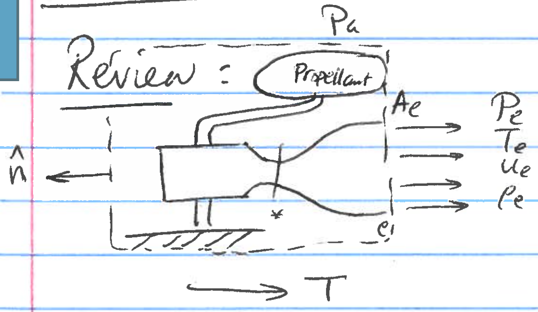
AE5535

NON-CHEMICAL ROCKETS

Review :



Recall : $\displaystyle\int_{S\,(\text{closed control volume})} P_a \hat{n}\, dS = 0$

momentum for c.v. : $\rho_e V_e^2 A_e + (P_e - P_a) A_e = F$ = Thrust magnitude

$\dot{m}_e u_e + (P_e - P_a) A_e = F$

define an 'effective exhaust velocity' $\triangleq c$

where $F = \dot{m} c \Rightarrow c = u_e \left[1 + \dfrac{(P_e - P_a)}{\gamma M_e^2 P_e} \right]$

if the nozzle is 'fully expanded' : $P_e = P_a \Rightarrow c = u_e$

we define a 'characteristic velocity' $\triangleq c^*$

$\dot{m}^* = \dfrac{P_t^* A^*}{c^*} \Rightarrow c^* = \left(\dfrac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \sqrt{\dfrac{R T_t^*}{\gamma}}$

I_{sp} = specific impulse = $\dfrac{\text{Thrust}}{\text{weight flow rate of propellant (at sea level)}}$

$I_{sp} = \dfrac{F}{\dot{m}_p g_0}$ [sec]

$I_{sp} = \dfrac{c}{g_0}$; note I_{sp} doesn't say anything about the overall required mass of the system.

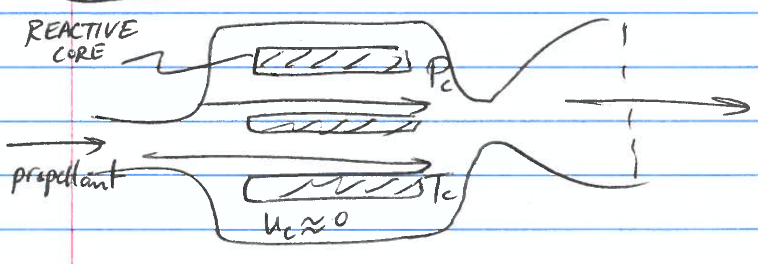
Total overall mass = payload + propellant + engine masses (structure)

→ for chemical rockets, the engine (structural) mass is relatively small compared to the propellant mass. (Space shuttle : ~5% ends up in orbit)

→ I_{sp} for chemical rockets wanted maxed, ($I_{sp\,max}$ for chemical ~ 460 sec).

→ for non-chemical rockets, high I_{sp} requires a massive power-generating equipment hence highest I_{sp} is not the best choice.

CONVENTIONAL NUCLEAR-THERMAL ROCKETS



Propellant is heated through convective heat transfer from hot tube walls plus some radiative heating.

* Approximate max performance of such things :

Assume P_a very low (space-based)

$C_p T_c = C_p T_e + \frac{u_e^2}{2}$ ⟹ P_e is very low (large A_e/A^*)

∴ T_e is very low ⟿ $C_p T_c \cong \frac{u_e^2}{2}$ and $c = u_e$

$c = \sqrt{2 C_p T_c}$; $C_p = \frac{R\gamma}{\gamma-1} = \frac{R_u}{MW}\frac{\gamma}{\gamma-1}$

$$\therefore I_{sp} = \frac{c}{g_0} = \sqrt{\frac{2\gamma}{\gamma-1}\frac{R_u}{MW}\frac{T_c}{g_0^2}}$$

for max I_{sp} ⟹ want MW as low as possible (Hydrogen) and T_c as high as possible (which is limited because unlike in chemical rockets, the heat is convective

So the material limits on walls is the controlling factor). $T_{c\,max} \approx 3000K$

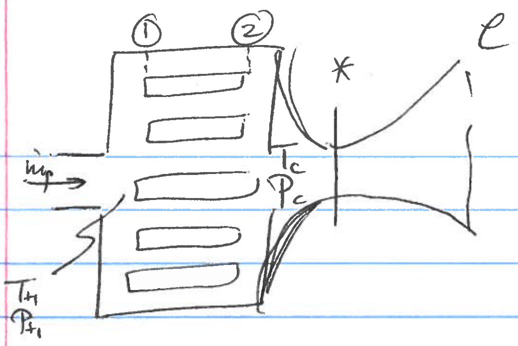
$\therefore\ I_{sp\,max} \cong 900$ sec

A nuclear reaction :

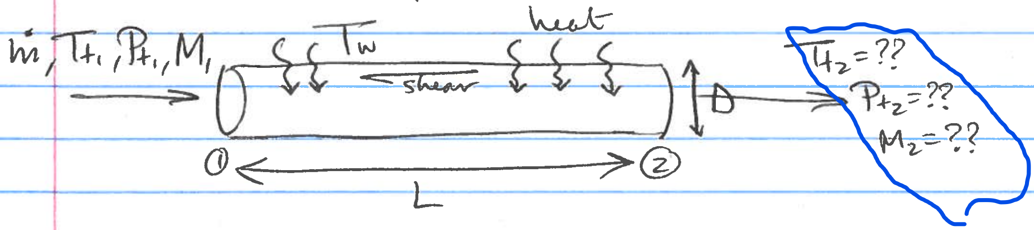
$\Delta E = C^2 \Delta m$; C = speed of light here $= 3 \times 10^8$ m/s

whereas, for chemical :

$$\frac{\Delta E_{chem}}{\Delta E_{nuke}} \sim 10^{-7} \text{ or } 10^{-8}$$



Consider a flow of propellant thru a circular tube (you can modify the geometry as neccessary) of circumference 'c' with wall temp T_w



The goal is to find the exit conditions (station 2).
We know the following (subsonic)

* 1-D heat addition / invescid (Rayleigh flow)
  $\Rightarrow$ M & T_t go up and P_t drops

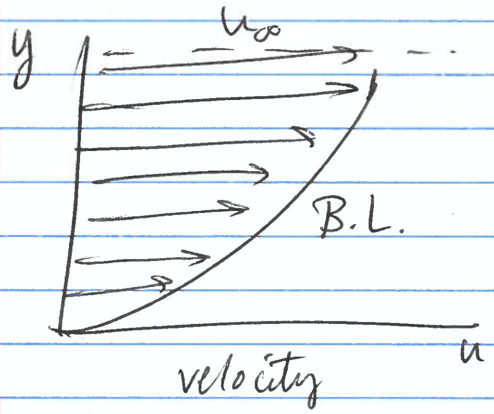
* 1-D viscous/adiabatic (Fanno flow)
  $\Rightarrow$ M goes up, P_t drops, T_t doesn't change

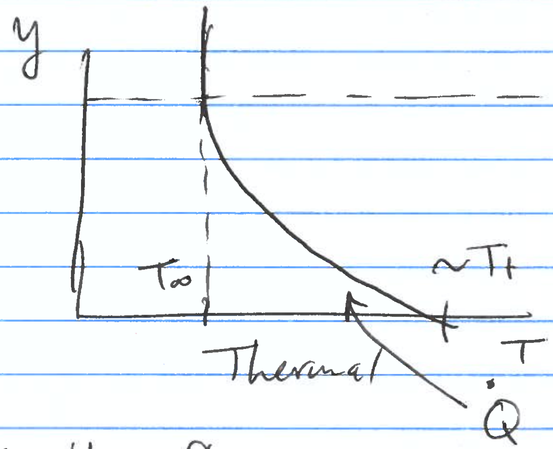
In our case here, we have both convective heat transfer and friction (in fact you can't seperate them due to the physical mechanism of convective heat transfer; molecular intercation).
We want to use the quasi-1D approach but we need to understand convective heat transfer.

Convective heat transfer (ht):

hot or cold wall

consider a 'thermal boundary layer'





if $T_w > T_t$ $\quad \dot{Q}$ into the flow

if $T_w < T_t$ $\quad \dot{Q}$ out of the flow

* In actuality, the situation is more complex; the fluid is at wall temperature at $y=0$.

write $\dot{Q}$ per unit area of wetted surface $= h(T_w - T_t)$

$\dot{Q}'' = h(T_w - T_t)$: h = heat transfer coefficient

h is a function of type of flow, fluid, temp, etc.

$[h] = [W/m^2K]$

The differential heat transfered across a dx per unit mass due to convective heat process $\delta q_{conv.}$

$$\delta q_{conv.} = \frac{h(T_w - T_t)\, c\, dx}{\dot{m}}$$

C = Circumfrence

Define the Stanton Number, N_{st}, as:

$$N_{ST} = \frac{hA}{\dot{m}C_p}$$ ← X-Area

So,

$$\frac{\delta q_{conv}}{\underbrace{C_p T_t}_{\text{non-dimensional}}} = \frac{N_{ST}(T_w - T_t)}{T_t}\left(\frac{C}{A}\right)dx$$

for a circular duct; $C = \pi D$, $A = \pi \frac{D^2}{4}$

we also know that $C_p\,dT_t = \delta q_{conv}$ (energy equation)

$$dT_t = 4 N_{ST}(T_w - T_t)\frac{dx}{D}$$

Now use the 'Reynolds Analogy', which relates skin friction to convective heat transfer.

$C_f = 2 N_{ST}$ where C_f is the friction coefficient

and $C_f = \frac{\tau_w}{\frac{1}{2}\rho u^2}$; τ_w = shear stress at wall

or $C_f = \frac{\tau_w}{\frac{1}{2}\rho_1 u_1^2}$ ← Entrance to be ~~used~~ specified in the problem

Then: $dT_t = 2 C_f (T_w - T_t)\frac{dx}{D}$

or: $\boxed{\delta q_{conv} = 2 C_p C_f (T_w - T_t)\frac{dx}{D}}$

for steady, stable operation of a nuke reactor and propellant flow, T_w, and fluid temp, $(T_w - T_t)$

$T_w - T_t$ must balance at every axial location along the tube to ensure the thermal balance in = out

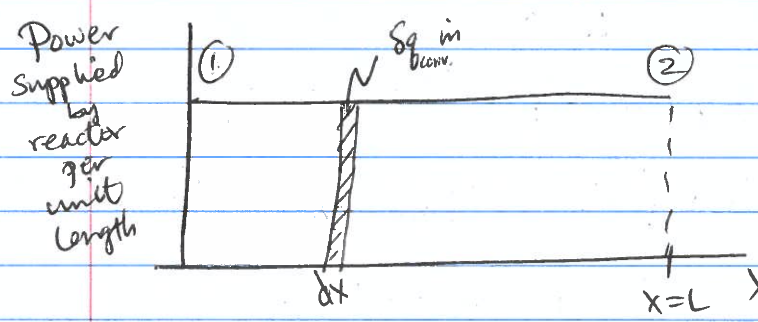
in = out

So, ~~we need~~ the determining factor is the so called power density of the reactor

$$|\delta q_{reactor}| = |\delta q_{conv}| = 2 C_p C_f (T_w - T_f) \frac{dx}{D}$$

We will look analytically at 2 special cases for power density of the reactor. Finding T_{f2}.

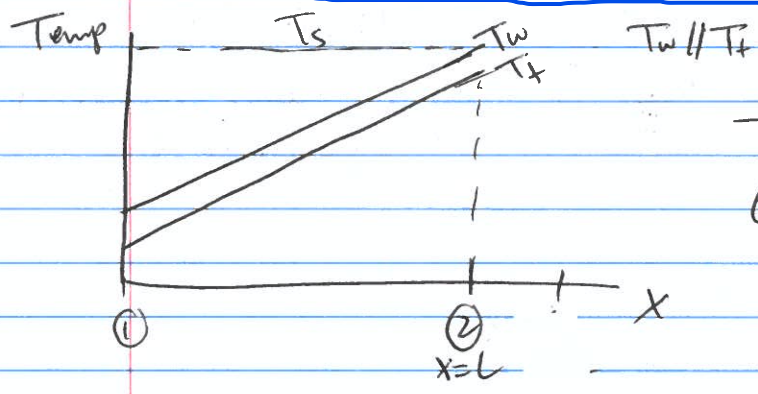
CASE A   constant power density distribution



⇒ Corresponds to thermally shielded ends of the reactor

since $\delta q_{conv.}$ is constant, then $\frac{dT_f}{dx}$ is also constant,

then $T_w - T_f$ is also constant along x



$T_w // T_f$

→ Based on material

T_s is design max wall temp (occurs at ②)

so, $\underbrace{T_w - T_f}_{\text{along } x} = \underbrace{T_s - T_{f2}}_{\text{at } ②}$

$$dT_f = 2C_f (T_s - T_{f2}) \frac{dx}{D}, \text{ integrating:}$$

$$T_{f2} - T_{f1} = 2C_f \frac{L}{D} (T_s - T_{f2})$$

$$\boxed{\frac{T_{f2}}{T_s} = \frac{\frac{T_{f1}}{T_s} + K}{1+K}} \Rightarrow \boxed{K = \frac{2C_f L}{D}}$$

