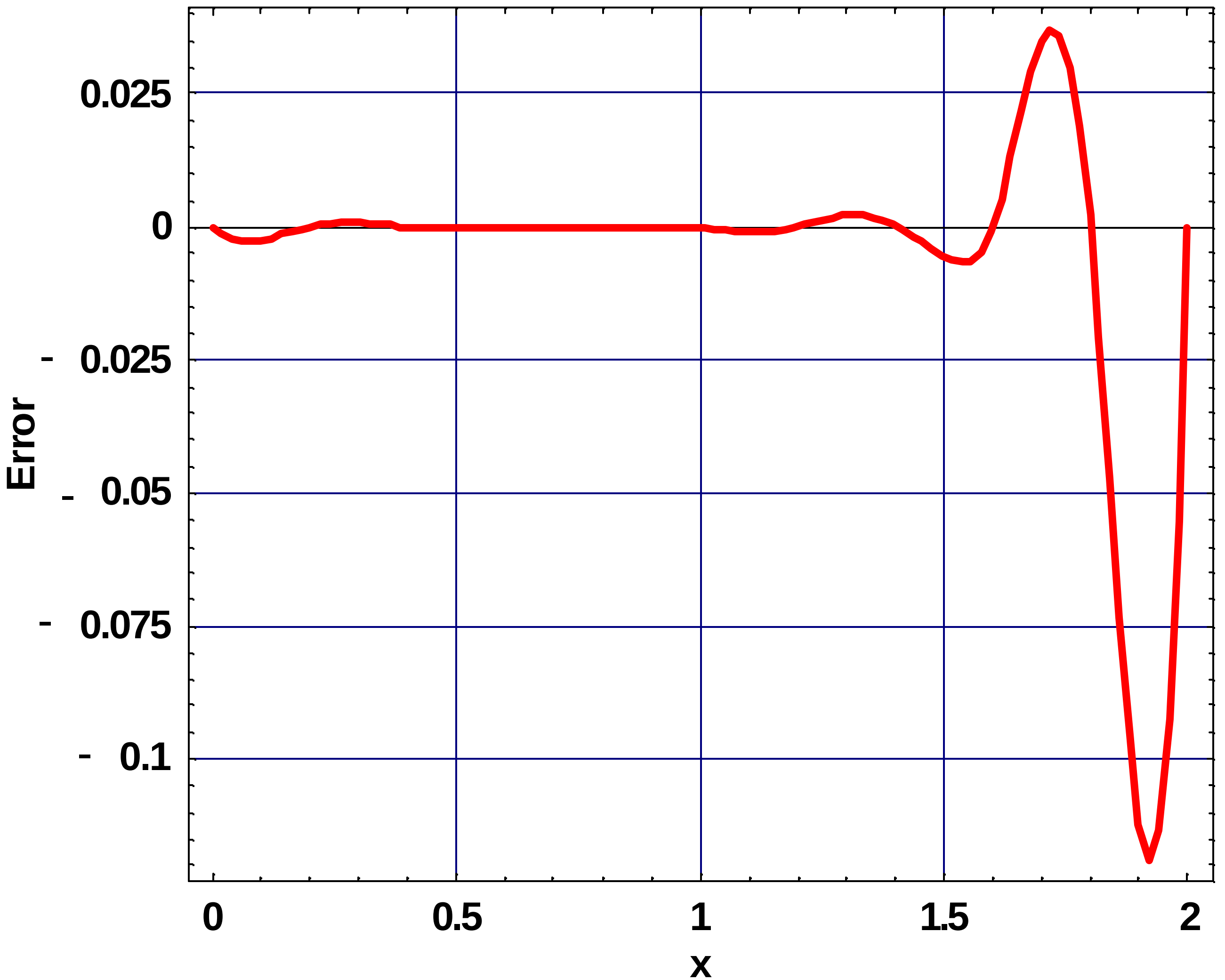
$f(x) = -0.841471 - 0.836756x + 10.4029x^2 - 24.5576x^3 + 30.0517x^4 - 16.9888x^5 + 3.42808x^6$

Question 2.

