Inferring community characteristics in labelled networks

IIB Project

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Overview

Introduction

Let's define some terms

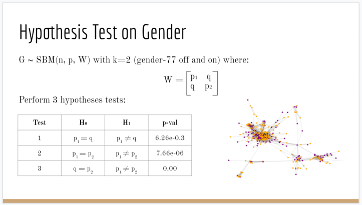
Inferring community characteristics in labelled networks

Network	set of vertices (aka nodes) connected by edges.
Graph	same as a network.
Labelled	information about each vertex. We call these *features*.
Community	Densely connected subset of nodes

Aim to identify what *features* impact graphical structure

How we got here

- Started by performing hypothesis tests on features



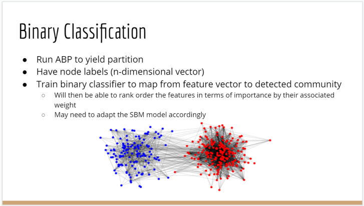


Figure: Slides from Michaelmas presentation

- This proved unhelpful as:
  - Almost all features statistically significant
  - Hard to rank order features against one another
  - Often poor model fit

Instead: find most “natural” partition and see if features can explain

Preliminaries

The stochastic block model (SBM)

We adopt the degree-corrected (DC) microcanonical (MC) formulation [7].

- N – number of vertices
- B – number of blocks

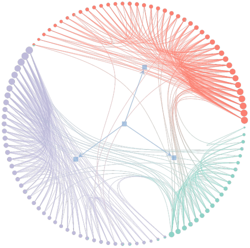


Figure: Typical SBM graph

DC-SBM$_{\mathrm{MC}}$ parameters:

- b – block membership vector
- e – block connectivity matrix
- k – degree sequence

$$A \sim \mathrm{DC\text{-}SBM_{MC}}(b, e, k)$$

With edges placed uniformly at random but respecting:

$$e_{rs} = \sum_{i,j \in [N]} A_{ij} \mathbb{1}\{b_i = r\} \mathbb{1}\{b_j = s\} \qquad \text{and} \qquad k_i = \sum_{j \in [N]} A_{ij}.$$

The feature-first block model

The feature-first block model (FFBM)

Define feature matrix $X \in \{0,1\}^{N \times D}$, this gives us that:

$$X \xrightarrow{\theta} \boxed{b} \xrightarrow{\psi} A$$

Figure: The feature-first block model (FFBM)

Likelihoods

$$p(b|X;\theta) = \prod_{i \in [N]} \phi_{b_i}(x_i;\theta)$$

$$p(A|b;\psi) \sim \mathrm{DC\text{-}SBM}_{\mathrm{MC}}(b, \psi_e, \psi_k)$$

Priors

$$p(\theta) = \mathcal{N}(\theta; 0, \sigma_\theta^2 I)$$

$$p(\psi|b) = p(\psi_e|b)p(\psi_k|\psi_e, b)$$

$$\phi_j(x;\theta) := \frac{\exp(w_j^T x_i)}{\sum_{k \in [B]} \exp(w_k^T x_i)}$$

Important property of the FFBM

Theorem

Our prior choice for $p(\theta)$ gives us that,

$$p(b|X) = B^{-N}.$$

Proof:

$$p(b|X) = \int p(b|X,\theta)p(\theta)d\theta = \int \prod_{i\in[N]} \phi_{b_i}(x_i;\theta)p(\theta)d\theta$$

$$= \prod_{i\in[N]} \int \frac{\exp(w_{b_i}^T x_i) \prod_{j\in[B]} \mathcal{N}(w_j; 0, \sigma_\theta^2 I)}{\sum_{k\in[B]} \exp(w_k^T x_i)} dw_{1:B}.$$

Which is a constant w.r.t. b.

