Polar Pi

The Best Algebra, Geometry, & “SAT” Math Book Ever Written

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(A 16 year old future Math legend☺)

Dedication to:

Jesus Christ (*The only God!!!*)

Paul the Apostle

and

A *cool dude* working for NASA who said*, "my best days productive days ☺"*

As you might already know, any summative high school assessment (i.e., the SAT/ACT) is really a comprehensive test on Algebra 1 and Geometry. So, this book is written with the purpose of preparing you for these tests. However, this book is really a self-sufficient summary of the first 2.5 years of high school mathematics. While some topics seem to have been treated briefly, we have not sacrificed rigor and were instructed more by the idea of, sometimes, less is more. Equally importantly, as you should already have built some familiarity with most of the topics in this book (if you're preparing for a standardized exam), this book should be a source to add rigor and iron out any kinks in your understanding.

Despite this book not being just a test prep book, we have heavily consulted the College board for most of the content found in the book. In fact, if you look at the two full length practice tests in this book, they look identical to a real SAT test in every way. It is for this reason that we have retained the letters S(tandard) A(ptitude) T(est) in the title of the book.

To the end of using the SAT as a template, the overall bucket of topics is as in the SAT, in the following three main categories.

- Heart of Algebra
- Problem Solving and Data Analysis
- Passport to Advanced Mathematics

For anyone with adequate exposure to high school mathematics, these categories don't come as surprises. For example, *equations of lines* fall in the "Heart of Algebra" whereas *Long Division of Polynomials* falls in "Passport to Advanced Mathematics."

Now, there is a fourth category that we have created. This Category is called, "Other.," for reasons that should be obvious. For instance, things like *Complex* (*Imaginary*) numbers fall in the "Other" category.

While the current SAT (2022) has a section where you are allowed a calculator, the "Digital SAT" to come in 2024 will allow students to use calculators on all sections. Now, it goes without saying that if you don't know how to reason mathematically, your calculator alone is no good. This is to say, the questions and ideas in this book are things you will need to master regardless of your ability on a calculator and whether the assessment you face is digital or pencil and paper.

Finally, each topic is accompanied by sufficient practice. We have carefully chosen questions to show the diversity of mastery required for you to own the topics whether your end goal is the SAT or otherwise. For example, to practice "the *Equation of a Circle*," we have created about 30 questions but to practice *Complex Numbers*, we have created fewer questions. The practice questions you face are very well aligned to the lessons and equally importantly, they are scaffolded (presented in order of difficulty, as difficulty can be subjective depending on the type of learner you are and the topic.)

Should you need additional practice beyond the practice questions following every lesson, we have included, as we already mentioned, 2 full length Practice Math Tests as well as two Mini Tests and one short summative test which appear in the book in the reverse of this specific listing we just provided. As with every question in this book, the tests are all accompanied by detailed solutions where you will find novel items, reinforcement of the lessons, and in a few places, new self-contained lessons within the solutions. Taken together with whatever other resource you choose what we have in this book should be all you need. Praying to God (*Jesus Christ*) WILL also help☺ The invisible hand is always a good bet☺

Let us begin, *shall we*?

Other

The 2 Standardized Exam Strategies

So, as a place to start, these two ideas are good when your back is against the walls so to speak, "*picking numbers* and *back-solving*."

These are two strategies that you can employ on any multiple-choice standardized exam (so, GRE, GMAT, Online or otherwise, not just SAT.) But the two strategies don't work for every question. "*Back-solving*" is a more robust strategy than *picking numbers* (that is to say, *back-solving* works on more questions than *picking numbers*.) *Picking numbers* works on problems where the question is posed with a lot of undefined variables. The examples to follow will make what we are saying a lot clearer.

Let's begin with "*back-solving*."

Back-solving

what value(s) of x form a solution to the equation below?

$$\frac{x+3}{2}=\frac{-1}{x}$$

A. -2
B. -1
C. -1 and -2
D. 2

Solution

These types of equations (*rational equations* to be discussed in more detail later), are best solved by cross-multiplying at the start.

Doing so, we have

$$\to x(x+3)=-1(2)$$

$$\to x^2+3x=-2$$

$$\to x^2+3x+2=0$$

Now, you should know how to solve the quadratic equation in the last step above. *Quadratic equations* like the one on the last step above will be discussed in greater detail, later in this book.

Since we know $x^2+3x+2=(x+2)(x+1)$, we have

$\to (x+2)(x+1)=0$, meaning $x=-2$ and $x=-1$ are the *solutions* to our equation.

But obviously, to get to this conclusion, you would have needed to take your Algebra class seriously. So, *how about if you had no clue how to even start solving this question*? Well, you can use "*back-solving*" to bail yourself out.

Here is how it works! Plug back each of the values of x from the given answer choices into the given equation right above them.

Now, while we have heard of speculation that "C" is usually the correct answer choice. "C"laims like these are unfounded. As such, we will start with answer choice A. Although it is important to note that if a question says, "*what is the biggest value?*" it should make sense to start with the answer choice that has *the biggest value*.

Answer choice A claims that $x=-2$. So, to test this answer choice, we will plug-in this value of x back into the original equation and see if it makes the equality hold.

If $x=-2$, then we have

$$\frac{x+3}{2}=\frac{-1}{x} \to \frac{-2+3}{2}=\frac{-1}{-2} \to \frac{1}{2}=\frac{1}{2}$$

Ah, that works. Unfortunately, we cannot say we are done because there is an answer choice that says that $x=-1$ is also solution (answer choice C. to be specific.) The fact that $x=-1$ is stated as the only solution in answer choice B should give you a nagging suspicion that the correct answer maybe C.

So we proceed to plug-in $x=-1$, and we have

$\frac{x+3}{2}=\frac{-1}{x} \rightarrow \frac{-1+3}{2}=\frac{-1}{-1} \rightarrow \frac{2}{2}=\frac{1}{1}$

Ah, our suspicion was not misplaced. We are done now! C is in fact the correct answer. As you can *C*, this strategy is one that can be used on many questions (especially if the questions involve equations of any sort.)

Picking-Numbers

While much more limited, *Picking numbers* is the other great strategy you can use to get unstuck. As *Picking-numbers* is a strategy suitable to some of the more "mind-bending" questions, it is as valuable as *back-solving* despite its limitations.

Let's get on with the example.

For a trip to *Boston*, n people were to pay a total of d dollars each. If two of the n people chose to drop out of the trip on the last day, in terms of n and d, how much more would each of the remaining people have to pay to make the trip possible?

A. $\frac{d}{(n-2)}$

B. $\frac{d}{(n-2)}-\frac{1}{n}$

C. $\frac{d}{(n-2)}-\frac{1}{n}$

D. $\frac{2d}{n(n-2)}$

Solution

If you know what you are doing, you will know that initially, each of the n people had to pay $\frac{d}{n}$ dollars. Then, since 2 people dropped out, there are now $n-2$ people and the cost per person has now gone up to $\frac{d}{(n-2)}$. So, the additional contribution from each person is the positive difference between these two quantities.

That is

$$\frac{d}{(n-2)}-\frac{d}{n}$$

Since there is no answer choice that looks like this, we get common denominators and simplify.

$$\rightarrow \frac{dn}{(n-2)n}-\frac{d(n-2)}{n(n-2)}=\frac{dn-(dn-2d)}{n(n-2)}$$

$$\rightarrow \frac{dn-dn+2d}{n(n-2)}=\frac{2d}{n(n-2)}$$

Ah, it is obvious now that the correct answer is D.

But wait, what about "*Picking-numbers*." Ah, yes, yes. Let's get to it.

So, to execute "*Picking-numbers*," we will pick numbers for the variables. In doing so, we should pick numbers that are friendly with one another.

So, here, for the number of people n, we can pick 10. To pick numbers that are friendly with one another, we can pick $\$100$ for the total amount of dollars. So, now, we see that initially, each person had to pay, $\frac{\$100}{10}=\$10$. But, after 2 people drop out, we have $10-2=8$ people. So, the 8 people would pay $\frac{\$100}{8}=\$12.50$. So, the additional contribution from each person is $\$12.50-\$10=\$2.50$.

Substituting the picked numbers $n=10$ and $d=\$100$ in each of the answer choices, what we are looking for is the answer choice that gives us $\$2.50$. Caution: Sometimes, when we do this, there may be more than one answer choice that works. So, what you ought to do in that situation is make a new set of choices for the numbers (again, friendly numbers.) Usually, just two sets of "*picked numbers*" is sufficient to rule out all other answer choices except the correct one.

For the sake of brevity here, let's only check that the correct answer will give us $\$2.50$. The reader can check the other answer choices to see that the picked numbers will fail to give $\$2.50$.

$$\frac{2d}{n(n-2)} \rightarrow \frac{2(\$100)}{10(10-2)}=\frac{2(\$100)}{10(8)}=\frac{\$100}{10(4)}=\frac{\$10}{4}=\$2.50$$

Let us now commence with the meat and bones of the Algebra and Geometry content.

The Heart of Algebra

Linear Equations

Algebra really starts with a discussion of lines. As such, why buck the trend?

Let us start with lines.

By definition, *a line is determined by two points*. Therefore, all you need to write the equation of a line are two points.

Alternatively, *a point on the line* and the *Slope* (*Gradient* or *Trajectory*) of the line are equally sufficient to determine the *equation of the line*.

The Point-Slope, Slope-Intercept, the Standard Forms

The Point-Slope form

You will get a repetition of what we say here in the problems and solutions to follow. Therefore, we will keep the discussion in this section (what follows) briefer than would otherwise be.

The *Point-Slope* form of a line <u>should be your go to way of writing the equation of a line</u>. This is so because, from it (the *Point-Slope* form), you can efficiently get to the other two forms −¿the *Slope-Intercept* and the *Standard Forms*.

The *point-Slope form* is
$y - y_1 = m(x - x_1)$.

As we already forecasted, in this form, all you need is a *slope* and *a point* (it is in the name☺) Obviously, if you are given *two points*, you can get to the *slope*, and you will have *two points to choose from* for *the point* in the *Point-Slope* form.

Therefore, every line can be written in this form.

Example

What is the equation of the line $\acute{AB}$ with points $A(-2, -1)$ and $B(2,3)$?

Everyone reading this book should already know the *slope* formula for a line.

$$m=\frac{y_2-y_1}{x_2-x_1}$$

Therefore, choosing one of the two points to be the *ordered pair* (x_1,y_1) and choosing the other to be (x_2,y_2), we write the following.

It doesn't matter which of the two points we choose to be (x_1,y_1) and which we choose to be (x_2,y_2).

$$\rightarrow m=\frac{3-(-1)}{2-(-2)}=\frac{4}{4}=1$$

While we randomly chose the two points, *the slope* simplified nicely. Since the point (no pun) is to demonstrate the approach and not getting stuck in cumbersome arithmetic, this will do.

Now that we have the slope, for the *point* in the *point-slope form*, since we have two points to choose from, we are at luxury. Just pick one of them!

Using $m=1$ and $(x_1,y_1)=(-2,-1)$

$$y-y_1=m(x-x_1).$$

$$\rightarrow y-(-1)=1(x-(-2)).$$

$$\rightarrow y+1=1(x+2).$$

$$\rightarrow y+1=x+2$$

As some people have trust issues, let's verify that we would get to the same place if we used the other point.

Again, using $m=1$ but this time, the other point $(x_1,y_1)=(2,3)$

$$y-y_1=m(x-x_1).$$

$$\rightarrow y-3=1(x-2).$$

$$\rightarrow y-3=x-2.$$

Now, it is not immediately clear that the equation using the first point is the same as the equation using the second point. We can easily check that they are the same!

In both cases, if we isolate y, we get what is called the *Slope-Intercept form*.

Isolating y in the first equation, we get $y=x+2-1 \rightarrow y=x+1$

Doing the same in the second equation, we get $y=x-2+3 \rightarrow y=x+1$

Ah, our faith is restored. Our faith in math anyway! We've always had undying faith in God (Jesus Christ☺)

Notice that when y is isolated, we have $y=mx+b$ where in the case of $y=x+1$, $m=1$ and $b=1$. This is what we call the *Slope Intercept Form* of the equation of a line.

At this point, you have learned how to go from the *Point-Slope* form to the *Slope-Intercept* form. Just start with the *Point-Slope form* and isolate y to get the *Slope-Intercept form*. But to get to the *Standard form*, a different example where the *slope* is not one may be more helpful.

The *Standard-Form* is $ax+by=c$.

We can get to this form either from the *Point-Slope* form or from the *Slope-Intercept form*.

For example, returning to $y=x+1$ (the example we had just looked at), we can get to the *standard-form* easily. Bring back the x to the left side so that the constant on the right side is isolated.

Doing so merely requires that we subtract x from both sides of $y=x+1$ to write, $-x+y=1$

Now, we should multiply both sides of this last equation by -1 to get a positive x, for better math grammar. <u>Yes, there is such a thing as math grammar</u>. But if we multiply both sides by negative one, we will have a negative constant on the right side. That is even pooper math-grammar. So, we will consider it done with $-x+y=1$ where we would claim that $a=-1$, $b=1$, and $c=1$ in the *standard-form* writing, $ax+by=c$.

Before we conclude this section and commence with the practice, let us look at one more example.

Example

Consider the point $(-2,3)$ and the *slope* $m=\frac{7}{2}$

The *Point-Slope* form of this line is easy to get to. It is literally a matter of plugging in the point and the slope into the following formula, a formula that we have already seen.

$y-y_1=m(x-x_1)$.

Here, $(x_1,x_2)=(-2,3)$ and therefore, we have

$y-3=\frac{7}{2}(x-(-2))$

$\rightarrow y-3=\frac{7}{2}(x+2)$

As far as the *Point-Slope* form, we are done here.

Now, as additional practice, to get to the *Slope – Intercept* form from here, all we must do is isolate y.

To do this, it requires that we first distribute $\frac{7}{2}$ to the two terms inside the parenthesis (on the right side of the equal sign in $y-3=\frac{7}{2}(x+2)$)

Thereafter, we should add 3 to both sides and so we have the following:

$\rightarrow y-3=\frac{7}{2}x+7$

$\rightarrow y=\frac{7}{2}x+7+3$

$\rightarrow y=\frac{7}{2}x+10$

Now onto the *Slope-Intercept* form.

While we can get to the *Standard Form* from the *Point-Slope Form*, it is easiest to get to the *Standard Form* the *Slope-Intercept Form*.

If you must see how to do it both ways, let's start with the *Point-Slope form* and get to the *Standard Form* first.

Remember, the *Standard Form* is $ax+by=c$.

In *Point-Slope Form*, at the conclusion of the last problem, we had

$y-3=\frac{7}{2}(x+2)$

Multiplying both side by 2*, we have*

$2(y-3)=2\left(\frac{7}{2}(x+2)\right)$

$\rightarrow 2y-6=7(x+2)$

$\rightarrow 2y-6=7x+14$

Getting the variables on one side and the constant on the other side, we have:

$\rightarrow -7x+2y=20$

We are now done!

We have gotten to $ax+by=c$, where $a=-7$, $b=2$ and $c=20$.

So then, let's get to the *Standard Form* from *Slope-Intercept form*. This as we said already is the preferable place to start and get to the *Standard Form*.

In *Slope-Intercept* form, we had $y=\frac{7}{2}x+10$.

Multiplying both sides by 2, we get $2y=7x+20$. Then, subtracting $7x$ from both sides, we have:

$\rightarrow -7x+2y=20$.

Getting to the *Slope-Intercept* form from the *Standard Form* is not hard. It is a matter of isolating y.

But there is rarely, if ever, a need for us to move in this direction. As we have already said, the *Point-Slope* form should be your "go to" and perhaps also deserves being the most popular.

Each form has *its own advantage*. Let us talk in more detail to illustrate the advantage of each.

The *Point-Slope Form*

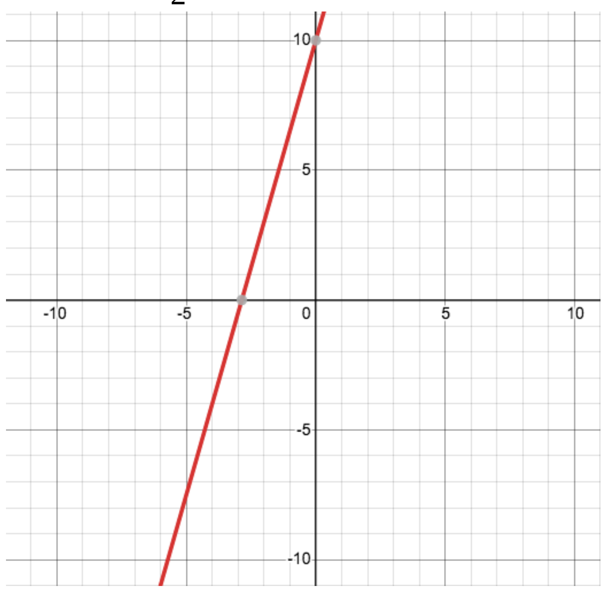
- Easiest form to get to. You only need to know *the slope* and *a point* but otherwise, it is immediate.
- From this form, it is also easy to get to the other forms efficiently.

The *Slope-Intercept Form*

- Form this form, we can *graph* a line very efficiently.
- Easy to get to the *Standard Form* from this form.

Why it is easy to get to the *graph* of a line from this form is worth discussing a bit more.

Let us look at the *graph* of $y=\frac{7}{2}x+10$ (the line we were just discussing☺)



As each square is of width 1 unit, you can see that 10 represents the $y-$¿*intercept*. That is where the graph crosses the $y-$¿*axis*. Additionally, from any *lattice point* [(x,y) pair where both x and y are integers], we can add to the y the *numerator* of our *slope* and add to the x the *denominator* of our *slope*, to find another *lattice point*. Alternatively, from any lattice point, we can subtract from the $y-$¿*coordiante* the numerator of our *slope* and subtract from the $x-$¿coordinate the denominator of our *slope*, to find another lattice point.

So, looking at the graph above, it is easy to see that $(0,10)$ is a point on our line. Now, doing the latter of the two in what we just said, we would write $(0-2,10-7)=(-2,3)$. You should be able to eyeball the point $(-2,3)$ as a point on our line.

We can also verify that this point is on our line because it must satisfy the *equation* of our line.

$$y=\frac{7}{2}x+10 \rightarrow 3=\frac{7}{2}(-2)+10 \rightarrow 3=-7+10=3.$$

So then, let's conclude by making a case for the *Standard Fdform*.

Earlier, we already showed how we get from $y=\frac{7}{2}x+10$ to $-7x+2y=20$, the *Slope-Intercept* and the *Standard Form* respectively.

Now then, what is the St*andard Form* good for?

- Just plug in $x=0$ to get the $y-¿intercept$. This looks as follows.

$x=0 \rightarrow -7(0)+2y=20 \rightarrow 2y=20 \rightarrow y=10$

$(0,10)$ is therefore the point where the line crosses the y-axis ($y-¿intercept$.)

- Equally convenient is getting to the *x-intercept* (the point where the line crosses the *x-axis*. This in fact might be the real bonus from the *Standard Form*.)

  This time, to get the *x-intercept*, we plug in 0 for y.

$y=0 \rightarrow -7x+2(0)=20 \rightarrow -7x=20 \rightarrow x=\frac{-20}{7}$

So, we see that the line needs to cross the $x-¿axis$ at $\left(\frac{-20}{7},0\right)$. If you look up at the graph on the previous page, you can see that this is that near the point $(-3,0)$ that I was convinced was actually $(-3,0)$.

- And finally, getting to *the slope* is not hard even in this form, although easier in the other forms as we can quickly eyeball it. We have *two ways*; the latter is useful when considering a *system of equations*. Remember a system of *linear* (straight line) *equations* is almost always written with the *standard form* $ax+by=c$.

  One way to get to the *Slope* from the *Standard-form* is as follows.

We can use the *x-intercept* and *y-intercepts* as two points and apply the *Slope* equation.

Another way to get to the slope $m=\frac{-a}{b}$ from $ax+by=c$.

For example, we know the *Slope* of $y=\frac{7}{2}x+10$ is $m=\frac{7}{2}$.

Now, if we are given the *Standard-form* of this same line, it would be $-7x+2y=20$. So, using the *slope formula* we just stated, we have

$$m=\frac{-a}{b} \to m=\frac{-(-7)}{2}=\frac{7}{2}.$$

Now a few examples*!!!! !! !!!!!!!*

Is this *point on the line*?

- A line in the xy- plane passes through the point $(1,-1)$ and has a *slope* of $\frac{3}{7}$. Which of the following points lies on the line?

A) $\left(\frac{10}{3},-\frac{10}{7}\right)$

B) $\left(\frac{10}{3},-\frac{10}{3}\right)$

C) $\left(\frac{10}{3},0\right)$

D) $\left(0,\frac{10}{7}\right)$

Solution

We need to first write the equation of the line. As we have said, the friendliest form for the *equation of a line* is point slope form (also most efficient) if you have a point (x_1,y_1) and the *slope* m.

Recall that the *Point-Slope form* of a line is given by
$y-y_1=m(x-x_1)$

We know $(x_1,y_1)=(1,-1)$ and $m=\frac{3}{7}$. So, plug and play.

$\to y-(-1)=\frac{3}{7}(x-1)$

$\to y+1=\frac{3}{7}(x-1)$

Now, to finish answering the question, we need to decide which of the points is on the line. *But what does it mean for a point to be on a line? It means, the "x" and "y" coordinates of that point satisfy the equation of the line*. Because the equation of the line defines the relationship of infinitely many collections of points, any of these points on the line must make the equation true.

As such, unfortunately, we need to as quickly as possible test each of the points into the equation of the line (the last equation above.)

If we do just that (test each pair of x and y values), we should find that the correct answer is:

B. $\left(\frac{10}{3}, 0\right)$.

$\rightarrow 0+1=\frac{3}{7}\left(\frac{10}{7}-1\right)$

$\rightarrow 1=\frac{3}{7}\left(\frac{10}{3}-1\right)$

$\rightarrow 1=\frac{3}{7}\left(\frac{10}{3}-\frac{3}{3}\right)=\frac{3}{7}\left(\frac{7}{3}\right)=1$

-

t	-2	3	5	9
$g(t)$	-1	-6	-8	-12

The table above shows some values of the linear function $g(x)$. Which of the following defines g?

