Supplementary Appendix C: Not for Publication

This appendix complements Section 3 of the paper. The first part investigates, using a simplified version of the model, (i) the conditions under which rest unemployment arises and (ii) the cyclical properties of workers' job separation and occupational mobility decisions. The second part presents the equations describing worker flows in a BRE. The third part shows that the sub-market structure we impose in paper endogenously arises from a competitive search model. The fourth part provides the definition of a BRE, the proof of Proposition 2 (in the main text), the proof of existence of the separation and reallocation cutoffs as well as of the results presented in Sections 1 and 3 of this appendix.

1. Model Implications and Comparative Statics

We start by exploring the main implications of our theory: the occurrence of rest unemployment and the cyclical properties of workers' decisions to search across occupations and to separate from jobs. To keep the intuition as clear as possible we study a simplified version of our model without occupational human capital accumulation, setting $x_h = 1$ for all h, and without occupational-wide productivity shocks, setting $p_o = 1$ for all o. These restrictions imply that all occupations are identical, worker mobility across occupations is fully undirected and purely driven by occupation-worker idiosyncratic shocks such that gross mobility equals excess mobility. Further, within an occupation workers differ only in their z-productivities and labor market segmentation is done along this dimension. Agents' value functions are still given by the Bellman equations described in Section 3.2 of the main text, but now with state space (z, A) instead of (z, x, o, A, p). The net value of searching across occupations then simplifies to

$$R(A) = -c + \int_{\underline{z}}^{\overline{z}} W^U(A, \tilde{z}) dF(\tilde{z}). \tag{1}$$

In Carrillo-Tudela and Visschers (2013) we show that in this setting and assuming $F(z'|z) < F(z'|\tilde{z})$ for all z, z' if $z > \tilde{z}$, the value functions $W^U(A, z)$, $W^E(A, z)$, $J(A, z)$ and $M(A, z)$, exist, are unique and increase in z. This implies that $\theta(A, z)$ also exists, is unique and increases with z. Further we show that if $\delta + \lambda(\theta(A, z)) < 1$ for all A, z, in equilibrium there exists a unique cutoff function z^s that depends only on A, such that $d(A, z) = \sigma(A, z) = 1$ if and only if $z < z^s(A)$, and $d(A, z) = \sigma(A, z) = 0$ otherwise. Since $R(A)$ is constant in z, $W^U(A, z)$ is increasing in z, and by the existence of a unique z^s so is $\max\{\lambda(\theta(A, z^r(A)))(W^E(A, z^r(A)) - W^U(A, z^r(A))), 0\}$, there also exists a reallocation cutoff function $z^r(A)$ such that workers decide to search across occupations if and only if $z < z^r(A)$ for every A, where $z^r(A)$ satisfies

$$R(A) = W^U(A, z^r(A)) + \max\{\lambda(\theta(A, z^r(A)))(W^E(A, z^r(A)) - W^U(A, z^r(A))), 0\}. \tag{2}$$

Using this simplified framework we first study the relative positions of the job separation and reallocation cut-off functions and hence gain insights on the conditions under which rest unemployment arises. We then study the slopes of these cut-off functions and gain insights into the cyclicality of separations and (excess) occupational mobility in our model. Using our calibrated model we have verified that the same properties as derived below apply to the more general setup considered there.

1.1 The Occurrence of Rest Unemployment

We first analyse how the value of waiting in unemployment and in employment changes with c, b and the persistence of the z-productivity process, and how these changes determine the relative position of z^r and z^s.

The simplest setting that captures a motive for waiting is one in which the z-productivity is redrawn randomly with probability $0 < (1-\gamma) < 1$ each period from cdf $F(z)$ and A is held constant. Time-variation in z is essential here because a worker can decide to stay unemployed in his occupation, even though there are no jobs currently available for him, when there is a high enough probability that his z-productivity will become sufficiently high in the future. All other features of the model remain the same, with the exception that we do not consider human capital accumulation or occupational-wide productivity differences. In Section 3.2 of this appendix we formulate the value functions for this stationary environment and provide the proofs of Lemmas 1 and 2, below.

In this stationary setting, the expected value of an unemployed worker with productivity z, measured at the production stage, is given by

$$\begin{aligned} W^U(A,z) &= \gamma\Big(b+\beta\max\Big\{R(A), W^U(A,z)+\max\{\lambda(\theta(A,z))(1-\eta)(M(A,z)-W^U(A,z)),0\}\Big\}\Big) \\ &+ (1-\gamma)\mathbb{E}_z[W^U(A,z)]. \end{aligned} \tag{3}$$

Equation (3) shows that there are two ways in which an unemployed worker with a $z < z^s$ can return to production. *Passively*, he can wait until his z-productivity increases exogenously. Or, *actively*, by paying c and sampling a new z in a different occupation. In the case in which the worker prefers to wait, the inner $\max\{.\} = 0$ as the workers is below the separation cutoff and the outer $\max\{.\} = W^U(A,z) = W^U(A,z^s(A))$, where the latter equality follows as in this simplified environment the z-productivity process is assumed to be iid. In the case in which the worker prefers to search across occupations the outer $\max\{.\} = R(A) = W^U(A,z^r(A))$. The difference $W^U(A,z^s(A)) - R(A)$, then captures the relative gain of waiting for one period over actively sampling a new z immediately. If $W^U(A,z^s(A)) - R(A) \geq 0$, then $z^s \geq z^r$ and there is rest unemployment. If $W^U(A,z^s(A)) - R(A) < 0$, then $z^r > z^s$, and endogenously separated workers immediately search across occupations.

Changing c, b, or γ will affect the relative gains of waiting, in employment and in unemployment. In the following lemma we derive the direction of the change in $W^U(A,z^s(A)) - R(A)$, where we take fully into account the feedback effect of changes in c, b, or γ on the match surplus $M(A,z) - W^U(A,z)$ that arises due to the presence of search frictions as discussed in Section 3.3 in the main text.

Lemma 1. *Changes in c, b or γ imply*

$$\frac{d(W^U(A,z^s(A))-R(A))}{dc} > 0, \frac{d(W^U(A,z^s(A))-R(A))}{db} > 0, \frac{d(W^U(A,z^s(A))-R(A))}{d\gamma} < 0.$$

It is intuitive that raising c directly increases the relative gains of waiting as it make occupational mobility more costly. An increase in c, however, also leads to a larger match surplus because it reduces $W^U(A,z)$, making employed workers less likely to separate and hence reducing rest unemployment. The lemma shows that, overall, the first effect dominates. A rise in b lowers the effective cost of waiting, while at the same time decreasing the match surplus by increasing $W^U(A,z)$, pushing towards more rest unemployment. An increase in γ, decreases the gains of waiting because it decreases the probability of experiencing a z-shock without paying c and hence increases the value of sampling a good z-productivity. In the proof of Lemma 1 we further show that an increase in $W^U(A,z^s(A)) - R(A)$ leads to an increase in $z^s(A) - z^r(A)$. This implies that for a sufficiently large c, b or $1-\gamma$ rest unemployment arises.

Rest Unemployment and Occupational Human Capital Occupational human capital accumulation makes a worker more productive in his current occupation. This implies that workers are willing to stay longer unemployed in their occupations because, for a given z, they can find jobs faster and receive higher wages. At the same time, a higher x makes the employed worker less likely to quit into unemployment, generating a force against rest unemployment. Taken together, however, the first effect dominates as the next result shows.

Lemma 2. *Consider a setting where A is fixed, z redrawn with probability $(1-\gamma)$, and production is given by $y = Axz$. Consider an unexpected, one-time, permanent increase in occupation-specific human capital, x, from $x = 1$. Then*

$$\frac{d(W^U(A, z^s(A, x), x) - R(A))}{dx} > 0.$$

This result implies that the difference $z^s(A, x) - z^r(A, x)$ becomes larger when human capital increases. Thus, more occupational human capital leads to rest unemployment.

1.2 The Cyclicality of Occupational Mobility and Job Separation Decisions

In Section 2.5 of the main text we documented that occupational mobility through unemployment (non-employment) is procyclical, while it is well established that job separations into unemployment (non-employment) are countercyclical (see Section 5 in the main text). In the model, the cyclicalities of workers' occupational mobility and job separation decisions are characterised by the slopes of the cutoff functions z^r and z^s with respect to A. As discussed in Section 3.3 of the main text, occupational mobility decisions are procyclical and job separation decisions are countercyclical when $dz^r/dA > 0$ and $dz^s/dA < 0$, respectively. We now explore the conditions under which such slopes arise endogenously in our model.

Occupational Mobility Decisions We start by investigating the impact of rest unemployment on the slope of z^r using the simplified version of the model (i.e. $x_h = 1$ for all h and $p_o = 1$ for all o). Using (2) and noting that $R(A) - W^U(A, z^r(A)) = 0$, we obtain

$$\frac{dz^r}{dA} = \frac{\int_{z^r}^{\overline{z}} \left(\frac{\partial W^U(A,z)}{\partial A} - \frac{\partial W^U(A,z^r)}{\partial A}\right) dF(z) - \frac{\partial}{\partial A}\Big(\lambda(\theta(A, z^r))(W^E(A, z^r) - W^U(A, z^r))\Big)}{\frac{\partial W^U(A,z^r)}{\partial A} + \frac{\partial}{\partial z^r}\Big(\lambda(\theta(A, z^r))(W^E(A, z^r) - W^U(A, z^r))\Big)}. \tag{4}$$

Since workers who decided to search across occupations must sit out one period unemployed, the term

$$\lambda(\theta(A, z^r))(W^E(A, z^r) - W^U(A, z^r))$$

captures the expected loss associated with the time cost of this decision: by deciding not to search across occupations, the worker could match with vacancies this period. When $z^r > z^s$, $\lambda(\theta(A, z^r))(W^E(A, z^r) - W^U(A, z^r)) > 0$ and is increasing in A and z^r. Therefore, an increase in A in this case, increases the loss associated with the time cost of searching across occupations and decreases dz^r/dA. However, when rest unemployment occurs ($z^s > z^r$), $\lambda(\theta(A, z^r))(W^E(A, z^r) - W^U(A, z^r)) = 0$ and this effect disappears. This follows as during rest unemployment periods workers have a contemporaneous job finding rate of zero and hence by searching across occupations, the worker does not lose out on the possibility of matching this period. In this case, the cyclicality of occupational mobility decisions purely depends on the remaining terms, in particular $\int_{z^r}^{\overline{z}} \left(\frac{\partial W^U(A,z)}{\partial A} - \frac{\partial W^U(A,z^r)}{\partial A}\right) dF(z)$. Thus, the presence of rest unemployment adds a procyclical force to occupational mobility decisions.

Now consider the impact of search frictions on the slope of z^r. We focus on the more general case of $z^r > z^s$, which includes the additional countercyclical force discussed above. To gain analytical tractability we consider the stationary environment used in the previous subsection and set $\gamma = 1$ such that both A and workers' z-productivities are permanent. This allows us to link wages and labor tightness to $y(A, z)$ in closed form. We then analyse the effects of a one-time, unexpected, and permanent change in A on z^r.[1] To isolate the role of search frictions, we compare this case with one without search frictions in which workers (who are currently not changing occupations) can match instantaneously with firms and are paid $y(A, z)$. In both cases we keep in place the same reallocation frictions. Let z^r_c denote the reallocation cutoff in the case without search frictions. The details of both cases, including the corresponding value functions and the proof of the following lemma can be found in Section 4 of this appendix.

Lemma 3. *Consider a one-time, unexpected, permanent increase in A. With search frictions the cyclical response of the decision to search across occupations is given by*

$$\frac{dz^r}{dA} = \frac{\beta \int_{z^r}^{\bar{z}} \left(\left(\frac{C_s(A,z)}{C_s(A,z^r)}\right) y_A(A,z) - y_A(A,z^r) \right) dF(z) - (1-\beta) y_A(A,z^r)}{(1-\beta F(z^r)) y_z(A,z^r)}, \tag{5}$$

where $y_i(A,z) = \partial y(A,z)/\partial i$ for $i = A, z$ and $C_s(A,z) = \frac{\beta\lambda(\theta(A,z))}{(1-\beta)(1-\beta+\beta\lambda(\theta(A,z)))}$. Without search frictions the cyclical response is given by

$$\frac{dz^r_c}{dA} = \frac{\beta \int_{z^r_c}^{\bar{z}} (y_A(A,z) - y_A(A,z^r_c)) dF(z) - (1-\beta) y_A(A,z^r_c)}{(1-\beta F(z^r_c)) y_z(A,z^r_c)}. \tag{6}$$

The first term in the numerator of (5) and (6) relates to $\int_{z^r}^{\bar{z}} \left(\frac{\partial W^U(A,z)}{\partial A} - \frac{\partial W^U(A,z^r)}{\partial A} \right) dF(z)$ in equation (4), while the second term captures the opportunity cost of the reallocation time. The proof of Lemma 3 shows that $C_s(A,z)/C_s(A,z^r)$ is increasing in z and $\frac{dz^r}{dA} > \frac{dz^r_c}{dA}$ at $z^r = z^r_c$. Thus, the presence of search frictions also adds procyclicality to the decision to search across occupations. Search frictions lead to a steeper reallocation cutoff function because, in this case, an increase in A increases $W^U(A,z)$ through *both* the wage and the job finding probability. In contrast, an increase in A in the frictionless case only affects $W^U(A,z)$ through wages (with a proportionally smaller effect on $w - b$), as workers always find jobs with probability one. The fact that $C_s(A,z)/C_s(A,z^r)$ is increasing in z for $z > z^r$, reflects that the impact of $y(A,z)$ on $W^U(A,z)$ is increasing in z. Indeed, from the proof of Lemma 3 one obtains $\frac{\partial W^U(A,z)}{\partial A} \Big/ \frac{\partial W^U(A,z^r)}{\partial A} = \frac{C_s(A,z) y_A(A,z)}{C_s(A,z^r) y_A(A,z^r)} > \frac{y_A(A,z)}{y_A(A,z^r)}$ for $z > z^r$.

One can get further intuition by considering the planners' problem (for details see Carrillo-Tudela and Visschers, 2013). The envelope condition implies that the planner, at the optimum allocation, does not need to change labor market tightness at each z for a infinitesimal change in A to still obtain the maximum net increase in expected output. At a given z, this means that an increase of dA creates

$$dW^U(A,z) = \frac{\beta\lambda(\theta(A,z))}{(1-\beta)(1-\beta+\beta\lambda(\theta(A,z)))} y_A(A,z) dA = C_s(A,z) y_A(A,z) dA$$

in additional life-time expected discounted output for the planner. Since our economy is constrained efficient, the change in A also creates the same lifetime expected discounted income for an unemployed worker with such z.

In addition, (5) and (6) show that the cyclicality of z^r in either case depends on the production function $y(A,z)$, in particular on the sign of $y_A(A,z) - y_A(A,z^r)$ for $z > z^r$. In the proof of Lemma 3 we show that with search frictions, the decisions to search across occupations will already be procyclical when the production function is modular and z^r is sufficiently close to z^s. This follows because the opportunity cost of searching across occupations becomes smaller as z^r approaches z^s from above. With rest unemployment this opportunity

[1] This approach follows Shimer (2005), Mortensen and Nagypal (2007), and Hagedorn and Manovskii (2008). Since the equilibrium value and policy functions only depend on A and z, analysing the change in the expected value of unemployment and joint value of the match after a one-time productivity shock is equivalent to compare those values at the steady states associated with each productivity level. This is because in our model the value and policy functions jump immediately to their steady state level, while the distribution of unemployed and employed over occupations takes time to adjust.

cost is zero. If z^r is substantially above z^s, we will need sufficient complementarities between A and z in the production function to obtain a procyclical z^r. Without search frictions, in contrast, a supermodular production function is only a necessary condition to generate procyclicality in z^r .

Job Separations The main aspect of having endogenous job separation and occupational mobility decisions is that the two can potentially interact. In particular, if $z^r > z^s$, workers separate endogenously to search across occupations and this could lead to z^r and z^s having the same cyclical behavior. For example, in the setting of Lemma 3 were both A and workers' z-productivities are permanent, we show at the end of Section 3.2 of this appendix that dz^s/dA depends directly on dz^r/dA. Namely,

$$\frac{dz^s(A)}{dA} = -\frac{y_A(A, z^s(A))}{y_z(A, z^s(A))} + \frac{\beta\lambda(\theta(A, z^r(A)))}{1-\beta(1-\delta)+\beta\lambda(\theta(A, z^r(A)))}\frac{y_A(A, z^r(A))}{y_z(A, z^s(A))}\left(1 + \frac{y_z(A, z^r(A))}{y_A(A, z^r(A))}\frac{dz^r(A)}{dA}\right).$$

The second term makes explicit the interaction between the decisions to separate from a job and to search across occupations when there is no rest unemployment. It captures the change in the gains of search across occupations, $dR(A)/dA$, and shows that z^r and z^s can have the same cyclicality. When instead $z^s > z^r$, workers endogenously separate into a period of rest unemployment. In this case, $R(A)$ has a smaller impact on the value of unemployment at the moment of separation. This is because searching across occupations would only occur further in the future, and then only if a worker's z-productivity would deteriorate below z^r. Thus, the presence of rest unemployment weakens any feedback of a procyclical z^r onto z^s. Indeed, by setting $\lambda(.) = 0$ in the above expression we get that z^s is always countercyclical.

