Charles Havens
Dr. Matthews
STAT 488-001
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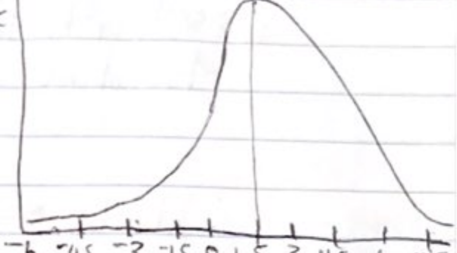
1 1.1A) $p(y) = P_r(\theta=1)p(y|\theta=1) + P_r(\theta=2)p(y|\theta=2) =$
$(0.5)N(y|1, 2^2) + (0.5)N(y|2, 2^2)$
We can see the kernel of this distribution has mean $\mu = (0.5)(1) + (0.5)(2) = 1.5$.



1.1B) $P_r(\theta=1|y=1) = \frac{P(y=1|\theta=1)P(\theta=1)}{P(y=1)} = \frac{N(1|1,2^2)\cdot(0.5)}{Pr(\theta=1)P(y=1|\theta=1)+Pr(\theta=2)P(y=1|\theta=2)} = \frac{0.5\,N(1|1,2^2)}{(0.5)\cdot N(1|1,2^2)+(0.5)\cdot N(1|2,2^2)} \approx \boxed{0.5312}$ Bayes' Theorem

1.1C) By testing different values of σ in the equation in 1.1B, we can see that $P_r(\theta=1|y=1)$ approaches $\frac{1}{2}$ as σ increases and $P_r(\theta=1|y=1)$ approaches 1 as σ decreases.

2 1.4A) $P_r(F>0|ps=8) = P_r(F>0) = \frac{8}{12} = \boxed{\frac{2}{3}}$ -7 -5 -3 -3 1 6 7 13 15 16 20 21
$P_r(F>8|ps=8) = P_r(F>8) = \boxed{\frac{5}{12}}$ (F>8)
$P_r(F>8|ps=8 \cap F>0) = P_r(F>8|F>0) = \boxed{\frac{5}{8}}$ (F>0)

1.4B) $N(0.07, 13.86)$ (From page 15, example 1.6)
$P_r(F>0|ps=8) = N(8.5|0.07, 13.86^2) \approx \boxed{0.7285}$ Normal approximation
$P_r(F>8|ps=8) = N(0.5|0.07, 13.86^2) \approx \boxed{0.5124}$ continuity correction
$P_r(F>8|ps=8 \cap F>0) = P_r(F>8|F>0) = \frac{Pr(F>8 \cap F>0)}{Pr(F>0)} = \frac{Pr(F>8)}{Pr(F>0)} \approx \frac{0.5124}{0.7285} \approx \boxed{0.7033}$ Bayes' Theorem

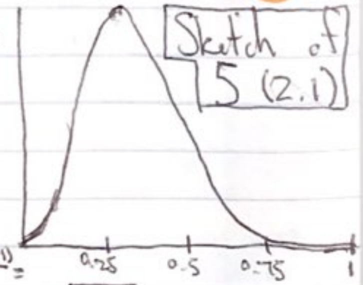
3 1.6) $P(I|T) = \frac{P(I \cap T)}{P(T)} = \frac{P(T|I)Pr(I)}{P(T|I)Pr(I)+P(T|F)Pr(F)} = \frac{(\frac{1}{2})(\frac{1}{300})}{(\frac{1}{2})(\frac{1}{300})+(\frac{1}{4})(\frac{1}{125})} = \frac{\frac{1}{600}}{\frac{1}{600}+\frac{1}{500}} = \frac{\frac{5}{3000}}{\frac{11}{3000}} = \boxed{\frac{5}{11}}$

4 1.7) $P(W|C) = P(L)P(W|2) + P(W)P(W|2) = (\frac{2}{3})(1) + (\frac{1}{3})(0) = \boxed{\frac{2}{3}}$
$P(W|S) = P(L)P(W|2) + P(W)P(W|2) = (\frac{2}{3})(0) + (\frac{1}{3})(1) = \boxed{\frac{1}{3}}$
Case 1: ☑ ☐ ☐
Case 2: ☑ ☑ ☐
Case 3: ☐ ☐ ☑
Monty Hall problem. We can see that the best prize is in the doors not chosen $\frac{2}{3}$ of the time.