Inference

Inference procedure

We want to draw:

$$\theta^{(t)} \sim p(\theta|A, X).$$

But computing $p(A|\theta, X)$ is $O(B^N) \Rightarrow$ split into:

$$b^{(t)} \sim p\Big(b|A, X\Big)$$
$$\theta^{(t)} \sim p\Big(\theta|X, b^{(t)}\Big)$$

Both steps implemented through Metropolis-Hastings, with:

$$\pi_z(z) - \text{target} \qquad q_z(z, z') - \text{proposal} \qquad \alpha_z(z, z') - \text{accept prob}$$

This is a pseudo-marginal approach. Andrieu and Roberts [1] show that:

$$\theta^{(t)} \sim \mathbb{E}_{b^{(t)}}\left[p(\theta|X, b^{(t)})\right] = p(\theta|A, X)$$

Sampling sequence

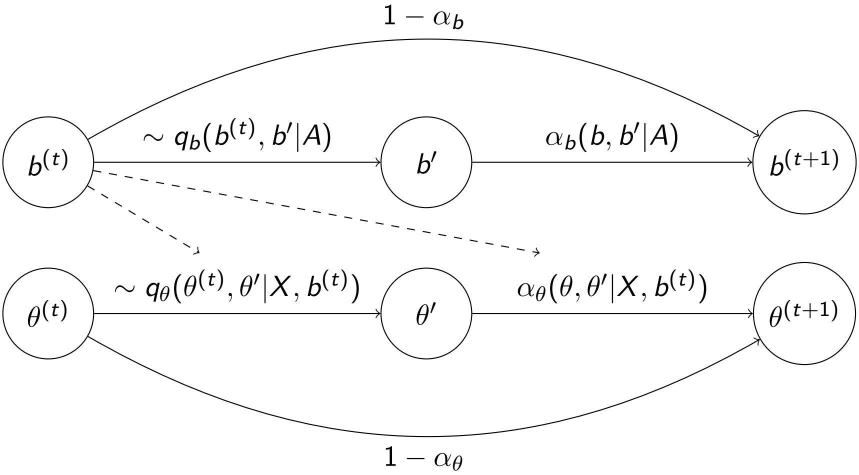
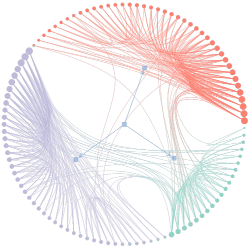


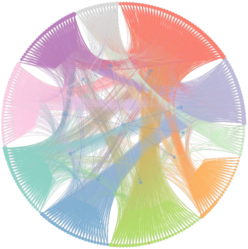
Figure: Sampling sequence.

Experiments

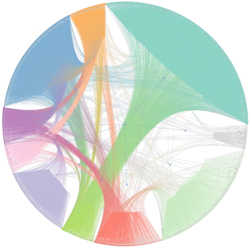
The datasets



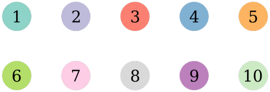
(a) Polbooks $D = 3$



(b) school $D = 13$



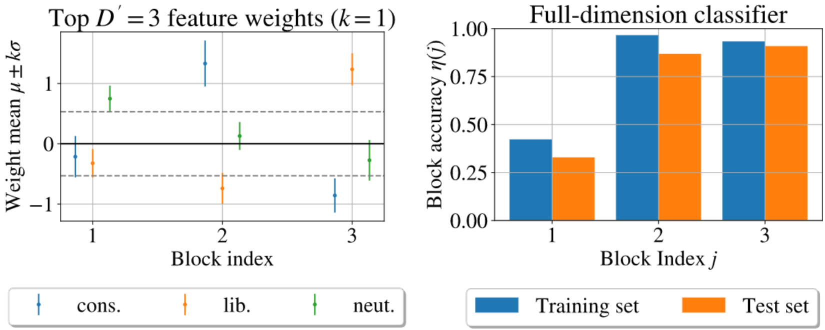
(c) FB egonet $d = 480$



(d) Legend

Figure: Networks laid out and coloured according to inferred block memberships $\hat{y}$ for a given experiment iteration. Visualisation performed using *graph-tool* [6].

