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STAT 405-001
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1A) $H_0: \mu = 30$, $H_A: \mu < 30$, $\alpha = 0.05$ $n=16$, $\bar{y}=24$, $s=10.24$

Because $n = 16 < 30$, we should use a t-test with $16-1=15$ df.

$t = \frac{\bar{y}-\mu}{\frac{s}{\sqrt{n}}} = \frac{24-30}{\frac{10.24}{\sqrt{16}}} = \frac{-6(4)}{10.24} = \frac{-75}{32} \approx -2.3438$ $|-2.3438| > 1.7531 = t_{0.05,15}$

We reject H_0 at the $\alpha=0.05$ level. There is sufficient evidence that $\mu < 30$ minutes. $\left|\frac{\bar{y}-30}{\frac{10.24}{\sqrt{16}}}\right| > 1.7531 = t_{0.05,15}$

Reject H_0 when $\bar{y} < 25.5122$

$p = 0.0166$

1B) $H_0: \sigma^2 = 144$, $H_A: \sigma^2 \neq 144$, $\alpha = 0.05$ $n-1=15$, $s^2 = 104.8576$

We can conduct a χ^2-test for population variance with $16-1=15$ df.

$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{(15)(104.8576)}{(144)} = \frac{1572.864}{144} = \frac{4096}{375} \approx 10.9226$ $10.9227 > 6.2621 = \chi^2_{0.975,15}$

We fail to reject H_0 at the $\alpha=0.05$ level. There is insufficient evidence that $\sigma^2 \neq 144$ minutes. $10.9227 < 27.4884 = \chi^2_{0.025,15}$

$6.2621 < \frac{(16-1)s^2}{144} < 27.4884$

Reject H_0 when $s^2 < 60.1165$ or $s^2 > 263.8885$

$p = 2(0.2419) \approx 0.4839$ $s < 7.7535$ or $s > 16.2446$

2) $H_0: \mu_1 - \mu_2 = 0$, $H_A: \mu_1 - \mu_2 \neq 0$, $\alpha = 0.02$ $n_1=12$, $\bar{y}_1=13.8$, $s_1=1.2$; $n_2=15$, $\bar{y}_2=12.9$, $s_2=1.5$

$S_p = \frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2} = \frac{(12-1)(1.2)^2+(15-1)(1.5)^2}{(12)+(15)-2} = 1.8936$

Because $n_1 = 12 < 30$ and $n_2 = 15 < 30$, we should use a t-test with $12+15-2=25$ df.

$t = \frac{\bar{y}_1-\bar{y}_2-0}{\sqrt{s_p(\frac{1}{n_1}+\frac{1}{n_2})}} = \frac{(13.8)-(12.9)-0}{\sqrt{(1.8936)(\frac{1}{12}+\frac{1}{15})}} \approx 1.6887$

$|1.6887| < 2.4851 = t_{0.01,25}$ We fail to reject H_0 at the $\alpha=0.02$ level. There is insufficient evidence that the mean heights are different.

$\left|\frac{\bar{y}_1-\bar{y}_2-0}{\sqrt{(1.6887)(\frac{1}{12}+\frac{1}{15})}}\right| < 2.4851$

$p = 2(0.0519) \approx 0.1037$

Reject H_0 when $|\bar{y}_1 - \bar{y}_2| > 1.3244$

3) $H_0: p = 0.6$, $H_A: p > 0.6$, $\alpha = 0.01$ $n = 250$, $\hat{p} = \frac{190}{250} = 0.76$

Because $np = (250)(0.76) = 190 > 5$ and $n(1-p) = (250)(0.24) = 60 > 5$, we can use a one-proportion z-test.

