Introduction to Image and Video Processing

Lab 4: Frequency domain properties, filtering

Spring 2022

1 FT symmetry for real images

For real images $f(x,y) = f^*(x,y)$, show that

$$|F(u,v)| = |F(-u,-v)|.$$

Use the fact that $f = f^*$ and the definition of the DFT:

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-jux}e^{-jvy}$$

Solution:
One way to solve this is to look at the magnitude squared and use the fact that $|z|^2 = z \cdot z^*$ for complex numbers, so $|F(u,v)|^2 = F(u,v)F^*(u,v)$. We know that:

$$\begin{aligned}
F^*(u,v) &= \left(\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-jux}e^{-jvy}\right)^* \\
&= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left(f(x,y)e^{-jux}e^{-jvy}\right)^* \\
&= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f^*(x,y)e^{+jux}e^{+jvy} = F(-u,-v)
\end{aligned}$$

where we used the fact that $f(x,y) = f^*(x,y)$ because the image is real, and $(e^{j\phi})^* = e^{-j\phi}$. Then:

$$\begin{aligned}
|F(u,v)|^2 &= F(u,v)F^*(u,v) = F(u,v)F(-u,-v) \\
&= F(-u,-v)F^*(-u,-v)
\end{aligned}$$

where we substitute $F(u,v) = F^*(-u,-v)$. Thus:

$$|F(u,v)|^2 = F(-u,-v)F^*(-u,-v) = |F(-u,-v)|^2$$

so $|F(u,v)| = |F(-u,-v)|$

2 FT translation (shifting) property

Prove the shifting property of the FT:

$$f(x-x_0, y-y_0) \leftrightarrow F(u,v)e^{-j(ux_0+vy_0)}$$

Use the definition of the FT: $F(u,v)=\int_{-\infty}^{+\infty} f(x,y)e^{-jux}e^{-jvy}dxdy$

Solution:

$$\mathcal{F}[f(x-x_0,y-y_0)]=\int_{-\infty}^{+\infty} f(x-x_0,y-y_0)e^{-jux}e^{-jvy}dxdy \tag{1}$$

We set:

$$x'=x-x_0,\ y'=y-y_0$$

Then:

$$\begin{aligned}
\mathcal{F}[f(x-x_0,y-y_0)] &= \int_{-\infty}^{+\infty} f(x',y')e^{-ju(x'+x_0)}e^{-jv(y'+y_0)}dx'dy' \\
&= \int_{-\infty}^{+\infty} f(x',y')e^{-jux'}e^{-jvy'}e^{-jux_0}e^{-jvy_0}dx'dy' \\
&= e^{-jux_0}e^{-jvy_0}\int_{-\infty}^{+\infty} f(x',y')e^{-jux'}e^{-jvy'}dx'dy' \\
&= e^{-jux_0}e^{-jvy_0}F(u,v)
\end{aligned}$$

3 FT of shifted delta $\delta(x-x_0,y-y_0)$

Find the FT of the shifted delta $\delta(x-x_0,y-y_0)$.

Solution:

$$\begin{aligned}
\mathcal{F}[\delta(x-x_0,y-y_0)] &= \int_{-\infty}^{+\infty} \delta(x-x_0,y-y_0)e^{-jux}e^{-jvy}dxdy \\
&= e^{-jux_0}e^{-jvy_0}
\end{aligned} \tag{2}$$

We set:

$$x'=x-x_0,\ y'=y-y_0$$

Then:

$$\begin{aligned}
\mathcal{F}[f(x-x_0,y-y_0)] &= \int_{-\infty}^{+\infty} f(x',y')e^{-ju(x'+x_0)}e^{-jv(y'+y_0)}dx'dy' \\
&= \int_{-\infty}^{+\infty} f(x',y')e^{-jux'}e^{-jvy'}e^{-jux_0}e^{-jvy_0}dx'dy' \\
&= e^{-jux_0}e^{-jvy_0}\int_{-\infty}^{+\infty} f(x',y')e^{-jux'}e^{-jvy'}dx'dy' \\
&= e^{-jux_0}e^{-jvy_0}F(u,v)
\end{aligned}$$

4 FT of cosine $\cos(x_0, y_0)$

Find the FT of the 1D cosine $\cos(u_0 x)$, given that:

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

and that the FT of a complex exponential is a delta function:

$$\mathcal{F}[e^{ju_0x}] = 2\pi\delta(u - u_0)$$

.

Solution:

$$\begin{aligned}
\mathcal{F}[\cos(u_0x)] &= \int_{-\infty}^{+\infty} \cos(u_0x)e^{-jux}dx \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{ju_0x} + e^{-ju_0x}}{2} e^{-jux}dx \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{-j(u-u_0)x} + e^{-j(u+u_0)x}}{2} dx \\
&= \pi[\delta(u - u_0) + \delta(u + u_0)]
\end{aligned}$$

Then we have:

$$\mathcal{F}[\cos(u_0x)] = \pi[\delta(u - u_0) + \delta(u + u_0).]$$

What will this look like on a 1D plot? What would the equivalent 2D cosine's FT look like ?

