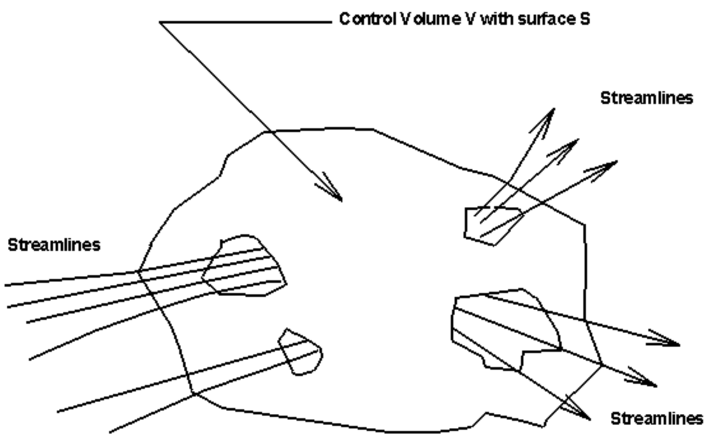
AE 5535

Spring 2021

Define an enclosed control volume (C.V.) containing only fluid which is fixed in space with arbitrary shape and volume (V) and with surface S (see sketch below). Fluid is contained within this control volume; fluid is also passing through at least some of the surfaces (into and/or out of the C.V.) As an illustration from the standpoint of the analysis of an entire aerospace engine flowpath, the surface boundaries of the control volume of interest would then be most usefully defined as enveloping all internal wetted surfaces (interfaces between solid structure and fluid) of the flowpath and cutting through all inlet and exit planes and fuel injection ports opening into the flowpath. Alternatively, the wetted surfaces of the defined control volume could extend back into the fuel/propellant system (lines and tanks). This is (or can be) an inclusive definition of a fluid control volume also encompassing (possibly) turbomachinery within the flowpath, etc.



I. Subdivide this overall C.V. into many *differential* elements (or differential control volumes) such that the mass of each differential C.V. is, by definition, ρdV, where dV is the volume of the differential C.V. and ρ is the local density of the fluid at the center of the *differential* C.V.

Therefore, the total mass of the fluid inside the overall C.V. at an instant, t, is

$$\iiint_V \rho dV$$

II. Consider a differential surface element with area *dS* on the overall surface, S, of the overall C.V. This surface element has a unit directional vector, $\hat{n}$, defined as directed outward from the C.V. Note that $\hat{n}$ is by definition perpendicular $\left(\perp\right)$ to the local surface area, *dS.*

Recall that may or may not be fluid passing through this differential surface element with local velocity vector, $\vec{V}$. This is because the side surface of the defined control volume may lie along a solid surface (say of a vehicle) or the fluid may be flowing parallel to the surface at that particular point.

The component of the velocity vector perpendicular ($\perp$) to the local surface area, dS, is, by definition, $\vec{V} \cdot \hat{n}$ (dot product). It is this component of $\vec{V}$ which determines the amount of mass sweeping through dS (obviously, the component of $\vec{V}$ parallel or tangential to the local surface area does not contribute to any mass sweeping in or out of local surface per unit time). Hence, the differential mass flow rate (kg/sec) passing out of the C.V. through dS is, by definition, equal to $\rho\vec{V} \cdot \hat{n}dS$. Therefore, the total net mass per second passing out of the control volume through S is the sum of all differential mass flow rates through all differential surface elements or

$$\iint_S \rho\vec{V} \cdot \hat{n}dS$$

Note that any solid surfaces do not contribute at all to this term since either 1) the velocity is zero on the surface (true no-slip condition on solid surface) or 2) the fluid velocity is by definition tangential to the solid surface (an 'inviscid' wall boundary condition sometimes used in modeling fluid flow where friction is neglected) such that $\vec{V} \cdot \hat{n} = 0$.

III. Consider a statement that must be true which describes the principle of 'conservation of mass' for this C.V.:

"The change in mass inside the overall C.V. per second must be equal to the net mass per second passing <u>into</u> the C.V through the surface S PLUS any mass being created per second inside the C.V."

This statement simply says that the mass that goes into the C.V. must either come out or is being collected inside the C.V. The last contribution (mass being created inside the C.V. is obviously zero in this universe.

Now, apply the expressions developed previously to this 'word' statement of conservation of mass and obtain the equation representation of conservation of mass:

$$\frac{\partial}{\partial t}\iiint_V \rho dV + \iint_S \rho\vec{V} \cdot \hat{n}dS = 0$$

IV. Momentum is a vector, i.e. $m\vec{V}$ is the vector representation of the momentum of a particle with mass, m, and velocity $\overline{V}$. (Recall $\overline{V} = u\hat{i} + v\hat{j} + w\hat{k}$ where u, v, and w are the velocity components in the x,y, and z directions, respectively. $\hat{i}, \hat{j}, \text{ and } \hat{k}$ are the unit normals in the x, y, and z coordinate directions, respectively).

Just as was done with the conservation of mass, formulate a 'word statement' which describes Newton's second law for the fluid in the overall C.V.:

"The change in momentum associated with the fluid inside the C.V. per second must be equal to the net momentum per second of the fluid passing into the C.V. through the surface plus the sum of all forces acting on the fluid inside the C.V."

V. The differential momentum of a differential element inside the C.V. is, by definition,

$$\rho\vec{V}dV.$$

The total momentum associated with the C.V. at an instant, t, is then

$$\iiint_V \rho\vec{V}dV .$$

Therefore the change in momentum associated with the fluid inside the C.V. per second is:

$$\frac{\partial}{\partial t}\iiint_V \rho\vec{V}dV .$$

Following the development in the conservation of mass equation in which the differential flow rate of mass through dS was defined as $\rho\vec{V}\cdot\hat{n}dS,$ the differential flow rate of momentum through dS out of the C.V. is defined as

$$\rho\vec{V}\left(\vec{V}\cdot\hat{n}\right)dS.$$

Hence, the net momentum per second of the fluid passing into the control volume is given by

$$-\iint_S \rho\vec{V}\left(\vec{V}\cdot\hat{n}\right)dS.$$

VI. Forces acting on the C.V.:

These are two basic types of forces acting on the fluid inside the C.V.: 1) body forces which affect the collective mass contained within the control volume (i.e. gravity, electro-magnetic, etc.) and 2) surface forces (composed of pressure forces which are perpendicular to the local surface element and hence collinear but opposite in sign from the $\hat{n}$ vector and shear forces which are tangential to the local surface). Note that pressure and shear forces exist on any defined surfaces *inside* the C.V. but internally cancel everywhere such that they need only be considered on the surface (S) of the overall C.V. Depending on the selection of the C.V., there could also be structural (hard-point) forces ($\vec{F}_{shaft}$) due to the control volume possibly cutting through shafts, etc. In such a case, the wetted area surface would not actually include the internal blade surfaces, etc.

A. Consider the differential surface *dS* with pressure, P, acting inward (by definition). The differential force, $d\vec{F}_P$ associated with this pressure is:

$d\vec{F}_P = -P\hat{n}dS$ (The minus sign arises because $\hat{n}$ is directed outward by definition while pressure is directed inward).

The overall force due to pressure on surface *S* is thus equal to

$$\iint_S d\vec{F}_P = -\iint_S P\hat{n}dS$$

B. Now consider the shear force on the fluid at the control volume surface due to shear stress, $\vec{\tau}_s$, acting on the fluid tangential to the surface (by definition). This shear stress is thought of usually as acting in a direction opposite to the velocity vector (since a solid surface pulls back on the fluid), however in the general case represents a general shearing force that could be either direction with respect to the local velocity vector. Note then that this is usually defined as negative of the wall shear stress referenced to a solid boundary, i.e. $\vec{\tau}_s = -\vec{\tau}_w$ in common notation.

The differential force, $d\vec{F}_S$, associated with the shear stress vector $(\vec{\tau}_W)$, is $d\vec{F}_S = \vec{\tau}_S dS$, so the overall force due to shear on surface *S* is equal to

$$\iint_S d\vec{F}_S = \iint_S \vec{\tau}_S dS.$$

C. Consider body forces as composed of forces due to gravity, electro-magnetic fields, etc. They can be defined <u>per unit mass</u> so that the differential force due to body forces acting on a differential volume element inside the overall C.V. is:

$d\overline{F_B} = \rho \vec{f}_B dV$ where $\vec{f}_B$ is the body force vector per unit mass

Hence, the overall force due to body forces acting on the mass inside the C.V. is:

$$\iiint_V d\overline{F_B} = \iiint_V \rho \overline{f_B} dV$$

VII. Now, applying all expressions in V and VI to the 'word statement' of Newton's momentum equation (see IV), the following equation representation of the vector momentum equation (describing 3 components) is obtained:

$$\frac{\partial}{\partial t}\iiint_V \rho\vec{V}dV + \iint_S \rho\vec{V}(\vec{V}\cdot\hat{n})dS = -\iint_S P\hat{n}\,dS + \iint_S \vec{\tau}_s dS + \iiint_V \rho\vec{f}_B\ dV + \vec{F}_{shaft}$$

VIII. The first law of thermodynamics states that the change in energy of a system is equal to the sum of the work done to and the heat added to that system. Note that the work and heat interactions are the sum of the energy received from the surroundings of the system (i.e. across the boundaries of the system). The total energy of the fluid inside the control volume has both internal and directed-kinetic energy components.

Internal energy is composed of contributions due to

a) molecular translation (random, i.e. thermal, kinetic energy 'KE' of individual fluid molecules contained within a fluid element itself moving at a directed velocity $\overline{V}$),
b) vibrational energy,
c) rotational energy,
d) electronic energy.