$z = \frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} = \frac{(0.76)-(0.6)}{\sqrt{\frac{(0.6)(1-0.6)}{(250)}}} = \frac{4\sqrt{15}}{3} \approx 5.164$ $|5.164| > 2.3264 = z_{0.01}$

We reject H_0 at the $\alpha=0.01$ level. There is sufficient evidence that $p > 0.6$. $p < 0.0001$ $2.3264 < \frac{\hat{p}-0.6}{\sqrt{\frac{(0.6)(1-0.6)}{(250)}}}$

Reject H_0 when $\hat{p} > 0.6721$

4) $H_0: \mu_{20} - \mu_0 = 0$, $H_A: \mu_{20} - \mu_0 \neq 0$, $\alpha = 0.01$ $n_0=400$, $\bar{y}_0=53.8$, $s_0=2.4$; $n_{20}=500$, $\bar{y}_{20}=54.5$, $s_{20}=2.8$

Because $n_0 = 400 > 30$ and $n_{20} = 500 > 30$, we can conduct a two-sample z-test.

$z = \frac{\bar{y}_{20}-\bar{y}_0-0}{\sqrt{\frac{s_0^2}{n_0}+\frac{s_{20}^2}{n_{20}}}} = \frac{(54.5)-(53.8)-0}{\sqrt{\frac{(2.4)^2}{(400)}+\frac{(2.8)^2}{(500)}}} = \frac{35\sqrt{470}}{188} \approx 4.0361$ $|4.0361| > 2.5758 = z_{0.005}$

We reject H_0 at the $\alpha=0.01$ level. There is sufficient evidence of a height difference. $\left|\frac{\bar{y}_{20}-\bar{y}_0-0}{\sqrt{\frac{(2.4)^2}{(400)}+\frac{(2.8)^2}{(500)}}}\right| > 2.5758$

Reject H_0 when $|\bar{y}_{20} - \bar{y}_0| > 0.4467$

$p < 0.0001$

5A) $H_0: p = 0.5$, $H_A: p > 0.5$ $\quad n = 10$, $\sum_{i=1}^{10} Y_i \geq 8 \rightarrow n\bar{Y} \geq 8 \rightarrow \bar{Y} \geq 0.8 \rightarrow \hat{p} = 0.8$

Because $np = (10)(0.5) = 5 \geq 5$ and $n(1-p) = (10)(1-0.5) = 5 \geq 5$, we can use a proportion z-test.

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} = \frac{0.8 - 0.5}{\sqrt{\frac{(0.5)(1-0.5)}{10}}} = \frac{3\sqrt{10}}{5} \approx 1.8974 = Z_{0.02889} \longrightarrow \boxed{\alpha = 0.0289}$$

5B) $f(Y_i | p) = p^{Y_i}(1-p)^{1-Y_i}$, $\quad L(Y_1, \ldots, Y_{10} | p) = \prod_{i=1}^{10} p^{Y_i}(1-p)^{1-Y_i} = p^{\sum_{i=1}^{10} Y_i}(1-p)^{10 - \sum_{i=1}^{10} Y_i}$

$$\frac{L(p_0)}{L(p_A)} = \frac{L(0.5)}{L(0.75)} = \frac{0.5^{\sum Y_i}(1-0.5)^{10-\sum Y_i}}{0.25^{\sum Y_i}(1-0.25)^{10-\sum Y_i}} = \frac{0.5^{10}}{U} = \left(\frac{0.5 \cdot 3^{\bar{Y}} \cdot 4}{3}\right)^{10} = \left(\frac{2 \cdot 3^{\bar{Y}}}{3}\right)^{10} < k \rightarrow$$

$$\frac{2 \cdot 3^{\bar{Y}}}{3} < k^{\frac{1}{10}} \rightarrow$$

$$3^{\bar{Y}} < \frac{3}{2}k^{\frac{1}{10}} \rightarrow$$

$$\boxed{\bar{Y} < \log_3\left(\frac{3}{2}k^{\frac{1}{10}}\right)} = k'$$

$U = 0.25^{\sum_{i=1}^{10} Y_i} \, 0.75^{10 - \sum_{i=1}^{10} Y_i} = \left(\frac{1}{4}\right)^{\sum_{i=1}^{10} Y_i}\left(\frac{3}{4}\right)^{10 - \sum_{i=1}^{10} Y_i} = \left(\frac{1}{4^{\sum Y_i}}\right)\left(\frac{3^{10-\sum Y_i}}{4^{10-\sum Y_i}}\right) = \frac{3^{10-n\bar{Y}}}{4^{10}} = \frac{3^{10-(10)\bar{Y}}}{4^{10}} = \frac{3^{10}}{3^{10\bar{Y}}4^{10}} = \left(\frac{3}{3^{\bar{Y}} \cdot 4}\right)^{10}$