A) $y-1=-1(x+2)$
B) $y+1=x+2$
C) $y+1=-x+2$
D) $y+1=-1(x+2)$

Solution

This is a standard basic Algebra question. We are basically being asked to write the equation of a line. We should do this either in *Standard Form* or in *Point-Slope form*.

As I have said with abusive repetition, the *Point-Slope* form is the most efficient.

For the *slope*, we can use any two pair of (x, y) that we have available. Since we have four points, we are at luxury. For no other reason than the fact that they appear first, we will use $(-2,-1)$ and $(3,-6)$. Even though it says $(t, g(t))$, we don't let that confuse us. It is just like (x, y).

Now, with $(x_1, y_1)=(-2,-1)$ and $(x_2, y_2)=(3,-6)$

$$m=\frac{-6-(-1)}{3-(-2)}=\frac{-5}{5}=-1$$

Once again, recall that the *Point-Slope* form is:

$y-y_1=m(x-x_1)$

$\rightarrow y-(-1)=-1(x-(-2))$

$\rightarrow y+1=-1(x+2)$

So, we can immediately see that the *correct answer* is D.

- © In the xy- plane, the line determined by the points $(a,3)$ and $(-12,a)$ passes through *the point* $(1,-1)$. What is one possible value of a?

Solution

This is a straightforward question even though it seems sophisticated.

First, we need to come up with the *Equation of the Line*. The only thing tricky about this task is that we don't know the value of a. But this is not a big deal, we can proceed as usual.

Find the *slope* from the two points given. We can use $(x_1,y_1)=(a,3)$ and $(x_2,y_2)=(-12,a)$

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{a-3}{-12-a}$$

Now, we can use this m but we can do better (simplify it) by multiplying both the *numerator* and *denominator* by negative one.

$$m=\frac{(a-3)(-1)}{(-12-a)(-1)}$$

$$m=\frac{3-a}{12+a}$$

Now, as always, it is best and quickest to write the equation of a line in *Point-Slope* form.

Using $(a,3)$ as point and the slope that we just found, we remind ourselves of the *Point-Slope* form and plug and play.

$y-y_1=m(x-x_1)$

$\rightarrow y-3=\frac{3-a}{12+a}(x-a)$

Now, we can try to simplify this, but it is not worth it and more importantly, doesn't help much if we do so. We should instead look at what else we are given to help us solve this problem.

We are told that $(1,-1)$ is *a point on this line*, so we can plug in $y=-1$ and $x=1$ to write:

$$-1-3=\frac{3-a}{12+a}(1-a)$$

$$\rightarrow -4=\frac{(3-a)(1-a)}{(12+a)}$$

Multiplying both sides of this last equation by $12+a$, we have

$$\rightarrow -4(12+a)=(3-a)(1-a)$$

$$-48-4a=3-3a-a+a^2$$

Getting everything on one side, we have:

$$0=48+3+4a-4a+a^2$$

$$\rightarrow 0=51+a^2$$

Here, to finish the job, we must solve a *quadratic*. While we haven't yet covered *quadratics*, the task here is straight forward, subtracting 51 from both sides of this last equation, we have

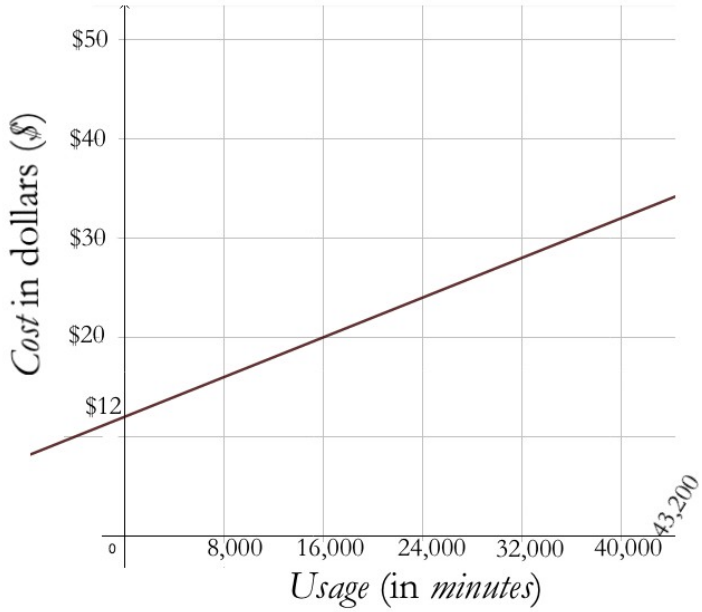
$$\rightarrow -51=a^2$$

$$\rightarrow a=\pm\sqrt{-51}=\pm\sqrt{-1}\sqrt{51}=\pm i\sqrt{51}$$

Problem Solving and Data Analysis

Interpretation of Slope and the Intercept

The graph below shows the graph of *usage* (*in minutes*) vs *price (in dollars)* for a customer's phone bill at a specific cell-phone company (this was like 7 years ago☺)



So, in a real-life situation as represented by the graph above, what is the meaning of the slope?

But first, let's have the slope handy. Using two safely chosen (from the graph above) points $(x_1, y_1)=(0,12)$ and $(x_2, y_2)=(1600,20)$, we can get a pretty accurate slope for the line graphed.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{20-12}{16000-0}=\frac{8}{16000}=\frac{8}{8\cdot 2\cdot 1000}=\frac{1}{2000}$$

We claim that $m=\frac{1}{200}$ is the change along the $y-$¿axis (*the vertical change*) for a unit change (*change of length one*) along the $x-$¿axis (*horizontal axis*.)

We can verify this. We can see that if we move 16000 units (whatever units maybe, but in this case minutes) along the horizontal axis, we should move 8 units north if the line is accurate.

Look at the line, if you go horizontally from 0 to 16000, you change vertically from "$y=12$" at "$x=0$" to "$y=20$" at "$x=16000$." Meaning, a net change along the vertical axis of 8.

But mathematically, this should mean that $16{,}000m=8$

$$\rightarrow 16{,}000\left(\frac{1}{2000}\right)=16\left(\frac{1}{2}\right)=8$$

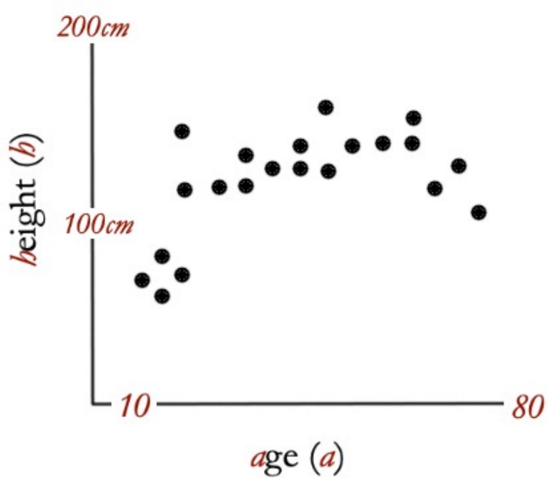
So, this is how you should go about *Slope-interpretation* questions. This is an important skill to have other than the basic skill you need Algebra-wise. Equipped with this, you need to adopt to the situation (the real-life linear model you're looking at.) For instance, having said all that we did, we can now easily answer a question about *the meaning of the slope* on the graph of *Cost* vs *Usage*.

The meaning is, "*it costs* $\frac{1}{200}$ *dollars or* 0.005*cents to talk for* 1 *minute*" in this cellphone plan.

Line of Best Fit and the Correlation Coefficient

If you are at all familiar with *correlation* between two variables, one of the first things that you pick up is the idea that "*correlation is not causation*." This is very true. We will see what is precisely meant by this in the discussion to follow shortly.

Consider the following two variables, "*height* (h)" and *age* $(a.)$ Often times, we take great care to put on the *horizontal axis* what is called the "*independent variable*" and on the *vertical axis*, the "*dependent variable*." In the two aforementioned variables, the roles can be swapped without affecting the correlation analysis or the general shape of the *Scatter Plot* that we expect to see. A *Scatter Plot* for these two variables would look as follows.

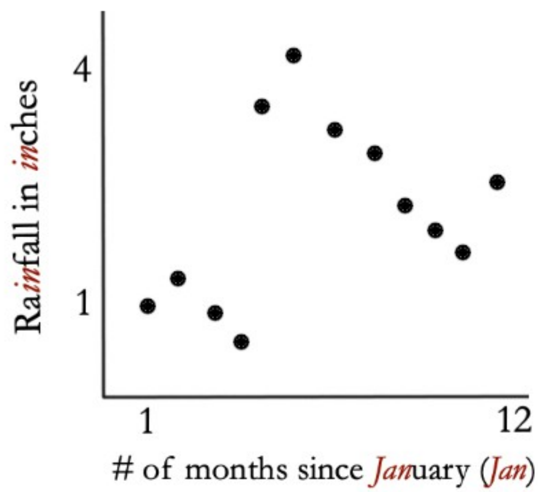


Looking at the *Scatter Plot* above, as we expect, the older that one (a person) is, the taller they are. But the association is also clearly not linear (*meaning, it is not always the case that the older you get, the taller you are.* For example, there are some 15-year-olds that are taller than 60-year-olds or more mature we might add on a philosophical tangent.)

If it was clearly the case that getting older means that you are taller, we would say that the relationship between *age* and *height* is linear and direct.

Direct means that an increase in one variable causes an increase in the other or that a decrease in one causes a decrease in the other. Equally importantly, we cannot say from this Scattered Plot (an accurate depiction of real life) that there is *a causal relationship* (that adage that correlation is causation that we had previously mentioned.) Finally, as *correlation* frequently comes with a *correlation coefficient* (you can calculate this coefficient using most scientific calculators), we would estimate the correlation coefficient on the plot above to be $\approx r = 0.6$.

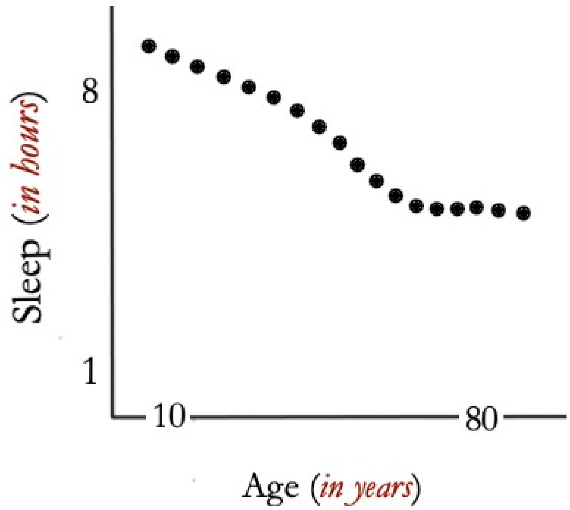
What we should take away from this *correlation coefficient* are 2 things. First, the fact that it is a positive number means that the association between the two variables would fall along a *positively sloped* line if we were to *fit a line* to the Scattered Plot (we will look at what it means to do *fit* a line using one of the four plots here.) We should also not forget to mention, since this first plot concerns people, you can count the dots to figure out how many people were surveyed (asked them their *age* and *height*) to gather the information displayed above.



In this second Scatter Plot, we have a very accurate representation of *Rainfall* in the US. Obviously, *Rainfall* vs which *month of the year* (*January* is 1 and *December* is 12) depends on Geography among other factors.

In the US, *June* is the rainiest month and *April* is the least rainy month. The 12 dots above represent each month of the year and as such, we shouldn't have expected more dots than what we see. It is hard to say what the association is, it is clearly *nonlinear*.

From the discussion we had earlier, you should gather that the *correlation coefficient* maybe negative in certain circumstances. As such, it is tempting to read a strong *negative association* between the *months* and *Rainfall* seeing the cluster of 8 months on the top right. Unfortunately, the *January* through *April* on the bottom right throw off such a claim. With these latter months included, we can try to consider a *positive correlation* by imaging how we can *fit* a *positively sloped* line from the bottom left to the upper right. This however is unreasonable as the dots are too spread out. Once we discuss how we can fit a line to a *Scatter Plot*, you will see precisely why the spread of the dots is of significance. In fact, *the line of best fit* will follow the analysis of our next to last *Scatter Plot* just below.



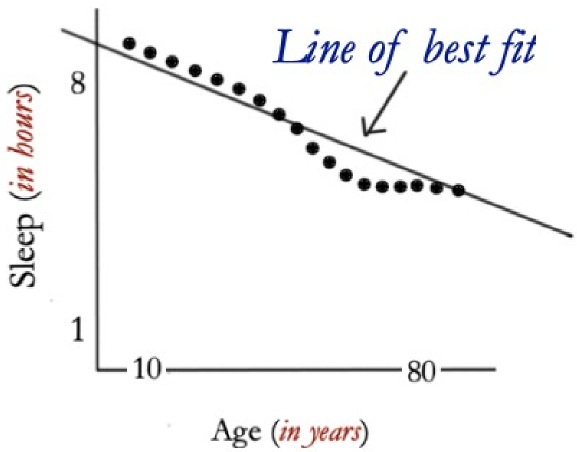
Have you ever gone to your local park at 6AM to jog or practice soccer, it seems like everyone there is 50 years or older. I suppose it maybe because they are at an age where they must take care of their body. But I suppose it is also because, scientifically, older people need less sleep. There is the popular suggestion that one needs 8 hours of sleep a night, but at the age of 60, it is more like six hours. This is all to say that the *Scatter Plot* above is supported by science.

It is *not strictly linear*, and it is not expected to be, but it is linear. If we had to assign a *correlation coefficient* to the plot without using a device, we estimate $r \approx -0.87$. What this tells us is that if we *fit a line* (*a line of best fit*), the line would be *negatively sloped* and most of the dots on the *Scatter Plot* would land on the line. You should also be able to understand that the tapering off that is visible towards the end is suggestive of the fact that past the age about 70 (super seniors), one requires only a steady five or six hours of sleep.

Having said all of this, this feels like a natural place to discuss a *line of best fit* before we move on to our last and final *Scatter Plot*, so, let's get to it.

Line of Best Fit

Let us *fit a line* to the third *Scatter Plot* that we just looked at.



A *line of best fit* (the *line we fit to our Scatter Plot*) can be done without a Calculator. It is a tad bit imprecise if we do it by hand, but it will still get the job done. To understand how a calculator does it, we would need a lengthy discussion.

In a nutshell, the calculator tries to "minimize the vertical distance between the line and the dots." When you do it by hand, here is the objective: you want your line to go through at least two points, but the main goal is to leave as many points above the line as you have below the line.

If you consider the number of dots above the line we have drawn and the number of dots below it (in the figure above), you can see that we have succeed in this goal.

Now, as the *line of best fit* is first and foremost a line, you can find *a Slope-Intercept* form of the line. As a matter of fact, some questions you face may ask you to do, precisely this.

Despite the lack of precise labels on our axes, it is easy to guess that one of our points is at about $(40,7)$ and the other at about $(90,6)$. Using these two points, we can figure out the equation to *the line of best fit* (the line of best fit is frequently called *a regression line*.) Finding the equation to this line is just like finding the equation to other lines. We have done it before, so, you should know what to anticipate.

As always, we need the *slope*.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{6-7}{90-40}=\frac{-1}{50}$$

Using either point, we can now write the equation to the *Line of Best Fit* in *Point-Slope* form.

$$y-7=\frac{-1}{50}(x-40)$$

Because the *Line of Best Fit* is usually presented in the *Slope-Intercept form*, we should work on transforming the *Point-Slope* form to the *Slope-Intercept* form (we have already discussed how we should do this in a previous lesson, so what follows is just an extra exercise/practice.)

$y-7=\frac{-1}{50}x+\frac{40}{50}$

$\rightarrow y-7=\frac{-1}{50}x+\frac{4}{5}$

$\rightarrow y=\frac{-1}{50}x+\frac{4}{5}+7 \rightarrow y=\frac{-1}{50}x+\frac{4}{5}+\frac{35}{7}$

$\rightarrow y=\frac{-1}{50}x+\frac{39}{5}$

Now, in certain situations, the *line of best fit* is used to *predict a future outcomes*. We can even do it here. Since out two variables are $a$ge and $s$leep, we can use a and s instead of x and $y$in the last of the equations above.

$\rightarrow s=\frac{-1}{50}a+\frac{39}{5}$

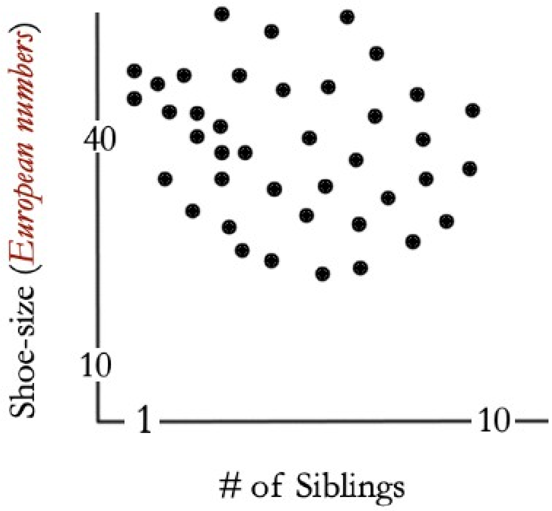
Although the variable renaming is only for a more accurate reflection of the *Scatter Plot* more than practical reasons, we can now use this adopted equation to figure out, for example, how much sleep one needs at the age of 100. We use $a=100$ to learn that

$\rightarrow s=\frac{-1}{50}100+\frac{39}{5}$

$\rightarrow s=\frac{-100}{50}+\frac{39}{5}=-2+7.8=5.8$

So, our model (*the line of best fit*) predicts that if you live to be 100 years old, you will need 5.8 hours of sleep a night. This kind of *predictive estimation* is expected of you on some problems.

And finally, onto the last of our *Scatter Plots* below.



This last plot is merely an example of a mess of a *Scatter Plot* that not even Jesus can save☺ In other words, there is no line you can fit to it and the *correlation coefficient* (r) would have to be zero.

The Heart of Algebra

Linear Equations

The **golden** *rule* – we *need* as many *equations as the number of variables we see in an equation.*

A first few set of examples

- © If

$\frac{7}{5}u=\frac{3}{13}$

What is the value of u?

Solution

This is a simple problem. What most students want (will) to do is the following.

$\left(\frac{7}{5}u=\frac{3}{13}\right)5$

Multiply by 5 on both sides to get rid of the denominator on the left side.

$\rightarrow 7u=\frac{15}{13}$

$\frac{1}{7}\left(7u=\frac{15}{13}\right)$

Now divide by 7 on both sides right?

$\rightarrow u=\frac{15}{13}\cdot\frac{1}{7}\rightarrow u=\frac{15}{91}$

But here is a more efficient path and what students ought to do.

The whole point is to get a coefficient of 1 on x on the left side. So, from the get-go, multiply both sides by the *multiplicative inverse* of $\frac{7}{5}$.

$\rightarrow\left(\frac{7}{5}u=\frac{3}{13}\right)\frac{5}{7}$

$1u=\frac{3}{13}\cdot\frac{5}{7}\rightarrow u=\frac{15}{91}$

- What value of v is a *solution* to the equation $3(v-1)+4(v+1)=12v$

Solution

First, we will distribute as follows:

$3v-3+4v+4=12v$

And now, combining like terms, we have:

$7v+1=12v$

$1=5v$

$v=\frac{1}{5}$

- If $7t=15$, what is the value of $21t-3$?

Solution

This is one of those question where you can either solve by finding t using the given equation and then plugging in the expression or solve efficiently by making a careful observation.

Remember, on a timed test, clever solutions are rewarded in efficiency. They are built in by God (Jesus Christ☺)

Now, if we go with the more efficient (clever) route, since we know

$7t=15 \rightarrow 3(7t)=3(15)=45$

$\rightarrow 21t=45$

$21t-3=45-3=42$

- An equation to a line is $y=kx-4$. If the line contains the point (a,b) where $a\neq 0$ and $b\neq 0$, then, what is the *slope* of the line in terms of a and b?

We are given the equation of the line in the *Slope-Intercept* form. Since we know that the slope is k, so far, we have: $y=kx-4$. To get the value of the *slope* (k in this case) in terms of a and b, we must plug into the equation of the line, the point (a,b), where a is the $x-¿$coordinate and b is the $y-¿$coordinate of the given ordered pair. So, replacing x with $a$and y with b in our start so far, we solve for k as follows:

$b=ka-4$

$\rightarrow b+4=ak$

$\rightarrow \frac{b+4}{a}=k$

Since k is what we are after, $\frac{b+4}{a}$ is the answer.

A follow up example

$2(4x-22)-5(4x-22)=7$

What value of x satisfies the equation above?

Solution

Notice the $4x-22$ is repeating, so we can factor it out.

$2(4x-22)-5(4x-22)=7 \rightarrow (4x-22)(2-5)=7$

Reducing, we have

$4x-22=\frac{-7}{3}$

$4x=22-\frac{7}{3}$

$x=\frac{1}{4}\left(22-\frac{7}{3}\right)=\frac{59}{12}$

A peculiar *Example*

$\frac{1-x}{4}=t$

If $t=5$, what is the value of x?

Solution

We should start by replacing the value of t in the expression. Doing so, we get the following linear equation:

$\frac{1-x}{4}=5$

Now, we can multiply both sides by 4 and subtract 1 to both sides to write:

$\rightarrow -x=19$

Finally, multiplying both sides by -1 gets us there:

$x=-19$

A *Final example*!

If $3(2a-6b)=10$, what is the value of $a-3b$?

Solution

To solve this problem, we will try to produce $a-3b$ from the given expression. Notice that we cannot solve for a or be individually. This is so because if you go back and read the Golden Rule stated at the start of this section, we need as many equations as variables. But we have one equation and two variables. Therefore, the only way to answer the question is to manipulate the given equation.

$3(2a-6b)=1$

Dividing both sides by 3*, we write*:

$2a-6b=\frac{10}{3}$

Now, we can factor out a 2 *from the left-hand side, and we have:*

$2(a-3b)=\frac{10}{3}$

Dividing both sides by 2*, we get to the desired destination*.

