

Inferring community characteristics in labelled networks

IIB Project

Lawrence Tray

Ioannis Kontoyiannis

June 6, 2021

Overview

1 Introduction

2 Preliminaries

3 The feature-first block model

4 Inference

5 Experiments

6 Conclusion

Introduction

Let's define some terms

Inferring community characteristics in labelled networks

Network set of vertices (aka nodes) connected by edges.

Graph same as a network.

Labelled information about each vertex. We call these *features*.

Community Densely connected subset of nodes

Aim to identify what *features* impact graphical structure

How we got here

- Started by performing hypothesis tests on features

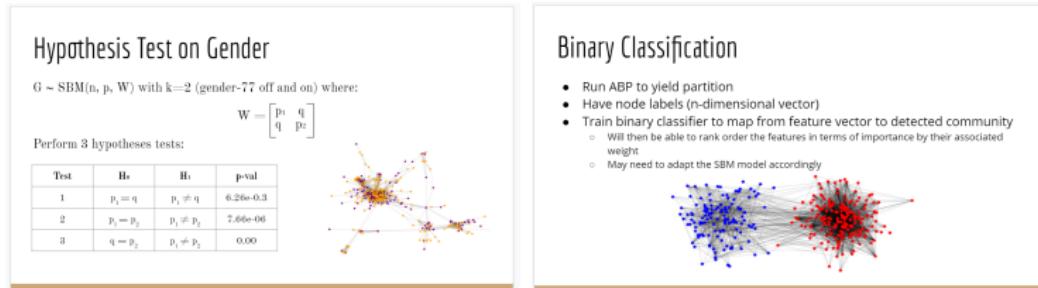


Figure: Slides from Michaelmas presentation

- This proved unhelpful as:
 - Almost all features statistically significant
 - Hard to rank order features against one another
 - Often poor model fit

Instead: find most “natural” partition and see if features can explain

Preliminaries

The stochastic block model (SBM)

We adopt the degree-corrected (DC) microcanonical (MC) formulation [5].

- N – number of vertices
- B – number of blocks

DC-SBM_{MC} parameters:

- b – block membership vector
- e – block connectivity matrix
- k – degree sequence

$$A \sim \text{DC-SBM}_{\text{MC}}(b, e, k)$$

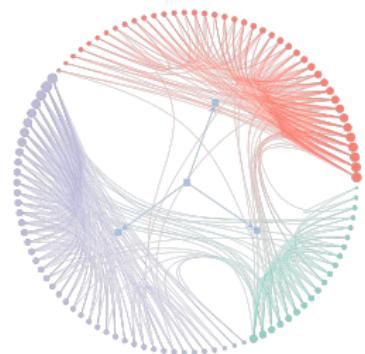


Figure: Typical SBM graph

With edges placed uniformly at random but respecting:

$$e_{rs} = \sum_{i,j \in [N]} A_{ij} \mathbb{1}\{b_i = r\} \mathbb{1}\{b_j = s\} \quad \text{and} \quad k_i = \sum_{j \in [N]} A_{ij}.$$

Metropolis-Hastings [3]

We want to draw samples $\{z^{(t)}\}$ from some distribution,

$$\pi^*(z) \propto \pi(z).$$

Just need to be able to evaluate $\pi(z)$ point-wise and simulate from a proposal $q(z, z')$. If we accept each proposal with probability,

$$\alpha(z, z') = \min \left(\frac{\pi(z') q(z', z)}{\pi(z) q(z, z')}, 1 \right),$$

then the resulting Markov chain is in detailed balance with $\pi(z)$.

The feature-first block model

The feature-first block model (FFBM)

Define feature matrix $X \in \{0, 1\}^{N \times D}$, this gives us that:

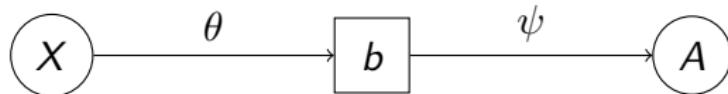


Figure: The feature-first block model (FFBM)

Likelihoods

$$p(b|X; \theta) = \prod_{i \in [N]} \phi_{b_i}(x_i; \theta)$$

$$p(A|b; \psi) \sim \text{DC-SBM}_{\text{MC}}(b, \psi_e, \psi_k)$$

Priors

$$p(\theta) = \mathcal{N}(\theta; 0, \sigma_\theta^2 I)$$

$$p(\psi|b) = p(\psi_e|b)p(\psi_k|\psi_e, b)$$

$$\phi_j(x; \theta) := \frac{\exp(w_j^T x_i)}{\sum_{k \in [B]} \exp(w_k^T x_i)}$$

Important property of the FFBM

Theorem

Our prior choice for $p(\theta)$ gives us that,

$$p(b|X) = B^{-N}.$$

Proof:

$$\begin{aligned} p(b|X) &= \int p(b|X, \theta)p(\theta)d\theta = \int \prod_{i \in [N]} \phi_{b_i}(x_i; \theta)p(\theta)d\theta \\ &= \prod_{i \in [N]} \int \frac{\exp(w_{b_i}^T x_i) \prod_{j \in [B]} \mathcal{N}(w_j; 0, \sigma_\theta^2 I)}{\sum_{k \in [B]} \exp(w_k^T x_i)} dw_{1:B}. \end{aligned}$$

Which is a constant w.r.t. b .

