Introduction to Image and Video Processing

Lab3: FT properties

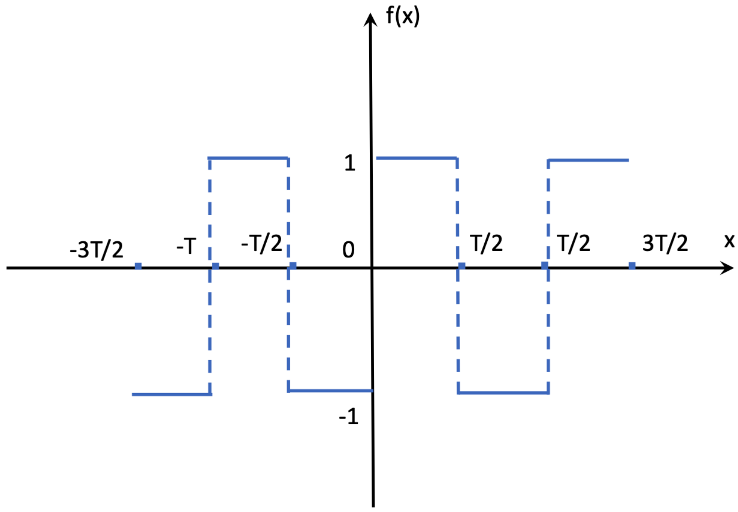
Spring 2022

How to use this document:
Solutions are given, and the corresponding Matlab code for most exercises is provided on the website. For some of the exercises, the solution is not given, but can be found in a similar way to the solved exercises.

1 Fourier Series of Square Wave

Consider a square wave with period T. Calculate its Fourier Series coefficients.

$$f(x) = \begin{cases} -1 \text{ for } -T/2 \leq x \leq 0, \\ 1 \ for\ 0 \leq x \leq T/2 \end{cases}$$



Solution:

1.1 Estimation of $a_0 = 0$

$$a_0 = \frac{2}{T}\int_{x=-T/2}^{T/2} f(x)dx = -\frac{2}{T}\int_{x=-T/2}^{0} dx + \frac{2}{T}\int_{x=0}^{T/2} dx$$
$$= -\frac{2}{T}\left(0 + \frac{T}{2}\right) + \frac{2}{T}\left(\frac{T}{2} - 0\right) = -1 + 1 = 0 \Rightarrow$$
$$\text{Then } a_0 = 0.$$

1.2 Estimation of a_n cosine coefficients

$$\begin{aligned}
a_n &= \frac{2}{T}\int_{x=-T/2}^{T/2} f(x)\cos\left(\frac{2\pi n}{T}x\right)dx \\
&= -\frac{2}{T}\int_{x=-T/2}^{0} \cos\left(\frac{2\pi n}{T}x\right)dx + \frac{2}{T}\int_{x=0}^{T/2} \cos\left(\frac{2\pi n}{T}x\right)dx \\
&= (1) + (2)
\end{aligned}$$

We set:

$$y = \frac{2\pi n}{T}x,$$

so

$$dy = \frac{2\pi n}{T}dx \Rightarrow dx = \frac{T}{2\pi n}dy$$

Then, for $x \in [-T/2, 0]$, we have:

$$y \in [-\pi n, 0].$$

For $x \in [0, T/2]$, we have:

$$y \in [0, \pi].$$

$$\begin{aligned}
(1) &= -\frac{1}{\pi n}\int_{-\pi n}^{0} \cos(y)dy \\
&= -\frac{1}{\pi n}[\sin(y)]_{-\pi n}^{0} \\
&= -\frac{1}{\pi n}\sin(\pi n) = 0
\end{aligned}$$

$$\begin{aligned}
(2) &= \frac{1}{\pi n}\int_{0}^{\pi n} \cos(y)dy \\
&= \frac{1}{\pi n}[\sin(y)]_{0}^{\pi n} \\
&= \frac{1}{\pi n}\sin(\pi n) = 0
\end{aligned}$$

Then $a_n = 0$.