The directed-kinetic energy is due to the overall superimposed bulk velocity $\overline{V}$ of a given fluid element. Therefore the total energy of a differential fluid element, dE_t , is the sum of the internal and this directed kinetic energy:

$$dE_t = \rho e_t dV = \rho e dV + \rho\left(\frac{\vec{V}\cdot\vec{V}}{2}\right)dV$$

Word statement of the 1st Law for the C.V. with fluid flow:

The (instantaneous) change in total energy of the fluid inside the C.V. per second is equal to:

i) net total energy per second in the fluid passing into the C.V. through surfaces S +
ii) the heat per second added to the fluid across S by conduction/convection +

iii) heat added per second to the fluid in the C.V. by radiation +
iv) work done on fluid at C.V surfaces per second by pressure and shear forces +
v) shaft work done per second on fluid inside the C.V.+
vi) work done per second by body forces

IX. The time rate term is (see momentum, continuity equations)

$$\frac{\partial}{\partial t}\iiint_V \rho\left\{e+\frac{\vec{V}\cdot\vec{V}}{2}\right\}dV$$

i) The energy flow rate term (out of C.V.) is (see momentum, continuity):

$$\iint_S \rho\left\{e+\frac{\vec{V}\cdot\vec{V}}{2}\right\}(\vec{V}\cdot\hat{n})dS$$

ii) The conduction/convection term is $\iint_S \dot{Q}_{cond} dS$ where $\dot{Q}_{cond}$ is the local conduction heat transfer rate per unit area (into the C.V.)

iii) The radiation term is $\dot{Q}_{rad}$; this is the heat interaction to fluid mass in C.V. per second due to radiation.

iv) a) work rate done on the fluid on surfaces of the control volume by pressure forces is

$\iint_S d\dot{W}_P$. Here $d\dot{W}_P$ is the force magnitude associated with pressure multiplied by the distance the fluid element moves in direction of pressure (per second), or

$$d\dot{W}_P = -(PdS)\cdot(\vec{V}\cdot\hat{n})$$

or

$$\iint_S d\dot{W}_P = -\iint_S P(\vec{V}\cdot\hat{n})dS.$$

b) work rate done on the fluid on surfaces of C.V. by shearing forces is

$\iint_S d\dot{W}_{shear}$. Here $d\dot{W}_{shear} = (\vec{\tau}_s\cdot\vec{V})dS$.

Therefore $\iint_S d\dot{W}_{shear} = \iint_S (\vec{\tau}_s\cdot\vec{V})dS$.

v) $\dot{W}_{shaft}$ (work interaction per second to fluid mass in C.V. due to shaft work from the surroundings) – actually equal to $\vec{F}_{shaft} \cdot \vec{V}_{rotation\ at\ S}$ for mechanical turbomachinery.

vi) Body force work rate term = $\iiint_V \rho \vec{f}_b \cdot \vec{V} dV$.

X. Applying these expressions to the 'word' definition of the 1st Law (called also the energy equation), the following is obtained:

$$\frac{\partial}{\partial t}\iiint_V \rho\left\{e+\frac{\vec{V}\cdot\vec{V}}{2}\right\}dV+\iint_S \rho\left\{e+\frac{\vec{V}\cdot\vec{V}}{2}\right\}(\vec{V}\cdot\hat{n})dS =$$
$$-\iint_S P(\vec{V}\cdot\hat{n})dS+\iint_S (\vec{\tau}_s\cdot\vec{V})dS+\iiint_V \rho\vec{f}_b\cdot\vec{V}dV+\dot{Q}_{rad}+\dot{W}_{shaft}+$$
$$\iint_S \dot{Q}_{cond}\,dS.$$

Now, $e+\frac{P}{\rho}=h$ (enthalpy) and $h_{total}=h+\frac{\vec{V}\cdot\vec{V}}{2}= h_t$; hence another version of this equation (neglecting body forces) is:

$$\frac{\partial}{\partial t}\iiint_V \rho\left\{e+\frac{\vec{V}\cdot\vec{V}}{2}\right\}dV+\iint_S \rho h_t(\vec{V}\cdot\hat{n})dS = \iint_S (\vec{\tau}_s\cdot\vec{V})dS+\dot{Q}_{tot}+\dot{W}_{shaft}$$

where $\dot{Q}_{tot} = \dot{Q}_{rad} + \iint_S \dot{Q}_{cond}\,dS$.

If the defined C.V. surfaces actually 'wrap' turbomachinery blades, etc., then $\vec{V}\cdot\hat{n}$ and $\vec{\tau}_s\cdot\vec{V}$ terms on the blade surfaces use the velocity relative to the boundaries (i.e. these terms are zero). Furthermore, for this case the shaft work rate must be modeled on the blade surface as:

$$\dot{W}_{shaft} = -\iint_S P(\vec{V}_{bound}\cdot\hat{n})\,ds+\iint_S \vec{\tau}_w\cdot\vec{V}_{bound}\,dS$$

SUMMARY:

For an arbitrary 'engine-based' control volume, 3 basic (integral) equations have been developed:

Continuity:

$$\frac{\partial}{\partial t}\iiint_V \rho dV + \iint_S \rho\vec{V}\cdot\hat{n}dS = 0$$

Momentum (vector equation with three components, without shaft force considerations):

$$\frac{\partial}{\partial t}\iiint_V \rho\vec{V}dV + \iint_S \rho\vec{V}(\vec{V}\cdot\hat{n})dS = -\iint_S P\hat{n}\,dS + \iint_S \vec{\tau}_s dS + \iiint_V \rho\vec{f}_B\,dV$$

Energy:

$$\frac{\partial}{\partial t}\iiint_V \rho\left\{e + \frac{\vec{V}\cdot\vec{V}}{2}\right\}dV + \iint_S \rho h_t\left(\vec{V}\cdot\hat{n}\right)dS = \iint_S \left(\vec{\tau}_s\cdot\vec{V}\right)dS + \dot{Q}_{tot} + \dot{W}_{shaft}$$

Also, note that the equation of state for a thermally perfect gas ($P = \rho RT$) is valid everywhere in the fluid.

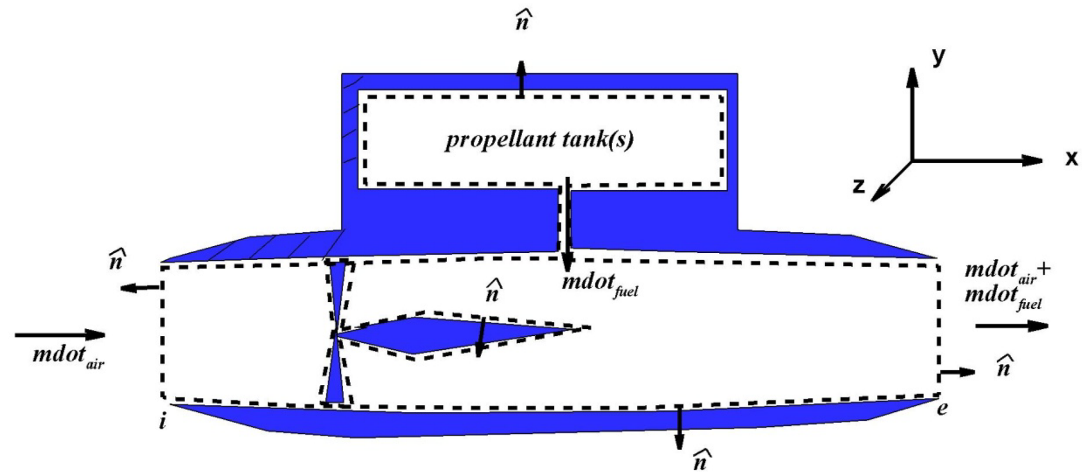
For discretized control volumes (differential control volumes), the energy equation needs to be modified to include additional species and thermal diffusion terms. Individual species continuity equations would also have to be described and would include species production rates and diffusion flux terms. For further details see Williams [1] and for a summary of the full differential equations (written in partial differential form) see [2].

[1] Williams, Combustion Theory
[2] Anderson, Computational Fluid Dynamics

Development of Thrust Relationships - 2021

What is the axial force (force in the x direction, see figure below) on all internal solid wetted surfaces in the engine including vehicle surfaces inside fuel lines and propellant tanks? Let this axial force be designated F_A.

Define a control volume (CV) as shown in which the boundaries of the CV 'wrap' all internal wetted surfaces of the engine AND the inlet and exit planes (i and e) and define $\hat{n}$ as the unit normal directed outward from the fluid on all boundaries of the control volume. For convenience here (in the following discussion), let $\hat{n}$ at stations i and e (entrance and exit) be co-linear with the x axis such that $\hat{n}_x$ across the control surface portion at station i is -1.0 and $\hat{n}_x$ across the control surface portion at station e is 1.0. (In other words, the inlet and exit faces of the engine are aligned with the y-z plane). Wetted surfaces refer to all boundaries between solid (engine structure) and fluid.



So, by definition:

$$F_A = \iint_{S_{wetted}} P\hat{n}_x dS + \iint_{S_{wetted}} \tau_{wx} dS$$

τ_{wx} is the axial component of $\vec{\tau}_w$ (force vector on solid walls due to fluid motion (shear). (1)

Important note: S_{wetted} does NOT include the portions of the control volume surface at inlet and exit planes of the engine (i and e).

General momentum equation for fluid flow in a control volume with volume V and surface S is:

$$\frac{\partial}{\partial t}\iiint_V \rho\vec{V}dV + \iint_{S_{cv}} \rho\vec{V}(\vec{V}\cdot\hat{n})dS = -\iint_{S_{cv}} P\hat{n}dS - \iint_{S_{cv}} \vec{\tau}_w dS \qquad (2)$$

The velocity vector, the unit normal vector, and the shear stress vector are defined as follows:
$\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$
$\hat{n} = \hat{n}_x\hat{\imath} + \hat{n}_y\hat{\jmath} + \hat{n}_z\hat{k}$
$\vec{\tau}_w = \tau_{w,x}\hat{\imath} + \tau_{w,y}\hat{\jmath} + \tau_{w,z}\hat{k}$

There are three components of this general momentum equation:

$$\frac{\partial}{\partial t}\iiint_V \rho u dV + \iint_{S_{cv}} \rho u(\vec{V}\cdot\hat{n})dS = -\iint_{S_{cv}} P\hat{n}_x dS - \iint_{S_{cv}} \tau_{w,x} dS \qquad \text{x component}$$

$$\frac{\partial}{\partial t}\iiint_V \rho v dV + \iint_{S_{cv}} \rho v(\vec{V}\cdot\hat{n})dS = -\iint_{S_{cv}} P\hat{n}_y dS - \iint_{S_{cv}} \tau_{w,y} dS \qquad \text{y component}$$

$$\frac{\partial}{\partial t}\iiint_V \rho w dV + \iint_{S_{cv}} \rho w(\vec{V}\cdot\hat{n})dS = -\iint_{S_{cv}} P\hat{n}_z dS - \iint_{S_{cv}} \tau_{w,z} dS \qquad \text{z component}$$

Concentrating on the x momentum equation here, 'break' the surface integrals into three contributions: inflow, outflow, and wetted surfaces inside the engine.