5 Low pass filtering

- Create Matlab code for (1) Ideal, (2) Butterworth, (3) Gaussian Low Pass Filters, with cutoff frequency D_0 determined by the user.
- Vary the parameters of the BLPF, GLPF so the cut-off frequency effect is the same, i.e. if they have cut-off frequencies D_0^{BLPF}, D_0^{GLPF}:

$$H_{BLPF}(D_0^{BLPF}) = H_{GLPF}(D_0^{GLPF})$$

- Display the frequency domain filters in a 2D plot (e.g. imshow, imagesc) and a 3D plot (e.g. mesh, surf, surfc). Don't forget to use a logarithmic scale (why?).
- Read an image and apply these three filters to it.
- Plot a slice of the DFT magnitude (for a $M \times N$ image f with DFT F, plot slice $|F(M/2, :)|$)of the original and filtered images. What do you observe?
- Display the resulting images in the spatial domain. What do you observe?
- Repeat for higher and lower cut-off and for more blurry images and more sharp imags. What do you observe in each case?

6 High pass filtering

- Create Matlab code for (1) Ideal, (2) Butterworth, (3) Gaussian High Pass Filters, with cutoff frequency D_0 determined by the user.
- Vary the parameters of the BLPF, GLPF so the cut-off frequency effect is the same, i.e. if they have cut-off frequencies D_0^{BLPF}, D_0^{GLPF}:

$$H_{BLPF}(D_0^{BLPF}) = H_{GLPF}(D_0^{GLPF})$$

- Display the frequency domain filters in a 2D plot (e.g. imshow, imagesc) and a 3D plot (e.g. mesh, surf, surfc). Don't forget to use a logarithmic scale (why?).
- Read an image and apply these three filters to it.
- Plot a slice of the DFT magnitude (e.g. for a $M \times N$ image f with DFT F, plot slice $|F(M/2,:)|$)of the original and filtered images. What do you observe?
- Display the resulting images in the spatial domain. What do you observe?
- Repeat for higher and lower cut-off and for more blurry images and more sharp images. What do you observe in each case?

7 Pencil Sketch

The pencil sketch effect produces an output image that looks like a pencil sketch of an input image. To achieve this, the following procedure takes place, for an input image I:

- A blurred version I_{blur} of the input image I is produced.
- For all coordinates, create: $I_{out}(x, y) \leftarrow I./I_{blur}(x, y)$
- Return I_{out}

8 1D Gaussian Filter

- Create Matlab code for two 1D Gaussian filters

$$H = e^{-ju^2/D_0^2},\ H = e^{-jv^2/D_0^2}$$

- Apply this filter to the FT F of an image f, to get the fittered image g's FT $G = F \cdot H$
- Display the frequency domain filters in a 2D plot (e.g. imshow, imagesc) and a 3D plot (e.g. mesh, surf, surfc). Don't forget to use a logarithmic scale (why?).
- Read an image and apply these two filters to it.
- Plot a slice of the DFT magnitude (e.g. for a $M\ timesN$ image f with DFT F, plot slice $|F(M/2,:)|$)of the original and filtered images. What do you observe?
- Display the resulting images in the spatial domain. What do you observe?
- Repeat for higher and lower cut-off and for more blurry images and more sharp images. What do you observe in each case?

9 Frequency transform of spatial filters

- Calculate the DFT of the mask below.

$$w(x, y) = \frac{1}{5}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Use the definition of the DFT:

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-jux}e^{-jvy}$$

In order to find $W(u,v)$, use the DFT definition and replace x, y in the exponentials with the values of the corresponding coordinates' positions, given by:

$$pos(x,y) = \begin{bmatrix} (-1,-1) & (-1,0) & (-1,1) \\ (0,-1) & (0,0) & (0,1) \\ (1,-1) & (1,0) & (1,1) \end{bmatrix}$$

You will need to use the following identity:

$$cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2}$$

Solution: The summation in the DFT formula is from 0 to M-1 or N-1, but in practice it can also be from -(M-1)/2 to (M-1)/2-1, -(N-1)/2 to (N-1)/2 +1 - or any interval of length M or N. Here we will sum from -1 to 1, since the mask given is 3×3. Thus, we have:

$$F(u,v) = \frac{1}{9}(1 \cdot e^{ju} + 1 \cdot e^{jv} + 1 + 1 \cdot e^{-jv} + 1 \cdot e^{-ju}) \tag{3}$$
$$= \frac{1}{9}(2\cos(u) + 2\cos(v) + 1) \tag{4}$$

Explain how $w(x,y)$ affects an image, based on (a) its spatial representation, (b) its frequency domain representation.

Solution: We can tell it is a smoothing/low pass filter from its spatial representation, because it contains only positive numbers in the spatial mask, so it will essentially average out the local pixel intensity values. From its frequency representation we can tell it's low pass because cosines are equal to 1 near 0, and in this case are also added to 1 (see previous question). This can also be visualized with a simple matlab representation, shown below in a 2D and 3D plot.