Mathematical Background

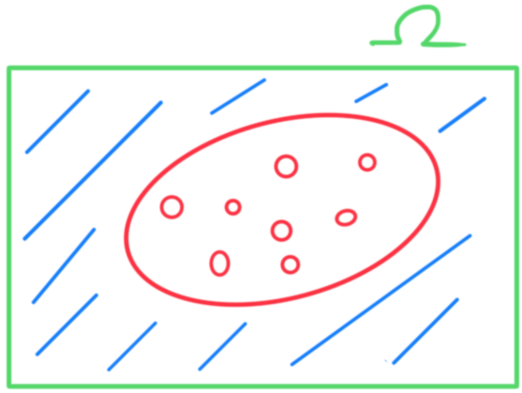
- Sets + DeMorgans laws
- Sequences and their limits
- Infinite series
  — The geometric series
- Sums with multiple indices
- Countable + uncountable sets.

Sets

- A collection of distinct elements.

$\{a, b, c, d\}$ finite

$\mathbb{R}$: real numbers infinite



$x \in S$ x is in a set

$x \notin S$ x is NOT in a set

$\{x \in \mathbb{R} : \cos(x) > 1/2\}$

x is in the set of real numbers and we restrict the set further by looking only at values where $\cos(x) > \frac{1}{2}$

Ω: Universal Set

$\emptyset$: Empty set

$(S^c)^c = S$

If $x \in \Omega$, $x \notin S$

$S \subset T: x \in S \Rightarrow x \in T$

If x is in a subset of T, which in this case is S. Then x also belongs to T and is either $\geq$ T. * Can also be written $\subseteq$.

$S \cup T: x \in S \cup T \iff$ $x \in S$ OR $x \in T$

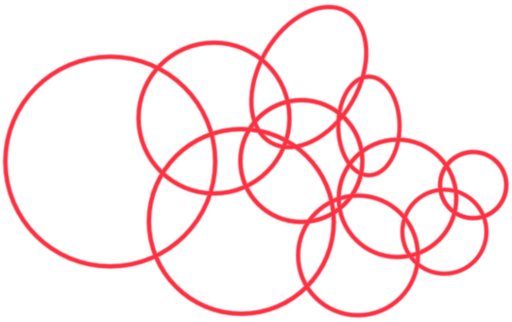
— Union

$S \cap T$: $x \in S \cap T \iff x \in S$ AND $x \in T$.

— Intersection

Infinite Sets

$S_n \quad n = 1, 2, 3, 4, 5 \ldots$



$x \in \bigcup_n S_n \iff x \in S_n$ For Some n

$x \in \bigcap_n S_n \iff x \in S_n$ For All n

Set properties

$S \cup T = T \cup S$

$S \cap (T \cup L) = (S \cap T) \cup (S \cap L)$

$(S^c)^c = S$

$S \cup \Omega = \Omega$

$S \cup (T \cup L) = (S \cup T) \cup L$

$S \cup (T \cap L) = (S \cup T) \cap (S \cup L)$

$S \cap S^c = \emptyset$ — There is no intersection

$S \cap \Omega = S$

$S \cup S^c = \Omega$

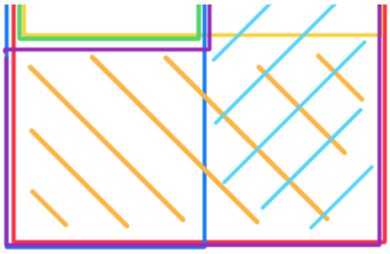
De Morgan's laws

$(S \cap T)^c = S^c \cup T^c$

$\cup$ = Union = OR

$\cap$ = Intersection = AND

$x \in (S \cap T)^c \iff x \notin S \cap T$



$$\iff \left\{ \begin{matrix} x \notin S \\ \text{OR} \\ x \notin T \end{matrix} \right\} \iff \left\{ \begin{matrix} x \in S^c \\ \text{OR} \\ x \in T^c \end{matrix} \right\}$$

$$\iff x \in S^c \cup T^c$$

2nd Law

The compliment of a union is the same as the intersection of the compliments.

Which is the same as the compliment of the compliments intersection is the same as union of the sets.

$$(S \cup T)^c = S^c \cap T^c =$$

$$= (S^c \cap T^c)^c = S \cup T = \left((S^c \cup T^c)^c\right)^c$$

* Basically flip from union to intersection and add or remove the compliment. • If there are two compliments they cancel out. otherwise.

$$\boxed{\begin{aligned} \left(\bigcap_n S_n\right)^c &= \bigcup_n S_n^c \\ \left(\bigcup_n S_n\right)^c &= \bigcap_n S_n^c \end{aligned}}$$

Sequences and their limits

$\{\ \}$ = A sequence.

