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Overall assessment (circle grade)	A*	A	B	C	D
Guideline standard	>75%	65-75%	55-65%	40-55%	<40%
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4F13 Probabilistic Machine Learning - True Skill Ranking

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Abstract

This report outlines the results of the second coursework for 4F13. We aim to rank professional

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1 Questions

a Gibbs Sampling

Gibbs sampling is used to sample from a multi-variate distribution by sequentially sampling from univariate conditionals. In the True-Skill model, we can approximate the conditional distribution with a 1-D Gaussian of a certain mean and variance - which are functions of the other variables. A portion of the modified code to compute these samples is given in listing 1.

Listing 1: Gibbs sampling additions

```
m = np.zeros((M, 1))
for p in range(M):
    # fill in m[p] prediction (natural param conditional)
    wins_array = np.array(G[:, 0] == p).astype(int)
    loss_array = np.array(G[:, 1] == p).astype(int)
    m[p] = np.dot(t[:, 0], (wins_array - loss_array))
```

We plot the sampled player skills for a few players (figure 1). These data do appear noisy (in some sense random) but it appears that neighbouring samples are strongly correlated. Furthermore it is hard to tell how soon the Gibbs sampler transitions into a stable probability density region.

To investigate the burn-in time (iterations until the Gibbs is in a stationary state), we can plot the population mean and standard deviation at each Gibbs iteration - figure 2. The population mean gives us little information but we see that the standard deviation converges to a steady value fairly quickly (only after 10 iterations). We can therefore set the burn-in time to $b = 10$ and we would discard all preceding samples.

However, there is an additional wrinkle, the plot in figure 1 is smoother than it should be; neighbouring samples are not independent. To test this hypothesis, we plot the auto-correlation of each player's skill samples (figure 3). For samples to be approximately independent, their correlation should be close to 0. This is only true for all players for an offset of at least $k = 10$. Nevertheless, we get good enough for $k = 5$ - excepting one player: Djokovic.

In summary, to run our Gibbs sampler efficiently and reliably, we only use samples after a burn-in period of $b = 10$ iterations. Even then, we thin the samples such that we only keep every t 'th ($t = 5$) sample to ensure that they are approximately independent. We use these values $b = 10$ and $t = 5$ for the rest of the report - such that for 1100 samples we only keep 218 of them.