1.3 Estimation of b_n sinusoid coefficients

$$\begin{aligned}
b_n &= \frac{2}{T}\int_{x=-T/2}^{T/2} f(x)\sin\left(\frac{2\pi n}{T}x\right)dx \\
&= -\frac{2}{T}\int_{x=-T/2}^{0} \sin\left(\frac{2\pi n}{T}x\right)dx + \frac{2}{T}\int_{x=0}^{T/2} \sin\left(\frac{2\pi n}{T}x\right)dx \Rightarrow \\
b_n &= (3) + (4)
\end{aligned}$$

We set:

$$y = \frac{2\pi n}{T}x$$

So:

$$dy = \frac{2\pi n}{T}dx \Rightarrow dx = \frac{T}{2\pi n}dy$$

For $x \in [-T/2, 0]$, we have:

$$y \in [-\pi n, 0]$$

For $x \in [0, T/2]$, we have:

$$y \in [0, \pi n]$$

$$\begin{aligned}
(3) &= -\frac{1}{\pi n}\int_{-\pi n}^{0} \sin(y)dy \\
&= -\frac{1}{\pi n}[-\cos(y)]_{-\pi n}^{0} \\
&= -\frac{1}{\pi n}(\cos(\pi n) - 1) = 1 - (-1)^n,
\end{aligned}$$

where we used the fact that $\cos(\pi n) = (-1)^n$.

$$\begin{aligned}
(4) &= \frac{1}{\pi n}\int_{0}^{\pi n} \sin(y)dy \\
&= \frac{1}{\pi n}[-\cos(y)]_{0}^{\pi n} \\
&= \frac{1}{\pi n}(1 - \cos(\pi n) = 1 - (-1)^n
\end{aligned}$$

Then:

$$b_n = \frac{2}{\pi n} \cdot (1 - (-1)^n) = \begin{cases} \frac{4}{\pi n}, & \text{for } n = 2\lambda + 1 \text{ (n odd)}, \\ 0 \text{ for } n = 2\lambda, & \text{(n even)} \end{cases}$$

As a result, the periodic square wave is only a sum of **sinusoidal components at odd frequencies**. This is written as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos(2\pi n x/T) + \sum_{n=1}^{+\infty} b_n \sin(2\pi n x/T) \Rightarrow$$

$$f(x) = \sum_{\lambda=0}^{\infty} \frac{4}{\pi(2\lambda+1)} \sin\left(\frac{2\pi(2\lambda+1)x}{T}\right)$$

2 Square Wave Fourier Series Approximation

1. Create a function for calculating the Fourier Series approximation of the square wave. Use the FS coefficients calculated in the previous exercise. Plot the square wave FS approximation by its FS for a different number of frequency components: e.g. plot it for one frequency component, for 3, 5 and 10.
2. Solution: see squarewave_FS.m, testFS.m

3 Distributivity

3.1 Proof:

Prove the distributivity property using the definition of the DFT:

$$\mathcal{F}[f_1(x,y) + f_2(x,y)] = F_1(u,v) + F_2(u,v)$$

Solution:

The DFT is given by:

$$F(u,v) = \frac{1}{M}\frac{1}{N}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi ux/M}e^{-j2\pi vy/N}$$

so we have:

$$\begin{aligned}
\mathcal{F}[f_1(x,y) + f_2(x,y)] &= \frac{1}{M}\frac{1}{N}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}[f_1(x,y) + f_2(x,y)]e^{-j2\pi ux/M}e^{-j2\pi vy/N} \\
&= \frac{1}{M}\frac{1}{N}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f_1(x,y)e^{-j2\pi ux/M}e^{-j2\pi vy/N} \\
&+ \frac{1}{M}\frac{1}{N}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f_2(x,y)e^{-j2\pi ux/M}e^{-j2\pi vy/N} \\
&= F_1(u,v) + F_2(u,v)
\end{aligned}$$

3.2 Programming:

Read two images with very different appearance (for visualization reasons), for example with very different edge orientations, shapes in them or blurriness/sharpness.