$a-3b=\frac{10}{6}=\frac{5}{3}$

An *example* on a number line for 5^{th}?

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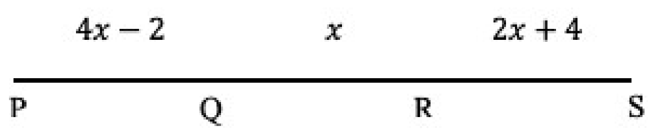

Note: Figure not drawn to scale

Solution

On $\overline{PS}$ above, $PQ=RS$. What is the length of $\overline{PS}$?

Since $PQ=RS$, we can set up the following equation:

$4x-2=2x+4$

$\rightarrow 2x=6 \rightarrow x=3$

$PS=4x-2+x+2x+4=7x+2$. Since $x=3$, $PS=7(3)+2=23$

The *Word Problem example*

A rectangle was altered by *decreasing its length* by 15 percent and *increasing its width* by $p$percent. If these alterations decreased the area of the rectangle by 10 percent, what is the value of p ?

Solution

A quick and easy way to solve this kind of problems is to take 100 as a reference number, because at the beginning every value will be 100%, and then we make the required adjustments. In this case, let’s take the original width and height to be 100 each. The area of the rectangle would be $100 \cdot 100 = 10000$. Notice this is the *Picking Numbers* strategy mentioned at the start of this book.

We are told that the length is now 15% *shorter*, and the *width has increased* by p%. After these alterations, the rectangle has sides 85 and $100 + p$. The formula for the *area of a rectangle* is *width times height*,

so, we can set up the following equation and solve for p.

$$85 \cdot (100 + p) = 9000$$

The 9000 on the right side is because we started off with an area of 10000 based on our picked numbers and we are told there is a 10 percent decrease in the area after the sides have been altered as instructed.

$$8500 + 85p = 900$$

$$85p = 500$$

$$p = \frac{500}{85} = \frac{100}{17}$$

Number Line Algebra Practice Problems

1\.
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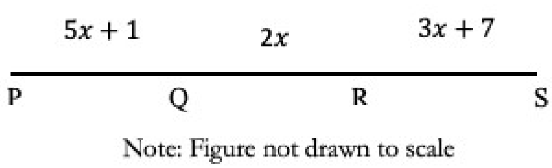

On $\overline{PS}$ above, if $PQ = RS$, what is the length of $\overline{PS}$?

2\.
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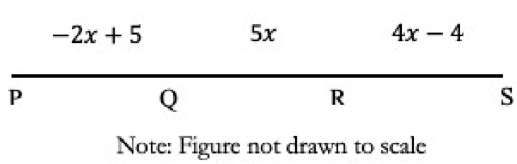

On $\overline{PS}$ above, if $PQ = RS$, what is the length of $\overline{RQ}$?

3.
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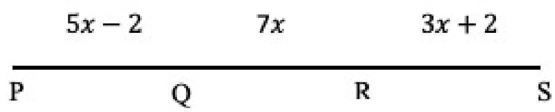


Note: Figure not drawn to scale

On $\overline{PS}$ above, if $PQ = RS$, what is the length of $\overline{PS} - \overline{RP}$?

Solutions to *Number Line* Algebra

1. The correct answer is 38. This is a simple Algebra problem. Since we are told that $PQ = RS$, we can write the equation:
$5x + 1 = 3x + 7$

$\rightarrow 5x - 3x = 7 - 1$

$\rightarrow 2x = 6$

$\rightarrow x = 3$

Substituting these for the three segments that add up to $\overline{PS}$ (*Betweeness of Points*), we see that

$PQ = 5(3) + 1 = 16$, $QR = 2(3) = 6$, and $RS = 3(3) + 7 = 16$

$\therefore \overline{PS} = PQ + QR + RS = 16 + 6 + 16 = 38$

2. The correct answer is 7.5. Since we are told that $PQ = RS$, we can write the equation:

$-2x + 5 = 4x - 4$

$\rightarrow 5 + 4 = 4x + 2x$

$\rightarrow 9 = 6x$

$\rightarrow x = \frac{9}{6} = 1.5$

Now, noticing that $\overline{RQ}$ is the same as $\overline{QR}$, we see that

$\overline{RQ} = 5x = 5(1.5) = 7.5$

3. The correct answer is 30. Once again, since we are told that $PQ = RS$, so, we can write the equation:

$5x-2=3x+2$

$\rightarrow 5x-3x=2+2$

$\rightarrow 2x=4$

$\rightarrow x=2$

Now, we need the difference $\overline{PS}-\overline{RP}$. We can easily see that:

$\overline{PS}=PQ+QR+RS=8+14+8=30$

$\overline{RP}=\overline{PR}=12x-2=12(2)-2=2(12-1)=22$

Linear Inequalities

Solving *linear inequalities* is the same as solving *linear equations* with two major differences.

- Anytime one must divide or multiply by a negative number in solving *an inequality*, one must *change the orientation of the inequality* (¿ *turns into* ¿ and visa versa, and the same goes for *non-strict inequalities*.)
- Instead of just one $x$value, *the solution is an infinite set of numbers* (an interval on a number line.)

But wait, we dropped a lingo that you may be unfamiliar with. What do we mean by *non-strict inequality*?
Let's address the following then we will answer this question!

The Trichotomy Laws

Every *real quantity* (so, *any real number*) is one of the following 3 things.

- *Positive*
- *Negative*
- Or *Zero*

Stated differently, we can say, for *any real number* x, $x>0$, $x<0$, or $x=0$.

It may seem trivial to state this, but it is actually very helpful in further studies in mathematics to understand this separation of a *real number** x into one of the three categories (The Trichotomy Laws) we just stated.

*Most numbers you deal with in middle through high school math are *real. Complex numbers,* which make for a bigger set (every real number is a complex number) are the only ones that are *not real,* and they cannot be ordered (meaning, *one complex number cannot be compared to another as being greater or lesser*.)

So then, what about this *strict* vs *non-strict* thing? You should already be familiar!

Great than (¿) and *lesser than* (or *less than*) (¿) are called *strict inequalities* whereas *greater than or equal to* $(\geq)$ or *lesser than or equal to* (*less than or equal to*) $(\leq)$ are called *non-strict inequalities*.

As such, when we write $x>1$, we mean, *all real numbers bigger than one starting with the first number adjacent to one* and *to the right of one on a number line* (whatever that is right?) Whereas when we write $x \geq 1$, we mean the same thing as $x>1$ but with the crucial difference that one is included (when we say $x \geq 1$.)

Why labor so hard to make such a distinction? <u>Because it matters</u>, even more so in advanced mathematics. If you are the curious type, the set of numbers belonging to $x>1$ don't have a *greatest lower bound* (also known as, *an infimum* or *the infimum*) whereas $x \geq 1$ do have *an infimum*, $x=1$.

Now, as this topic is simple, and so a couple of examples will do!

Examples

- Which of the following is in the solution set of the inequality $3x-5 \geq 5-4x$?

Solution

To start, we will proceed as we would with equations. So, we will add $5+4x$ to both sides. We are doing this to isolate the *combine like-terms*. And so, we will get the following:

$7x \geq 10$

Now, we can divide both sides by 7. Since 7 is a positive number, we don't have to worry about the inequality sign, but whenever we divide (or multiply) by a negative number, we must reverse the inequality.

$x \geq \frac{10}{7}$

The solution set is $\left[\frac{10}{7},+\infty\right)$.

$4x-12y>16$

- Which that follow is equivalent to the following inequality?

$-4x-12y>16$

A. $3x-y>4$
B. $x-3y<-4$
C. $3y-x>4$
D. $x+3y<-4$

Solution

We can simplify the inequality by dividing every term by -4. Since $-4<0$, we must worry about changing the inequality sign.

$-4x-12y>16$

$\rightarrow x+3y<-4$

So, answer choice D.

A System of Linear Inequalities

Let's dive right in and start the discussion in this section with an example!

- If the *system of inequalities* $y\leq-2x-1$ and $y<-5x$ is *graphed* in the xy- plane below, which quadrant contains no solutions to the system?

Solution

This is an easy enough question to answer. All you need to know is how to *graph linear inequalities*.

We have used "*Desmos,*" an online graphing tool to graph the two inequalities (the graph to follow.)

You should know from your Algebra class that ≤ and ≥ require solid lines and that dotted/dashed lines are for *strict inequalities* (¿ and ¿.)

Now, clearly, as opposed to *linear equations* (equations of lines), *linear inequalities* require shading. We usually shade "*above*" and "*to the right*" of the line if it is ¿ or ≥. If you are ever in doubt, just pick a point (x, y) in the prospective "solution set." Testing the picked ordered pair (x, y) in the original inequality, you can decide if your guessed ordered pair satisfies the inequality (letting you know that you shaded correctly.)

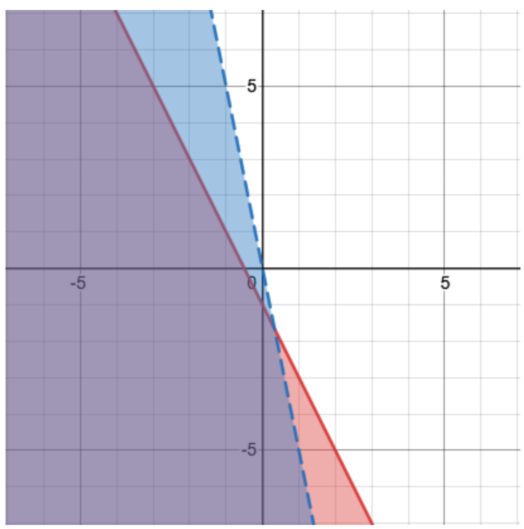
For example, to graph the linear inequality $y \leq -2x-1$, first graph $y=-2x-1$. Thereafter, all you need to do is decide which way you should shade. We will pick $(0{,}0)$ to test in the inequality because it should be clear that the origin is on one side of the line $y=-2x-1$.

So then, plugging in $(x, y)=(0{,}0)$ into the inequality $y \leq -2x-1$, you write:

$0 \leq -2(0)-1$

$\rightarrow 0 \leq -1$

That is a false statement. 0 is not *less than or equal* to negative one. So, you know you're supposed to shade on the other side of the line [away from $(0{,}0)$¿.



Obviously, once you have correctly graphed both inequalities, all that you must do is look at the *overlapping shaded area* to identify your solution set.

Now, looking at the *overlapping shaded area*, it is obvious that the only quadrant where we don't have a solution set is the first quadrant.

•

$y > -x+c$
$y \leq x+k$

© In the xy- plane, if $(0,0)$ is a solution to the system of inequalities above, which of the following relationship between c and k is true?

A. $c>k$
B. $c<k$
C. $c \geq k$
D. $c \leq k$

Solution

So, on a problem like this, you're kind of at a dead end. Meaning there is only one thing to do to start: plugging in $(x,y)=(0,0)$.

$y>-x+c \rightarrow 0>-0+c$

$\rightarrow 0>c \rightarrow c<0$

And on the second inequality
$y \leq x+k \rightarrow 0 \leq 0+k$

$0 \leq k$

$\rightarrow k \geq 0$

Now, since we have $c<0$ and $k \geq 0$, the correct conclusion is $c \leq k$. This is so because $c$is always negative and k is at minimum zero.

Polynomials

Combining *Like-Terms* (*Adding/Subtracting* Polynomials)

Technically, as linear equations are first degree *polynomials*, our discussion of *polynomials* started much earlier, but let us pretend like this little, small discussion here is the real start.

Let's dive in with a problem in this section as we did in the last topic we just finished discussing.

$f(x)=3x^2-5x+4$

$g(x)=5x-7x^2+2$

- Which of the following is the difference $f-g$between the two *polynomials* above?

Solution

This is a simple problem. The key word is "*difference*." "*Difference*" means "*subtraction*." So, we write:

$f-g=3x^2-5x+4-(5x-7x^2+2)$

The most important two things to do from here are, in order, (1) distribute the "mins sign" in front of the parenthesis and then (2) *combine like-terms*.

$3x^2-5x+4-5x+7x^2-2$

$3x^2+7x^2-5x-5x+4-2$

$\rightarrow 10x^2-10x+2$

- Which of the following is an equivalent form of?

$(2.5x-1.6)^2-(2.4-4.5x^2)$

Solution

You may encounter a problem like this which appears to be made for a calculator. But you can tackle it as follows. All that is required is that you be able to square $2.5 \rightarrow 2.5^2$ and square $1.6 \rightarrow 1.6^2$.

Neither of these tasks are very hard. If you can square 25 and 16, you can also deal with the task at hand with similar comfort. You should know even without paper and pencil that $25^2=625$ and $16^2=256$. Therefore, $2.5^2=6.25$ and $1.6^2=2.56$.

With this and knowing *Binomial Expansion* (although we are yet to discuss this, this is something you should already know), you have all the tools to successfully solve this problem as follows.

$(2.5x-1.6)^2=(2.5x-1.6)(2.5x-1.6)$

$$\rightarrow = 6.25x^2 - 2(2.5x)(1.6) + 2.56$$

Now, notice that we didn't discuss the multiplication job we have to deal with in the middle. We can deal with it cleverly:

$$2(2.5x)(1.6) = 5(1.6)x = \frac{10(1.6x)}{2} = \frac{16x}{2} = 8x$$

Now, we have:

$$(2.5x - 1.6)^2 = 6.25x^2 - 8x + 2.56$$

Putting it all together, we can write that:

$(2.5x - 1.6)^2 - (2.4 - 4.5x^2)$

¿ $6.25x^2 - 8x + 2.56 - 2.4 + 4.5x^2$

Parallel and Perpendicular Lines

This section should be completed with just the definitions. Because, we have sufficiently discussed everything else that would have been necessary here.

Two lines are *parallel* if they have the *same slope*. Two lines are *perpendicular* if their *slopes are additive inverse reciprocals*.

Alright, alright. Just so someone doesn't accuse us of being too lazy here, let us give an example.

As the *slopes* are so easy to pick out in the *Slope-Intercept* and *Point-Slope forms*, we should consider examples where the lines are given in the *Standard form*.

- Consider lines $l_1, l_1,$and l_1. Suppose they (these lines) had the three equations:

$l_1 \rightarrow 4x+2y=42$

$l_2 \rightarrow -2x-y=7$

and

$l_3 \rightarrow 7x-14y=16$

The task is now figuring out which is *parallel* (*perpendicular*) to the other?

Remember, in the standard form $ax+by=c$, the slope is given by:

$$m=\frac{-a}{b}$$

So, the *slope* of the first line l_1 is:

$$m=\frac{-4}{2}=-2$$

The *slope* of the first line l_2 is:

$$m = \frac{-(-2)}{(-1)} = -2$$

And the *slope* of the first line l_3 is:

$$m = \frac{-7}{(-14)} = \frac{1}{2}$$

So, we see readily (upon calculating the *slopes* in most reduced form) that lines l_1 and l_2 are *parallel* and l_3 is *perpendicular* to both lines.

It is kinda like parallel lines cut by a *transversal* where l_3 is the *transversal*.

Systems of *2* Equations

Knowing how to *solve a system of equations* is a very important for success in high school math and beyond. This cannot be overstated if your goal is to do very well on the SAT Math, or any other comprehensive high school mathematics exam. Everything you need to know about this topic is succinctly summarized here.

First note, when you're solving *a system of linear equations*, all you are doing is figuring out the *point of intersection* of two lines. This big picture in solving systems is invaluable as you tackle a variety of questions testing your understanding.

Now, to understand the big picture, we need to visualize what *could* happen with the two lines (*system of 2 equations*.)

There are only three possibilities.

- The lines *don't meet* (they *are parallel*.) And hence, *NO Solution*.
- The lines are *exactly on top of each other*. Hence, *infinitely many solutions*.
- The lines are *not parallel* and or *on top of each other* (they intersect at one point.) This is the *Unique* (*ONE*) *solution* case.

Let us examine each one.

No Solution

A system of *linear equations* has *No solution* if the lines are *parallel*.

But what does this mean visually? It means that the lines have the *same slope* but different y-*intercepts*.

Now, frequently, a *system of equations* for two lines is given in *the standard form* of the *equation of a line*. The *standard form of a line* is $ax+by=c$.

But, as you should already know, the form that allows us to quickly pick out the *slope* is the *Slope-Intercept form*. Recall that the *slope intercept form* is $y=mx+b$.

Now, suppose you were given the *standard form* of the equation of a line:

$$2x+5y=10$$

Solving for y, we can get this equation into the *slope intercept form*. (It should be clear that we do this *subtracting* $2x$ from both sides and subsequently *dividing* by 5.)

$$\rightarrow 5y=-2x+10$$

$$\rightarrow y=\frac{-2}{5}x+2$$

Now, it is very easy to tell that *the slope* is $\frac{-2}{5}$.

More importantly, notice that all that was important for us to focus on is $\frac{-a}{b}$ from the *standard form* $ax+by=c$. In other words, if we are given the standard form, we only need

$$\frac{-a}{b}$$

to decide *the slope*.

- So, if we are given the system,

$$2x+5y=10$$
$$-6x-15y=20$$

Solution

We can without doing much work see that the lines are *parallel*. This is because

$$\frac{-a}{b}=\frac{-(-6)}{(-15)}=\frac{-6}{15}=\frac{-2}{5}$$

We needed to simplify for the slope of the second equation (just above) but for the first equation, we simply write:

$\frac{-a}{b}=\frac{-2}{5}$

Now, we didn't have to calculate $\frac{-a}{b}$. We can see that the coefficients of x and y in the second equation are just -3 times the ones in the first equation. So, the ratio of the *corresponding coefficients* was bound to be the same.

Now, notice we had to make sure that 20 is not $-3(10)=-30$. Because, if instead of 20, if the second equation was $-6x-15y=-30$, then, we have the *infinitely many solutions* case. This is so because

$-6x-15y=-30 \rightarrow -3(2x+5y)=-3(10)$

Dividing both sides of $-3(2x+5y)=-3(10)$ by negative 3, we have:

$2x+5y=10.$

This equation is the same as the first equation. So, the line given by $2x+5y=10$ and the one given by $-6x-15y=-30$ are one and the same!

With this, we have created the runway to the *infinitely many solutions* case.

Infinitely many solutions

If one equation in a system is $ax+by=c$, then, we have *infinitely many solutions* if the second equation in the system is $kax+kby=kc$.

A First Example

If we have

$2x+3y=5$

$-8x-12y=-20$

Solution

Quick inspection lets us know that we have *infinitely many solutions*. This is so because we can easily see $-4(2x+3y=5) \rightarrow -8x-12y=-20$. As such, the equations $2x+3y=5$ and $-8x-12y=-20$ both define the same line!

So, the two equations when graphed would give two lines that are *exactly on top of each other*. As such, any (x, y) that satisfies the equation of one of the two lines also satisfies the other. Hence *infinitely many solutions* exist to the two equations as a system.

Remember, when we solve *a system of equations*, we are trying to find the *point of intersection* of the two lines, and therefore, here, the two lines intersect at *infinitely many points* since they are on top of one another!

Onto the next Example

$-3x+9y=12$
$x+ky=-4$

©In the *system of equations* above, what must be the value of k so that there are *infinitely many solutions* (x, y) ?

Solution

We can see that the two equations would make the same line (*infinitely many solutions*) if we divide all the coefficients in the first equation by -3. Doing that, we can see that $k=-3$.

Now, try this on your own

$-5x+y=-3$
$3x-8y=24$

- What is the *solution* (x, y) to the system of equations above?

Solution

We can start by multiplying the first equation by 8.

$\rightarrow 8(-5x+y)=8(-3)$

$\rightarrow -40x+8y=-24$

Now, we can add the two equations:

$3x-8y=24+$¿
$-40x+8y=-24$
$\downarrow$
$-37x=0$
$x=0$

Now that we have the value of x, we can use it to find the value of y by replacing x by zero in either of the original two equations.

$-5x+y=-3$

$-5(0)+y=-3$

$y=-3$

Hence, the *solution* is $(0,-3)$.

Here is another layup you can try on your own

$$\frac{2}{3}(5x-3y)=\frac{16}{3}$$

$$y=3x$$

The *system of equations* above has *solution* (x,y). What is the value of y?

Solution

We are told that $y=3x$, which means $\frac{y}{3}=x$. Knowing this, we can replace all x's in the first equation with $\frac{y}{3}$.

$$\rightarrow\frac{2}{3}\left(\frac{5y}{3}-3y\right)=\frac{16}{3}$$

Now, we only must solve this linear equation.

$$\frac{5y}{3}-3y=8$$

$$\rightarrow\frac{5y-9y}{3}=8$$

$$\rightarrow-4y=24$$

$$y=-6$$

Gursha (*Another for you to practice with*)

$8x+3y=13$
$3x+2y=11$

If (x,y) is a solution to the *system of equations above*, what is the value of $y-x$?

Solution

First, we will find the *solution to the system*, and then we will evaluate the value of $y-x$.

Multiplying the first equation by 2, and the second equation by 3, we get the following:

$16x+6y=26$
$9x+6y=33$

Now, *subtracting* the second equation from the first one:

$16x+6y=26$
$-(9x+6y=33)$
$\downarrow$
$7x=-7$
$x=-1$

Now, replacing the value of $x$in the second original equation, we have:

$3(-1)+2y=11$

$\rightarrow -3+2y=11$

$\rightarrow 2y=14$

$\rightarrow y=7$

And so, the *solution* to the system is $(-1,7)$and the value of $y-x$ is $7-(-1)=8$.

A Nonlinear System

$y=-x^2-x$
$2(4x+5)+y=0$

If (x, y) is a solution to the nonlinear *system of equations* above and $x>0$, what is the value of $-2xy$?

Solution

Pay attention to some details like, $x>0$. Otherwise, this is not very hard. The reason it is a *nonlinear system* is because one of the equations is *quadratic*. But that doesn't make it significantly more challenging than solving *a linear system*.

We still take the same approach, it is just that, for *nonlinear systems*, after simplifying the system to just one equation in one variable, we might end up having to solve *a quadratic* equation.

But, just as with *a linear system*, what we must do is isolate one variable in one of the equations and substitute in the other equation, to the end of, as we have already said, making one of the equations all in terms of one variable.