Because k' does not depend on p_A, this is the uniformly most powerful test.

5C) $\alpha = P(\bar{Y} < k' | \theta = \frac{1}{2}) = 0.0546875$ $\quad$ Per page 540, $\text{power}(\theta_0) = \alpha$.

6A) $f(y|\theta) = \frac{y^2}{2\theta^3}e^{\frac{-y}{\theta}}$, $\quad L(Y_1, \ldots, Y_n | \theta) = \prod_{i=1}^{n} \frac{Y^2}{2\theta^3}e^{\frac{-Y}{\theta}} = \left(\frac{1}{2\theta^3}\right)^n\left(\prod_{i=1}^{n} Y^2\right)e^{-\frac{\sum Y_i}{\theta^n}} = \frac{\prod Y^2}{2^n\theta^{3n}e^{(\sum Y_i)/\theta^n}}$

$$\frac{L(\theta_0)}{L(\theta_A)} = \frac{2^n\theta_0^{3n}e^{(\sum Y_i)/\theta_0^n}}{2^n\theta_A^{3n}e^{(\sum Y_i)/\theta_A^n}} = \left(\frac{\theta_0}{\theta_A}\right)^{3n}\exp\left(\frac{\sum_{i=1}^n Y_i}{\theta_0^n} - \frac{\sum_{i=1}^n Y_i}{\theta_A^n}\right) =$$

$$\left(\frac{\theta_0}{\theta_A}\right)^{3n}\exp\left(\frac{\theta_A^n \sum Y_i - \theta_0^n \sum Y_i}{(\theta_0\theta_A)^n}\right) =$$

$$\left(\frac{\theta_0}{\theta_A}\right)^{3n}\exp\left(\frac{(\sum Y_i)(\theta_A^n - \theta_0^n)}{(\theta_0\theta_A)^n}\right) =$$

$$\left(\frac{\theta_0}{\theta_A}\right)^{3n}\exp\left(\frac{n\bar{Y}(\theta_A^n - \theta_0^n)}{(\theta_0\theta_A)^n}\right) =$$

$$\left(\frac{\theta_0}{\theta_A}\right)^{3n}\exp\left(\frac{\bar{Y}(\theta_A^n - \theta_0^n)}{(\theta_0\theta_A)^n}\right)^n =$$

$$\left(\left(\frac{\theta_0}{\theta_A}\right)^3\exp\left(\frac{\bar{Y}(\theta_A^n - \theta_0^n)}{(\theta_0\theta_A)^n}\right)\right)^n < k \rightarrow$$

$$\left(\frac{\theta_0}{\theta_A}\right)^3\exp\left(\frac{\bar{Y}(\theta_A^n - \theta_0^n)}{(\theta_0\theta_A)^n}\right) < k^{\frac{1}{n}} \rightarrow$$

$$\exp\left(\frac{\bar{Y}(\theta_A^n - \theta_0^n)}{(\theta_0\theta_A)^n}\right) < \left(\frac{\theta_0}{\theta_A}\right)^{-3}k^{\frac{1}{n}} \rightarrow$$

$$\frac{\bar{Y}(\theta_A^n - \theta_0^n)}{(\theta_0\theta_A)^n} < \ln\left(\left(\frac{\theta_0}{\theta_A}\right)^{-3}k^{\frac{1}{n}}\right) \rightarrow$$

$$\boxed{\bar{Y} < \frac{(\theta_0\theta_A)^n}{(\theta_A^n - \theta_0^n)}\left(\frac{\ln k}{n} + 3(\ln\theta_A - \ln\theta_0)\right)} = k'$$

6B) No, because k' depends on θ_A. It is not the uniformly most powerful test.