2. Worker Flows

In a BRE the evolution of the distribution $\mathcal{G}$ of employed and unemployed workers across labor markets (z, x), occupations o and employment status es is a result of (i) optimal vacancy posting $\theta(.)$, job separation decisions $d(.)$ and occupational mobility decisions $\rho(.)$ and $\mathcal{S}(.)$, all depending on the state vector $\omega = (A, p, z, x, o)$; and (ii) the exogenous retiring probability μ. To obtain the laws of motions of unemployed and employed workers it is then useful to derive the measure of unemployed and employed workers at each stage j within a period, where $j = s, r, m, p$ represent separations, reallocations, search and matching and production as described in the main text. It is also useful to consider the following Markov Chain: in period t an employed worker with human capital level x_h increases his human capital to x_{h+1} with probability $\chi^e(x_{h+1}|x_h)$, where $\chi^e(x_{h+1}|x_h) = 1 - \chi^e(x_h|x_h)$, $x_h < x_{h+1}$, $h = 1, \ldots, H$ and $x_H < \infty$. Human capital depreciation occurs during unemployment with probability $\chi^u(x_{h-1}|x_h)$, where $\chi^u(x_{h-1}|x_h) = 1 - \chi^u(x_h|x_h)$, $x_h > x_{h-1}$, $h = 1, \ldots, H$. Let $u_t^j(z, x_h, o)$ denote the measure of unemployed workers in labor market (z, x_h) in occupation o at the beginning of stage j in period t. Similarly, let $e_t^j(z, x_h, o)$ denote the measure of employed workers in labor market (z, x_h) in occupations o at the beginning of stage j in period t.

2.1 Unemployed Workers

Given the initial conditions $(A_0, p_0, \mathcal{G}_0^p)$, the measure of unemployed workers characterised by (z, x_h) in occupation o at the beginning of next period's separation stage is

$$\begin{aligned} u_{t+1}^s(z,x_h,o)dz &= (1-\mu)\Bigg[\chi^u(x_h|x_h)\int_{\underline{z}}^{\overline{z}} u_t^p(\tilde{z},x_h,o)dF(z|\tilde{z})d\tilde{z} + \chi^u(x_h|x_{h+1})\int_{\underline{z}}^{\overline{z}} u_t^p(\tilde{z},x_{h+1},o)dF(z|\tilde{z})d\tilde{z}\Bigg] \\ &+ \mu\Bigg[\sum_{\tilde{o}=1}^{O}\sum_{\tilde{h}=1}^{H}\int_{\underline{z}}^{\overline{z}}\Big[u_t^p(\tilde{z},x_{\tilde{h}},\tilde{o}) + e_t^p(\tilde{z},x_{\tilde{h}},\tilde{o})\Big]d\tilde{z}\Bigg]\psi_o(\mathbf{1}_{h=1})dF(z). \end{aligned} \tag{7}$$

Conditional on not retiring from the labor market, the terms inside the first squared bracket show the probability that unemployed workers in labor markets $(\tilde{z}, x_h, o)$ and $(\tilde{z}, x_{h+1}, o)$ in the previous period's production stage will be in labor market (z, x_h, o) immediately after the z and x_h shocks occur. The term in the second squared bracket refers to the measure of new workers who entered the economy to replace those who left at the beginning of the period due to the μ-shock. We assume that the population of workers is constant over time, making the inflow equal to the outflow of workers. New workers are assumed to enter unemployed with a $\tilde{z}$ randomly drawn from $F(.)$ and with the lowest human capital level x_1. The above equation considers the inflow who has been assigned productivity z and occupation o, where ψ_o denotes the probability that workers in the inflow are assigned occupation o and $\mathbf{1}_{h=1}$ denotes an indicator function which takes the value of one when the labor market (z, x_h) is associated with x_1 and zero otherwise.

During the separation stage some employed workers will become unemployed. Since by assumption these newly unemployed workers do not participate in the current period's reallocation or search and matching stages, it is convenient to count them at the production stage. This implies that $u_{t+1}^s(z, x_h, o)dz = u_{t+1}^r(z, x_h, o)dz$. Similarly, we will count at the production stage those unemployed workers who arrived from other occupations during the reallocation stage, as they also do not participate in the current period's search and matching stage. This implies that the measure of unemployed workers characterised by (z, x_h) in occupation o at the beginning of the search and matching stage is given by

$$u_{t+1}^m(z,x_h,o)dz = (1-\rho(A,p,z,x_h,o))u_{t+1}^r(z,x_h,o)dz.$$

Noting that $(1-\lambda(\theta(A,p,z,x_h,o)))u_{t+1}^m(z,x_h,o)dz$ workers remain unemployed after the search and matching stage, the above assumptions on when do we count occupational movers and those who separated from their employers imply that the measure of unemployed workers characterised by (z, x_h) in occupation o during the production stage is given by

$$\begin{aligned} u_{t+1}^p(z,x_h,o)dz &= (1-\lambda(\theta(A,p,z,x_h,o)))u_{t+1}^m(z,x_h,o)dz + d(z,x_h,o,A,p)e_{t+1}^s(z,x_h,o)dz \\ &+ (\mathbf{1}_{h=1})\Bigg[\sum_{\tilde{o}\neq o}\sum_{\tilde{h}=1}^{H}\Big[\int_{\underline{z}}\rho(\tilde{z},\tilde{x}_h,\tilde{o},A,p)\alpha(s_o(\tilde{z},\tilde{x}_h,\tilde{o},A,p),\tilde{o})u_{t+1}^r(\tilde{z},x_{\tilde{h}},\tilde{o})d\tilde{z}\Big]\Big]dF(z)\Bigg]. \end{aligned} \tag{8}$$

2.2 Employed Workers

Next we turn to describe the laws of motion for employed workers. Given the initial conditions $(A_0, p_0, \mathcal{G}_0^p)$, the measure of employed workers characterised by (z, x_h) in occupation o at the beginning of next period's separation stage is given by

$$\begin{aligned} e^s_{t+1}(z, x_h, o)dz &= (1-\mu)\Bigg[\chi^e(x_h|x_h)\int_{\underline{z}}^{\overline{z}} e^p_t(\tilde{z}, x_h, o)dF(z|\tilde{z})d\tilde{z} \\ &+ (\mathbf{1}_{h>1})\chi^e(x_h|x_{h-1})\int_{\underline{z}}^{\overline{z}} e^p_t(\tilde{z}, x_{h+1}, o)dF(z|\tilde{z})d\tilde{z}\Bigg]. \end{aligned} \tag{9}$$

Conditional on not retiring from the labor market, the terms inside the squared bracket show the probability that employed workers in labor markets $(\tilde{z}, x_h, o)$ and $(\tilde{z}, x_{h-1}, o)$ in the previous period's production stage will be in labor market (z, x_h, o) immediately after the z and x_h shocks occur. In this case, the indicator function $\mathbf{1}_{h>1}$ takes the value of one when the labor market (z, x_h) is associated with a value of $x_h > x_1$ and zero otherwise.

Since we count those employed workers who separated from their employers in the production stage and employed workers do not participate in the reallocation or the search and matching stages, it follows that $e^s_{t+1}(z, x_h, o)dz = e^r_{t+1}(z, x_h, o)dz = e^m_{t+1}(z, x_h, o)dz$. This implies that the measure of employed workers characterised by (z, x_h) in occupation o during the production stage is given by

$$e^p_{t+1}(z, x_h, o)dz = (1 - d(z, x_h, o, A, p))e^s_{t+1}(z, x_h, o)dz + \lambda(\theta(\omega))u^m_{t+1}(z, x_h, o)dz, \tag{10}$$

where the last term describes those unemployed workers in labor market (z, x_h) who found a job in their same occupation o.

3. Competitive Search Model

In the model described in the main text we exogenously segment an occupation into many sub-markets, one for each pair (z, x). We assumed that workers with current productivities (z, x) in occupation o only participate in the sub-market (z, x) in such an occupation. We now show that this sub-market structure endogenously arises from a competitive search model in the spirit of Moen (1997) and Menzio and Shi (2010). To show this property in the simplest way, we focus on the case in which all occupations have the same productivities (only excess mobility) and workers only differ in their z-productivities within an occupation. This is the same simplification we used in Section 1 of this appendix. Adding differences in occupation productivities and occupational human capital is a straightforward extension. The full theoretical and quantitative analysis of this competitive search model can be found in our earlier working paper Carrillo-Tudela and Visschers (2013).

3.1 Basic Setup

As in the main text, we look for an equilibrium in which the value functions and decisions of workers and firms in any occupation only depend on the productivities $\omega = (A, z)$ and workers' employment status. Following Menzio and Shi (2010) and Menzio, Telyukova and Visschers (2016) we divide the analysis in two steps. The first step shows that at most one sub-market is active for workers with current productivity z. The second step shows that in equilibrium firms will post wage contracts such that a worker with current productivity z does not find it optimal to visit any other sub-market other than the one opened to target workers of this productivity.

Assume that in each occupation firms post wage contracts to which they are committed. For each value of z in an occupation o there is a continuum of sub-markets, one for each expected lifetime value $\tilde{W}$ that could potentially be offered by a vacant firm. After firms have posted a contract in the sub-market of their choice, unemployed workers with productivity z (henceforth type z workers) can choose which appropriate sub-market to visit. Once type z workers visit their preferred sub-market j, workers and firms meet according to a constant returns to scale matching function $m(u_j, v_j)$, where u_j is the measure of workers searching in sub-market j, and v_j the measure of firms which have posted a contract in this sub-market. From the above matching function one can easily derived the workers' job finding rate $\lambda(\theta_j) = m(\theta_j)$ and the vacancy filling rate $q(\theta_j) = m(1/\theta_j)$, where labor market tightness is given by $\theta_j = v_j/u_j$. The matching function and the job finding and vacancy filling rates are assumed to have the following properties: (i) they are twice-differentiable functions, (ii) non-negative on the relevant domain, (iii) $m(0, 0) = 0$, (iv) $q(\theta)$ is strictly decreasing, and (v) $\lambda(\theta)$ is strictly increasing and concave.

We impose two restrictions on beliefs off-the-equilibrium path. Workers believe that, if they go to a sub-market that is inactive on the equilibrium path, firms will show up in such measure to have zero profit in expectation. Firms believe that, if they post in an inactive sub-market, a measure of workers will show up, to make the measure of deviating firms indifferent between entering or not. We assume, for convenience, that the zero-profit condition also holds for deviations of a single agent: loosely, the number of vacancies or unemployed, and therefore the tightness will be believed to adjust to make the zero-profit equation hold.

3.2 Agents' Problem

Workers Consider the value function of an unemployed worker having productivity z in occupation o at the beginning of the production stage, $W^U(\omega) = b + \beta\mathbb{E}[W^R(\omega')]$. The value of unemployment consists of the flow benefit of unemployment b this period, plus the discounted expected value of being unemployed at the beginning of next period's reallocation stage,

$$W^R(\omega') = \max_{\rho(\omega')}\{\rho(\omega')R(\omega') + (1 - \rho(\omega'))\mathbb{E}[S(\omega') + W^U(\omega')]\}, \tag{11}$$

where $\rho(\omega')$ takes the value of one when the worker decides to reallocate and take the value of zero otherwise and recall R is the expected value of reallocation. In this case,

$$R(\omega) = -c + \sum_{o' \neq o} \int W^U(A', \tilde{z}) \frac{dF(\tilde{z})}{O - 1}.$$

The worker's expected value of staying and searching in his current occupation is given by $\mathbb{E}[S(\omega') + W^U(\omega')]$. In this case, $W^U(\omega') = \mathbb{E}[W^U(\omega')]$ describes the expected value of not finding a job, while $S(\omega')$ summarizes the expected value added of finding a new job. The reallocation decision is captured by the choice between $R(\omega')$ and the expected payoff of search in the current occupation.

To derive $S(.)$ note that $\lambda(\theta(\omega, W_f))$ denotes the probability with which a type z worker meets a firm f in the sub-market associated with the promised value W_f and tightness $\theta(\omega)$. Further, let $\alpha(W_f)$ denote the probability of visiting such a sub-market. From the set $\mathcal{W}$ of promised values which are offered in equilibrium by firms for a given z, workers only visit with positive probability those sub-markets for which the associated W_f satisfies

$$W_f \in \arg\max_{\mathcal{W}} \lambda(\theta(\omega', W_f))(W_f - W^U(\omega')) \equiv S(\omega'). \tag{12}$$

When the set $\mathcal{W}$ is empty, the expected value added of finding a job is zero and the worker is indifferent between visiting any sub-market.

Now consider the value function at the beginning of the production stage of an employed worker with productivity z in a contract that currently has a value $\tilde{W}_f(\omega)$. Similar arguments as before imply that

$$\tilde{W}_f(\omega) = w_f + \beta\mathbb{E}\left[\max_{d(\omega')}\{(1 - d(\omega'))\tilde{W}_f(\omega') + d(\omega')W^U(\omega')\}\right], \tag{13}$$

where $d(\omega')$ take the value of δ when $\tilde{W}_f(\omega') \geq W^U(\omega')$ and the value of one otherwise. In equation (13), the wage payment w_f at firm f is contingent on state ω, while the second term describes the worker's option to quit into unemployment in the separation stage the next period. Note that $W^U(\omega') = \mathbb{E}[W^U(\omega')]$ as a worker who separates must stay unemployed for the rest of the period and $\tilde{W}_f(\omega') = \mathbb{E}[\tilde{W}_f(\omega')]$ as the match will be preserved after the separation stage.

Firms Consider a firm f in occupation o, currently employing a worker with productivity z who has been promised a value $\tilde{W}_f(\omega) \geq W^U(\omega)$. Noting that the state space for this firm is the same as for the worker and given by ω, the expected lifetime discounted profit of the firm can be described recursively as

$$J(\omega; \tilde{W}_f(\omega)) = \max_{w_f, \tilde{W}_f(\omega')} \Big\{ y(A,z) - w_f + \beta\mathbb{E}\Big[\max_{\sigma(\omega')} \Big\{ (1-\sigma(\omega'))J(\omega'; \tilde{W}_f(\omega')) + \sigma(\omega')\tilde{V}(\omega') \Big\} \Big] \Big\}, \quad (14)$$

where $\sigma(\omega')$ takes the value of δ when $J(\omega'; \tilde{W}_f(\omega')) \geq \tilde{V}(\omega')$ and the value of one otherwise, $\tilde{V}(\omega') = \max\left\{\overline{V}(\omega'), 0\right\}$ and $\overline{V}(\omega')$ denotes the maximum value of an unfilled vacancy in occupation o at the beginning of next period. Hence (14) takes into account that the firm could decide to target its vacancy to workers of with a different productivity in the same occupation or withdraw the vacancy from the economy and obtain zero profits.

The first maximisation in (14) is over the wage payment w_f and the promised lifetime utility to the worker $\tilde{W}_f(\omega')$. The second maximisation refers to the firm's layoff decision. The solution to (14) then gives the wage payments during the match (for each realisation of ω for all t). In turn these wages determine the expected lifetime profits at any moment during the relation, and importantly also at the start of the relationship, where the promised value to the worker is $\tilde{W}_f$.

Equation (14) is subject to the restriction that the wage paid today and tomorrow's promised values have to add up to today's promised value $\tilde{W}_f(\omega)$, according to equation (13). Moreover, the workers' option to quit into unemployment, and the firm's option to lay off the worker imply the following participation constraints

$$\Big(J(\omega'; \tilde{W}_f(\omega')) - \tilde{V}(\omega')\Big) \geq 0 \quad \text{and} \quad \Big(\tilde{W}_f(\omega') - W^U(\omega')\Big) \geq 0. \quad (15)$$

Now consider a firm posting a vacancy in occupation o. Given cost k and knowing ω, a firm must choose which unemployed workers to target. In particular, for each z a firm has to decide which $\tilde{W}_f$ to post given the associated job filling probability, $q(\theta(\omega, \tilde{W}_f))$. This probability summarises the pricing behaviour of other firms and the visiting strategies of workers. Along the same lines as above, the expected value of a vacancy targeting workers of productivity z solves the Bellman equation

$$V(\omega) = -k + \max_{\tilde{W}_f} \Big\{ q(\theta(\omega, \tilde{W}_f))J(\omega, \tilde{W}_f) + (1 - q(\theta(\omega, \tilde{W}_f)))\tilde{V}(\omega) \Big\}. \quad (16)$$

We assume that there is free entry of firms posting vacancies within any occupation. This implies that $V(\omega) = 0$ and $\tilde{W}_f$ that yield a $\theta(\omega, \tilde{W}_f) > 0$, and $V(\omega) \leq 0$ for all those ω and $\tilde{W}_f$ that yield a $\theta(\omega, \tilde{W}_f) \leq 0$. In the former case, the free entry condition then simplifies (16) to $k = \max_{\tilde{W}_f} q(\theta(\omega, \tilde{W}_f))J(\omega, \tilde{W}_f)$.

3.3 Endogenous Market Segmentation

We now show that if there are positive gains to form a productive match with a worker of type z, firms offer a unique $\tilde{W}_f$ with associate tightness $\tilde{\theta}(A, z)$ in the matching stage. This implies that only one sub-market opens for a given value of z and workers of productivity z optimally chose to search in such a sub-market. Consider a value of z. For any promised value W^E, the joint value of the match is defined as $W^E + J(A, z, W^E) \equiv \tilde{M}(A, z, W^E)$. Lemma 4 shows that under risk neutrality the value of a job match is constant in W^E and J decreases one-to-one with W^E.

Lemma 4. *The joint value $\tilde{M}(A, z, W^E)$ is constant in $W^E \geq W^U(A, z)$ and hence we can uniquely define $M(A, z) \stackrel{def}{=} \tilde{M}(A, z, W^E)$, $\forall\, M(A, z) \geq W^E \geq W^U(A, z)$ on this domain. Further, $J_W(A, z, W^E) = -1$, $\forall\, M(A, z) > W^E > W^U(A, z)$.*

The proof of Lemma 4 is presented in Section 4.4. It crucially relies on the firms' ability to offer workers inter-temporal wage transfers such that the value of the job match is not affected by the (initial) promised value. Note that Lemma 4 implies that no firm will post vacancies for z values such that $M(A, z) - W^U(A, z) \leq 0$. Lemma 5 now shows that for a given z, for which $M(A, z) - W^U(A, z) > 0$, firms offer a unique $\tilde{W}_f$ in the matching stage and there is a unique θ associated with it.