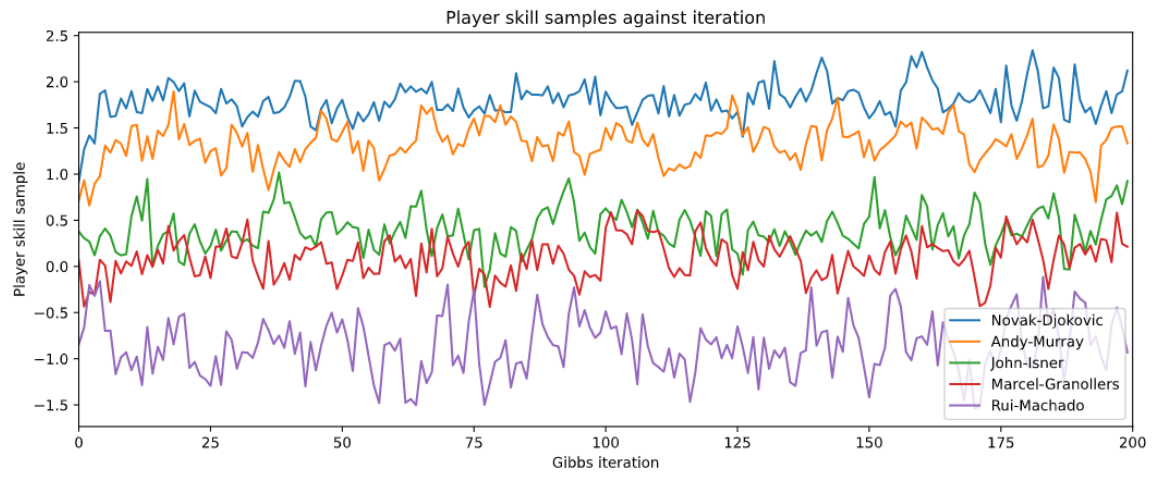


Figure 1: Gibbs skill samples for various players

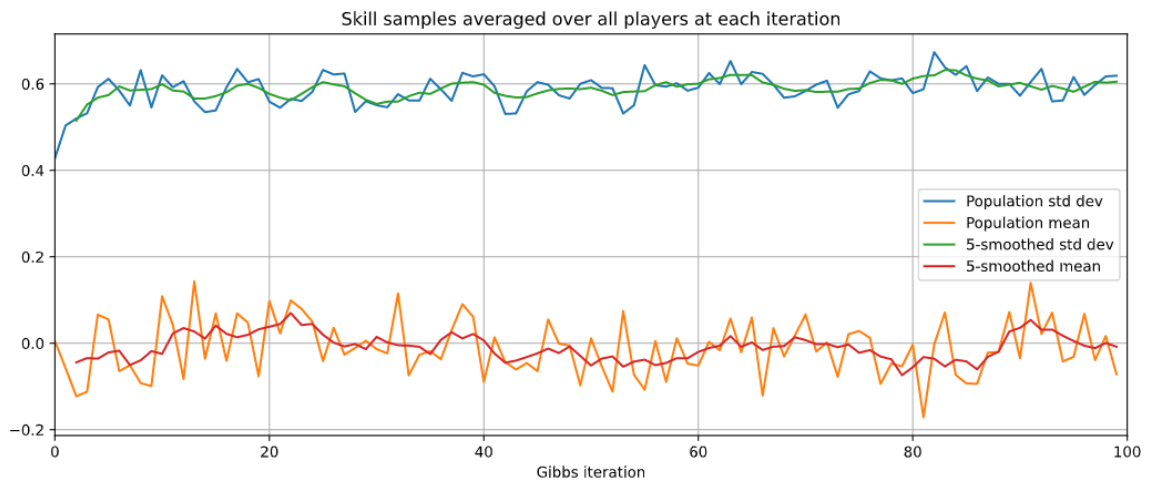


Figure 2: Gibbs skill samples averaged over all players at each iteration

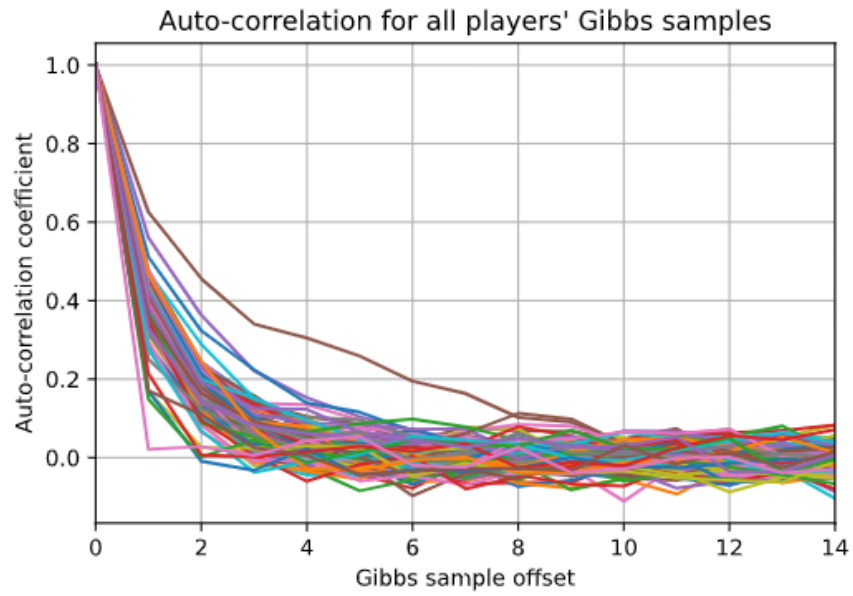


Figure 3: Auto-correlation for Gibbs sampler for all players

b EP - Message Passing

Nevertheless, Gibbs sampling is rather computationally intensive. An alternative is message-passing through the Expectation Propagation (EP algorithm). This treats each player as a vertex on a graph, connected by directed edges representing game outcomes. We can use message passing (a form of Belief Propagation) to iteratively improve on our guess for each player's marginal skill. We approximate the marginal skill by a Gaussian with two parameters: mean and precision (inverse variance). These parameters have the advantage of combining simply to form the messages.

In Gibbs sampling, we run a Markov chain and aim to converge to a stationary probability distribution. This target distribution is the joint player skill distribution. However, in EP we do not converge to a distribution but rather a stable graph object; further iterations do not alter the parameters of each vertex: mean and precision.

In the previous section, we saw the Gibbs sampler only takes $b = 10$ samples to converge to a stationary distribution. For the EP algorithm, we can plot the player parameters against iteration number for a few players to gauge convergence (figure 4).