$\theta_A > \theta_0$

$\ln\theta_A > \ln\theta_0$

$0 > \ln(\theta_0 - \theta_A)$

7A) $f(x|\theta_1)=\theta_1 x^{\theta_1-1}$, $f(x|\theta_2)=\theta_2 y^{\theta_2-1}$, $L(x_1,\dots,x_n|\theta_1)=\prod_{i=1}^{n}\theta_1 x_i^{\theta_1-1} = \theta_1^n\prod_{i=1}^{n} x_i^{\theta_1-1}$

$L(\theta_1,\theta_2|x,y)=\theta_1^n\theta_2^m\prod_{i=1}^{n} x_i^{\theta_1-1}\prod_{j=1}^{m} y_j^{\theta_2-1}$

$L(y_1,\dots,y_m|\theta_2)=\prod_{j=1}^{m}\theta_2 y_j^{\theta_2-1} = \theta_2^m\prod_{j=1}^{m} y_j^{\theta_2-1}$

$\ln(L(\theta_1,\theta_2))=\ln\left(\theta_1^n\theta_2^m\prod_{i=1}^{n} x_i^{\theta_1-1}\prod_{j=1}^{m} y_j^{\theta_2-1}\right)=$

$n\ln\theta_1+m\ln\theta_2+\sum_{i=1}^{n} x_i^{\theta_1-1}+\sum_{j=1}^{m} y_j^{\theta_2-1}=$

$n\ln\theta_1+m\ln\theta_2+(\theta-1)\sum_{i=1}^{n}\ln x_i+(\theta-1)\sum_{j=1}^{m}\ln y_j$

$\frac{d(L(\theta_1,\theta_2))}{d\theta_1}=\frac{n}{\theta_1}+\sum_{i=1}^{n}\ln x_i=0\rightarrow$

$\frac{n}{\theta_1}=-\sum_{i=1}^{n}\ln x_i\rightarrow$

$\hat{\theta}_1=\frac{-n}{\sum\ln x_i}=\frac{-n}{\ln\prod x_i}$

$\frac{d(L(\theta_1,\theta_2))}{d\theta_2}=\frac{m}{\theta_2}+\sum_{j=1}^{m}\ln y_j=0\rightarrow$

$\frac{m}{\theta_2}=-\sum_{j=1}^{m}\ln y_j\rightarrow$

$\hat{\theta}_2=\frac{-m}{\sum\ln y_j}=\frac{-m}{\ln\prod y_j}$

$\ln(L(\theta|x,y))=(n+m)\ln\theta+(\theta-1)\left(\sum_{i=1}^{n}\ln x_i+\sum_{j=1}^{m}\ln y_j\right)$

$\frac{d(L(\theta))}{d\theta}=\frac{n+m}{\theta}+\sum_{i=1}^{n}\ln x_i+\sum_{j=1}^{m}\ln y_j=0\rightarrow$

$\frac{n+m}{\theta}=-\left(\sum_{i=1}^{n}\ln x_i+\sum_{j=1}^{m}\ln y_j\right)\rightarrow$

$\hat{\theta}=\frac{-(n+m)}{\sum\ln x_i+\sum\ln y_j}=\frac{-(n+m)}{\ln\prod x_i+\ln\prod y_j}=\frac{-(n+m)}{\ln\prod x_i\prod y_j}$

$L(\theta|x,y)=\theta^{n+m}\prod_{i=1}^{n} x_i^{\theta-1}\prod_{j=1}^{m} y_j^{\theta-1}$