Inference

Inference procedure

We want to draw:

$$\theta^{(t)} \sim p(\theta|A, X).$$

But computing $p(A|\theta, X)$ is $O(B^N) \Rightarrow$ split into:

$$b^{(t)} \sim p(b|A, X)$$
$$\theta^{(t)} \sim p(\theta|X, b^{(t)})$$

Both steps implemented through Metropolis-Hastings, with:

$$\pi_z(z) - \text{target} \quad q_z(z, z') - \text{proposal} \quad \alpha_z(z, z') - \text{accept prob}$$

This is a pseudo-marginal approach. Andrieu and Roberts [1] show that:

$$\theta^{(t)} \sim \mathbb{E}_{b^{(t)}} \left[p(\theta|X, b^{(t)}) \right] = p(\theta|A, X)$$

Sampling sequence

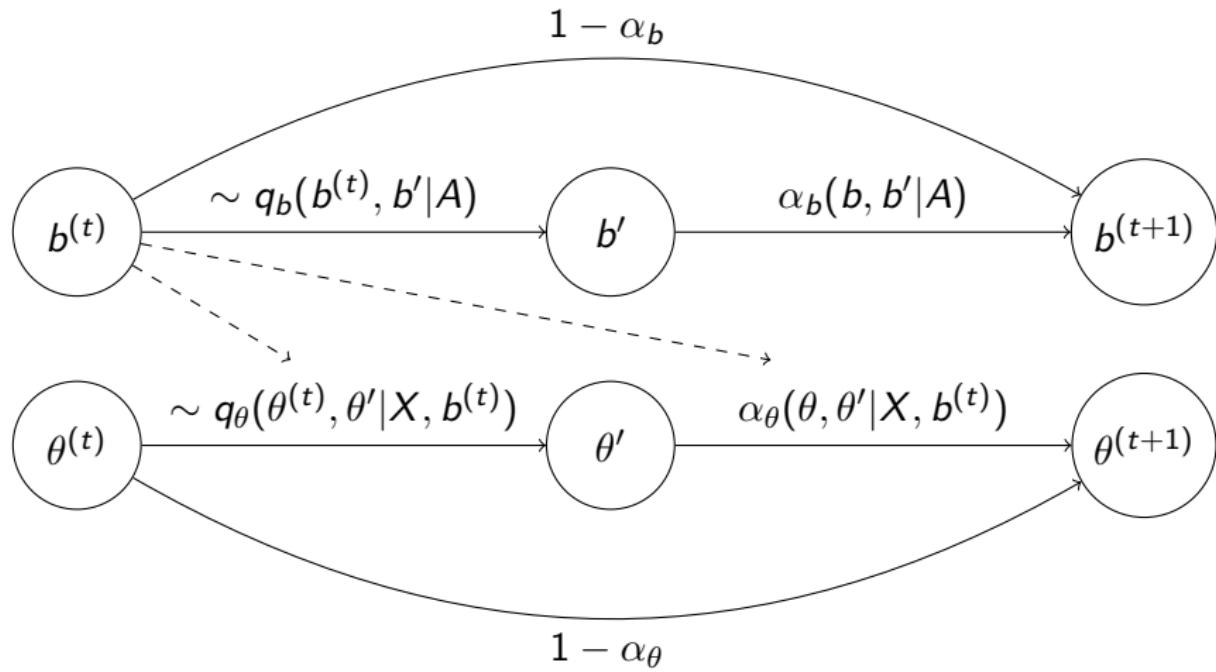


Figure: Sampling sequence.

b -step

Our target,

$$p(b|A, X) \propto p(b|X)p(A|b) = \pi_b(b),$$

can be evaluated as,

$$\begin{aligned}\pi_b(b) &= p(b|X) \sum_{\psi} p(A, \psi|b) \\ &= p(b|X)p(A, \psi^*|b) \\ &= p(A|\psi^*, b)p(\psi^*|b)p(b|X),\end{aligned}$$

where ψ^* is the only value compatible with (A, b) . Big win for the microcanonical formulation.

We borrow $q_b(b, b')$ from Peixoto [5].