- Calculate their DFTs and add the DFTs. Compute the inverse DFT and display the result.
- Calculate their sum and its DFT. Compute the inverse DFT of their sum and display the result.
- What do you observe ? Why?
- Solution: see FT_distributivity.m

4 Scaling

4.1 Magnitude Scaling

Prove the magnitude scaling property *for the continuous FT*:

$$\alpha f(x,y) \Leftrightarrow \alpha F(u,v)$$

Solution - Magnitude scaling proof:

The FT is given by:

$$F(u,v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-j2\pi ux}e^{-j2\pi vy}dxdy$$

so for $\alpha f(x,y)$ we have:

$$\begin{aligned}
\mathcal{F}[\alpha f(u,v)] &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \alpha f(x,y)e^{-j2\pi ux}e^{-j2\pi vy}dxdy \\
&= \alpha \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-j2\pi ux}e^{-j2\pi vy}dxdy \\
&= \alpha F(u,v)
\end{aligned}$$

4.2 Programming:

- Calculate the FT of an image and its scaled version by a factor s. Compare the magnitudes of the resulting FT's. What do you observe?
- Solution: see FT_scaling.m

4.3 Coordinates Scaling

Prove the coordinates scaling property *for the continuous FT*:

$$f(\alpha x, \beta y) \Leftrightarrow \frac{1}{|\alpha\beta|}F(u/\alpha, v/\beta)$$

For the change of variables by scaling, you should use:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\alpha x,\beta y)dxdy = \frac{1}{|\alpha\beta|}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x',y')dx'dy'$$

Solution - Coordinates scaling proof:

The FT is given by:

$$F(u,v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-j2\pi ux}e^{-j2\pi vy}dxdy$$

so we have:

$$\mathcal{F}[f(\alpha x,\beta y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\alpha x,\beta y)e^{-j2\pi ux}e^{-j2\pi vy}dxdy$$

Here, we set $k_1 = \alpha x, k_2 = \beta y$, where $\alpha \neq 0,\ \beta \neq 0$, so:

$$x = k_1/\alpha, y = k_2/\beta$$

and

$$dk_1 = \alpha dx, dk_2 = \beta dy$$

we have:

$$\mathcal{F}[f(\alpha x, \beta y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(k_1, k_2) e^{-j2\pi u k_1/\alpha} e^{-j2\pi v k_2/\beta} dk_1 dk_2$$

$$\mathcal{F}[f(\alpha x, \beta y)] = \frac{1}{|\alpha\beta|} F\left(\frac{u}{\alpha}, \frac{v}{\beta}\right)$$

5 Shift of FT to the center of the coordinate system

5.1 Proof:

Prove the following, using the equality $e^{j\pi} = -1$:

$$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$$

Solution - Coordinates shift to center:

$$\begin{aligned}
\mathcal{F}[f(x,y)(-1)^{x+y}] &= \frac{1}{M}\frac{1}{N}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)(-1)^{x+y} e^{-j2\pi ux/M} e^{-j2\pi vy/N} \\
&= \frac{1}{M}\frac{1}{N}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)(e^{j\pi})^{x+y} e^{-j2\pi ux/M} e^{-j2\pi vy/N} \\
&= \frac{1}{M}\frac{1}{N}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) e^{-j2\pi ux/M + j\pi x} e^{-j2\pi vy/N + j\pi y} \\
&= \frac{1}{M}\frac{1}{N}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) e^{-j\pi x(2u/M-1)} e^{-j\pi y(2v/N-1)} \\
&= \frac{1}{M}\frac{1}{N}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) e^{-j2\pi x(u-M/2)/M} e^{-j2\pi y(v-N/2)/N} \\
&= F(u - M/2, v - N/2)
\end{aligned}$$

5.2 Programming:

- Write a program that multiplies each image pixel's intensity value $f(x, y)$ with $(-1)^{x+y}$. Display the resulting image.
- Calculate the FTs of the original and transformed image. What do you observe?
- In Matlab, compare the result with fftshift.
- Solution: see FT_shift_sq.m

6 Conjugate Symmetry

6.1 Proof:

Prove that the FT of real images $f(x, y)$ is conjugate symmetric:

$$F(u, v) = F^*(-u, -v)$$

Hint: Use the facts that (1) for real numbers z, we have $z^* = z$ and (2) the complex conjugate of a complex exponential $y = e^{j\phi}$ is $y^* = e^{-j\phi}$.