$\lambda=\frac{L(\theta|x,y)}{L(\theta_1,\theta_2|x,y)}=\frac{\theta^{n+m}\prod_{i=1}^{n} x_i^{\theta-1}\prod_{j=1}^{m} y_j^{\theta-1}}{\theta_1^n\theta_2^m\prod x_i^{\theta_1-1}\prod y_j^{\theta_2-1}}=\frac{\theta^{n+m}}{\theta_1^n\theta_2^m}\prod_{i=1}^{n} x_i^{\theta-\theta_1}\prod_{j=1}^{m} y_j^{\theta-\theta_2}=$

$\frac{\theta^{n+m}}{\theta_1^n\theta_2^m}\prod x_i^{\frac{n}{\ln\prod x_i}-\frac{n+m}{\ln\prod x_i+\ln\prod y_j}}\prod y_j^{\frac{m}{\ln\prod y_j}-\frac{n+m}{\ln\prod x_i+\ln\prod y_j}}=$

$\frac{\theta^{n+m}}{\theta_1^n\theta_2^m}e^n\prod x_i^{\frac{-(n+m)}{\ln\prod x_i+\ln\prod y_j}}e^m\prod y_j^{\frac{-(n+m)}{\ln\prod x_i+\ln\prod y_j}}$

$\prod x_i^{\frac{n}{\ln\prod x_i}}=e^n$, $\prod y_i^{\frac{m}{\ln\prod y_i}}=e^m$

$e^{n+m}\frac{\theta^{n+m}}{\theta_1^n\theta_2^m}\left(\prod x_i\prod y_j\right)^{\frac{-(n+m)}{\ln(\prod x_i\prod y_j)}}=$

$e^{n+m}\frac{\theta^{n+m}}{\theta_1^n\theta_2^m}e^{-(n+m)}=$

$\boxed{\frac{\theta^{n+m}}{\theta_1^n\theta_2^m}}=$

7B) $\left(\frac{-(n+m)}{\sum\ln x_i+\sum\ln y_j}\right)^{n+m}\left(\frac{-n}{\sum\ln x_i}\right)^{-n}\left(\frac{-m}{\sum\ln y_j}\right)^{-m}=$

$\left(\frac{-(n+m)}{\sum\ln x_i+\sum\ln y_j}\right)^{n}\left(\frac{-(n+m)}{\sum\ln x_i+\sum\ln y_j}\right)^{m}\left(\frac{-n}{\sum\ln x_i}\right)^{-n}\left(\frac{-m}{\sum\ln y_j}\right)^{-m}=$

$\left(\frac{(n+m)\sum\ln x_i}{n(\sum\ln x_i+\sum\ln y_j)}\right)^{n}\left(\frac{(n+m)\sum\ln y_j}{m(\sum\ln x_i+\sum\ln y_j)}\right)^{m}=$

$\left(\frac{n+m}{n}\right)^{n}\left(\frac{n+m}{m}\right)^{m}\left(\frac{\sum\ln x_i}{\sum\ln x_i+\sum\ln y_j}\right)^{n}\left(\frac{\sum\ln y_j}{\sum\ln x_i+\sum\ln y_j}\right)^{m}=$

$T^n(1-T)^m$

$\frac{\sum\ln y_j}{\sum\ln x_i+\sum\ln y_j}=\frac{\sum\ln x_i+\sum\ln y_j-\sum\ln x_i}{\sum\ln x_i+\sum\ln y_j}=1-T$

7C) $T\sim Beta(n,m)=\frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)}T^{n-1}(1-T)^{m-1}$

8A) $f(\theta|y_i)=\theta e^{-(\theta+1)\ln y_i}$, $L(\theta|y_1,\cdots,y_n)=\prod_{i=1}^{n}\theta e^{-(\theta+1)\ln y_i}=\theta^{n}e^{-(\theta+1)\sum_{i=1}^{n}\ln y_i}$

$p(\theta)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\theta^{\alpha-1}e^{-\theta/\beta}$, $L(\theta|y_1,\cdots,y_n)p(\theta)=\theta^{n}e^{-(\theta+1)\sum_{i=1}^{n}\ln y_i}\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\theta^{\alpha-1}e^{-\theta/\beta}=$