Lemma 5. *If the elasticity of the vacancy filling rate is weakly negative in θ, there exists a unique $\theta^*(A, z)$ and $W^*(A, z)$ that solve* (12)*, subject to* (16)*.*

The proof of Lemma 5 is also presented in Section 4.4. The requirement that the elasticity of the job filling rate with respect to θ is non-positive is automatically satisfied when $q(\theta)$ is log concave, as is the case with the Cobb-Douglas and urn ball matching functions. Both matching functions imply that the job finding and vacancy filling rates have the properties described in Lemma 5 and hence guarantee a unique pair $\tilde{W}_f, \theta$. Consider a Cobb-Douglas matching function as it implies a constant $\varepsilon_{q,\theta}(\theta)$. Using η to denote the (constant) elasticity of the job finding rate with respect to θ, we find the well-known division of the surplus according to the Hosios' (1991) rule

$$\eta(W^E - W^U(A, z)) - (1 - \eta)J(A, z, W^E) = 0. \tag{17}$$

Since for every value of z there is at most one $\tilde{W}_f$ offered in the matching stage, the visiting strategy of an unemployed worker is to visit the sub-market associated with $\tilde{W}_f$ with probability one when $S(A, z) > 0$ and to randomly visit any sub-market when $S(A, z) = 0$ (or not visit any submarket at all). Let $\tilde{W}_f^z$ denote the unique expected value offered to workers of productivity z in equilibrium.

The final step is to verify that if we allow firms to post a menu of contracts which specify for each z a different expected value, then the above equilibrium allocation and payoffs can be sustained in this more general setting. To show this we closely follow Menzio and Shi (2010) with the addition of endogenous reallocation. Since the expected value of reallocation R is a constant across z-productivities, it is easy to verify that the proof developed in Menzio and Shi (2010) applies here as well.

Intuitively, consider a BRE in which firms that want to only attract a worker of type z offer a contract with expected value $\tilde{W}_f^z$, as derived above, to any worker of type z or higher; and for lower worker types, the firm offers contracts with expected values strictly below these types' value of unemployment. Since $\tilde{W}_f^z$ and θ^z in this candidate equilibrium are increasing in z, this implies that only the type z workers visit these firms. To show that this is indeed optimal for firms, first consider the deviation in which a firm opens a new sub-market that attracts workers of different types z and $\hat{z}$ by offering expected values W^z and $W^{\hat{z}}$. Without loss of generality assume that firms obtain higher profit from matching with a worker of type z instead of type $\hat{z}$. Next consider an alternative deviation in which a firm opens a sub-market that only attract workers of type z by offering W^z and the value of unemployment to any other worker type. Given the off-equilibrium beliefs, a higher tightness is associated with this alternative deviation which makes it more attractive for type z workers to visit relative to the original deviation. That is, if there is a strictly profitable deviation in which more than one type visits the same sub-market, there is always another strictly more profitable deviation in which a sub-market is visited by only one type. However, Lemma 5 shows that the latter deviation cannot exist and therefore there also cannot be a profitable deviation in which more than type visits the same sub-market in equilibrium.

4. Proofs

Definition A Block Recursive Equilibrium (BRE) is a set of value functions $W^U(\omega)$, $W^E(\omega)$, $J(\omega)$, workers' policy functions $d(\omega)$, $\rho(\omega)$, $\mathcal{S}(\omega)$, firms' policy function $\sigma(\omega)$, tightness function $\theta(\omega)$, wages $w(\omega)$, laws of motion of A, p, z and x for all occupations, and laws of motion for the distribution of unemployed and employed workers over all occupations, such that: *(i)* the value functions and decision rules follow from the firm's and worker's problems described in equations (1)-(5) in the main text; *(ii)* labor market tightness $\theta(\omega)$ is consistent with free entry on each labor market, with zero expected profits determining $\theta(\omega)$ on labor markets at which positive ex-post profits exist; $\theta(\omega) = 0$ otherwise; *(iii)* wages solve equation (6) in the main text; *(iv)* the worker flow equations map initial distributions of unemployed and employed workers (respectively) over labor markets and occupations into next period's distribution of unemployed and employed workers over labor markets and occupations, according to the above policy functions and exogenous separations.

Proposition 2 Given $F(z'|z) < F(z'|\tilde{z})$ for all z, z' when $z > \tilde{z}$: (i) a BRE exists and it is the unique equilibrium, and (ii) the BRE is constrained efficient.

4.1 Proof of Proposition 2

We divide the proof into two parts. In the first part we show existence of equilibrium by deriving the operator T, showing it is a contraction and then verifying that the candidate equilibrium functions from the fixed point of T satisfy all equilibrium conditions. The second part shows efficiency of equilibrium.

4.1.1 Existence

Step 1: Let $M(\omega) \equiv W^E(\omega) + J(\omega)$ denote the value of the match. We want to show that the value functions $M(\omega)$, $W^U(\omega)$ and $R(\omega)$ exist. This leads to a three dimensional fixed point problem. It is then useful to define the operator T that maps the value function $\Gamma(\omega, n)$ for $n = 0, 1, 2$ into the same functional space, such that $\Gamma(\omega, 0) = M(\omega)$, $\Gamma(\omega, 1) = W^U(\omega)$, $\Gamma(\omega, 2) = R(\omega)$ and

$$T(\Gamma(\omega, 0)) = y(A, p_o, z, x) + \beta \mathbb{E}_{\omega'} \Big[\max_{d^T} \{(1 - d^T) M(\omega') + d^T W^U(\omega')\} \Big],$$

$$T(\Gamma(\omega, 1)) = b + \beta \mathbb{E}_{\omega'} \Big[\max_{\rho^T} \{\rho^T R(\omega') + (1 - \rho^T)(D^T(\omega') + W^U(\omega'))\} \Big],$$

$$T(\Gamma(\omega, 2)) = \max_{\mathcal{S}(\omega)} \Big(\sum_{\tilde{o} \in \mathbf{O}^-} \alpha(s^T_{\tilde{o}}) \int_{\underline{z}}^{\overline{z}} W^U(\tilde{z}, x_1, \tilde{o}, A, p) dF(\tilde{z}) + (1 - \sum_{\tilde{o} \in \mathbf{O}^-} \alpha(s^T_{\tilde{o}}))[b + \beta \mathbb{E}_{\omega'} R(\omega')] - c \Big),$$

where the latter maximization is subject to $s_o \in [0, 1]$ and $\sum_{o \in \mathbf{O}^-} s_o = 1$, and

$$D^T(\omega') \equiv \lambda(\theta(\omega'))(1 - \eta) \Big(M(\omega') - W^U(\omega') \Big), \text{ with } \theta(\omega') = \Big(\frac{\eta (M(\omega') - W^U(\omega'))}{k} \Big)^{\frac{1}{1-\eta}}. \tag{18}$$

Lemma A.1: T is (i) a well-defined operator mapping functions from the closed space of bounded continuous functions $\mathcal{E}$ into $\mathcal{E}$, and (ii) a contraction.

First we show that the operator T maps bounded continuous functions into bounded continuous functions. Let $W^U(\omega)$, $R(\omega)$ and $M(\omega)$ be bounded continuous functions. It then follows that $\lambda(\theta(\omega))$ and $D(\omega)$ are continuous functions. It also follows that $\max\{M(\omega), W^U(\omega)\}$ and $\max\{R(\omega), D(\omega) + W^U(\omega)\}$ are continuous. Further, since the constraint set for $\mathcal{S}$ is compact-valued and does not depend on ω, functions $\alpha(.)$

are continuous, and the integral of continuous function W^U is continuous, the theorem of the maximum then implies that the expression

$$\max_{\mathcal{S}(\omega)}\Big(\sum_{\tilde{o}\in\mathbf{O}^-}\alpha(s^T_{\tilde{o}})\int_{\underline{z}}^{\overline{z}}W^U(\tilde{z},x_1,\tilde{o},A,p)dF(\tilde{z})+(1-\sum_{\tilde{o}\in\mathbf{O}^-}\alpha(s^T_{\tilde{o}}))[b+\beta\mathbb{E}_{\omega'}R(\omega')]-c\Big) \tag{19}$$

is continuous. Therefore T maps continuous functions into continuous functions. Moreover, since the domain of ω is bounded, and $\alpha(.)$ and $\lambda(.)$ are bounded on bounded domains, T maps the space of bounded continuous functions into itself.

Second we show that T defines a contraction. Consider two functions $\Gamma,\Gamma'\in\mathcal{E}$, such that $\|\Gamma-\Gamma'\|_{\text{sup}}<\varepsilon$. It then follows that $\|W^U-W^{U\prime}\|_{\text{sup}}<\varepsilon$, $\|R-R'\|_{sup}<\varepsilon$ and $\|M-M'\|_{\text{sup}}<\varepsilon$, where W^U, R and M are part of Γ as defined above. We first establish that

$$\|D+W^U-D'-W^{U\prime}\|_{sup}<\varepsilon, \tag{20}$$

when function-tuples D and D' are derived from (M,W^U), and $(M',W^{U\prime})$ respectively. With this aim consider, without loss of generality, the case in which $M(\omega)-W^U(\omega)>M^{'}(\omega)-W^{U\prime}(\omega)$ at a given ω. Instead of $D(\omega)$, write $D(M(\omega)-W(\omega))$ to make explicit the dependence of D on M and W. To reduce notation we suppress the dependence on ω for this part of the proof and further condense $W^U=W$. From $M-W>M^{'}-W^{'}$, it follows that $\varepsilon>W^{'}-W\geq M^{'}-M>-\varepsilon$. Construct $M^{''}=W^{'}+(M-W)>M^{'}$ and $W^{''}=M^{'}-(M-W)<W^{'}$. Equation (18) implies $D(M-W)+W$ is increasing in M and in W. To verify this, hold M constant and note that

$$\frac{d(D(M-W)+W)}{dW}=-\lambda(\theta(.))+1\geq 0 \tag{21}$$

by virtue of $d(D(M-W))/d(M-W)=\lambda$ and that $\lambda\in[0,1]$, where the inequality in (21) is strict when $\lambda\in[0,1)$ and weak when $\lambda=1$. Then, it follows that

$$\begin{aligned}-\varepsilon<D(M^{'}-W^{''})+W^{''}-D(M-W)-W&\leq D(M^{'}-W^{'})+W^{'}-D(M-W)-W\\&\leq D(M^{''}-W^{'})+W^{'}-D(M-W)-W<\varepsilon\end{aligned}$$

where $D(M^{'}-W^{''})=D(M-W)=D(M^{''}-W^{'})$ by construction. Note that the outer inequalities follow because $M-M^{'}>-\varepsilon$ and $W^{'}-W<\varepsilon$. Given that D, M and W are bounded continuous functions on a compact domain and the above holds for every ω, it then must be that $\|D+W^U-D^{'}-W^{U^{'}}\|_{\text{sup}}<\varepsilon$. Since $\|\max\{a,b\}-\max\{a',b'\}\|<\max\{\|a-a'\|,\|b-b'\|\}$, as long as the terms over which to maximize do not change by more than ε in absolute value, the maximized value does not change by more ε, it then follows that $\|T(\Gamma(\omega,n))-T(\Gamma^{'}(\omega,n)\|<\beta\varepsilon$ for all ω; $n=0,1$.

To show that $\|T(\Gamma(\omega,3))-T(\Gamma^{'}(\omega,3)\|<\beta\varepsilon$ for all ω, with a slight abuse of notation, define $TR(\omega,\mathcal{S})$ as

$$\begin{aligned}TR(\omega,\mathcal{S})=&\sum_{\tilde{o}\in\mathbf{O}^-}\alpha(s_{\tilde{o}}(\omega))\int_{\underline{z}}^{\overline{z}}\Big[b+\beta\mathbb{E}_{\omega'}\Big[\max_{\rho^T}\{\rho^TR(\omega')+(1-\rho^T)(D^T(\omega')+W^U(\omega'))\}\Big]\Big]dF(\tilde{z})\\&+(1-\sum_{\tilde{o}\in\mathbf{O}^-}\alpha(s_{\tilde{o}}(\omega)))[b+\beta\mathbb{E}_{\omega'}R(\omega')]-c.\end{aligned} \tag{22}$$

Let $\mathcal{S}_*(\omega)=\{s^*_1,\ldots,s^*_{o-1},s^*_{o+1},\ldots,s^*_O\}$ be the maximizer of TR at ω and $\mathcal{S}_*{}'(\omega)$ of TR'. Without loss of generality assume that $TR(\omega,\mathcal{S})>TR'(\omega,\mathcal{S}')$. Since our previous arguments imply that

$$\|\Big[\max_{\rho}\{\rho R+(1-\rho)(D+W^U)\}\Big]-\Big[\max_{\rho}\{\rho R'+(1-\rho)(D'+W')\}\Big]\|_{sup}<\beta\varepsilon$$

it then follows that $\|TR(\omega,\mathcal{S}_*)-TR'(\omega,\mathcal{S}_*)\|_{sup}<\beta\varepsilon$. Further, since $TR(\omega,\mathcal{S}_*)>TR'(\omega,\mathcal{S}^{'}_*)>TR'(\omega,\mathcal{S}_*)$ we have that $0<TR(\omega,\mathcal{S}_*)-TR'(\omega,\mathcal{S}_*)<\beta\varepsilon$. Using the latter inequality to obtain $0<$

$TR(\omega,\mathcal{S}_*) - TR'(\omega,\mathcal{S}_*^{'}) + TR'(\omega,\mathcal{S}_*^{'}) - TR'(\omega,\mathcal{S}_*) < \beta\varepsilon$ and noting that $TR'(\omega,\mathcal{S}_*^{'}) - TR'(\omega,\mathcal{S}_*) > 0$, it follows that $0 < TR(\omega,\mathcal{S}_*) - TR'(\omega,\mathcal{S}_*^{'}) < \beta\varepsilon$. This last step is valid for all ω, then it implies that $\|T(\Gamma(\omega,3)) - T(\Gamma^{'}(\omega,3)\|_{sup} < \beta\varepsilon$ and hence the operator T is a contraction and has a unique fixed point.

Step 2 (Linking the Mapping T and BRE Objects): From the fixed point functions $M(\omega)$, $W^U(\omega)$ and $R(\omega)$ define the function $J(\omega) = \max\{(1-\zeta)[M(\omega) - W^U(\omega)], 0\}$, and the functions $\theta(\omega)$ and $V(\omega)$ from $0 = V(\omega) = -k + q(\theta(\omega))J(\omega)$. Also define $W^E(\omega) = M(\omega) - J(\omega)$ if $M(\omega) > W^U(\omega)$, and $W^E(\omega) = M(\omega)$ if $M(\omega) \leq W^U(\omega)$. Finally, define $d(\omega) = d^T(\omega)$, $\sigma(\omega) = \sigma^T(\omega)$, $\rho(\omega) = \rho^T(\omega)$, $\mathcal{S}(\omega) = \mathcal{S}^T(\omega)$ and a $w(\omega)$ derived using the Nash bargaining equation in the main text given all other functions.

Given $1-\zeta = \eta$ and provided that the job separation decisions between workers and firms coincide, which they are as a match is broken up if and only if it is bilaterally efficient to do so according to $M(\omega)$ and $W^U(\omega)$, then equations (5) and (6) in the main text (describing $J(\omega)$ and surplus sharing) are satisfied. Further, equation (3) in the main text (describing $W^E(\omega)$) is satisfied by construction, $\theta(\omega)$ satisfies the free-entry condition and $w(\omega)$ satisfies equation (6) in the main text. Hence, the constructed value functions and decision rules satisfy all conditions of the equilibrium and the implied evolution of the distribution of employed and unemployed workers also satisfies the equilibrium conditions.

Uniqueness follows from the same procedure in the opposite direction and using a contradiction argument. Suppose the BRE is not unique. Then a second set of functions exists that satisfy all the equilibrium conditions. Construct $\hat{M}$, $\hat{W}^U$ and $\hat{R}$ from these conditions. Since in any equilibrium the job separation decisions have to be bilaterally efficient and the occupational mobility decisions (ρ and $\mathcal{S}$) are captured in T, then $\hat{M}$, $\hat{W}^U$ and $\hat{R}$ must be a fixed point of T, contradicting the uniqueness of the fixed point established by Banach's Fixed Point Theorem. Hence, there is a unique BRE.

4.1.2 Efficiency

The social planner, currently in the production stage, solves the problem of maximizing total discounted output by choosing job separations decisions $d(.)$, occupational mobility decisions $\rho(.)$ and $\mathcal{S}(.)$, as well as vacancy creation decisions $v(.)$ for each pair (z, x_h) across all occupations $o \in O$ in any period t. The first key aspect of the planner's choices is that they could potentially depend on the entire state space $\Omega^j = \{z, x, o, A, p, \mathcal{G}^j\}$ for each of the four within period stages $j = s, r, m, p$ (separation, reallocation, search and matching, production) and workers' employment status, such that its maximization problem is given by

$$\max_{\{d(\Omega^s),\rho(\Omega^r),\mathcal{S}(\Omega^r),v(\Omega^m)\}} \mathbb{E}\sum_{t=0}^{\infty}\Bigg(\sum_{o=1}^{O}\sum_{h=1}^{H}\int_{\underline{z}}^{\overline{z}} \beta^t\left[u_t^p(z,x_h,o)b + e_t^p(z,x_h,o)y(A_t,p_{o,t},z,x_h)\right]dz \qquad (23)$$
$$- \sum_{o'=1}^{O}\sum_{h'=1}^{H}\int_{\underline{z}}^{\overline{z}} \beta^{t+1}\Big[c\rho(z',x'_h,o',A_{t+1},p_{t+1},\mathcal{G}^r_{t+1})u^r_{t+1}(z',x'_h,o')$$
$$+ kv_{t+1}(z',x'_h,o',A_{t+1},p_{t+1},\mathcal{G}^m_{t+1})\Big]dz'\Bigg),$$

subject to the initial conditions $(A_0, p_0, \mathcal{G}_0^p)$, the laws of motion for unemployed and employed workers described in Section 2 of this appendix (with corresponding state space Ω^j for the decision rules), and the choice variables $\rho(.)$ and $d(.)$ being continuous variables in $[0,1]$, as the planner can decide on the proportion of workers in labor market (z, x_h) to separate from their jobs or to change occupations.