For the 'momentum flux' integral:

$$\iint_{S_{cv}} \rho u(\vec{V}\circ\hat{n})dS = \iint_{A_e} \rho u^2\, dA_e - \iint_{A_i} \rho u^2\, dA_i + \underbrace{\iint_{S_{wetted}} \rho u(\vec{V}\circ\hat{n})dS}_{=0}$$

The last term is zero since that term is integrated on all solid (internal) wetted surfaces such that u and $\vec{V}$ are zero on that part of the control surface (as the control volume is constructed).

For the pressure term surface integral:

$$\iint_{S_{cv}} P\hat{n}_x\, dS = \iint_{A_e} P\hat{n}_x\, dA_e + \iint_{A_i} P\hat{n}_x\, dA_i + \iint_{S_{wetted}} P\hat{n}_x\, dS$$

Realizing that $\hat{n}_x$ at station i is -1 and that $\hat{n}_x$ at station e is 1, this becomes:

$$\iint_{S_{cv}} P\hat{n}_x\, dS = \iint_{A_e} P\, dA_e - \iint_{A_i} P\, dA_i + \iint_{S_{wetted}} P\hat{n}_x\, dS$$

Since generally $\tau_{w,x}$ terms are zero on the inflow and outflow (especially if flow is considered 'uniform' at these stations – i.e., no shear), write

$$\iint_{S_{cv}} \tau_{w,x}\, dS = \iint_{S_{wetted}} \tau_{w,x}\, dS$$

Combine and collect the momentum and pressure terms on inflow and outflow (and for convenience from here on use general single integral to denote surface integrals); obtain the following version of the x component of the momentum equation:

$$\frac{\partial}{\partial t}\iiint_V \rho u dV + \int_{A_e} (\rho u^2 + P) dA_e - \int_{A_i} (\rho u^2 + P) dA_i = -\int_{S_{wetted}} P\hat{n}_x dS - \int_{S_{wetted}} \tau_{w,x} dS \qquad (3)$$

Hence

$$-F_A = \frac{\partial}{\partial t}\iiint_V \rho u dV + \int_{A_e} (\rho u^2 + P) dA_e - \int_{A_i} (\rho u^2 + P) dA_i \qquad (4)$$

For steady engine operation (including back into the fuel/propellant lines and at least to the propellant tanks):

$$\frac{\partial}{\partial t}\iiint_V \rho u dV \approx 0 \qquad (5)$$

This is exactly true everywhere for steady pumping (pressurization) of fuel from non-pressurized tanks and even approximately true for pressurized tank systems since $\vec{V} \approx 0$ inside the tank anyway.

So (for steady engine operation) we can define the F_A (recall this is the axial force on all internal solid wetted surfaces of engine including inside the fuel/propellant system) as:

$$-F_A = \int_{A_e} (\rho u^2 + P) dA_e - \int_{A_i} (\rho u^2 + P) dA_i \qquad (6)$$

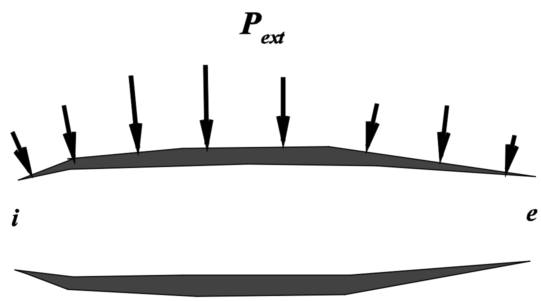
and since $Thrust_{internal-wetted-surfaces}$ is defined by convention as force developed in negative (-) x direction:

$$Thrust_{internal-wetted-surfaces} = \int_{A_e} (\rho u^2 + P) dA_e - \int_{A_i} (\rho u^2 + P) dA_i \qquad (7)$$

Further, if we assume that ρ, u, P are 'uniform' (do not vary) across inlet and exit planes, then

$$Thrust_{internal-wetted-surfaces} = (\rho_e u_e^2 + P_e) A_e - (\rho_i u_i^2 + P_i) A_i \qquad (8)$$

Now, there is an additional force **due to pressure** on the external surface of the 'cowl' of the engine (see sketch below), so if we expand the axial force definition to include this additional contribution then we can say that



F_A (external cowl and internal wetted surfaces)

$$= F_{A(ext+\text{int})} = F_A + \int_{\substack{external\\wetted\\surfaces}} P_{ext}\hat{n}_{x(ext)}dS \qquad (9)$$

$$= -\int_{A_e} (\rho u^2 + P)dA_e + \int_{A_i} (\rho u^2 + P)dA_i + \int_{\substack{external\\wetted\\surfaces}} P_{ext}\hat{n}_{x(ext)}dS$$

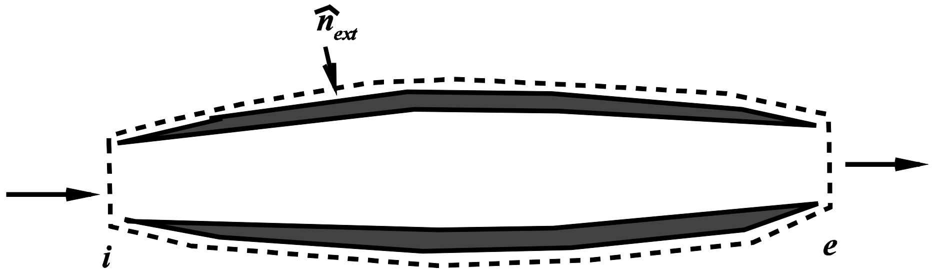
Also note that if $P_{ext} = P_i$ and uniform flow at i and e, then it can be shown that

$$F_{A(ext+\text{int})} = -\rho_e u_e^{\,2} A_e + \rho_i u_i^{\,2} A_i - (P_e - P_i)A_e \qquad (10)$$

Aside: How is this shown?

If we define a control volume which simply 'wraps' the external surfaces of the engine and covers the inlet and the exit plane (see sketch), fluid statics demands that

$$\iint_{S_{cv}} P_i \hat{n}_{x(ext)} dS = 0 \qquad (11)$$



(i.e. the force due to a fixed (non-varying) pressure on any closed surface = 0 by definition).

So we can write:

$$0 = \iint_{S_{cv}} P_i \hat{n}_{x(ext)} dS = P_i A_i - P_i A_e + \int_{\substack{ext \\ wetted \\ surfaces}} P_i \hat{n}_{x(ext)} dS \qquad (12)$$

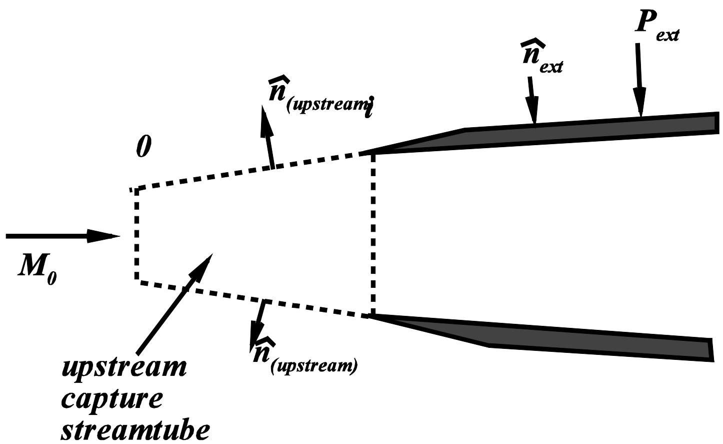
and

$$\begin{aligned} &F_{A(ext+\text{int})}(\text{if } P_{ext} = P_i \text{ and uniform flow at } i \text{ and } e) \\ &\qquad = -(\rho_e u_e^{\,2} + P_e) A_e + (\rho_i u_i^{\,2} + P_i) A_i - P_i A_i + P_i A_e \qquad (13) \\ &\qquad = \rho_i u_i^{\,2} A_i - \rho_e u_e^{\,2} A_e - (P_e - P_i) A_e \end{aligned}$$

so write

$$\begin{aligned} &Thrust(\text{uniform flow at } i \text{ and } e \text{ and } P_{ext} = P_i) \\ &\qquad = \rho_e u_e^{\,2} A_e - \rho_i u_i^{\,2} A_i + (P_e - P_i) A_e \end{aligned} \qquad (14)$$

Now, look at the ‘capture streamtube’ in front of the engine (in subsonic flow, captured fluid can be accelerating or decelerating from freestream, 0, to the engine face at i) – see Figure below:



Define a control volume in the fluid flow which encloses the captured streamtube as shown and then apply the steady x momentum equation just to the fluid in that (upstream) control volume.

$$\int_{A_i} (\rho u^2 + P)dA_i - \int_{A_0} (\rho u^2 + P)dA_0 = -\int_{\substack{sides\\ of\\ upstream-cv}} P_{side}\hat{n}_{x(upstream)}dS \tag{15}$$

and noting that we can define $\hat{n}_{x(ext)}$ from 0 to e (instead of just i to e) such that $\hat{n}_{x(upstream)} = -\hat{n}_{x(ext)}$ then

$$\int_{A_i} (\rho u^2 + P)dA_i = \int_{A_0} (\rho u^2 + P)dA_0 + \int_{\substack{sides\\ of\\ upstream-cv}} P_{side}\hat{n}_{x(ext)}dS \tag{16}$$