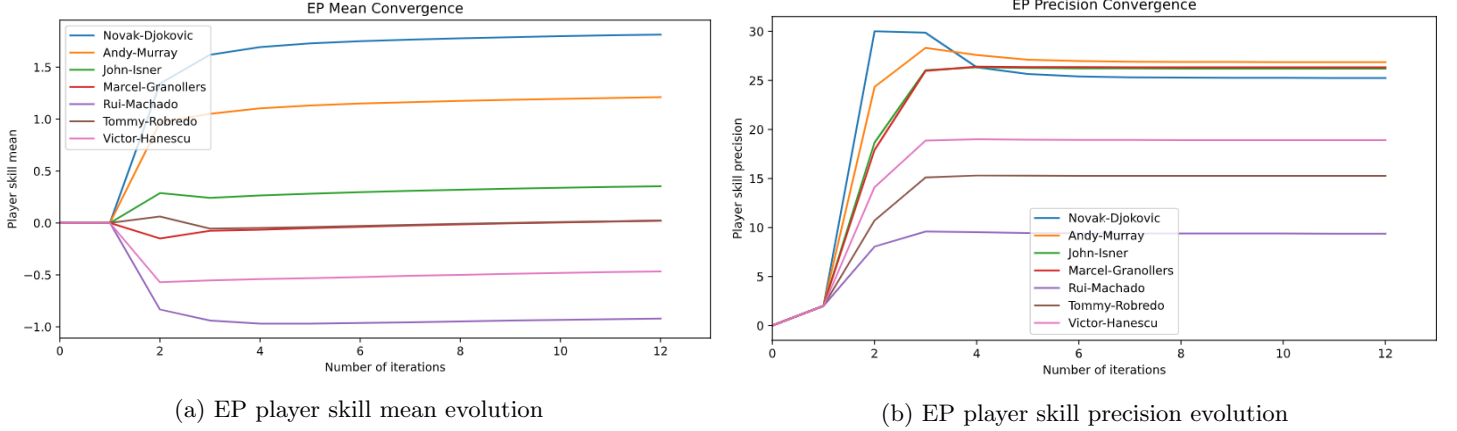


Figure 4: EP player skill parameter evolution against iteration

From the plots we can tell that we converge quickly for this set of players. Setting the number of iterations to $T = 5$ seems reasonable to get a good estimate of the rankings as the mean and precision do not change much after this point. This value of $T = 5$ will be used throughout the rest of the report.

c EP - Player Comparison

Suppose that a player i goes up against j . In the True-Skill model, we denote their skill by $w_i \sim \mathcal{N}(m_i, \sigma_i^2)$ and $w_j \sim \mathcal{N}(m_j, \sigma_j^2)$ respectively. The mean and variance parameters are estimated through message passing (the variance is just the inverse of the precision).

The performance difference $s_{ij} = w_i - w_j$ is corrupted by Gaussian noise of unit variance $t_{ij} = s_{ij} + n$ to account for performance inconsistency ($n \sim \mathcal{N}(0, 1)$). The match result is then given by the sign of t_{ij} such that $t_{ij} > 0 \Rightarrow i$ wins and j wins otherwise.

Having run message-passing, we wish to compute the probability that i is more skilful than j : $P(s_{ij} > 0)$. This is different from the probability that i beats j in a head-to-head: $P(t_{ij} > 0)$. These probabilities are easy to compute by assuming that $w_i \perp\!\!\!\perp w_j \perp\!\!\!\perp n \perp\!\!\!\perp w_i$, so means and variances combine additively:

$$s_{ij} = w_i - w_j \sim \mathcal{N}(m_i - m_j, \sigma_i^2 + \sigma_j^2) \quad (1)$$

$$t_{ij} = s_{ij} + n \sim \mathcal{N}(m_i - m_j, \sigma_i^2 + \sigma_j^2 + 1) \quad (2)$$

For any Gaussian r.v. $X \sim \mathcal{N}(\mu, \sigma^2)$, $P(X > 0) = \Phi(\mu/\sigma)$ - where $\Phi(\cdot)$ is the standard Gaussian c.d.f. Applying this result to equations 1 and 2 for the top four ranked players, we obtain table 1.

$P(s_{ij} > 0)$	Djokovic	Federer	Nadal	Murray	$P(t_{ij} > 0)$	Djokovic	Federer	Nadal	Murray
Djokovic	-	0.91	0.94	0.99	Djokovic	-	0.64	0.66	0.72
Federer	0.09	-	0.58	0.81	Federer	0.36	-	0.52	0.59
Nadal	0.06	0.42	-	0.76	Nadal	0.34	0.48	-	0.57
Murray	0.01	0.19	0.24	-	Murray	0.28	0.41	0.43	-

(a) Prob. row player is more skilful
(b) Prob. row player wins a head-to-head

Table 1: Top four players comparison based on EP (5 iterations)

We are more confident in who is better than in who will win: $|P(s_{ij} > 0) - 0.5| > |P(t_{ij}) - 0.5| \quad \forall i, j$. This makes sense mathematically as the variance $\sigma_i^2 > \sigma_s^2$. It also makes sense intuitively because an individual match is subject to performance variation. I would bet my house that Djokovic is a better tennis player than Andy Murray, but I'm not betting that much on Djokovic winning his match against him tomorrow.

d Gibbs - Player Comparison

However, we can also compare players through Gibbs sampling. Restricting ourselves to just Djokovic and Nadal, we compare them in three ways:

1. Approximate marginals by Gaussian
2. Approximate joint by bivariate Gaussian
3. Empirical comparison of joint samples

The first method is shown in figure 5. We estimate the parameters of each Gaussian by taking the mean and variance of each player's Gibbs samples (after accounting for burn-in and thinning). Indeed a Gaussian does seem to be a good fit for the data. We see that Djokovic has higher mean and lower variance.