Know T_{f1}

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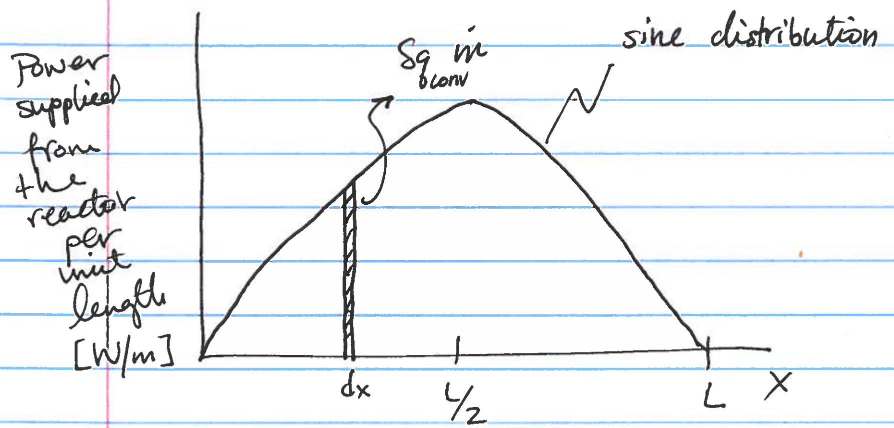
CASE B sine reactor power distribution

– corresponds to thermally unshielded ends of the reactor



The actual case is somewhere between constant and sine wave distribution. They are limiting cases.

$$C_p\, dT_t = \delta q_{b\,conv} = \delta_{react} = 2 C_f C_p (T_w - T_t)\frac{dx}{D}$$

max heating input into fluid ($\delta q_{b\,react\,max}$) will be at $x = \frac{L}{2}$

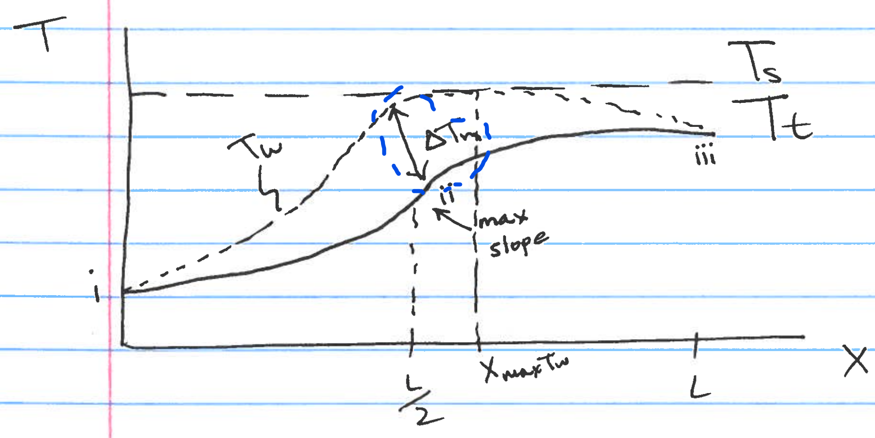
corresponds

$$\Delta T_m = T_w - T_t \quad \text{at} \quad x = \frac{L}{2} \ldots$$

$$T_w - T_t = \Delta T_m \sin\left(\frac{\pi x}{L}\right)$$

ΔT_m has to follow a sine distribution

T_s occurs somewhere between $\frac{L}{2}$ & L



i) zero slope of T_t at $x = 0$
ii) max slope of T_t at $x = \frac{L}{2}$
iii) zero slope of T_t at $x = L$

$T_s \longrightarrow$ max T_w occurs at some x between $x = \frac{L}{2}$ & $x = L$

– we want to find ~~x_s~~ T_{t_2} , $T_t(x)$, $T_w(x)$

use x' as the integration variable :

$$dT_t = 2C_f(T_w - T_t)\frac{x'}{D} = 2C_f \frac{\Delta T_m}{D} \sin\left(\frac{\pi x'}{L}\right) dx'$$

integrate from $x'=0$ to $x'=x$

e.g. $\boxed{C}$ $T_t - T_{t_1} = \frac{2C_f}{D}\Delta T_m \frac{L}{\pi}\left[1-\cos\left(\frac{\pi x}{L}\right)\right]$

now apply equation $\boxed{C}$ at station ② $(x=L)$

$T_{t_2} - T_{t_1} = \left(\frac{2C_f L}{D\pi}\right)\Delta T_m \cdot 2$ or : equation $\boxed{B}$:

$\boxed{B}$ $\Delta T_m = (T_{t_2} - T_{t_1})\left(\frac{\pi D}{4C_f L}\right)$

sub the ΔT_m from $\boxed{B}$ into $\boxed{C}$ and solve for T_t and then sub that T_t into : $T_w = T_t + \Delta T_m \sin\left(\frac{\pi x}{L}\right)$ [def. of ΔT_m]

yields → equation $\boxed{D}$: $T_w = T_{t_1} + \frac{2C_f}{\pi}\left(\frac{L}{D}\right)\Delta T_m\left[1-\cos\left(\frac{\pi x}{L}\right)\right] + \Delta T_m \sin\left(\frac{\pi x}{L}\right)$

Now $\frac{dT_w}{dx} = 0$ at max $T_w = T_s$

differentiate $\boxed{D}$, set it to zero, you get :

$$\left(\frac{x}{L}\right)_{T_w=T_s} = 1 + \frac{1}{\pi}\arctan\left(\frac{-\pi D}{2C_f L}\right)$$ [[in radians]]

sub back into $\boxed{D}$:

equation $\boxed{A}$ $T_s = T_{t_1} + \left[\frac{2C_f L}{\pi D}\right]\Delta T_m\left\{1 + \sqrt{1 + \left(\frac{\pi D}{2C_f L}\right)^2}\right\}$

prior to the analysis, we know : T_s, T_{t_1}, C_f, L, D

So the procedure to analytically solve the case of unshielded reactor:

from $\boxed{A}$ find ΔT_m

from $\boxed{B}$ find T_{t_2}

from $\boxed{C}$ find $T_t(x)$

from $\boxed{D}$ find $T_w(x)$

$$\Delta T_m = T_w - T_t$$

* For both cases A & B,

$$A: \quad \frac{T_{t_2}}{T_s} = \frac{\frac{T_{t_1}}{T_s} + K}{1+K} \quad ; \quad K = \frac{2C_f L}{D}$$

B: also a function of K

as $\frac{C_f L}{D}$ increases $\longrightarrow$ T_{t_2} increases

The problem is that as $\frac{C_f L}{D}$ goes up, P_{t_2} drops!

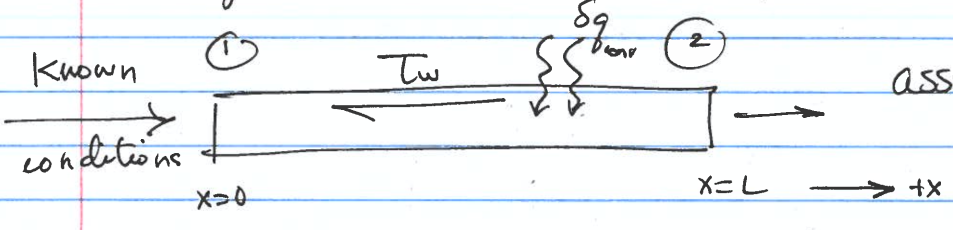
* How can we estimate / calculate P_{t_2} ??

Two WAYS:

I. Estimate P_t drop by using 'average shear' method.

II. Solve the full quasi-1-D differential equations thru channels from ① to ②; thru a reactor core; simultaneous convective heat transfer and wall friction (you can even add radiation heat transfer if you want).

'Average shear' method (approximate)



assume T_{t2} is known

approximate the total shear force on tube walls to be:

$$F_{s_x} = -\frac{1}{2}(\tau_1 + \tau_2)\pi D L$$

let $\tau = \frac{1}{2}\rho V^2 C_f$, $\therefore$ average shear stress $\frac{\tau_1+\tau_2}{2} = \frac{C_f}{4}(\rho_2 u_2^2 + \rho_1 u_1^2)$

X-momentum :

$$\rho_2 u_2^2 A_2 - \rho_1 u_1^2 A_1 + P_2 A_2 - P_1 A_1 = \text{total shear force} \quad : A_1 = A_2$$

$$\rho_2 u_2^2 - \rho_1 u_1^2 + (P_2 - P_1) = \frac{-C_f L}{D}(\rho_2 u_2^2 + \rho_1 u_1^2)$$

rearrange and realize $P = \rho R T$

$$\left(1+\frac{C_f L}{D}\right)\frac{P_2}{RT_2}\frac{\gamma u_2^2}{\gamma} - \frac{P_1 \gamma u_1^2}{RT_1 \gamma}\left(1 - \frac{C_f L}{D}\right) + P_2 - P_1 = 0$$

$$P_2\left(1+\gamma M_2^2\left(1+\frac{C_f L}{D}\right)\right) - P_1\left(1+\gamma M_1^2\left(1-\frac{C_f L}{D}\right)\right) = 0$$

So :

$$\boxed{A} \quad \frac{P_2}{P_1} = \frac{1+\gamma M_1^2\left(1-\frac{C_f L}{D}\right)}{1+\gamma M_2^2\left(1+\frac{C_f L}{D}\right)}$$

$$\boxed{B} \quad \frac{P_{t_2}}{P_{t_1}} = \frac{P_2}{P_1}\left\{\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right\}^{\frac{\gamma}{\gamma-1}}$$

continuity : $\rho_2 u_2 = \rho_1 u_1 \Rightarrow \frac{P_2}{RT_2} M_2 \sqrt{\gamma R T_2} = \frac{P_1}{RT_1} M_1 \sqrt{\gamma R T_1}$

$$\frac{P_2}{P_1}\frac{M_2}{M_1} = \sqrt{\frac{T_2}{T_1}} = \sqrt{\frac{T_{t_2}}{T_{t_1}}}\left[\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right]^{1/2}$$

$$\boxed{C} \quad \sqrt{\frac{T_{t_2}}{T_{t_1}}} = \frac{M_2}{M_1}\left[\frac{1+\gamma M_1^2\left[1-\frac{C_f L}{D}\right]}{1+\gamma M_2^2\left[1+\frac{C_f L}{D}\right]}\right]\left[\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right]^{1/2} \rightarrow \text{find } M_2$$

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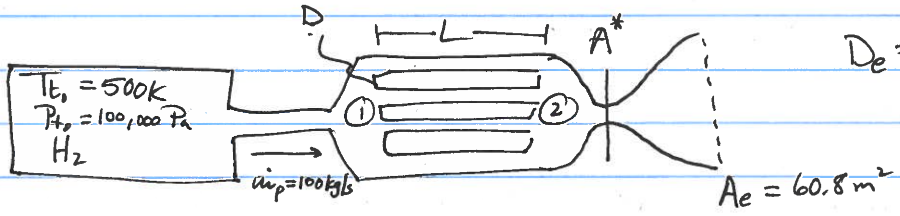
EXAMPLE: Space based nuclear thermal rocket (using the average shear method)



let $C_f = 0.005$, $L = 5m$, $D = 0.01m$ constant power distribution
$T_s = 2950K$, $M_1 = 0.1$

$T_{t_1} \sim T_{t_0}$ (adiabatic no shaft work), $P_{t_1} \sim P_{t_0}$ (isentropic)

$$\frac{T_{t_2}}{T_s} = \frac{\frac{T_{t_1}}{T_s} + \frac{2C_fL}{D}}{1 + \frac{2C_fL}{D}} = 2542\ K$$

$\boxed{C}$

from $$\frac{M_2}{M_1}\left\{\frac{1+\gamma M_1^2\left(1-\frac{C_fL}{D}\right)}{1+\gamma M_2^2\left(1+\frac{C_fL}{D}\right)}\right\}\left\{\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right\}^{1/2} = \sqrt{\frac{T_{t_2}}{T_{t_1}}}$$

$$\Rightarrow M_2 = 0.47 \quad ; \quad \frac{P_2}{P_1} = \frac{1+\gamma M_1^2\left(1-\frac{C_fL}{D}\right)}{1+\gamma M_2^2\left(1+\frac{C_fL}{D}\right)} = 0.4538$$

lowest

$$\therefore P_{t_2} = P_{t_1}\left\{\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right\}^{\frac{\gamma}{\gamma-1}} \cdot \frac{P_2}{P_1} = 52\ 964\ Pa$$

Let $P_{te} \cong P_{t_2}$ & $T_{te} \cong T_{t_2}$ (isentropic nozzle)

from $$\dot{m}_p = \frac{P_{te}A_e}{\sqrt{T_{te}}}\sqrt{\frac{\gamma}{R}}\,M_e\left(1+\frac{\gamma-1}{2}M_e^2\right)^{\frac{-(\gamma+1)}{2(\gamma-1)}}$$

$$\Rightarrow M_e = 3.5$$

$$P_e = \frac{P_{te}}{\left(1+\frac{\gamma-1}{2}M_e^2\right)^{\frac{\gamma}{\gamma-1}}} = 695.4\ Pa$$

$$T_e = \frac{T_{te}}{1+\frac{\gamma-1}{2}M_e^2} = 737K \Rightarrow u_e = M_e\sqrt{\gamma R T_e} = 7247\ m/s$$

$$\text{Thrust} = \dot{m}_p u_e + (P_e - \cancel{P_a}^{\,0})A_e = 766.955\ kN$$

$$I_{sp} = \frac{\text{Thrust}}{\dot{m}_p g_0} = 782\ s$$

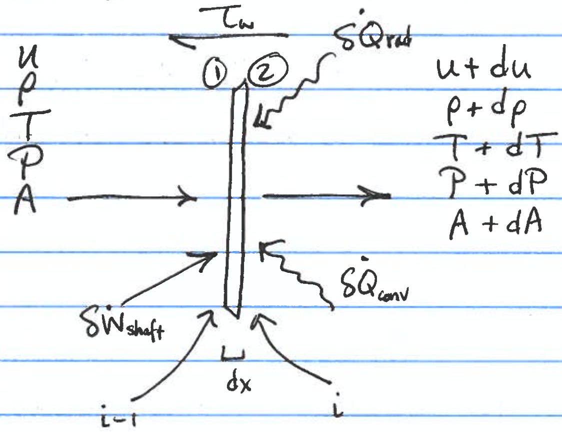
$$\dot{Q}_{react} = \dot{m}_p C_{p_{H_2}} (T_{t_2} - T_{t_1}) = 2.92 \times 10^9 \text{ Watts}$$

$$C_{p_{H_2}} = \frac{R\gamma}{\gamma - 1} = \frac{R_u \gamma}{M_w (\gamma - 1)} = 14438 \text{ J/kg}\cdot\text{K}$$

The rocket is very very big!