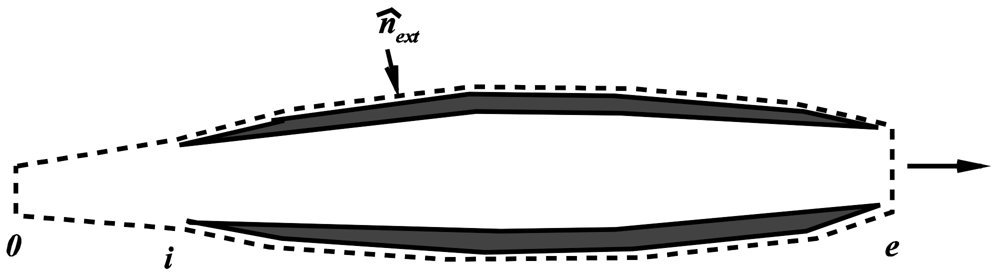
So, substituting into the former expression for F_A (external cowl and internal wetted surfaces)

$$F_{A(ext+\text{int})} = -\int_{A_e} (\rho u^2 + P)dA_e + \int_{A_0} (\rho u^2 + P)dA_0 + \int_{\substack{external\\ wetted\\ surfaces}} P_{ext}\hat{n}_x dS + \int_{\substack{sides\\ of\\ upstream-cv}} P_{side}\hat{n}_{x(ext)}dS \tag{17}$$

INSTALLED ENGINE THRUST:

Installed Engine Thrust = - $F_{A(ext+\text{int})}$ (18)

Noting finally that a control volume encompassing the freestream capture area (0), the sides of the upstream capture streamtube, the external surfaces of the cowl, and the exit plane (e) of the



engine can be drawn and that the net resultant force due to a constant P_0 on the surface of this control volume is zero via fluid statics (see sketch), or

$$\iint_{\substack{S_{closed}\\volume}} P_0 \hat{n}_x dS = 0\,. \tag{19}$$

This integral can then be added to the above expression for $F_{A(ext+\mathrm{int})}$ such that

Installed Engine Thrust =

$$\int_{A_e} [\rho u^2 + (P - P_0)] dA_e - \int_{A_0} [\rho u^2 + (P - P_0)] dA_0 - \int_{\substack{external\\wetted\\surfaces}} (P_{ext} - P_0)\hat{n}_x dS - \int_{\substack{sides\\of\\upstream\\cv}} (P_{side} - P_0)\hat{n}_{x(ext)} dS \tag{20}$$

Now let

$$\text{'External drag'} = D_{ext} = \int_{\substack{external\\wetted\\surfaces}} (P_{ext} - P_0)\hat{n}_x dS \tag{21}$$

$$\text{'Additive drag'} = D_{add} = \int_{\substack{sides\\of\\upstream-cv}} (P_{side} - P_0)\hat{n}_{x(ext)} dS \tag{22}$$

So, for uniform flow at engine exit, e, and free-stream, 0

Installed thrust (uniform flow *e* and *0*) $= = \rho_e u_e^{\,2} A_e - \rho_0 u_0^{\,2} A_0 + (P_e - P_0) A_e - D_{ext} - D_{add}$ (23)

If we neglect $D_{ext} + D_{add}$ then

UNINSTALLED THRUST = $\rho_e u_e^2 A_e - \rho_0 u_0^2 A_0 + (P_e - P_0) A_e = \dot{m}_e u_e - \dot{m}_0 u_0 + (P_e - P_0) A_e$ (24)

Note that D_{add} = 0 when no curvature on the upstream capture streamtube (i.e. no acceleration or deceleration upstream)

Furthermore it can be shown that D_{add} is always positive (+) whether capture streamtube is accelerating or decelerating AND

It can be shown that D_{add} = - D_{ext} for perfect external flow (shockless, inviscid) and for $P_e = P_0$.

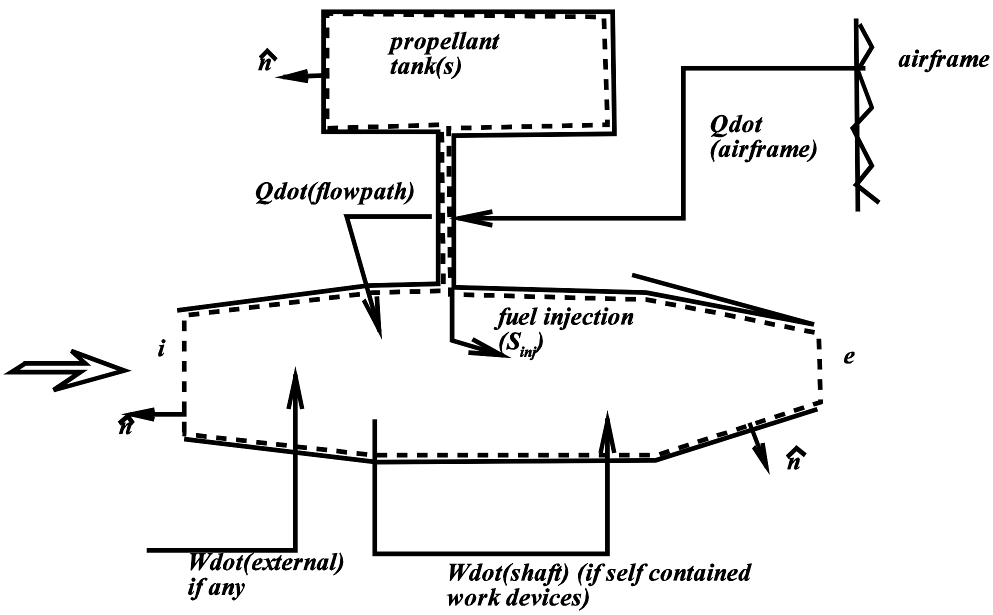
So,

Installed Thrust = Uninstalled Thrust if external flow is shockless and inviscid and if $P_e = P_0$.

<u>Development of Energy Relationships</u>

What is the heat rate added to the flowpath from i to e (in terms of fluid dynamics)?

Define a control volume (CV) comprising ALL internal wetted surfaces (including fuel and propellant systems) + the inlet and exit plane of the engine (same as originally done for the thrust relationships). This control volume and energy relationships would look like:



Note that $\dot{Q}_{air-frame}$ is the heat rate which is (possibly) removed from the airframe and deposited into the fuel (i.e. the fuel is here looked at a heat sink for the airframe although heat could flow the other direction such that the term would be negative). $\dot{Q}_{flow-path}$ is the energy added to the flow through the engine (i to e) from the fuel, however, usually this term would be negative, i.e. the fuel could be used to cool the airframe. $\dot{W}_{shaft}$ is the NET work rate from self-contained shaft, i.e. for ideal turbojet with equal work extracted from downstream turbine and put back into flow by upstream compressor, $\dot{W}_{shaft}$ would be zero. $\dot{W}_{external}$ would cover any other external to fluid work interaction. This by no means is a completely comprehensive schematic on possible energy flows, but represents a reasonable configuration.

Apply the control volume form of the energy equation to the fluid in the control volume and obtain:

$$\frac{\partial}{\partial t}\iiint_V \rho e_t dV + \iint_{A_e} \rho u h_t dA_e - \iint_{A_i} \rho u h_t dA_i = \dot{Q}_{air-frame} + \dot{W}_{external} \qquad (25)$$

Now examine control volume associated with just fuel system by itself (for convenience keeping $\hat{n}_{inj}$ the same direction as $\hat{n}$ in figure for flowpath:

$$\frac{\partial}{\partial t}\iiint_{V_{\substack{fuel\\system}}} \rho e_t dV - \iint_{S_{inj}} \rho_{inj} h_{t(inj)} (\vec{V}_{inj} \cdot \hat{n}_{inj}) dS_{inj} = \dot{Q}_{air-frame} - \dot{Q}_{flowpath} \qquad (26)$$

And since the flow is steady in the flowpath of the engine

$$\frac{\partial}{\partial t}\iiint_V \rho e_t dV = \frac{\partial}{\partial t}\iiint_{V_{\substack{fuel\\system}}} \rho e_t dV \qquad (27)$$

then

$$\iint_{A_e} \rho u h_t dA_e - \iint_{A_i} \rho u h_t dA_i + \iint_{S_{inj}} \rho_{inj} h_{t(inj)} (\vec{V}_{inj} \cdot \hat{n}_{inj}) dS_{inj} = \dot{Q}_{flowpath} + \dot{W}_{external} \qquad (28)$$

or, for uniform flow at e and i:

$$\dot{m}_e h_{te} - \dot{m}_i h_{ti} - m_{inj} h_{t(inj)} = \dot{Q}_{flowpath} + \dot{W}_{external} \qquad (29)$$

Note that $\dot{W}_{external}$ = 0 for 'ideal' engines (i.e. it is any – possible - additional work supplied from the surroundings not taken out of flow in engine).

Recall also that $h_t = C_P T + \frac{u^2}{2}$ for a perfect gas.

Development of Mass Flow Rate Relationships

What is the mass flow rate of fuel added between i and e?

First examine the overall control volume comprising the entire flowpath and fuel system (see Thrust and Energy sections) and apply the conservation of mass to this CV:

$$\iint_{A_e} \rho u dA_e - \iint_{A_i} \rho u dA_i = -\frac{\;}{\partial t}\iiint_V \rho dV \qquad (30)$$

Now examine a control volume that comprises just the fuel system and propellant tanks alone:

$$\frac{\partial}{\partial t}\iiint\limits_{V_{\substack{fuel\\system}}} \rho dV = \iint\limits_{S_{inj}} \rho_{inj}(\vec{V}_{inj}\cdot\hat{n}_{inj})dS_{inj} \tag{31}$$

And note that for steady flow in the flowpath

$$\frac{\partial}{\partial t}\iiint\limits_{V} \rho dV = \frac{\partial}{\partial t}\iiint\limits_{V_{\substack{fuel\\system}}} \rho dV \tag{32}$$

Hence:

$$\iint\limits_{A_e} \rho u dA_e - \iint\limits_{A_i} \rho u dA_i = -\iint\limits_{S_{inj}} \rho_{inj}(\vec{V}_{inj}\cdot\hat{n}_{inj})dS_{inj} \tag{33}$$

Here refer back to the Energy Relationship Section for directionality of $\hat{n}_{inj}$. It is defined referenced to the flowpath such that it is oriented INTO the fuel line, i.e. $\vec{V}_{inj}\cdot\hat{n}_{inj}$ is negative by definition.