A quick look at the two equations reveals that isolating the variable y in the second equation is the path of least resistance.

So, we have $\rightarrow y=-2(4x-5)$

Now, substituting for y in the first equation with the right-hand side of what we wrote just above, we have

$y=-x^2-x \rightarrow$

$-2(4x-5)=-x^2-x$

$\rightarrow -8x+10=-x^2-x$

$\rightarrow 0=-x^2-x+8x-10$

$\rightarrow 0=-x^2+7x-10$

As it is easier on the eye to have zero on the right, we can write:

$-x^2+7x-10=0 \rightarrow -(x^2-7x+10)=0$

Dividing or *multiplying* both sides of the last equation by -1, we have:

$x^2-7x+10=0$

You should already know how to solve *quadratics* by factoring, that is what we must do here to finish the job.

We can see that $x^2-7x+10=(x-5)(x-2)$

As such, we have $(x-5)(x-2)=0$

$\rightarrow x=5, x=2$

So, we see that the *two solutions* are $x=5$ and $x=2$.

Tie the Knot with this one that you *try on your own*!

$-3x-4y=2$
$3x+3y=-3$

What is the *solution* (x,y) to the system of equations above?

Solution

We observe that in the system, the *coefficients* of x are opposite. This informs us to add the two given equations.

$-3x-4y=2$
$+(3x+3y=-3)$
$\downarrow$
$-y=-1$

$\rightarrow y=1$

Now, we can use the value of y to find the value of x.

$3x+3(1)=-3 \rightarrow 3x=-6$

$\rightarrow x=-2$

$(x,y) \rightarrow (-2,1)$.

Another Gursha (*A mouthful* to conclude)

$-3x+9y=12$
$x+ky=-4$

- In the *system of equations* above, what must be the value of k so that the solution (x,y) is undefined?

Solution

With quick inspection, we notice that the first equation can be simplified. Let's divide every term in the first equation by 3. Doing so, we have:

$-x+3y=4$
$x+ky=-4$

Notice that if we multiply the first equation by -1, we will get something looking very much like the second equation.

$x-3y=-4$

$x+ky=-4$

They only differ between the two transformed equations is the value of k. If both equations were the same, we would have *infinitely many solutions*. So, we choose $k=-3$.

Absolute Value

A good number of problems on *Absolute Values* are conceptual, as they should be. Meaning, you need to *understand what an absolute value is* at the core.

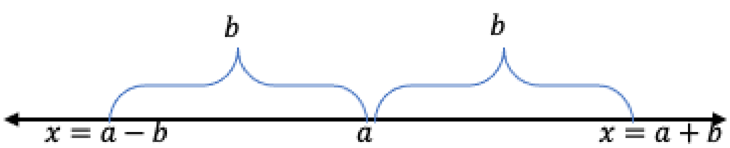
The most important item for you to keep in mind is the following *geometric interpretation* of *absolute values*.

$|x-a|=b$

This means,

"On a number line, center at a*,* x *is* b *units away."*

The visual that should accompany this definition is the following.



Equipped with this *geometric interpretation*, we can readily solve many *absolute value* questions efficiently.

This definition also gives us the ability to interpret some word problems as *absolute values* (wherever it is fitting.)

A first Example [*Abol*]

What are both *solutions* to the equation:

$|x+4|=2$

Solution

First, note that to use the *geometric interpretation* we had written earlier, we must write this above *absolute value equation* in the exact form of the definition, that is $|x-a|=b$. Doing so is a simple task.

$|x+4|=2$

$\rightarrow |x-(-4)|=2$

Now, using this last equation, we see that the solutions x we are seeking must satisfy:

On a number line, centered at $x=-4$ *on a number line, we are 2 units away.*

As such, the two solutions are:

$$x=-4+2=-2 \text{ and } x=-4-2=-6$$

Notice that we can use this *interpretation* even on more sophisticated *absolute value equations.*

For instance, if we were instead tasked with:

And now a Second [Tona]

$|2x+4|=7$

Solution

First, as before, we need the form $|x-a|=b$ for our *interpretation*.

$\rightarrow |2(x+2)|=7 \rightarrow 2|x+2|=7$

$\rightarrow |x+2|=3.5$

$\rightarrow |x-(-2)|=3.5$

So now we mean, *on a number line centered at* $x=-2$*, we are* 3.5 *units away.*

So, $x=-2-3.5=-5.5$ or x¿$-2+3.5=1.5$

A third and final Example [*Bereka*]

Two different points on a *number line* are both 7 units from the point with coordinate $x=-2$. The solution to which of the following equations gives the coordinates of both points?

A. $|x-2|=7$
B. $|x+2|=7$

C. $|x-7|=2$
D. $|x+7|=2$

Solution

The *absolute value* of a number represents its distance from the origin. In this case, we are told the distance is 7, but not from the origin, instead from the point x=−2.

Because of this, we need to shift the number inside the *absolute value* by 2.

$|x+2|=7$

If this doesn't make sense, look at the final answer here and go back and read the earlier presentation on Absolute Values that we discussed at the start.

An alternate Definitions *of the Absolute Value*

$$|x|=\begin{cases} x & x\geq 0 \\ -x & x<0 \end{cases}$$

It is an application of this second definition of the *absolute value* that students usually apply in solving problems like the ones in the examples above.

For instance, if we returned to the equation $|x+4|=2$

We see that using the alternate definition we just provided, either $x+4=2$ *or* $-(x+4)=2$.

$$\rightarrow x+4=2 \textit{ or } x+4=-2$$

Now, $x+4=2 \rightarrow x=2-4=-2$ *or* $x+4=-2 \rightarrow x=-2-4=-6$

Now then, before we conclude, let us look at two more things.

First, notice that the *geometric interpretation* we had given to the *absolute value* atop this section can be easily adopted to *absolute value inequalities*.

This is what we are saying,

$|x-a|<b$

means,

"O a number line, centered at a*, we are within* b *units."*

Where, $|x-a| \leq b$ would mean the same thing but including $x=b$ in the *solution set*.

A Quick example on Abs Inequalities

Solve for x if
$|x-4|<2$

Solution

$|x-4|<2$ means *"on a number line, centered at* $x=4$*, we are within* 2."
So, this time, instead of the *solution set* just being two x-values, we have infinitely many values. Specifically, any number from $4-2=2$ to $4+2=6$ would fit this inequality, excluding the numbers 2 and 6 themselves.

Finally, a fancy but worthwhile definition of the absolute value of x.

Yes, we can define the absolute value a third way.

The Third and Final *Meaning of Absolute Values*

$|x|=\sqrt{x^2}$

Personally, while the efficacy provided by the *geometry interpretation* is sweet, this last definition is my favorite. Because look at what it would say:

$|-2|=\sqrt{(-2)^2}=\sqrt{4}=2$
And

$|2|=\sqrt{(2)^2}=\sqrt{4}=2$

It is well defined because, well, inside the square root, *you first square whatever number you're given*. In squaring it, you make it bigger, but most importantly, *you make it positive*. But then, you "*unsquared it*" by taking *the positive square root*. As such, you get back to the original number but the positive version.

Solving for a Variable

Let us dive into this section with a problem, the following:

$$q=\frac{\left[\frac{t}{42}\right]\left[1-\frac{t}{42}\right]^{N}}{\left[1-\frac{t}{42}\right]^{N}-1}p$$

- In the formula above, write p in terms of the other variables.

Solution

This looks like a hard question. It is intimidating looking for Sure. But it is an easy question. This is a skill that is tested quite a bit – *Solving for a Variables* – so it is important that you practice, should you need more practice. We have a set of questions designed to help you practice specifically this skill.

Notice, the expression is like:

$$q=\frac{a}{b}p$$

So, on cast in this manner, it becomes clear that it is a much simpler looking equation. That is, in this simplified equation just above, if we asked you to solve for p, it wouldn't be such a huge ask.

Multiplying both sides of the equation by:

$$\frac{b}{a}$$

$$\rightarrow p=\frac{b}{a}q$$

This is exactly what we must do in the main task given us at the start. It is like:

$$a=\left[\frac{t}{42}\right]\left[1-\frac{t}{42}\right]^N$$

$$b=\left[1-\frac{t}{42}\right]^N-1$$

$$\rightarrow p=\frac{\left[1-\frac{t}{42}\right]^N-1}{\left[\frac{t}{42}\right]\left[1-\frac{t}{42}\right]^N}q$$

Here is a problem that you can *try on your own*!

- Solve for y, in the equation $xy^2-3y^2=x+1$

Solution

Here we just must be careful with the order of operations.
First, we will factor out a y^2 from the left-hand side:

$$y^2(x-3)=x+1$$

Now, we divide both sides by $x-3$:

$$y^2=\frac{x+1}{x-3}$$

Taking the *square root* on both sides, we get:

$$y=\pm\sqrt{\frac{x+1}{x-3}}$$

Problem Solving and Data Analysis

Data Analysis

Data maybe presented in a variety of different ways. The most common ways to present data are: *Pie Charts*, *Histograms*, *Scatterplots*, *Box and Whisker Plots*, and *Line Graphs*.

All of these are good *visual representations* of statistics. These *visual representations* allow us to *summarize data set in a way that is easy to consume*, even on a cursory look.

Data Analysis is important beyond high school mathematics and or standardized exams. For example, if you are buying a house or anything that requires long term investment, what is presented in front of you is frequently data given to you in one of the ways we have mentioned above (perhaps differently as well.)

Before we give a closer examination to the different ways of presenting data visually, we should also not fail to mention that there are more novel and interesting ways to represent data in the modern world. That is, the discussion here might not suffice for you to cover all the ways in which data maybe presented. But, if you spend the time to learn what is here, it will give you the necessary background to be able apply to what you have learned here to other perhaps more novel data presentation methods you might encounter.

Having said all of this, let's have a closer look, in shall we?

Data through Tables

The following sets of visual representation of Data come from the University of X and they contain information about the 2024 entering Freshman class.

On the University's website, we first find the following.

Number of Applicants	34,372
Number Accepted	2,511
Number Enrolled	1,848

As what we are given are just raw numbers, it (what we see) doesn't constitute a "*visual representation*." To get a better feel for what is being communicated here, we would have to do a bit of work. For example, the following:

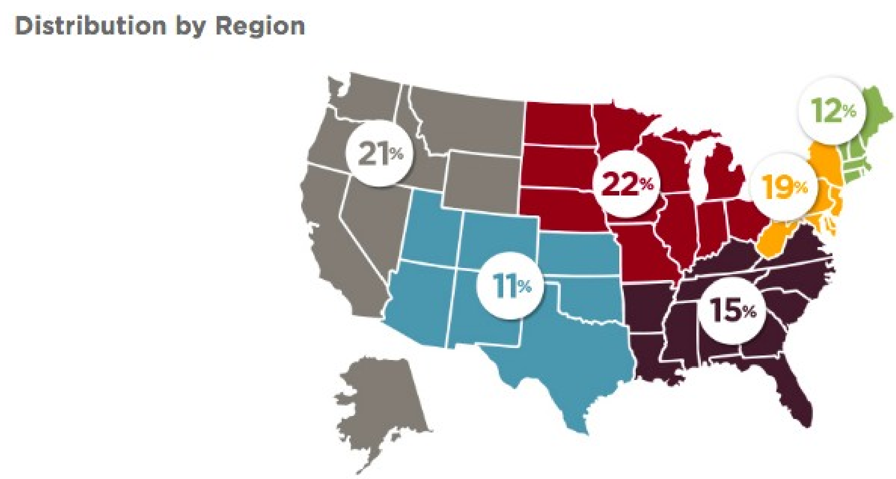
$$\frac{2{,}511}{34{,}372} = 0.73055$$

This tells us that 7.31% of the applicants were *accepted* to this university. Do you see how 7.31% gives a much better feel for what happened in the application process a lot more than just looking at the given numbers?

Looking at the visual below, we can see why *visual representations* of data circumvent the extra work that we just had to do to get a better feel for what we are seeing.

Data through Novel Visual Representations

At the same university, the following "novel" way of presenting data shows the *geographic distribution of accepted students*. We say "novel" not because this is the first time someone has presented data in the following manner, but more because what you see below is not a *Pie Chart*, a *Histogram*, a *Scatterplot*, or a *Line Graph*. But notice that if you have familiarity with a *Pie Chart*, you will not have much trouble understanding what you are seeing here.



So, what kind of question could you encounter as a good question if you were provided both the enrollment statistics we saw in the table above, and this map of the US with numbers just above? How about the following?

A possible question

- How many students were *accepted* from the *Western states of America*?

Solution

We know from the table on the previous page that 2,511 students were accepted in total. Of those, we know that 21% come from the *Western states of America*. So, putting these two together, we have:

$0.21 \times 2{,}511 = 527.31$

Therefore, we conclude that there must have been 527 students *accepted* from the *Western states of America*. Note that we ignore the decimal portion of 0.31 because there is no such thing as 0.31 person. So, we conclude that the math worked out this way most likely because 21% was a rounded number as far as the percentage of *accepted* students that came from the *Western united states*.

More Data through Tables

More information about this same University follows in the presentation below.

Involvement in High School Activities	
Community Service	81%
Editorial	22%
Music	51%
Religious Organization	15%
Student Government	38%
Theater	22%
Varsity Athletics	65%

Data through linking Tables to Box and Whisker Plots

What is interesting to note in what you see just above is that the percentages don't add up to 100%. Do you understand why?

If you didn't understand why, it is because the categories overlap. For example, the same student maybe involved in *Community Service* and *Varsity Athletics*.

In this sense, the way the data is represented in the previous table is very different from how it is represented in US map we previously saw. Both presentations involve percentages, however, in the US map presentation, since the percentages cannot overlap (the same student cannot be from Texas and California), we expect the percentages on the US map to add to 100%. Let us do a sanity check by adding up the percentages from left to right.

$$21\% + 11\% + 22\% + 15\% + 19\% + 12\% = 100\%.$$

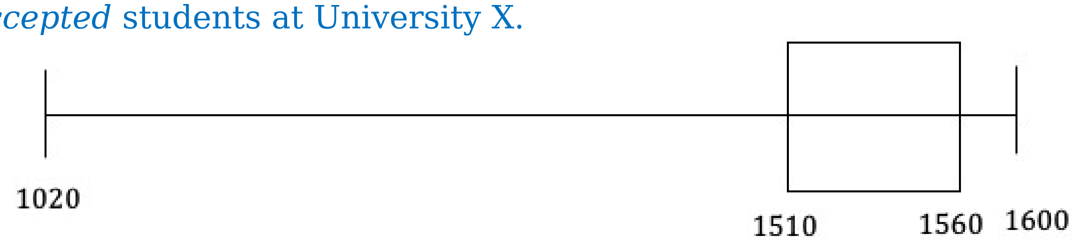
Let us look at the following data set about scores on Standardize exams for the same entering freshman class at University X.

Standardized Testing	
ACT Middle 50%	34 – 35
SAT Middle 50%	1510 – 1560
ACT Score Range (admitted students)	20 – 36
SAT Score Range (Admitted students)	1020 – 1600

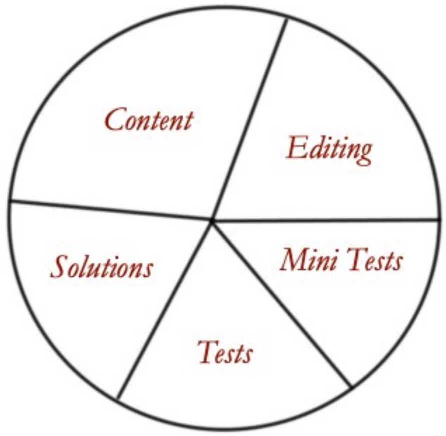
This is like a *Box and Whisker Plot*, if you are not familiar with Box and Whisker plots, don't worry. We can without the requirement of such familiarity still make sense of what we see above.

The table says, in the category of SAT middle 50%,the range of scores given is 1510 to 1560. What this means is that 25% of the students *accepted* to University X scored below 1510 on their SAT (if they reported SAT scores instead of ACT scores) and 25% of the students *accepted* to University X scored above 1560. If you look at the very bottom of the chart above, you'd know that the lowest SAT score for *accepted* students at this University is 1020 and the highest is 1600. What we cannot say from the information given just above is, for example, how many students got a 1600.

Now, since we mentioned *Box-and-Whisker* plots already and this book is intended for a broader use than just the SAT, here is how a *Box-and-Whisker* plot would visually summarize what we just said about SAT scores for *accepted* students at University X.



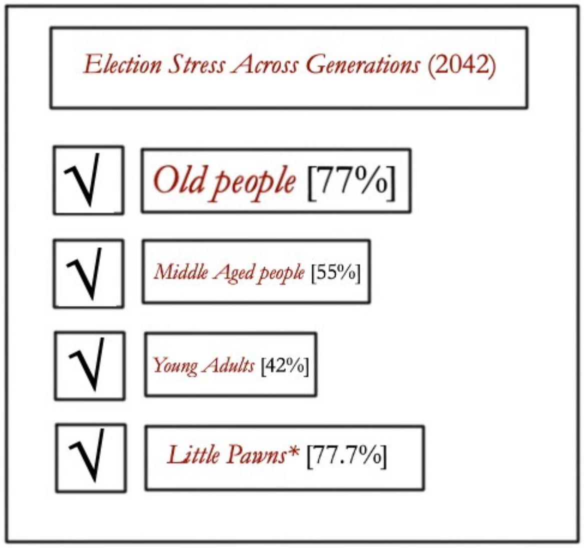
The *Whiskers* at leftmost and rightmost ends tell us the smallest and the largest SAT scores and therefore, they allow us to calculate the *range* of SAT scores quick. The *range* is the difference between the largest and the smallest numbers in a particular data set. 1510, the left end of the box is called *Q1* or *Quartile 1* (25%) mark, *Q3* or *Quartile 3* is the 75% mark and *Q2* the 50% mark not shown in the *Box and Whisker plot* is exactly between *Q1* and *Q3*. Usually, *a Box and Whisker plot* has a vertical line somewhere between *Q1* and Q3 showing the *median* (*Q2*.) As the table which the *Box and Whisker plot* is to depict doesn't show the *median*, *Q2* (*the median*) is omitted in our *Box and Whisker plot*. Additionally, there is the idea of the *Inter Quartile Range* (aka *IQR*) which is *the difference between Q3 and Q2* ($Q3-Q2$ is the IQR.)



The chart above is what we call a *Pie Chart*. Notice that while *Pie Charts* are usually color coded, we have chosen not to do so here as the color coding doesn't add much to the data analysis. Additionally, each sector (a portion of the area of the circle) usually comes with percentages (numbers), we have also omitted numbers here to make clear that the *Pie Chart* is still revealing without these extra details.

The chart is supposed to communicate the rough allotment of time to different tasks in writing this book. So, for example, with relative ease we can look at the *Pie Chart* above and say that the content writing and the editing took up roughly half of the authors' time. We can also with similar efficacy say that writing up solutions took as much as writing up tests (the two full length practice tests contained in this book.) As further analysis doesn't add to the summary for the *Pie Chart* above, we will elect to conclude the discussion on *Pie Charts*, right here.

Bar Graphs



**Little Pawns* because while there are some very smart young folks coming up, most of them jump on it if it is new and mostly don't know what they are talking about.

While the checkmarks on the left side of the *Bar Graph* are fancy addons, what one needs to pay attention to is the length of the bar for the range of *ages* each bar represents (this is called *categorical data*.)

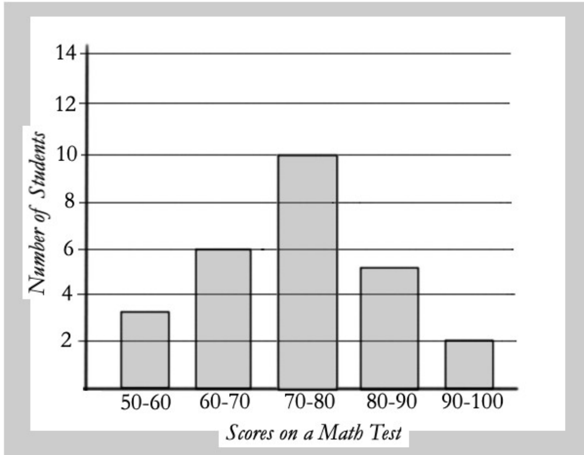
A quick look at the *Bar Graph* should reveal that the *old people* and *millennials* will have the biggest concern about the *204*2election outcomes. As the *age ranges* are assigned names, we can consider each as a category. The percentages displayed to the right of each bar aids us in making a quick conclusion about what is communicated by the *Bar Graph* as a whole.

We should also understand that the percentages are not supposed to add to 100 as each percentage is within itself (for example, of 100% of matures, 59% said they were stressed so the other 41% was not stressed or didn't respond to stress about the election.)

Now, a specific type of *Bar Graph* is called a *Histogram*. So then, let's get to it!

Histograms

Below is an example.



In the *Histogram* above, we have "*Scores on a Math Test*" vs "*Number of Students*" on the *horizontal* and *vertical axes* respectively. While the scores range from 50 to 100, they are presented in the bars of the *Histogram* in intervals of 10s.

We can draw several conclusions from this *Histogram*. For example, we can count the total number of students by adding up the heights of each of the vertical bars. So, the total number of students in in this class would be $3+6+10+5+2=26$.

We can also see that the math scores in this class are "*normally distributed*." This means that the *Histogram* is shaped like *an upside-down bell peaking in the middle and falling at either end*. While we couldn't avoid mentioning this fact about the "*normalness*" of *the distribution of scores* above, "*normal distributions*" are not an item you need to concern yourself with on a high school standardized exam.

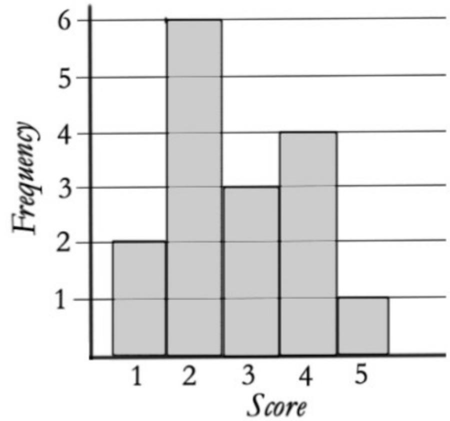
Using the *Histogram*, we can also ask and answer other questions like, how many students scored 70 or above? 80 or above? If a passing score is 60, how many students failed?