Solution:

$$\begin{aligned}
F^*(u,v) &= \left(\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-jux}e^{-jvy}dxdy\right)^* \\
&= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left(f(x,y)e^{-jux}e^{-jvy}\right)^* dxdy \\
&= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f^*(x,y)(e^{-jux}e^{-jvy})^* dxdy
\end{aligned}$$

Here $(e^{-jux}e^{-jvy})^* = (e^{-jux})^*(e^{-jvy})^* = e^{-jux}e^{-jvy}$, and $f(x, y) = f^*(x, y)$, so we have:

$$\begin{aligned}
F^*(u,v) &= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f^*(x,y)e^{+jux}e^{+jvy}dxdy \\
&= F(-u,-v)
\end{aligned}$$

Thus $(F(u, v))^* = F(-u, -v)$, i.e. $F(u, v) = F^*(-u, -v)$

6.2 Programming:

- Calculate the DFT of an image and find its real and imaginary components, R and I respectively.
- Display the magnitude of the DFT. What do you observe regarding its symmetry?
- Find the inverse DFT of the **real part** of the 2D DFT and display it in a 2D plot. What do you observe?
- Find the inverse DFT of the complex conjugate of the 2D DFT and display it in a 2D plot. What do you observe?
- Solution: see FT_symmetry.m

Hint: Use the following properties of the DFT and of complex conjugate numbers:

- *for a complex number $F = R + jI$ (like our DFT), we have:*

$$\text{complex conjugate } F^* = R - jI$$

- *the real part of a complex number F can be found from:*

$$Real(F) = \frac{F + F^*}{2}$$

- *the complex conjugate of an FT is:*

$$F^*(u,v) = F(-u,-v)$$

- *the complex conjugate FT is also the FT of the image flipped:*

$$F^*(u,v) = \mathcal{F}[f(-x,-y)]$$

7 Separability

- Show that the 2D DFT can be estimated as two successive 1D DFT's in the horizonal (u) and vertical (v) directions respectively. *Hint: See eqs. 4.6.14, 4.6.15, Fig. 4.35 in 2nd edition Gonzalez.* Use the definitions:

$$F(u,v) = \frac{1}{M}\sum_{x=0}^{M-1} e^{-j2\pi ux/M} \frac{1}{N}\sum_{y=0}^{N-1} e^{-j2\pi vy/N}$$

$$F(x,v) = \frac{1}{N}\sum_{y=0}^{N-1} e^{-j2\pi vy/N}, \; F(u,y) = \frac{1}{M}\sum_{x=0}^{M-1} e^{-j2\pi ux/M}$$

- Create a program that (1) reads an image, (2) calculates its 2D FFT with fft2 in Matlab, (3) calculates its 2D FFT by applying two 1D FFT's successively, based on the Separability property. *Note: In Matlab you will need to be find the transpose using the Matlab operator ".' ".*
- Compare the 2D FFT calculated with fft2 and with the 1D fft's visually (with a 2D or 3D plotting function). It's up to you to ensure their difference is clearly visualized.
- Calculate the mean squared difference between the 2D FFT calculated with fft2 and the 2D FFT calculated with successive 1D fft's. Comment on it.
- In Matlab the operator ".' " calculates the transpose, and "' " calculates the conjugate transpose [1]. Repeat the previous step using the conjugate transpose instead of the transpose and visualize the resulting inverse 2D FFTs. Explain what you observe mathematically, using the definitions of the 2D DFT and 1D DFT's in the first question.
- Solution: see FT_separability.m

[1] z = x+ jy, z* = x-jy