θ -step

Our target can be written as:

$$p(\theta|X, b) \propto p(b|X, \theta)p(\theta) = \pi_\theta(\theta) \propto \exp(-U(\theta)).$$

\therefore can write -ve log target as:

$$U(\theta) = \sum_{i,j} y_{ij} \log \frac{1}{a_{ij}} + \frac{1}{2\sigma_\theta^2} \|\theta\|^2 = \mathcal{NL}(\theta) + \frac{1}{2\sigma_\theta^2} \|\theta\|^2,$$

where $y_{ij} := \mathbb{1}\{b_i = j\}$ and $a_{ij} = \phi_j(x_i; \theta)$. This form looks very familiar?
We can use ∇U to bias our proposal:

$$\theta' = \theta^{(t)} - h_t \nabla U \left(\theta^{(t)} \right) + \sqrt{2h_t} \cdot \xi$$

This is now called the Metropolis-adjusted Langevin algorithm (MALA).

Serialise the chains

The $b^{(t)}$ does not use $\theta^{(t)}$.

The $\theta^{(t)}$ update uses $b^{(t)}$ through $y_{ij}^{(t)} := \mathbb{1}\{b_i^{(t)} = j\}$.

Why not use empirical mean of this quantity:

$$\hat{y}_{ij} := \frac{1}{|\mathcal{T}_b|} \sum_{t \in \mathcal{T}_b} y_{ij}^{(t)}.$$

This is an unbiased estimate of $p(b_i = j | A, X)$.

Using \hat{y} instead of $y^{(t)}$ for the θ -step:

- Reduced variance in evaluation of U and ∇U
- Can run b and θ -chains sequentially rather than in parallel \Rightarrow different lengths.

Dimensionality Reduction

Want to know which features we can discard:

- Write θ as matrix W , so that W_{ij} is weight for block i and feature j .
- Compute mean $\hat{\mu}_{ij}$ and std dev $\hat{\sigma}_{ij}$ of the $W_{ij}^{(t)}$ -samples

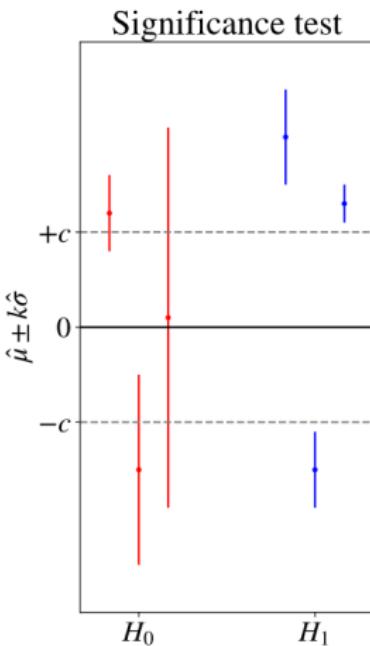
Imagine a test on W_{ij} such that:

$$H_0 : |W_{ij}| \leq c \quad H_1 : |W_{ij}| > c$$

If we use Laplace approximation can come up with simple decision rule:

$$h_{ij} = H_1 \iff (\hat{\mu}_{ij} - k\hat{\sigma}_{ij}, \hat{\mu}_{ij} + k\hat{\sigma}_{ij}) \cap (-c, +c) = \emptyset$$

With $k > 0$ controlling degree of significance of result



Dimensionality Reduction cont...

If we specify cut-off $c > 0$ and multiplier $k > 0$ can only retain features d such that:

$$\mathcal{D}' := \{d \in [D] : \exists i \in [B] \text{ s.t. } h_{id} = H_1\}$$

Instead it is often more practical to fix $|\mathcal{D}'| = D'$ and $k = k_0$, then find the maximal cut-off:

$$c^* = \arg \max_{c>0} \{c : |\mathcal{D}'| = D', k = k_0\}$$

Experiments

The datasets

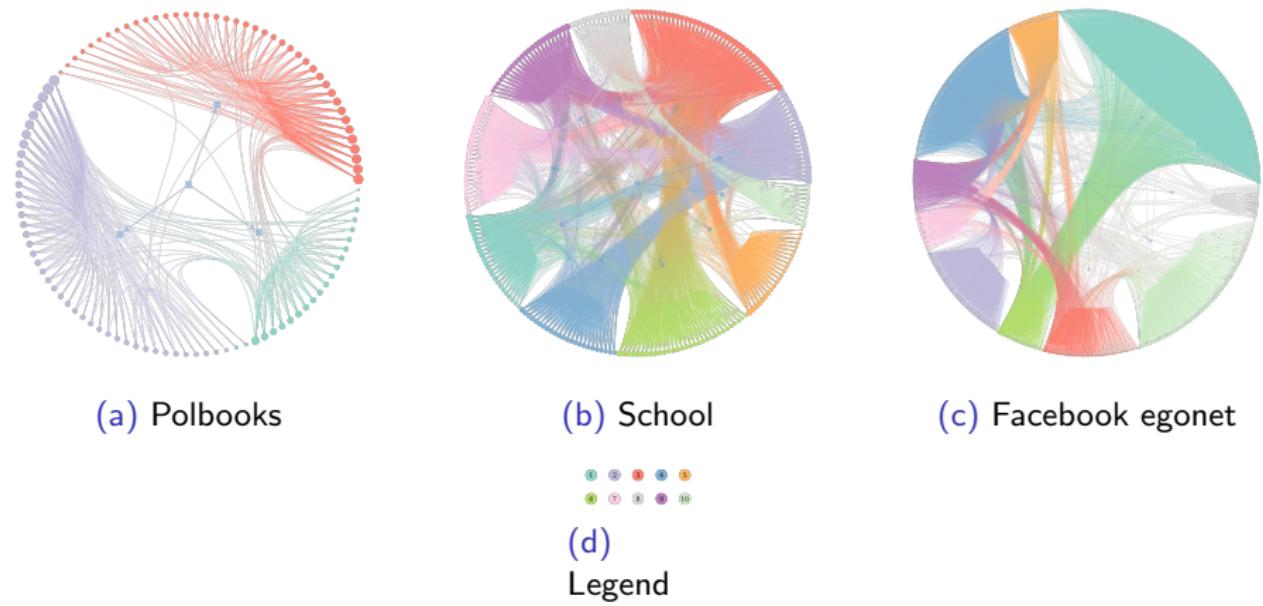
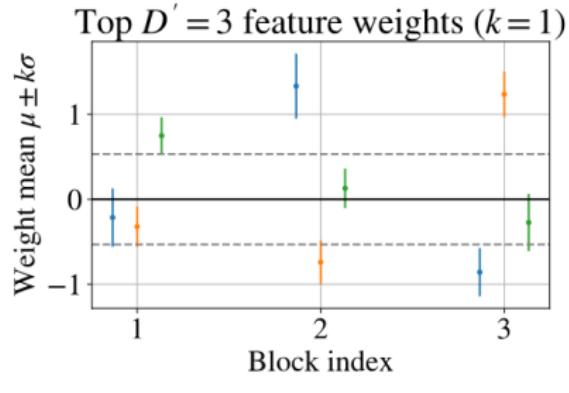
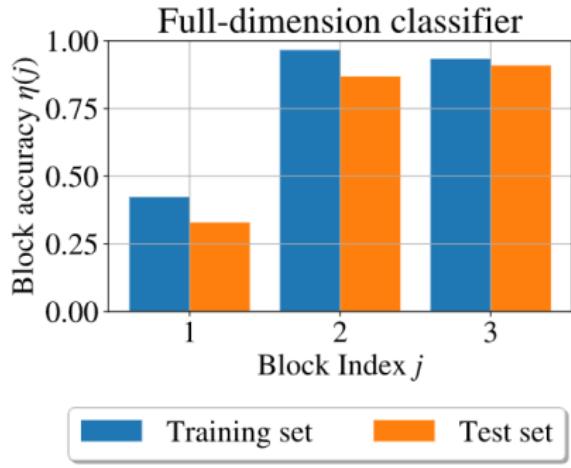


Figure: Networks laid out and coloured according to inferred block memberships \hat{y} for a given experiment iteration. Visualisation performed using *graph-tool* [4].

Political Books



(a) θ -samples. Dotted line is $\pm c^*$.



(b) Per-block accuracy $\eta(j)$.

Figure: Political books dataset.

Conclusion

Thanks for listening



Figure: Source, Floyd [2]

References

- [1] Christophe Andrieu and Gareth O. Roberts. The pseudo-marginal approach for efficient Monte Carlo computations. *The Annals of Statistics*, 37(2):697 – 725, 2009. doi: 10.1214/07-AOS574. URL <https://doi.org/10.1214/07-AOS574>.
- [2] The Australian Pink Floyd. Any questions. URL <https://www.facebook.com/australianpinkfloyd/posts/if-youve-got-any-questions-youd-like-to-ask-the-band-well-be-doing-a-qa-session-1643211382381105/>.
- [3] W. K. Hastings. Monte carlo sampling methods using markov chains and their applications. *Biometrika*, 57(1):97–109, 1970. ISSN 00063444. URL <http://www.jstor.org/stable/2334940>.
- [4] Tiago P. Peixoto. The graph-tool python library. *figshare*, 2014. doi: 10.6084/m9.figshare.1164194. URL http://figshare.com/articles/graph_tool/1164194.
- [5] Tiago P. Peixoto. Nonparametric bayesian inference of the microcanonical stochastic block model. *Physical Review E*, 95(1), Jan 2017. ISSN 2470-0053. doi: 10.1103/physreve.95.012317. URL <http://dx.doi.org/10.1103/PhysRevE.95.012317>.