II. Solving General Differential Quasi-1D Flow Equations with Friction, Convective Heat Transfer (Steady Flow) constant γ & c_p.

$\longrightarrow$ This will find all fluid parameters including P_{t_2}



the differential area that shear and convective heat transfer act on is the circumference by dx

$$c \cdot dx = dS_{wetted}$$

from the general C.V. form of governing equations:

① $\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$ (continuity)

② $\frac{dP}{\rho} + u\,du = \frac{-\tau_w c\,dx}{\rho A} + \eta\, \delta \dot{W}_{shaft}$ (momentum)

③ $c_p dT + u\,du = \delta q_{\substack{across \\ boundary}} + \delta W_{shaft}$ (energy)

$$u\,du = d\left(\frac{u^2}{2}\right)$$

④ $\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$ (from $P = \rho R T$) ∴ R = const.

$\delta W_{shaft} \longrightarrow [J/kg]$

it is $\frac{\delta \dot{W}_{shaft}}{\dot{m}}$, similarly, $\delta q = \frac{\delta \dot{Q}_{rad}}{\dot{m}} + \frac{\delta \dot{Q}_{conv}}{\dot{m}}$

η = a 2nd Law effectiveness of the work interaction process

External work addition + to flow :

- for a positive $+\delta W_{shaft}$ (like compressors) η ranges from 0 to 1.0

if $\eta < 1.0 \Rightarrow$ non-isentropic (non-ideal) work
if $\eta = 0 \Rightarrow$ all external work turns into heat :(

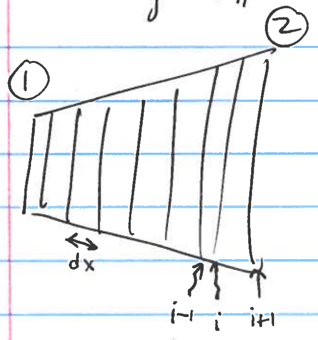
- for a negative $-\delta W_{shaft}$ out of flow (like turbines), work extraction, η is greater than 1.0 (effectiveness, not efficiency)

$\eta > 1.0$

if $\eta = 1.0 \Rightarrow$ isentropic

* How do we solve the four differential equations for a flow?

- first, 'discretize' the physical domain (the channel) into many differential elements $= N_{TOT}$



i = step number counter, ranges from 1 to N_{tot}
the simplest approach is to use the direct 'explicit' method.
i.e. $du = u_i - u_{i-1}$
$i \longrightarrow$ current step (unknown parameters)
$i-1 \longrightarrow$ previous step at $x - dx$ (known parameters)

So, $dP = P_i - P_{i-1}$

so, in general, four differential equations, have four unknowns : P_i, ρ_i, u_i, T_i

* handling the nonlinear terms:

$$\frac{d\rho}{\rho} = \frac{\rho_i - \rho_{i-1}}{\rho_{i-1}}$$

$\rightarrow$ reduce, algebriacally, to one equation, one unknown.
(say, solve for u_i, or even du)

$\rightarrow$ solve for u_i, then back substitute for the other parameters for T_i, ρ_i, P_i

$$\rho_i = \frac{\rho_{i-1} u_{i-1} A_{i-1}}{u_i A_i}$$

& $T_i \Rightarrow$ from $c_p T_i + \frac{u_i^2}{2} = c_p T_{i-1} + \frac{u_{i-1}^2}{2} + \delta q_{conv(i)} + \delta W_{shaft(i)} + \delta q_{rad(i)}$

& $P_i = \rho_i R T_i$

iterate from i to $i+1$ $\longrightarrow$ repeat until $i = N_{Tot} + 1$

Some issues:

a) Step size

b) work for subsonic & supersonic (will not work through a throat)

c) Check your special cases

d) Check overall conservation at the end (from ① to ②)

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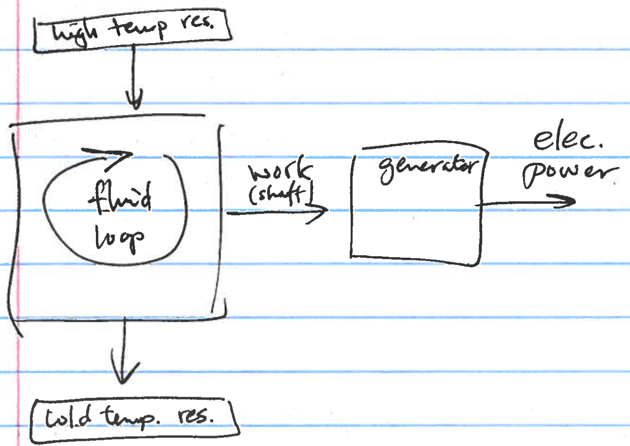
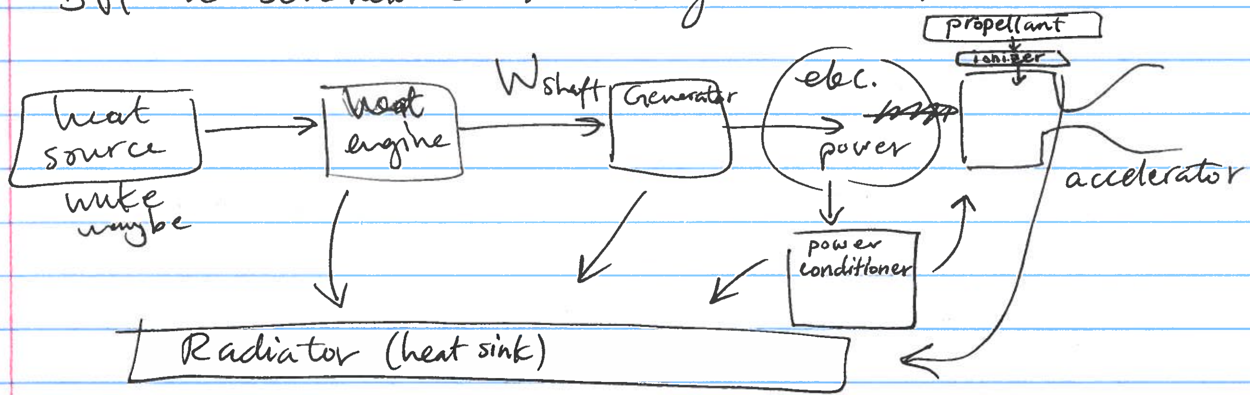
Electrical Rockets

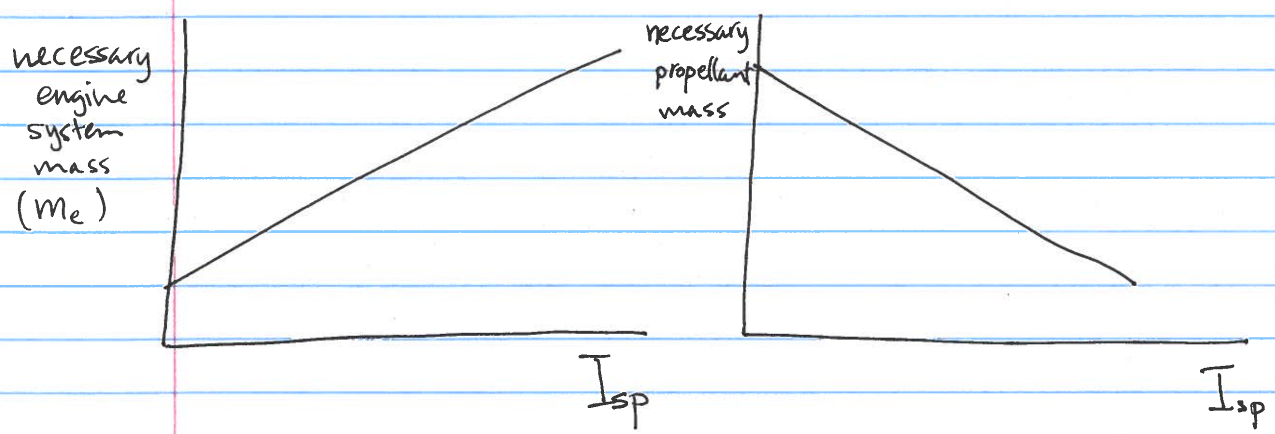
The electrical equipment necessary to get a high I_{sp} can be massive; the higher the I_{sp}, the higher the support mass (engine / accelerator), m_e



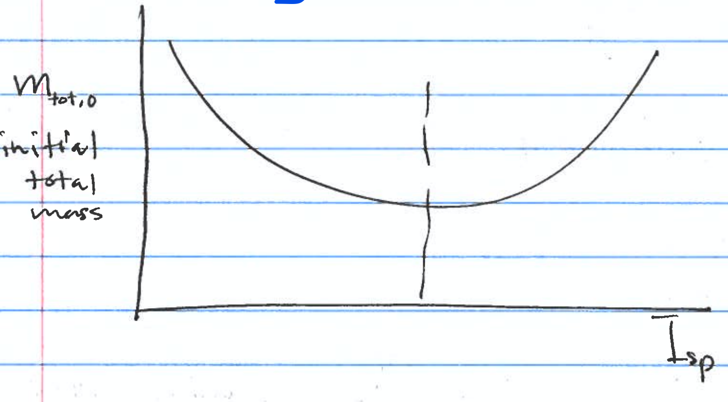
But a schematic for a generic elec. rocket:

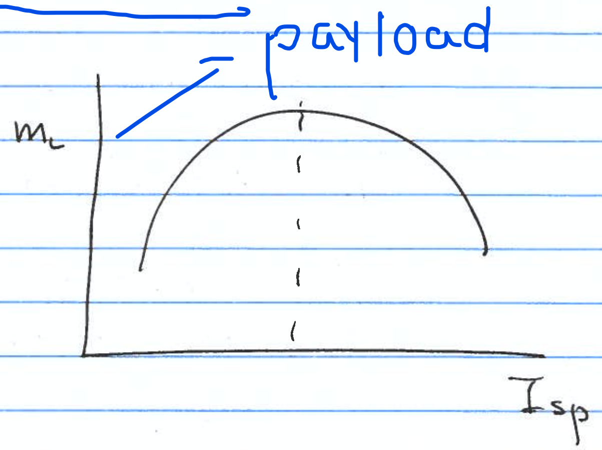


* Selection of the optimum I_{sp} for a generic electric rocket



- Develop a method to define the optimum $I_{sp})_{opt}$ which minimizes the overall initial mass.

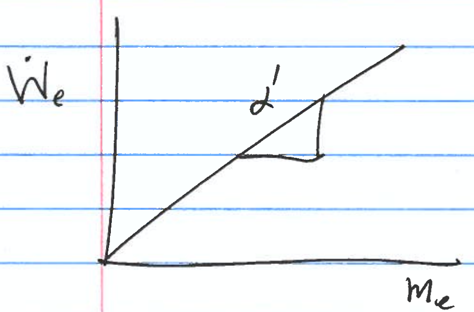




where $m_{tot,o} = m_e + m_p + m_L$

- as a simple approximation, say power delivered to the accelerator, $\dot{W}_e$ is proportional to the system's mass.

$$\dot{W}_e = \alpha' m_e \quad \text{(linear approximation)}$$



α' is technology dependent

also, assume that the kinetic energy of the propellant at the exit plane is much greater than the thermal internal energy.

effective exhaust velocity $c = u_e$ actual velocity

$$\eta_a \dot{W}_e = \dot{m}_p \frac{c^2}{2} \quad : \eta_a = \text{accelerator efficiency}$$

$$\therefore \; m_e = \frac{\dot{m}_p}{\alpha' \eta_a}\left(\frac{c^2}{2}\right)$$

remember, $I_{sp} = \frac{c}{g_0}$

optimizing c is optimizing I_{sp}

Total mass of the vehicle:

$m_{tot,0} = m_e + m_p + m_L$

The firing time, τ; $\tau = \frac{m_p}{\dot{m}_p}$ assuming constant $\dot{m}$

$$m_e = \frac{m_p}{\alpha' \eta_a \tau}\left(\frac{c^2}{2}\right)$$

let $m_{tot,0} = m_0$ total initial mass of the vehicle.

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$$\frac{m_L}{m_0} = 1 - \frac{m_p}{m_0}\left[1 + \frac{c^2}{2\alpha' \eta_a \tau}\right]$$

– we know for free-space rockets (no gravity, no drag)

$$\frac{m_0 - m_p}{m_0} = e^{\frac{-\Delta V}{c}}$$

ΔV = absolute change in velocity of rocket during firing

OR $$1 - e^{\frac{-\Delta V}{c}} = \frac{m_p}{m_0}$$

for convenience, $\alpha \triangleq \frac{\Delta V}{c}$

$$\therefore \frac{m_L}{m_0} = 1 - (1 - e^{-\alpha})\left[1 + \frac{\Delta V^2}{2\alpha^2 \alpha' \eta_a \tau}\right]$$

also for convenience, $\beta \triangleq \frac{\Delta V}{\sqrt{2\alpha' \eta_a \tau}}$

$$\therefore \boxed{\frac{m_L}{m_0} = 1 - (1 - e^{-\alpha})\left[1 + \left(\frac{\beta}{\alpha}\right)^2\right]}$$

this is valid whether or not the rocket is optimized

– We would like to maximize $\frac{m_L}{m_0}$; differentiate $\frac{m_L}{m_0}$ with respect to α; set equal to zero.

recall, $\alpha = \frac{\Delta V}{c}$, $I_{sp} = \frac{c}{g_0}$

$$0 = -e^{-\alpha}\left[1 + \left(\frac{\beta}{\alpha}\right)^2\right] - (1 - e^{-\alpha})\left[-2\beta^2 \alpha^{-3}\right]$$

$$\therefore \boxed{\left(\frac{\beta}{\alpha}\right)^2 = \frac{\alpha}{2(e^{\alpha} - 1) - \alpha}}$$ only for optimal rocket

The previous equation relates τ_{opt} & c_{opt}

hence at α_{opt} :

$$\left.\frac{M_L}{M_0}\right)_{opt} = 1 - \left[1 - e^{-\alpha_{opt}}\right]\left\{1 + \frac{\alpha_{opt}}{2(e^{\alpha_{opt}} - 1) - \alpha_{opt}}\right\}$$

conversly, for a targeted / perscribed $\frac{m_L}{m_0}$; we can find α_{opt}.

$$\therefore \alpha = \frac{\Delta V}{c} = \frac{\Delta V}{I_{sp}\, g_0}$$

If we have required ΔV, a desired $\frac{m_L}{m_0}$, $\eta_a \alpha'$ [W/kg]

we can solve for α_{opt}, then $I_{sp_{opt}}$, then β_{opt}, then τ_{opt}

Example: Electric Rocket Selection.