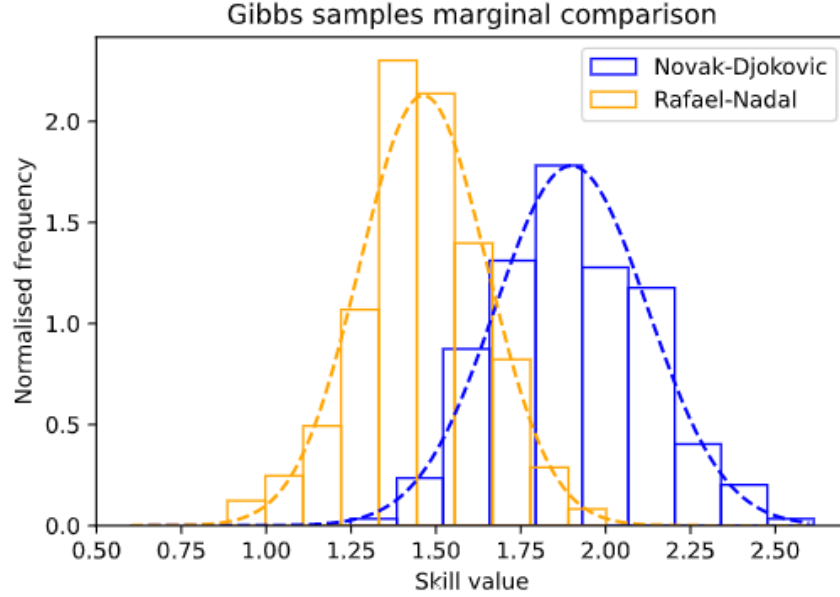


Figure 5: Gibbs samples marginal comparison - Djokovic v Nadal

Instead we can fit a bivariate Gaussian to the pair-wise samples $\{(w_i^{(n)}, w_j^{(n)})\}_{n=1}^N$ (figure 6). We approximate the parameters as before but now noting that we have an additional covariance term σ_{ij}^2 off the diagonal of the 2×2 variance matrix. This covariance term is non-zero (in fact positive) meaning that the principal axes of the Gaussian are slightly skewed.

The last method involves simply plotting the difference of the two skill samples (figure 7). Any bars coloured red have Nadal higher skill than Djokovic.

Each of these methods give a different way of estimating the probability player i is more skilful than j : $P(s_{ij} > 0)$. They are listed below:

1. Independent Gaussians - $\Phi((m_i - m_j)/(\sigma_i^2 + \sigma_j^2))$
2. Correlated Gaussians - $\Phi((m_i - m_j)/(\sigma_i^2 - 2\sigma_{ij}^2 + \sigma_j^2))$
3. Empirical comparison - $N^{-1} \sum_{n=1}^N \mathbb{1}(w_i^{(n)} > w_j^{(n)})$

Figure 6 shows that the samples are not independent so method 2 is preferred over 1. In addition, as the probabilities can be very small, the empirical estimate (though unbiased) will need a very high number of samples N to be accurate (the red area in figure 7 is very small). Therefore we prefer method 2 for comparing player skills. Doing this for the top four players yields table 2.

Table 2 agrees with 1a very well. If anything the predictions tend to be more confident for the joint Gaussian model as the covariance between each of the top four players' skills is positive. Therefore, this decreases the variance of the difference, leading to more confident predictions.