Political books [5]



B	D	$\bar{\mathcal{L}}_0$	$\bar{\mathcal{L}}_1$
3	3	0.563 ± 0.042	0.595 ± 0.089

Conclusion

Conclusion

What have we achieved:

- Flip thinking of how we test for a feature's impact on graphical structure
- Developed efficient inference algorithm

What is yet to come:

- FFBM can only explain macro-structure $\Rightarrow$ extend to hierarchical structure

If I could start again

Thanks for listening

Figure: Source, Floyd [2]

References

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[7] Tiago P. Peixoto. Nonparametric bayesian inference of the microcanonical stochastic block model. *Physical Review E*, 95 (1), Jan 2017. ISSN 2470-0053. doi: 10.1103/physreve.95.012317. URL http://dx.doi.org/10.1103/PhysRevE.95.012317.

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Metropolis-Hastings [3]

We want to draw samples $\{z^{(t)}\}$ from some distribution,

$$\pi^*(z) \propto \pi(z).$$

Just need to be able to evaluate $\pi(z)$ point-wise and simulate from a proposal $q(z, z')$. If we accept each proposal with probability,

$$\alpha(z, z') = \min\left(\frac{\pi(z')q(z', z)}{\pi(z)q(z, z')}, 1\right),$$

then the resulting Markov chain is in detailed balance with $\pi(z)$.

b-step

Our target,

$$p(b|A, X) \propto p(b|X)p(A|b) = \pi_b(b),$$

can be evaluated as,

$$\begin{aligned}\pi_b(b) &= p(b|X) \sum_{\psi} p(A, \psi|b) \\ &= p(b|X)p(A, \psi^*|b) \\ &= p(A|\psi^*, b)p(\psi^*|b)p(b|X),\end{aligned}$$

where ψ^* is the only value compatible with (A, b). Big win for the microcanonical formulation.

We borrow $q_b(b, b')$ from Peixoto [7].

θ-step

Our target can be written as:

$$p(\theta|X,b) \propto p(b|X,\theta)p(\theta) = \pi_\theta(\theta) \propto \exp(-U(\theta)).$$

$\therefore$ can write -ve log target as:

$$U(\theta) = \sum_{i,j} y_{ij} \log \frac{1}{a_{ij}} + \frac{1}{2\sigma_\theta^2}||\theta||^2 = N\mathcal{L}(\theta) + \frac{1}{2\sigma_\theta^2}||\theta||^2,$$

where $y_{ij} := \mathbb{1}\{b_i = j\}$ and $a_{ij} = \phi_j(x_i;\theta)$. This form looks very familiar? We can use ∇U to bias our proposal:

$$\theta' = \theta^{(t)} - h_t \nabla U\left(\theta^{(t)}\right) + \sqrt{2h_t} \cdot \xi$$

This is now called the Metropolis-adjusted Langevin algorithm (MALA).

Serialise the chains

The $b^{(t)}$ does not use $\theta^{(t)}$.

The $\theta^{(t)}$ update uses $b^{(t)}$ through $y_{ij}^{(t)} := \mathbb{1}\{b_i^{(t)} = j\}$.

Why not use empirical mean of this quantity:

$$\hat{y}_{ij} := \frac{1}{|\mathcal{T}_b|} \sum_{t \in \mathcal{T}_b} y_{ij}^{(t)}.$$

This is an unbiased estimate of $p(b_i = j | A, X)$.

Using $\hat{y}$ instead of $y^{(t)}$ for the θ-step:

- Reduced variance in evaluation of U and ∇U
- Can run b and θ-chains sequentially rather than in parallel $\Rightarrow$ different lengths.