Note that implicitly the social planner is constrained in the search technology across occupations: it faces the same restrictions as an individual worker (in occupation $o \in O$), on the proportion of time that can be devoted to obtain a z-productivity from occupation $\tilde{o} \neq o$. Namely, $s_{\tilde{o}}(.) \in [0,1]$, $\sum_{\tilde{o}\in\mathbf{O}_o^-} s_{\tilde{o}}(.) = 1$ and $\sum_{\tilde{o}\in\mathbf{O}_o^-} \alpha(s_{\tilde{o}}(.), o) \leq 1$, where $\mathbf{O}_o^-$ denotes the set of remaining occupations relative to o. The latter notation highlights that, as in the decentralised problem, once the occupational mobility decision has been taken the new z-productivity cannot be obtain from the departing occupation.

Rewriting the planners' problem in recursive form as the fixed point of the mapping T^{SP} and letting next period's values be denoted by a prime yields

$$T^{SP}W^{SP}(\Omega^p) = \max_{\substack{\{d(\Omega^{s'}),\rho(\Omega^{r'}),\\ \mathcal{S}(\Omega^{r'}),v(\Omega^{m'})\}}} \sum_{o=1}^{O}\sum_{h=1}^{H}\int_{\underline{z}}^{\overline{z}} (u^p(z,x_h,o)b + e^p(z,x_h,o)y(A,p_o,z,x_h))dz \tag{24}$$
$$+ \beta\mathbb{E}_{\Omega^{s'}|\Omega^p}\left[-\left(c\sum_{o=1}^{O}\sum_{h=1}^{H}\int_{\underline{z}}^{\overline{z}}\rho(\Omega^{r'})u^{r\prime}(z',x_h',o)dz' + k\sum_{o=1}^{O}\sum_{h=1}^{H}\int_{\underline{z}}^{\overline{z}} v(\Omega^{m'})dz'\right) + W^{SP}(\Omega^{p\prime})\right],$$

subject to the same restrictions described above. Our aim is to show that this mapping is a contraction that maps functions $W^{SP}(.)$ from the space of functions linear with respect to the distribution $\mathcal{G}^p$ into itself, such that

$$W^{SP}(\Omega^p) = \sum_{o=1}^{O}\sum_{h=1}^{H}\int_{\underline{z}}^{\overline{z}}\big(u^p(z,x_h,o)W^{u,SP}(\omega) + e^p(z,x_h,o)M^{SP}(\omega)\big)\, dz, \tag{25}$$

for some functions $W^{u,SP}(\omega)$ and $M^{SP}(\omega)$, where $\omega = (z, x_h, o, A, p)$, and $u^p(.)$ and $e^p(.)$ are implied by $\mathcal{G}^p$.

In Section 2 of this appendix, we decomposed the next period's measure of unemployed workers at the production stage, $u^{p'}$, into three additive terms (see equation (8)). The first term refers to those unemployed workers that were unsuccessful in matching, either due to not finding a posted vacancy in their submarket or because the planner chose not to post vacancies in their submarket. The second term refers to those workers who separated from employment and hence were restricted from the search and matching stage within the period. The third term refers to those who changed occupations and came from other markets into submarket (z, x_h, o) and were restricted from the search and matching stage within the period. Likewise, we decomposed the next period's measure of employed workers at the production stage into two additive terms (see equation (10)). The first one refers to the survivors in employment from the previous separation stage, while the second terms refers to new hires. Given a continuation value function $W^{SP}(\Omega^{p'})$ that is linear in $\mathcal{G}^p$, as in (25), we now show that we can also decompose the expression in (24) into different additive components.

Consider first those unemployed workers who are eligible to participate in the search and matching stage. Note that the matching technology implies $v(\Omega^m) = \theta(\Omega^m)(1 - \rho(\Omega^r))u^m(z, x_h, o)$. Therefore we can consider $\theta(\Omega^{m'})$ as the planner's choice in equation (24) instead of $v(\Omega^{m'})$. Next we isolate the terms of $W^{SP}(\Omega^{p'})$ that involve workers who went through the search and matching stage: the first term of the unemployed worker flow equation (8) and the second term of the employed worker flow equation (10). We then combine these terms with the cost of vacancy posting in (24). Noting that the dependence of $u^m(\Omega^{m'})$ captures a potential dependence of the planner's occupational mobility decisions made in the previous reallocation stage, we can express the terms in the mapping (24) that involve $\theta(\Omega^{m'})$ as

$$\sum_{o=1}^{O}\sum_{h=1}^{H}\int_{\underline{z}}^{\overline{z}} u^{m\prime}(\Omega^{m'})\Big\{-k\theta(.) + \lambda(\theta(.))M^{SP}(\omega') + (1-\lambda(\theta(.))W^{u,SP}(\omega')\Big\}dz'. \tag{26}$$

Note that by backwards induction, the planner's optimal decisions for $\theta(\Omega^{m'})$ maximize (26). Given that the terms within the curly brackets only depend on ω' when the continuation value is linear as in (25), maximizing with respect to θ implies an optimal $\theta(\omega')$. Using this result we let $W^{m,SP}(\omega')$ denote the sum of the maximized terms inside the curly brackets in (26) .

Next consider the search-direction choice across occupations for those workers who the planner has decided to move across occupations. The following expression summarizes the terms of (24) that involve this choice,

$$\sum_{o'}\sum_{h'=1}^{H}\int_{\underline{z}}^{\overline{z}}\Big[u^r_{t+1}(\Omega^{r'})\rho(\Omega^{r'})\Big\{\sum_{\hat{o}\neq o'}\alpha(s_{\hat{o}}(.),o')\int_{\underline{z}}^{\overline{z}}\Big[W^{u,SP}(\hat{z},x_1,\hat{o},A',p')\Big]dF(\hat{z})$$
$$+(1-\sum_{\hat{o}\neq o'}\alpha(s_{\hat{o}}(.),o'))[b+\beta\mathbb{E}_{\omega''}R^{SP}(\omega'')]\Big\}\Big]dz', \tag{27}$$

where we have applied the properties of the z-productivity process to change the order of integration relative to the third term of flow equation (8) and have integrated over all of next period's production stage states. Since we allow previous decisions to depend on the entire state space, $u^r_{t+1}(.)$ and $\rho(.)$ are written as a function of $\Omega^{r'}$ as they potentially depend on the distribution $\mathcal{G}^r$. Note, however, that the term inside the curly brackets in (27) can be maximized separately for each $(z', {x_h}', o')$ and can therefore be summarized by $R^{gross,SP}(\omega') = R^{SP}(\omega') - c$, while the search intensity decision $\mathcal{S}(.)$ has solution vector $\{s(\omega', \tilde{o})\}$.

Noting that $u^s(.) = u^r(.)$ and $u^m(.) = (1 - \rho(.))u^r(.)$ (see Section 2 of this appendix), the above results imply we can express the term on the second line of equation (24) as

$$\beta\mathbb{E}_{\Omega^{s'}|\Omega^p}\Bigg[\Bigg(\sum_{o=1}^{O}\sum_{h=1}^{H}\int_{\underline{z}}^{\overline{z}}\Big[u^{s'}(z',x'_h,o)\Big\{\rho(.)R^{SP}(\omega')+(1-\rho(.))W^{m,SP}(\omega')\Big\} \tag{28}$$
$$+e^{s'}(z',x'_h,o)\Big\{(1-d(.))M^{SP}(\omega')+d(.)W^{u,SP}(\omega')\Big\}\Big]dz\Bigg)\Bigg],$$

where it follows that the maximizing decisions $\rho(.)$ and $d(.)$ are also functions of ω', as these are the only other dependencies within the curly brackets.

Finally, given the properties of the shock processes, note that $u^s_{t+1}(.)$ and $e^s_{t+1}(.)$ are linear functions of $u^p(.)$ and $e^p(.)$, as demonstrated by equations (7) and (9) in Section 2 of this appendix. This implies that we can express $T^{SP}W^{SP}(\Omega^p)$ as

$$T^{SP}W^{SP}(\Omega^p)=\sum_{o=1}^{O}\sum_{h=1}^{H}\int_{\underline{z}}^{\overline{z}}\left[TW^{U,SP}_{max}(z,x_h,o,A,p)u(z,x_h,o)+TM^{SP}_{max}(z,x_h,o,A,p)e(z,x_h,o)\right]dz, \tag{29}$$

where $TW^{U,SP}_{max}$ is given by

$$TW^{U,SP}_{max}(\omega)=\max_{\rho(\omega'),\theta(\omega')}\Big\{b+\beta\mathbb{E}_{\omega'|\omega}\Big[\rho(\omega')\Big(\int_{\underline{z}}^{\overline{z}}\max_{\mathcal{S}(\omega')}\Big[\sum_{\tilde{o}\neq o}\alpha(s_{\tilde{o}}(\omega',o))W^{U,SP}_{max}(\tilde{z},x_1,\tilde{o},A',p')dF(\tilde{z})$$
$$+(1-\sum_{\tilde{o}\neq o}\alpha(s_{\tilde{o}}(\omega',o)))[b+\beta\mathbb{E}_{\omega''|\omega'}R^{U,SP}_{max}(\omega'')\Big]-c\Big) \tag{30}$$
$$+(1-\rho(\omega'))\Big[\lambda(\theta(\omega'))(M^{SP}_{max}(\omega')-W^{U,SP}_{max}(\omega'))-\theta(\omega')k+W^{U,SP}_{max}(\omega')\Big]\Big]\Big\},$$

and TM^{SP}_{max} is given by

$$TM^{SP}_{max}(p,x,z)=\max_{d(\omega')}\Big\{y(z,x_h,A,p_o)+\beta\mathbb{E}_{\omega'|\omega}\Big[\Big(d(\omega')W^{U,SP}_{max}(\omega')+(1-d(\omega'))M^{SP}_{max}(\omega')\Big)\Big]\Big\}. \tag{31}$$

Hence we have established that the mapping T^{SP} described in equation (24) maps functions $W^{SP}(.)$ from the space of functions linear with respect to the distribution $\mathcal{G}^p$ (of the form (25)) into itself.

It is now straightforward to show that if $TW^{U,SP}_{max}$, TR^{SP}_{max} and TM^{SP}_{max} are contraction mappings, then (29) (and thereby (24)) is also a contraction mapping. Given the regularity properties assumed on the shock processes and following the proof of the decentralised case in Section 4.1.1 of this appendix, we can show that a fixed point exists for (30) and (31). Using equation (29), we then can construct the fixed point of the expression in (24). It then follows from (30) and (31) that if the Hosios' condition holds, allocations of the fixed point of T are allocations of the fixed point of T^{SP}, and hence the equilibrium allocations in the decentralised setting are also the efficient allocations.

4.1.3 BRE gives the *unique* equilibrium allocation

We now show that in any equilibrium, decisions and value functions only depend on $\omega = (z, x_h, o, A, p)$. We proceed using a contradiction argument. Suppose there is an alternative equilibrium in which values and decisions do not depend only on ω, but also on an additional factor like the entire distribution of workers over employment status and z-productivities $\mathcal{G}$, or its entire history of observables, H_t. Consider the associated value functions in this alternative equilibrium, where the relevant state vector of the alternative equilibrium is given by (ω, H_t). Our aim is to show that such an equilibrium cannot exist.

First suppose that in the alternative equilibrium all value functions are the same as in the BRE, but decisions differ at the same ω. This violates the property that, in our setting, all maximizers in the BRE value functions are unique, leading to a contradiction. Now suppose that in the alternative equilibrium at least one value function differs from the corresponding BRE value function at the same ω. It is straightforward to show that the expected values of unemployment must differ in both equilibria. Let $W^U(\omega, H_t)$ denote the value function for unemployed workers in the alternative equilibrium, and let $W^U(\omega)$ denote the corresponding value function in the BRE for the same ω. Since in the proof of efficiency of a BRE we did not rely on the uniqueness of the BRE in the broader set of all equilibria, we can use the proved results of Section 3.1.2 of this appendix here.

In particular, recall that the social planner's problem is entirely linear in the distribution of workers across states and hence $W^U(\omega)$ is the best the unemployed worker with ω can do (without transfers), including in the market equilibrium. Likewise, $M(\omega)$ is the highest value of the joint value of a match, including in the market equilibrium. Since value functions are bounded from above and from below and are continuous in their state variables, there exists a supremum of the difference between $W^U(\omega)$ and the candidate market equilibrium's $W^U(\omega, H_t)$, $\sup(W^U(\omega) - W^U(\omega, H_t)) = \epsilon_u > 0$. Similarly, there also exists a supremum for the difference between $M(\omega)$ and $M(\omega, H_t)$, $\sup(M(\omega) - M(\omega, H_t)) = \epsilon_m > 0$. In what follows, we will show that a difference in the value functions for unemployed workers (or the value functions for the joint value of the match) arbitrarily close to $\epsilon_u > 0$ (or $\epsilon_m > 0$) cannot occur. Otherwise, this will require that the difference in *tomorrow's* values to be larger than ϵ_u (or ϵ_m). In turn, this implies that an alternative equilibrium cannot exist.

From the above definition of supremum, it follows that

$$\max\{M(\omega), W^U(\omega)\} - \max\{M(\omega, H_t), W^U(\omega, H_t)\} < \max\{\epsilon_u, \epsilon_m\}.$$

Since

$$M(\omega) = y(z, x_h, A, p_o) + \beta\mathbb{E}[\max\{M(\omega'), W^U(\omega')\}],$$

and likewise for $M(\omega, H_t)$ it follows that at any (ω, H_t),

$$M(\omega) - M(\omega, H_t)) < \beta \max\{\epsilon_u, \epsilon_m\}. \tag{32}$$

Consider first the case in which $\epsilon_m \geq \epsilon_u$. For (ω, H_t) achieving a difference $M(\omega) - M(\omega, H_t) > \beta\epsilon_m$ is not possible since this will lead to a contradiction in equation (32) when $\epsilon_m > 0$.

Next consider the case in which $\epsilon_m < \epsilon_u$. Simplifying notation by dropping ω and using the prime instead of (ω, H_t), we first establish an intermediate step. At *any* (ω, H_t) it holds that

$$\lambda(\theta)(1-\eta)M + (1-\lambda(\theta)(1-\eta))W < \lambda(\theta')(1-\eta)M' + (1-\lambda(\theta')(1-\eta))W' + \epsilon_u. \tag{33}$$

There are two cases to be analysed to show the above relationship.

Case 1: Suppose that $(M'-W') \geq M-W$, then $\lambda(\theta') \geq \lambda(\theta)$. Define $K = (1-\eta)(\lambda(\theta')-\lambda(\theta))(M'-W') \geq 0$. Combining the latter with $\lambda(\theta)(1-\eta)(M-M') + (1-\lambda(\theta)(1-\eta))(W-W') \leq \epsilon_u$, it must be true that

$$\lambda(\theta)(1-\eta)(M-M') + (1-\lambda(\theta)(1-\eta))(W-W') - K \leq \epsilon_u, \tag{34}$$

from which (33) follows.

Case 2: Suppose that $(M-W) > (M'-W')$, then $\lambda(\theta) > \lambda(\theta')$. From the derivative of $\frac{d}{d(M-W)}(\lambda(\theta(M-W))(1-\eta)(M-W)) = \lambda(\theta(M-W))$, we can establish that, if $(M-W) > (M'-W')$,

$$\lambda(\theta)((M-W)-(M'-W')) > \lambda(\theta)(1-\eta)(M-W) - \lambda(\theta')(1-\eta)(M'-W') > \lambda(\theta')((M-W)-(M'-W')).$$

Since in any equilibrium (not only in the BRE) θ' depends only on $(M'-W')$ and constant parameters, we can use this relationship to establish that

$$W + \lambda(\theta)(M-W) - (W' + \lambda(\theta)(M'-W')) < \epsilon_u,$$

from which (33) follows.

The final step is to consider an (ω, H_t) such that $\max\{\epsilon_m, \beta^{-1}\epsilon_u\} < W^U(\omega) - W^U(\omega, H_t) < \epsilon_u$, where such a (ω, H_t) exists by the definition of supremum. With this in hand it is straightforward to check that the difference in *tomorrow*'s value (under the expectation sign), between $W^U(\omega)$ and $W^U(\omega, H_t)$ will not exceed ϵ_u, since term-by-term, the difference is bounded by ϵ_u. This also implies that today's difference, $W^U(\omega) - W^U(\omega, H_t)$, cannot be more than $\beta\epsilon_u > 0$, which contradicts our premise. This establishes that a difference in the value functions for unemployed workers (or the value functions for the joint value of the match) arbitrarily close to $\epsilon_u > 0$ (or $\epsilon_m > 0$) cannot occur. Hence the BRE is the unique equilibrium.

This completes the proof of Proposition 2.