Finally, if uniform flow is at i and e:

$$\rho_e u_e A_e - \rho_i u_i A_i = \dot{m}_{inj(fuel)} \tag{34}$$

or

$$\dot{m}_e - \dot{m}_i = \dot{m}_{inj-fuel} \tag{35}$$

<u>Summary of Mass, Energy, and Force Relationships</u>

A. $$\iint\limits_{A_e} \rho u dA_e - \iint\limits_{A_i} \rho u dA_i = -\iint\limits_{S_{inj}} \rho_{inj}(\vec{V}_{inj}\cdot\hat{n}_{inj})dS_{inj} = \dot{m}_{fuel} \tag{36}$$

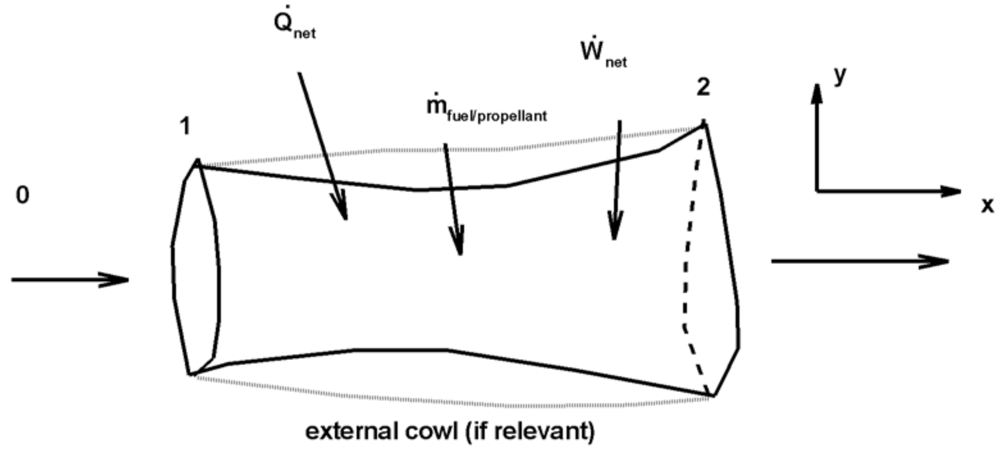
B. $$\iint\limits_{A_e} \rho u h_t dA_e - \iint\limits_{A_i} \rho u h_t dA_i + \iint\limits_{S_{inj}} \rho_{inj} h_{t(inj)}(\vec{V}_{inj}\cdot\hat{n}_{inj})dS_{inj} = \dot{Q}_{flow-path} \tag{37}$$

C. $$\iint\limits_{A_e} (\rho u^2 + P)dA_e - \iint\limits_{A_i} (\rho u^2 + P)dA_i = THRUST_{internal-surfaces} \tag{38}$$

And recall that $THRUST_{internal-surfaces}$ is the axial force in the negative x direction on **all** internal wetted surfaces due to **all** forces due to pressure and shear.

Summary – Simple wall-bounded duct (streamtube):

Consider a duct (say an engine or a piece of an engine); assume uniform flow at stations 1 and 2 and cross-sectional areas at 1 and 2 are perpendicular to the x axis and flow velocity aligned with x axis at stations 1 and 2:



1. Net AXIAL force (+ in negative x direction) on all internal wetted surfaces of the duct or streamtube:

$$\rho_2 u_2^2 A_2 + P_2 A_2 - (\rho_1 u_1^2 A_1 + P_1 A_1) \qquad (39)$$

2. Net AXIAL force (here defined + in negative x direction) on all internal wetted surfaces of the duct PLUS axial component of force on external cowl due to constant pressure = P_1:

$$\rho_2 u_2^2 A_2 + (P_2 - P_1) A_2 - \rho_1 u_1^2 A_1 \qquad (40)$$

3. 'Conventional uninstalled engine thrust' definition (assumes no additive drag and constant pressure on external cowl surface = P_0 where station '0' is upstream of engine - freestream):

$$\rho_2 u_2^2 A_2 + (P_2 - P_0) A_2 - \rho_0 u_0^2 A_0 \qquad (41)$$

4. Energy equation (neglecting fuel/propellant total enthalpies flow rate – or lumping it into $\dot{Q}_{net}$ term:

$$\rho_2 u_2 A_2 \left(C_P T_2 + \frac{u_2^2}{2}\right) - \rho_1 u_1 A_1 \left(C_P T_1 + \frac{u_1^2}{2}\right) = \dot{Q}_{net} + \dot{W}_{net} \qquad (42)$$

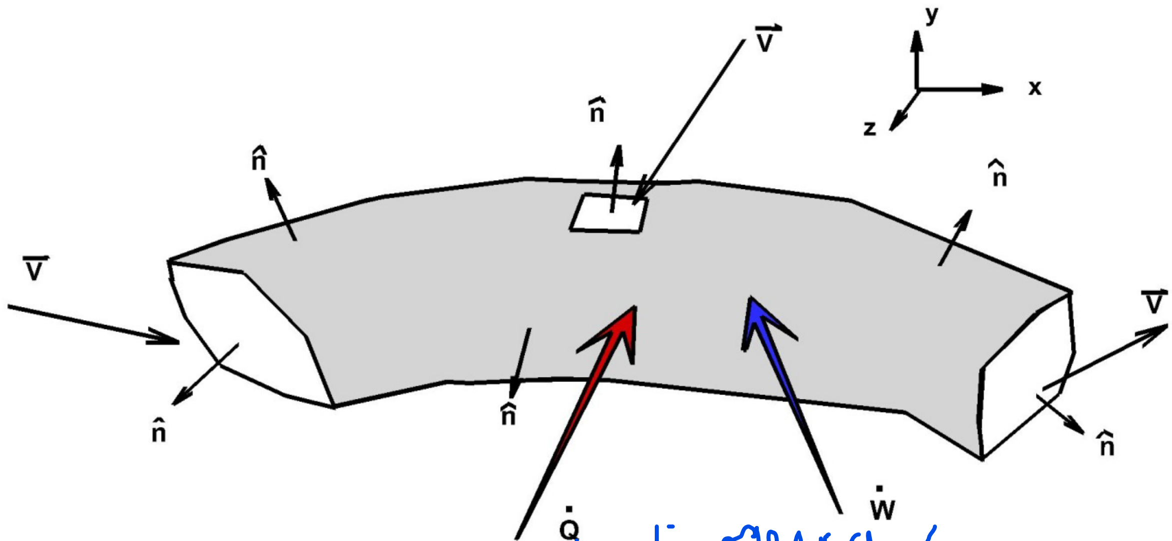
5. Mass flow rate (continuity relationship):

$$\rho_2 u_2 A_2 - \rho_1 u_1 A_1 = \dot{m}_{propellant/fuel} \qquad (43)$$

also
$$P_1 = \rho_1 R T_1$$
$$P_2 = \rho_2 R T_2 \qquad (44)$$

Utilizing control volume analysis for aerospace engine flow-fields (defining forces experienced by engine wetted surfaces in terms of fluid properties on inlets/exits) - AE 5535

Define a control volume (cv), preferentially enclosing all fluid within a defined/desired engine flow-path. This control volume could also enclose a segment of interest of an engine flow-path, rather than an entire engine flow-path. Generally, for the present development, the control volume is defined such that its 'side' surfaces S_{wetted} lie along 'internal' engine solid walls (i.e., along fluid/solid interfaces internal to the engine flow-path). However, in order to have a continuous closed control surface, the control surface also necessarily cuts through defined/desired inlets/exits of the engine (or engine segment) where fluid is exiting or entering the control volume.



Ext energy

Hence, by definition, the overall control volume surface can be subdivided as follow:

$$S_{cv} = S_{wetted} + \sum_{all\ inlets/exits} S_{inlet/exit} \tag{1}$$

Note also that the fluid velocity vector is $\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$ by definition.

I. *x,y,z* components of the (vector) momentum equation:

The steady control volume form of the (vector) momentum equation for fluid in the defined c.v. (neglecting body forces) is (see control volume handout):

$$\iint_{S_{cv}} \rho\vec{V}(\vec{V}\cdot\hat{n})dS = -\iint_{S_{cv}} P\hat{n}\,dS - \iint_{S_{cv}} \vec{\tau_w}\,dS \tag{2}$$

The *x* component of eq. (2) is:

$$\iint_{S_{cv}} \rho\,u(\vec{V}\cdot\hat{n})dS = -\iint_{S_{cv}} P\hat{n}_x\,dS - \iint_{S_{cv}} \tau_{w,x}\,dS \tag{3}$$

The *y* component of eq. (2) is:

$$\iint_{S_{cv}} \rho\, v(\vec{V}\cdot\hat{n})dS = -\iint_{S_{cv}} P\hat{n}_y\, dS - \iint_{S_{cv}} \tau_{w,y}\, dS \tag{4}$$

The *z* component of eq. (2) is:

$$\iint_{S_{cv}} \rho w\,(\vec{V}\cdot\hat{n})dS = -\iint_{S_{cv}} P\hat{n}_z\, dS - \iint_{S_{cv}} \tau_{w,z}\, dS \tag{5}$$

<u>II. The force experienced by the (internal) wetted surfaces of the engine:</u>

By definition of fluid forces due to pressure and shear stress on the wetted (solid) surfaces of the engine ('internal' flowpath, as shown in figure), the net overall force vector on the (internal) wetted surfaces of the engine is:

$$\vec{F} = F_x\hat{n}_x + F_y\hat{n}_y + F_z\hat{n}_z \tag{6}$$

Here:

In eq. (6), the net *x* component of force experienced by all (internal) engine walls (wetted surfaces adjacent to fluids) is (by definition):

$$F_x = \iint_{S_{wetted}} P\hat{n}_x\, dS + \iint_{S_{wetted}} \tau_{w,x}\, dS \tag{7}$$

In eq. (6), the net *y* component of force experienced by all (internal) engine walls (wetted surfaces adjacent to fluids) is (by definition):

$$F_y = \iint_{S_{wetted}} P\hat{n}_y\, dS + \iint_{S_{wetted}} \tau_{w,y}\, dS \tag{8}$$

In eq. (6), the net *z* component of force experienced by all (internal) engine walls (wetted surfaces adjacent to fluids) is (by definition):

$$F_z = \iint_{S_{wetted}} P\hat{n}_z\, dS + \iint_{S_{wetted}} \tau_{w,z}\, dS \tag{9}$$

<u>III. Using the fluid momentum equation to define the net force components experienced by all (internal) engine walls in terms of flow properties at inlets/exits. (Done for the x component here).</u>

Now relate the x component of the fluid momentum equation (equation (3)) to the force component in the x direction experienced by the engine walls (wetted surfaces), i.e. equation (7). This is done as follows:

Repeat for convenience here equation (3) which is the x component of the fluid momentum equation for the control volume:

$$\iint_{S_{cv}} \rho\, u(\vec{V} \cdot \hat{n}) dS = -\iint_{S_{cv}} P\hat{n}_x\, dS - \iint_{S_{cv}} \tau_{w,x}\, dS \qquad (3)$$

Note again (eq. (1) the division of S_{cv} into S_{wetted} and the summation of inlet/exit areas, i.e.:

$$S_{cv} = S_{wetted} + \sum_{all\ inlets/exits} (S_{inlet/exit}) \qquad (1)$$

Realize also that on all internal wetted surfaces S_{wetted} (interfaces between internal engine solid walls and fluid) the following conditions are necessary:

$\vec{V} \cdot \hat{n} = 0$ and furthermore (for no-slip viscous solid wall boundary condition everywhere on S_{wetted}), $\vec{V} = 0$ as well.