All these questions we just asked are very similar and easy to answer looking at the histogram applying the same kind of reasoning as we did when we counted the total number of students.

If you need a clue, we will answer one of the questions here. 10 students scored between 70 and 80. But there are questions we cannot answer such as, how many kids scored below a 72 or below a 75? The fact that 75 is right between 70 and 80 is no more helpful than asking the similar question about 72.

A different kind of *Histogram* called *a Frequency Histogram* is one where we can answer a few more questions that we were able to do here. Let us look at this so called "*Frequency Histogram*" using the figure that follows.

A Frequency Histogram



Here, in the *Frequency Histogram* above, we once again have "*Scores*" on the *horizontal axis* but on the *vertical axis* we have "*Frequency*." Even on the previous *Histogram*, the *vertical axis* which we labeled "*Number of Students*" can still be considered *frequency*, but it is more useful in this *Histogram* which is aptly named.

While it is not clear what "*Score*" represents, if we represented the above data set without a *Histogram*, it would look as follows, $1,1,2,2,2,2,2,2,3,3,3,4,4,4,4,5$.

However, we don't need to list the "*Scores*" in the manner we just showed for us to work with the represented data, the *Histogram* is sufficient.

For example, we can find the average score as follows.

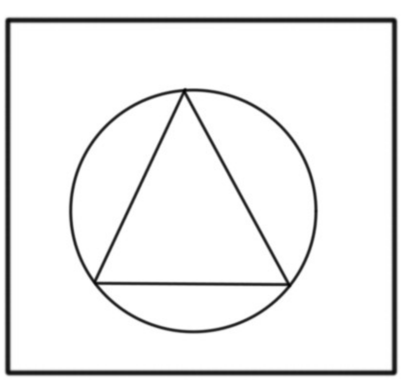
$$\frac{12+2\cdot 6+3\cdot 3+4\cdot 4+5\cdot 1}{17}=2.588$$

What is important that you understand is why we must divide by 17.
This is so because there is a total of 17scores.

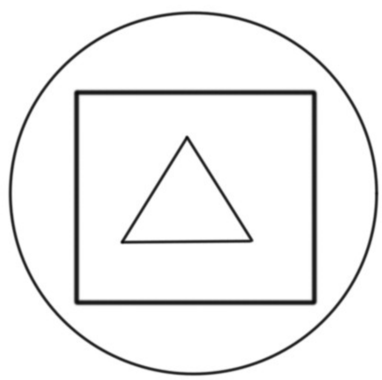
In Conclusion

Here is a novel way to represent data that we once saw somewhere. We have put our own touch on it.

Reus



Luis



Doesn't make sense, right?

Does a *legend* help?

Rectangle is *Age*, *Circle* is *Height*, and *triangle* is *Shoe Size*.

Luis is 16years old, and *Reus* is ageless but obviously older as the rectangle is much bigger under *Reus* when compared to that of *Luis*. But *Luis* is taller cause the circle for *Luis* is much bigger than the circle for *Reus*. Finally, *Reus* wears bigger shoes than *Luis* as his triangle is bigger. Now, for those of you intellectuals going, "shouldn't the taller person have the bigger shoes?" The answer is, we learned somewhere that your feet and nose don't stop growing (elsewhere like *Height* apparently the age of 26 is it.)

Clearly, the advantage to *visual representation of data* as opposed to representing these three *categories* with numbers is that as with all *visual representations*, one can quickly look and learn a lot. It is like a *Box and Whisker* plot.

In addition, we have put this in the conclusion of this section for a reason. That is, there is no way that we can cover you for all the different ways that data may be provided you. There are many interesting ways to do it (present data) beyond *Bar graphs* and *Pie charts*. As such, with enough exposure, it is up to the student to recognize the different scenarios and interpret the data accordingly (equipped with what one may have learned here and elsewhere, one should be able to succeed in such tasks.)

Passport to Advanced Mathematics

The *Binomial Square* & the *difference of squares*

The importance of the *Binomial Square* cannot be understated for your success in high school math. You should already be familiar with the *Binomial Square* from your Algebra class.

If you need a reminder, it says:

$$(a+b)^2=a^2+2ab+b^2$$

and

$$(a-b)^2=a^2-2ab+b^2$$

It is important that you recognize the *Binomial Square* in both forms.

Now, as an easy example, let us start with the following:

Example (the Layup)

$$9x^2+12x+4$$

Solution

You should suspect that this is the "*Binomial Square*." The triggers for this suspicion are the perfect squares on the *first term* and *last term*.

We can see the given expression is of the form:

$$u^2+2uv+v^2$$

To understand this, we must claim $9x^2=u^2 \rightarrow 3x=u$ and $v^2=4 \rightarrow v=2$.

Now, all we have left to do is check that the middle term is in fact $2uv=2(3x)(2)=12x$.

Using the *Binomial Square*, since $u^2+2uv+v^2=(u+v)^2$, we see that:

$$9x^2+12x+4=(3x+2)^2$$

On a harder problem, you might be tasked with making the following.

Example (the No Look Pass)

$$16u^4-72u^2v^2+81v^4$$

Solution

The first thing you must do here is recognize that it is the second of the two *Binomial Squares* (the one with a minus sign.)

$(a-b)^2 = a^2 - 2ab + b^2$

Otherwise, the task is not that much more difficult than the previous problem.

The given expression starts with *a perfect square* and ends with *a perfect square*. Therefore, we can see that

$$a^2 = 16u^4 = (4u^2)^2 \rightarrow a = 4u^2 \text{ and } b^2 = 81v^4 = (9v^2)^2$$

Just like the previous example, at this stage, all we must do is check that the middle term
$-2ab$ is equal to $-72u^2v^2$.

That is in fact the case,
Look, $-2ab = -2(4u^2)(9v^2) = -72u^2v^2$

Since our check is done, we appropriately claim that:

$16u^4 - 72u^2v^2 + 81v^4 = (4u^2 - 9v^2)^2$

On the right side of this last equation, we could have done more if we were equipped with the "*Difference of Squares*."

So then, without further ado, let us discuss the *Difference of Squares*.

I had once said that I like the "*Difference of Squares*" more than ice cream. I scream sporadically anyway cause life can be hard. But Jesus (God, the only God) always rescues me, so I guess I don't need to scream so much. I just believe like you should also☺

Once again, you should have built familiarity with the difference of squares from your Algebra class. It says:
$a^2 - b^2 = (a+b)(a-b)$

So, as already mentioned, if we had this tool earlier in the second example of this section, we could have done more factoring using the "*difference of squares*."

At the very end of that previous example, since we had:

$16u^4 - 72u^2v^2 + 81v^4 = (4u^2 - 9v^2)^2$

Looking inside of the parentheses, we could have seen that:

$$4u^2 - 9v^2 = a^2 - b^2 = (2u)^2 - (3v)^2 \text{ with:}$$

$a=2u$ and $b=3v$
$\rightarrow (2u)^2-(3v)^2=4u^2-9v^2=(2u+3v)(2u-3v)$

So then since we now know that $a^2-b^2=(a+b)(a-b)$.

As this doesn't count as a legitimate example, let us conclude this section with one more example.

Example (the No Look Reverse Dunk)

Consider x^4-81

Solution

First, note that $x^4-81=(x^2)^2-9^2=a^2-b^2$ with $a=x^2$ and $b=9$

$\rightarrow (x^2)^2-9^2=(x^2+9)(x^2-9)$

Now, we can once more appeal to the *difference of squares* on
$x^2-9=(x+3)(x-3)$

So, completely factored, $x^4-81=(x^2+9)(x+3)(x-3)$.

Now, notice that we are unable to factor x^2+9.

Technically, we can. We just need to use *Complex Numbers* (*Imaginary Numbers*.)

Recall the *imaginary unit* $i=\sqrt{-1}$?
$\rightarrow i^2=-1$.

Using this, we can write that $x^2+9=x^2-(-1)9$

$\rightarrow x^2-i^2 9=x^2-(3i)^2$

$\rightarrow x^2-(3i)^2=a^2-b^2=(x-3i)(x+3i)$

Yes, once we have the form a^2-b^2, we can always claim it as $(a+b)(a-b)$.

Now, this last example is very important to our discussion of *Complex Numbers* to follow.

Notice, we started with $x^2+9=a^2+b^2$ because $9=3^2$ so we have a *perfect square* in 9.

Having started with a^2+b^2, we ended up with $(x-3i)(x+3i)=(a-bi)(a+bi)$

In fact, the following is always true if we are allowed *complex numbers*.

$\rightarrow a^2+b^2=(a-bi)(a+bi)$

This last equation is monumentally important in *Complex Division* in the next section.

In summary, we are saying, we can force a "*difference of squares*" out of any expression of the form a^2+b^2 as long as we involve "*Complex Numbers*."

Polynomials Continued...

Graphs

So, the *Fundamental Theorem of Algebra* says:

"An nth degree *polynomial* has <u>at most</u> n zeroes and $n-1$ *bends* (*turning points*.) *Turning Points* are formally known as "*local minima*" and "*local maxima*" in Calculus.

Now, pay attention. It says, <u>at most</u>.

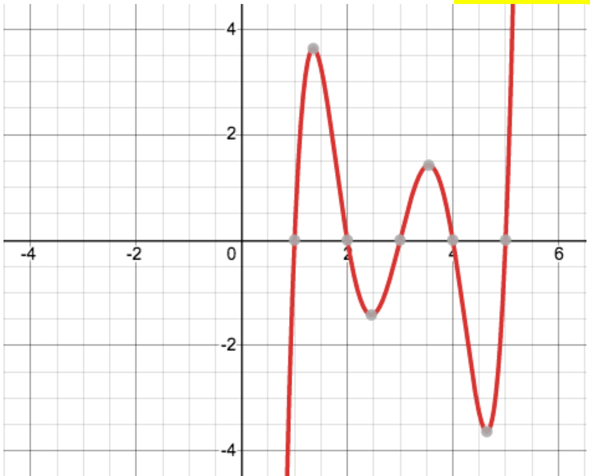
Let us look at the following two *polynomials*.

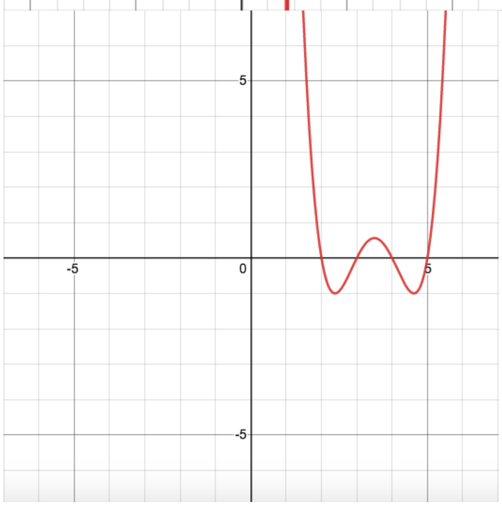
$f(x)=(x-1)(x-2)(x-3)(x-4)(x-5)$

and

$g(x)=(x^2+1)(x-3)(x-4)(x-5)$

The *Graph* of f The *Graph* of g





Now, notice that both *polynomials* are of *degree* 5. Just multiply all the xs that appear in the two polynomials and you will see that the highest power of x will be x^5. This means, both *polynomials* are *Quintics*.

Now, one of them crosses the x-*axis* (meaning it has x-*intercept, root, zero, solution*) *five* (5) times while the other crosses the x-*axis* (meaning it has x-*intercept, root, zero, solution*) *three* (*3*) time.

This highlights the key phrase in the *Fundamental Theorem of Algebra*, "<u>at most</u>."

Now, going back to the two functions above, f has five *real zeroes* (*solutions*, $x-$¿*intercepts*, *roots*), namely $x=1$, $x=2$, $x=3$, $x=4$, and $x=5$.

Whereas g has three *real zeroes*, $x=3$, $x=4$, and $x=5$ and two *imaginary zeroes*. The two *imaginary zeroes* are the reason why we only see <u>three places</u> where the function g crosses the $x-$¿axis.

Now, the two *imaginary zeroes* come from the factor of g, x^2+1. This should not be surprising given the brief (more to come later) discussion that we have just had on *Complex* (*Imaginary*) numbers.

In concluding that section, we said, $a^2+b^2=(a-bi)(a+bi)$.

$\rightarrow x^2+1=(x-1i)(x+1i)$

So, the two *imaginary zeroes* come from $(x-1i)(x+1i)=0$

$\rightarrow x=\pm i$.

But *imaginary zeroes* can only be drawn on the *real- imaginary* axis. We cannot see them on the $xy-¿$plane.

Equally importantly, if $a-bi$ is a *zero* of a *polynomial*, then it is automatic that so $a+bi$is also (and vice versa.) It is also worth noting that $a-bi$ and $a+bi$ are called "*Conjugates*."

Proportions

Given the following *proportion*, all that follow is equivalent to it.

$\frac{a}{b}=\frac{x}{y}[¿]$

$\rightarrow \frac{b}{a} = \frac{y}{x}$

This second equation is the same as [¿].

It is easy to see, we just took the reciprocal of the given original *proportion* *

$\rightarrow \frac{a+b}{b} = \frac{x+y}{y}$

This one above is also easy to see because all we have done is add 1 to both sides of the original *proportion* [¿]
If you need more clarity, here are the steps that get you there:

$\rightarrow \frac{a}{b} + 1 = \frac{x}{y} + 1$

$\rightarrow \frac{a}{b} + \frac{b}{b} = \frac{x}{y} + \frac{y}{y}$

$\rightarrow \frac{a+b}{b} = \frac{x+y}{y}$

The following is also the same as the original because we just took the *reciprocal* of the last step above.
$\rightarrow \frac{b}{a+b} = \frac{y}{x+y}$

When in doubt, you can always cross multiply to check if a given *proportion* is equivalent to another. For example, if we want to check 3 against *, we can do the following.

We start with the original [¿]and cross multiply:

$\frac{a}{b} = \frac{x}{y}$

$\rightarrow ay = bx$

And so then,
$\rightarrow \frac{b}{a+b} = \frac{y}{x+y}$

We cross multiply and write:

$\rightarrow b(x+y)=y(a+b)$

$\rightarrow bx+by=ya+yb$

$\rightarrow bx=ya$

This is now the same as [¿] QED

Rational Functions

Understanding how to deal with *Rational Functions* is essential for a good foundation in mathematics. You need to be adept at dealing with *Rational Functions* in all of the following ways.

- *Adding* and *Subtracting Rational Functions*
- *Multiplying* and *Diving Rational Functions*

  There are very few questions that deal with multiplying *Rational Functions*. But Division of *Rational Functions* is featured prominently on standardized exams or all other summative assessments. The way one deals with *Dividing Rational Functions* is using "*Polynomial Long Division*."

  With all of this said, if you don't have this skill yet, don't fret. There are ways to get around having to master *Long Division* and, in the discussion to follow, we give you plenty of demonstration on exactly how to do this.

- *Solving equations involving Rational Functions*.

Solving *Rational Equations* (equations involving *Rational Functions*) is a matter of just cross multiplying and dealing with the resulting equation the same way one would deal with any other equation. At times, after cross multiplying, we end up with a *quadratic* to solve. And thus, it goes without saying that your mastery of *quadratics* is one of the backbones to your success on this topic as well as many others beyond Algebra.

Before we offer you a ton of practice on the ideas mentioned above, let's support you with a handful of helpful examples as in the following!

$\frac{5x-15}{(x+5)^2}-\frac{5}{(x+5)}$

- We know that the expression above is equivalent to

$$\frac{k}{(x+5)^2}$$

where k is a constant and $x \neq -5$. What is the value of k?

Solution

The first thing we should do is get *common denominators* on the given expression. We do this as follows:

$$\frac{5x-15}{(x+5)^2}-\frac{5(x+5)}{(x+5)(x+5)}$$

$$\rightarrow \frac{5x-15-5(x+5)}{(x+5)^2}=\frac{5x-15-5x-25}{(x+5)^2}$$

$$¿\frac{-40}{(x+5)^2}$$

Since what we are told at the end means:

$$\frac{k}{(x+5)^2}=\frac{-40}{(x+5)^2}$$

It is abundantly clear at this point that $k=-40$!

- If $x>-2$, find a reduced equivalent form to:

$$\frac{1}{\frac{3}{x-2}-\frac{5}{x+4}}$$

Solution

Once again, we must get common *denominators* but first, let's isolate the *denominator* on 1 and simplify it.

$$\frac{3}{x-2}-\frac{5}{x+4}=\frac{3(x+4)}{(x-2)(x+4)}-\frac{5(x-2)}{(x+4)(x-2)}$$

$$\rightarrow \frac{3(x+4)-5(x-2)}{(x-2)(x+4)} = \frac{3x+12-5x+10}{(x-2)(x+4)}$$

$$¿ \frac{-2x+22}{(x-2)(x+4)} = \frac{-2(x+11)}{(x-2)(x+4)}$$

Now, notice that we kept the *common denominator* $(x-2)(x+4)$ in *factored form* all the way through. This is generally good practice because, in the end, although $(x+11)$ was not a factor that cancels either $x-2$ or $x+4$, sometimes, such is the case.

$$\frac{x^2-9}{x-3} = 2$$

- What are the values of x that satisfy the equation above?
  A. 3
  B. 0
  C. -1
  D. -1 and 3

Solution

The first thing we must notice is that if $x=3$, the denominator would be equal to 0, and division by 0 is not defined, so $x \neq 3$. Now we can solve the *Rational Equation* keeping this in mind. First, we will multiply both sides of the equation by $x-3$.

$$x^2-9=2(x-3)$$

$$x^2-9=2x-6$$

$$\rightarrow \frac{8x^2+10x-3}{2x+3} = \frac{(2x+3)(4x-1)}{2x+3} = 4x-1$$

This is so because

$$\frac{x^2-9}{x-3}=2 \rightarrow \frac{(x+3)(x-3)}{(x-3)}=2$$

You should look for this kind of convenience when you first look at a *Rational Equation*.

Polynomial *Long Division*

- The expression

$$\frac{4x-2}{x+5}$$

is equivalent to which of the following?

A. $\frac{4-2}{5}$

B. $4-\frac{2}{5}$

C. $4-\frac{22}{x+5}$

D. $4-\frac{2}{x+5}$

Solution

We are given the following expression: $\frac{4x-2}{x+5}$

Now, we can use *Long Division*, but we will apply a smart trick. We will rewrite the expression as follows:

$$\frac{4x-2}{x+5}=\frac{4x+20-20-2}{x+5}=\frac{4x+20-22}{x+5}$$

¿ $$\frac{4x+20}{x+5}-\frac{22}{x+5}=\frac{4(x+5)}{x+5}-\frac{22}{x+5}$$

$$\rightarrow =4-\frac{22}{x+5}$$

Which is answer choice C.

And this last example should have come up earlier in the “difficulty” *pecking order but as it is leads to a quadratic, that slight complication* (not hard actually) *has earned it its placement!*

- If

$$\frac{4}{2x}=\frac{4x-2}{8}$$

What is the value of?

$$\frac{3x}{2}$$

Solution

To solve for x, we will cross multiply and write:

$$\frac{4}{2x}=\frac{4x-2}{8}$$

$$\rightarrow 32=2x(4x-2)$$

$$\rightarrow 32=8x^2-4x$$

Now, we can *divide both sides by* 4.

$$\rightarrow 8=2x^2-x$$

$$\rightarrow 8=2x^2-x$$

$$2x^2-x-8=0$$

Finally, using the *quadratic formula*, we can get the values of x that satisfy the equation.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\rightarrow x=\frac{1\pm\sqrt{1+64}}{4}$$

After simplifying, we get that $x=\frac{1+\sqrt{65}}{4}$,or that $x=\frac{1-\sqrt{65}}{4}$ which means that

$$\frac{3x}{2}=\frac{3-3\sqrt{65}}{8}$$

Or

$$\frac{3x}{2}=\frac{3+3\sqrt{65}}{8}$$

Rational Equations (Passport to Advanced Math)

1. The expression

$$\frac{4x-1}{x+7}$$

is equivalent to which of the following?

A. $\frac{4-1}{7}$

B. $4-\frac{1}{7}$

C. $4-\frac{7}{x+7}$

D. $4-\frac{29}{x+7}$

2. If

$$\frac{4}{3}=\frac{4x-2}{(2-4x)}$$

What is the value of?

$$\frac{5x}{2}$$

A. $\frac{1}{2}$ B. $\frac{5}{2}$ C. $\frac{5}{4}$ D. $\frac{2}{5}$

3.
$$\frac{x^2-4}{x+2}=7$$

What are the values of x that satisfy the equation above?

A. 9 B. 0 C. -2 D. -2 *and* 9

4.
The expression
$$\frac{4x+2}{x+7}$$
is equivalent to which of the following?

A. $\frac{4+2}{7}$ B. $4+\frac{2}{7}$ C. $4+\frac{2}{x+7}$ D. $4-\frac{26}{x+7}$

Solutions *Rational Functions* & *Equations* (Passport to Adv. Math)

1. The correct answer is D. $4-\frac{29}{x+7}$

This is a question on *polynomial long division*. But we don't recommend doing *long division*. It is an overkill. Instead, observe the following clever solution.

$$\frac{4x+28-28-1}{x+7}$$

The all-important clever trick above is adding and subtracting 28 in the numerator. In total, we have added 0 because $+28-28=0$. Looking at the next few steps, you should be able to see why we chose this specific number.

$$\frac{4x+28-28-1}{x+7}=\frac{4x+28-29}{x+7}=\frac{4(x+7)-29}{x+7}$$

$$\rightarrow \frac{4(x+7)}{x+7}-\frac{29}{x+7}=4-\frac{29}{x+7}$$

2. The correct answer is C. $\frac{5}{4}$

Multiplying both sides of the equation by $3(2-4x)$

We have

$\rightarrow 4(2-4x)=3(4x-2)$

$\rightarrow 8-16x=12x-6$

$\rightarrow 14=28x$

$\rightarrow x=\frac{1}{2}$

Now that we have the value of x, we can simply multiply is by $\frac{5}{2}$.