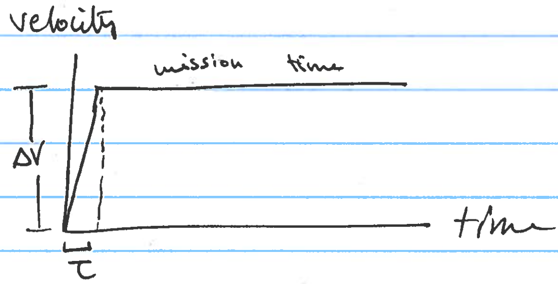
_ we want a 6-month mission, travel distance of 1.56×10^{11} m

ion drive $\rightarrow$ $I_{sp} = 5000$ s

let $\alpha' \eta_a = 100$ W/kg ; ignore any deceleration required (velocity match at destination).

Let $m_L = 100{,}000$ kg ; find a 'reasonable' rocket config.

also ignore accel / decel legs in ΔV calculations



$$\therefore \Delta V_{req} = 10{,}000 \text{ m/s}$$

$\boxed{A}$ What is the smallest possible m_0 and corresponding τ_{opt} for such mission and configuration of rocket?

given m_L, ΔV, I_{sp}

$m_L \longrightarrow \tau$ will fall out

$\alpha = \frac{\Delta V}{c}$: $c = I_{sp}\, g_0 = 49{,}050$ m/s

$\therefore\ \alpha = .204$

for optimal rocket : $\left(\frac{\beta}{\alpha}\right)^2_{opt} = \left[\frac{\alpha}{2(e^{\alpha}-1)\alpha}\right]_{opt}$

$\therefore\ \beta_{opt} = .1848$

and $\beta = \frac{\Delta V}{\sqrt{2\alpha'\eta_a \tau}}$ $\Longrightarrow$ $\tau_{opt} = 169$ days! problem!

$$\frac{m_L}{m_0} = 1 - (1-e^{-\alpha})\left[1 + \left(\frac{\beta}{\alpha}\right)^2\right]$$

$\therefore\ \left.\frac{m_L}{m_0}\right)_{opt} = 0.664$ $\Longrightarrow$ $m_{0,min} = 150{,}602$ kg

$\frac{m_P}{m_0} = 1 - e^{-\frac{\Delta V}{c}} = 0.18454$ $\Longrightarrow$ $m_p = 27{,}792$ kg

$\frac{m_e}{m_0} = \frac{m_P}{m_0}\frac{1}{\alpha'\eta_a\tau}\left(\frac{c}{2}\right)^2 = .15203$ $\Longrightarrow$ $m_e = 22{,}896$ kg

$\boxed{B}$ τ is an issue (as seen in $\boxed{A}$); so what's the minimum τ_b?
[won't be optimum]

let $\frac{m_L}{m_0} \longrightarrow 0$ (very large m_0)

$$0 = 1 - (1-e^{-\alpha})\left[1 + \left(\frac{\beta}{\alpha}\right)^2\right]$$

Solve for $\beta_{limit} = .4288$ $\Longrightarrow$ $\tau_{b,min} = 31.5$ days

that's with the given ΔV, I_{sp}, $\alpha'\eta_a$ $\longrightarrow$ very large m_0!!

$\boxed{c}$ Allow 2 months firing time ; $m_L = 110{,}000$ kg
(still non optimal) using the same ΔV

$\therefore \ \tau_b = 62$ days , $\beta = .3055$

$\frac{m_L}{m_o} = .4016$, $\frac{m_p}{m_o} = .1845$, $\frac{m_e}{m_o} = .4144$

$m_o = 273{,}904$ kg
$m_p = 50{,}546$ kg
$m_e = 113{,}506$ kg

$\dot{m}_p = \frac{m_p}{\tau_b} = 0.01$ kg/s

Thrust $= \dot{m}c = 491$ N

$a_{initial} = \frac{\text{Thrust}}{m_o} = 0.0018$ m/s^2 nobody's in danger of getting smashed.

$a_{final} = \frac{\text{Thrust}}{m_o - m_p} = 0.002$ m/s^2

Example: Electric rocket from LEO to Moon, $\Delta V_{req} = 6154\ \frac{m}{s}$

arcjet $\longrightarrow$ use H_2O as a propellant ; $I_{sp} = 1500\ s$

$m_p = 1815\ kg$. Estimate the:

a) Overall vehicle mass, m_0

b) Maximum payload mass possible, m_L

c) τ_b, assume $\alpha' \eta_a = 100\ W/kg$

d) Thrust

Solution:

a) $e^{\frac{\Delta V}{c}} = \frac{m_0}{m_0 - m_p}$; $c = I_{sp} g_0 = 14{,}715\ m/s$

$\therefore e^{\frac{\Delta V}{c}} = \frac{1}{1 - \frac{m_p}{m_0}} \Rightarrow \frac{m_p}{m_0} = .342 \Rightarrow m_0 = 5310.5\ kg$

b) & c) want maximum m_L (optimal)

$\alpha = \frac{\Delta V}{c} = .4182 = \alpha_{opt}$

$\beta_{opt} \Rightarrow \left(\frac{\beta}{\alpha}\right)^2_{opt} = \left[\frac{\alpha}{2(e^{\alpha}-1)-\alpha}\right]_{opt} \Rightarrow \beta_{opt} = .3434$

$\therefore \tau_{opt} = 446\ hrs$

$\left.\frac{m_L}{m_0}\right)_{opt} = 1 - (1 - e^{-\alpha_{opt}})\left[1 + \left(\frac{\beta}{\alpha}\right)^2_{opt}\right] = .4278$

$\therefore m_{L_{opt}} = 2272\ kg$ maximum

d) Thrust $= \dot{m} c = \frac{m_p}{\tau} c = 16.6\ N$

$m_e = 1224\ kg$

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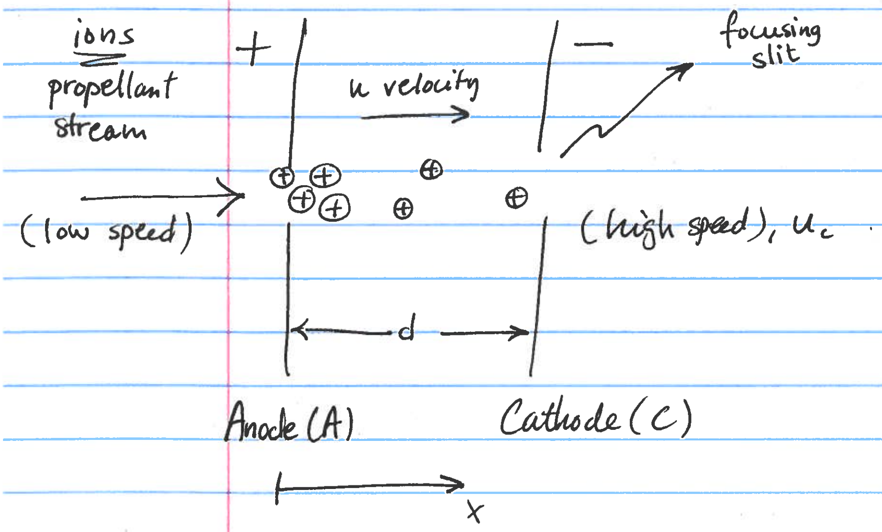
Ion Rocket (Electro-Static)

Consider an ion rocket as modeled by 'plane diode'



– An electric field between the plates is established due to the voltage difference betwe[en] the plates. The field exists whether or not there are propellant ions between the plates.

$u_c \gg u_a$

By definition, electric work done on a unit charge [1 Coulomb] in moving it from point 1 to point 2 is the potential difference [in Volts], $\Delta\phi$ between those to points.

$$\therefore \quad \text{Volt} = \frac{\text{Joule}}{\text{Coulomb}}$$

So for a charge q in Coulombs, the work done is

$q(\phi_1 - \phi_2) \longrightarrow$ Joules

Also, local Electric force, F, at any point between the plates is defined since $F \cdot dx = q\, d\phi$

$$F = \text{local accelerative force} = q\,\frac{d\phi}{dx}$$

for an ion of mass m and charge q, the work done is the change in kinetic energy from Anode to the x-location of interest.

$$q(\phi_A - \phi) = \tfrac{1}{2}mu^2 - \tfrac{1}{2}mu_A^2$$ Anode

At the Cathode, then,

$$u_c^2 = 2\left(\frac{q}{m}\right)(\phi_A - \phi_c) + \cancel{u_A^2}^{\sim 0}$$

$$\therefore u_c = \sqrt{2\left(\frac{q}{m}\right)(\phi_A - \phi_c)}$$

$[\,\approx u$

$$I_{sp} = \frac{u_c}{g_0} = \frac{1}{g_0}\sqrt{2\left(\frac{q}{m}\right)(\phi_A - \phi_c)}$$

$$\text{Thrust} = \dot{m}_c u_c \;;\; \dot{m}_c = \rho_c u_c A_c = m\eta_{ic} u_c A_c$$

where, m = mass of one ion

η_{ic} = Number of ions / volume (ion number density) at the cathode

A_c = cross-sectional area of the focusing slit

$$\therefore \text{Thrust} = m\eta_{ic} u_c^2 A_c$$

we want to put this in terms of electric current

[Amps] - Current is the amount of charge flow per second $= q\eta_{ic}u_c A_c$

[Amps/m²] - Current per cross-sectional area (flux) $= j = u_c q \eta_{ic}$

$$\eta_{ic} = \frac{j}{u_c q} \;;\; \text{Thrust per area} = 2m\eta_{ic}\left(\frac{q}{m}\right)(\phi_A - \phi_c)$$

Combine with the u_c definition:

$$\boxed{\frac{\text{Thrust}}{\text{Area}} = \left(\frac{m}{q}\right) j \sqrt{2\left(\frac{q}{m}\right)(\phi_A - \phi_c)}^{\,1/2}}$$

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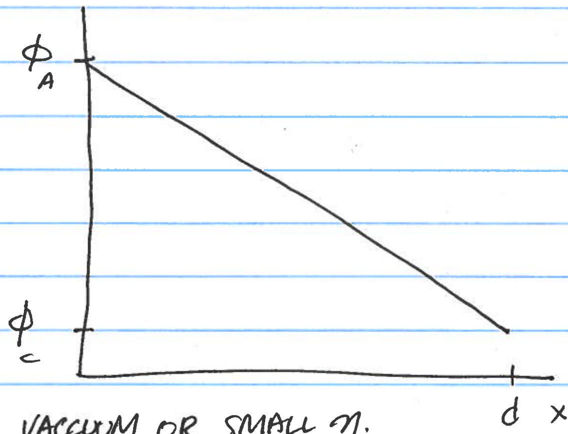


IN VACCUM OR SMALL η_i

* $\frac{\text{Thrust}}{\text{Area}}$ depends on $\left(\frac{m}{q}\right)$, Voltage, and supplied current per unit area.

* Question: Can we make the current per unit area as large as we want? i.e. can we provide as many ions per second to the anode as we want and assume that they will accelerate between the anode and cathode?

– No, the presence of many positively ~~ion~~ charged ions between the plates affects the local electric field (potential difference distribution) and lowers the accelerative force that a given ion experiences. Recall, $F_{local} = q\frac{d\phi}{dx}$

* Maxwell's Second Equation describes the variation of voltage ~~due~~ with η_i at any x.

$$\frac{d^2\phi}{dx^2} = \frac{-\eta_i q}{\varepsilon_0}$$

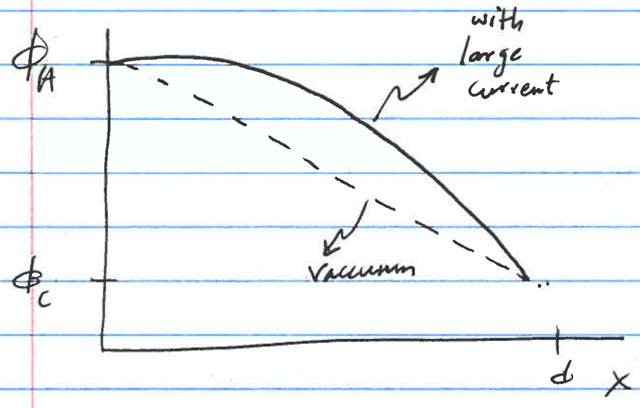
: ε_0 = universal constant of 'permitivity of free space' $= 8.85 \times 10^{-12}\ C^2/N.m^2$

– describes the concavity of ϕ vs. x between the plates

– recall $j = q\eta_i u$

$$\varepsilon_0 u \frac{d^2\phi}{dx^2} = -j \quad \text{OR} \quad j = \left[-\varepsilon_0\sqrt{2\left(\frac{q}{m}\right)(\phi_A - \phi)}\right]\frac{d^2\phi}{dx^2}$$

The presence of the positively charged ions between the plates tends to 'pull up' the potential across x, especially largely near the anode where n_i is large.



- eventually, at high enough j; j_{max}, $\frac{d\phi}{dx} = 0$ which is at the anode

NO Accel.

recall, $F_{local} = q\frac{d\phi}{dx} \Rightarrow$ when $\frac{d\phi}{dx} = 0$ at anode $\Rightarrow F = 0$
i.e. no accelerative force at the anode! So, there is a natural physical limit on the ions per second that can be supplied (max current per unit area).