	X	Y
X	XX	XY
X	XX	XY

5 2.1) $Beta(4,4) = \frac{x^{4-1}(1-x)^{4-1}}{\frac{\Gamma(4)\Gamma(4)}{\Gamma(4+4)}} = \frac{\Gamma(8)}{\Gamma(4)\Gamma(4)}x^3(1-x)^3 = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot3\cdot2\cdot1}x^3(1-x)^3 = 140x^3(1-x)^3 \propto x^3(1-x)^3$
$P_r(X<3) = P_r(X=0) + P_r(X=1) + P_r(X=2) = \binom{10}{0}x^0(1-x)^{10-0} + \binom{10}{1}x^1(1-x)^{10-1} + \binom{10}{2}x^2(1-x)^{10-2} = (1-x)^{10} + 10x(1-x)^9 + 45x^2(1-x)^8$
$p(x|x<3) = p(x)P_r(X<3) = 140x^3(1-x)^3((1-x)^{10} + 10x(1-x)^9 + 45x^2(1-x)^8) \propto \boxed{x^3(1-x)^{13} + 10x^4(1-x)^{12} + 45x^5(1-x)^{11}}$
Sketch on following page

6 2.5A) $P_r(y=k)=\int_0^1 P_r(y=k|\theta)\,d\theta=\int_0^1\binom{n}{k}\theta^k(1-\theta)^{n-k}\,d\theta=\binom{n}{k}\int_0^1\theta^k(1-\theta)^{n-k}d\theta=$



-We observe that $\int_0^1 \mathrm{Beta}(\alpha,\beta)\,d\theta=1$:

$$\int_0^1 \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}\,d\theta=1$$

$$\int_0^1 \theta^{\alpha-1}(1-\theta)^{\beta-1}=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Substitute $\alpha=k+1$ and $\beta=n-k+1$:

$$\binom{n}{k}\frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(k+1+n-k+1)}=\binom{n}{k}\frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)}=\frac{n!}{k!(n-k)!}\left(\frac{k!(n-k)!}{(n+1)!}\right)=\boxed{\frac{1}{n+1}}$$

2.5B) We see the posterior mean $E(\theta|y)=\frac{\alpha+y}{\alpha+\beta+n}$, and we observe that $\frac{\alpha}{\alpha+\beta}\le E(\theta|y)\le\frac{y}{n}$.

2.5C) $\mathrm{Var}(\theta)=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}=\frac{(1)(1)}{((1)+(1))^2((1)+(1)+1)}=\frac{1}{12}$

$\mathrm{Var}(\theta|y)=\frac{(\alpha+y)(\beta+n-y)}{(\alpha+\beta+n)^2(\alpha+\beta+n+1)}=\frac{((1)+y)((1)+n-y)}{((1)+(1)+n)^2((1)+(1)+n+1)}=\frac{(y+1)(n-y+1)}{(n+2)^2(n+3)}$

We can see when graphing $\mathrm{Var}(\theta|y)$ that the maximum value is $\frac{1}{16}$ (and the maximum value with integer inputs is $\frac{5}{90}$) when $y=n=1$, which is less than $\frac{1}{12}$.

c)

2.5D) $\mathrm{Beta}(1,3)$, $y=1$, $n=1$; $\mathrm{Var}(\theta|y)>\mathrm{Var}(\theta)\rightarrow\frac{(\alpha+y)(\beta+n-y)}{(\alpha+\beta+n)^2(\alpha+\beta+n+1)}>\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}\rightarrow$

$\frac{((1)+(1))((3)+(1)-(1))}{((1)+(3)+(1))^2((1)+(3)+(1)+(1))}>\frac{(1)(3)}{((1)+(3))^2((1)+(3)+1)}\rightarrow$

$\frac{(2)(3)}{(5)^2(6)}=\frac{1}{25}=\boxed{\frac{16}{400}>\frac{15}{400}}=\frac{3}{80}=\frac{(3)}{(4)^2(5)}$

7 2.11) See following page

8 2.12) $p(y|\theta)\sim\mathrm{Poisson}(\theta)=\frac{\theta^y e^{-\theta}}{y!}$

$\ln p(y|\theta)=\ln\frac{\theta^y e^{-\theta}}{y!}=\ln\theta^y+\ln e^{-\theta}-\ln y!=$

$y\ln\theta-\theta-\ln y!$

$\frac{d}{d\theta}[\ln p(y|\theta)]=\frac{y}{\theta}-1=\frac{y-\theta}{\theta}$

$\sqrt{J(\theta)}=\sqrt{E\left[\left(\frac{d}{d\theta}\ln p(y|\theta)\right)^2\right]}=\sqrt{E\left[\left(\frac{y}{\theta}-1\right)^2\right]}\propto\boxed{\sqrt{\frac{1}{\theta}}}$

$\mathrm{Gamma}(\alpha,\beta)=\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}$

$\mathrm{Gamma}(\frac{1}{2},1)=\frac{(1)^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}\theta^{(\frac{1}{2})-1}e^{-(1)\theta}=\frac{1}{e^\theta\Gamma(\frac{1}{2})}\sqrt{\frac{1}{\theta}}\propto\frac{1}{e^\theta}\sqrt{\frac{1}{\theta}}$

Unfortunately, it does not appear there is a direct match between the Jeffreys prior density for θ in $\mathrm{Poisson}(\theta)$ and the gamma distribution, but after graphing it $\boxed{\mathrm{Gamma}(\frac{1}{2},1)}$ appears to be a close fit.