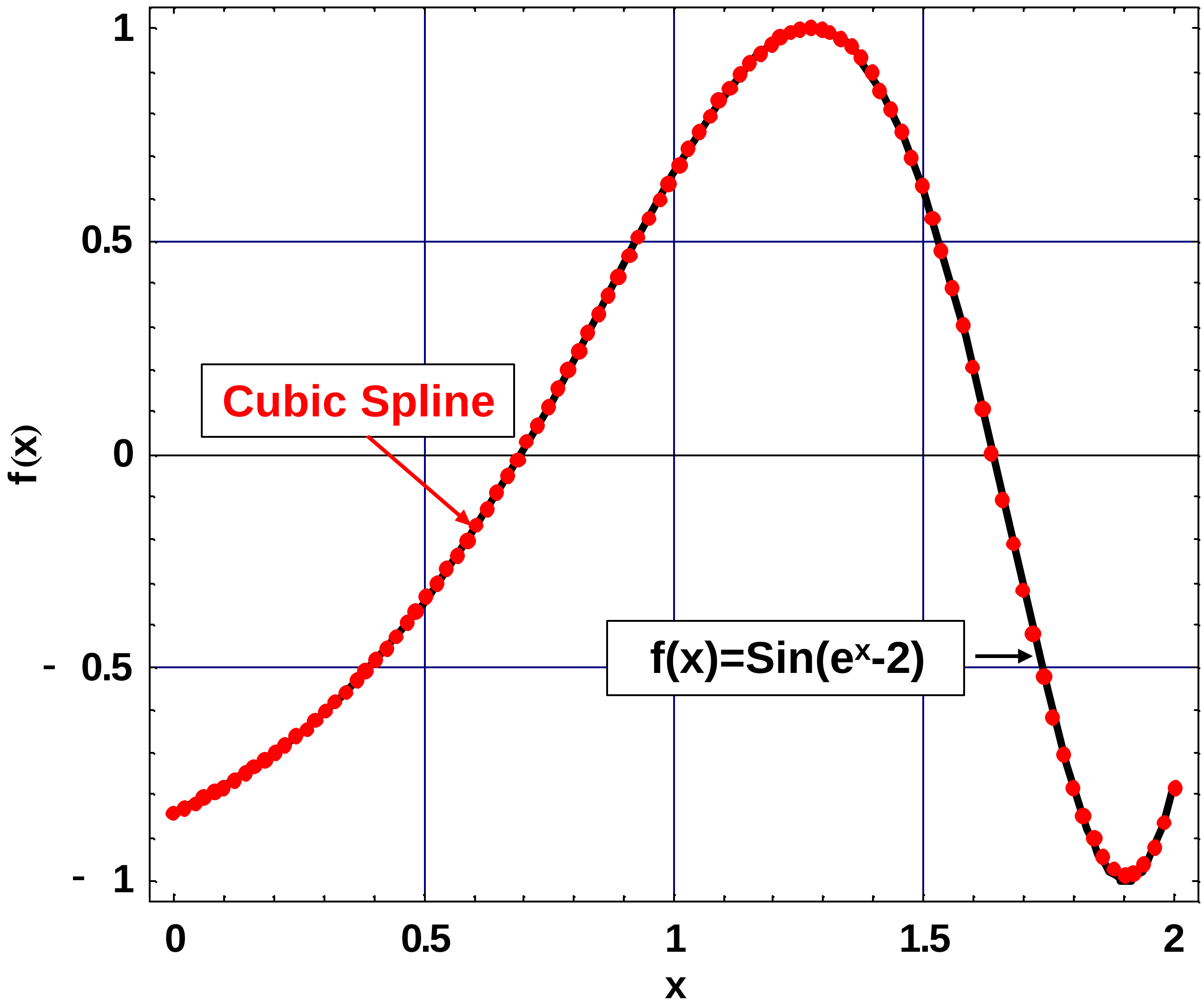


Natural Cubic Spline
1
0.5
0
- 0.5
- 1
f(x)
Cubic Spline
f(x)=Sin(e^x-2)
0
0.5
1
1.5
2
x

