ON BZS COMPLETELY 0-SIMPLE SEMIGROUPS

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Abstract Following the work of [6], we investigate a class of semigroups whose elements are either idempotent or nilpotent of index 2. These semigroups are known as *Boolean Zero-Square* semigroups, or *BZS* semigroups. In particular, we present a method to count the number of idempotents and nilpotents in any completely 0-simple semigroup, or $C0S$ semigroup. We developed a program in C++ to generate *BZS* $C0S$ semigroups.

1. Introduction

A *semigroup* S is a set with a single binary associative operation. An *ideal* I in a semigroup S is a subset of S which is is closed under multiplication by elements of S; that is, if $s \in S$ and $i \in I$, then $si \in S$ and $is \in S$. We denote these conditions by writing $sI \subseteq I$ and $Is \subseteq I$. A semigroup S which contains a zero element 0 is *0-simple* if S and 0 are the only ideals in S.

An element $e \in S$ is *idempotent* if if $e^2 = e$. Let E denote the set of idempotents in E. We can define a partial order on E as follows: if $e, f \in S$, then $e \leq f$ iff $e = ef = fe$. This partial order is *trivial* if $e \leq f$ implies $e = f$.

A semigroup is *completely 0-simple* if S is 0-simple and the partial order on E is trivial. Such semigroups have been characterized by the following theorem, known as the *Rees Representation Theorem*.

Theorem 1.1. [1, 5] Let I, Λ be two sets. Let S be the set of all $I \times \Lambda$ matrices with at most one non-zero element from a group G, together with the zero matrix. A non-zero matrix is denoted (i, g, λ) while the matrix consisting entirely of zeros is denoted 0.

Let P be a $\Lambda \times I$ matrix with at least one entry from G in each row and column and zeros elsewhere. The matrix P is referred to as the *sandwich matrix*. Define multiplication $*$ on S as follows: if $(i, g, \lambda), (j, h, \mu) \in S$ then

$$(i, g, \lambda) * (j, h, \mu) = (i, g, \lambda)(\lambda, k, j)(j, h, \mu)$$

This product is (i, gkh, μ) if $k \neq 0$ and 0 otherwise. Then S is a $C0S$ semigroup. Conversely, any $C0S$ semigroup can be represented by such a set of matrices.

A semigroup S is *Boolean Zero Square* or *BZS* if every element $s \in S$ satisfies either $s^2 = s$ or $s^2 = 0$. Semigroups and rings with these properties are studied in [2, 3, 4, 6]. In this paper we characterize those $C0S$ semigroups which are *BZS*.

In this paper we focus on $C0S$ semigroup whose entries are either 0 or 1. In Section 2 we present the main results of this paper. In Section 3 we give examples of our results. Finally, in Section 4 we present the pseudo-code for the computer program which we wrote that generates *BZS* $C0S$ semigroups and serves to verify our results. We also provide a link to the actual program.

2. Main Results

Notation We investigate $C0S$ semigroups whose matrices have entries from $\{0, 1\}$. If $A = (i, 1, \lambda)$, we denote A as $E_{i,\lambda}$. The set of idempotents in S is denoted by E, and the set of nilpotents in S is denoted by N. The cardinality of these sets is denoted by $|E|$ and $|N|$.

Lemma 2.1. The matrix $E_{i,\lambda}$ is nilpotent if and only if the (λ, i) entry in P is 0.
The matrix $E_{i,\lambda}$ is idempotent if and only if the (λ, i) entry in P is 1.

Proof. $E_{i,\lambda} * E_{i,\lambda} = E_{i,\lambda} P E_{i,\lambda} = E_{i,\lambda} E_{\lambda,i} E_{i,\lambda}$. This element is zero iff the (λ, i) entry in P is zero, and this element is 1 iff the (λ, i) entry in P is 1. $\square$

Theorem 2.2. If S is a $C0S$ semigroup, then S is BZS iff the entries in S and the sandwich matrix P are from $\{0, 1\}$.

Proof. ($\Rightarrow$) By contrapositive. Let $(i, g, \lambda) \in S$ where $g \neq 0$ and $g \neq 1$. Let the (λ, i) entry in P be h where $h \neq 0$ and $h \neq g^{-1}$. Then $(i, g, \lambda) * (i, g, \lambda) = (i, g, \lambda)(\lambda, h, i)(i, g, \lambda) = (i, ghg, \lambda)$ and $ghg \neq g$ and $ghg \neq 0$.
($\Leftarrow$) This follows from Lemma 2.1. $\square$

Theorem 2.3. Let S be a $C0S$ semigroup over $\{0, 1\}$. Then

(1) every element of N is nilpotent of index 2;
(2) $|E|$ equals the number of $1's$ in P;
(3) $|N|$ equals the number of $0's$ in P.

Moreover, the matrix $A = E_{i,\lambda}$ is idempotent iff the the $(\lambda, 1)$ entry in P is 1; otherwise, $A^2 = 0$.

Proof. This follows from Lemma 2.1 and Theorem 2.2. $\square$

Corollary 2.4. $max\{|I|, |\Lambda|\} \leq |E| \leq |I||\Lambda|$

Proof. If every element of P is 1, then P contains $|I||\Lambda|$ 1's. Hence $|E| \leq |I||\Lambda|$ by Theorem 2.3(2).

