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STAT 405-001
31 March 2021

1A) $L(Y_1, \dots, Y_n \mid \theta) = f(Y_1, \dots, Y_n \mid \theta) = \prod_{i=1}^{n} f(Y_i \mid \theta) = \prod_{i=1}^{n} 2\theta Y_i e^{-\theta Y^2} \rightarrow$

$\ln(L) = \ln\left(\prod_{i=1}^{n} 2\theta Y_i e^{-\theta Y^2}\right) = \sum_{i=1}^{n} \ln\left(2\theta Y_i e^{-\theta Y^2}\right) =$

$\sum_{i=1}^{n} \ln(2\theta) + \sum_{i=1}^{n} \ln(Y_i) + \sum_{i=1}^{n} \ln(e^{-\theta Y^2}) =$

$n \ln(2\theta) + \sum_{i=1}^{n} \ln(Y_i) - \theta \sum_{i=1}^{n} Y_i^2$

$\sum_{i=1}^{n} \ln(e^{-\theta Y^2}) = \sum_{i=1}^{n} \theta Y_i^2 = -\theta \sum_{i=1}^{n} Y_i^2$

$u = 2\theta$
$du = 2 \rightarrow n \cdot \frac{1}{2\theta}(2)$

$\frac{d(\ln(L))}{d(\theta)} = \frac{n}{\theta} + 0 - \sum_{i=1}^{n} Y_i^2 = 0 \rightarrow$

$\frac{n}{\theta} = \sum_{i=1}^{n} Y_i^2 \rightarrow$

$\boxed{\hat{\theta} = \frac{n}{\sum_{i=1}^{n} Y_i^2}}$

1B) $F(w) = P(W \le w) = P(Y^2 \le w) = P(Y \le \sqrt{w}) = \int_0^{\sqrt{w}} 2\theta y e^{-\theta y^2}\, dy =$

$u = -\theta y^2$
$du = -2\theta y\, dy$

$\int_0^{\sqrt{w}} -e^{u}\, du = \Big|_0^{\sqrt{w}} -e^{u} = \Big|_0^{\sqrt{w}} -e^{-\theta y^2} =$

$-e^{-\theta(\sqrt{w})^2} - -e^{-\theta(0)^2} =$

$-e^{-\theta w} + 1 =$

$\boxed{1 - e^{-\theta w}}$ We can see this is the CDF of $W \sim Exp\left(\frac{1}{\theta}\right)$.

1C) Because $Y \sim Rayleigh\left(\sqrt{\frac{1}{2\theta}}\right)$, $Var(Y) = \frac{4-\pi}{2}\left(\frac{1}{2\theta}\right)$ and $\lim_{\theta \to \infty} \frac{4-\pi}{2}\left(\frac{1}{2\theta}\right) = 0$. Per Theorem 9.1, $\hat{\theta}$ is a consistent estimator for θ.

1D) $L(Y_1, \dots, Y_n \mid \theta) = \prod_{i=1}^{n} 2\theta Y_i e^{-\theta Y_i^2} = (2\theta)^n \prod_{i=1}^{n} Y_i e^{-\theta Y_i^2} =$

$(2\theta)^n \left(\prod_{i=1}^{n} Y_i\right) e^{\sum -\theta Y_i^2} =$

$(2\theta)^n \left(\prod_{i=1}^{n} Y_i\right) e^{-\theta \sum Y_i^2} =$

$(2\theta)^n \left(\prod_{i=1}^{n} Y_i\right) e^{-\theta n \bar{Y^2}} =$

$(2\theta)^n \left(\prod_{i=1}^{n} Y_i\right) e^{\frac{-\theta n}{U}} \rightarrow$

$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} Y_i^2} \rightarrow U = \frac{n}{\sum_{i=1}^{n} Y_i^2} \rightarrow \frac{1}{U} = \frac{\sum_{i=1}^{n} Y_i^2}{n} \rightarrow \frac{1}{U} = \bar{Y^2}$

$\boxed{g(U, \theta) = (2\theta)^n e^{\frac{-\theta n}{U}} \quad h(Y_1, \dots, Y_n) = \prod_{i=1}^{n} Y_i}$

$\therefore$, $\hat{\theta}$ is a sufficient statistic for θ.