$i \in \mathbb{N} = \{1,2,3...\}$ $\mathbb{R}^n$ ← Euclidean space in n dimensions, such

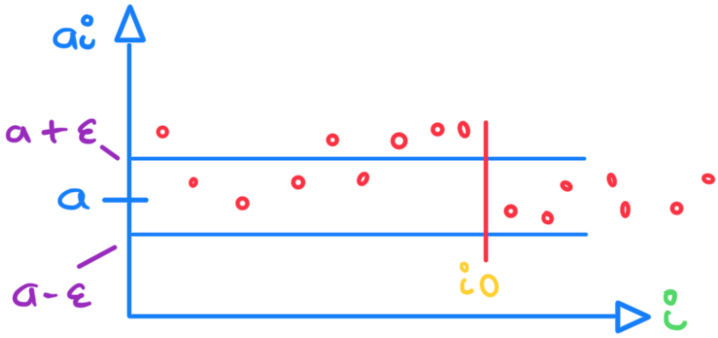
$a_i \in S \quad S = \mathbb{R}$ or $\mathbb{R}$ as vectors.

* Formally a sequence is a function that associates an element of S to any natural number. $f: \mathbb{N} \to S$.

$f_i = a_i$

$a_i \xrightarrow[i \to \infty]{} a$

$\lim_{i \to \infty} a_i = a$



Convergence of a sequence

For any $\varepsilon > 0$, there exists a time (i_0) where if $i \geq i_0$, then the element of the sequence is within epsilon of a.

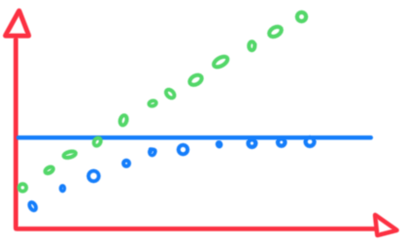
$$|a_i - a| < \varepsilon$$

$$\left.\begin{array}{l} a_i \to a \\ b_i \to b \end{array}\right\} \Rightarrow \begin{array}{l} a_i + b_i \to a+b \\ a_i b_i \to ab \end{array} \Rightarrow g(a_i) \to g(a)$$

only if a continuous function

- If $a_i \leq a_{i+1}$ for all i.
  - The sequence converges to ∞
  - The sequence converges to a real number

- If $|a_i - a| \leq b_i$, for all i AND $b_i \to 0$, then $a_i \to a$.



- If the bound (b_i) goes to 0, then a_i will converge to a.

Infinite series

$$\sum_{i=1}^{\infty} a_i = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$$ If a limit exists!

- A limit exists
  - If $a_i \geq 0$: (All terms are non-negative)

  * Monotonic sequence (always increasing)

- If a_i terms do not all have the same sign (+ or −)
  - Limit may not exist
  - Limit may exist, but change depending on sum order.

* Solution: Take the sum of the absolute values.

- Basically disregard the sign.

$$\sum_{i=1}^{\infty} |a_i| < \infty$$

Geometric Series

$$S = \sum_{i=0}^{\infty} a^i = 1 + a + a^2 + \dots \qquad |a| < 1$$

Infinite sum

$a < 1$ means the terms go to 0

$$(1-a)(1+a+\dots+a^n) = 1 - a^{n+1}$$

limit

The limit of the finite series is the infinite series

- We take the limit as n goes to infinity.

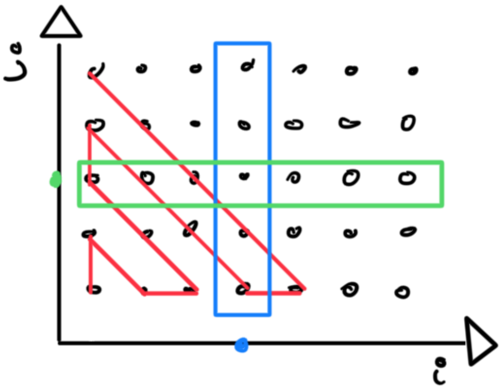
$n \to \infty$

$$(1-a)S = 1 \cancel{a^{n+1}}$$ (Converges to 0)

$$\boxed{S = \frac{1}{1-a}}$$ Formula for the infinite geometric series

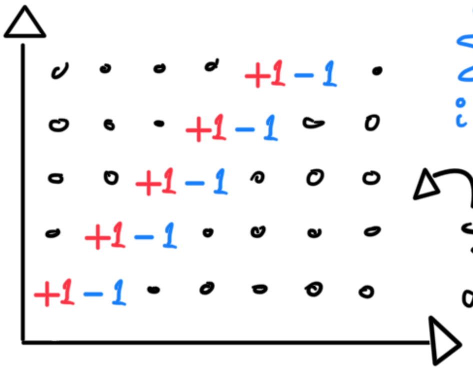
Order of Multiple indices

$$\sum_{i \geq 1\, j \geq 1} a_{ij}$$



$$\sum |a_{ij}| < \infty$$

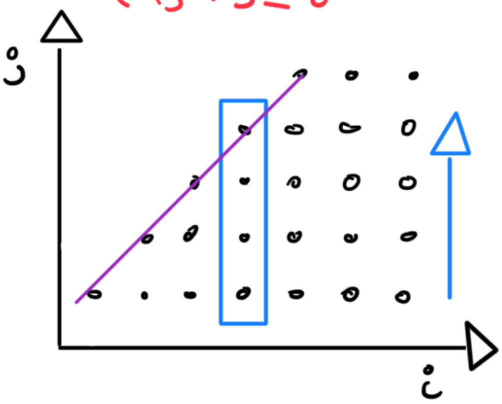
If values are absolute and less than infinity order does NOT matter.