$\frac{5x}{2}=\frac{5}{2}(x)=\frac{5}{2}\cdot\frac{1}{2}=\frac{5}{4}$

3. The correct answer is A. 9

This is a simple rational equation. Multiplying both sides of the equation by $x+2$

We have

$\rightarrow x^2-4=7(x+2)$

$\rightarrow x^2-4=7x+14$

$\rightarrow x^2-7x-18=0$

Now, $x^2-7x-18=(x-9)(x+2)$, this is a simple exercise in factoring *quadratics*.

$\rightarrow (x-9)(x+2)=0 \quad \rightarrow x=9$ or $x=-2$

Now, $x=-2$ doesn't work in the original equation because it would make the denominator 0. So, the only acceptable solution is $x=9$.

We could have also seen early on that $x=-2$ is an extraneous solution if we go back to the original equation we were given:

$\frac{x^2-4}{x+2}=7 \rightarrow \frac{(x+2)(x-2)}{x+2}=7 \rightarrow x-2=7$

$\rightarrow x=9$

You should always look to see if there is a common factor to cancel (such as what we just demonstrated) before you proceed to solve a *Rational Equation*.

4. The correct answer is D. $4-\frac{26}{x+7}$

This is a question on *Polynomial Long Division*. But we don't recommend doing *Long Division*. It is an overkill. Instead, we will apply the same clever trick we used in problem 2.

$$\frac{4x+28-28+2}{x+7}$$

$$\frac{4x+28-28+2}{x+7}=\frac{4x+28-26}{x+7}=\frac{4(x+7)-26}{x+7}$$

$$\rightarrow \frac{4(x+7)}{x+7}-\frac{26}{x+7}=4-\frac{26}{x+7}$$

More on Rational Equations (Passport to Advanced Math)

1. If

$$g(x)=\frac{x^2-4x+2}{x-2}$$

What is $g(-2)$?

A. $\frac{7}{2}$

B. $\frac{-5}{2}$

C. $\frac{5}{2}$

D. $\frac{-7}{2}$

2. Which of the following expressions is equivalent to?

$\frac{x^2+4x+2}{x-2}$

A. $x+6+\frac{14}{x-2}$ B. $x+6-\frac{8}{x-2}$ C. $x+3-\frac{7}{x-2}$ D. $x+1-\frac{8}{x-2}$

3. Which of the following expressions is equivalent to?

$\frac{x^2-4x-2}{x+4}$

A. $x-8+\frac{30}{x-2}$ B. $x+8-\frac{30}{x-2}$ C. $x-8-\frac{30}{x-2}$ D. $x+8-\frac{30}{x-2}$

4. Which of the following expressions is equivalent to?

$\frac{x^2+15x+20}{x+15}$

A. $x-\frac{20}{x+15}$ B. $x+\frac{20}{x+15}$ C. $x+1-\frac{20}{x+5}$ D. $x+\frac{20}{x+15}$

5. Which of the following expressions is equivalent to?

$$\frac{2x^2-x-15}{x-3}$$

A. $2x+5$

B. $2x+\frac{5}{x-3}$

C. $2x-5$

D. $2x-\frac{5}{x-3}$

6. If

$$h(x)=\frac{4x^2+2x+1}{x-1}$$

What is $h(-1)$?

A. Undefined B. $\frac{-3}{2}$ C. $\frac{5}{2}$ D. $\frac{-5}{2}$

7. Which of the following expressions is equivalent to?

$$\frac{8x^2+10x-3}{2x+3}$$

A. $4x-2$ B. $4x+2$ C. $4x+1$ D. $4x-1$

Solutions to *Rational* Equations (Passport to Advanced Math)

1. The correct answer is D. $\frac{-7}{2}$

   We are given the function $g(x)=\frac{x^2-4x+2}{x-2}$. Since we need to find $g(-2)$, all we must do is replace x with -2. Doing so, we have,

$$g(-2)=\frac{(-2)^2-4(-2)+2}{-2-2}=\frac{4+8+2}{-4}=\frac{14}{-4}=\frac{-7}{2}$$

2. The correct answer is A.
   The given expression is

$$\frac{x^2+4x+2}{x-2}$$

   We can do *polynomial long division*. But it would be like killing a fly with a shotgun. We can instead use the following clever trick on all similar questions on the SAT Math to avoid labor.

   Notice, the given expression is the same as

$$\frac{x^2-2x+2x+4x+2}{x-2}$$

   What we have done to the original expression is subtracted and added $2x$. That way, we have in net added 0 since $-2x+2x=0$.

   Why did we add and subtract $2x$. It will be clear soon enough so let's continue. Combing the two *like-terms* after "$-2x$" terms, we write:

$$\frac{x^2-2x+6x+2}{x-2}$$

$$\rightarrow \frac{x^2-2x}{x-2}+\frac{6x+2}{x-2}$$

Now, we can factor out an x from the first quotient in what we have written just above.

$$\rightarrow \frac{x(x-2)}{x-2}+\frac{6x+2}{x-2}$$

$$\rightarrow x+\frac{6x+2}{x-2}$$

Once again, more clever rewriting.

$$\rightarrow x+\frac{6x-12+12+2}{x-2}$$

$$\rightarrow x+\frac{6x-12+14}{x-2}$$

$$\rightarrow x+\frac{6x-12}{x-2}+\frac{14}{x-2}$$

$$\rightarrow x+\frac{6(x-2)}{x-2}+\frac{14}{x-2}$$

$$\rightarrow x+6+\frac{14}{x-2}$$

3. The correct answer is A.

We start with:

$$\frac{x^2-4x-2}{x+4}$$

We will apply the same clever trick on this problem as we did in the previous solutions (problem 2.)

Notice, the given expression is the same as

$$\frac{x^2+4x-4x-4x-2}{x+4}$$

What we have done is added and subtracted $4x$. That way, we have in net added 0 since $4x-4x=0$.

Let's continue with the solution.

$$\frac{x^2+4x-8x-2}{x+4}$$

$$\rightarrow \frac{x^2+4x}{x+4}+\frac{-8x-2}{x+4}$$

Now, we can factor out an x from the first quotient in what we have written just above.

$$\to \frac{x(x+4)}{x+4}+\frac{-8x-2}{x-2}$$

$$\to x+\frac{-8x-2}{x-2}$$

Once again, more clever rewriting, the reason for this should become clear, once again!

$$\to x+\frac{-8x-32+32-2}{x+4}$$

$$\to x+\frac{-8x-32+30}{x+4}$$

$$\to x+\frac{-8x-32}{x+4}+\frac{30}{x+4}$$

$$\to x+\frac{-8(x+4)}{x+4}+\frac{30}{x+4}$$

$$\to x-8+\frac{30}{x+4}$$

4. The correct answer is D.

The given expression is

$$\frac{x^2+15x+20}{x+15}$$

We will apply the same clever trick on this problem that we had previously used.

Notice, the given expression is the same as

$$\frac{x^2+15x}{x+15}+\frac{20}{x+15}=\frac{x(x+15)}{x+15}+\frac{20}{x+15}$$

Now, after simplifying, we get our answer:

$$x+\frac{20}{x+15}$$

5. The correct answer is A.

The given expression is

$$\frac{2x^2-x-15}{x-3}$$

We will apply the same clever factoring trick on this problem.

Notice, the given expression is the same as:

$$\frac{2x^2-6x+6x-x-15}{x-3}$$

$$\frac{2x^2-6x+5x-15}{x-3}$$

$$\rightarrow \frac{2x^2-6x}{x-3}+\frac{5x-15}{x-3}$$

Now, we can factor out a $2x$ from the first quotient, and a 5 from the second one to write:

$$\rightarrow \frac{2x(x-3)}{x-3}+\frac{5(x-3)}{x-3}$$

Canceling the $x-3$ in both quotients, we get our answer:

$2x+5$

6. The correct answer is B.
   We are given the function

   $$h(x)=\frac{4x^2+2x+1}{x-1}$$

   Since we want to find $h(-1)$, all we must do is replace $x$with -1.

   Doing this, we have, $h(-1)=\frac{4(-1)^2+2(-1)+1}{-1-1}=\frac{4-2+1}{-2}=\frac{3}{-2}=\frac{-3}{2}$

7. The correct answer is D.
   This can also be done using *Polynomial Long Division*. But once again, that would be an overkill. But, this time, instead of using clever rewriting of the given expression as we had previously done, it is better if we just factor the *quadratic* in the numerator.
   The *quadratic* in the numerator can be factored using the "*ac method*."

   $8x^2+10x-3$ is of the form ax^2+bx+c

   So, we multiply a and c. $ac=8(-3)=-24$

   Now, we must figure out what two numbers multiply to -24 and add to 10.

   It should be clear that the two numbers are 12 and -2.

   So, using these two numbers as *coefficients*, we split the middle term bx into two terms as follows

$$8x^2+10x-3=8x^2+12x-2x-3$$

Now, we group factor the four-termed *quadratic*, group the <u>first two</u> terms and the <u>last two</u> terms together.

$$8x^2+12x-2x-3=4x(2x+3)-1(2x+3)$$

$$\rightarrow 8x^2+10x-3=4x(2x+3)-1(2x+3)$$

$$¿(2x+3)(4x-1)$$

$$\rightarrow \frac{8x^2+10x-3}{2x+3}=\frac{(2x+3)(4x-1)}{2x+3}=4x-1$$

More on Rational Functions (Other)

1. For what value(s) of x is the following function *undefined*?

$$g(x)=\frac{x+2}{(x+2)^2+7(x+2)+12}$$

   A. -2 and -5
   B. -5 and -6
   C. -6 and -2
   D. -2 only

2. For what value(s) of x is the following function *undefined*?

$$f(x)=\frac{x+5}{(x+5)^2+12}$$

   A. -5
   B. -5 and 0
   C. 0 only
   D. It is defined for all values of x.

3. *Salvador* correctly performed *Polynomial Long Division* as written in the following equation where a and b are positive integers. What is the value of $a+b$?

$$\frac{4x^3+12x^2+20}{4x-8}=x^2+x+\frac{ax+b}{x-2}$$

A. 15
B. 5
C. 10
D. 12

4. In the rational equation below, c and d are constants. What is the value of $c+d$?

$$\frac{x^2-c}{x^2+d}=\frac{x^4-1}{x^4+2x^2+1}$$

A. -2
B. 2
C. 0
D. 1

5. In the expression below, if a, b, and c are real constants, what is the value of $a+b+c$?

$$f(x)=\frac{1}{\frac{4}{x-2}+\frac{2}{x^2-4}}=\frac{(x+a)(x+b)}{4x+c}$$

A. 14
B. -10
C. -14
D. 10

6. In the equation below, if the value of $k=7$, then what is the value of $\frac{k}{x}$?

$$\frac{k}{x^2}=\frac{x}{49}$$

A. 1

B. 7
C. 7^2
D. 0

7.
$$\frac{4x-2}{(x+3)^2}-\frac{x}{(x+3)}$$

The expression above is equal to
$$\frac{-ax^2+bx+c}{(x+3)^2}$$
What is the value of $a+b+c$?

A. -2
B. 2
C. -3
D. -1

Solutions to More on Rational Functions (Other)

1. The correct answer is B.
   Since we have a *Rational Function*, it is not defined when the denominator is equal to zero. Because of this, we should set the denominator equal to zero, and solve for the value of x that makes it undefined. To get there, we must solve the following *quadratic* equation:

$$(x+2)^2+7(x+2)+12=0$$

First, we will *expand the binomial* (the squared quantity) and *distribute the seven*. Doing so, we have:

$$x^2+4x+4+7x+14+12=0$$

Now, we will *combine-like-terms*.

$$x^2+11x+30=0$$

We can solve this *quadratic* equation by factoring. Thus, we write:

$$(x+5)(x+6)=0$$

So, we see that the two x values that make the denominator undefined are $x=-5$ and $x=-6$, which is answer choice B.

2. The correct answer is D. It is defined for all values of x.

Once again, because it is a Rational Function, it is not defined when the denominator is equal to zero. But when we attempt to find values that make the denominator equal to zero, we must start as follows.

$$(x+5)^2+12=0$$

First, we will subtract 12 from both sides to write:

$$(x+5)^2=-12$$

Now, before you even move forward from the last step, notice that we have a squared quantity on the left of the equal sign above and a negative number on the right. This is impossible.

3. The correct answer is A.

Our task is to find the value of $a+b$ in the following expression:

$$\frac{4x^3+12x^2+20}{4x-8}=x^2+x+\frac{ax+b}{x-2}$$

We will manipulate the left-hand side of the equality to make it look like the right-hand side, and then we will compare both sides. We'll start by splitting the middle term in the numerator of the left-hand side as follows.

$$¿\frac{4x^3-8x^2+20x^2+20}{4x-8}$$

Now, we will split this left-hand side into two fractions:

$$\frac{4x^3-8x^2}{4x-8}+\frac{20x^2+20}{4x-8}=¿$$

Now we will factor and simplify:

$$\frac{x^2(4x-8)}{4x-8}+\frac{20x^2+20}{4x-8}=x^2+5x+\frac{ax+b}{x-2}$$

$$x^2+\frac{20x^2+20}{4x-8}=x^2+5x+\frac{ax+b}{x-2}$$

Looking good eh? We will apply the same strategy on the remaining part.

$$x^2+\frac{20x^2+20}{4(x-2)}=x^2+5x+\frac{ax+b}{x-2}$$

$$x^2+20\frac{(x¿¿2+1)}{4(x-2)}=¿¿$$

$$x^2+5\frac{(x¿¿2+1)}{(x-2)}=x^2+5x+\frac{ax+b}{x-2}¿$$

$$x^2+5\frac{(x¿¿2-2x+2x+1)}{(x-2)}=x^2+5x+\frac{ax+b}{x-2}¿$$

$$¿x^2+5\frac{(x¿¿2-2x)}{(x-2)}+\frac{5(2x+1)}{(x-2)}=x^2+\frac{5x(x-2)}{(x-2)}+\frac{5(2x+1)}{(x-2)}¿$$

$$¿x^2+5x+\frac{5(2x+1)}{(x-2)}=x^2+5x+\frac{ax+b}{x-2}$$

$$x^2+5x+\frac{10x+5}{x-2}=x^2+5x+\frac{ax+b}{x-2}$$

Aaha, finally. Looking closely to the position of terms on both sides, we can conclude that $a=10$and $b=5$. Hence, the value of $a+b$is 15.

4. The correct answer is C.

This is just a fun exercise in factoring if you like factoring like we do. Notice that both denominator and numerator on the quotient to the left of the equal sign are quadratics.

$$\frac{x^2-c}{x^2+d}=\frac{x^4-1}{x^4+2x^2+1}$$

So, the *quartics* on the right-hand side (both numerator and denominator) must have a common quadratic as a factor. Otherwise, this question doesn't make sense!

$$\frac{x^2-c}{x^2+d}=\frac{(x^2+1)(x^2-1)}{\text{¿¿}}$$

We used the *Difference of Squares* [$x^4-1=(x^2)^2-1^2=a^2-b^2=(a+b)(a-b)=(x^2+1)(x^2-1)$] in the numerator and the *Binomial Square* [$x^4+2x^2+1=(x^2)^2-2x^2(1)+1^2=a^2+2a(b)+b^2=(a+b)^2$¿ in the denominator.

Cancelling the repeated factor on the far right in **, we write:

$$\frac{x^2-c}{x^2+d}=\frac{x^2-1}{x^2+1}$$

By comparing both sides of the equation, we can see that $c=1$and $d=1$, which means that the value of $c+d$ is equal to $-1+1=0$.

5. The correct answer is D. 10

This is so because, as we did in the third example in the section on Rational Functions, here, we focused on the denominator of 1. So, we write:

$$\frac{4}{x-2}+\frac{2}{x^2-4}=\frac{4}{x-2}+\frac{2}{(x-2)(x+2)}$$

$$\text{¿}\frac{4(x+2)}{(x-2)(x+2)}+\frac{2}{(x-2)(x+2)}$$

$$\text{¿}\frac{4(x+2)+2}{(x-2)(x+2)}=\frac{4x+8+2}{(x-2)(x+2)}$$

$$\text{¿}\frac{4x+10}{(x-2)(x+2)}$$

$$\rightarrow\frac{1}{\frac{4}{x-2}+\frac{2}{x^2-4}}=\frac{1}{\frac{4x+10}{(x-2)(x+2)}}=\frac{(x-2)(x+2)}{4x+10}$$

Since we have:

$$\frac{(x+a)(x+b)}{4x+c}=\frac{(x-2)(x+2)}{4x+10}$$

$\rightarrow a+b+c=-2+2+10=10$

6. The correct answer is A.
   We will apply an interesting trick to solve this question quickly. We are given the following expression:

$$\frac{k}{x^2}=\frac{x}{49}$$

First, we will substitute the value of $k=7$.

$$\frac{7}{x^2}=\frac{x}{49}$$

First, we will multiply both sides by x^2.We will get the following equation:

$$7=\frac{x^3}{49}$$

Now, notice that 49 is 7^2. So we will write:

$$7=\frac{x^3}{7^2}$$

Multiplying both sides by 7^2, we have:

$$7^3=x^3$$

This leads to the conclusion that $x=7$. Now, we can substitute this value in the expression we have to find.

$$\frac{k}{x}=\frac{7}{7}=1$$

7. The correct answer is C.
   First, we observe that the expression in which a, b , and c are present has a specific form. We have a *second-degree polynomial* in the numerator, and a squared binomial in the denominator. We will try to give this form to the given expression:

$$\frac{4x-2}{(x+3)^2}-\frac{x}{(x+3)}$$

First, we will combine the fractions by multiply by $x+3$, the *numerator* and *denominator* of the quotient to the right of the minus sign.

$$\frac{4x-2}{(x+3)^2}-\frac{x(x+3)}{(x+3)^2}$$

$$\frac{4x-2}{(x+3)^2}-\frac{x^2+3x}{(x+3)^2}$$

Since both quotients have a *common denominator* now, we can simply add the numerators under a single denominator.

$$\frac{4x-2-(x^2+3x)}{(x+3)^2}$$

Only thing left to do is simplifying the *numerator*, and we will be able to find what we need.

$$\frac{4x-2-x^2-3x}{(x+3)^2}$$

$$\frac{-x^2+x-2}{(x+3)^2}$$

By comparing both expressions, we can see that $a=1, b=1, c=-2$. Hence, the value of $a+b+c$ is $1+1-2=0$. The correct answer is C.

Work Problems (Other)

The approach to solving work problems is the same. Let us consider the following problem as our first example. Talking about these types of problems doesn't make sense without giving an example.

Example 1

Let us say *Reus can shovel snow off a lot in* 40 *minutes. Luis can shovel the same lot of snow in* 30 *minutes. Working together, how long would Reus and Luis take to shovel the lot?*

Solution

Since they are shoveling one lot, we assign the value 1 to the *numerator* and write the following equation.

$$\frac{1}{40}+\frac{1}{30}=\frac{1}{x}$$

In the solution to this equation, the value of x, will tell us how long it will take them working together.

Of course, solving this equation is a simple exercise in solving a basic rational equation.

$$\frac{1}{40}+\frac{1}{30}=\frac{1}{x}\rightarrow\frac{1(3)}{40(3)}+\frac{1(4)}{30(4)}=\frac{1}{x}$$

$$\rightarrow\frac{3}{120}+\frac{4}{120}=\frac{1}{x}$$

$$\rightarrow\frac{7}{120}=\frac{1}{x}$$

Here, you should know to cross multiply. Doing so, we have:

$$\rightarrow\frac{7}{120}=\frac{1}{x}\rightarrow 7x=120$$

$$\rightarrow x=\frac{120}{7}=17.4$$

So, clearly, they would get the job done faster if they worked together even though they work at different paces (this is usually what happens in these problems.)

Example 2

This time, *Reus is stuffing 100 envelopes and he can do it in 45 minutes. If Luis and Reus can stuff 100 envelopes in 30 minutes working together, how many minutes does it take Luis to stuff 100 envelopes?*

Solution

We start with the following equation:

$$\frac{1}{45}+\frac{1}{x}=\frac{1}{30}$$

Now, notice that we might want to make the *numerators* 100 to account for the 100 envelopes. But look at what happens if we do.

$$\frac{100}{45}+\frac{100}{x}=\frac{100}{30}$$

We can divide both sides of the equation by 100, to reduce it back to the original equation we had. So, we consider stuffing 100 envelopes as one job. If it was 150envelopes for say Luis, we would make the numerator on x,1.5. If it was 200 envelopes for Luis, the *numerator* on x would be 2 etc.

But since we are working with the original equation, we work to get *common denominators* and proceed as before.

$$\frac{1(x)}{45(x)}+\frac{1(45)}{x(45)}=\frac{1}{30}$$

$$\rightarrow \frac{x}{45x}+\frac{45}{45x}=\frac{1}{30} \rightarrow \frac{x+45}{45x}=\frac{1}{30}$$

$$\rightarrow 30(x+45)=45x$$

$$\rightarrow 30x + (30)(45) = 45x$$

$$\rightarrow 30 \cdot 45 = 15x$$

$$\rightarrow \frac{30 \cdot 45}{15} = x$$

$$\rightarrow 30 \cdot 3 = x = 90$$

Notice, we kept $(30)(45)$ instead of 1350. *This is so because we should anticipate the need to divide*. As such, *it doesn't make sense to figure out* 30*times* 45 *and then figure out how to divide that by* 15. This kind of move usually pays dividends in doing arithmetic efficiently.

There is really no point in providing any more examples, but we will give one more example for any student who might have felt one more example was all they needed to fully master this topic.

Example 3

Reus can pack 42 boxes in 30 minutes. If *Luis* helps him, *Reus* and *Luis* can pack 84 boxes in 20 minutes. How quickly can *Luis* pack 126 boxes?

Solution

To start, notice $84$is double 42 and 126 is triple 42.

Therefore, if we can figure out how many *minutes* it takes *Luis* to pack 42 boxes, we can just triple that number of *minutes* to figure out how long it takes him to pack 126 boxes.