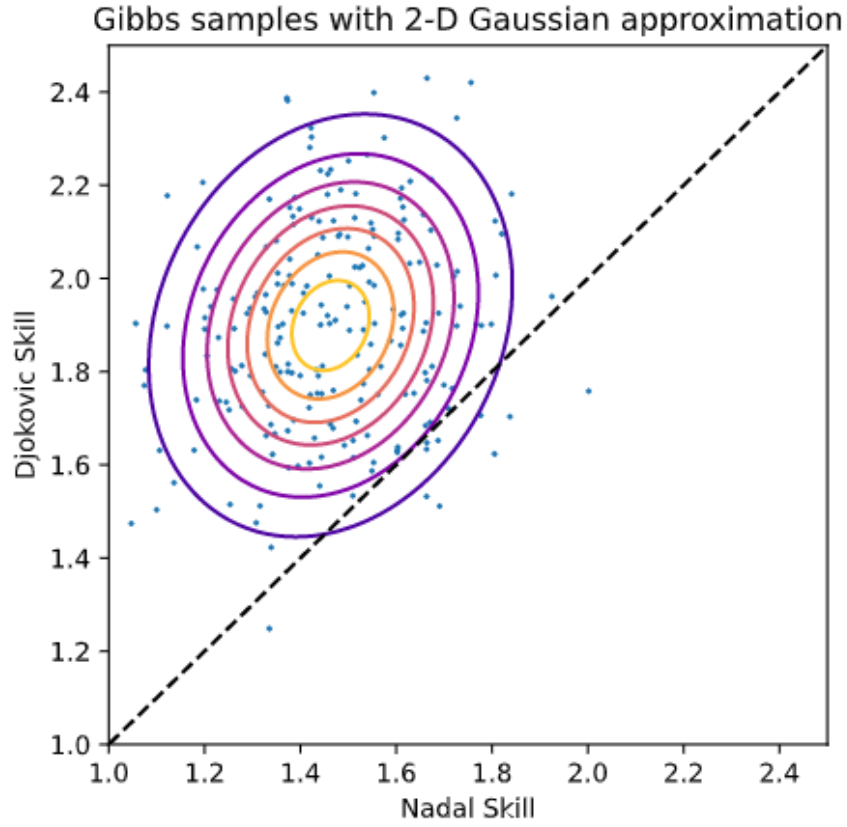


Figure 6: Gibbs samples joint comparison - Djokovic v Nadal

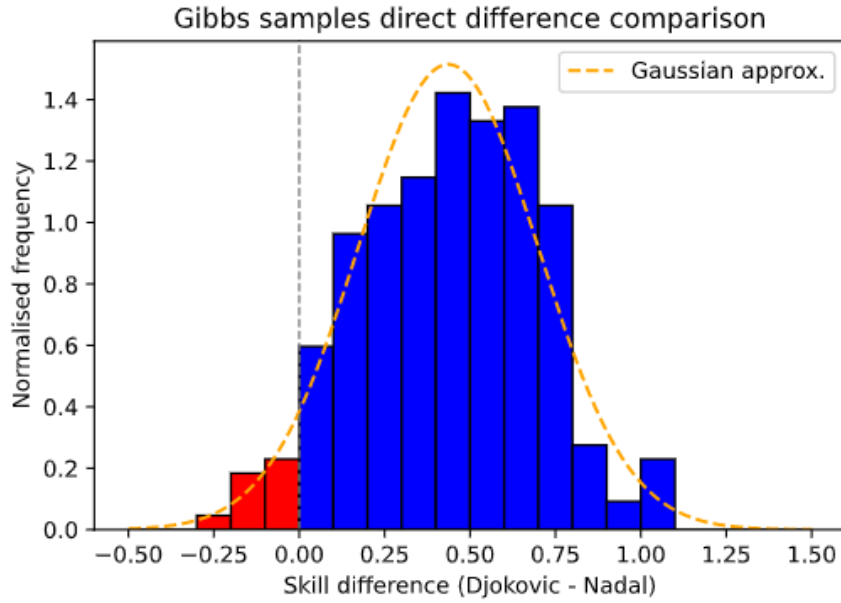


Figure 7: Gibbs samples direct difference comparison - Djokovic v Nadal

$P(s_{ij} > 0)$	Djokovic	Federer	Nadal	Murray
Djokovic	-	0.92	0.95	0.98
Federer	0.08	-	0.63	0.81
Nadal	0.05	0.37	-	0.75
Murray	0.02	0.19	0.25	-

Table 2: Player skills head-to-head - $P(s_{ij} > 0) = \Phi((m_i - m_j)/(\sigma_i^2 - 2\sigma_{ij}^2 + \sigma_j^2))$

e Method Comparison: Win ratio, Gibbs and EP

We go about ranking players using different methods: basic win ratio model; Gibbs sampling for True-Skill model; and EP estimates for True-Skill.

The win ratio model assumes that each player has a fixed probability of winning their match irrespective of their opponent. If a player has played n matches and won k of them, we know their win ratio is distributed as $p \sim \mathcal{N}(k/n, k(n-k)/n^2)$; this applies standard results for combining n independent Bernoulli r.v.'s. We plot the mean and standard deviation of p on figure 9.

With Gibbs sampling, we sample the player skill from conditional distributions and update the sample for each player at every iteration such that we mimic sampling from the joint. After burn-in and thinning, we have a set of independent samples for the skill of each player and we can calculate the mean and variance of this set empirically (plotted on figure 10).

For Expectation Propagation, we run the Message Passing algorithm for 5 iterations to achieve convergence. This returns the expected means and precisions (inverse variance) for the skill of each player and so we can plot these directly (figure 11)

