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Homework 4

9.1.

$\mathrm{eff}(\hat\theta_1, \hat\theta_5) = \mathrm{Var}(\hat\theta_5) / \mathrm{Var}(\hat\theta_1) = (\theta^2 / 3) / (\theta^2) = \mathbf{1 / 3}$

$\mathrm{eff}(\hat\theta_2, \hat\theta_5) = \mathrm{Var}(\hat\theta_5) / \mathrm{Var}(\hat\theta_2) = (\theta^2 / 3) / (\theta^2 / 2) = \mathbf{2 / 3}$

$\mathrm{eff}(\hat\theta_3, \hat\theta_5) = \mathrm{Var}(\hat\theta_5) / \mathrm{Var}(\hat\theta_3) = (\theta^2 / 3) / (5\theta^2 / 9) = \mathbf{3 / 5}$

$\mathrm{Var}(\hat\theta_1) = \mathrm{Var}(Y_1) = \theta^2$
$\mathrm{Var}(\hat\theta_2) = (\mathrm{Var}(Y_1) + \mathrm{Var}(Y_2)) / 2^2 = \theta^2 / 2$
$\mathrm{Var}(\hat\theta_3) = (\mathrm{Var}(Y_1) + 4\mathrm{Var}(Y_2)) / 3^2 = 5\theta^2 / 9$
$\mathrm{Var}(\hat\theta_5) = (\mathrm{Var}(Y_1)+\mathrm{Var}(Y_2)+\mathrm{Var}(Y_3)) / 3^2 = (\theta^2 + \theta^2 + \theta^2) / 9 = 3\theta^2 / 9 = \theta^2 / 3$

9.6.

$\mathrm{eff}(\hat\lambda_1, \hat\lambda_2) = \mathrm{Var}(\hat\lambda_2) / \mathrm{Var}(\hat\lambda_1) = (\lambda / n) / (\lambda / 2) = \mathbf{2 / \lambda}$

$\mathrm{Var}(\hat\lambda_1) = (\mathrm{Var}(Y_1) + \mathrm{Var}(Y_2)) / 2^2 = \lambda / 2$
$\mathrm{Var}(\hat\lambda_2) = \Sigma_{i=1}^n \mathrm{Var}(Y_i) / n^2 = n\lambda / n^2 = \lambda / n$

9.39.

$$P(Y_1 = y_1, \ldots, Y_n = y_n | U = u) = \frac{P(Y_1 = y_1, \ldots, Y_n = y_n)}{P(U = u)} = \frac{\Pi_{i=1}^n \lambda^{y_i}e^{-\lambda} / y_i!}{(n\lambda)^u e^{-n\lambda} / u!} = \frac{\lambda^{\Sigma y_i} e^{-n\lambda} / \Pi_{i=1}^n y_i!}{(n\lambda)^u e^{-n\lambda} / u!} = \frac{\lambda^u e^{-n\lambda} / \Pi_{i=1}^n y_i!}{n^u \lambda^u e^{-n\lambda} / u!} = \frac{1 / \Pi_{i=1}^n y_i!}{n^u / u!} = \boldsymbol{\frac{u!}{n^u\, \Pi_{i=1}^n y_i!}}$$

Because $Y_1, Y_2, \ldots, Y_n$ are independent, we can say that $U = \Sigma_{i=1}^n Y_i \sim \mathrm{Poisson}(n\lambda)$.

Because this distribution does not depend on λ, $\Sigma_{i=1}^n Y_i$ is sufficient for λ.

9.45.

$L(y_1, \ldots, y_n|\theta) = \Pi_{i=1}^n f(y_i|\theta) =$
$\Pi_{i=1}^n a(\theta)b(y_i)e^{-c(\theta)d(y_i)} =$
$(a(\theta))^n(\Pi_{i=1}^n b(y_i))e^{-c(\theta)\,\Sigma d(y_i)} =$
$(a(\theta))^n e^{-c(\theta)u}\, \Pi_{i=1}^n b(y_i) \rightarrow$
$g(u,\theta) = (a(\theta))^n e^{-c(\theta)u}$, $h(y) = \Pi_{i=1}^n b(y_i)$

Substitute $U = \Sigma_{i=1}^n d(Y_i)$
Per Theorem 9.4, $\Sigma_{i=1}^n d(Y_i)$ is sufficient for θ.

9.60.

a. $L(y_1, \ldots, y_n|\theta) = \Pi_{i=1}^n f(y_i|\theta) =$
$\Pi_{i=1}^n \theta y_i^{\theta-1} = \theta^n(\Pi_{i=1}^n y_i)^{\theta-1} =$
$\theta^n \exp(-U)^{\theta-1} = \theta^n(e^{-(\theta-1)U}) =$
$\theta^n(e^{-(\theta-1)(-\ln(\Pi Y_i))}) =$
$\boldsymbol{\theta^n(e^{(\theta-1)\ln(\Pi Y_i)})}$

$U = \Sigma_{i=1}^n -\ln(Y_i) = -\Sigma_{i=1}^n \ln(Y_i) = -\ln(\Pi_{i=1}^n Y_i) \rightarrow$
$-U = \ln(\Pi_{i=1}^n Y_i) \rightarrow \exp(-U) = \Pi_{i=1}^n Y_i$

Thus, the density function is in exponential form and $\Sigma_{i=1}^n -\ln(Y_i)$ is sufficient for θ.

b. $F_W(w) = P(W \le w) =$
$P(-\ln(Y) \le w) = P(-Y \le e^w) =$
$1 - P(Y \le -e^w) = 1 - \int_0^{\exp(-w)} \theta y^{\theta-1}\, dy =$
$1 - |_0^{\exp(-w)}\, y^\theta = 1 - (e^{-\theta w} - 0) = \mathbf{1 - e^{-\theta w}}$

$W_i = -\ln(Y_i)$

We can see that this distribution is $\mathrm{Exp}(1/\theta)$.

c. $F_U(u) = P(U \le u) = P(2\theta W \le u) =$
$P(W \le u / 2\theta) = F_W(u / 2\theta) =$
$1 - e^{-\theta(u/2\theta)} = 1-e^{-u/2} \sim \mathrm{Exp}(2) = \chi^2(2) \rightarrow$
$P(2\theta\Sigma_{i=1}^n W_i \le u) \sim \chi^2(2n)$.

$U = 2\theta\Sigma_{i=1}^n W_i$
We can see that this distribution with $2\theta W$ is $\mathrm{Exp}(2)$, which is equivalent to $\chi^2(2)$. Thus, $P(2\theta\Sigma_{i=1}^n W_i \le u) \sim \chi^2(2n)$.

d. $E(1/(2\theta\Sigma_{i=1}^{n} W_i)) = E(U^{-1})$ E(InverseChi($2n$)) = 1 / ($2n$-2) = **1 / 2(n-1)**

e.