$\longrightarrow$ Max possible j for a given device.

* The Child-Langmuir Law maximizes j subject to constraint that $\frac{d\phi}{dx} = 0$ at $x = 0$, obtains:

$$j_{max} = 0.4444\, \varepsilon_0 \sqrt{2\left(\frac{q}{m}\right)}\, \frac{(\phi_A - \phi_C)^{3/2}}{d^2}$$

$$\left.\frac{\text{Thrust}}{\text{Area}}\right)_{j_{max}} = 0.8888\, \varepsilon_0 \left(\frac{\phi_A - \phi_C}{d}\right)^2 = 0.2222\, \varepsilon_0\, g_0^4 \left(\frac{m}{qd}\right)^2 I_{sp}^4$$

$\therefore$ The thrust per unit area for the ion drive is limited by 'space charge'; limitation in rate at which ions are attracted off anode due to ~~pressure~~ presence of ions downstream.

* I_{sp}'s can be quite large > 7000 sec

But Thrust magnitudes are very small!!

Example: An electrostatic (ion drive) uses heavy particles with charge-to-mass, $\left(\frac{q}{m}\right) = 500\ C/kg$ providing an I_{sp} of 3000 s.

a) What is the acceleration voltage, $(Q_A - Q_c)$ will be necessary

$- I_{sp} = \frac{u_c}{g_0}$: $u_c = 29{,}340\ m/s$

$$I_{sp} = \frac{1}{g_0}\sqrt{2\left(\frac{q}{m}\right)}(Q_A - Q_c)^{1/2} \Rightarrow (\phi_A - \phi_c) = 866{,}125 \text{ Volts}$$

and for $I_{sp} = 6000s$, $(\phi_A - \phi_c) = 3{,}464{,}500$ Volts (not linear variation)

b) With space charge limitation current and maximum allowable gradient of 10^5 V/cm, what is the accel. distance, d, and the diameter of a round beam providing 2.23 N of thrust?

$d > 8.66 \rightarrow$ Arcing!

$$\frac{\phi_A - \phi_c}{d} = 10^5\ V/cm \quad \text{so} \quad d = 8.66\ cm \quad (\text{minimum})$$

$$\left.\frac{\text{thrust}}{\text{Area}}\right)_{max\,j} = .2222\,\varepsilon_0\, g_0^{\,4}\left(\frac{m}{q\,d}\right)^2 I_{sp}^{\,4} = 786.9\ N/m^2$$

or for thrust $= 2.23\ N \Rightarrow$ Area $= .00283\ m^2$

$\therefore$ diameter of round beam $= 0.06\ m$

$$\dot{m}_{propellant} = \dot{m}_c = \frac{\text{Thrust}}{u_c} = 7.6 \times 10^{-5}\ kg/s$$

$$c)\ j = j_{max} = .\cancel{4444}\,\varepsilon_0\sqrt{2\left(\frac{q}{m}\right)}\,\frac{(\phi_A - \phi_c)^{3/2}}{d^2} = 13.37\ \text{Amps}/m^2$$

$$\text{Power} = VI = (866125\ V)\left(13.37\ \frac{\text{Amps}}{m^2}\right)(\text{Area of beam})$$

Power = 32.7 kW

It is possible to increase $\frac{\text{Thrust}}{\text{Area}}$ by decreasing d

but limited 'd' due to arcing or shorting. [2m]

d) Use colloids (heavy particles) like cesium ions
(have large $\left(\frac{m}{q}\right)$)

→ difficult to focus the beam directly thru slits or hole in cathode because of their large mass.

[redacted]

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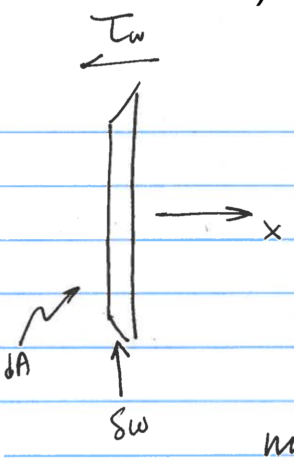
6.3

Brief review of quasi-one-dimensional differential force on wetted surface (due to pressure, shear, external work interaction)



$$\underset{\substack{\text{internal}\\\text{wetted}\\\text{surface}}}{dF_x} = -P\,dA + \tau_w c\,dx - \eta\,\delta w\,\rho A$$

$$\frac{dP}{\rho} + u\,du = -\frac{\tau_w c\,dx}{\rho A} + \eta\,\delta w$$

multiply by ρA :

$$A\,dP + \dot{m}\,du = -\tau_w c\,dx + \eta\,\delta w\,\rho A$$

$$-P\,dA + d(PA) + \dot{m}\,du = \quad '' \quad ''$$

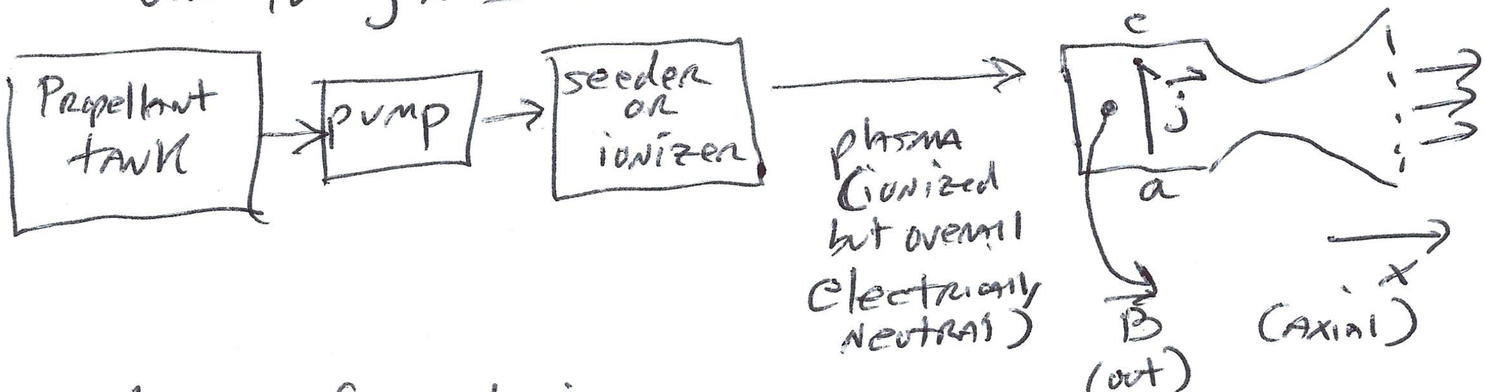
$$\underline{d(PA) + \dot{m}\,du} = P\,dA - \tau_w c\,dx + \eta\,\delta w\,\rho A$$

STREAM THRUST $= \dot{m}u + PA$

Electromagnetic Rocket example (Quasi-1-D Approximate model)

'Lorentz force Accelerator'

or Magneto-hydrodynamic (MHD) Accelerator:
Relies on 'body force' (force/mass in propellant) generated due to $\vec{j} \times \vec{B}$ Lorentz force in a plasma



Lorentz force devices:

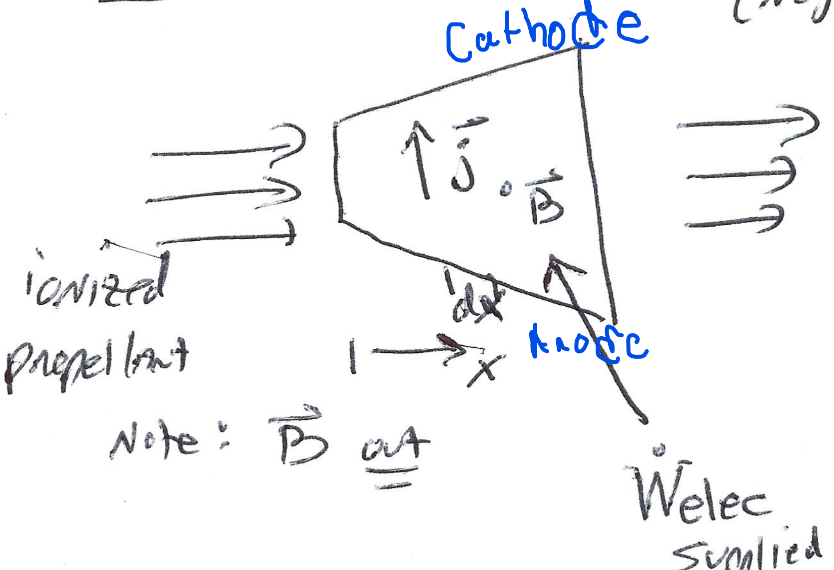
* For an accelerator (as shown above) free electrons present in the moving plasma carry the current through the plasma between anode & cathode (a & c). Due to the $\vec{j} \times \vec{B}$ Lorentz force in the direction of plasma flow, they accelerate in x direction. This requires electrical energy input. The electrons transmit this force via collisional processes to the bulk of the plasma; i.e., there is a net conversion of supplied electrical energy from surroundings to kinetic energy of propellant.
* Since the amount/availability of free electrons per sec is dependent on $\dot{m}$, u of propellant flow, the actual established current $\vec{j}$ is then itself dependent on $\dot{m}$, u of plasma, i.e. electric power requirements depend on flow

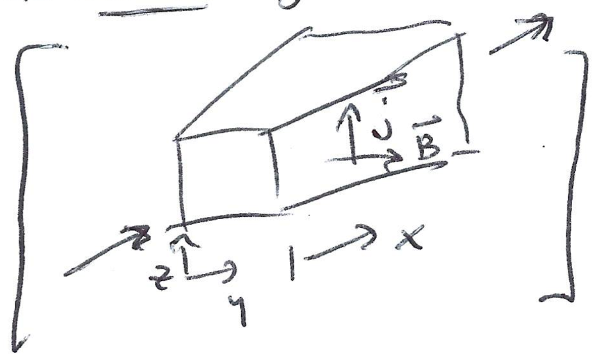
properties (& vice versa).

* Note on power generation. For a MHD generator, electric work is produced at expense of kinetic energy of plasma ($\vec{B}$ field reversed)

Quasi-1-D differential model of Lorentz force plasma accelerator

(Neglects induced magnetic field effects!)





Define a local accelerator 'effectiveness' at a station x (descriptive of work/force interactions across a dx)

$$\eta = \frac{dF_{x(body)} \cdot U}{\delta \dot{W}_{elec\ supplied} \cdot \dot{m}}$$

(η is sometimes called the 'load factor')

$dF_{x(body)}$ at a station x = body force in x direction per unit mass of plasma, that is acting on plasma (propellant) across a dx. (Units Newtons)

v = local axial velocity of plasma (m/s)

$\delta w_{elec\ supplied}$ = electrical work per mass supplied from surroundings to maintain the current, i.e., establish the applied electric field from anode to cathode.

I = current (flow of charge/sec) (Coulombs/sec) Coulomb → unit of charge

σ_0 = electric conductivity of propellant (degree of ionization) units → mho's Ω Ω

B = strength of applied magnetic field (in Teslas) (say associated with permanent magnet)

A = cross-sectional area of duct ($= h \times w$) (m^2)

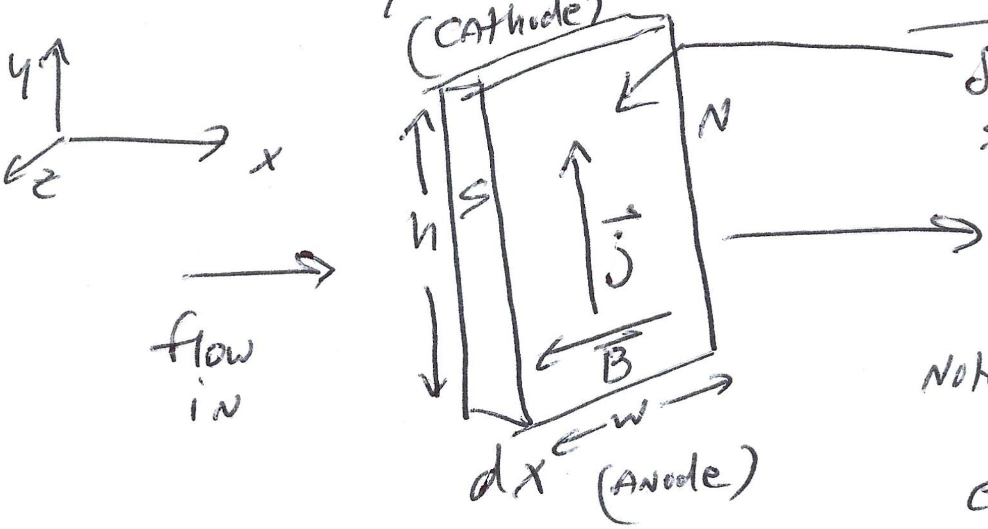
V = applied potential difference; work done by electric field per unit charge (Volts or J/coulomb)

j = current per cross-sectional area of conductor (not duct!); so for a differential slab of plasma, area current j going through is $w \cdot dx$ → $\frac{\text{Coulombs}}{\text{sec}}$ per area

⇐ Note 'dropping' of vector notation

E_y = applied electric field (N/coulomb)

Electrical / force considerations : (differential 'slab' of plasma)



$\delta \dot{W}_{elec\ supplied} = \delta W_{elec\ supplied} \cdot \dot{m}$

Note current j {Coulombs/sec m^2} established in vertical direction due to electric field (Force/Coulomb)

$A = h \times w$ (cross-sectional Area of duct)

(1) B has units of Teslas $\left\{ \text{or } \dfrac{N \cdot S}{\text{Coulomb} \cdot m} \right\}$

j has units of $C/sec/m^2$ C → Coulombs

(electric current per cross-sectional area of conductor → cross-sectional in direction of current flow!)