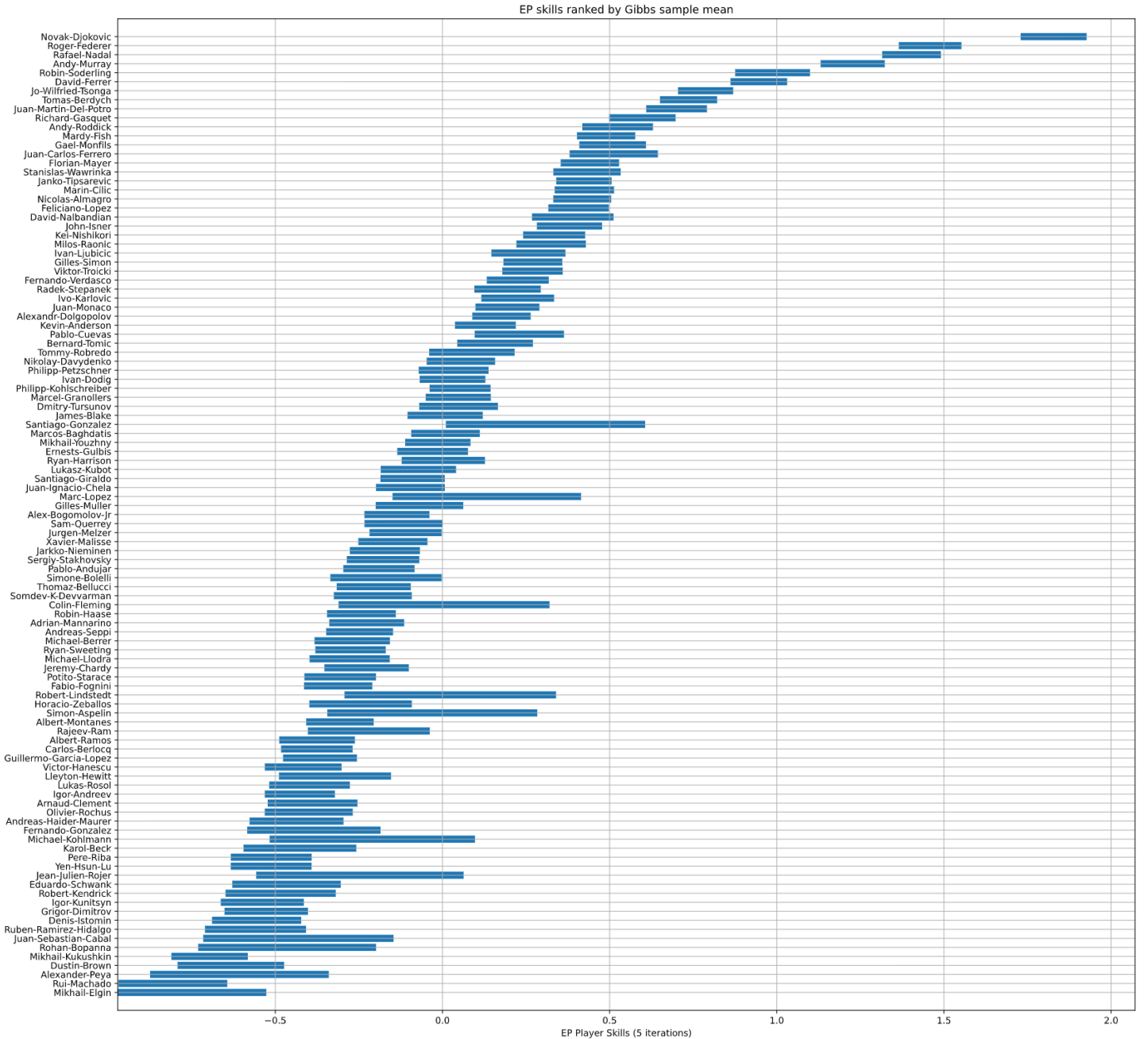


Figure 8: Player skills calculated by EP, ranked by Gibbs sample mean

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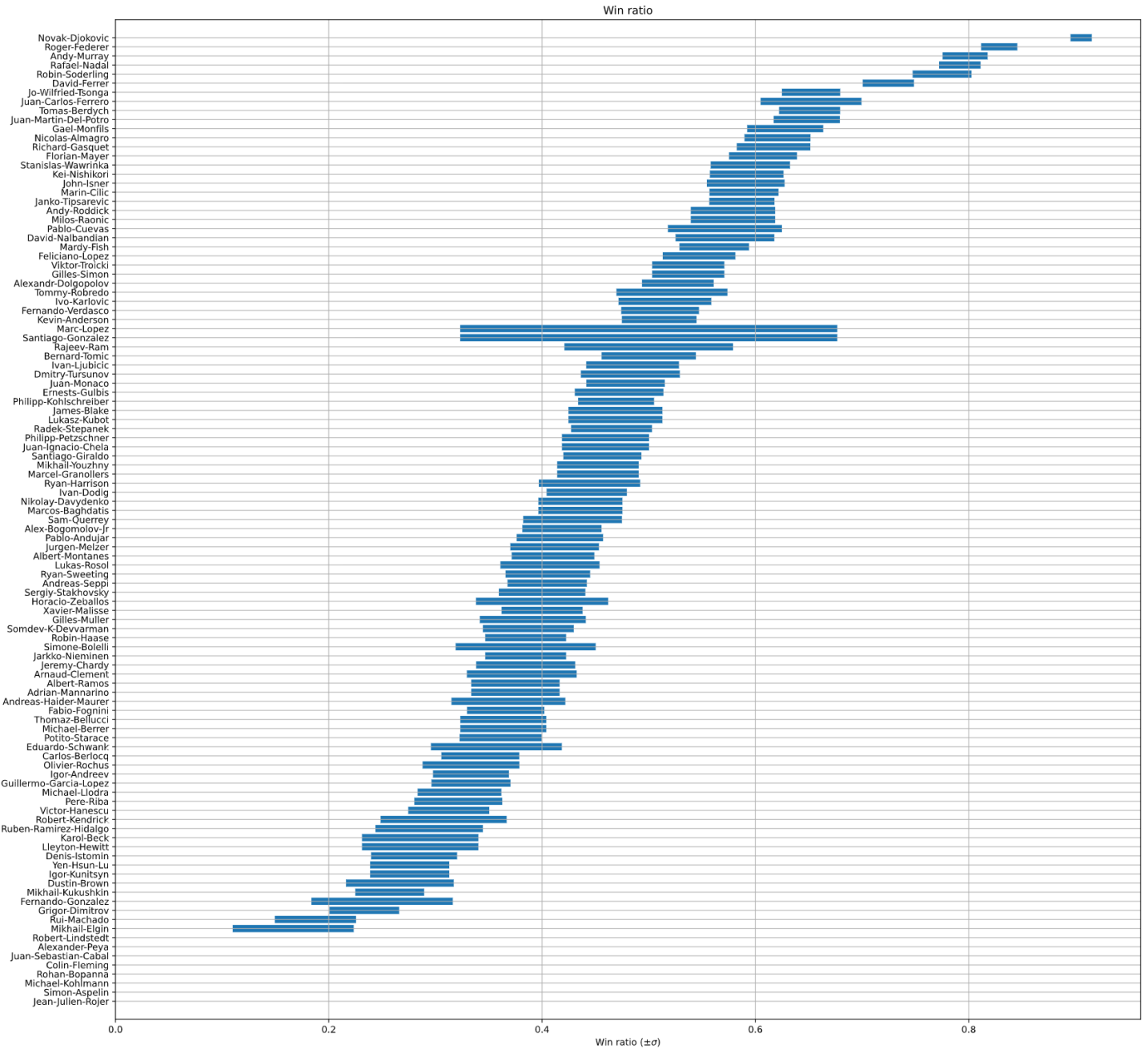