Error for Natural Cubic Spline

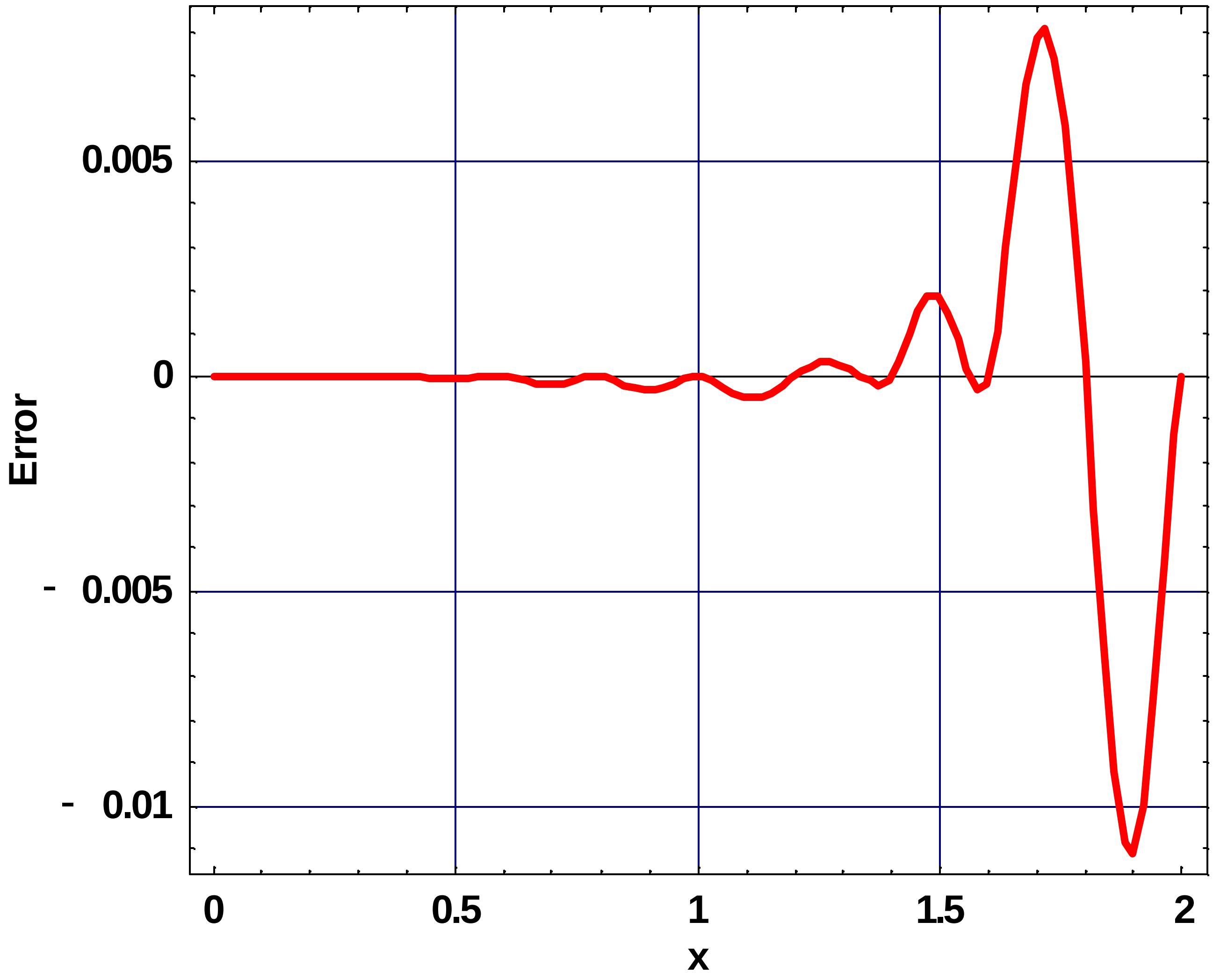


Cubic Spline with Clamped Ends (Extra info)



Note that when you use the clamped end boundary conditions, you get a better approximation at the right boundary. However, the derivative of the function may or may not be available in other problems

Error for Cubic Spline with Clamped Ends (Extra info)



Error values for the cubic spline approximation :

Boundary Condition	**x=0.1**	**x=0.9**	**x=1.5**	**x=1.9**
natural	-0.00250403	-0.00022346	-0.00553973	-0.112989
clamped	0.0000108384	-0.000351246	0.0017765	-0.0110839

QUESTION 3 : Use method of undetermined coefficients:

Expand $f_{i+1}, f_{i-1}, f_{i+2}$, and f_{i-2} into Taylor series about x_i :

$$(1) \quad f_{i+1} = f_i + hf_i' + \frac{h^2}{2}f_i'' + \frac{h^3}{6}f_i''' + \frac{h^4}{24}f_i^{(4)} + \frac{h^5}{120}f_i^{(5)} + O(h^6)$$

$$(2) \quad f_{i-1} = f_i - hf_i' + \frac{h^2}{2}f_i'' - \frac{h^3}{6}f_i''' + \frac{h^4}{24}f_i^{(4)} - \frac{h^5}{120}f_i^{(5)} + O(h^6)$$

$$(3) \quad f_{i+2} = f_i + 2hf_i' + 2h^2f_i'' + \frac{4h^3}{3}f_i''' + \frac{2h^4}{3}f_i^{(4)} + \frac{4h^5}{15}f_i^{(5)} + O(h^6)$$

$$(4) \quad f_{i-2} = f_i - 2hf_i' + 2h^2f_i'' - \frac{4h^3}{3}f_i''' + \frac{2h^4}{3}f_i^{(4)} - \frac{4h^5}{15}f_i^{(5)} + O(h^6)$$

Now multiply (2) by α ; (3) by γ and (4) by β.

$(1) + \alpha(2) + \gamma(3) + \beta(4) \Rightarrow$

$$f_{i+1} + \alpha f_{i-1} + \gamma f_{i+2} + \beta f_{i-2} = (1+\alpha+\gamma+\beta) f_i +$$

$$(1-\alpha+2\gamma-2\beta)hf_i' + \left(\frac{1}{2}+\frac{\alpha}{2}+2\gamma+2\beta\right)h^2f_i'' +$$

$$\left(\frac{1}{6}-\frac{\alpha}{6}+\frac{4\gamma}{3}-\frac{4\beta}{3}\right)h^3f_i''' + \left(\frac{1}{24}+\frac{\alpha}{24}+\frac{2\gamma}{3}+\frac{2\beta}{3}\right)h^4f^{(}$$

$$+\left(\frac{1}{120}-\frac{\alpha}{120}+\frac{4\gamma}{15}-\frac{4\beta}{15}\right)h^5f_i^{(5)} + O(h^6) \qquad (5)$$

In order to have a finite difference approximation to f_i' with an order of accuracy of $O(h^4)$, the coefficients of f_i'', f_i''', and $f_i^{(4)}$ should be equal to zero:

So $$\frac{1}{2}+\frac{\alpha}{2}+2\gamma+2\beta=0 \qquad (6)$$

$$\frac{1}{6}-\frac{\alpha}{6}+\frac{4\gamma}{3}-\frac{4\beta}{3}=0 \qquad (7)$$

$$\frac{1}{24}+\frac{\alpha}{24}+\frac{2\gamma}{3}+\frac{2\beta}{3}=0 \qquad (8)$$

From (6) $$\alpha+1=-4\gamma-4\beta \qquad (9)$$

Re-writing (8) using (9)

$$\frac{1}{24}+\frac{\alpha}{24}+\frac{2}{3}\left(\frac{-\alpha-1}{4}\right)=0$$

$$\frac{1}{24}+\frac{\alpha}{24}-\frac{\alpha}{6}-\frac{1}{6}=0$$

$$-\frac{3}{24}\alpha=\frac{3}{24} \Rightarrow \boxed{\alpha=-1} \qquad (10)$$

Re-writing (6) $\Rightarrow$ $\gamma+\beta=0 \Rightarrow \gamma=-\beta$

(7) $\Rightarrow$ $$\frac{1}{3}+\frac{8\gamma}{3}=0 \Rightarrow \boxed{\begin{array}{l}\gamma=-1/8\\ \beta=1/8\end{array}} \qquad (11)$$

Substituting the values of

α, γ, and β in equation (5):

$$f_{i+1} - f_{i-1} - 1/8\, f_{i+2} + \frac{1}{8} f_{i-2} = \left(1 + 1 - \frac{1}{4} - \frac{1}{4}\right) h f_i'$$

$$+ \left(\frac{1}{120} + \frac{1}{120} - \frac{4}{120} - \frac{4}{120}\right) h^5 f_i^{(5)} + O(h^6)$$

$$\frac{-f_{i+2} + 8 f_{i+1} - 8 f_{i-1} + f_{i-2}}{8} = \frac{6}{4} h f_i' - \frac{h^5}{20} f_i^5 + \cdots$$