$j \times B$ then has units of force / m^3

so... axial (differential) Lorentz force $dF_{x(body)}$ on a differential 'slab' of plasma with volume $A\,dx$ is

$$(A\,dx)\,j\,B = A\,j\,B\,dx = dF_{x(body)}$$

{with $\vec{j} \perp$ to $\vec{B}$ as shown & assumed for quasi-1-D model}

2 externally supplied power required across a differential element dx = $\underline{V \cdot I = \delta W_{\substack{elec \\ supplied}} \cdot \dot{m}}$

where $I = j \times \underbrace{w \times dx}$ → cross-sectional area in direction of current flow

(This is current through differential volume $dx \cdot h$)

& $\underline{V = E_y \cdot h}$ E_y → force / Coulomb

(this by definition of electric field between anode & cathode)

so

$$V \cdot I = \delta W_{\substack{elec \\ supplied}} \cdot \dot{m} = E_y \cdot A \cdot j \cdot dx \qquad A \to h \times w$$

3 Then since η (acceleration efficiency) $= \dfrac{dF_{x(body)} \cdot U}{\delta W_{\substack{elec \\ supplied}} \cdot \dot{m}}$

(U = axial velocity of plasma)

$$\eta = \frac{(A j B dx) U}{E_y A j dx}$$

($AjBdx$ from 1)

or $\boxed{\eta = \dfrac{UB}{E_y}}$ 'load factor' or γ for MHD!

really 2nd law effectiveness (No induced effects considered)

Note: this η is exactly representative of the general η used earlier in the 'general' characterization of work interaction in diff. equations!

4 If plasma velocity $= 0$ (just anode/cathode with stationary but conductive plasma between them)

$$j = \sigma_0 E_y \text{ by definition of } \sigma_0 \text{ (conductivity)}$$

But due to plasma flow ($u \neq 0$!)

$$\boxed{j = \sigma_0 (E_y - \underbrace{uB}_{}) = \sigma_0 E_y (1 - \eta)}$$

contribution due to plasma flow effect (i.e., $u \neq 0$)

or $j = \sigma_0 E_{y\,(\text{effective})} = \sigma_0 (E_y - E_{y\,\text{down}})$

applied field

correction on field (seen by plasma)

Note: This plasma flow-induced 'correction' on j (or $E_{y\,\text{effective}}$) is due to the fact that free electrons in the plasma are moving with the plasma at velocity u in x direction. Thus there is also an 'effective' axial current 'j_x' (contribution), which in turn produces a 'negative' E_y (down) due to the $\vec{j_x} \times \vec{B}$ Lorentz effect.

→ This E_y (down) reduces the effect of the applied E_y by uB!

Description of 'UB' effect:

Can be obtained using same analysis done in 1 on previous pages but now dealing specifically with this j_x current contribution.

Specifically there is a downward Lorentz force contribution associated with this j_x on the differential 'slab' of plasma with volume $A\,dx$;

i.e. (see 1)

$$\Rightarrow \quad dF_{y_{elec\,(down)}} = A\left(\frac{\text{Coulombs in a differential volume}}{dt \quad A}\right) B\,dx$$

From same analysis in 1 but applied to j_x !!

↑ diff. force (down) (due to plasma movement in x direction) acting on plasma in a diff. volume $A\,dx$

← cross-sectional area of duct! ($j_x \rightarrow x$ direction)

$\boxed{\text{\& } U = \frac{dx}{dt} \text{ !}}$

or

$$\boxed{E_{y_{down}} = \frac{dF_{elec\,(down)}}{\text{Coulombs in a diff. volume } A \cdot dx} = BU}$$

so

$$\boxed{j = \sigma_0 (E_y - UB) = \sigma_0 E_y (1 - \eta)}$$

Note: This is based on the quasi-1-D model with assumptions as noted!

5 can also show that since

$$dF_{x(body)} = A\,dx\,B\,j \quad (\text{from } 1)$$

& $$j = \sigma_0 (E_y - UB) \quad (\text{from } 4)$$

then $$\boxed{dF_{x(body)} = A\sigma_0 B\,dx\,(E_y - UB)}$$

Thus $$\frac{U\,dF_{x(body)}}{\dot{m}} = \frac{A\sigma_0 B^2 U^2 dx}{\dot{m}}\left(\frac{E_y}{UB} - 1\right)$$

$$= \frac{A\sigma_0 B^2 U^2 dx}{\dot{m}}\left(\frac{1}{\eta} - 1\right)$$

Let $$G = \frac{A\sigma_0 B^2 U^2\,dx}{\dot{m}}$$ so

$$\boxed{\frac{U\,dF_{x(body)}}{\dot{m}} = G\left(\frac{1}{\eta} - 1\right)}$$

but since $$\eta = \frac{dF_{x(body)} \cdot U}{\delta W_{\substack{elec\\supplied}} \cdot \dot{m}}$$

$$= \eta\, \delta W_{\substack{elec\\supplied}}$$

& $$\boxed{\delta W_{\substack{elec\\supplied}} = G\left(\frac{1}{\eta}\right)\left(\frac{1}{\eta} - 1\right)}$$

[Note: $\eta = \frac{UB}{E_y}$ (B is applied magnetic field, E_y is applied electric field where $V = E_y \cdot h$)

so local $\eta(x)$ determined by B, E_y <u>And</u> local velocity U !!

Now write the steady quasi-1-D differential equations for 'MHD' flow (assuming frictionless, adiabatic)
(Note → you could include friction / convective / radiative terms if desired!)

Continuity: $$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

X momentum: $$\frac{dP}{\rho} + u\,du = \frac{dF_{x(body)} \cdot u}{\dot{m}} = \eta\, \delta W_{\substack{elec \\ supplied}}$$
$$= G\left(\frac{1}{\eta} - 1\right)$$

energy: $$C_p\,dT + u\,du = \delta W_{\substack{elec \\ supplied}} = G\left(\frac{1}{\eta}\right)\left(\frac{1}{\eta} - 1\right)$$

Eq. of state: $$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$ where $$G = \frac{A\sigma_0 B^2 u^2\,dx}{\dot{m}}$$

Note: this implies if you try to increase $\delta W_{\substack{elec \\ supplied}}$ at constant G, then η will change!!

Since $\eta = \dfrac{dF_{x(body)} \cdot U}{\delta W_{\substack{elec \\ supplied}} \cdot \dot{m}}$ $\left(= \dfrac{UB}{E_y} \right)$

Note that as $\eta \to 1.0$, the ratio of elec. pwr. to force pwr delivered to flow goes to 'unity' which sounds good but note diff. equations → magnitude of power supplied & hence current & force-based power into flow goes to zero as well!

For $\eta < 1$ $\left\{ \begin{array}{l} \text{work addition, } \delta W_{\substack{elec \\ supplied}} \oplus, \\ \text{'compressor' analogy} \end{array} \right\}$

As η decreases further below 1.0, increasing work interaction actually occurs but is less & less efficient at producing an accelerative body force $dF_{x(body)}$ in the momentum equation. The balance of power supplied goes to internal ~~heat~~ heat (or 'Joule heating'), hence total pressure losses mount, etc.

For $\eta > 1$ $\left\{ \begin{array}{l} \text{work extraction, } \delta W_{\substack{elec \\ supplied}} \ominus, \\ \text{'turbine' analogy} \end{array} \right\}$

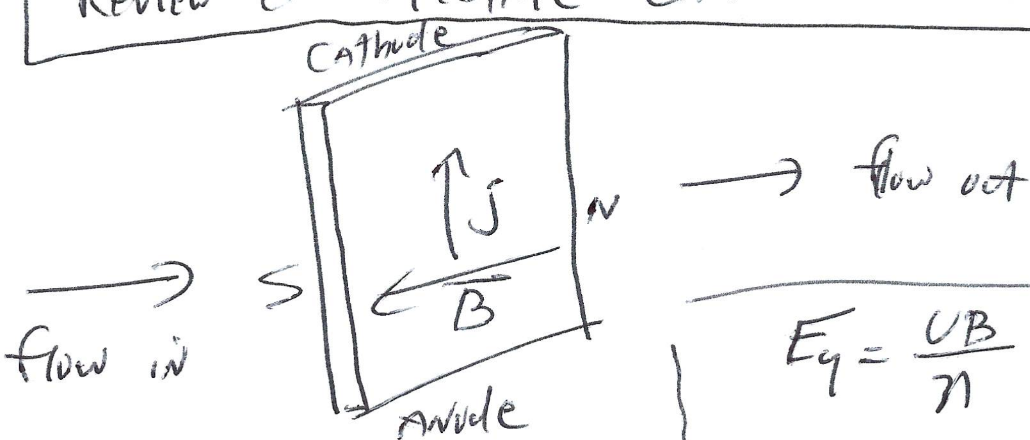
For production of on-board electric power

(power extracted by flow) η must be > 1.0

(Recall $\delta W_{elec. \, supplied}$ is negative)

larger η means more power extracted but with ever-increasing flow losses ...

Review of electric considerations :



$$E_y = \frac{UB}{\eta}$$

E_y is applied electric field between anode & cathode (force / Coulomb)

$j = \sigma_0 (E_y - UB)$ $C/_{s\,m^2}$

current $I = j (w\,dx)$ $C/s \rightarrow$ Amps $C/_s \rightarrow$ amps

{current through a differential element, anode to cathode}

$$\text{Voltage} = V = E_y \cdot h \rightarrow \frac{\text{Work}}{\text{Coulomb}} = \frac{UB}{\eta} \cdot h$$

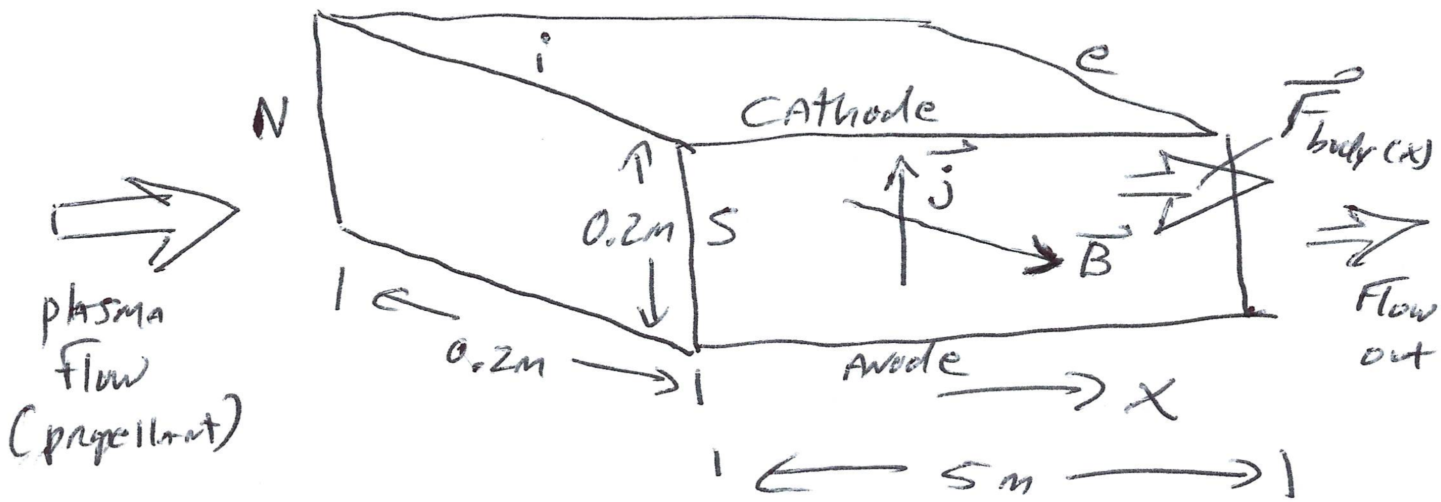
Differential elec. pwr. supplied across a $dx = V \cdot I$

$$\text{Total elec. power supplied in duct} = \int_{\text{Length of duct}} V \cdot I$$

Example of MHD Accelerator :

H_2 propellant, Rectangular duct (no area change)

Choose $\eta = 0.8$, Neglect skin friction; Adiabatic walls



$U_i = 1500$ m/s

$P_i = 100000$ N/m^2

$T_i = 300$ K

$\sigma_0 = 100$ mho's } really

$B = 1$ Tesla } large!

$H_2 \rightarrow R = 4125$ J/kg-K

$\gamma = 1.4$

[Note since $\eta = 0.8$ & B is given $\Rightarrow E_y$ 'falls out' (for local u)

if E_y & B given, η 'falls out' (for local u)]

$M_i = 1.14 \quad T_{t_i} = 378$ K $\quad C_p = R\left(\frac{\gamma}{\gamma-1}\right) = 14438$ J/kg-K

$\dot{m}_i = \dot{m}_e = 4.85$ kg/sec

Integrate through duct (use a computer & solve differential equations via iteration taking small dx, etc.) & obtain:

$M_e = 3.365$

$V_e = 5352.4$ m/s

$T_e = 438$ K

$T_{t_e} = 1430$ K

Thrust = 16322 N

I_{sp} = 343 sec

Note I_{sp} is NOT a measure of thrust efficiency!

$$\text{Thrust efficiency} = \frac{F U_0}{W_{elec}}$$

Overall power into duct across duct length = electric power required = 73.7×10^6 Watts

(also Fluids!) $= \dot{m} C_p (T_{t_e} - T_{t_i})$!!

∗ Note: you can get I_{sp} up alot by expanding in a nozzle! (This is a square duct!)

Now try $\eta = 0.7$: will be less efficient but will put more power into system.

... get

Electric power required

$= \cancel{3}\ 3.66 \times 10^8$ Watts

$V_e = 10191$ m/s $T_e = 2006$ K $P_e = 98399$ N/m²

Thrust = 42076 N $I_{sp} = 885$ sec (even with NO area expansion)

* From $\eta = 0.8$ to $\eta = 0.7$, electric power required went up by a factor of 5 but I_{sp} & thrust went up by ~2.5.

* Obviously electric power requirements will factor into m_e (engine/system mass)

* when skin friction/heat transfer considered, you want to go to larger scale ducts

* Crucially, 3-D effects always reduce performance with other effects neglected.