Figure 9: Player rankings by win ratio

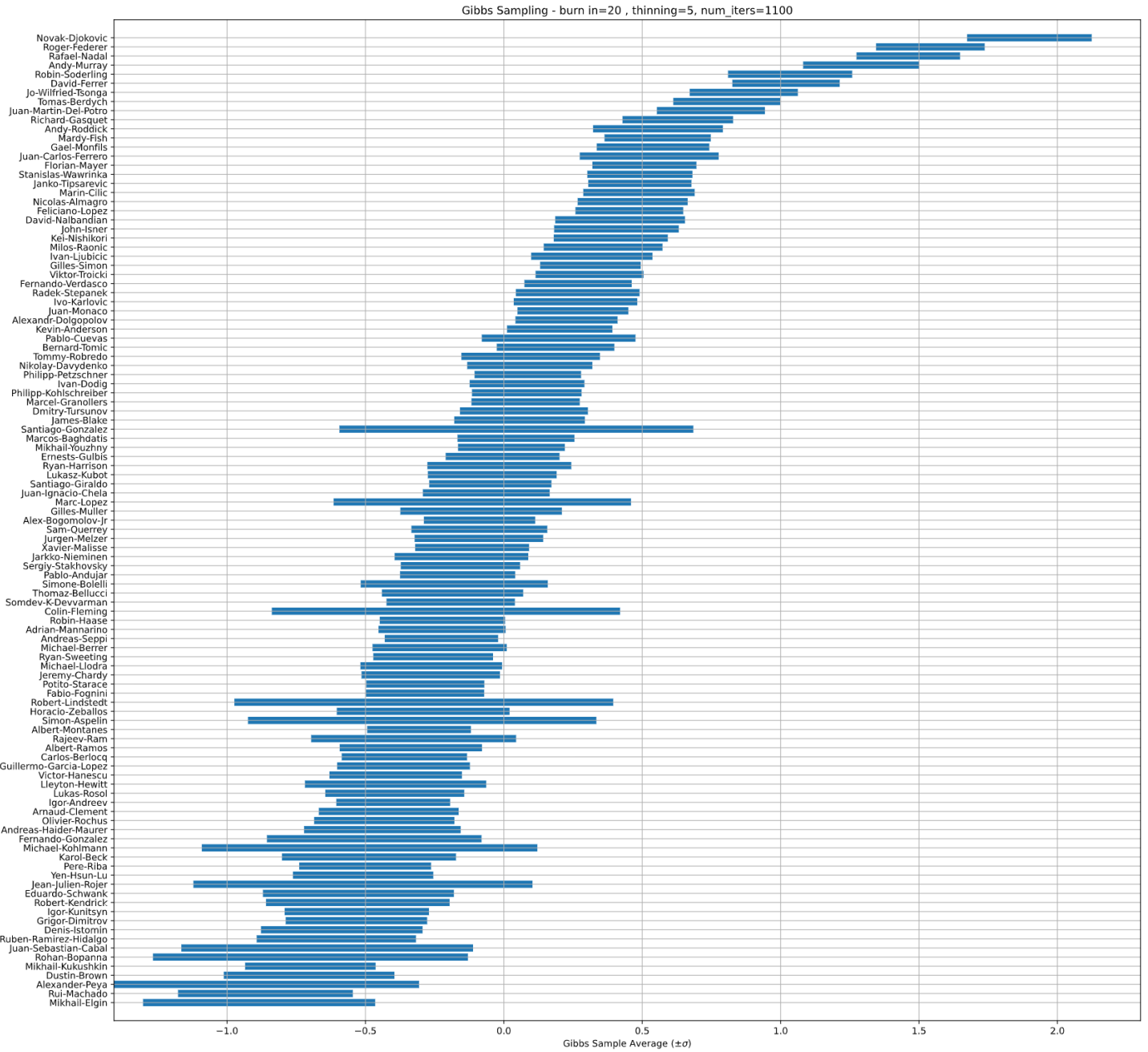


Figure 10: Players rankings by mean of