4.2 Proof of Existence of a Reallocation and Separation cutoff

3.2.1 Reservation property of occupational mobility decisions, z^r

Here we show that $M(\omega)$ and $W^U(\omega)$ as derived in the proof of Proposition 2 are increasing in z. If $M(\omega)$ and $W^U(\omega)$ are continuous and bounded functions increasing in z, T maps them into increasing (bounded and continuous) functions. For employed workers $(T(\Gamma(\omega, 0))$, this follows since both $\max\{M(\omega'), W^U(\omega')\}$ and $y(.)$ are increasing in z, while the stochastic dominance of the z-productivity transition law implies higher expected z's tomorrow. For unemployed workers $(T(\Gamma(\omega, 1))$, note that the value of changing occupations $R(\omega)$ does not depend on the current z of the worker, while equation (21) implies that $D(M(\omega) - W^U(\omega)) + W^U(\omega)$ is increasing in z. Again, given stochastic dominance of the tomorrow's z when today's z is higher, $T(\Gamma(\omega, 1))$ is also increasing in z. The reservation property follows immediately, since $R(\omega)$ is constant in z and $D(M(\omega) - W^U(\omega)) + W^U(\omega)$ is increasing in z.

3.2.2 Reservation property of job separation decisions, z^s

We now show that $M(\omega) - W^U(\omega)$ is increasing in z when $\delta + \lambda(\theta(\omega)) < 1$ for the case of no human capital accumulation and occupational-wide shocks. In the calibration we show that this property holds also for the case of human capital accumulation and occupational-wide shocks.

Consider the same operator T defined in the proof of Proposition 2, but now the relevant state space is given by (A, z). Note that the value functions describing the worker's and the firm's problem, do not change, except for the fact that we are using a smaller state space. It is straightforward to verify that the derived properties of T in Lemma A.1 also apply in this case. We now want to show that this operator maps the subspace of functions Γ into itself with $M(A, z)$ increasing weakly faster in z than $W^U(A, z)$. To show this, take $M(A, z)$ and $W^U(A, z)$ such that $M(A, z) - W^U(A, z)$ is weakly increasing in z and let z^s denote a reservation productivity such that for $z < z^s$ a firm-worker match decide to terminate the match. Using $\lambda(\theta)(M - W^U) - \theta k = \lambda(\theta)(M - W^U) - \lambda'(\theta)(M - W^U)\theta = \lambda(\theta)(1 - \eta)(M - W^U)$, we construct the following difference

$$\begin{aligned} &T\Gamma(A, z, 0) - T\Gamma(A, z, 1) = \\ &\quad y(A, z) - b + \beta \mathbb{E}_{A,z}\Big[(1 - \delta)\max\{M(A', z') - W^U(A', z', 0\} - \\ &\quad \max\Big\{\int W^U(A', \tilde{z})dF(\tilde{z}) - c - W^U(A', z'), \lambda(\theta)(1 - \eta)\big(M(A', z') - W^U(A', z')\big)\Big\}\Big]. \end{aligned} \tag{35}$$

The first part of the proof shows the conditions under which $T\Gamma(A, z, 0) - T\Gamma(A, z, 1)$ is weakly increasing in z. Because the elements of the our relevant domain are restricted to have $W^U(A, z)$ increasing in z, and $M(A, z) - W^U(A, z)$ increasing in z, we can start to study the value of the term under the expectation sign, by cutting a number of different cases to consider depending on where z' is relative to the implied reservation cutoffs.

– *Case 1.* Consider the range of tomorrow's $z' \in [\underline{z}(A'), z^r(A'))$, where $z^r(A') < z^s(A')$. In this case, the term under the expectation sign in the above equation reduces to $-\int W^U(A', \tilde{z})dF(\tilde{z}) + c + W^U(A', z')$, which is increasing in z'.

– *Case 2.* Now suppose tomorrow's $z' \in [z^r(A'), z^s(A'))$. In this case, the term under the expectation sign becomes zero (as $M(A', z') - W^U(A', z') = 0$), and is therefore constant in z'.

– *Case 3.* Next suppose that $z' \in [z^s(A'), z^r(A'))$. In this case, the entire term under the expectation sign reduces to

$$(1 - \delta)(M(A', z') - W^U(A', z')) - \int W^U(A', \tilde{z})dF(\tilde{z}) + c + W^U(A', z'),$$

and, once again, is weakly increasing in z', because by supposition $M(A', z') - W^U(A', z')$ is weakly increasing in z', and so is $W^U(A', z')$ by Lemma A.1.

– *Case 4.* Finally consider the range of $z' \geq \max\{z^r(A'), z^s(A')\}$, such that in this range employed workers do not quit nor reallocate. In this case the term under the expectation sign equals

$$(1 - \delta)[M(A', z') - W^U(A', z')] - \lambda(\theta(A', z'))(1 - \eta)[M(A', z') - W^U(A', z')]. \tag{36}$$

It is easy to show using the free entry condition that $\frac{d(\lambda(\theta^*(A', z'))(1 - \eta)[M(A', z') - W^U(A', z')])}{d(M - W)} = \lambda(\theta(A', z'))$, and hence that the derivative of (36) with respect to z' is positive whenever $1 - \delta - \lambda(\theta) \geq 0$.

Given $F(z'|z) < F(z'|\tilde{z})$ for all z, z' when $z > \tilde{z}$, the independence of z of A, and that the term under the expectation sign are increasing in z', given any A', it follows that the integral in (35) is increasing in today's z. Together with $y(A, z)$ increasing in z, it must be that $T\Gamma(A, z, 1) - T\Gamma(A, z, 0)$ is also increasing in z.

To establish that the fixed point also has increasing differences in z between the first and second coordinate, we have to show that the space of this functions is closed in the space of bounded and continuous functions. In particular, consider the set of functions $\mathbb{F} \stackrel{def}{=} \{f \in \mathcal{C} | f : X \times Y \to \mathbb{R}^2, |f(x, y, 1) - f(x, y, 2)| \text{ increasing in } y\}$, where $f(.,.,1), f(.,.,2)$ denote the first and second coordinate, respectively, and $\mathcal{C}$ the metric space of bounded and continuous functions endowed with the sup-norm.

The next step in the proof is to show that fixed point of $T\Gamma(A,z,0)-T\Gamma(A,z,1)$ is also weakly increasing in z. To show we first establish the following result.

Lemma B.1: *$\mathbb{F}$ is a closed set in $\mathcal{C}$*

Proof. Consider an $f' \notin \mathbb{F}$ that is the limit of a sequence $\{f_n\}, f_n \in \mathbb{F}, \forall n \in \mathbb{N}$. Then there exists an $y_1 < y$ such that $f'(x,y_1,1)-f'(x,y_1,2) > f'(x,y,1)-f'(x,y,2)$, while $f_n(x,y_1,1)-f_n(x,y_1,2) \leq f_n(x,y,1)-f_n(x,y,2)$, for every n. Define a sequence $\{s_n\}$ with $s_n = f_n(x,y_1,1)-f_n(x,y_1,2)-(f_n(x,y,1)-f_n(x,y,2))$. Then $s_n \geq 0$, $\forall n \in \mathbb{N}$. A standard result in real analysis guarantees that for any limit s of this sequence, $s_n \to s$, it holds that $s \geq 0$. Hence $f'(x,y_1,1)-f'(x,y_1,2) \leq f'(x,y,1)-f'(x,y,2)$, contradicting the premise. □

Thus, the fixed point exhibits this property as well and the optimal quit policy is a reservation-z policy given $1-\delta-\lambda(\theta) > 0$. Since $y(A,z)$ is strictly increasing in z, the fixed point difference $M - W^U$ must also be strictly increasing in z. Furthermore, since $\lambda(\theta)$ is concave and positively valued, $\lambda'(\theta)(M-W^U) = k$ implies that job finding rate is also (weakly) increasing in z.

4.3 Proofs of the "Model Implications and Comparative Statics"

We now turn to the proofs of Lemmas 1, 2 and 3 presented in Section 1. Recall that these results were derived using a simplified version of the model without occupational human capital accumulation, setting $x_h = 1$ for all h, and without occupational-wide productivity shocks, setting $p_o = 1$ for all o. These restrictions imply that the relevant state space is (z, A). We further assumed that the z-productivity is redrawn randomly with probability $0 < (1-\gamma) < 1$ each period from cdf $F(z)$ and A is held constant. In this stationary environment, the expected value of unemployment for a worker currently with productivity z in occupation o is given by

$$\begin{aligned} W^U(A,z) &= \gamma\Big(b + \beta \max\Big\{R(A), W^U(A,z) + \max\{\lambda(\theta(A,z))(1-\eta)(M(A,z)-W^U(A,z)),0\}\Big\}\Big) \\ &+ (1-\gamma)\mathbb{E}_z[W^U(A,z)]. \end{aligned} \tag{37}$$

where the expected value of occupational mobility is given by $R(A) = -c + \mathbb{E}_z[W^U(A,z)]$. The values of employment at wage $w(A,z)$ for a worker currently with productivity z and a firm employing this worker are given by

$$W^E(A,z) = \gamma\big[w(A,z) + \beta[(1-\delta)W^E(A,z) + \delta W^U(A,z)]\big] + (1-\gamma)\mathbb{E}_z[W^E(A,z)], \tag{38}$$

$$J(A,z) = \gamma\big[y(A,z) - w(A,z) + \beta[(1-\delta)J(A,z))]\big] + (1-\gamma)\mathbb{E}_z[J(A,z)]. \tag{39}$$

The joint value of a match is then

$$M(A,z) = \gamma\big[y(A,z) + \beta[(1-\delta)M(A,z) + \delta W^U(A,z)]\big] + (1-\gamma)\mathbb{E}_z[M(A,z)]. \tag{40}$$

Proof of Lemma 1 We state the detailed version of this lemma as Lemma H.1. To simplify the analysis and without loss of generality, in what follows we let $\delta = 0$. Since we do not vary aggregate productivity to proof this lemma, we let $A = 1$ and abusing notation we refer to z as total output. This is done without loss of generality as now output only depends (and is strictly increasing and continuous) on the worker-occupation match-specific productivity. To abbreviate notation defined $W^s \equiv W^U(z^s)$.

Lemma H.1: *The differences $W^s - R$ and $z^r - z^s$, respond to changes in c, b and γ as follows*

1. *(i) $W^s - R$ is strictly increasing in c and (ii) $z^r - z^s$ is decreasing in c, strictly if $z^r > z^s$.*
2. *(i) $W^s - R$ is strictly increasing in b and (ii) $z^r - z^s$ is strictly decreasing in b.*
3. *(i) $W^s - R$ is strictly decreasing in γ and (ii) $z^r - z^s$ is strictly increasing in γ.*

We first consider the link between $W^s - R$ and $z^r - z^s$ as the parameter of interest κ (i.e. c, b or γ) changes. The reservation productivities for job separation and occupational mobility implicitly satisfy

$$M(\kappa, z^s(\kappa)) - W^s(\kappa) = 0 \tag{41}$$

$$W^U(\kappa, z^r(\kappa)) - R(\kappa) = \lambda(\theta(\kappa, z^r(\kappa)))(1-\eta)(M(\kappa, z^r(\kappa)) - W^s(\kappa)) + (W^s(\kappa) - R(\kappa)) = 0, \tag{42}$$

where (42) only applies when $R(\kappa) \geq W^s(\kappa)$ and hence only when $z^r(\kappa) \geq z^s(\kappa)$. In the case of $W^s(\kappa) > R(\kappa)$, the assumed stochastic process for z implies that $z^r(\kappa) = \underline{z}$. The latter follows as $W^U(z) = W^s(\kappa)$ for all $z < z^s$. Also note that we can use $W^s(\kappa)$ instead of $W(\kappa, z^r(\kappa))$ in (42) as the assumed stochastic process for z also implies that $W^U(z) = W^s(\kappa)$ for all $z < z^r$ when $R(\kappa) > W^s(\kappa)$.

Note that by defining

$$Z(\kappa) \equiv \mathbb{E}_z[\max\{M(\kappa, z), W^s(\kappa)\} - \max\{W^U(\kappa, z) + \lambda(\theta(\kappa, z))(1-\eta)(M(\kappa, z) - W^U(\kappa, z)), R(\kappa)\}],$$

we can rewrite equations (41) and (42) as

$$M(\kappa, z^s(\kappa)) - W^s(\kappa) = z^s - b + \beta(1-\gamma)Z(\kappa) + \beta\gamma(W^s(\kappa) - R(\kappa)) = 0.$$

Further note that

$$M(\kappa, z^r(\kappa)) - W^s(\kappa) = z^r - b + \beta(1-\gamma)Z(\kappa) + \beta\gamma(1 - \lambda(\theta(\kappa, z^r))(1-\eta))(M(\kappa, z^r(\kappa)) - W^s(\kappa)),$$

which is strictly positive and changes with κ. Using (42) leads to

$$(1-\beta\gamma)(M(\kappa, z^r(\kappa)) - W^s(\kappa)) = z^r - b + \beta(1-\gamma)Z(\kappa) + \beta\gamma(W^s(\kappa) - R(\kappa))$$

Leaving implicit the dependency on κ, we obtain that

$$\frac{d(M - W^s)}{d(W^s - R)} = \frac{d(M - W^s)}{d(\lambda(\theta)(1-\eta)(M - W^s))} \cdot \frac{d(\lambda)(1-\eta)(M - W^s)}{d(W^s - R)},$$

where the latter term equals -1, and the former term equals $\lambda^{-1}(\theta)$. This follows as Nash Bargaining and free-entry imply $q(\theta)\eta(M - W^U) = c$ and hence $\frac{d\theta}{dM - W^U} = \frac{\theta}{(1-\eta)(M - W^U)}$. Then $\lambda'(\theta)\frac{d\theta}{dM - W^U}(1-\eta)(M - W^U) + \lambda(\theta)(1-\eta)$ reduces to $\lambda(\theta)$. It then follows that when $z^r(\kappa) \geq z^s(\kappa)$, the derivative of z^r with respect to κ is given by

$$\frac{dz^r(\kappa)}{d\kappa} = -\frac{d}{d\kappa}\Big[-b + \beta(1-\gamma)Z(\kappa)\Big] - \left(\beta\gamma + \frac{1-\beta\gamma}{\lambda(\theta(\kappa, z^r(\kappa)))}\right)\frac{d(W^s(\kappa) - R(\kappa))}{d\kappa}. \tag{43}$$

We can similarly obtain that when $z^r(\kappa) \geq z^s(\kappa)$, the derivative of z^s with respect to κ is given by

$$\frac{dz^s(\kappa)}{d\kappa} = -\frac{d}{d\kappa}\Big[-b + \beta(1-\gamma)Z(\kappa)\Big] - \beta\gamma\frac{d(W^s(\kappa) - R(\kappa))}{d\kappa}. \tag{44}$$

Equations (43) and (44) then imply that when $z^r(\kappa) \geq z^s(\kappa)$ (i.e. z^r and $z^s(\kappa)$ are interior), the derivative of $z^r(\kappa) - z^s(\kappa)$ with respect to κ has the opposite sign to the derivative of $W^s(\kappa) - R(\kappa)$ with respect to κ.

As mentioned above, in the case of $W^s(\kappa) > R(\kappa)$ our simplified model implies $z^r(\kappa) = \underline{z}$. Hence changes in κ can only affect z^s, although will affect $W^s(\kappa)$ and $R(\kappa)$. Using (37) and (40) we obtain that for $W^s(\kappa) > R(\kappa)$ the reservation separation cutoff is given by

$$z^s(\kappa) = b - \frac{1-\gamma}{\gamma}\mathbb{E}_z[M(\kappa, z) - W^U(\kappa, z)].$$

Taking derivative with respect to κ then lead to

$$\frac{dz^s(\kappa)}{d\kappa} = \frac{db}{d\kappa} - \frac{d((1-\gamma)/\gamma)}{d\kappa}\mathbb{E}_z[M(\kappa,z) - W^U(\kappa,z)] + \frac{1-\gamma}{\gamma}\frac{d\mathbb{E}_z[M(\kappa,z) - W^U(\kappa,z)]}{d\kappa}. \tag{45}$$

To complete the lemma we now obtain the derivatives of $W^s(\kappa) - R(\kappa)$ and $\mathbb{E}_z[M(\kappa,z) - W^U(\kappa,z)]$ with respect to κ.

Comparative Statics with respect to c: Consider the difference $W^s - R$ and values of c such that $R \geq W^s$. In this case we have that

$$W^s = (1-\gamma)(R+c) + \gamma(b + \beta R),$$
$$W^s - R = -\gamma(1-\beta)R + (1-\gamma)c + \gamma b.$$

Suppose towards a contradiction that $d(W^s - R)/dc < 0$. The above equations imply that $\frac{dR}{dc} > \frac{(1-\gamma)}{\gamma(1-\beta)} > 0$. We will proceed by showing that under $d(W^s - R)/dc < 0$ both the expected match surplus (after a z-shock) and the match surplus for active labor markets (those with productivities that entail positive match surplus) decrease with c, which implies that the value of unemployment decreases with c, which in turn implies $\frac{dR}{dc} < 0$, which is our contradiction.