Here is how we set up our equation.

$$\frac{1}{30} + \frac{1}{x} = \frac{2}{20}$$

So, the two *numerators* that are 1 are saying 42boxes for *Reus* and *Luis* respectively, the 2on the right-hand side is saying 84 boxes with the 20 in the *denominator* representing how long it takes both working together to pack 84 boxes.

Of course, we can reduce $\frac{2}{20}$ so we should do that before we continue solving the equation.

$$\frac{1}{30}+\frac{1}{x}=\frac{2}{20} \rightarrow \frac{1}{30}+\frac{1}{x}=\frac{1}{10}$$

From here, proceeding as before, we have:

$$\frac{1(x)}{30(x)}+\frac{1(30)}{x(30)}=\frac{1}{10}$$

$$\rightarrow \frac{x}{30x}+\frac{30}{30x}=\frac{1}{10}$$

$$\rightarrow \frac{x+30}{30x}=\frac{1}{10}$$

Cross multiplying, we have

$$\rightarrow 10(x+30)=30x$$

$$\rightarrow 10x+300=30x$$

$$\rightarrow 300=20x$$

$$\rightarrow x=\frac{300}{20}$$

$$\rightarrow x=\frac{30}{2}=15$$

So, since *Luis* can pack 42 boxes in 15 minutes, it would take him $45=(3)(15)$ minutes to pack 126 boxes.

Following this discussion are a few problems for you to practice. While these examples should have adequately prepared you for the examples to follow, as some examples require you to use hours, either feel free to convert the hours into minutes or keep track of the units given in the problem and in the answer choices.

Work Practice Problems (Other)

1. *Peter* can tie 20 shoelaces in 1 hour. It takes *Joshua* twice as long as *Peter* to tie the same 20shoelaces. If they worked together, how many minutes would it take *Peter* and *Joshua* to tie the 20 shoelaces?

   A. $\frac{2}{3}$
   B. 40
   C. 120
   D. 80

2. At a hotdog eating contest, *Joey* can eat 70 hotdogs in 10 minutes and *Miki* can eat 140 hotdogs in 25 minutes. If *Joey* and *Miki* started eating hotdogs together, how many hotdogs (*to the nearest hotdog*) could they eat in 12minutes?

   A. 210
   B. $\frac{42}{25}$
   C. 118
   D. 151

3. *Natalie* can bake an *Injera* 3 times faster than *Hanan*. Working together, the two women can bake 40 *injeras* in 5 hours. How long would it take *Natalie* to bake 20*injeras*?

   A. 10 minutes
   B. 200 minutes
   C. 600 minutes
   D. 30 minutes

4. *Jessica* and *Alba* can put make-up on 14 clients in 42 minutes. *Jessica* can put make-up on x clients in $6x$ minutes and *Alba* can put make-up on y clients in $7x$ minutes. If $y=kx$, what is the value of k ?

Solutions to Work Practice Problems (Other)

1. The correct answer is B. 40.

Since *Joshua* takes twice as long as *Peter* to tie 20 shoelaces, and *Peter* takes 1 hour, *Joshua* takes 2 hours. To find out how long they would take to tie the same 20 shoelaces working together, we must solve for x in the following equation:

$$\frac{20}{1}+\frac{20}{2}=\frac{20}{x}$$

$$30=\frac{20}{x}$$

$$x = \frac{2}{3}$$

Since we have been working in hours and we are asked to find how many minutes they would take, we need to convert this value $\left(\frac{2}{3}\right)$into minutes. We can do so by multiplying $\frac{2}{3}$ by 60.

$$60\left(\frac{2}{3}\right) = 40$$

2. The correct answer is D. 151.2

We can write the following equation to find the answer:

$$\frac{70}{10} + \frac{140}{25} = \frac{x}{12}$$

The left side of the equation represents the number of hotdogs they both can eat in 1 *minute*. Since we are told that they can eat x hotdogs in 12 minutes, $\frac{x}{12}$ is also the number of hotdogs they can eat in one *minute*, so we proceed to solve for x in the equation.

$$\frac{70}{10} + \frac{140}{25} = \frac{x}{12}$$

$$12(7 + 5.6) = x$$

$$x = (12)(12.6)$$

$$x = 151$$

3. The correct answer is B. 200.

Since *Natalie* can bake an *Injera* 3 times faster than *Hanan*, this means that in the time Natalie bakes $3x$ *Injeras*, *Hanan* bakes x *Injeras*. We can combine this information with the next fact. Working together, they can bake 40 *Injeras* in 5 hours. We can set up the following equation:

$$x + 3x = 40 \quad \rightarrow x = 10$$

In 5 hours, *Hanan* can bake 10 *Injeras*, while *Natalie* can bake 30. Because 5 hours is equivalent to 300 *minutes*, *Natalie* takes 300 *minutes* to bake 30 *Injeras*, so she takes 10 *minutes* to bake a single *Injera*. We can set up the following relationship:

10 *minutes* ¿ 1 *Injera*

We can multiply by 20 on both sides

$$200\ minutes = 20\ Injera$$

4. The correct answer: $\frac{7}{6}=k$

To start, we know that we must write the following equation (consult the previous solutions and the examples in the book to figure out why.)

$$\frac{14}{42}=\frac{x}{6x}+\frac{y}{7x}$$

$$\rightarrow \frac{1}{3}=\frac{7x}{42x}+\frac{6y}{42x}$$

$$\rightarrow \frac{1}{3}=\frac{7x+6y}{42x}$$

Now, we are additionally told that $y=kx$. So, making this substitution, our equation above transforms to:

$$\rightarrow \frac{1}{3}=\frac{7x+6(kx)}{42x}$$

$$\rightarrow \frac{1}{3}=\frac{(7+6k)x}{42x} \rightarrow \frac{1}{3}=\frac{(7+6k)}{42}$$

$$\rightarrow \frac{42}{3}=7+6k \rightarrow 14=7+6k$$

$$\rightarrow 7=6k \rightarrow \frac{7}{6}=k$$

Direct and *Inverse* Relations

Remember *a relation* is a *function*. As such, *Direct* and *Inverse* relations are *functions* between two variables where the relationship is, as the names say, either *Direct* or *Inverse*. Wait, but what does that mean?

A *Direct* relation is one where if the variable along the *horizontal* axis goes up, the variable along the *vertical* axis goes up also. The natural model for this relationship is a line with a positive slope. The reason why a line with a negative slope doesn't make sense is because as $x-$¿value go up, a negatively slopped line would have the $y-$¿values going down and that is not a Direct relationship that is more like an *Inverse* relation.

Now, we can give a lot of natural examples as these relations come up naturally, but one will suffice for each.

For a *Direct* relation, the variable on the horizontal axis can be age or shoe size, and the variable on the y-axis can be the other. In other words, it doesn't matter which variable is along the $x-$¿axis in this case, it is naturally understood that as your age goes up, so does your shoe size. Many other examples abound but again, this will do as a demonstration of *Direct* relations.

The quintessential example of an *Inverse* relation is <u>the relation between the *width* and *length* of a rectangle with *a fixed area*</u>. Let us say that the rectangle has an *area* of 24. Then, here are a table of integer values of what the *width* and *length* have to bee to keep the area 24.

Width (w)	*Length* ¿)	*Area* $A=24$
24	1	24
12	2	24
8	3	24
3	8	24
2	12	24
1	24	24

What is abundantly clear here is that as the width decreases, the length increases. Or, as the length decreases, the width increases, etc. This is what we call an *Inverse* relation.

Whereas *Direct* relations are modeled by

$$y=kx$$

Inverse relations are modeled by

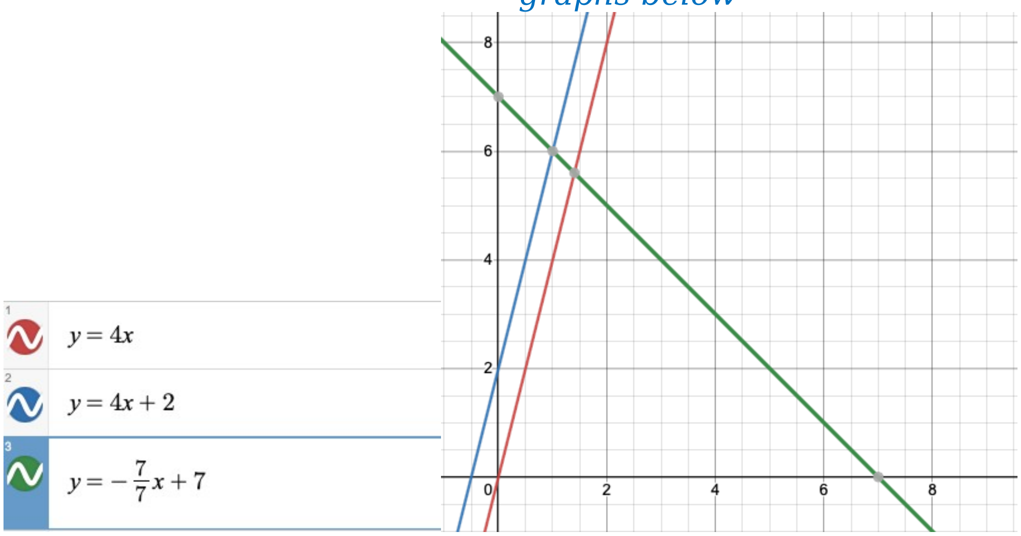
$y = \frac{k}{x}$

It is possible to consider

$$y = kx + b \qquad \& \qquad y = \frac{k}{x} + b$$

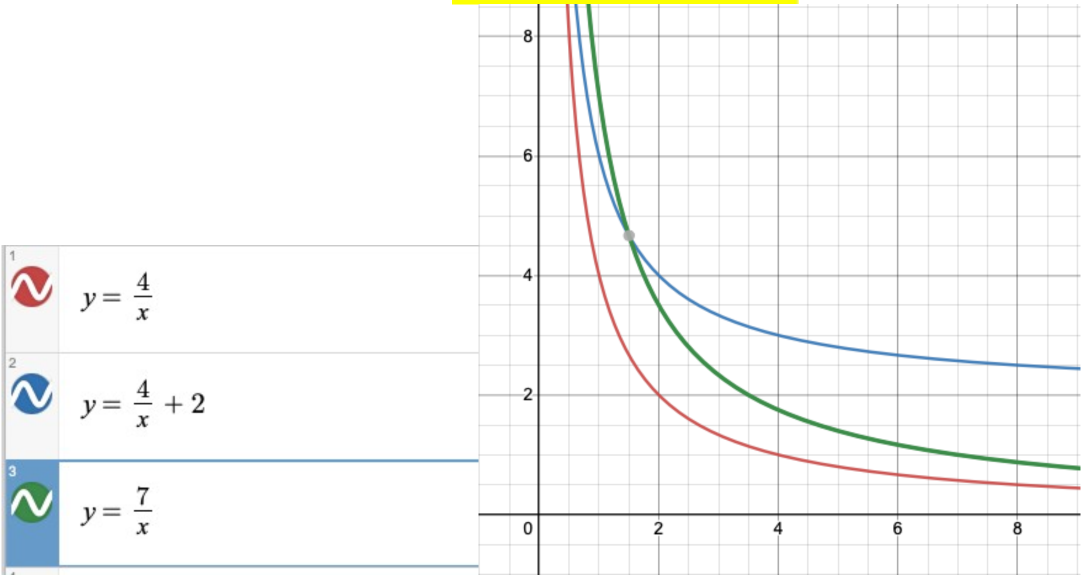
Direct and *Inverse* relations because they are vertical shifts of the original two equations we wrote above. And as you can see from the "*graphs below"* and your understanding of what vertical shifts do to functions, the core relation we want is preserved despite a vertical shift. That is, even if $y = kx$ is shifted by b to turn into $y = kx + b$, we still have that as x increases, y increases so long as k is positive. But, in practice, we only deal with $b = 0$ when we are talking about inverse and direct relations.

"graphs below"



Direct relations graphed

Two of the three graphs above show a direct relation. Notice that from the first line $y = 4x$ to the second $y = 4x + 2$, the only change is the value of b (the *vertical shift amount*.) But, in both cases, we have *a Direct* relation. It is in the negatively slopped line of the third and last equation on the left above that is not a direct relation. This is so because as the *horizontal* variable increases, the *vertical* variable is decreasing. Finally, notice it only makes sense to consider positive values whether it is the *shoes* and *age* or other pair of variables we might want to model.



Inverse relations graphed

All three graphs this time show an *Inverse* relation between the variable on the *horizontal* axis and the variable on the *vertical* axis. Just as *Direct* relations, it only naturally makes sense to consider positive values along both axes. The values of k are changed from $4$to 7 but notice that a change in the value of k doesn't change the overall picture in the relation.

Before we conclude, let's turn the area *Inverse* relation we already saw into a word problem so that you get one actual practice on a question about these ideas just how it might be presented to you!

Example

The *width* of a rectangle is 21 when the *length* is 2. What is the *length* of the rectangle when the *width* is 6?

Solution

We know that the *area* of a rectangle is

$$A = lw$$

$$\rightarrow 42 = (2)(21) = A$$

$$lw = 42$$

$$\rightarrow l = \frac{42}{w}$$

Now we just substitute the given *width*, and we can find the *length* with this last equation above.

$$\rightarrow l=\frac{42}{6}=7$$

Other

Complex Numbers (*Imaginary Numbers*)

A number in mathematics is called *Imaginary* (*Complex*) if it is of the form

$a+bi$

where a and b are *real numbers* and $i=\sqrt{-1}$.

Now, $i=\sqrt{-1} \to i^2=-1$.

From this, we can see that $i^3=i\cdot i^2=i(-1)=-i$

Subsequently, $i^4=i^2\cdot i^2=(-1)(-1)=1$, etc.

Specifically, it should be clear that $i^{65}=i^{64}\cdot i=(i^4)^{16}\cdot i=(1)^{16}\cdot i=i$

Adding and Subtracting Complex Numbers

As this is not much different than combining-like-terms, no lengthy discussion should be required. In fact, we should be able to dive right into the practice problems which are all accompanied by solutions. But a quick reminder perhaps won't hurt.

In $a+bi$, a is called, "the *real part*." And b is called, "the *imaginary part*."

Note: Not a typo, b is the *imaginary part* (just the constant that multiplies the i.)

So, to add two *Complex Numbers*, "add *real part* to *real part* and add *imaginary part* to *imaginary part*."

Example

$4-2i-(4+2i)$

¿$4-2i-4-2i=4-4-2i-2i=-4i$

Now, looking at $-4i$, first, note that $-4i=0+(-4)i$

Therefore, the *real part* of our answer is $a=0$ and the *imaginary part* of our answer is -4.

Notice, as we said, combining *imaginary parts* is like "combining like-terms."

Compare $4-4-2i-2i=-4i$ to $4-4-2x-2x=-4x$.

Multiplying Complex Numbers

Multiplying two *complex numbers* is like the **FOIL** process for multiplying two *Binomials* in our work on *Polynomials*.

Let us provide one example:

$(4-2i)(7+2i)$

F $\rightarrow (4)(7)$, **O** $\rightarrow (4)(2i)$, **I** $\rightarrow (-2i)(7)$, **L** $\rightarrow (-2i)(2i)$.

When we simplify, we get:

$(4)(7)+(4)(2i)+(-2i)(7)+(-2i)(2i)$

¿ $28+8i-14i-4i^2$

¿ $28+8i-14i-4(-1)$

¿ $28+8i-14i+4$

¿ $32-6i$

Dividing Complex Number

Recall $a^2+b^2=(a-bi)(a+bi)$ from our discussion of *complex numbers* on an example on the *Binomial Square* and *Difference of squares* and then again at the end of *Polynomials Continued….*

There, we said that it was going to be monumentally important in "*Complex Division*." And so, that moment has arrived.

- Consider the quotient:

$$\frac{4+2i}{1-2i}$$

The goal in *Complex Division* is to get a result at the end that is of the form $x+iy$.

To get there, what we must do is

multiply both the numerator and denominator of the quotient by the conjugate of the denominator.

This is always what you should do when you are given a problem on dividing two *complex numbers*.

$$\frac{(4+2i)(3+2i)}{(3-2i)(3+2i)}$$

Now, on the numerator, our earlier discussion about multiplying *complex numbers* should have served you well.

Specifically, $(4+2i)(3+2i)=12+8i+6i+(2i)^2=12+14i+4i^2$

Now, keeping in mind that $i^2=-1$, we have $12+14i+4i^2=12+14i-4=8+14i$.

So, we are done with the numerator. You should make easy work of the denominator.

Since it is a product of *conjugates*, we can exploit our earlier result that

$$(a-bi)(a+bi)=a^2+b^2$$

to write: $(3-2i)(3+2i)=3^2+2^2=9+4=13$.

Putting all of this together, we have:

$$\frac{4+2i}{1-2i}=\frac{8+14i}{13}=\frac{8}{13}+\frac{14}{13}i$$

In what follows, you will get sufficient practice on *Complex Numbers* and their arithmetic.

Complex Numbers (Other)

$(2-3i)-(-2+5i)$

1. Which of the following *complex numbers* is equal to the expression above? (Recall that $i=\sqrt{-1}$.)
   A. $4-2i$
   B. $4+8i$
   C. $4+2i$
   D. $4-8i$

2. Which of the following *complex numbers* is equal to $(-4-2i^3)-(-4i^2+2i)$ with $i=\sqrt{-1}$?
   A. $-4i$
   B. $4i$
   C. -8
   D. 0

3. Which of the following *complex numbers* is equal to $(2-3i)(-2+5i)$ for $i=\sqrt{-1}$?
   A. 19
   B. $2i$
   C. $-4-2i$
   D. $11+16i$

4. Which of the following *complex numbers* is equal to $(-4-2i^3)(-4i^2+2i)$ with $i=\sqrt{-1}$?
   A. $-16+16i$
   B. -20
   C. 16
   D. $16+4i$

5. What of *complex numbers* in Standard Form is equal to the following quotient? (With $i=\sqrt{-1}$)

$\frac{2-3i}{-2+5i}$?

6. What of *complex numbers* in Standard Form is equal to the following quotient? (With $i=\sqrt{-1}$)

$\frac{2-3i}{i^3}$?

7. What of *complex numbers* in Standard Form is equal to the following quotient? (With $i=\sqrt{-1}$)

$\frac{-4-2i^3}{-4i^2+2i}$?

Solution to *Complex Numbers* (Other)

1. The correct answer is D. $4-8i$

    All we must do really is *combine-like-terms*. First, distributing the minus sign, we have:

$(2-3i)-(-2+5i)=2-3i+2-5i$

$\rightarrow 2-3i+2-5i=4-8i$

2. The correct answer is C. -8

    We must simplify the expression $(-4-2i^3)-(-4i^2+2i)$. Since $i=\sqrt{-1}$, it follows that $i^2=-1$. So then $i^3=i^2\cdot i=(-1)i=-i$.

    Using these, the given expression simplifies as follows.

    $(-4-2i^3)-(-4i^2+2i)=-4-2i^3+4i^2-2i$

    ¿$-4-2(-i)+4(-1)-2i$

$\rightarrow = -4+2i-4-2i=-8$

3. The correct answer is D. $11+16i$

   This is just a simple exercise in multiplying two *binomials*.

   $(2-3i)(-2+5i)=2(-2)+(2)(5i)+(-3i)(-2)+(-3i)(5i)$

   ¿$-4+10i+6i-15i^2$
   Now, recalling $i^2=-1$, we have:

   ¿$-4+10i+6i-15(-1)$

   ¿$-4+15+10i+6i=11+16i$

4. The correct answer is B. -20.
   We are given the expression $(-4-2i^3)(-4i^2+2i)$,and we are tasked with simplifying it.
   We already know that $i^2=-1$ and that $i^3=-i$. Using this, we can rewrite the expression as follows:
   $(-4-2i^3)(-4i^2+2i)=[-4-2(-i)][-4(-1)+2i$¿$]$

   $\rightarrow (-4+2i)(4+2i)=(2i-4)(2i+4)$

   Now, we can use the *Difference of Squares*!

   $(2i-4)(2i+4)=(2i)^2-4^2=4i^2-16=-4-16=-20$

5. The correct answer is $\frac{-19}{29}-\frac{4i}{19}$.
   As you should know, all *complex numbers* can be written as $a+bi$, where a and b are *real numbers*. Now, when you are *dividing two complex numbers*, the most important thing is to multiply the numerator and denominator by the conjugate of the denominator, $a-bi$.But first, we need to make sure the denominator is of the form $a+bi$.
   Notice in the quotient

$$\frac{2-3i}{-2+5i}$$

   The *denominator* is of the form $a+bi$. Too make it easier, we will rearrange the denominator. We will write the denominator as $5i-2$, and its conjugate will be $bi-a=5i-(-2)=5i+2$.
   Thus, multiplying both denominator and numerator of the given quotient by this conjugate, we write:

$$\rightarrow \frac{(2-3i)(5i+2)}{(5i-2)(5i+2)}$$

Now, we can apply difference of squares in the denominator and expanding the product in the numerator, we will have to write:

$$\frac{(2-3i)(5i+2)}{(5i-2)(5i+2)}=\frac{10i+4-15i^2-6i}{(5i)^2-2^2}=\frac{4i+19}{-29}$$

Writing our answer in standard form ($a+bi$¿, we have:

$$\frac{4i+19}{-29}=\frac{-19}{29}-\frac{4i}{19}$$

6. The correct answer is A.
   If you need more details, consult the solution to problem 5.
   Notice in the given quotient:

$$\frac{2-3i}{i^3}$$

The denominator i^3 is not of the form $a+bi$. But it is an easy fix.

$$i^3=-i=0-1i$$

So, the conjugate of the denominator is $0+1i=i$

Thus, we should multiply both denominator and numerator by this conjugate!

$$\rightarrow\frac{(2-3i)i}{i^3 i}\rightarrow\frac{2i-3i^2}{i^4}$$

$i^2=-1$ and therefore, $i^4=i^2 i^2=(-1)(-1)=1$

$$\rightarrow\frac{2i-3(-1)}{1}$$

Simplifying completely and writing our answer in the standard form $a+bi$, we have

$$3+2i$$

7. The correct answer is $\frac{-3}{5}+\frac{4}{5}i$
   To start, we are given the quotient

$$\frac{-4-2i^3}{-4i^2+2i}$$

We know that $i^2=-1$ and that $i^3=-i$. And so the given quotient just above is the same as the following:

$$\frac{-4-2(-i)}{-4(-1)+2i}=\frac{-4+2i}{4+2i}=\frac{2i-4}{2i+4}$$

The *conjugate* of the denominator will be $2i-4$.