* No induced current/B fields considered here

* short-circuiting, etc. is an issue (problem)

* expand in a nozzle for higher I_{sp}'s.

* Can also compute E_y, T, I versus x (since η 'fixed')

  & integrate VI over length to get electric power requirements

  (should of course be equal to $\dot{m} c_p (T_{t_e} - T_{t_i})$)

Relationship of the second law effectiveness to the classic polytropic compressor and turbine efficiencies for quasi-one-dimensional modeling of work interaction flow-fields

Write the governing differential equation for quasi-one-dimensional fluid motion for perfect gas with a differential work per mass interaction, δw_{surr} (adiabatic though – no heat interaction across boundary from surroundings and inviscid such there is no shear term in the momentum equation). δw_{surr} is provided across the boundary (i.e. is the work interaction per mass coming from the surroundings) and the flow is assume to be steady This formulation is then as follows:

$$\frac{d\rho}{\rho}+\frac{du}{u}+\frac{dA}{A}=0$$

$$\frac{dP}{\rho}+udu=\eta\delta w_{surr}$$

$$C_P T+udu=\delta w_{surr}$$

$$\frac{dP}{P}=\frac{d\rho}{\rho}+\frac{dT}{T}$$

Note that the second law effectiveness η is the ratio of work actually (effectively) arriving in the flow as a work interaction δw_{eff} to the work provided (work price paid) across the boundary δw_{surr}. The difference $\delta w_{surr}-\delta w_{eff}$ is obviously the lost work (which arrives in the flow as a heat interaction rather than a work interaction due to internal irreversibilities). That 'lost work' increment of course does not appear in the momentum equation since it DOES arrive as heat, not work.

So (from the above) the differential change in entropy (here due to internal irreversibilities within the overall work interaction process) is hence

$$ds=\frac{\delta w_{surr}}{T}(1-\eta)$$

This is a very fundamental formulation from the 2nd law standpoint and works as well for work removal (negative δw_{surr}) as for work input (positive δw_{surr}). For the case of work removal η is necessarily greater than 1.0, since for every joule of energy removed across the boundary from the flow as work, more than one joule of work potential is lost in the flow itself – with the balance of course showing up as an INTERNAL (waste) heat 'addition' (thus not appearing in the energy equation which simply is concerned with the interaction across the overall system boundary).

The classic definition of the polytropic compressor efficiency, e_c, is differential and is based on total pressure concerns.

$e_c = \dfrac{\delta w_{ideal}}{\delta w_{actual}}$ where the differential work interactions are defined as that required to obtain a given δP_t (positive).

Similarly the polytropic turbine efficiency, e_t, is defined as follows:

$e_t = \dfrac{\delta w_{actual}}{\delta w_{ideal}}$ where again the differential work interactions are defined as that required to obtain a given δP_t (albeit negative).

Note the inverse relationship of these quantities e_c and e_t in terms of work actual versus work ideal. The effectiveness η is, however, a continuous function from 0 to infinity.

With some manipulation the following relationships between e_c, e_t and η can be shown:

$$\eta = (e_c - 1)\left(\frac{1}{\frac{T_t}{T}}\right) + 1$$

$$\eta = (\frac{1}{e_t} - 1)\left(\frac{1}{\frac{T_t}{T}}\right) + 1$$

$$e_c = 1 + \frac{T_t}{T}(\eta - 1)$$

$$e_t = \frac{1}{1 + \frac{T_t}{T}(\eta - 1)}$$

Recall also that by definition

$$\frac{T_t}{T} = \left(1 + \frac{\gamma - 1}{2} M^2\right)$$

The fundamental difference between e_t/e_c formulation and the η formulation is that e_t/e_c are based on ideal/actual work increments total pressure change (differential) whereas η is based on work increment (work interaction actually received productively by flow and price paid in terms of work that surroundings provide (before the losses gobble some of it up internally and spit it out as waste heat).

The presence of the T_t / T term is a result of the fact that that part of the initial work coming across the boundary that instead arrives in the flow as a heat interaction (the 'lost work' increment), is seen by the flow as a 'Rayleigh style' heat interaction and hence there is a total pressure loss associated with that lost work (waste heat) – recall that for Rayleigh type interactions you get a total pressure decrease for nonzero Mach number, and it goes up with Mach number.

As an example of this if one considers a flow locally at M=2.236 where for some given differential work coming into the flow across the boundary, $e_c = 0$ (implying that there is no total pressure change at all). For this case $\eta = 0.5$, which says that the one half of the total differential work coming across the boundary (δw_{surr}) DOES arrive as a work interaction in the flow but the other half as a pure heat interaction. The sum of the total pressure increase associated with that which actually appears in the flow as work, $\eta \delta w_{surr} = \delta w_{eff}$, and the total pressure DECREASE associated with the waste heat arriving at finite Mach number (2.236) is exactly ZERO (since $e_c = 0$). This can be shown by subdividing a differential step into these two differential processes itself (work + heat interactions) and doing the analysis.

The bottom line is that the second law effectiveness η is directly and rigorously related to the polytropic efficiencies e_c and e_t. Furthermore both the polytropic efficiencies and the η (all three differentially defined) can be related directly (see textbooks) to the overall general compressor and turbine efficiencies η_C, η_t with the assumption of constant differential efficiencies (see, for instance, Gordon Oates).

This of course results in the familiar

$$\frac{P_{te}}{P_{ti}} = \left(\frac{T_{te}}{T_{ti}}\right)^{\frac{(\gamma)e_C}{\gamma - 1}}$$ for a compressor (i to e across the compressor)

and

$$\frac{P_{te}}{P_{ti}} = \left(\frac{T_{te}}{T_{ti}}\right)^{\frac{(\gamma)}{(\gamma-1)e_t}}$$ for a turbine (i to e across the turbine).

Note also that for low Mach numbers (say 0.5 and under) applicable to conventional turbomachinery, the following is approximately true:

$$\eta \approx e_C$$

$$\eta \approx \frac{1}{e_t}$$

for compressors and turbines, respectively. For Mach approaching zero, these become identities.

Electric rockets – survey of major types   - AE 5335

I. **Electrothermal** – propellant is heated directly (electrically) either by charge relief or by filaments, then thermodynamically expanded; heated gas is accelerated to supersonic speeds through a nozzle as in a chemical rocket.

A. **resistojets** – electrical energy heats filaments resistively which then heats propellant.

Simple design

Filaments are electrically heated surfaces such as

1) coils of heated wire

2) hollow heated tubes

3)heated knife blades

4) heated solid cylinders

- power requirements range between 1 W and 5 KW.

-Thrust can be steady state or intermittent

-Propellants used – H_2, O_2, H_2O, CO_2, NH_3 (ammonia), N_2, gases (NH_3 & H_2) which result from catalytic decomposition of hydrazine (N_2H_4), a monopropellant. (Such a hydrazine-based system has advantage of pre-heating gases to about 1000K before electrical energy addition – i.e. reduces electrical power needed.)

-typical thrust ranges: 2-100 milliNewtons

- Isp (specific impulse) 200-300 seconds (thrust per weight flow of propellant)

- Thrust efficiency = thrust power output/electric power input ~ 65% - 90%

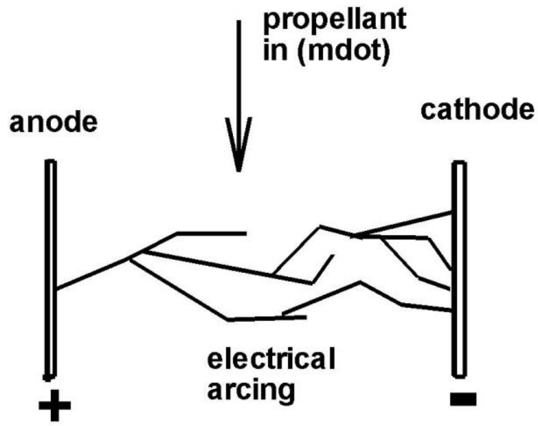
- Thrust duration – months

Advantages – simple, easily controlled, simple power conditioning, can use inert propellant, low cost, augmentation easy, relatively high efficiency, 2700K max temperature due to material limits

Disadvantages – lowest Isp, heat loss, gas disassociation, erosion

Chamber pressure effects: high pressure reduces losses due to disassociation and improves heat transfer to propellant, reduces size of chamber and nozzle for given mass flow rate of propellant. But high pressure increases wall heat transfer rate (material issues), erosion, etc. P chamber ~ 15 to 200 psi practical.

B. **Arcjets** - central portion of gas flow heated directly by electric arc discharges. Direct deposition of energy into propellant (can locally reach temperatures > 20,000K); energy appears as random ‘Ohmic’ heating (thermal). Problem is that arc current concentrates in filaments, temperatures high, disassociation and ionization occurs (energy absorbed in these processes)



- Some improvements can be made by swirling the propellant to lengthen the filaments and applying magnetic fields such that filaments are lengthened ('spun').
- Typical thrust ranges: 2 – 700 milliNewtons
- Isp: 400 – 1500 seconds
- Thrust efficiency (thrust pwr/elec pwr): 60% - 80%
- Propellant s H_2, N_2, N_2H_4, NH_3
- Advantages – same as resistojets, low voltage
- Disadvantages – high current, lower efficiency, heavy wiring, erosion, disassociation, etc

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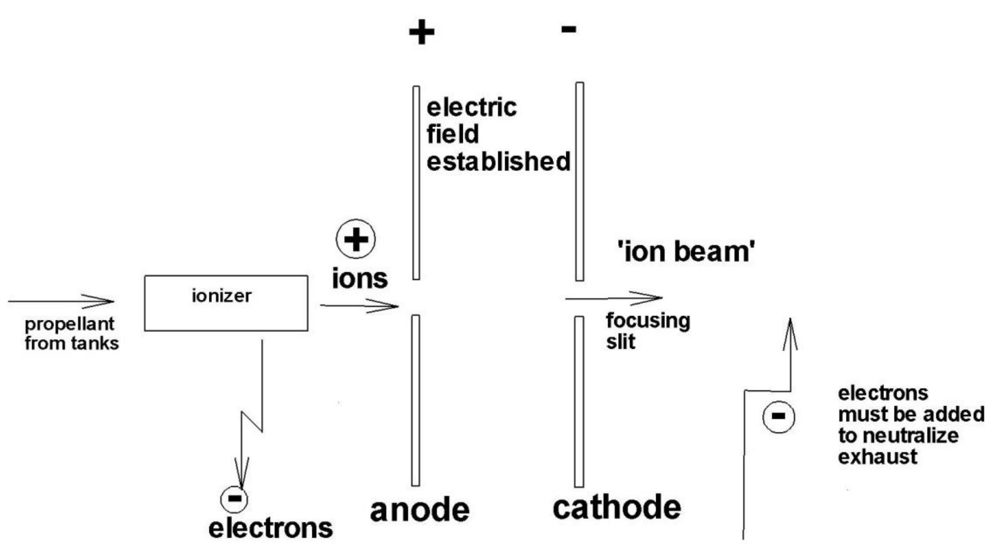
II. Electrostatic – acceleration achieved by application of electrostatic fields on charged propellant/particles, such as ions or colloids (propellant is stream of discrete charged particles)

a) ion (~240,000 times as massive as an electron) – charged single atoms (positive charge)

b) colloid – relatively massive charged multi-atom particles (10,000 times as massive as ion)

- electrostatic thrusters are categorized by their source of charged particles. Once charged particles produced, they are accelerated in a focusing region with a strong electric field.



Xenon often preferred as a propellant (heavy, inert, monoatomic , expensive though)

(cesium and mercury used to used but hazardous, corrosive, reactive)

Takes considerable electricity to generate the ions!

-typical thrust ranges: 0.001 – 200 milliNewtons

- Isp: 1500 – 5000 seconds

Thrust efficiency: 60% - 80%

-duration: months

Propellant: Xe, Hg, Ce

- Advantages – high Isp, Xenon is inert, okay efficiency
- Disadvantages – complex power conditioning (high voltages), low thrust per unit area, heavy power supply

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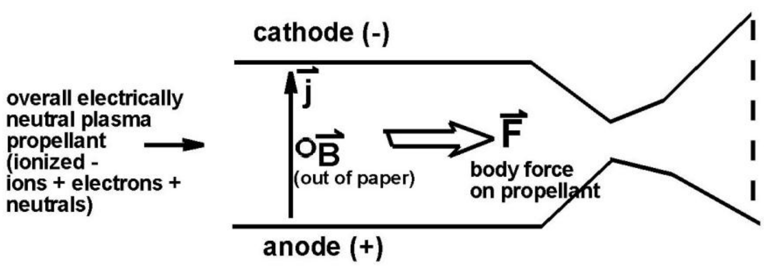
<u>III. Electromagnetic</u> – acceleration is achieved by interaction of electric and magnetic fields on highly ionized plasma. A plasma is a high temperature electrically neutral gas containing electrons, ions, and neutral species (very electrically conductive though) – can be up to and above 6000K.

Electromagnetic theory – when a conductor located in a magnetic field carries a current, a force will be exerted on the conductor in a direction at right angles to the current and the magnetic field. This force will accelerate a plasma conducting a current in a thruster and give a neutral exhaust beam.