The minimal value of $|E|$ equals the minimal number of 1's in P by Theorem 2.3. Recall that each element of P has a least one 1 in each row and column. Therefore, to find the minimal value of $|E|$ we need to count the minimal number of 1's in any sandwich matrix P. Take as matrix P the matrix with elements in positions $(1, 1), (2, 2), \ldots, (|I|, |I|)$. There are three cases to consider.

If $|I| = |\Lambda|$, then this matrix has $|I| = |\Lambda|$ 1's, and there is a single 1 in each row and column. Any other matrix with $|I| = |\Lambda|$ can have fewer 1's. Hence $|E| = |I| = |\Lambda|$.

If $|I| > |\Lambda|$, then we can arbitrarily put a single 1 anywhere in each of the remaining rows. In this case $|E| = |I|$.

If $|I| < |\Lambda|$ then we can arbitrarily put a single 1 in each of the remaining columns. In this case $|E| = |\Lambda|$. $\square$

We generalize the previous result to arbitrary $C0S$ semigroups. The proof is virtually identical to that of Theorem 2.3.

Theorem 2.5. Let S be a $C0S$ semigroup over $G \cup 0$ for an arbitrary group G. Then

(1) $|E|$ equals the number of nonzero entries in P;
(2) if k is the number of $0's$ in P, then $|N| = |G|k$.

Moreover, the matrix (i, g, λ) is idempotent iff the (λ, i) entry of P is g^{-1}, and (i, g, λ) is nilpotent of index 2 iff the (λ, i) entry of P is 0.

3. Examples

In this section we provide examples of the semigroups in the previous section.

Example 3.1. Let S be the set of all 2×2 matrices over $\{0, 1\}$ with at most one nonzero entry, and let the sandwich matrix P be the identity matrix, $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $S = \{E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}\} \cup \{0\}$. Denote these matrices by A, B, C, D respectively. The semigroup S is given in the following table.

	0	*A*	*B*	*C*	*D*
0	*0*	*0*	*0*	*0*	*0*
A	*0*	*A*	*B*	*0*	*0*
B	*0*	*0*	*0*	*A*	*B*
C	*0*	*C*	*D*	*0*	*0*
D	*0*	*0*	*0*	*C*	*D*

The sandwich matrix has two 0's and two 1's. As from Theorem 2.3 we have two idempotents where $E = \{A, D\}$, and two nilpotents where $N = \{B, C\}$.

Example 3.2. Let S be as in Example 3.1 and let $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. The semigroup S is given in the following table.

	0	*A*	*B*	*C*	*D*
0	*0*	*0*	*0*	*0*	*0*
A	*0*	*0*	*0*	*A*	*B*
B	*0*	*A*	*B*	*A*	*B*
C	*0*	*0*	*0*	*C*	*D*
D	*0*	*C*	*D*	*C*	*D*

The sandwich matrix has one 0 and three 1's. As from Theorem 2.3 we have three idempotents where $E = \{B, C, D\}$, and one nilpotent where $N = \{A\}$.

Example 3.3. Let S be the set of all 2×3 matrices over $\{0, 1\}$ with at most one nonzero entry, and let $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $S = \{E_{1,1}, E_{1,2}, E_{1,3}, E_{2,1}, E_{2,2}, E_{2,3}\} \cup \{0\}$. Denote these matrices by A, B, C, D, E, F respectively. The multiplication table is

	0	*A*	*B*	*C*	*D*	*E*	*F*
0	*0*	*0*	*0*	*0*	*0*	*0*	*0*
A	*0*	*A*	*B*	*C*	*A*	*B*	*C*
B	*0*	*A*	*B*	*C*	*0*	*0*	*0*
C	*0*	*0*	*0*	*0*	*A*	*B*	*C*
D	*0*	*D*	*E*	*F*	*D*	*E*	*F*
E	*0*	*D*	*E*	*F*	*0*	*0*	*0*
F	*0*	*D*	*E*	*F*	*D*	*E*	*F*

The sandwich matrix has two 0's and four 1's. As from Theorem 2.3 we have four idempotents where $E = \{A, B, D, F\}$, and two nilpotents where $N = \{C, E\}$.

4. Pseudo-code

Here we present pseudo-code for the program which we used to check our results. The program can be found at **include a link to the program**.

Add pseudo-code here

References

[1] Clifford, A. H. and G. B. Preston, *The Algebraic Theory of Semigroups*, American Mathematical Society, New York, 1961.

[2] Farag, Mark, and Ralph P. Tucci, BZS Rings, Palestine Journal of Mathematics, Vol. 8(2) (2019), 8 – 14.

[3] Farag, Mark, and Ralph P. Tucci, BZS Rings II, Journal of Algebra and Related Topics, Vol. 9, No 2, (2021), pp 29-37.

[4] Farag, Mark, and Ralph P. Tucci, BN Rings, submitted.

[5] Howie, J. M., *Fundamentals of semigroup theory*, London Mathematical Society Monographs No. 12, Clarendon Press, Oxford, 1995.

[6] Pinto, G. A., Boolean Square Zero (BZS) Semigroups, SQU Journal for Science, 2021, 26(1), 31 - 39.

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