

Detecting Causal Structure in Time Series Data

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Abstract

1 Introduction

2 Hypothesis Testing

2.1 Single Sample against known mean

We start with the simple case of determining whether or not a coin is fair. Each coin flip can be represented by a random variable with a Bernoulli distribution $X_i \sim \text{Bern}(p)$. Each coin can result in either a Tails or a Heads denoted by $X_i \in \{0, 1\}$. We toss the coin n times leading us to a set $\{X_i\}_{i=1}^n$. We wish to test the null hypothesis $p = p_0$ against the alternative. We shall keep the p_0 notation for generality, though for a fair coin we require $p_0 = 1/2$.

$$\begin{aligned} H_0 : & \quad p = p_0 \\ H_1 : & \quad p \neq p_0 \end{aligned}$$

We denote the number of heads with the random variable $K := \sum_{i=1}^n X_i \sim \text{Bern}(p, n)$. For a particular experiment we observe $K = k$. We employ a standard likelihood ratio test. The test statistic t_n is calculated as the log-likelihood ratio of observing $K = k$ under H_1 and H_0 .

$$t_n := \log \frac{\mathcal{L}(H_1)}{\mathcal{L}(H_0)} = \log \frac{\max_p P(K = k|p)}{P(K = k|p = p_0)} \quad (1)$$

We note that since K is distributed as a binomial, $P(K = k|p) = \binom{n}{k} p^k (1-p)^{n-k}$. If we can vary p , this probability is maximised for $p = \hat{p} := k/n$. Therefore, the test statistic is given by.

$$t_n = \log \frac{\binom{n}{k} \hat{p}^k (1-\hat{p})^{n-k}}{\binom{n}{k} p_0^k (1-p_0)^{n-k}} = \log \frac{\hat{p}^k (1-\hat{p})^{n-k}}{p_0^k (1-p_0)^{n-k}} \quad (2)$$

The combinatoric term $\binom{n}{k}$ cancels out. This implies that the order in which the heads land does not matter when determining the fairness of the coin. We can work the above expression into a more usable form:

$$\begin{aligned} t_n &= k \log \frac{\hat{p}}{p_0} + (n-k) \log \frac{1-\hat{p}}{1-p_0} \\ &= n \left(\hat{p} \log \frac{\hat{p}}{p_0} + (1-\hat{p}) \log \frac{1-\hat{p}}{1-p_0} \right) \\ &= n \mathcal{D}(\text{Bern}(\hat{p}) || \text{Bern}(p_0)) \end{aligned} \quad (3)$$

Where $\mathcal{D}(f||g) := \sum_{x \in \mathcal{X}} f(x) \log \frac{f(x)}{g(x)}$ is the Kullback-Leibler divergence between two arbitrary probability mass functions f and g . This is also called the relative entropy. The KL divergence has the property that:

$$\mathcal{D}(f||g) \geq 0 \quad \text{with equality iff} \quad f(x) = g(x) \quad \forall x \in \mathcal{X} \quad (4)$$

We can exploit this result for the case that $f \approx g$ to obtain a simplified expression for the KL divergence. We begin by defining $\delta(x) := f(x) - g(x)$. We are interested in the region where δ is small. We start by substituting for $f = \delta + g$ and

then taking the Taylor expansion of $\log 1 + x$.

$$\begin{aligned}
\mathcal{D}(f||g) &= \sum_{x \in \mathcal{X}} (\delta + g) \log \left(1 + \frac{\delta}{g} \right) \\
&= \sum_{x \in \mathcal{X}} (\delta + g) \left(\frac{\delta}{g} - \frac{\delta^2}{2g^2} + O(\delta^3) \right) \\
&= \sum_{x \in \mathcal{X}} \delta + \frac{1}{2} \sum_{x \in \mathcal{X}} \frac{\delta^2}{g} + O(\delta^3) \\
&= \frac{1}{2} \sum_{x \in \mathcal{X}} \frac{\delta^2}{g} + O(\delta^3) \\
&= \frac{1}{2} \chi^2(f||g) + O(\delta^3)
\end{aligned}$$