Gibbs skill samples

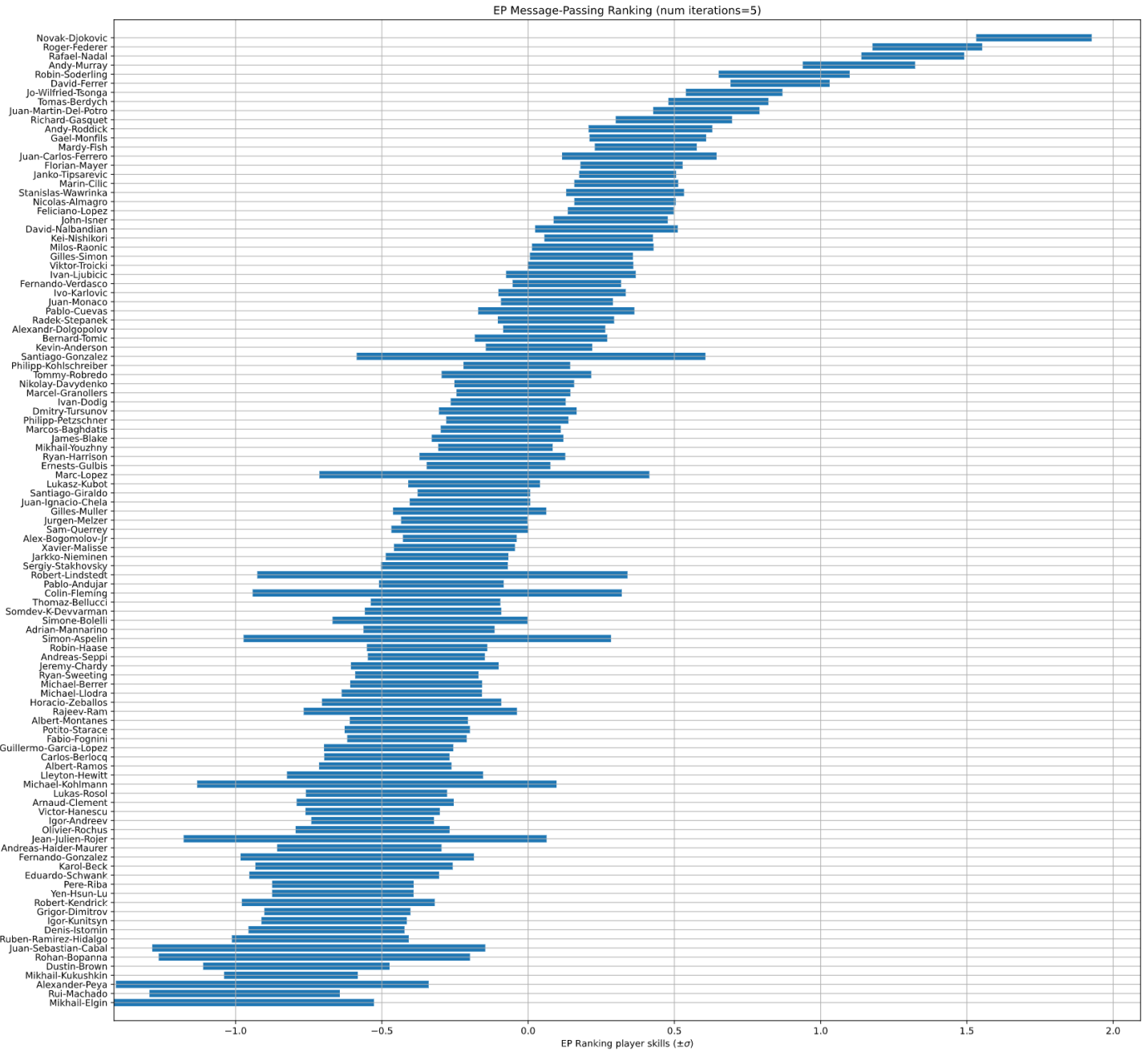


Figure 11: Player rankings by mean of EP-estimated skills