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STAT 405-001

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10.99 A) $P_{oisson}(\lambda) = \frac{e^{-\lambda}\lambda^{y}}{y!}$ $\quad L(Y_1, \dots, Y_n \mid \lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{y_i}}{y_i!} = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n}(y_i)!}$

$\frac{L(\lambda_0)}{L(\lambda_a)} < k \rightarrow \frac{e^{-n\lambda_0}\lambda_0^{\sum y_i}}{e^{-n\lambda_a}\lambda_a^{\sum y_i}} < k \rightarrow$

$e^{-n(\lambda_0-\lambda_a)}\left(\frac{\lambda_0}{\lambda_a}\right)^{\sum_{i=1}^{n} y_i} < k \rightarrow$

B) $\underline{P_{oisson}(k) = \frac{e^{-\lambda}\lambda^{k}}{k!}}$

$\left(\frac{\lambda_0}{\lambda_a}\right)^{\sum_{i=1}^{n} y_i} < k\, e^{n(\lambda_0-\lambda_a)} \rightarrow$

$\ln\left(\frac{\lambda_0}{\lambda_a}\right)^{\sum_{i=1}^{n} y_i} < \ln\left(k\, e^{n(\lambda_0-\lambda_a)}\right) \rightarrow$

$\sum_{i=1}^{n} Y_i \ln\left(\frac{\lambda_0}{\lambda_a}\right) < \ln k + n(\lambda_0-\lambda_a) \rightarrow$

$\boxed{\sum_{i=1}^{n} Y_i > \frac{\ln k + n(\lambda_0-\lambda_a)}{\ln \lambda_0 - \ln \lambda_a}}$ $\quad \lambda_a > \lambda_0 \rightarrow \ln\left(\frac{\lambda_0}{\lambda_a}\right) < 0$

$\sum_{i=1}^{n} Y_i > k' \rightarrow n\bar{Y} > k' \rightarrow \underline{n > \frac{k'}{\bar{Y}}}$

$\alpha = P\left(\sum_{i=1}^{n} Y_i > k' \mid H_0 \text{ is true}\right) \rightarrow$

$\alpha = 1 - P\left(\sum_{i=1}^{n} Y_i \le k'\right) \rightarrow$

$\underline{\alpha = 1 - \sum_{y=0}^{k'} \frac{e^{-n\lambda}(n\lambda)^{y}}{y!} \rightarrow \beta = 1-\alpha = \sum_{y=0}^{k'} \frac{e^{-n\lambda}(n\lambda)^{y}}{y!}}$

c) Yes, because of Theorem 10.1 and the fact that the same rejection region would have been obtained for any $\lambda_0$ and $\lambda_a \ni \lambda_a > \lambda_0$.

D) We can see from Problem 10.99A that all steps except the last would be the same, and the resulting rejection region is:

$\sum_{i=1}^{n} Y_i \ln\left(\frac{\lambda_0}{\lambda_a}\right) < \ln k + n(\lambda_0-\lambda_a) \rightarrow$

$\boxed{\sum_{i=1}^{n} < \frac{\ln k + n(\lambda_0-\lambda_a)}{\ln \lambda_0 - \ln \lambda_a}}$, $\quad \lambda_a < \lambda_0 \rightarrow \ln\left(\frac{\lambda_0}{\lambda_a}\right) > 0$

$$f(y|2,\theta)=\frac{\theta^2}{\Gamma(2)}y^{2-1}e^{-\theta y}=\theta^2 y e^{-\theta y}\longrightarrow L(y_1,\dots,y_n|2,\theta)=\prod_{i=1}^{n}\theta^2 y e^{-\theta y}=$$

$$\theta^{2n}\prod_{i=1}^{n}y_i\, e^{-\sum_{i=1}^{n}\theta y_i}\longrightarrow$$

$$\frac{L(2,\theta_0)}{L(2,\hat\theta)}=\frac{\theta_0^{2n}\prod_{i=1}^{n}y_i\, e^{-\theta_0\sum_{i=1}^{n}y_i}}{\hat\theta^{2n}\prod_{i=1}^{n}y_i\, e^{-\hat\theta\sum_{i=1}^{n}y_i}}=\left(\frac{\theta_0}{\hat\theta}\right)^{2n}e^{\sum_{i=1}^{n}y_i(\hat\theta-\theta_0)}=$$

$$\left(\frac{\theta_0}{\left(\frac{2}{\bar y}\right)}\right)^{2n}e^{(n\bar y)\left(\left(\frac{2}{\bar y}\right)-\theta_0\right)}=$$

$$\left(\frac{\theta_0\bar y}{2}\right)^{2n}e^{2n-\theta_0 n\bar y}=$$

$$\left(\frac{\theta_0\bar y}{2}\right)^{2n}\frac{e^{2n}}{e^{\theta_0\bar y n}}=$$

$$\left(\frac{(e\theta_0\bar y)^2}{4e^{\theta_0\bar y}}\right)^{n}$$

$$\ln(L)=\ln\left(\theta^{2n}\prod_{i=1}^{n}y_i\, e^{-\sum_{i=1}^{n}\theta y_i}\right)=$$

$$\ln(\theta^{2n})+\ln\left(\prod_{i=1}^{n}y_i\right)+\ln\left(e^{-\sum_{i=1}^{n}\theta y_i}\right)=$$