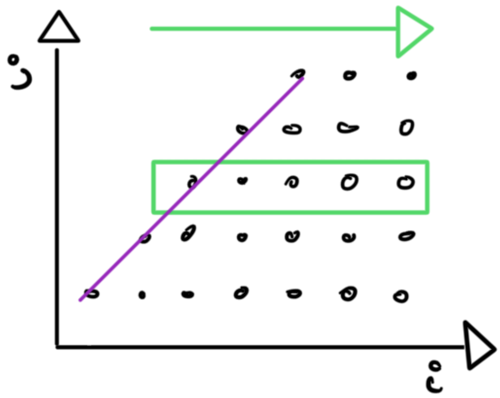
$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right)$$

In this example you cannot fix i or j as it will result in a different outcome. Furthermore $+1-1$ continue to infinity, so order does matter.

$$\sum_{(i,j):\, j \leq i} a_{ij}$$



$i = j$



$$\sum_{i=1}^{\infty}\sum_{j=1}^{i} a_{ij}$$

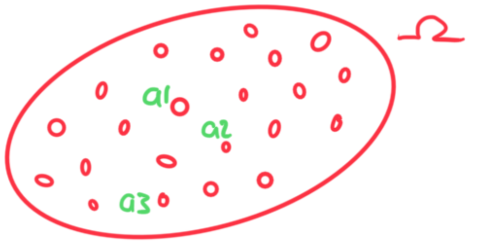
$$\sum_{j=1}^{\infty}\sum_{i=j}^{\infty} a_{ij}$$

* Sum over all j's then over all i's.

* Sum over all i's then over all j's.

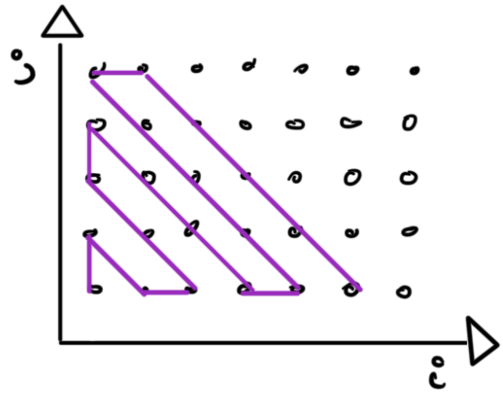
Countable vs uncountable infinite sets

- Countable: Can be put in 1-1 Correspondence with positive integers (in a sequence).



$\{a_1, a_2, a_3, \ldots\} = \Omega$

- Positive integers 1, 2, 3, . . .
- Integers 0, 1, -1, 2, -2, 3, -3 . . .
- Pairs of positive integers



- rational numbers q, with $0 < q < 1$

$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \cancel{\frac{2}{4}}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5} \ldots$

- uncountable:
- Continuous subsets of the real line.

E.g. the interval $[0, 1]$

* Basically anything continuous.

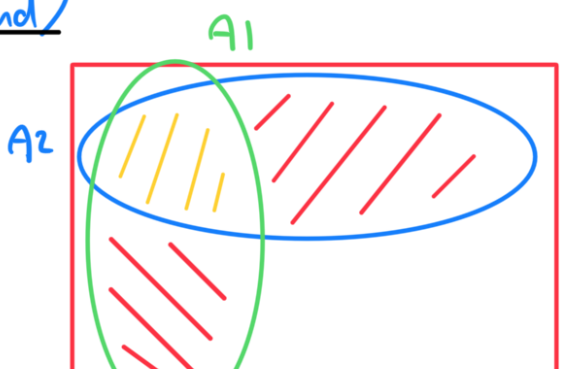
Cantor's diagonalisation argument

(uncountable series)

x_1: 2.7 4 3 2 1 5
x_2: 3.1 2 4 8 1 1
x_3: 1.8 9 2 4 7 3

The diagonal number does not appear in the sequence, and so all decimals cannot be counted.

Bonferroni inequality (union bound)



- very few students are smart
- very few students are beautiful

- Then very few are smart

OR beautiful.

$$P(A_1 \cup A_2) \le P(A_1) + P(A_2)$$

* If A_1 and A_2 are small, then their union will be small.

$$P(A_1 \cap A_2) \ge P(A_1) + P(A_2) - 1$$

$$P\left((A_1 \cap A_2)^c\right) = P(A_1^c + A_2^c) \le P(A_1^c) + P(A_2^c)$$

$$P\,\cancel{1} - P(A_1 \cap A_2) \le \cancel{1} - P(A_1) + P(A_2)$$

$$= P(A_1 \cap \dots \cap A_n) \ge P(A_1) + \dots + P(A_n) - (n-1)$$

* All this really shows is that there's a relationship between set size and set intersection or union, which can be proved using DeMorgan's laws, set theorical opperations and the union bound.