Finally;

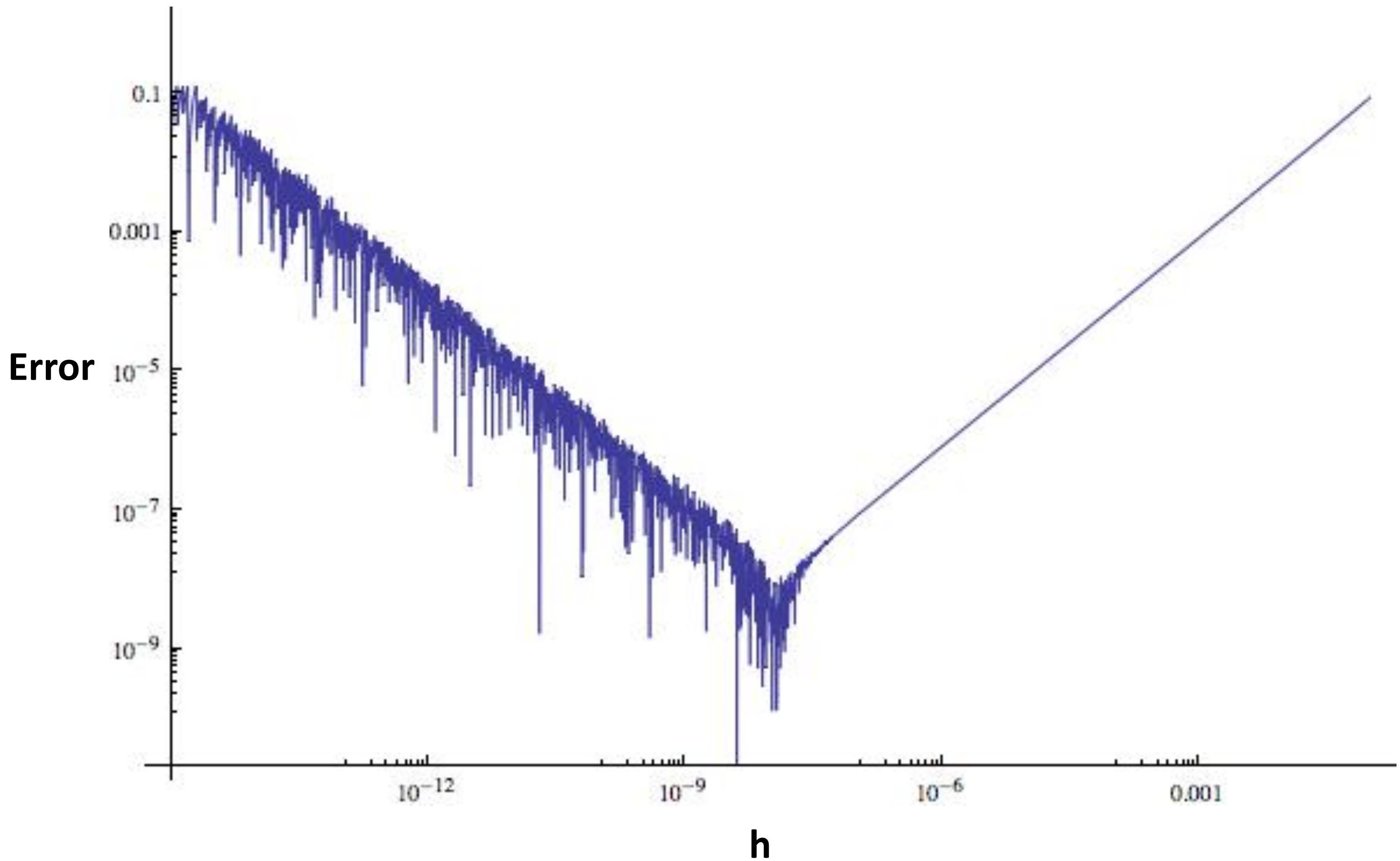
$$f_i' = \underbrace{\frac{-f_{i+2} + 8 f_{i+1} - 8 f_{i-1} + f_{i-2}}{12h}}_{\text{Approximation}} + \underbrace{\frac{h^4}{30} f_i^{(4)}}_{\text{Leading Term of the truncation error}} + \cdots$$