1E) $f_W(w)$ $= 2\theta \sqrt{w} e^{-\theta w^2}$ $\sim Rayleigh\left(\sqrt{\frac{1}{2\theta}}\right)$ Let $W = Y_i$

$E(W) = \frac{1}{2}\sqrt{\frac{\pi}{\theta}} \rightarrow E\left(\prod_{i=1}^{n} Y_i\right) = \left(\frac{1}{2}\sqrt{\frac{\pi}{\theta}}\right)^n \rightarrow \left(\prod_{i=1}^{n} Y_i\right)^{\frac{1}{n}} = \frac{1}{2}\sqrt{\frac{\pi}{\theta}} \rightarrow \left(2\left(\prod_{i=1}^{n} Y_i\right)^{\frac{1}{n}}\right)^2 = \frac{\pi}{\theta} \rightarrow$

$\boxed{\hat{\theta} = \frac{\pi}{4\left(\prod_{i=1}^{n} Y_i\right)^{2/n}}}$

1F) $E(Y) = \sigma\sqrt{\frac{\pi}{2}} = \sqrt{\frac{1}{8}}\sqrt{\frac{\pi}{2}} = \frac{\sqrt{\pi}}{4}$ $\quad \theta = 4$, $\sigma^2 = \frac{1}{2\theta} = \frac{1}{2(4)} = \frac{1}{8}$

$\frac{s}{\sqrt{n}} = \frac{\sqrt{\frac{4-\pi}{2}\sigma^2}}{\sqrt{24}} = \frac{\sqrt{\frac{4-\pi}{2}\left(\frac{1}{8}\right)}}{\sqrt{24}} = \sqrt{\frac{4-\pi}{384}}$

Per the Central Limit Theorem, $Y \sim N\left(\frac{\sqrt{\pi}}{4}, \sqrt{\frac{4-\pi}{384}}\right)$. $\boxed{\frac{\frac{\sqrt{\pi}}{4} - \frac{1}{\sqrt{11}}}{\sqrt{\frac{4-\pi}{384}}} \approx 3\ SD \rightarrow \boxed{0.001372}}$

$\hat{\theta} = 5.5$, $\sigma = \sqrt{\frac{1}{2\theta}} = \sqrt{\frac{1}{2(5.5)}} = \sqrt{\frac{1}{11}}$

2A) $\mu_1' = \int_0^\theta y\left(\frac{2}{\theta^2}y\right) dy =$

$\Big|_0^\theta \frac{2}{3\theta^2}y^3 =$

$\frac{2}{3\theta^2}(\theta^3 - 0^3) =$

$\frac{2}{3}\theta$

$m_1' = \frac{\sum_{i=1}^n Y_i}{n} = \bar{Y} = \frac{2}{3}\theta \to \boxed{\tilde{\theta} = \frac{3}{2}\bar{Y}}$

2B) $L(y_1, \dots, y_n|\theta) = f(y_1, \dots, y_n|\theta) = \prod_{i=1}^n f(y_i|\theta) = \prod_{i=1}^n \frac{2}{\theta^2}y_i =$

$\left(\frac{2}{\theta^2}\right)^n \prod_{i=1}^n y_i = 2^n \theta^{-2n} \prod_{i=1}^n y_i$

$\frac{d(L)}{d\theta} = 2^n(-2n\theta^{-2n-1})$ [redacted] $= 0 \to$

$(2^n)\frac{n}{\theta^{2n+1}}$ [redacted] $= 0 \to$

$(2^n)n$ [redacted] $= \theta^{2n+1} \to$ [redacted] $\boxed{\hat{\theta} = 2^{\frac{n}{2n+1}} \cdot n^{\frac{1}{2n+1}}}$

2C) $L(y_1, \dots, y_n|\theta) = 2^n\theta^{-2n}\prod_{i=1}^n y_i = U \prod_{i=1}^n y_i \to$ $U = \prod_{i=1}^n y_i$ [redacted]

[redacted] $\boxed{g(U, \theta) = 2^n \theta^{-2n} U, \ h(y_1, \dots, y_n) = 1}$

$U =$ [redacted] $\prod_{i=1}^n Y_i$ is a sufficient statistic for θ.

2D) $f_W(w)$ [redacted] $= \frac{2}{\theta^2}w$ Let $W = Y_i$

[redacted]

$E(W) = w\frac{2}{\theta^2}w \to E\left(\prod_{i=1}^n Y_i\right) = \frac{2}{\theta^2}w^{2n} \to$

$\theta^2 = \frac{2w^{2n}}{\prod_{i=1}^n Y_i} \to$

$\theta = w^n\sqrt{\frac{2}{\prod_{i=1}^n Y_i}} \to \boxed{\theta = w^2\sqrt{\frac{2}{U}}}$