Where the summation over δ evaluates to 0 because δ is the difference of two valid p.m.f's which each sum to 1 over $x \in \mathcal{X}$. We are able to neglect the $O(\delta^3)$ terms for f very close to g we shall see what this means later. $\chi^2(f||g)$ is known as the chi-squared distance between two distributions and is defined simply as $\chi^2(f||g) := \sum_{x \in \mathcal{X}} (f - g)^2 / g$. We now investigate the chi-squared divergence for $f = \text{Bern}(p)$ and $g = \text{Bern}(q)$.

$$\begin{aligned}
\chi^2(\text{Bern}(p)||\text{Bern}(q)) &= \frac{(p - q)^2}{q} + \frac{((1 - p) - (1 - q))^2}{1 - q} \\
&= \frac{(p - q)^2}{q(1 - q)} \\
&= \left(\frac{p - q}{\sqrt{q(1 - q)}} \right)^2
\end{aligned}$$

Now we can exploit these results to get a workable expression for the test statistic t_n .

$$\begin{aligned}
t_n &= n\mathcal{D}(\text{Bern}(\hat{p})||\text{Bern}(p_0)) \\
&= \frac{n}{2} \chi^2(\text{Bern}(\hat{p})||\text{Bern}(p_0)) + nO(\delta^3) \\
&= \frac{1}{2} \left(\frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}} \right)^2 + \epsilon
\end{aligned}$$

So far we have been treating t_n as deterministic but it is merely an observation a random variable. To make this distinction clear we shall use upper-case to refer to random variables. Therefore, we have that the random variable T_n is a function of $\hat{P} := K/n$. For large n , we can find the distribution of \hat{P} by the Central Limit Theorem.

$$\begin{aligned}
\hat{P} &= \frac{K}{n} = \frac{1}{n} \sum_{i=1}^n X_i \\
H_0 : \mathbb{E}[X_i] &= p_0, \text{Var}(X_i) = p_0(1 - p_0) \\
\text{CLT, as } n \rightarrow \infty : \hat{P} &\sim \mathcal{N}(\mu = p_0, \sigma^2 = p_0(1 - p_0)/n) \\
\therefore \frac{\hat{P} - p_0}{\sqrt{p_0(1 - p_0)/n}} &= Z \sim \mathcal{N}(\mu = 0, \sigma^2 = 1)
\end{aligned}$$

Therefore neglecting the error term ϵ^1 , we have that under H_0 , for sufficiently large n .

$$T_n = \frac{1}{2} Z^2 \sim \frac{1}{2} \chi_1^2 \quad (5)$$

By the definition of the chi-squared distribution with one degree of freedom. To reject the null hypothesis H_0 at the $100(1 - \alpha)\%$ confidence level, we require that $P(T_n \geq t_n | H_0) < \alpha$. In other words, a low probability of observing this result under the null hypothesis.

¹See the Appendix for proof of negligibility

2.2 Two samples equality of means

We now complicate things by introducing a second coin, with throws denoted by $\{Y_i\}_{i=1}^m$ where we assume each throw is i.i.d Bernoulli with parameter q ($Y_i \sim \text{Bern}(q)$). Note that the population sizes n and m may be different. We define $L := \sum_{i=1}^m Y_i$ which is the analogue of K . We now set up our hypotheses to be:

$$H_0 : p = q$$

$$H_1 : p \neq q$$

Proceeding as before we can derive a formula for the test statistic. This time we denote the test statistic by t_N where $N := n + m$.