Error Table

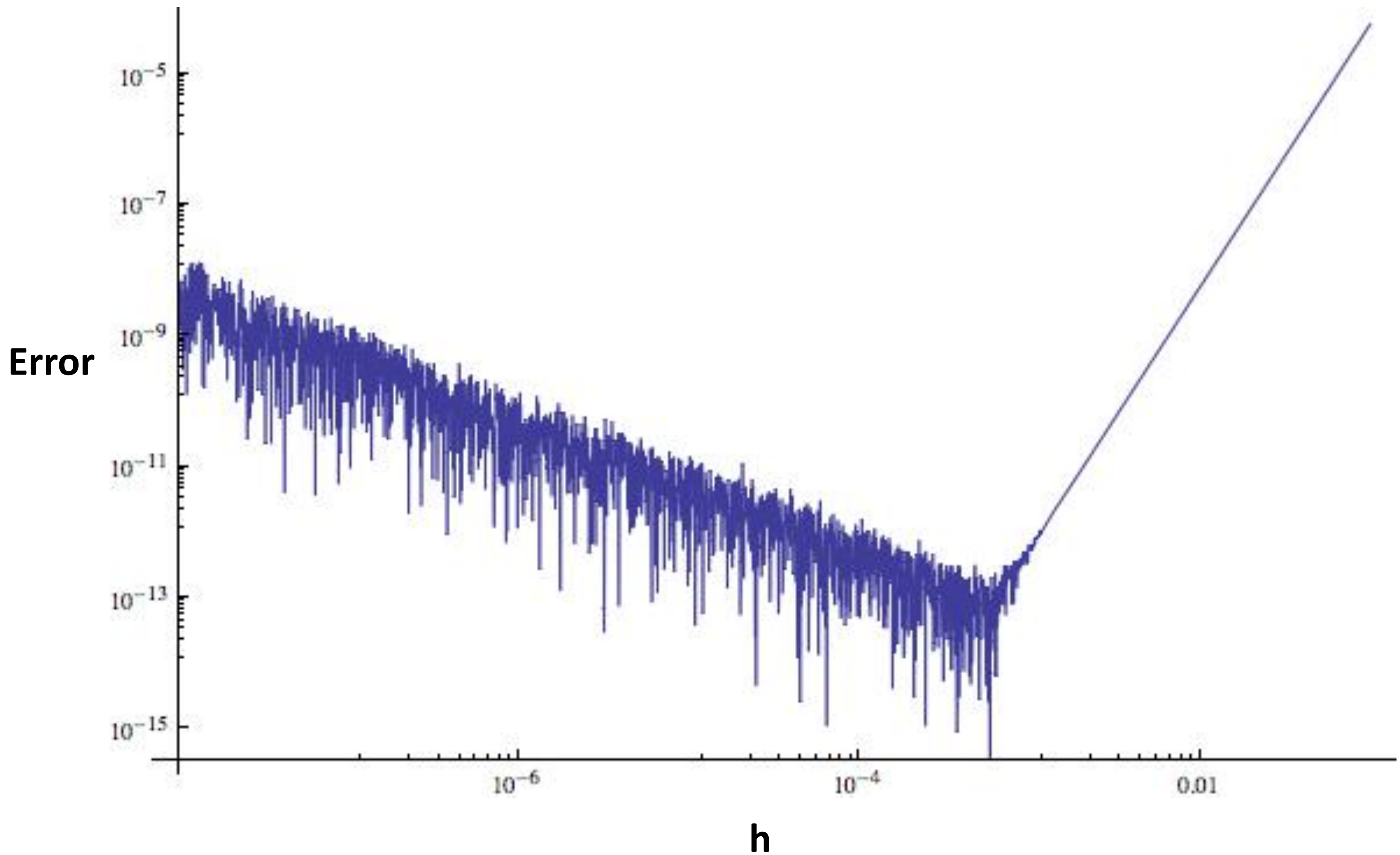
h	Error (1st Order)	Error (4^{th} Order)
10^{-1}	8.67541×10^{-2}	5.74867×10^{-5}
10^{-2}	8.26649×10^{-3}	5.58038×10^{-9}
10^{-3}	8.22587×10^{-4}	5.57554×10^{-13}
10^{-4}	8.22181×10^{-5}	3.35509×10^{-13}
10^{-5}	8.22139×10^{-6}	4.10538×10^{-12}
10^{-6}	8.22042×10^{-7}	6.99095×10^{-11}
10^{-7}	8.07461×10^{-8}	3.00165×10^{-10}
10^{-8}	9.18195×10^{-9}	9.18195×10^{-9}
10^{-9}	1.20204×10^{-7}	4.61894×10^{-8}
10^{-10}	1.23043×10^{-6}	2.4987×10^{-7}

