Computer Science Department
CS660 – Mathematical Foundations of Analytics (CRN# 22921)
Spring 2025

Project #2 / Due 03-Apr-2025

Let's create our own code to compute the **Eigenvalues**, **Eigenvectors**, **Principal Components**, and **Singular Values** of a Matrix.
We then compare the run time of our modules to Python's standard (built-in) libraries.

Write **Python** scripts by utilizing the **sklearn, NumPy** libraries in order to complete the following tasks:

1_ Computation of **Eigenvalues, Eigenvectors** of a Matrix:
 Compare your code's runtime vs Python's NumPy/eig:
 https://numpy.org/doc/stable/reference/generated/numpy.linalg.eig.html

2_ Computation of **Principal Components** of a Matrix:
 Compare your code's runtime vs Python's NumPy/pca:
 https://scikit-learn.org/stable/modules/generated/sklearn.decomposition.PCA.html

3_ Computation of **Singular Values** of a Matrix:
 Compare your code's runtime vs Python's NumPy/svd:
 https://numpy.org/doc/stable/reference/generated/numpy.linalg.svd.html

You should create modules (independent functions, with their respective parameters) for each task.

Important Notes:

1_ Output of eigenvalues/eigenvector computation should look like this:
Given as input the matrix below:

$$\begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The eigenvalues are:

$$\lambda_1 = \frac{3}{2} - \frac{\sqrt{1+4\sqrt{2}}}{2}, \lambda_2 = \frac{\sqrt{1+4\sqrt{2}}}{2} + \frac{3}{2}, \lambda_3 = 0$$

Then, find the corresponding eigenvector for each eigenvalue.

2_ Steps of a Matrix's **Principal Components** computation are:

- Standardize / Normalize data (mean centering)
- Compute the covariance matrix
- Compute the eigenvectors and eigenvalues of the covariance matrix to identify the PCs
- Compute the explained variance and select N PCs
- Create feature vector with all PCs, sorted by their importance

3_ Steps of a Matrix's **Singular Values** computation, creation of U, Σ, V* matrices are:

Calculating the SVD consists of finding the eigenvalues and eigenvectors of $\boldsymbol{AA^T}$ and $\boldsymbol{A^TA}$. The eigenvectors of A^TA make up the columns of $\boldsymbol{V}$, the eigenvectors of AA^T make up the columns of $\boldsymbol{U}$.
Also, the **singular values in Σ are square roots of eigenvalues from $\boldsymbol{AA^T}$ or $\boldsymbol{A^TA}$.** The singular values are the diagonal entries of the S matrix and are arranged in descending order. The singular values are always real numbers. If the matrix A is a real matrix, then U and V are also real.

Find the transpose matrix:

$$\begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Multiple matrix with its transpose:

$$W = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of matrix W:

$$\textbf{Eigenvalue: } 8\textbf{, eigenvector: } \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

$$\textbf{Eigenvalue: } 2\textbf{, eigenvector: } \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

$$\textbf{Eigenvalue: } 0\textbf{, eigenvector: } \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Find the square roots of the nonzero eigenvalues (σ_i):

$$\sigma_1 = 2\sqrt{2}$$

$$\sigma_2 = \sqrt{2}$$

The Σ matrix is a zero matrix with σ_i on its diagonal: $\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The columns of the matrix U are the normalized (unit) vectors: $U = \begin{bmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{3} & -\frac{\sqrt{3}}{3} & 0 \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} \end{bmatrix}$

$$V = \begin{bmatrix} \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ \frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{3} & \frac{1}{2} \end{bmatrix}$$

The matrices U, Σ, and V are such that the initial matrix $\begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} = U\Sigma V^T$

4_To time the execution of a command within a Notebook just put **%%time** at the top of the cell.

```
%%time

def fib(n):
    if n <= 1:
        return n
    return fib(n - 1) + fib(n - 2)
fib(32)
```

```
CPU times: user 421 ms, sys: 174 µs, total: 421 ms
Wall time: 420 ms
```

```
2178309
```

5_To time the execution of a command within Python code, do the following:

```
import time
start = time.time()
"the code you want to test stays here"
end = time.time()
print(end - start)
```