In addition, for 'uniform' flow on inlets/exits (meaning no gradients in velocity across any given inlet or exit area), the shear stress vector is zero by definition on inlets/exits:

$\overrightarrow{\tau_w} = 0$ (Note only on inlets and exits – obviously in general $\overrightarrow{\tau_w} \neq 0$ on S_{wetted}).

Equation 3) then becomes (with all these requirements/conditions for the cv as defined here):

$$\sum_{all\ inlets/exits} \{\rho u(\vec{V} \cdot \hat{n})S\} = -\sum_{all\ inlets/exits} \{P\hat{n}_x S\} - \iint_{S_{wetted}} P\hat{n}_x\, dS - \iint_{S_{wetted}} \tau_{w,x}\, dS \qquad (10)$$

However the last two (integral) terms on the right hand side of eq. (10) form, by definition, - F_x (from equation (7)).

Hence from eq. (10), the following can be written for the axial force component experienced by the internal (solid wetted) surfaces of the engine:

$$F_x = -\sum_{all\ inlets/exits} \{\rho u(\vec{V} \cdot \hat{n})S + P\hat{n}_x S\} \qquad (11)$$

Similarly the following can be derived for the *y* and *z* force components:

$$F_y = -\sum_{all\ inlets/exits} \{\rho v(\vec{V} \cdot \hat{n})S + P\hat{n}_y S\} \qquad (12)$$

$$F_z = -\sum_{all\ inlets/exits} \{\rho w(\vec{V} \cdot \hat{n})S + P\hat{n}_z S\} \qquad (13)$$

Equations (11), (12), and (13) then define the axial (x), transverse (y), and vertical (z) components of force experienced by the internal wetted surfaces of the engine (or engine segment) due to the fluid adjacent to it. **<u>This is in terms of the fluid properties (and orientation information) at inlets/exits only!</u>**

IV. Heat/work rates and mass flow rates utilizing control volume energy and continuity equations:

In a similar fashion and for the same control volume, the energy equation (single scalar equation!) can be reduced to:

mass. mo
cont

$$\sum_{all\ inlets/exits} \left\{ \rho(\vec{V} \cdot \hat{n})S\left(C_P T + \frac{|\vec{V}|^2}{2}\right) \right\} = \dot{Q} + \dot{W} \tag{14}$$

Similarly, the continuity equation (single scalar equation!) becomes:

$$\sum_{all\ inlets/exits} \{\rho(\vec{V} \cdot \hat{n})S\} = 0 \tag{15}$$

V. Some useful things to know about $\vec{V} \cdot \hat{n}$ (here given for simplicity in 2-D):

First, $\vec{V} \cdot \hat{n}$ is a scalar (i.e. NOT a vector)!

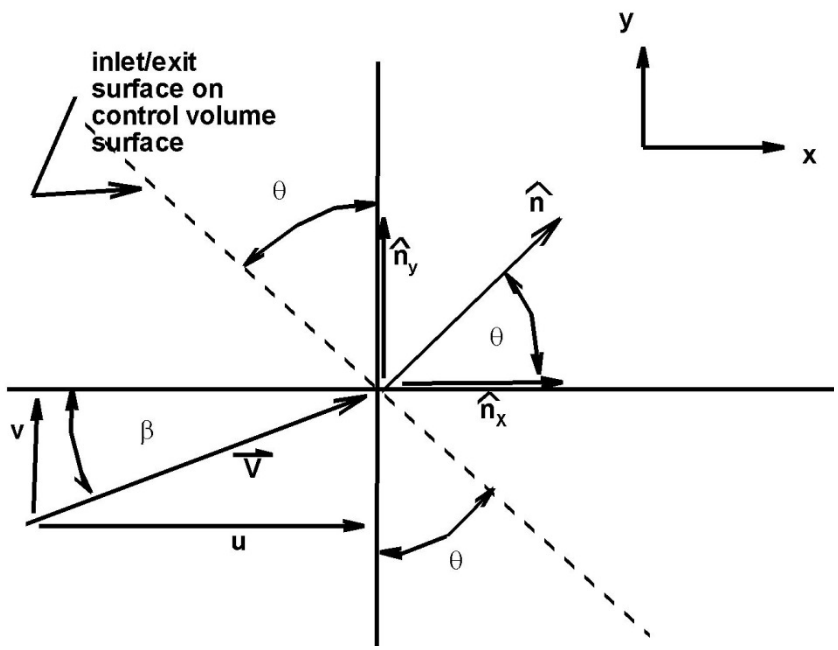
In 2-D: $\vec{V} = u\hat{\imath} + v\hat{\jmath}$ and $\hat{n} = \hat{n}_x\hat{\imath} + \hat{n}_y\hat{\jmath}$.

Note that $\hat{n}_x$ and $\hat{n}_y$ since $\hat{n}$ is a unit vector (magnitude 1.0) will be the cosine and the sine respectively of an orientation angle (or vice-versa, depending on the exact orientation angle definition in a given application/problem).

Definition of scalar dot product is as follows: $\vec{V} \cdot \hat{n} = (u\hat{\imath} + v\hat{\jmath}) \cdot (\hat{n}_x\hat{\imath} + \hat{n}_y\hat{\jmath}) = u\hat{n}_x + v\hat{n}_y$

VI. EXAMPLE of the dot product characteristics in terms of orientation angles

As an illustrative example (**only** an example since angle definitions – and orientations – may and almost always vary from application to application!), consider the $\hat{n}$ and $\vec{V}$ characteristics for an arbitrary inflow or outflow plane in the control volume as shown in the following sketch:



SO . . . FOR THIS ORIENTATION AND ANGLE DEFINITON ONLY!

$\vec{V} = |\vec{V}|cos\beta\hat{\imath} + |\vec{V}|sin\beta\hat{\jmath}$ and $\hat{n} = cos\theta\hat{\imath} + sin\theta\hat{\jmath}$

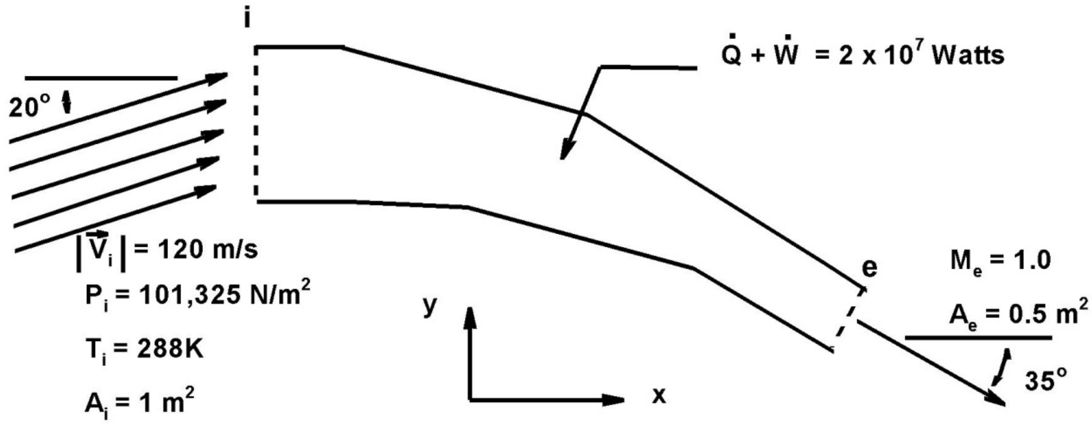
Hence $\vec{V} \cdot \hat{n} = |\vec{V}|cos\beta\, cos\theta + |\vec{V}|sin\beta\, sin\theta$

But <u>to emphasize once again</u> – the signs - and whether terms are sin/cos -depend entirely on the trig and geometry of the specific problem (i.e., angle definition, orientation of inflow/outflow plane, direction of velocity vector, etc.)

In-Class Example AE 5335: Control Volume Analysis and Forces

A thrust augmentation system for a fighter engine (sketched below) is operated at a high angle of attack (20^o). The air at the inlet face (station i) has static pressure and static temperature of 101,325 N/m^2 and 288K, respectively, and a velocity magnitude of 120 m/s. This upstream air approaches the inlet at an (upward) angle of 20 degrees, as shown (i.e., the angle of attack is 20 degrees). The inlet face is aligned with the y axis and has a cross-sectional area of 1 m^2. Inside the inlet, a net work AND heat rate of 2 x 10^7 Watts are added to the flow (net positive to the flow). At the exit (station e), the flow is choked (M_e = 1) and is exhausted at an angle 35 degrees directed downward from the x axis, as shown. Exit cross-sectional area is 0.5 m^2.

Calculate the axial and vertical forces (F_x and F_y) on the internal wetted surfaces of this engine section.