Comment: As expected, the reduction in error is consistent with the order of each method. The round-off error becomes important at small mesh sizes, so the error starts to increase after a certain grid size for each method. The optimal grid size for the first order approximation is approximately 10^{-8} whereas the optimal grid size for the fourth order is around 10^{-3} (see also the plots in the next two pages).

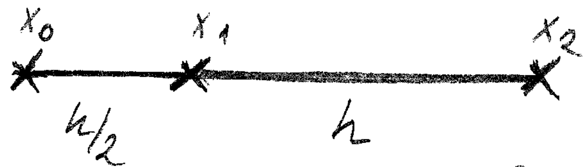
Error vs. Step Size (1st Order Approximation)



Error vs. Step Size (4th Order Approximation)



QUESTION 4)



Let $f(x_0) = f_i$ $\quad f(x_0 + h/2) = f_{i+1}$ & $f(x_0 + \frac{3h}{2}) = f_{i+2}$

Write Taylor Series expansion for f_{i+1} and f_{i+2} about $x_i = x_0$:

$$f_{i+1} = f_i + \frac{h}{2} f_i' + \frac{h^2}{8} f_i'' + \frac{h^3}{48} f_i''' + \ldots \quad (1)$$

$$f_{i+2} = f_i + \frac{3h}{2} f_i' + \frac{9h^2}{8} f_i'' + \frac{27h^3}{48} f_i''' + \ldots \quad (2)$$

Part a) To obtain a 2^{nd} order accurate one-sided finite difference formula to approximate $\left(\frac{df}{dx}\right)_i$, multiply (1) with -9 and add to (2) $\Rightarrow$

$$-9(\text{Eqn } 1) + \text{Eqn} 2 \Rightarrow$$

$$-9f_{i+1} = -9f_i - \frac{9h}{2} f_i' - \frac{9h^2}{8} f_i'' - \frac{9h^3}{48} f_i''' - \ldots$$

$$+ \quad f_{i+2} = f_i + \frac{3h}{2} f_i' + \frac{9h^2}{8} f_i'' + \frac{27h^3}{48} f_i''' + \ldots$$

$$-9f_{i+1} + f_{i+2} = -8f_i - 3h f_i' + \frac{3}{8} h^3 f_i''' + \ldots \quad (3)$$

$$f_i' = \underbrace{\frac{-f_{i+2} + 9f_{i+1} - 8f_i}{3h}}_{\text{Approximation}} + \underbrace{\frac{1}{8} h^2 f_i'''}_{\text{Leading term of the TE}} + \ldots \quad (4)$$

Part b) To obtain a first order accurate one-sided finite difference formula to approximate $\left(\frac{d^2f}{dx^2}\right)_i$ multiply Eqn(1) with -3 and add to (2) $\Rightarrow$

$-3(\text{Eqn } 1) + \text{Eqn } 2 \Rightarrow$

$$-3f_{i+1} = -3f_i - \frac{3h}{2}f_i' - \frac{3h^2}{8}f_i'' - \frac{3h^3}{48}f_i''' - \cdots \quad (5)$$

$$f_{i+2} = f_i + \frac{3h}{2}f_i' + \frac{9h^2}{8}f_i'' + \frac{27h^3}{48}f_i''' + \cdots \quad (6)$$

$+$

$$-3f_{i+1} + f_{i+2} = -2f_i + \frac{3h^2}{4}f_i'' + \frac{h^3}{2}f_i''' + \cdots \quad (7)$$

$$f_i'' = \underbrace{\frac{4f_{i+2} - 12f_{i+1} + 8f_i}{3h^2}}_{\text{Approximation}} \underbrace{- \frac{2}{3}hf_i''' - \cdots}_{\text{Leading term of the TE.}} \quad (8)$$