This $\vec{J} X \vec{B}$ ('Lorentz force') appears directly as a body force in the propellant

$\bar{J}$ = electrical current $\vec{B}$ = magnetic field



-this conceptually can 'get around' the disassociation problem. One problem is high thermal heat transfer to the walls

- typical thrust ranges: 0.001 – 2000 N +

- Isp: 1000 – 8000 seconds

- thruster efficiency: 20% - 50%

- duration: weeks,years

-typical propellants: Xe, Ar, H2, Cs, N2

-Advantages: can be simple, high thrust per unit area

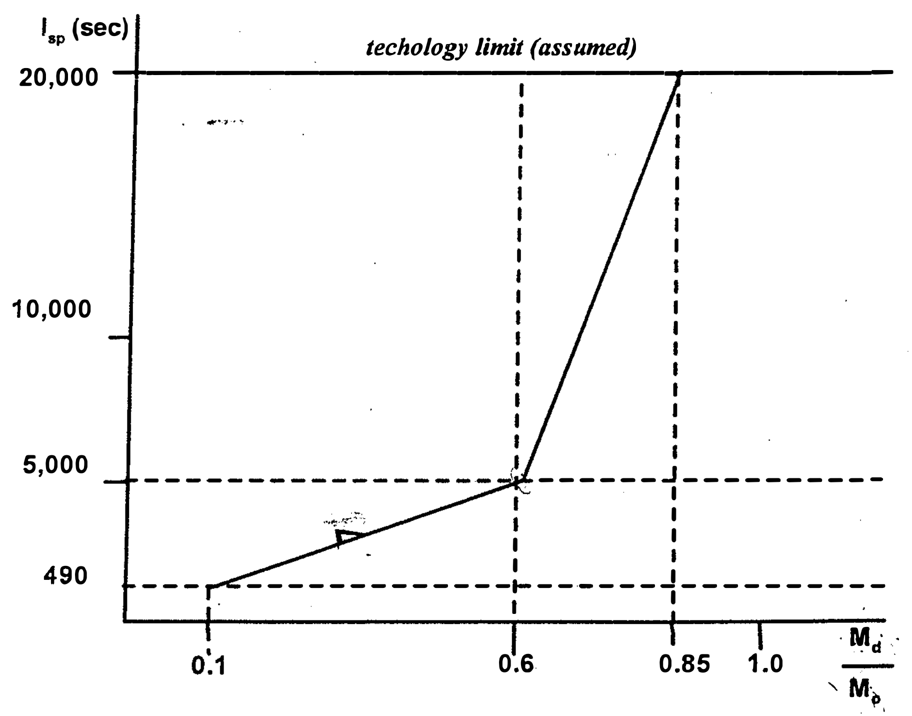
-Disadvantages: heavy power supply, low efficiency, other issues

[redacted]

~~The following table provides rough information regarding the types of rockets for which~~ the indicated specific impulse ranges are achievable.

Chemical rockets	$I_{sp} < 460$ seconds
Nuclear rockets	$460 < I_{sp} < 1000$ seconds
Arc-jet rockets	$1000 < I_{sp} < 3000$ seconds
Electro-Magnetic rockets	$3000 < I_{sp} < 5000$ seconds
Ion rockets	$5000 < I_{sp} < 10000$ seconds
Exotic rockets	$10000 < I_{sp}$

As specific impulse goes up, the propulsion system (engine) mass necessary to provide a given thrust generally increases, i.e. for a given initial overall mass, the dead weight mass generally increases as I_{sp} increases. (So, for instance, an ion rocket – although having a higher I_{sp} – would also have more structural mass associated with its propulsion system than a nuclear rocket.) Assume that the following plot provides an approximation of the I_{sp} versus dead weight relationship:



$\frac{dy}{dx} =$

$\frac{5{,}000 - 496}{.6 - .1}$

Using the information given, find the best type of rocket ('best' meaning maximizing payload mass to initial overall mass) for each of the following missions (with initial velocity equal to zero for each mission):

a) four day mission from Earth to Moon,

b) 180 day mission from Earth to Mars,
c) 700 day mission from Earth to Saturn.

Also, what is the minimum possible time to get to Alpha Centuria (four light years) based on the above information?

Do NOT consider the deceleration leg of the mission at all (i.e. just get there in the allotted time – don't try to decelerate to match velocity with the target). Assume negligible gravity and no atmospheric drag. Also, for calculations of 'velocity required' ignore any time or distance associated with the acceleration legs of the missions.

$$\frac{\Delta V}{c} = \ln\left(\frac{M_o}{M_d + M_L}\right)$$

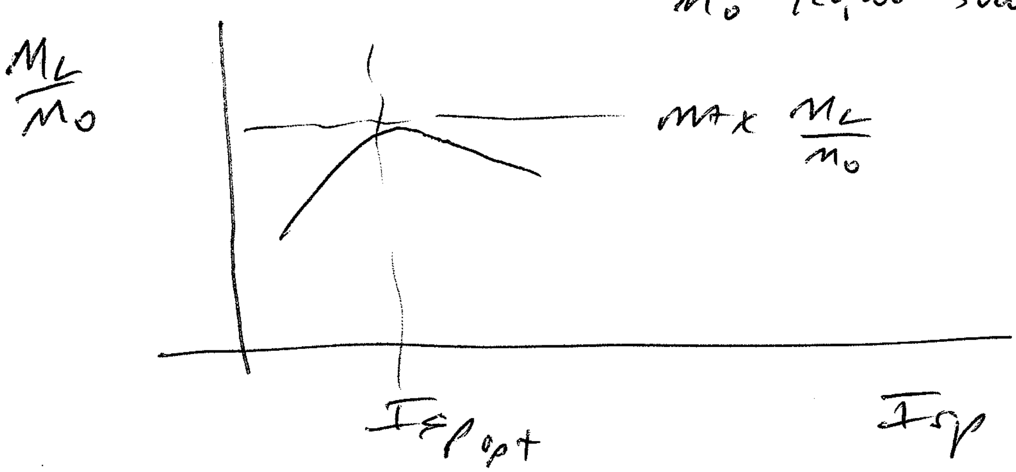
since $M_o = M_d + M_L + M_p$

$$\boxed{\Delta V = I_{sp} \cdot g_o \left\{ \frac{1}{\frac{M_d}{M_o} + \frac{M_L}{M_o}} \right\}}$$

If $I_{sp} < 490$ $\quad \frac{M_d}{M_o} = 0.1$

$490 < I_{sp} < 5000$ $\quad \frac{M_d}{M_o} = \frac{.5}{(5000 - 490)}(I_{sp} - 490) + .1$

$5000 < I_{sp} < 20{,}000$ $\quad \frac{M_d}{M_o} = \frac{.35}{(20{,}000 - 5000)}(I_{sp} - 5000) + .6$



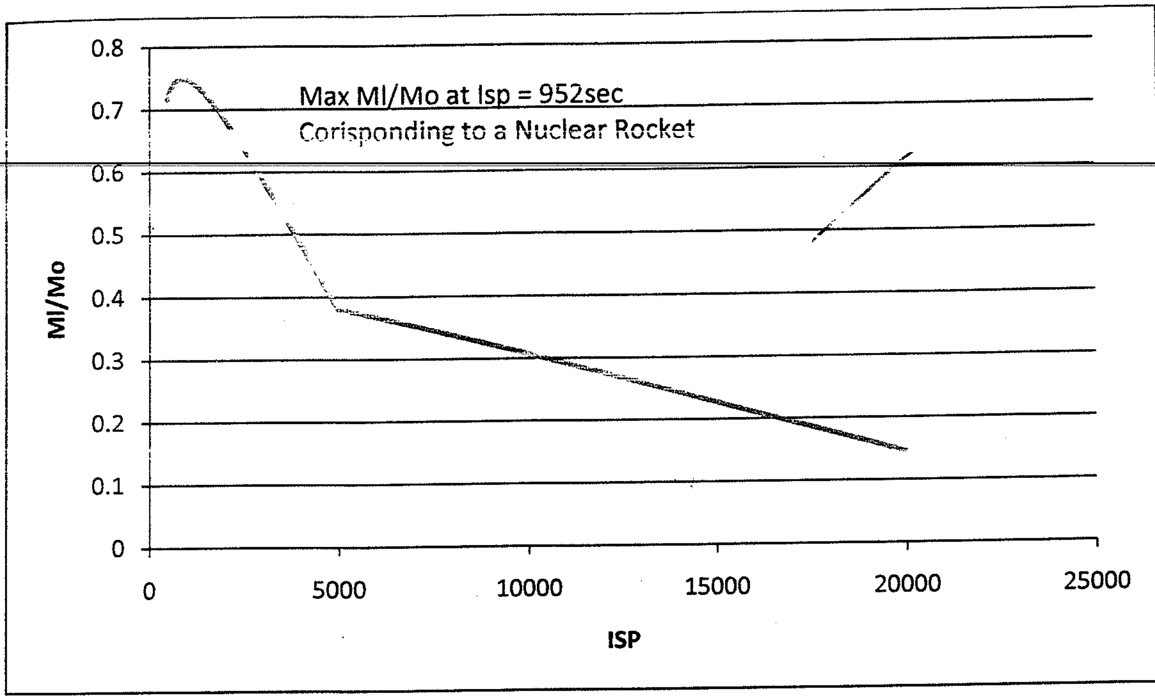
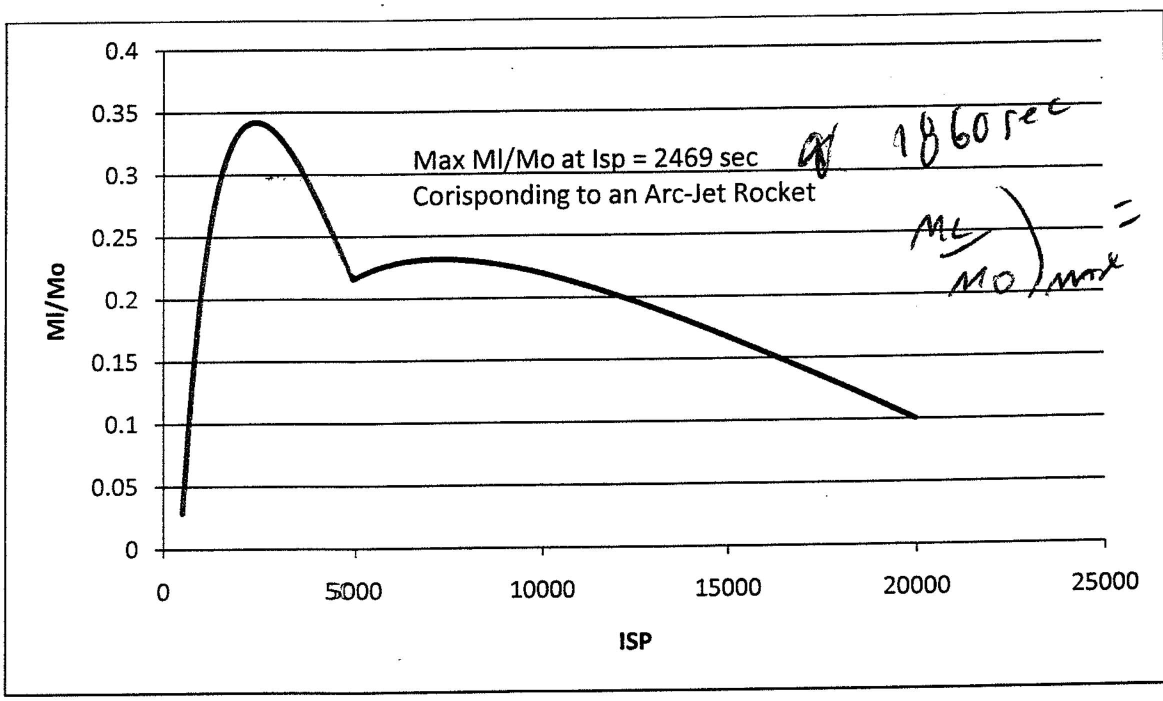


Figure 1: Case A to the Moon



@ 1860 sec

$\left.\frac{M_L}{M_O}\right)_{max} = .504$

[redacted]

Figure 2: Case B to Mars

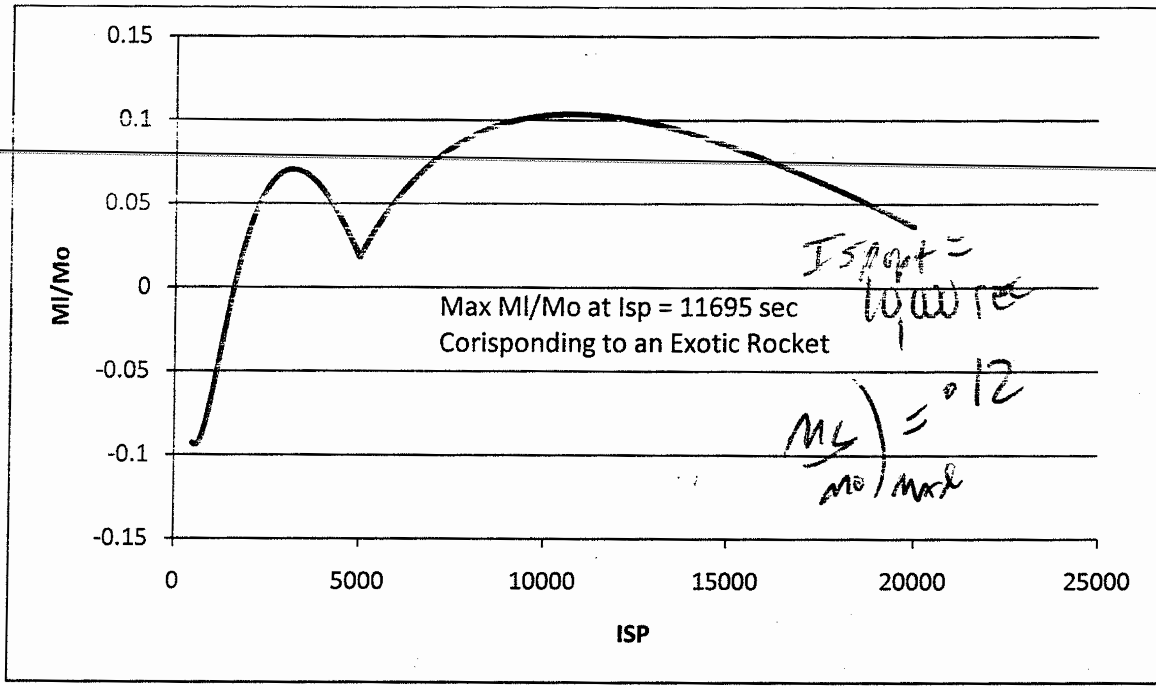


Figure 3: Case C to Saturn