Dimensionality Reduction

Want to know which features we can discard:

- Write θ as matrix W, so that W_{ij} is weight for block i and feature j.
- Compute mean $\hat{\mu}_{ij}$ and std dev $\hat{\sigma}_{ij}$ of the $W_{ij}^{(t)}$-samples

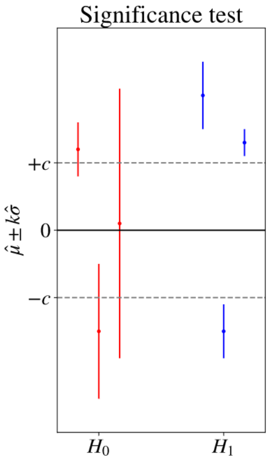
Imagine a test on W_{ij} such that:

$$H_0 : |W_{ij}| \leq c \qquad H_1 : |W_{ij}| > c$$

If we use Laplace approximation can come up with simple decision rule:

$$h_{ij} = H_1 \iff (\hat{\mu}_{ij} - k\hat{\sigma}_{ij}, \hat{\mu}_{ij} + k\hat{\sigma}_{ij}) \cap (-c, +c) = \emptyset$$

With $k > 0$ controlling degree of significance of result



Dimensionality Reduction cont...

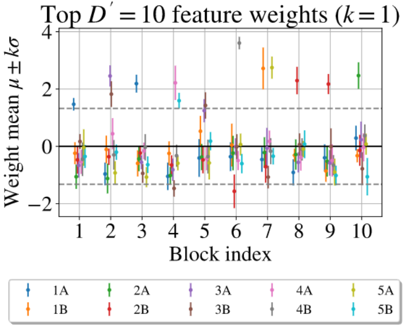
If we specify cut-off $c > 0$ and multiplier $k > 0$ can only retain features d such that:

$$\mathcal{D}' := \{d \in [D] : \exists i \in [B] \text{ s.t. } h_{id} = H_1\}$$

Instead it is often more practical to fix $|\mathcal{D}'| = D'$ and $k = k_0$, then find the maximal cut-off:

$$c^* = \arg\max_{c>0}\{c : |\mathcal{D}'| = D', k = k_0\}$$

Primary school dynamic contacts [8]

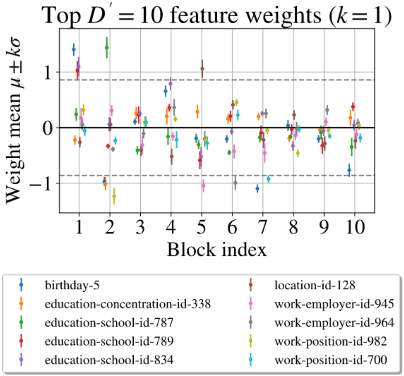


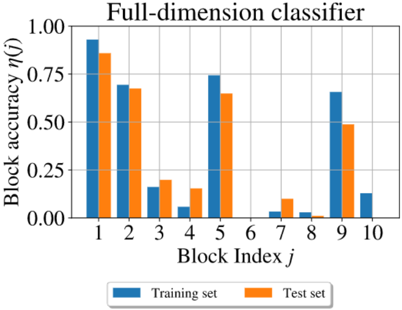
(a) θ-samples. Dotted line is $\pm c^*$.

B	D	D'	$\bar{\mathcal{L}}_0$	$\bar{\mathcal{L}}_1$	$\bar{\mathcal{L}}'_0$	$\bar{\mathcal{L}}'_1$
10	13	10	0.787 ± 0.127	0.885 ± 0.129	0.793 ± 0.132	0.853 ± 0.132

Table: Goodness of fit

Facebook egonet [4]





B	D	D'	$\bar{\mathcal{L}}_0$	$\bar{\mathcal{L}}_1$	$\bar{\mathcal{L}}'_0$	$\bar{\mathcal{L}}'_1$
10	480	10	1.326 ± 0.043	1.538 ± 0.069	1.580 ± 0.150	1.605 ± 0.106

Table: Goodness of fit