Exercise 15

Exercise 17

(i)

$$X_f \cap X_g = V(f)^c \cap V(g)^c = (V(f) \cup V(g))^c = (V(fg))^c = X_{fg}$$

(ii)

$$X_f = \emptyset \Rightarrow V(f) = X \Rightarrow f^n = 0$$

f is in all primes, (nilradical) so was nilpotent.
f is nilpotent so in the nilradical so in all primes, so no primes do not contain f.

(iii)

$$X_f = X \Rightarrow V(f) = \emptyset$$

no primes contain the ideal that f generates. $yf = 1$
no primes can contain a unit. so every prime will not contain f.

(iv) by equality in Exercise 15

(v)

(vi)

(vii)

Exercise 19

If R = nilradical, if $f, g \in R$, $X_f, X_g \neq \emptyset$, $X_f \cap X_g = X_{fg}$.
Since R is prime, $X_{fg} \neq \emptyset$, every open set can be written as basic open sets.
Spec(A) is irreducible, non empty open sets intersect non trivially, $X_{fg} \neq \emptyset$ so $fg \in R$ so R is prime.

Exercise 20

Exercise 21

(i) $\Rightarrow$ a prime not containing f will map to a prime not containing $\phi(f)$
$\Leftarrow$ a prime not containing $\phi(f)$ will map to a prime not containing f
Hence, $\to \phi^{\star-1}(X_f) = Y_{\phi(f)}$, so $\phi^{\star-1}$ is continuous.

(ii) Same as above.

(iii)

(iv) ϕ is surjective, so $A/\ker\phi \cong B$, primes containing the kernel are primes in the A quotient, paricularly they are primes in B.
So $\phi^\star$ is bijective. Remains to show two sided continuity.

(v)

(vi)

(vii)

Exercise 23

(i) X_f is open. We show that $(X_f)^c = X_{1-f} \to X_f = V(1-f)$.

$$(X_f)^c = V(f) = (V(1-f))^c.$$

is prime contains f and $1-f$ then $f = 0$ and $f = 1$ in A/P which contradicts P being a prime. $f(1-f) = f^2 - f = f - f = 0$.

(ii)

$$X_{f_1} \cup \cdots \cup X_{f_n} = V(1-f_1) \cup \cdots \cup V(1-f_n) = V((1-f_1)\cdots(1-f_n)) = X_{1-(1-f_1)\cdots(1-f_n)}$$

By part (i)

(iii) The hint gives the solution.

(iv)