$$\begin{aligned} t_N &:= \log \frac{\mathcal{L}(H_1)}{\mathcal{L}(H_0)} = \log \frac{\max_{p,q} P(K = k|p)P(L = l|q)}{\max_p P(K = k|p)P(L = l|q = p)} \\ &= \log \frac{\binom{n}{k} \hat{p}^k (1 - \hat{p})^{n-k} \binom{m}{l} \hat{q}^l (1 - \hat{q})^{m-l}}{\binom{n}{k} \hat{r}^k (1 - \hat{r})^{n-k} \binom{m}{l} \hat{r}^l (1 - \hat{r})^{m-l}} \\ &= \log \frac{\hat{p}^k (1 - \hat{p})^{n-k}}{\hat{r}^k (1 - \hat{r})^{n-k}} + \log \frac{\hat{q}^l (1 - \hat{q})^{m-l}}{\hat{r}^l (1 - \hat{r})^{m-l}} \end{aligned}$$

Where, $\hat{p} := k/n, \hat{q} := l/m, \hat{r} := (k + l)/(n + m) = (n\hat{p} + m\hat{q})/(n + m)$. Using the same tricks as before, we can express this in terms of the chi-squared distance between the various parameters:

$$\begin{aligned} t_N &= n\mathcal{D}(\text{Bern}(\hat{p})||\text{Bern}(\hat{r})) + m\mathcal{D}(\text{Bern}(\hat{q})||\text{Bern}(\hat{r})) \\ &\approx \frac{n}{2}\chi^2(\text{Bern}(\hat{p})||\text{Bern}(\hat{r})) + \frac{m}{2}\chi^2(\text{Bern}(\hat{q})||\text{Bern}(\hat{r})) \\ &= \frac{1}{2\hat{r}(1 - \hat{r})} (n(\hat{p} - \hat{r})^2 + m(\hat{q} - \hat{r})^2) \\ &= \frac{1}{2\hat{r}(1 - \hat{r})} \left(n \left(\frac{m(\hat{p} - \hat{q})}{n + m} \right)^2 + m \left(\frac{n(\hat{q} - \hat{p})}{n + m} \right)^2 \right) \\ &= \frac{nm(\hat{p} - \hat{q})^2}{2\hat{r}(1 - \hat{r})(n + m)} \\ &= \frac{1}{2} \left(\frac{(\hat{p} - \hat{q})}{\sqrt{\hat{r}(1 - \hat{r})(1/n + 1/m)}} \right)^2 \end{aligned}$$

Under the null hypothesis H_0 , we require $p = q (= \mu)$; we introduce this third variable μ to refer to the true mean to avoid ambiguity. Applying the central limit theorem (for sufficiently large n and m) and combining Gaussians in the standard way, we have that:

$$\begin{aligned} \hat{P} &\sim \mathcal{N}\left(\mu, \frac{\mu(1 - \mu)}{n}\right) \\ \hat{Q} &\sim \mathcal{N}\left(\mu, \frac{\mu(1 - \mu)}{m}\right) \\ \hat{R} &\sim \mathcal{N}\left(\mu, \frac{\mu(1 - \mu)}{n + m}\right) \\ \therefore \sqrt{\frac{nm}{\mu(1 - \mu)(n + m)}}(\hat{P} - \hat{Q}) &= Z \sim \mathcal{N}(0, 1) \\ \delta\hat{R} := \hat{R} - \mu &\sim \mathcal{N}\left(0, \frac{\mu(1 - \mu)}{n + m}\right) \end{aligned}$$

We almost have T_n we just need to demonstrate that $\hat{R}(1 - \hat{R})$ is sufficiently close to $\mu(1 - \mu)$ for our purposes.

$$\begin{aligned} \hat{R}(1 - \hat{R}) &= (\mu + \delta\hat{R})(1 - (\mu + \delta\hat{R})) \\ &= \mu(1 - \mu) + O(\delta\hat{R}) \\ \therefore \frac{1}{\hat{R}(1 - \hat{R})} &= \frac{1}{\mu(1 - \mu)} \left(\frac{1}{1 + O(\delta\hat{R})} \right) \\ &= \frac{1}{\mu(1 - \mu)} (1 + O(\delta\hat{R})) \\ &\approx \frac{1}{\mu(1 - \mu)} \end{aligned}$$