Consider an active labor market with $W^U(z) > R$, the surplus on this labor market is then given by

$$\begin{aligned} M(z) - W^U(z) = &\gamma(z - b + \beta(1 - \lambda(\theta(z))(1-\eta))(M(z) - W^U(z))) \\ &+ (1-\gamma)(\mathbb{E}[M(z) - W^U(z)] + (z - \mathbb{E}[z])), \end{aligned} \tag{46}$$

where $\mathbb{E}[M(z) - W^U(z)]$ describes the expected surplus after a z-shock (after the search stage). Note that

$$\frac{d}{d(M(z) - W^U(z))}(\lambda(\theta(z))(1-\eta)(M(z) - W^U(z))) = \lambda(\theta(z)). \tag{47}$$

As described before, this result follows as (dropping the z argument for brevity) our assumptions of Nash Bargaining and free-entry imply $(1-\eta)(M - W^U) = \frac{(1-\eta)}{\eta}J = \frac{1-\eta}{\eta}\frac{k}{q(\theta)}$, and hence $\lambda(\theta)(1-\eta)(M - W^U) = \frac{1-\eta}{\eta}k\theta$. Further, since $\frac{d\theta}{d(M - W^U)} = \frac{\eta}{1-\eta}\frac{\lambda(\theta)}{k}$, the chain rule then yields a derivative equal to $\lambda(\theta)$. This result implies that from (46) we obtain

$$0 < \frac{d(M(z) - W^U(z))}{d(\mathbb{E}[M(z) - W^U(z)])} = \frac{1-\gamma}{1 - \gamma\beta(1 - \lambda(\theta(z)))} < 1. \tag{48}$$

We now show that under $d(W^s - R)/dc < 0$, the expected match surplus decreases in c. Note that the expected match surplus measured after the search stage is given by

$$\begin{aligned} \mathbb{E}[M(z) - W^U(z)] = &\int_{z^r} z - b + \beta(1 - \lambda(\theta(z))(1-\eta))(M(z) - W^U(z))dF(z) \\ &+ \int_{z^s}^{z^r} z - b + \beta(M(z) - R)dF(z) + \int^{z^s} z - b + \beta(W^s - R)dF(z), \end{aligned} \tag{49}$$

where the $(1-\gamma)$ shock integrates out. Our contradiction supposition implies that the third term of this expression is decreasing in c. The second term, $\int_{z^s}^{z^r}[M(z) - W^U(z)]dF(z)$, can be rewritten as

$$M(z) - W^s = \gamma(z - b + \beta(M(z) - W^s + W^s - R)) + (1-\gamma)(\mathbb{E}[M(z) - W^U(z)] + z - \mathbb{E}[z]),$$

and rearranging yields

$$M(z) - W^s = \frac{\gamma}{1-\gamma\beta}(z - b + \beta(W^s - R)) + \frac{1-\gamma}{1-\gamma\beta}\mathbb{E}[M(z) - W^U(z)],$$

where $\frac{\gamma}{1-\gamma\beta}(z - b + \beta(W^s - R))$ is decreasing in c under our contradiction supposition. In the case of the first term in (49), note that (48) implies $M(z) - W^U(z)$ depends on c through $\mathbb{E}[M(z) - W^U(z)]$. Combining all the elements, (48), (49) and the last two equations, we find that

$$
\begin{aligned}
\frac{d\mathbb{E}[M(z)-W^U(z)]}{dc} &= \int_{z^r} \frac{(1-\gamma)\beta(1-\lambda(\theta(z)))}{1-\gamma\beta(1-\lambda(\theta(z)))} dF(z) \frac{d\mathbb{E}[M(z)-W^U(z)]}{dc} \\
&\quad + (F(z^r)-F(z^s)) \left(\frac{\gamma\beta}{1-\gamma\beta} \frac{d(W^s-R)}{dc} + \frac{1-\gamma}{1-\gamma\beta} \frac{d\mathbb{E}[M(z)-W^U(z)]}{dc} \right) \\
&\quad + F(z^s)\beta \frac{d(W^s-R)}{dc} \\
\iff \quad \frac{d\mathbb{E}[M(z)-W^U(z)]}{dc} &= C \cdot \frac{d(W^s-R)}{dc} < 0, \qquad (50)
\end{aligned}
$$

where C is a positive constant. Equation (48) then implies that $\frac{d[M(z)-W^U(z)]}{dc} < 0$.

Next consider $\frac{dW^U(z)}{dc}$ and $\frac{d\mathbb{E}[W^U(z)]}{dc}$. Note that for the case in which $z \leq z^r$, $W^U(z) = W^s = (1-\gamma)\mathbb{E}[W^U(z)] + \gamma(b + \beta\mathbb{E}[W^U(z)] - \beta c)$; while for $z > z^r$, $W^U(z) = (1-\gamma)\mathbb{E}[W^U(z)] + \gamma(b + \beta(\lambda(\theta(z))(1-\eta)(M(z)-W^U(z)) + \beta W^U(z)))$. It then follows that

$$
\mathbb{E}[W^U(z)] = F(z^r)(b + \beta\mathbb{E}[W^U(z)] - \beta c) + \int_{z^r} (b + \beta\lambda(\theta(z))(1-\eta)(M(z)-W^U(z)) + \beta W^U(z)) dF(z),
$$

which combined with

$$
W^U = \frac{1-\gamma}{1-\beta\gamma}\mathbb{E}[W^U(z)] + \frac{\gamma}{1-\beta\gamma}(b + \beta\lambda(\theta(z))(1-\eta)(M(z)-W^U(z))),
$$

leads to

$$
\begin{aligned}
&\left(1 - \beta F(z^r) - \beta\frac{1-\gamma}{1-\beta\gamma}(1-F(z^r))\right)\mathbb{E}[W^U(z)] \\
&\qquad = F(z^r)(b-\beta c) + \int_{z^r} \frac{b + \beta\lambda(\theta(z))(1-\eta)(M(z)-W^U(z))}{1-\beta\gamma} dF(z).
\end{aligned}
$$

Taking the derivative with respect to c, we find that both the first and second terms on the RHS of the last expression decrease with c, the latter because we have established before that $\frac{d(M(z)-W^U(z))}{dc} < 0$. It then follows that $\frac{d\mathbb{E}[W^U(z)]}{dc} < 0$ and hence $\frac{dR}{dc} = \frac{d\mathbb{E}[W^U(z)]}{dc} - 1 < 0$, which yields our desired contradiction. Equations (43) and (44) then imply that $\frac{d(z^r(c)-z^s(c))}{dc} < 0$ when $z^r > z^s$.

Now consider the difference $W^s - R$ and values of c such that $R < W^s$. In this case rest unemployment implies $W^s = \gamma(b + \beta W^s) + (1-\gamma)\mathbb{E}[W^U(z)]$. Note that here $\frac{dW^s}{dc} = 0$, since workers with productivities $z \leq z^s$ will never change occupations. Doing so implies paying a cost $c > 0$ and randomly drawing a new productivity, while by waiting a worker obtains (with probability $1-\gamma$) a free draw from the productivity distribution. Hence, $d(W^s - R)/dc = -dR/dc$. Since workers with $z > z^s$ prefer employment in their current occupation, the above arguments imply that when $R < W^s$ the expected value of unemployment, $W^U(z)$, is independent of the value the worker obtains from sampling a new z in a different occupation. It then follows that $\frac{dR}{dc} = \frac{d\mathbb{E}[W^U(z)]}{dc} - 1 = -1 < 0$, which once again yields are desired contradiction. Further note that in this case $\frac{d\mathbb{E}_z[M(c,z)-W^U(c,z)]}{dc} = 0$ and hence (45) implies that $\frac{dz^s(c)}{dc} = 0$ and $\frac{d(z^r(c)-z^s(c))}{dc} = 0$ when $z^s > z^r$.

Comparative Statics with respect to b: We proceed in the same way as in the previous case. Consider the difference $W^s - R$ such that $R \geq W^s$. Expressing W^s and W^U, for $z > z^s$, as

$$
W^s = (1-\gamma)\mathbb{E}[W^U(z)] + \gamma(b + \beta(R - W^s)) + \gamma\beta W^s \qquad (51)
$$

$$
W^U(z) = (1-\gamma)\mathbb{E}[W^U(z)] + \gamma(b + \beta(\lambda(\theta)(1-\eta)(M(z)-W^U(z)))) + \gamma\beta W^U(z), \qquad (52)
$$

we find that $W^s - \mathbb{E}[W^U(z)] = \int_{z^r}(W^s - W^U(z))dF(z)$, which in turn implies

$$W^s - R = \frac{1}{1-\gamma\beta F(z^r)}\left(-\beta\gamma\int_{z^r}\lambda(\theta(z))(1-\eta)(M(z)-W^U(z))dF(z) + (1-\gamma\beta)c\right). \tag{53}$$

That is, the difference $W^s - R$ decomposes into (i) the forgone option of searching for a job in the new occupation next period (first term in the brackets) and (ii) a sampling cost that only has to be incurred next period with probability γ, and discounted at rate β (second term in the brackets).

Next consider the relationship between $M(z) - W^U(z)$ and $\mathbb{E}[M(z) - W^U(z)]$. From (47) and (48), we find that

$$\frac{d(M(z)-W^U(z))}{db} = \frac{1-\gamma}{1-\gamma\beta(1-\lambda(\theta(z)))}\frac{d\mathbb{E}[M(z)-W^U(z)]}{db} - \frac{\gamma}{1-\gamma\beta(1-\lambda(\theta(z)))}. \tag{54}$$

Note that $\frac{d(M(z)-W^U(z))}{db}$ must have the same sign for all z, which is positive if and only if

$$\frac{d\mathbb{E}[M(z)-W^U(z)]}{db} > \frac{\gamma}{1-\gamma}.$$

Towards a contradiction, suppose $d(W^s - R)/db < 0$. Then, we have $\frac{d(W^s-R)}{db} = \frac{d(W^s-\mathbb{E}[W^U(z)])}{db}$, which equals $\frac{d}{db}\left(-\int_{z^r}\max\{W^U(z)-W^s,0\}dF(z)\right)$. By the envelope condition, the effect $\frac{dz^r}{db}$ disappears. By the previous argument and (51) subtracted by (52), it follows that $\frac{d(W^s-R)}{db} < 0$ implies $\frac{d(M(z)-W^U(z))}{db} > 0$ and by (54) that $\frac{d\mathbb{E}[M(z)-W^U(z)]}{db} > 0$.

Using the the same arguments as in (50) we find that

$$\begin{aligned}\frac{d\mathbb{E}[M(z)-W^U(z)]}{db} &= -1 + \int_{z^r}\frac{\beta(1-\lambda(\theta(z))) - \gamma\beta(1-\lambda(\theta(z)))}{1-\gamma\beta(1-\lambda(\theta(z)))}dF(z)\frac{d\mathbb{E}[M(z)-W^U(z)]}{db} \\ &\quad - \int_{z^r}\frac{\gamma\beta(1-\lambda(\theta(z)))}{1-\gamma\beta(1-\lambda(\theta(z)))}dF(z)\frac{d\mathbb{E}[M(z)-W^U(z)]}{db} \\ &\quad + (F(z^r)-F(z^s))\left(\frac{\gamma\beta^2}{1-\gamma\beta}\frac{d(W^s-R)}{db} + \frac{\beta(1-\gamma)}{1-\gamma\beta}\frac{d\mathbb{E}[M(z)-W^U(z)]}{db}\right) \\ &\quad - (F(z^r)-F(z^s))\frac{\gamma\beta}{1-\gamma\beta} + F(z^s)\beta\frac{d(W^s-R)}{db} \qquad (55) \\ \implies \frac{d\mathbb{E}[M(z)-W^U(z)]}{db} &= C_2\cdot\frac{d(W^s-R)}{db} - C_3 < 0,\end{aligned}$$

where C_2 and C_3 are positive constants. The fact that the last expression leads to $\frac{d\mathbb{E}[M(z)-W^U(z)]}{db} < 0$ yields our desired contradiction. Equations (43) and (44) then imply that $\frac{d(z^r(b)-z^s(b))}{db} < 0$ when $z^r > z^s$.

Next we consider the case in which $W^s > R$ and note that in this case equation (53) becomes

$$W^s - \mathbb{E}[W^U(z)] = -\frac{\beta\gamma}{1-\beta\gamma}\int_{z^s}\lambda(\theta(z))(1-\eta)(M(z)-W^U(z))dF(z).$$

As before, if we start from the premise that $d(W^s - R)/db < 0$, this will imply, by virtue of (54), that $\frac{d\mathbb{E}[M(z)-W^U(z)]}{db} > 0$. Noting that in this case equation (49) reduces to

$$\mathbb{E}[M(z)-W^U(z)] = \int_{z^s} z - b + \beta\lambda(\theta(z))(1-\eta)(M(z)-W^U(z))dF(z),$$

and (55) reduces to

$$\begin{aligned}\frac{d\mathbb{E}[M(z)-W^U(z)]}{db} &= -1 + \int_{z^s}\frac{\beta(1-\lambda(\theta(z))) - \gamma\beta(1-\lambda(\theta(z)))}{1-\gamma\beta(1-\lambda(\theta(z)))}dF(z)\frac{d\mathbb{E}[M(z)-W^U(z)]}{db} \\ &\quad - \int_{z^s}\frac{\gamma\beta(1-\lambda(\theta(z)))}{1-\gamma\beta(1-\lambda(\theta(z)))}dF(z)\frac{d\mathbb{E}[M(z)-W^U(z)]}{db},\end{aligned}$$

we obtain that $\frac{d\mathbb{E}[M(z)-W^U(z)]}{db} < 0$, once again yielding our desired contradiction. Equation (45) then imply that $\frac{d(z^r(b)-z^s(b))}{db} < 0$ also when $z^s > z^r$.

Comparative Statics with respect to γ: Here we also proceed in the same way as before by assuming (towards a contradiction) that $d(W^s - R)/d\gamma > 0$. We start with the case in which $R \geq W^s$. Using equation (53) we find that

$$\frac{d(W^s - R)}{d\gamma} = \frac{\beta F(z^r)}{1-\beta\gamma}(W^s - R) - \frac{1}{1-\beta\gamma F(z^r)}\left(\int_{z^r} \lambda(\theta(z))(1-\eta)(M(z) - W^U(z))dF(z) + \beta c\right)$$
$$- \int_{z^r} \beta\gamma\lambda(\theta(z))\frac{d(M(z) - W^U(z))}{d\gamma}dF(z).$$

It then follows that

$$- \int_{z^r} \beta\gamma\lambda(\theta(z))\frac{d(M(z) - W^U(z))}{d\gamma}dF(z) \geq \frac{\beta F(z^r)}{1-\beta\gamma}(R - W^s)$$
$$+ \frac{1}{1-\beta\gamma F(z^r)}\left(\int_{z^r} \lambda(\theta(z))(1-\eta)(M(z) - W^U(z))dF(z) + \beta c\right) > 0. \tag{56}$$

Using the above expressions we turn to investigate the implications of assuming $d(W^s - R)/d\gamma > 0$ for $\frac{d\mathbb{E}[M(z)-W^U(z)]}{d\gamma}$. We can rewrite (49), bringing next period's continuation values to the LHS, as

$$(1-\beta)\mathbb{E}[M(z) - W^U(z)] = \int_{z^r} z - b - \beta\lambda(\theta(z))(1-\eta)(M(z) - W^U(z))dF(z)$$
$$+ \int_{z^s}^{z^r} z - b + \beta(W^s - R)dF(z)$$
$$+ \int^{z^s} z - b + \beta(W^s - R) - \beta(M(z) - W^s)dF(z).$$

Taking derivatives with respect to γ, we find

$$(1-\beta)\frac{d\mathbb{E}[M(z) - W^U(z)]}{d\gamma} = - \int_{z^r} \beta\lambda(\theta(z))(1-\eta)\frac{d(M(z) - W^U(z))}{d\gamma}dF(z)$$
$$+ \int_{z^s}^{z^r} \beta\frac{d(W^s - R)}{d\gamma}dF(z)$$
$$+ \int^{z^s} \beta\Big(\frac{d(W^s - R)}{d\gamma} - \frac{d(M(z) - W^s)}{d\gamma}\Big)dF(z), \tag{57}$$

where the first term is positive by virtue of (56) and the second term is positive by assumption. For the third term it holds that

$$\frac{d(M(z) - W^s)}{d\gamma} = (1-\gamma)\frac{d\mathbb{E}[M(z) - W^U(z)]}{d\gamma} + \gamma\beta\frac{d(W^s - R)}{d\gamma} + (z - b + \beta(W^s - R) - \mathbb{E}[M(z) - W^U(z)]).$$

Substituting out this expression in the third line of the RHS of (57) and re-arranging implies that $\frac{d\mathbb{E}[M(z)-W^U(z)]}{d\gamma} > 0$.

From $M(z) - W^U(z) = (1-\gamma)\mathbb{E}[M(z) - W^U(z)] + \gamma(z - b + \beta(1 - \lambda(\theta(z))(1-\eta))(M(z) - W^U(z))$, it follows that for $z > z^r$

$$\beta\gamma\lambda(\theta)\frac{dM(z) - W^U(z)}{d\gamma} = \frac{\beta\gamma\lambda(\theta(z))}{1-\beta\gamma(1-\lambda(\theta(z)))}\Bigg(\Big(z - b + \beta(1 - \lambda(\theta(z))(1-\eta))(M(z) - W^U(z))$$
$$- \mathbb{E}[M(z) - W^U(z)]\Big) + (1-\gamma)\frac{d\mathbb{E}[M(z) - W^U(z)]}{d\gamma}\Bigg), \tag{58}$$

where we have used (47) and (48). Integrating this term over all $z > z^r$, we have

$$\int_{z^r} \beta\gamma\lambda(\theta(z))\frac{d(M(z) - W^U(z))}{d\gamma}dF(z) \geq \frac{\beta\gamma\lambda(\theta(z^r))}{1 - \beta\gamma(1 - \lambda(\theta(z^r)))}\bigg(\int_{z^r}(1-\gamma)\frac{d\mathbb{E}[M(z) - W^U(z)]}{d\gamma}dF(z)$$
$$+ \frac{1}{\gamma}\int_{z^r}(M(z) - W^U(z) - \mathbb{E}[M(z) - W^U(z)])dF(z)\bigg) > 0, \qquad (59)$$

where the last inequality follows from the fact that $M(z) - W^U(z) - \mathbb{E}[M(z) - W^U(z)]$ and $\frac{\beta\lambda(\theta(z))}{1-\beta\gamma+\beta\gamma\lambda(\theta(z))}$ are increasing in z. Then $\int_{z^r}(M(z) - W^U(z) - \mathbb{E}[M(z) - W^U(z)])dF(z) > 0$ and hence the LHS of (59) is positive as stated, contradicting our premise in (56). Equations (43) and (44) then imply that $\frac{d(z^r(\gamma)-z^s(\gamma))}{d\gamma} > 0$ when $z^r > z^s$.