Homework 1

Charles Hwang

6/7/2022

Charles Hwang

Dr. Matthews

STAT 488

7 June 2022

Problem 2.11

Problem 2.11(a)

```
rm(list=ls())
f<-function (y,x){density<-NULL
for (i in 1:length(x))
density<-c(density,prod(dcauchy(y,x[i],1)))
density}
N<-f(c(43,44,45,46.5,47.5),seq(0,1,0.01))/(0.01*sum(f(c(43,44,45,46.5,47.5),seq(0,1,0.01))))
plot(seq(0,1,0.01),N,type="l",ylim=c(0,1.2),main="Problem 2.11(a) - Normalized Posterior Density Functi
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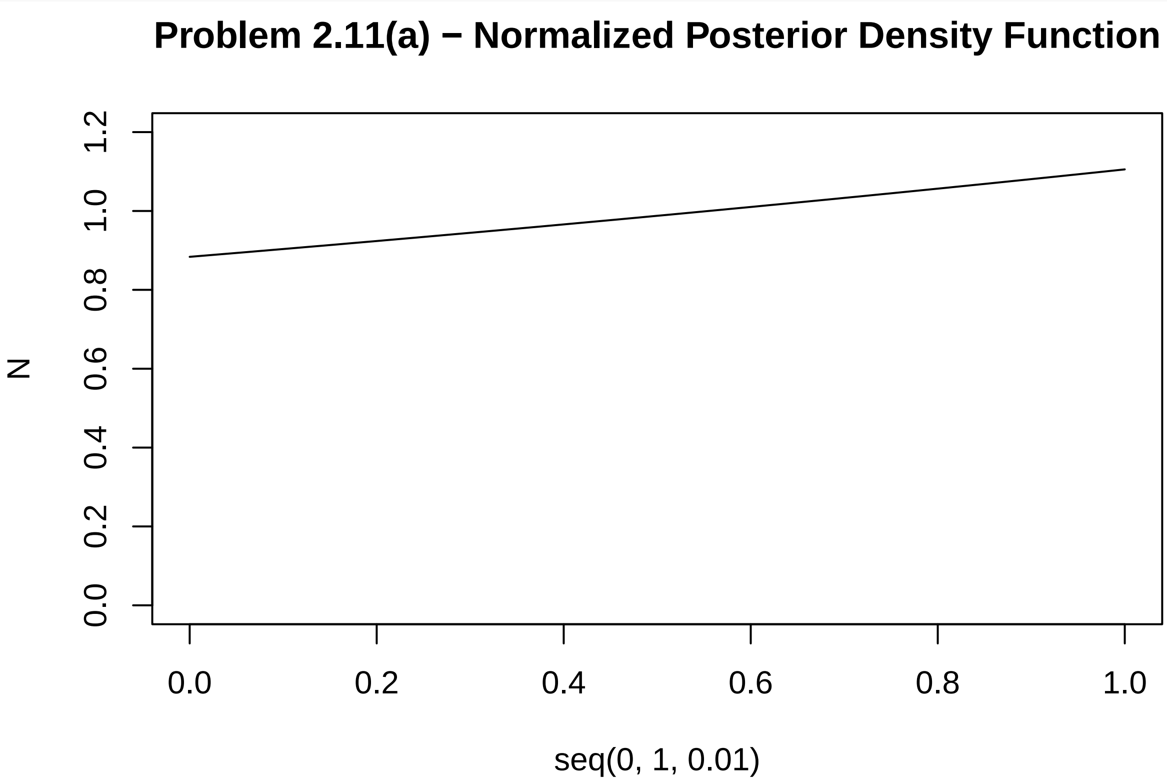

Problems 2.11(b-c)

```
set.seed(76)
Draws<-sample(seq(0,1,0.01),1000,0.01*N,replace=TRUE)
par(mfrow=c(2,1))
hist(Draws,breaks=100,main="Problem 2.11(b) - Sample (n=1000) from Posterior Density")
hist(rcauchy(length(Draws),Draws),breaks=100,main="Problem 2.11(c) - Sample (n=1000) from Predictive Di
```