$$2n\ln\theta+\ln\left(\prod_{i=1}^{n}y_i\right)-\sum_{i=1}^{n}\theta y_i$$

$$\frac{d(\ln(L))}{d\theta}=2n\left(\frac{1}{\theta}\right)+0-\sum_{i=1}^{n}y_i=0\longrightarrow$$

$$\frac{2n}{\theta}=\sum_{i=1}^{n}y_i\longrightarrow$$

$$\hat\theta=\frac{2n}{\sum_{i=1}^{n}y_i}\longrightarrow\hat\theta=\frac{2n}{n\bar y}\Longrightarrow\hat\theta=\frac{2}{\bar y}\longrightarrow\bar y=\frac{2}{\hat\theta}$$

Unbiased $E(\bar Y)=E\left(\frac{\sum_{i=1}^{n}Y_i}{n}\right)=\frac{1}{n}E\left(\sum_{i=1}^{n}Y_i\right)=\frac{1}{n}\sum_{i=1}^{n}E(Y_i)=\frac{1}{n}\sum_{i=1}^{n}\frac{\alpha}{\beta}=\frac{1}{n}\sum_{i=1}^{n}\frac{2}{\theta}=\frac{1}{n}\left(\frac{2n}{\theta}\right)=\frac{2}{\theta}=\bar y$

Consistent $Var(\bar Y)=Var\left(\frac{\sum_{i=1}^{n}Y_i}{n}\right)=\frac{1}{n^2}Var\left(\sum_{i=1}^{n}Y_i\right)=\frac{1}{n^2}\sum_{i=1}^{n}Var(Y_i)=\frac{1}{n^2}\sum_{i=1}^{n}\left(\frac{\alpha}{\beta^2}\right)=\frac{1}{n^2}\sum_{i=1}^{n}\frac{2}{\theta^2}=\frac{1}{n^2}\left(\frac{2n}{\theta_0^2}\right)=\frac{2}{n\theta_0^2}$

$$\lim_{n\to\infty}\left(\frac{2}{n\theta_0^2}\right)=0$$

Sufficient $g(\bar y,\theta_0)=\theta_0^{2n}e^{-\theta_0 n\bar y}$, $\quad h(y_1,\dots,y_n)=\prod_{i=1}^{n}y_i$

$H_0:\theta=\theta_0$

$H_A:\theta<\theta_0$

$$\int_0^\infty f(y|2,\theta)\,dy=\int_0^\infty\theta^2 y e^{-\theta y}\,dy=\Big|_0^\infty-(\theta y+1)e^{-\theta y}=$$

$$\frac{-\theta(\infty)-1}{e^{\theta(\infty)}}-\left(\frac{-\theta(0)+1}{e^{\theta(0)}}\right)=$$

$$0+\frac{0+1}{1}=1$$

The area under the curve of Gamma$(2,\theta)$ is 1 $\forall\ \theta>0$. We can use this integral to find the point at which there is $\alpha$ of the curve to the right (the critical point). Because this is a one-tailed test, $\alpha$ is only on one side.

$$\int_0^k f(y|2,\theta)\,dy=\frac{-\theta(k)-1}{e^{\theta(k)}}-\left(\frac{-\theta(0)+1}{e^{\theta(0)}}\right)=1-\alpha\longrightarrow$$

$$\frac{-\theta k-1}{e^{\theta k}}=-\alpha\longrightarrow$$

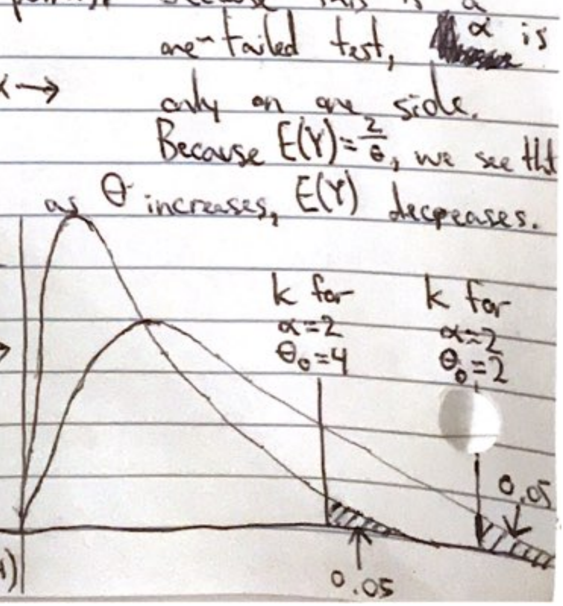
Critical point: $\boxed{\frac{\theta k+1}{e^{\theta k}}=\alpha}$

Because $E(Y)=\frac{2}{\theta}$, we see that as $\theta$ increases, $E(Y)$ decreases.

For example, if $\alpha=0.05$, we will see that $k\approx\frac{4.74386}{\theta_0}$. If $\bar y>k$, we can reject $H_0$ at $\alpha=0.05$. $\longrightarrow$

$\alpha=0.01 - k\approx\frac{6.63835}{\theta_0}$

$\alpha=0.1 - k\approx\frac{3.88972}{\theta_0}$




Graph for Gamma(2,2) and Gamma(2,4)