Next we turn to investigate the implications of assuming $d(W^s - R)/d\gamma > 0$ on $\frac{d\mathbb{E}[M(z)-W^U(y)]}{d\gamma}$ for the case in which $W^s > R$ to obtain a contradiction. Here a relationship between $\frac{d(M(z)-W^U(z))}{d\gamma}$ and $\frac{d\mathbb{E}[M(z)-W^U(z)]}{d\gamma}$ can be derived directly:

$$(1-\beta)\frac{d\mathbb{E}[M(z) - W^U(z)]}{d\gamma} = -\beta\int_{z^r}\lambda(\theta(z))\frac{d(M(z) - W^U(z))}{d\gamma}dF(z). \qquad (60)$$

Using the above and (58) we obtain that

$$-(1-\beta)\frac{d\mathbb{E}[M(z) - W^U(z)]}{d\gamma} = \int_{z^s}\bigg(\frac{\beta\gamma\lambda(\theta(z))}{1 - \beta\gamma(1 - \lambda(\theta(z)))}\bigg(\Big(z - b + \beta(1 - \lambda(\theta(z))(1-\eta))(M(z) - W^U(z))$$
$$- \mathbb{E}[M(z) - W^U(z)]\Big) + (1-\gamma)\frac{d\mathbb{E}[M(z) - W^U(z)]}{d\gamma}\bigg)\bigg)dF(z),$$

which in turn can be expressed as

$$\frac{d\mathbb{E}[M(z) - W^U(z)]}{d\gamma}\left((1-\beta) + \frac{F(z^s)\beta\gamma\lambda(\theta(z))(1-\gamma)}{1 - \beta\gamma + \beta\gamma\lambda(\theta(z))}\right) =$$
$$-\int_{z^s}\bigg(\frac{\beta\gamma\lambda(\theta(z))}{1 - \beta\gamma(1 - \lambda(\theta(z)))}\Big(z - b + \beta(1 - \lambda(\theta(z))(1-\eta))(M(z) - W^U(z))$$
$$- \mathbb{E}[M(z) - W^U(z)]\Big)\bigg)dF(z) < 0. \qquad (61)$$

From (61), it then follows that $\frac{d\mathbb{E}[M(z)-W^U(z)]}{d\gamma} < 0$. Using equation (60), we obtain that $W^s - R$ is decreasing in γ, where

$$\frac{d(W^s - R)}{d\gamma} = \frac{d(W^s - \mathbb{E}[W^U(z)])}{d\gamma} = -\frac{1}{1-\beta}\int_{z^s}\gamma\lambda(\theta(z))\frac{d(M(z) - W^U(z))}{d\gamma} < 0,$$

which leads to our required contradiction. Equation (45) then imply that $\frac{d(z^r(\gamma)-z^s(\gamma))}{d\gamma} > 0$ also when $z^s > z^r$. This complete the proof for Lemma H.1.

Proof of Lemma 2 Using the same setting as in Lemma H.1 we now introduce human capital x, assuming it enters in a multiplicative way in the production function. Keeping the same notation as in Lemma H.1, let $A = 1$ such that total output is given by zx. Without loss of generality for the results derived in this lemma, normalize $x = 1$. If we have an incremental improvement in x that is occupational specific, the value of sampling will stay constant, at $R = \mathbb{E}[W^U(1, z)] - c$. However, the value of $W^s(x)$ will increases with x. Since $W^s(x) = (1-\gamma)\mathbb{E}[W^U(x, z)] + \gamma(b + \beta\max\{R, W^s(x)\})$ it follow that $W^s(x)$ is increasing in x through $\mathbb{E}[W^U(x, z)]$.

To investigate the dependence between $W^s(x)$ and $\mathbb{E}[W^U(x, z)]$ when x changes we first suppose that

$R \geq W^s(x)$. The value of unemployment when $z \geq z^r(x)$ is given by

$$W^U(x,z) = (1-\gamma)\mathbb{E}[W^U(x,z)] + \gamma(b + \beta\lambda(\theta(x,z)))(1-\eta)(M(x,z) - W^U(x,z)) + \beta W^U(x,z)),$$

while for $z^r(x) > z$ it is given by

$$W^s(x) = (1-\gamma)\mathbb{E}[W^U(x,z)] + \gamma(b + \beta R).$$

When comparing the expected value of separating with the expected value of moving to another occupations (reseting $x = 1$), the difference is given by

$$W^s(x) - \mathbb{E}[W^U(1,z)] = (1-\gamma)\mathbb{E}[W^U(x,z)] + C_4,$$

where C_4 denotes those terms that are constant in x.

As a result, $\frac{d(W^s(x) - \mathbb{E}[W^U(1,z)])}{dx} = (1-\gamma)\frac{d\mathbb{E}[W^U(x,z)]}{dx}$, where $\mathbb{E}[W^U(x,z)]$ can be expressed as

$$\mathbb{E}[W^U(x,z)] = F(z^r(x))(b + \beta R) + \int_{z^r(x)} \Big[b + \beta\lambda(\theta(x,z))(1-\eta)(M(x,z) - W^U(x,z)) + \beta W^U(x,z)\Big] dF(z).$$

Using the expression $W^U(x,z)$, it then follows that

$$\left(1 - \frac{\beta(1-\gamma)}{1-\beta\gamma}(1 - F(z^r(x)))\right)\mathbb{E}[W^U(x,z)]$$
$$= F(z^r(x))(b + \beta R) + \frac{1 - F(z^r(x))}{1-\beta\gamma}b + \int_{z^r(x)} \frac{\beta\lambda(\theta(x,z))(1-\eta)(M(x,z) - W^U(x,z))}{1-\beta\gamma} dF(z).$$

Taking the derivative with respect to x and using the envelope condition, which implies that the term premultiplying $dz^r(x)/dx$ equal zero, yields

$$\frac{d\mathbb{E}[W^U(x,z)]}{dx} = \frac{\beta \frac{d}{dx}\big(\int_{z^r(x)} [\lambda(\theta(x,z))(1-\eta)(M(x,z) - W^U(x,z))] dF(z)\big)}{1 - \beta(1 - (1-\gamma)F(z^r(x)))}. \tag{62}$$

Since the denominator is positive, the sign of $d\mathbb{E}[W^U(x,z)]/dx$ is given by the sign of its numerator. By virtue of the envelope condition and equation (47) in the proof of Lemma 1, the latter is given

$$\int_{z^r(x)} \beta\lambda(\theta(x,z)) \frac{d(M(x,z) - W^U(x,z))}{dx} dF(z),$$

where

$$\frac{d(M(x,z) - W^U(x,z))}{dx} = (1-\gamma)\frac{d\mathbb{E}[M(x,z) - W^U(x,z)]}{dx} + (1-\gamma)(z - \mathbb{E}[z]) + \gamma z$$
$$+ \beta\gamma(1 - \lambda(\theta(x,z)))\frac{d(M(x,z) - W^U(x,z))}{dx}. \tag{63}$$

Hence the numerator of (62) is given by

$$\int_{z^r(x)} \left(\frac{\beta\lambda(\theta(x,z))(1-\gamma)}{1 - \beta\gamma(1 - \lambda(\theta(x,z)))} \left(\frac{d\mathbb{E}[M(x,z) - W^U(x,z)]}{dx} + z - \mathbb{E}[z]\right) + \frac{\beta\lambda(\theta(x,z))\gamma}{1 - \beta\gamma(1 - \lambda(\theta(x,z)))} z \right) dF(z). \tag{64}$$

It is then clear that the sign of $d\mathbb{E}[W^U(x,z)]/dx$ is the same as the sign of $d\mathbb{E}[M(x,z) - W^U(x,z)]/dx$. To investigate the latter note that

$$\mathbb{E}[M(x,z) - W^U(x,z)] = \int_{\underline{z}}^{\bar{z}} (xz - b)dF(z) + \int_{z^r(x)} \beta\lambda(\theta(x,z))(1-\eta)(M(x,z) - W^U(x,z))dF(z)$$
$$+ \int_{z^s(x)}^{z^r(x)} \beta(M(x,z) - W^s(z))dF(z) + \int^{z^r(x)} \beta(W^s(x) - R)dF(z). \tag{65}$$

We will now take the derivative of this expression with respect to x to investigate its sign. For this purpose it is useful to note that

$$\int_{z^s(x)}^{z^r(x)} \frac{\beta d(M(x,y)-W^s(x))}{dx} dF(z) = \beta \int_{z^s(x)}^{z^r(x)} \Big[(1-\gamma)\frac{d\mathbb{E}[M(x,y)-W^U(x,y)]}{dx} + (1-\gamma)(y-\mathbb{E}[y]) + \gamma y$$

$$+ \beta\gamma \frac{d(M(x,y)-W^s(x))}{dx} + \beta\gamma \frac{d(W^s(x)-R)}{dx}\Big] dF(z),$$

and that

$$\int^{z^r(x)} \frac{\beta d(W^s(x)-R))}{dx} dF(z) = \frac{\beta(1-\gamma)F(z^r(x))}{1-\beta(1-(1-\gamma)F(z^r(x)))} \times \tag{66}$$

$$\int_{z^r} \left[\frac{\beta\lambda(\theta(x,z))(1-\gamma)}{1-\beta\gamma(1-\lambda(\theta(x,z)))}\left(\frac{d\mathbb{E}[M(x,z)-W^U(x,z)]}{dx} + z - \mathbb{E}[z]\right)\right.$$

$$\left. + \frac{\beta\lambda(\theta(x,z))\gamma}{1-\beta\gamma(1-\lambda(\theta(x,z)))} z\right] dF(z).$$

Using (64) and the above equations we obtain that $\frac{d\mathbb{E}[M(x,z)-W^U(x,z)]}{dx} =$

$$\mathbb{E}(z) + \int_{z^r(x)} \left(\frac{\beta\lambda(\theta(x,z))(1-\gamma)}{1-\beta\gamma(1-\lambda(\theta(x,z)))}\left(\frac{d\mathbb{E}[M(x,z)-W^U(x,z)]}{dx} + z - \mathbb{E}[z]\right) + \frac{\beta\lambda(\theta(x,z))\gamma}{1-\beta\gamma(1-\lambda(\theta(x,z)))} z\right) dF(z)$$

$$+ \frac{\beta}{1-\beta\gamma} \int_{z^s(x)}^{z^r(x)} \left[(1-\gamma)\left(\frac{d\mathbb{E}[M(x,z)-W^U(x,z)]}{dx} + z - \mathbb{E}[z]\right) + \gamma z\right] dF(z)$$

$$+ \left(\frac{\beta}{1-\beta\gamma}\frac{(1-\gamma)F(z^r(x))}{1-\beta(1-(1-\gamma)F(z^r(x)))} \times \right. \tag{67}$$

$$\left. \int_{z^r(x)} \left(\frac{\beta\lambda(\theta(x,z))(1-\gamma)}{1-\beta\gamma(1-\lambda(\theta(x,z)))}\left(\frac{d\mathbb{E}[M(x,z)-W^U(x,z)]}{dx} + z - \mathbb{E}[z]\right) + \frac{\beta\lambda(\theta(x,z))\gamma}{1-\beta\gamma(1-\lambda(\theta(x,z)))} z\right) dF(z)\right).$$

The next step is to investigate whether the sum of all terms premultiplying $\frac{d\mathbb{E}[M(x,z)-W^U(x,z)]}{dx}$ in (67) is less than 1. If this is the case, grouping all these terms and solving for $d\mathbb{E}[M(x,z)-W^U(x,z)]/dx$ will imply that $d\mathbb{E}[M(x,z)-W^U(x,z)]/dx > 0$ as the reminder terms in the RHS of (67) are positive because the integrating terms $z-\mathbb{E}[z]$ will also yield a positive term.

We proceed by noting that the terms premultiplying $\frac{d\mathbb{E}[M(x,z)-W^U(x,z)]}{dx}$ in the last three lines of (67) are larger than in (66). By replacing the premultiplication term in (66) with the corresponding term in (67) we will show that the entire term premultiplying $\frac{d\mathbb{E}[M(x,z)-W^U(x,z)]}{dx}$ in (67) is less than one. Some algebra establishes that to show the latter we need to verify that

$$\beta\left(\frac{d(M(x,y)-W^U(x,y))}{dx} + \frac{d(W^s-R)}{dx}\right) F(z^r(x)) < 1-\beta(1-\gamma)(1-F(z^r(x))). \tag{68}$$

By collecting the terms premultiplying $\frac{d\mathbb{E}[M(x,z)-W^U(x,z)]}{dx} + z - \mathbb{E}[z]$, and substituting these into the LHS of (68), we obtain that

$$F(z^r)\left(\frac{\beta(1-\gamma)(1-\beta+\beta(1-\gamma)F(z^r(x)))}{(1-\gamma\beta)(1-\beta+\beta(1-\gamma)F(z^r(x)))} + \frac{\beta(1-\gamma)(\beta(1-\gamma)(1-F(z^r(x))))}{(1-\gamma\beta)(1-\beta+\beta(1-\gamma)F(z^r(x)))}\right)$$

$$= \frac{\beta(1-\gamma)(1-\beta+\beta(1-\gamma))}{(1-\gamma\beta)(1-\beta+\beta(1-\gamma)F(z^r(x)))} F(z^r(x)). \tag{69}$$

The RHS of (68) can be rewritten as $1-\beta+\beta\gamma+\beta(1-\gamma)F(z^r(x))$. Noting that

$$\beta\gamma > \frac{\beta\gamma(\beta(1-\gamma))\Big(1-\beta+\beta(1-\gamma)\Big)F(z^r(x))}{(1-\gamma\beta)(1-\beta+\beta(1-\gamma)F(z^r(x)))} \tag{70}$$

$$\beta(1-\gamma)F(z^r(x)) > \frac{\beta(1-\gamma)(1-\gamma\beta)\big(1-\beta+\beta(1-\gamma)\big)F(z^r(x))}{(1-\gamma\beta)(1-\beta+\beta F(z^r(x)))}, \tag{71}$$

and adding them up we find that the RHS is precisely the term in (69). Therefore $\beta\gamma+\beta(1-\gamma)F(z^r(x))$ is larger than (69), from which the desired result follows as the remaining term, $1-\beta$, is larger than zero and the desired inequality is slack. This yields $d\mathbb{E}[M(x,z)-W^U(x,z)]/dx > 0$. It then follows from (63) that $d(M(x,z)-W^U(x,z))/dx > 0$ and therefore, by (62), that $d\mathbb{E}[W^U(x,z)]/dx > 0$ and $d(W^s(x)-R)/dx > 0$.

We now need to consider the case in which $W^s(x) > R$. Here we once again obtain that

$$\frac{d(W^s(x)-R)}{dx} = (1-\gamma)\frac{d\mathbb{E}[W^U(x,z)]}{dx},$$

where

$$(1-\beta)\frac{d\mathbb{E}[W^U(x,z)]}{dx} = \int_{z^s(x)} \beta\lambda(\theta(x,z))\frac{d(M(x,z)-W^U(x,z))}{dx}dF(z) \tag{72}$$

and

$$\begin{aligned}\frac{d(M(x,z)-W^U(x,z))}{dx} &= (1-\gamma)\frac{d\mathbb{E}[M(x,z)-W^U(x,z)]}{dx} + (1-\gamma)(\mathbb{E}[z]-z)\\ &\quad + \gamma(z+\beta(1-\lambda(\theta(x,z))))\left(\frac{d(M(x,z)-W^U(x,z))}{dx}\right),\end{aligned} \tag{73}$$

while the expected surplus evolves according to

$$\frac{d\mathbb{E}[M(x,z)-W^U(x,z)]}{dx} = \int_{z^s(x)} z+\beta(1-\lambda(\theta(x,z)))\frac{d(M(x,z)-W^U(x,z))}{dx}. \tag{74}$$

Substituting (73) into (74), it follows that $\frac{d\mathbb{E}[M(x,z)-W^U(x,z)]}{dx} > 0$, from which in turn it follows that (72) is also positive.

Finally, we investigate the implications of a change in x on $z^s(x)$ and $z^r(x)$. When $z^r(x) > z^s(x)$, these reservation cutoff functions are given by

$$\begin{aligned} M(x,z^s(x)) - W^s(x) &= 0\\ \lambda(\theta(x,z^r(x)))(1-\eta)(M(x,z^r(x))-W^U(x,z^r(x))) + (W^s(x)-R) &= 0.\end{aligned}$$

Further we can obtain that

$$\begin{aligned} M(x,z^s(x)) - W^s(x) &= xz^s(x) - b + \beta(1-\gamma)\mathbb{E}[\max\{M(x,z)-W^U(x,z), W^s(x)-R\}]\\ &\quad + \beta\gamma(W^s(x)-R)\\ M(x,z^r(x)) - W^s(x) &= xz^r(x) - b + \beta(1-\gamma)\mathbb{E}[\max\{M(x,z)-W^U(x,z), W^s-R\}]\\ &\quad + \beta\gamma(1-\lambda(\theta(x,z^r(x))(1-\eta)(M(x,z^r(x))-W^s(x)).\end{aligned}$$

Taking derivatives with respect to x we find that

$$z^s(x) + \beta(1-\gamma)\frac{d}{dx}\Big(\mathbb{E}[\max\{M(x,z)-W^U(x,z), W^s(x)-R\}]\Big) + \beta\gamma\frac{d(W^s(x)-R)}{dx} + x\frac{dz^s(x)}{dx} = 0$$

$$\frac{\lambda(\theta(x,z))}{1-\beta\gamma(1-\lambda(\theta(x,z)))}\left(z^r(x) + \beta(1-\gamma)\frac{d}{dx}\Big(\mathbb{E}[\max\{M(x,z)-W^U(x,z), W^s(x)-R\}]\Big) + x\frac{dz^r(x)}{dx}\right)$$
$$+\frac{d(W^s(x)-R)}{dx} = 0$$

Since $\frac{d(W^s(x)-R)}{dx} > 0$, this implies that

$$z^s(x) + x\frac{dz^s(x)}{dx} \geq z^r(x) + x\frac{dz^r(x)}{dx} + \frac{1-\beta\gamma}{\lambda(\theta(x,z))}\frac{d(W^s(x)-R)}{dx}, \tag{75}$$

which implies that, evaluated at $x = 1$,

$$\frac{dz^s(x)}{dx} - \frac{dz^r(x)}{dx} > z^r(x) - z^s(x).$$

The above result then yields that for $z^r > z^s$, more occupational human capital brings closer together the two cutoffs. For $z^r < z^s$, it holds in this simplified setting that z^r jumps to the corner, $z^r = \underline{z}$, while z^s decreases with x. This completes the proof of Lemma 2.