AE 5535

– IN CLASS EXAMPLE (HAND OUT), SOLUTION:

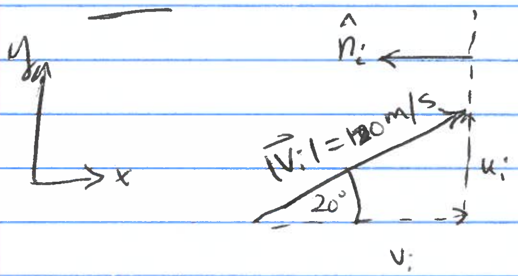
PRE-LIM:

$$\vec{V} = u\hat{i} + v\hat{j}$$
$$\hat{n} = \hat{n}_x\hat{i} + \hat{n}_y\hat{j}$$

$$\vec{V}_i = u_i\hat{i} + v_i\hat{j}$$
$$\hat{n}_i = \hat{n}_{x_i}\hat{i} + \hat{n}_{y_i}\hat{j}$$

$$\vec{V}_e = u_e\hat{i} + v_e\hat{j}$$
$$\hat{n}_e = \hat{n}_{xe}\hat{i} + \hat{n}_{ye}\hat{j}$$

INFLOW:



so, $\hat{n}_i = -\hat{i}$

$(\hat{n}_{x_i} = -1,\ \hat{n}_{y_i} = 0)$

$$\begin{cases} u_i = (120\,m/s)\cos 20^\circ = 112.8\,m/s \\ v_i = (120\,m/s)\sin 20^\circ = 41.04\,m/s \end{cases}$$

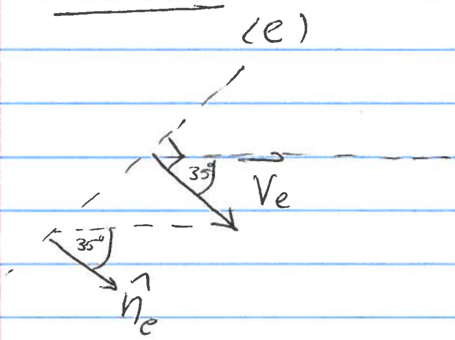
Static dens → $\rho_i = \dfrac{P_i}{RT_i} = 1.225\ kg/m^3$ ∴ $R_{AIR} = 287\ J/kg{\cdot}K$

TOTAL CONDITIONS:

$$T_{t_i} = \left(1 + \frac{\gamma - 1}{2} M_i^2\right) T_i \quad , \quad M_i = \frac{|\vec{V}_i|}{\sqrt{\gamma R T_i}}$$

$\therefore T_{t_i} = 295.17\ K$; $M_i = 0.353$

OUTFLOW:



$\hat{n}_{xe} = \cos 35°$

$\hat{n}_{ye} = -\sin 35°$

MASS FLOW RATE ($\dot{m}$) CONSIDERATIONS:

$$\dot{m}_i = \int_{A_i} \rho_i (\vec{V}_i \cdot \hat{n}_i)\, dA_i = \rho_i u_i A_i = 138.23 \text{ kg/s}$$

$$* \quad \dot{m}_e = \int_{A_e} \rho_e (\vec{V}_e \cdot \hat{n}_e)\, dA_e = \rho_e V_e A_e = \dot{m}_i = 138.23 \text{ kg/s}$$

ENERGY CONSIDERATIONS:

$$\int_{A_e} \rho_e (\vec{V}_e \cdot \hat{n}_e) h_{te}\, dA_e - \int_{A_i} \rho_i (\vec{V}_i \cdot \hat{n}_i) h_{t_i}\, dA_i$$

$$= \dot{Q} + \dot{W}$$

$$\because h_t = C_p T + \frac{V^2}{2} \quad [\text{J/kg}]$$

$$** \quad \left\{ \left(C_p T_e + \frac{\vec{V}_e \cdot \vec{V}_e}{2}\right) \dot{m}_e - \left(C_p T_i + \frac{\vec{V}_i \cdot \vec{V}_i}{2}\right) \dot{m}_i = \dot{Q} + \dot{W} \right\}$$

BUT WE KNOW $M_e = 1.0$ so $|\vec{V}_e| = \sqrt{\gamma R T_e}$

$\therefore$ SOLVE ** FOR $T_e = 366$ K

THEN $|\vec{V}_e| = 383.5$ m/s

$\therefore \rho_e = \dot{m}_e / V_e A_e$ FROM *

$\rho_e = 0.721$ kg/m^3

$\therefore P_e = \rho_e R T_e = 75{,}727$ N/m^2

Now analyze

FORCES:

(X-FORCE) AXIAL

BY DEFINITION, THE AXIAL FORCE ON INTERNAL INLET WALLS $= F_x$

$$= \int_{\text{wetted}} P\hat{n}_x \, dS + \int_{\text{wetted}} \vec{\tau}_{wall_x} \, dS$$

(SIDES OF C.V.)

BUT FROM X-MOMENTUM FOR C.V. EQN.

$$-F_x = -\int_{\text{wetted}} P\hat{n}_x \, dS - \int_{\text{wetted}} \vec{\tau}_{w_x} \, dS = \int_{A_e} \rho_e u_e (\vec{V}_e \cdot \hat{n}_e) \, dA_e + \int_{A_i} \rho_i u_i (\vec{V}_i \cdot \hat{n}_i) \, dA_i$$

$$+ \int_{A_e} P\hat{n}_{x_e} \, dA_e + \int_{A_i} P\hat{n}_{x_i} \, dA_i$$

$$\Rightarrow -F_x = \dot{m}_e u_e - \dot{m}_i u_i + P_e A_e \cos 35° - P_i A_i$$

$$\therefore F_x = 42{,}474 \text{ N} \quad \text{(DRAG) IN } +X\text{-DIRECTION} \longrightarrow$$

(Y-FORCE)

BY DEFINITION, THE VERTICAL FORCE ON INLET WALLS, $= F_y$

$$F_y = \int_{\text{wetted}} P\hat{n}_y \, dS + \int_{\text{wetted}} \vec{\tau}_{w_y} \, dS$$

FROM Y-MOMENTUM FOR C.V. EQN.

$$-F_y = -\int_{\text{wetted}} P\hat{n}_y \, dS - \int_{\text{wetted}} \vec{\tau}_{w_y} \, dS = \int_{A_e} \rho_e v_e (\vec{V}_e \cdot \hat{n}_e) \, dA_e + \int_{A_i} \rho_i v_i (\vec{V}_i \cdot \hat{n}_i) \, dA_i$$

$$+ \int_{A_e} P\hat{n}_{y_e} \, dA_e + \cancelto{0}{\int_{A_i} P\hat{n}_{y_i} \, dA_i}$$