$\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\theta^{n+\alpha-1}e^{-(\theta+1)\sum_{i=1}^{n}\ln y_i-\theta/\beta}$

$\cancel{\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\theta^{n+\alpha-1}\exp\left(-\theta\beta(\theta+1)\sum\ln y_i-\theta\right)}=$

$\cancel{\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\theta^{n+\alpha-1}\exp\left(\frac{-\theta(\beta(\theta+1)\sum\ln y_i+\theta)}{-\theta\,\theta\beta}\right)}=$

$\cancel{\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\theta^{n+\alpha-1}\exp\left(\frac{-\theta}{\frac{\theta\beta}{\beta(\theta+1)\sum\ln y_i+\theta}}\right)}$

~~We can see that this is the kernel for Gamma(n+α, $\frac{\theta\beta}{\beta(\theta+1)\sum\ln y_i+\theta}$).~~ ~~Gamma(α*, β*)~~

$\cancel{\int\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\theta^{(n+\alpha)-1}\exp\left(\frac{-\theta}{\frac{\theta\beta}{\beta(\theta+1)\sum\ln y_i+\theta}}\right)d\theta}=$

$\cancel{\boxed{\Gamma(n+\alpha)\left(\frac{\theta\beta}{\beta(\theta+1)\sum\ln y_i+\theta}\right)^{n+\alpha}\theta^{(n+\alpha)-1}\exp\left(\frac{-\theta}{\frac{\theta\beta}{\beta(\theta+1)\sum\ln y_i+\theta}}\right)}}$

~~8B)~~ $\cancel{E(\theta|y_1,\cdots,y_n)=\alpha^*\beta^*=\boxed{\frac{(n+\alpha)\theta\beta}{\beta(\theta+1)\sum\ln y_i+\theta}}}$

$\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\theta^{n+\alpha-1}e^{-\theta\sum\ln y_i-\cancel{\sum\ln y_i}-\theta/\beta}\propto\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\theta^{n+\alpha-1}e^{-\theta\sum\ln y_i-\theta/\beta}=$

$\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\theta^{n+\alpha-1}e^{-\theta(\sum\ln y_i-\frac{1}{\beta})}=$

$\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\theta^{n+\alpha-1}\exp\left(\frac{-\theta}{\frac{1}{\sum\ln y_i-1/\beta}}\right)$

We can see that this is the kernel for Gamma($n+\alpha$, $\frac{1}{\sum\ln y_i-1/\beta}$). Gamma($\alpha^*$, β^*)

$\int\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\theta^{(n+\alpha)-1}\exp\left(\frac{-\theta}{\frac{1}{\sum\ln y_i-1/\beta}}\right)d\theta=$

$\boxed{\Gamma(n+\alpha)\left(\frac{1}{\sum\ln y_i-1/\beta}\right)^{n+\alpha}\theta^{(n+\alpha)-1}\exp\left(\frac{-\theta}{\frac{1}{\sum\ln y_i-1/\beta}}\right)}$

8B) $E(\theta|y_1,\cdots,y_n)=\alpha^*\beta^*=\boxed{\frac{n+\alpha}{\sum\ln y_i-1/\beta}}$

8C) $\alpha^*=n+\alpha=(15)+(3)=18$

$\beta^*=\frac{1}{\sum\ln y_i-1/\beta}=\frac{1}{\ln(\prod y_i)-1/(2)}=\frac{1}{\ln(1164)-1/2}\approx 0.2174$

$\boxed{(2.3192, 5.9173)}$

8D) $H_0:\theta\leq 2$, $H_A:\theta>2$

$P(\theta<2)\rightarrow 0.00658$

$P(\theta>2)=1-P(\theta<2)=$

$1-0.00658\approx 0.9934>0.00658$

$P_r(\theta\in\Omega_A)>P_r(\theta\in\Omega_0)$ — We reject $H_0:\theta\leq 2$ in a Bayesian test. There is sufficient evidence that $\theta>2$.