Algebraic manipulations in maintaining `indyset`
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Algebraic explanation of what's happening:

- The n'th basis vector: $B_n =$ `basis[n][.]`, $B_{nr} =$ `basis[n][r]`, $B_{nP_r} = 0$ for $n > r$.
- the matrix to expand the basis vectors in terms of input vectors: $D_{nr} =$ `basexpand[n][r]`, $D_{np} = 0$ for $n < p$
- the permutation of the columns: $P_r =$ `ordervar[r]`
- the input gradients: $E_{nr} =$ n'th `candval->gradient[r]`
- the candleft: $C_{nr} =$ n'th `candleft[r]` at end of `isindy`.
  Note $C_{nP_r} = 0$ for $n > r$
- the candexpand: $K_{nr} =$ n'th `candexpand[r]` at end of `isindy`. This is the coefficient of B_r in the expansion of E_n, so $K_{nr} = 0$ for $n \leq r$

The n'th `isindy` (counting only those subsequently `placelast`'ed) calculates K_{nk} so that

$$C_{nr} = E_{nr} - \sum_k K_{nk} B_{kr}$$

will have no components in the columns $r = P_q$ for the first n q's (start n at 0), or

$$C_{nP_r} = 0 \text{ for } r < n.$$

Note as B_{k,P_r} is guaranteed 0 for $k > r$, this can be done by successive subtractions, with K_{nk} given by the P_k'th component of what is left, divided by B_{k,P_k}.

So

$$E_{nr} = B_{nr} + \sum_q^{n-1} K_{nk} B_{kr}.$$

For discussion, introduce $\tilde{K}_{nq} = K_{nq}$ for $n \neq q$, $\tilde{K}_{nq} = 1$ for $n = q$. $\tilde{K}$ does not appear in the program. Then if a new gradient E_n has been `placelast`'ed,

$$E_{nr} = \sum_q \tilde{K}_{nq} B_{qr}.$$

When it is decided to `placelast`, the C_{nr} becomes B_{nr}, and the matrix D_{nr} has its n'th row entered as

$$D_{nr} = \begin{cases} 0 & \text{for} \quad r > n \\ 1 & \text{for} \quad r = n \\ -\sum_q K_{nq} D_{qr} & \text{for} \quad r < n \end{cases}.$$

This matrix is the inverse of $\tilde{K}$, for consider

$$M_{\alpha\beta} = \left(\tilde{K} D\right)_{\alpha\beta} = \sum_q \tilde{K}_{\alpha q} D_{q\beta}.$$

As the matrix elements are 0 if $\alpha < q$ or $q < \beta$, there are no terms in the sum for $\alpha < \beta$, and then $M_{\alpha\beta} = 0$. When $\alpha = \beta$ there is only the $q = \alpha$ term and it is one. For $\alpha > \beta$, explicitly writing the $q = \alpha$ term separately, we have

$$M_{\alpha\beta} = D_{\alpha\beta} + \sum_{q=\beta}^{\alpha-1} K_{\alpha q} D_{q\beta} = 0$$

by the recursive definition of D. The purpose of this is that we can then convert an expansion for a new candidate gradient G, which we know in terms of the basis vectors B_k, into an expansion in terms of the original equations. As $E = \tilde{K} \cdot B$, we have $B = D \cdot E$, and

$$\begin{aligned} G_r &= \sum_k K_k B_{kr} = \sum_k K_k \sum_q D_{kq} E_{qr} = \sum_q \left(\sum_k K_k D_{kq} \right) E_{qr} \\ &= \sum_q R_q E_{qr}, \end{aligned}$$

the coefficients $R_q = \sum_k K_k D_{kq}$ are the expansion coefficients we need.