$$\therefore -F_y = \dot{m}_e v_e - \dot{m}_i v_i - P_e \sin 35° A_e$$

$$\therefore F_y = 57{,}801 \text{ N} \quad (+y\text{-DIRECTION}) \uparrow$$

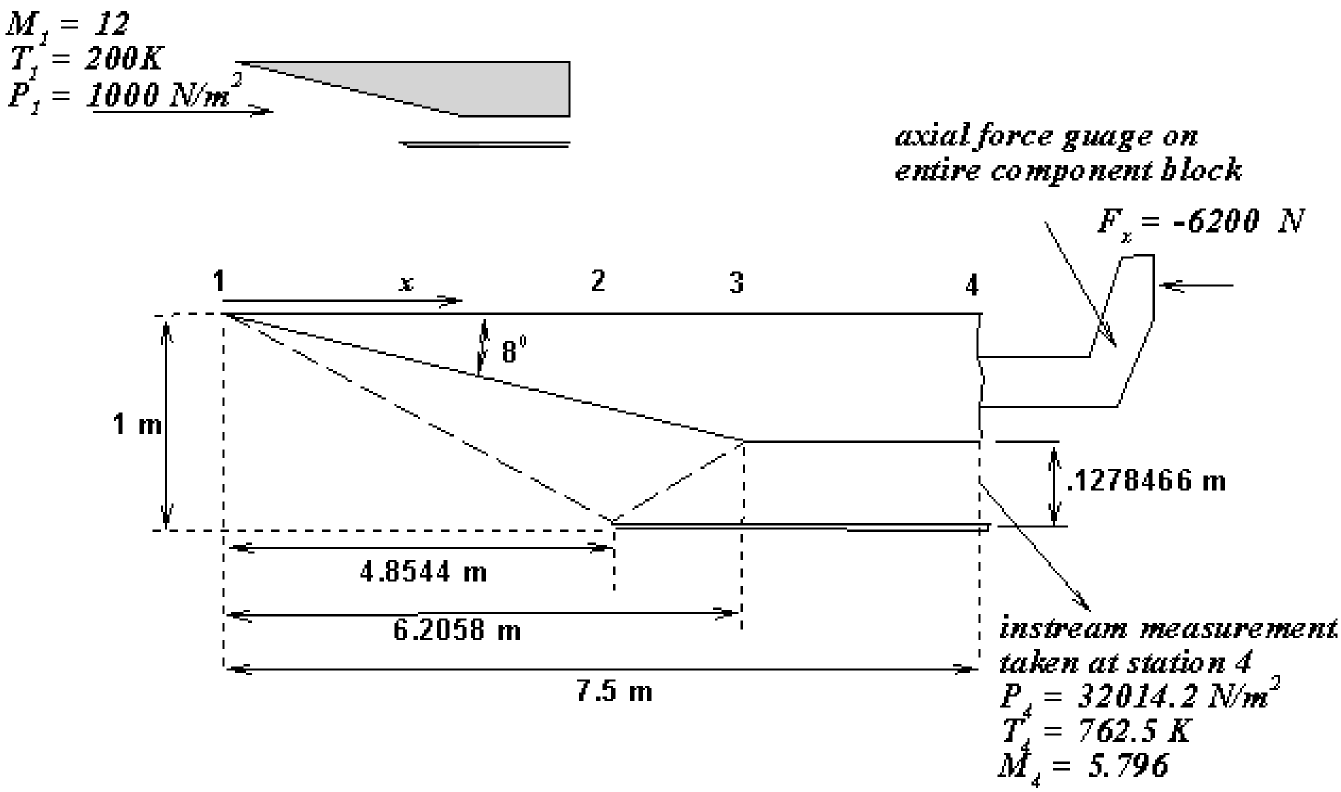
KEY

In class example: Control Volume Analysis **AE 5335**

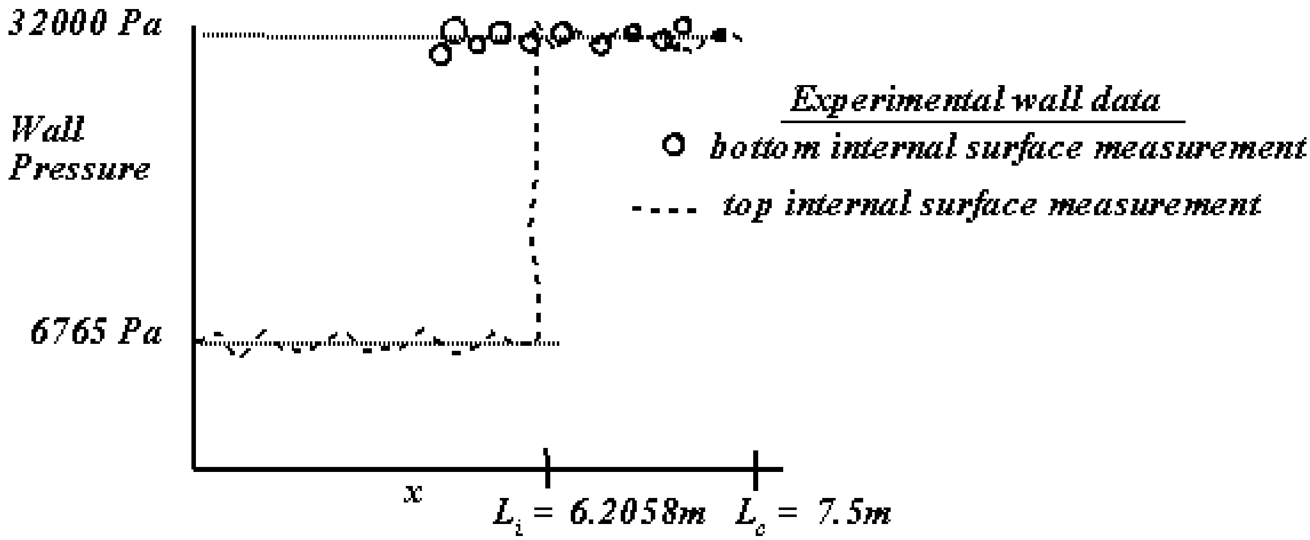
You are a wind-tunnel test engineer in charge of a test of an inlet-forebody component for a high-speed air-breathing vehicle. The configuration and conditions and data are as shown on the following pages. Show all work! You should use compressible flow relations/equations for ALL results (not tables/charts!) and keep significant figures in your numbers to ensure accuracy!

You have a good deal of instrumentation and data obtained during the experiment. This data includes wall (static) pressure distributions and heat transfer rate distributions on both upper and lower ***internal*** surfaces. These are shown below. You also have in-stream (core-flow) measurements of pressure, temperature, and Mach number at station 4 in the internal stream. In addition, you have an axial force measurement from a force stand on which the entire component block is mounted. All measurements and conditions are as shown in the figures below.

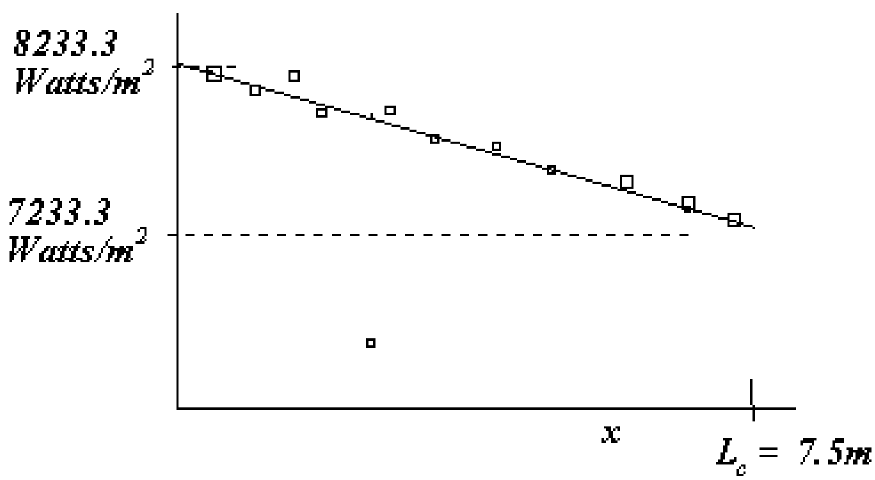
a) Find the inflow mass flow rate. Then estimate the mass flow rate associated ***with the in-stream measurements*** (at station 4).
b) Estimate the axial force on the component ***using inviscid flow analysis*** (oblique shock analysis - i.e. do the analysis). Directly integrate computed wall pressures to determine this estimate of the axial force.
c) Estimate the axial force on the component ***using inviscid flow analysis*** (oblique shock analysis – i.e. do the analysis). Now use the *momentum equation over the relevant control volume* to determine this estimate of the axial force.
d) Estimate the axial force on the component ***using the available experimental wall pressures***.
e) Estimate the axial force on the component ***using the in-stream measurements and control volume analysis (i.e. use the momentum equation over the relevant control volume)*** to determine this axial force**.**
f) Estimate the overall heat rate load removed from the flow ***using the available experimental wall heat transfer measurements***.
g) Estimate the overall heat rate load and error ***using the in-stream measurements*** (at station 4).



all internal surfaces instrumented with static pressure ports and heat transfer guages



overall heat transfer (cooling - heat removed) rate per unit area



Q 9.1 soln analysis

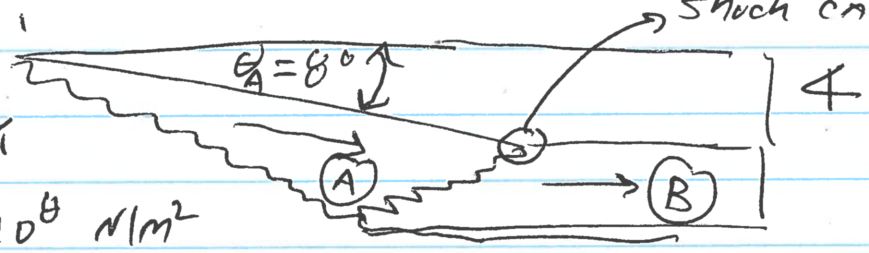
a) $\dot{m}_{inflow} = \rho_1 u_1 A_1 = \frac{P_1}{RT_1} M_1 \sqrt{\gamma R T_1} A_1 = 59.264$ kg/sec

$\dot{m}_{IN\text{-}stream\ measurements} = \rho_{4m} \, U_{4m} \, A_4 = 60$ kg/sec

$U_{4m} \rightarrow 3208.14$ m/s $\left(= M_{4m} \sqrt{\gamma R T_{4m}}\right)$

$\rho_{4m} \rightarrow .1463$ kg/m³ $\left(= \frac{P_{4m}}{RT_{4m}}\right)$

b) do inviscid shock analysis



$T_{t0} = 5960$ K

$P_{t0} = 1.444 \times 10^8$ N/m²

$U_0 = 3401.74$ m/s

1st shock

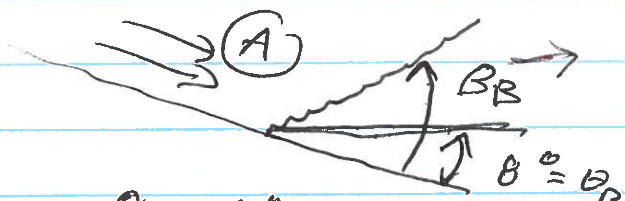
$\beta_A = 11.64°$

$M_A = 8.20456$

$P_A = 6672.24$ N/m²

$T_A = 412.08$ K

2nd shock



$\theta_B = 8°$

$\beta_B = 13.405°$

$M_B = 6.318904$

$P_B = 27050.9$ N/m²

$T_B = 663.274$ K

$U_B = 3262.073$ m/s

$\rho_B = .1421409$ kg/m³

$\dot{m}_B = 59.264$ kg/sec

$$\boxed{F_{px} = P_A \cdot \left(\frac{6.2058}{\cos 8°}\right) \sin 8° = 5819.2\,N}$$

c) $-F_x = \left(\rho_B U_B^2 A_4 + P_B A_4\right) - \left(\rho_0 U_0^2 A_0 + P_0 A_0\right) = 5819.5\,N$

d) $P_{measured\ experiment} \left(\frac{6.2058}{\cos 8°}\right) \sin 8° = 5900\,N$

e) $-\{(\dot{m}_{\text{in-stream measurements}} \, U_{4m} + P_{4m} A_4)$

$$-(\rho_0 U_0^2 A_0 + P_0 A_0)\} = 6019.4 \, N$$

f) ($\dot{q}$ linear) in experiment

$$\dot{Q}_{\text{WALL gauges}} = \left(\frac{8233.3 \, W/m^2 + 7233.3 \, W/m^2}{2}\right) \times 7.5 \, m$$

$$= -58{,}000 \text{ Watts}$$

g) $\dot{Q}_{\text{in-stream measurement}} = \dot{m}_{\text{in-stream}} \left(C_p T_{4\,\text{measured}} + \frac{U_{4\,\text{measured}}^2}{2} \right)$

$$- \dot{m}_{\text{inflow}} \left(C_p T_0 + \frac{U_0^2}{2} \right)$$

$$= -58{,}000 \text{ Watts}$$

This is a hard match → ~~But~~ Note there are big errors due to Accuracy (very high Mach #)

This entropy check is for PREVIOUS problem!!!

ALSO, LET'S CHECK ENTROPY

$$\dot{S} = \dot{m}\left(c_p \ln\frac{T_e}{T_i} - R\ln\frac{P_e}{P_i}\right) = 44{,}832 \text{ J/K}\cdot\text{s}$$

↗ ENTROPY GENERATION RATE (OR CHANGE RATE).

[redacted]

THROAT :

[redacted]