We can neglect the terms of order $\delta\hat{R}$ and higher powers, as it is zero mean and for sufficiently large $n + m$ the variance approaches 0. Therefore, we have the desired expression for T_N .

$$T_N \approx \frac{1}{2} \left(\sqrt{\frac{nm}{\mu(1-\mu)(n+m)}} (\hat{P} - \hat{Q}) \right)^2 = \frac{1}{2} Z^2 \sim \frac{1}{2} \chi_1^2 \quad (6)$$

For this test we can also use the z-statistic instead of the t-statistic. Since the former is distributed like a Gaussian it may be easier to deal with.

$$z_N = \frac{\hat{p} - \hat{q}}{\sqrt{\hat{r}(1-\hat{r})(1/n + 1/m)}} \sim \mathcal{N}(0, 1) \quad (7)$$

3 ABP

ABP(r, T) on a graph G with vertex set $V = V(G)$ and edge set $E = E(G)$

1. Initialise messages for (v, v') :

$$y_{v' \rightarrow v}^{(0)} \leftarrow \mathcal{N}(0, 1)$$

2. Iterate for $1 \leq t \leq T$ and for $(v, v') \in E$:

$$\text{compute average } s^{(t-1)} \leftarrow \frac{1}{2|E|} \sum_{(v, v') \in E} y_{v' \rightarrow v}^{(t-1)}$$

$$\text{recentre messages } z_{v' \rightarrow v}^{(t-1)} \leftarrow y_{v' \rightarrow v}^{(t-1)} - s^{(t-1)}$$

$$\text{sum incoming } y_{v' \rightarrow v}^{(t)} \leftarrow \sum_{(v', v'') \in E \setminus \{v\}} z_{v'' \rightarrow v'}^{(t-1)}$$

if $(v''' \rightarrow v \rightarrow v')$ on cycle of length $r' \leq r$ then correct:

$$y_{v' \rightarrow v}^{(t)} \leftarrow y_{v' \rightarrow v}^{(t)} - \sum_{(v, v'') \in E \setminus \{v', v'''\}} z_{v'' \rightarrow v'}^{(t-r')}$$

3. Assignment, for all $v \in V$:

$$\text{Sum incoming } y_v^{(T)} = \sum_{(v, v') \in E} y_{v' \rightarrow v}^{(T)}$$

Assign labels $\sigma_v = 1$ if $y_v^{(T)} > 0$ and 0 otherwise

4 Appendix

4.1 Proving H.O.T can be neglected

We have shown that under the null hypothesis, that for sufficiently large n , $(\hat{P} - p_0) \sim \mathcal{N}(0, \beta/n)$ for some positive finite constant β (in this case $\beta = p_0(1 - p_0)$ but we just require finiteness for this proof). Therefore, $Q := \sqrt{n}(\hat{P} - p_0) \sim \mathcal{N}(0, \beta)$. Manipulating our expression for the error term ϵ we can show that it can be expressed as a sum of

$$\begin{aligned} \epsilon &= nO(\delta^3) \\ \therefore |\epsilon| &\leq n \sum_{i=0}^{\infty} \alpha_i |\hat{P} - p_0|^{3+i} \quad \text{for some finite constants } \alpha_i \geq 0 \\ |\epsilon| &\leq \sum_{i=0}^{\infty} \alpha_i n^{-\frac{1+i}{2}} (\sqrt{n}|\hat{P} - p_0|)^{3+i} \\ |\epsilon| &\leq \sum_{i=0}^{\infty} \alpha_i n^{-\frac{1+i}{2}} |Q|^{3+i} \end{aligned} \quad (8)$$

We know that Q is a Gaussian of zero mean and finite variance, therefore $|Q|$ will be a finite value. However, we see that it is scaled by a negative power of n , therefore for sufficiently large n we have that the error term asymptotes to 0. To be precise:

$$\lim_{n \rightarrow \infty} P(|\epsilon| < \eta) = 1 \text{ for arbitrarily small } \eta \geq 0 \quad (9)$$