Proof of Lemma 3 To proof this lemma we use the equations (37) - (40), where we have assumed no human capital accumulation. Further, to simplify we let $\gamma = 1$ such that the z-productivity does not change. We also focus on the case in which $z^r > z^s$ such that $z^r, z^s \in (\underline{z}, \bar{z})$ and without loss of generality let $\delta = 0$. In this stationary environment, described by A and z, note that at labor markets whose z-productivities equal z^r it holds that

$$\int_{\underline{z}}^{\bar{z}} W^U(A,z)dF(z) - c = W^U(A,z^r) + \lambda(\theta(A,z^r))(W^E(A,z^r) - W^U(A,z^r)). \tag{76}$$

Further, the expected value of unemployment for workers with $z < z^r$ is given by $W^U(A,z) = W^U(A,z^r)$. This follows since over this range of z's,

$$\int_{\underline{z}}^{\bar{z}} W^U(A,z)dF(z) - c \geq W^U(A,z) + \lambda(\theta(A,z))(W^E(A,z) - W^U(A,z)))$$

and unemployed workers prefer change occupations the period after arrival. On the other hand, the value of unemployment for workers with $z \geq z^r$ is given by

$$W^U(A,z) = \frac{b + \beta\lambda(\theta(A,z))(W^E(A,z) - W^U(A,z))}{1-\beta}.$$

Equation (76) can then be expressed as

$$\beta\int_{\underline{z}}^{\bar{z}} \Big(\max\{\lambda(\theta(A,z))(W^E(A,z) - W^U(A,z)), \lambda(\theta(A,z^r))(W^E(A,z^r) - W^U(A,z^r))\}\Big)dF(z)$$
$$= \lambda(\theta(A,z^r))(W^E(A,z^r) - W^U(A,z^r)) + c(1-\beta).$$

Using $\eta\lambda(\theta(A,z))(W^E(A,z) - W^U(A,z)) = (1-\eta)\lambda(\theta(A,z))J(A,z) = (1-\eta)\theta(A,z)k$, we have that $R(A) = W^U(A,z^r(A))$ can be expressed as

$$\frac{(1-\eta)k}{\eta}\left(\beta\int_{\underline{z}}^{\bar{z}} \max\{\theta(A,z), \theta(A,z^r)\}dF(z)\right) - c(1-\beta) = \frac{(1-\eta)k}{\eta}\theta(A,z^r), \tag{77}$$

where the LHS describes the net benefit of moving to a different occupation and the RHS the benefit of staying in the same occupation. With this derivation we now analyse under what conditions $dz^r/dA > 0$ and compare it to the competitive case.

To obtain the dz^r/dA from (77) we first use the free-entry condition and the Cobb-Douglas specification for the matching function to obtain an implicit function that solves for θ,

$$\theta(A,z)^{\eta-1}\frac{\eta(y(A,z) - b) - \beta(1-\eta)\theta(A,z)k}{1-\beta} - k \equiv E(\theta; A, z) = 0,$$

where differentiation then implies that θ is increasing in both A and z,

$$\frac{d\theta(A,z)}{dj} = \frac{\theta(A,z)}{w(A,z) - b}\frac{dy_j(A,z)}{dj}, \quad \text{for } j = A, z,$$

and it is straightforward to show that in this stationary environment the wage equation is given by

$$w(A,z) = (1-\eta)y(A,z) + \eta b + \beta(1-\eta)\theta(A,z)k.$$

Next, to make precise the comparison with an economy in which occupations are segmented in many competitive labor markets, consider the same environment as above, with the exception that workers can match instantly with firms. As before, we assume free entry (without vacancy costs), and constant returns to scale production. This implies that every worker will earn his marginal product $y(A,z)$. Importantly, we keep the reallocation frictions the same: workers who change occupations have to forgo production for a period, and arrive at a random labor market in a different occupation at the end of the period. In the simple case of permanent productivity (A,z), the value of being in a labor market with z, conditional on $y(A,z) > b$, is $W^c(A,z) = y(A,z)/(1-\beta)$, where to simplify we have not considered job destruction shocks.

Block recursiveness, given the free entry condition, is preserved and decisions are only functions of (A,z). Unemployed workers optimally choose to change occupations, and the optimal policy is a reservation quality, z_c^r, characterised by the following equation

$$\beta \int \max\{y(A,z), y(A,z_c^r)\} dF(z) + (b-c)(1-\beta) = y(A,z_c^r).$$

The LHS describes the net benefit of switching occupations, while the RHS the value of of staying employed earning y in the (reservation) labor market.

Rearranging the above equations, the reservation z-productivities for the competitive and frictional case satisfy, respectively,

$$b + \beta \int_{\underline{z}}^{\bar{z}} \frac{\max\{y(A,z), y(A,z_c^r)\}}{1-\beta} dF(z) - \frac{y(A,z_c^r)}{1-\beta} - c_c = 0$$

$$\frac{(1-\eta)k}{\eta}\left(\beta \int_{\underline{z}}^{\bar{z}} \frac{\max\{\theta(A,z), \theta(A,z^r)\}}{1-\beta} dF(z) - \frac{\theta(A,z^r)}{1-\beta}\right) - c_s = 0.$$

Using these equations, the response of the reservation z-productivity, for the competitive, and the frictional case is then given by

$$\frac{dz_c^r}{dA} = \frac{\beta F(z_c^r)\frac{y_A(A,z_c^r)}{y_z(A,z_c^r)} + \beta \int_{z_c^r}^{\bar{z}} \frac{y_A(A,z)}{y_z(A,z_c^r)} dF(z) - \frac{y_A(A,z_c^r)}{y_z(A,z_c^r)}}{1-\beta F(z_c^r)}$$

$$\frac{dz^r}{dA} = \frac{\beta F(z^r)\frac{y_A(A,z^r)}{y_z(A,z^r)} + \beta \int_{z^r}^{\bar{z}} \frac{\theta(A,z)(w(A,z^r)-b)}{\theta(A,z^r)(w(A,z)-b)}\frac{y_A(A,z)}{y_z(A,z^r)} dF(z) - \frac{y_A(A,z^r)}{y_z(A,z^r)}}{1-\beta F(z^r)}$$

These are the expression shown in Lemma 3, where we have used the fact that $\frac{\theta(A,z)}{(w(A,z)-b)}\frac{(1-\eta)k}{\eta} = \frac{\lambda(\theta(A,z))}{1-\beta+\beta\lambda(\theta(A,z))}$ by virtue of

$$\eta \frac{w(A,z)-b}{1-\beta+\beta\lambda(\theta(A,z))} = \frac{(1-\eta)k}{q(\theta(p,z))},$$

which follows from the combination of the free entry condition and the Hosios. This completes the proof of Lemma 3.

Implications of Lemma 3 We now show two implications of Lemma 3. First we show that search frictions adds a procyclical force to occupational mobility decisions. Choosing c_c, c_s appropriately such that $z_c^r = z^r$, the above expressions imply that $\frac{dz^r}{dA} > \frac{dz_c^r}{dA}$ if $\frac{\theta(A,z)}{w(A,z)-b} > \frac{\theta(A,z^r)}{w(A,z^r)-b}$, $\forall\ z > z^r$. Hence we now need to show that $\frac{\theta(A,z)}{w(A,z)-b}$ is increasing in z.

$$\frac{d\left(\frac{\theta(A,z)}{w(A,z)-b}\right)}{dz} = \frac{\theta y_z(A,z)}{(w(A,z)-b)^2} - \theta\left(\frac{(1-\eta)+(1-\eta)\beta\frac{\theta}{w(A,z)-b}k}{(w(A,z)-b)^2}\right) y_z(A,z),$$

which has the same sign as $\eta - (1-\eta)\beta k \frac{\theta}{w(A,z)-b}$ and the same sign as

$$\eta(1-\eta)(y(A,z)-b) + \eta(1-\eta)\beta\theta k - (1-\eta)\beta\theta k$$
$$= (1-\eta)(\eta(y(A,z)-b) - (1-\eta)\beta\theta k).$$

But $\eta(y(A,z)-b)-(1-\eta)\beta\theta k = y(A,z) - w(A,z) > 0$ and we have established that search frictions within labor markets make occupational mobility decisions more procyclical relative to the competitive benchmark, given the same $F(z)$ and the same initial reservation productivity $z^r = z^r_c$.

Second, we show that impact of the production function on the procyclicality of occupational mobility decisions. Here we want to show that with search frictions, if the production function is modular or supermodular (i.e. $y_{Az} \geq 0$), there exists a $c \geq 0$ under which occupational mobility decisions are procyclical. With competitive markets, if the production function is modular, occupational mobility decisions are countercyclical, for any $\beta < 1$ and $c \geq 0$.

Note that modularity implies that $y_A(A,z) = y_A(A,\tilde{z}),\ \forall z > \tilde{z}$; while supermodularity implies $y_A(A,z) \geq y_A(A,\tilde{z}),\ \forall z > \tilde{z}$. Hence modularity implies

$$\frac{dz^r_c}{dA} = \frac{1}{1-\beta F(z^r_c)} \frac{y_A(A,z^r_c)}{y_z(A,z^r_c)} \left(\beta F(z^r_c) + \beta \int_{z^r_c}^{\bar{z}} \frac{y_A(A,z)}{y_A(A,z^r_c)} dF(z) - 1 \right) < 0,\ \forall\, \beta < 1.$$

In the case with frictions,

$$\frac{dz^r}{dA} = \frac{1}{1-\beta F(z^r)} \frac{y_A(A,z^r)}{y_z(A,z^r)} \left(\beta F(z^r) + \beta \int_{z^r}^{\bar{z}} \frac{\theta(A,z)(w(A,z^r)-b)}{\theta(A,z^r)(w(A,z)-b)} \frac{y_A(A,z)}{y_A(A,z^r)} dF(z) - 1 \right).$$

If we can show that the integral becomes large enough, for c large enough, to dominate the other terms, we have established the claim. First note that $\frac{y_A(A,z)}{y_A(A,z^r)}$ is weakly larger than 1, for $z > z^r$ by the (super)modularity of the production function. Next consider the term $\frac{\theta(A,z)(w(A,z^r)-b)}{\theta(A,z^r)(w(A,z)-b)}$. Note that

$$\lim_{z \downarrow y^{-1}(b;A)} \frac{\theta(A,z)}{w(A,z)-b} = \frac{\lambda(\theta(A,z))}{1-\beta+\beta\lambda(\theta(A,z))} = 0,$$

because $\theta(A,z) \downarrow 0$, as $y(A,z^r) \downarrow b$. Hence, fixing a z such that $y(A,z) > b$, $\frac{\theta(A,z)(w(A,z^r)-b)}{\theta(A,z^r)(w(A,z)-b)} \to \infty$, as $y(a,z^r) \downarrow b$. Since this holds for any z over which is integrated, the integral term becomes unboundedly large, making dz^r/dA strictly positive if reservation z^r is low enough. Since the integral rises continuously but slower in z^r than the also continuous term $\frac{\theta(A,z^r)}{1-\beta}$, it can be readily be established that z^r depends continuously on c, and strictly negatively so as long as $y(A,z^r) > b$ and $F(z)$ has full support. Moreover, for some $\bar{c}$ large enough, $y(A,\underline{z}^r) = b$, where $\underline{z}^r$ is a lower bound for z^r. Hence, as $c \uparrow \underline{z}^r$, $dz^r/dA > 0$.

Job Separations Here we show the derivation of the slope of z^s for the case $z^r(A) > z^s(A)$ for all A described in Section 1.2. Note that $R(A) = \frac{b+\beta\theta(A,z^r(A))k(1-\eta)/\eta}{1-\beta}$. The derivative of this function with respect to A equals

$$\frac{\beta k(1-\eta)}{(1-\beta)\eta} \frac{\theta}{w(A,z^r(A))-b} \left(y_A(A,z^r(A)) + y_z(A,z^r(A)) \frac{dz^r(A)}{dA} \right). \tag{78}$$

Since $w(A,z^r(A)) - b = (W^E(A,z^r(A)) - W^U(A,z^r(A)))(1-\beta(1-\delta)+\beta\lambda(\theta(A,z^r(A))))$ and $\frac{\theta\beta k(1-\eta)}{(1-\beta)\eta} = \beta\lambda(\theta(A,z^r(A)))(W^E(A,z^r(A)) - W^U(A,z^r(A))$, we find that (78) reduces to

$$\frac{\beta\lambda(\theta(A,z^r(A)))}{1-\beta(1-\delta)+\beta\lambda(\theta(A,z^r(A))} \left(y_A(A,z^r(A)) + y_z(A,z^r(A)) \frac{dz^r(A)}{dA} \right). \tag{79}$$

From the cutoff condition for separation, we find $(1-\beta)R(A) = y(A,z^s(A))$. Taking the derivative with respect to A implies the left side equals (79) and the right side equals $y_A(A,z^s(A)) + y_z(A,z^s(A)) \frac{dz^s(A)}{dA}$.

Rearranging yields the equation in Section 1.2.

4.4 Proofs of Competitive Search Model

Proof of Lemma 4 Fix any occupation o and consider a firm that promised $W \geq W^U(A,z)$ to the worker with productivity z, delivers this value in such a way that his profit $J(A,z,W)$ is maximized, i.e. solving (14). Now consider an alternative offer $\hat{W} \neq W$, which is also acceptable to the unemployed worker, and likewise maximizes the profit given $\hat{W}$ for the firm, $J(A,z,\hat{W})$. Then an alternative policy that delivers W by using the optimal policy for $\hat{W}$, but transfers additionally $W - \hat{W}$ to the worker in the first period must be weakly less optimal, which using the risk neutrality of the worker, results in

$$J(A,z,W) \geq J(A,z,\hat{W}) - (W - \hat{W})$$

Likewise, an analogue reasoning implies $J(A,z,\hat{W}) \geq J(A,z,W) - (\hat{W} - W)$, which together with the previous equation implies

$$J(A,z,W) \geq J(A,z,\hat{W}) - (W - \hat{W}) \geq J(A,z,W) - (\hat{W} - W) - (W - \hat{W}),$$

and hence it must be that $J(A,z,W) = J(A,z,\hat{W}) - (W - \hat{W})$, for *all* $M(A,z) \geq W, \hat{W} \geq W^U$. Differentiability of J with slope -1 follows immediately. Moreover, $M(A,z,W) = W + J(A,z,\hat{W}) + \hat{W} - W = M(A,z,\hat{W}) \equiv M(A,z)$. Finally, if $W'(A',z') < W^U(A',z')$ is offered tomorrow while $M(A',z') > W^U(A',z')$, it is a profitable deviation to offer $W^U(A',z')$, since $M(A',z') - W^U(A',z') = J(A',z',W^U(A',z')) > 0$ is feasible. This completes the proof of Lemma 4.

Proof of Lemma 5 Fix any occupation o and consider productivity z, such that $M(A,z) - W^U(A,z) > 0$. Since we confine ourselves to this productivity, with known continuation values $J(A,z,W)$ and $W^U(A,z)$ in the production stage, we drop the dependence on A,z for ease of notation. Free entry implies $k = q(\theta)J(W) \Rightarrow \frac{dW}{d\theta} < 0$. Notice that it follows that the maximand of workers in (12), subject to (16) is continuous in W, and provided $M > W^U$, has a zero at $W = M$ and at $W = W^U$, and a strictly positive value for intermediate W: hence the problem has an interior maximum on $[W^U, M]$. What remains to be shown is that the first order conditions are sufficient for the maximum, and the set of maximizers is singular.

Solving the worker's problem of posting an optimal value subject to tightness implied by the free entry condition yields the following first order conditions (with multiplier μ):

$$\begin{aligned} \lambda'(\theta)[W - W^U] - \mu q'(\theta)J(W) &= 0 \\ \lambda(\theta) - \mu q(\theta)J'(W) &= 0 \\ k - q(\theta)J(W) &= 0 \end{aligned}$$

Using the constant returns to scale property of the matching function, one has $q(\theta) = \lambda(\theta)/\theta$. This implies, combining the three equations above, to solve out μ and $J(W)$,

$$0 = \lambda'(\theta)[W(\theta) - W^U] + \frac{\theta q'(\theta)}{q(\theta)}k \equiv G(\theta),$$

where we have written W as a function of θ, as implied by the free entry condition. Then, one can derive $G'(\theta)$ as

$$G'(\theta) = \lambda''(\theta)[W(\theta) - W^U] + \lambda'(\theta)W'(\theta) + \frac{d\varepsilon_{q,\theta}(\theta)}{d\theta},$$

where $\varepsilon_{q,\theta}(\theta)$ denotes the elasticity of the vacancy filling rate with respect to θ and

$$\frac{d\varepsilon_{q,\theta}(\theta)}{d\theta} = \frac{q'(\theta)k}{q(\theta)} + \frac{\theta[q''(\theta)q(\theta) - q'(\theta)^2]k}{q(\theta)^2}.$$

Since the first two terms in the RHS are strictly negative, G' is strictly negative when $\varepsilon_{q,\theta}(\theta) \leq 0$. The latter then guarantees there is a unique $\tilde{W}_f$ and corresponding θ that maximizes the worker's problem. This completes the proof of Lemma 5.

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