Now as before, multiplying both denominator and numerator by this *conjugate* we write:

$$\rightarrow \frac{(2i-4)(2i-4)}{(2i+4)(2i-4)}=\frac{(2i-4)^2}{(2i)^2-4^2}=\frac{(2i)^2-2(2i)(4)+4^2}{-4-16}=\frac{-4-16i+16}{-20}$$

$$\rightarrow \frac{12-16i}{-20}$$

Simplifying completely and writing our answer in the *Standard Form* $a+bi$ we have

$$\frac{12-16i}{-20}=\frac{-3}{5}+\frac{4}{5}i$$

−−−−−−−¿

Numerical Analysis

The types of questions we consider in what follows are analysis questions. The questions might not all be of the following mold, but at the core basically they are questions where the key to unlocking them is "*thinking*." Of course, "*thinking*" mathematically but what we are saying is that you're not doing Algebra or searching for some theorem in Geometry to solve questions like these.

One type of these questions is what is called Numerical Analysis [Analysis (of numbers.)] It is best that we just show you an example, so let's get to it.

There are other types of questions that may fit this section, but we will just focus on this one specific type, leaving you with enough problems to try on your own along with solutions so that you are well practiced.

$\frac{a-b}{-b}=c$

- In the equation above, if a is positive and b is negative, which of the following must be true?

A. $c<0$
B. $c>1$
C. $0<c<1$
D. $-1<c<0$

Solution

The correct answer is B. $c>1$

Note, the given expression is $\frac{a-b}{-b}=c$

We are told that a is positive and b is negative. Which means, $-b$ is positive. Now, we can split the expression into two fractions:

$\frac{a-b}{-b}=\frac{a}{-b}+\frac{-b}{-b}=\frac{a}{-b}+1=c$

Since both a and $-b$ are positive, $\frac{a}{-b}$ is also positive. Now since we are adding we recognize that 1 plus a positive number is always greater than 1. Hence, $c>1$.

Number Analysis (Passport to Advanced Math)

$$\frac{-u+v}{v}=c$$

1. In the equation above, if both u and v are positive, then, which of the following must be true?

A. $c<1$
B. $c>1$
C. $0<c<1$
D. $-1<c<0$

$$\frac{-2p-3q}{4q}=c$$

2. In the equation above, if both p and q are positive, which of the following must be true?

A. $c>1$
B. $c<0$
C. $0<c<1$
D. $c=\frac{-5}{4}$

$$\frac{2p-3q}{4q}=c$$

3. In the equation above, if p is positive and q is negative, which of the following must be true?

A. $c>\frac{5}{4}$
B. $c>0$
C. $c<0$
D. $c<\frac{-1}{4}$

$$\frac{-p+4q}{2q}=c$$

4. In the equation above, if p is positive and q is negative, which of the following must be true?

A. $c>2$
B. $c>\frac{3}{2}$
C. $c<0$
D. $c>0$

Solutions to *Number Analysis* (Passport to Advanced Math)

1. The correct answer is A. $c<1$

   Note, the given expression is

$$\frac{-u+v}{v}=c$$

   We are told that both u and v are positive. Which means, $-u$ is negative. Now, we can split the expression into two fractions as follows:

$$\frac{-u+v}{v}=\frac{-u}{v}+\frac{v}{v}=\frac{-u}{v}+1$$

   Since $-u$ is negative and v is positive, $\frac{-u}{v}<0$ because a negative numerator divided by a positive denominator is a negative number. We can add 1 to both sides of the inequality $\frac{-u}{v}<0$, and we are left to conclude that $\frac{-u}{v}+1<1$. Hence, $c<1$

2. The correct answer is B.

   Note, the given expression is

$$\frac{-2p-3q}{4q}=c$$

   We are told that both p and q are positive. Which means, $-2p-3q$ is negative. Now, since q is positive, $4q$ in the denominator is positive. We cannot say what number c is since we don't know the value of p and q. But we can surely say that a negative numerator divided by a positive denominator is a negative number. Hence, $c<0$.

3. The correct answer is D. $c<\frac{-3}{4}$

   Note, the given expression is

$$\frac{2p-3q}{4q}=c$$

We are told that p is positive and q is negative. This means $2p$ and $-3q$ are both positive, so the numerator is positive. But because q is negative, the denominator is negative.

Splitting the fraction, first we write:

$$\frac{2p-3q}{4q}=\frac{2p}{4q}+\frac{-3q}{4q}=\frac{2p}{4q}+\frac{-3}{4}$$

Looking at the far right above, we can draw the conclusion that $\frac{2p}{4q}$ is negative, because a positive numerator divided by a negative denominator is a negative number. Adding $\frac{-3}{4}$ will keep $\frac{2p}{4q}$ both negative and $\leftarrow\frac{3}{4}$. Hence, $c<\frac{-3}{4}$.

4. The correct answer is A. $c>2$

We start with the expression:

$$\frac{-p+4q}{2q}=c$$

We are told that p is positive and q is negative. Splitting the expression into two fractions, we write:

$$\frac{-p+4q}{2q}=\frac{-p}{2q}+\frac{4q}{2q}=\frac{-p}{2q}+\frac{4}{2}=\frac{-p}{2q}+2$$

Since $-p$ is negative and $2q$ is also negative, $\frac{-p}{2q}$ is positive. And because $\frac{-p}{2q}>0$., adding 2 to both sides of the inequality will mean that $\frac{-p}{2q}+2>2$. Hence, $c>2$.

$-------¿$

Quadratics

A *quadratic* is a *polynomial* of the form ax^2+bx+c where $a\neq 0$. You should already know this but what we are asserting is that what makes a *quadratic* a *quadratic* is the fact that the highest power of x that appears in the *polynomial* is 2 (x^2.¿ Meaning, b and c can be zero.

Examples of quadratics are $y=-2x^2, f(x)=x^2-9$ and $g(x)=4x^2-2x$.

Now, ALL *quadratics* can be solved by the *quadratic formula*!

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

So, when in doubt, you can always power through a question about solving a *quadratic* by using this formula so long as you can identify a, b, and c (a task that only requires that you be awake☺)

Of course, questions about a *quadratic* can be more interesting than merely solving a *quadratic*. We have carefully designed questions in the exercises to expose you to all the varieties of questions you might encounter.

So, other than practicing, the following ideas about *quadratics* are also very important.

The *Discriminant*

The *Discriminant* is the part of the *quadratic formula* inside the square root. The *Discriminant* of a *quadratic* tells us a lot about the *quadratic*.

If you haven't yet figured it out, the *discriminant* is:

b^2-4ac

Now, let us consider the three possibilities of what could happen with this *Discriminant*.

By the *Trichotomy Laws*, we know that every number is either ¿0, ¿0, or ¿0.

Since b^2-4ac is just a number after it is simplified, it is either ¿0, ¿0, or ¿0.

As such, we have either $b^2-4ac>0$, $b^2-4ac<0$, or $b^2-4ac=0$.

Let us consider each one at a time.

Two Real Solutions

$b^2-4ac>0$

In this case (when the *Discriminant* is positive), the *quadratic* you're working with has *two real solutions*. This is so because, you have:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\to x=\frac{-b\pm\text{Positive Number}}{2a}$$

Obviously, <u>the square root of a positive number is a positive number</u>. Additionally, it should be clear that we get a different value when we add a "Positive Number" instead of a Subtracting the same "Positive Number" from $-b$. Therefore, giving us two different *solutions* both of which are *real numbers*. Hence, the *2-real solutions case*.

Two Imaginary Solutions

$b^2-4ac<0$

In this case (when the *Discriminant* is negative), the *quadratic* you're working with has *two imaginary solutions*. This is so because, we have

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\rightarrow x=\frac{-b\pm \textit{Square root of a negative number}}{2a}$$

As you know from our discussion of *Complex* (*imaginary*) numbers, the square root of a negative number involves the imaginary unit i.

For example, $\sqrt{-25}=\sqrt{25}\sqrt{i}=5i$

Thus, when the *discriminant* is less than 0, we have an *imaginary number* that we *add* and *subtract* from $-b$. This gives us two different *solutions* both of which are *imaginary numbers*. Hence, the 2-*imaginary* solutions case.

And finally, drum roll,

The One Solutions Case

$b^2-4ac=0$

Now, when the *Discriminant* is equal to zero as we have just above, look at what happens on the *quadratic formula*.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\rightarrow x=\frac{-b\pm 0}{2a}$$

$\rightarrow x = \frac{-b}{2a}$

So, there is *only one solution* in this case. In fact, the *one solution case* is very special as is the number $\frac{-b}{2a}$.

Now, since the *one solution case* means the *quadratic* has only *one real solution*, we can construct such a *quadratic*. It is a *quadratic* with *a repeated zero*.

A simple example can be $f(x) = (x-2)(x-2) = (x-2)^2$

$f(x) = 0 \rightarrow (x-2)^2 = 0$

$\rightarrow x = 2$

Now, if we use the *Binomial Square*, we have $(x-2)^2 = x^2 - 4x + 4$

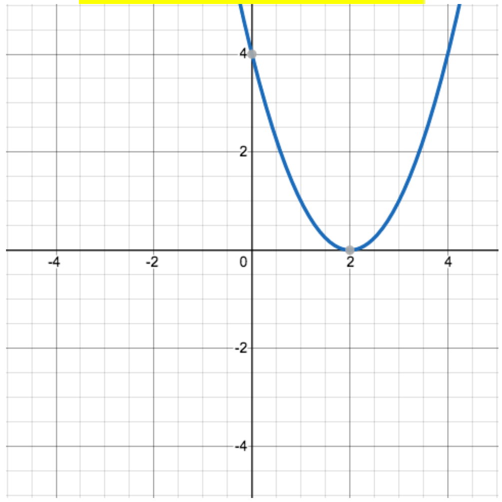
On the right-hand side, since we have the *quadratic in standard form*, as such, we can easily pick out that

$x = \frac{-b}{2a} = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2$

But wait, this is exactly the *root/solution/x-intercept/zero* to the *quadratic* $(x-2)^2 = 0$.

Yes! Let us look at the graph of this *quadratic* to understand why.

Graph of $f(x) = (x-2)^2 = x^2 - 4x + 4$



You see, as the graph shows, the *quadratic* bounces on the $x-$¿axis touching the $x-$¿axis at the *axis of symmetry*.

Oh wait, so, the *axis of symmetry* of a *quadratic* is

$x=\frac{-b}{2a}$

Yes!

In fact, if we look at the following other *quadratic*, we can find additional proof to this assertion.

Consider $g(x)=x^2-6x+8$

Note, this is the same as $g(x)=(x-4)(x-2)$

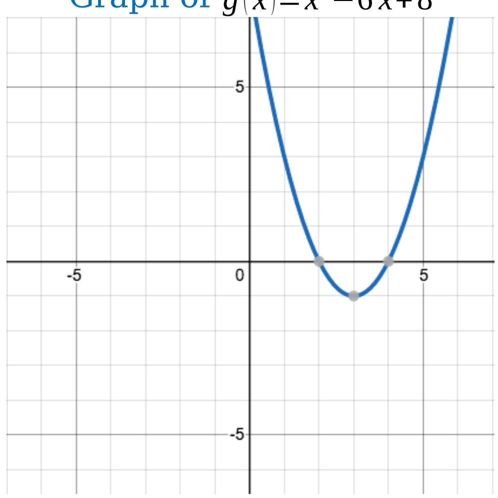
Now, if we try to find the *zeroes* of this *quadratic*, we see that:

$g(x)=0 \rightarrow (x-4)(x-2)=0$

$\rightarrow x=4$ *or* $x=2$.

Now let us graph this *quadratic* and get a helpful visual.

Graph of $g(x)=x^2-6x+8$



Looking at the graph above, the *axis of symmetry* is $x=3$. Notice that this is:

$$x=\frac{(4+2)}{2}=\frac{6}{2}=3$$

Ah, we get it, we get it! So, the *axis of symmetry* is the *average of the roots.*

In fact, we can generalize this. Check it out.

So, the *quadratic formula* gives us both *roots* to a *quadratic*. These are the *two roots*. Let us call them x_1 and, x_2.

One of them is

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

and another is

$$x_1=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Now, the *average of these two* is:

$$\frac{(x_1+x_2)}{2}=\frac{1}{2}(x_1+x_2)$$

$$\rightarrow x=\frac{1}{2}\left(\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$\rightarrow x=\frac{1}{2}\left(\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}\right)$$

Notice the highlighted parts are the same but *opposite* in sign. So, their sum is zero.

$$\rightarrow x=\frac{1}{2}\left(\frac{-2b}{2a}\right)$$

$$\rightarrow x=\frac{-b}{2a}$$

To*get*her (Or *try this* on your own)

- Let’s do a quick problem on this so that we are practiced a bit☺

$$3x^2+3x=p$$

In the equation above, p is a constant. If the equation has *No Real Solutions*, which of the following can be the value of p?

A. 1

B. $\frac{-1}{2}$

C. 3

D. -3

Solution

First, for us calculate the *Discriminant*, we need the *quadratic* to equal zero (We need the *quadratic* written in *Standard Form*.) That's easy, we just need to write:

$3x^2+3x-p=0$

Now, we know the *Discriminant* is:

$b^2-4ac \rightarrow 3^2-4(3)(-p)$

$\rightarrow 9+12p$

We need this to be *less than zero* (¿0) since we want *No Real Solutions*!!!

$\rightarrow 9 \leftarrow 12p$

Dividing both sides by -12 (*remember, we must change the direction of the inequality when we divide or multiply by a negative number on inequalities*), we have

$\frac{-9}{12} > p \rightarrow p \leftarrow \frac{3}{4}$

From the answer choices, the only one that makes sense is -3.

The *Zero*-Property

The *zero*-property is a little-known property that we frequently use in algebra without stating it.

It should be clear that if $ab=0$ then either $a=0$ or $b=0$.

So, for example, if $(x-2)(x+3)=0$ it follows that $x-2=0$ or $x+3=0$.

$\rightarrow x=2$ *or* x¿-3.

For a real example, let us look at the following test question:

- What is the sum of the solutions to $(x-1.7)(x-1.3)=0$

A. -3 B. -0.4 C. 0.4 D. 3

Solution

Because of the *zero-product property*, we must set each factor equal to 0.

$$x-1.7=0 \text{ or } x-1.3=0$$

$$x=1.7 \text{ or } x=1.3$$

Therefore, the sum of the solutions is $1.7+1.3=3$, which is answer choice D.

The *Factor* and *Remainder* Theorems

x	$g(x)$
-2	3
-1	0
1	0
4	7

The function g is defined by a *polynomial*. Some values of x and $g(x)$ are shown in the table above. Answer the two questions below based on this provided information.

- What are the factor of $g(x)$?

Solution

Remember, if say we have $x^2+7x+10$, we can factor this *polynomial* (*a quadratic*) as
$(x+2)(x+5)$.

Now, if we set the *quadratic equal to zero*, it is the same as setting the *factored version equal to zero*.

That is, $x^2+7x+10=0 \rightarrow (x+2)(x+5)=0$

$\rightarrow x=-2$ or $x=-5$.

Now, we can answer our question here. Because, for $x=-2$ or $x=-5$, the y-coordinates are zero. That is, we would have $(-2,0)$ and $(-5,0)$ as points.

More importantly, if we know $x=-2$ is a *zero* (aka, $x-$¿*intercept*, *root*, or *solution*) to a *polynomial*, we know that $(x-(-2))=(x+2)$ is a *factor*.

Since $g(x)$ is y, we look there, and we can see that $g(x)=y=0$ for $x=-1$ and $x=1$. Meaning $(x-(-1))=(x+1)$ and $(x-1)$ are the *factors*.

- What is the *remainder* when $g(x)$ is divided by $x+2$?

This is about the *Remainder Theorem*. You can learn about the *Remainder Theorem* in detail. And surely, you can do *long division* or *synthetic division* where appropriate if the *polynomial* is given. But here, we will simply state the *Remainder Theorem* and proceed as follows:

The Remainder Theorem

The *remainder* when a polynomial $p(x)$ is divided by $x-a$ is $p(a)$.

Solution

So, the *remainder* when $g(x)$ is divided by $x+2$ is $g(-2)$. This is so because, note we need $x+2$ in the form $x-a \rightarrow x-(-2)$. Thus, $a=-2$ in this case and $p(a)=g(-2)$.

*Comp*leting the Square

Completing the Square is an important skill that you should already have familiarity with. If that is not the case, it is the essence of quadratics and therefore, we must address in what follows

Remember, a *quadratic* in *standard form* is $y=ax^2+bx+c$. A *quadratic* in *vertex form* is $y=a(x-h)^2+k$.

The process of changing *a quadratic* in the *standard form* to *the vertex form* is what we call, "*Completing the Square*."

Standard form → After *Completing the Square* → *Vertex Form*

$y=ax^2+bx+c$

$y=a(x-h)^2+k$

But *let's get it* (this right here is the entire reason you should be happy you bought this book. It tries so hard to appeal to 15-year-olds☺)

At the core, *Completing the Square* is all about the *Binomial Square*. Let us adopt the *Binomial Squares* here so that we use the variable x together with a constant b.

That way, we have:

$(x+b)^2 = x^2 + 2bx + b^2$

and

$(x-b)^2 = x^2 - 2bx + b^2$

So, consider the following three *two-termed* quadratics.

$x^2+8x+¿¿$, $x^2-6x+¿¿$ and $x^2-7x+¿¿$

Now, if we look at the *middle term* and the *last term* in the two adopted Binomial Squares above, they are very revealing!!

$2bx$ vs b^2

$-2bx$ vs b^2

In both *Binomial Squares*, we can see that what happened is, <u>we took the *middle coefficient, squared it*, and made it *the last term*</u>. This is what is at the heart of *Completing the Square*.

So, it this this what we must do to complete the blanks (_____) above.

$x^2+8x+\left(\frac{8}{2}\right)^2$, $x^2-6x+\left(\frac{-6}{2}\right)^2$, and $x^2-7x+\left(\frac{-7}{2}\right)^2$

In order, simplifying, we have:

$x^2+8x+16=(x+4)^2$, $x^2-6x+9=(x-3)^2$, and $x^2-7x+\frac{49}{4}=\left(x-\frac{7}{2}\right)^2$

Now, usually, in *Completing the Square*, we must do a bit more than what we just saw☺

The task you will be faced with will be more like:

- The *quadratic* with equation $y=-2x^2-4x+7$ is changed into the *vertex form* $y=a(x-h)^2+k$, what is the value of $\frac{(h-a)}{k}$?

$y=-2¿$

Solution

To start, we had to make sure that the leading coefficient is 1. That is why we have factored out the -2 and set it outside of the parenthesis so that we can *complete the square* inside of the parenthesis.

$y=-2(x^2+2x+1)+7+2$

Now, pay attention! This is the most crucial part.

Yes, we added one ($+1$) inside the parenthesis so that we have $x^2+2x+1=(x+1)^2$. But technically, we didn't add one $(+1)$, because of the -2 in front of the parenthesis, we *subtracted* 2 because $-2(+1)=-2$.

It is for this reason that we had to add two at the end after the $+7$ ($+2$.) That way, *we have in net added zero*. We should worry if we *in net add anything different from zero as* it is no longer the same *quadratic*.

Here is a practice problem on *Completing the Square* of the sort you might encounter on a standardized exam.

x^2-8x+3

- Which of the following is equivalent to the expression above?

A. $(x-4)^2+13$
B. $(x+4)^2-13$
C. $(x-4)^2-13$
D. $(x+4)^2+13$

Solution

We will *Complete the Square* to make the given expression like the answer choices, and then compare. To *Complete the Square*, we will proceed as in the previous solution right before this question was posed

x^2-8x+3

$\rightarrow x^2-8x+16-16+3$

After doing this, we have formed *a perfect square* with the first three terms.

$(x-4)^2-13$

Which is equal to answer choice C.

The Cap *on Quadratics*

Before we conclude this section, the following two problems are a good summary of what we have discussed in this whole section on *quadratics*. So, try them on your own, consulting the solution as you need.

First

- © In the $xy-¿$plane, the graph of $y=4x^2-2x$intersects the graph of $y=2x$ at the points $(0,0)$ and $(a,2a)$. What is the value of a?

Solution

We can easily solve the *quadratic equation* $4x^2-2x=2x$ to find the $x-$¿values of the intersection points.

$4x^2-2x=2x$

$\rightarrow 4x^2-4x=0$

$\rightarrow 4x(x-1)=0$

$$4x=0 \text{ or } x-1=0$$

$$x=0 \text{ or } x=1.$$

Now, to find the $y-$¿value, we should plug in the $x-$¿value in either of the two equations. Let's use the simpler equation $y=2x$.

For $x=0$, we get that $y=0$, which is the first intersection point. For $x=1$, we get that $y=2(1)=2$. This is the other point of intersection.

$(a,2a)=(1,2)$. As you can see, the condition that the x-value is half of the y-value holds. We can see therefore that the value of a is 1.

And then

- If $(at-2)(bt+3)=8t^2-kt-7$ for all values of t and $a-b=2$, then what are the two values of k?

Solution

We can solve this problem by expanding the product on the left side of the given equation and comparing coefficients.

$(at-2)(bt+3)=8t^2-kt-7$

$\rightarrow abt^2+(3a-2b)t-6=8t^2-kt-7$

Comparing coefficients, we see that

$$ab=8 \text{ and } 3a-2b=-k$$

We can solve for a and b using the other condition: $a-b=2$.

Since $ab=8$, $b=\frac{8}{a}$. We will substitute this second equation in the given condition above.

$\rightarrow a-\frac{8}{a}=2$

Multiplying both sides by a, we have

$\rightarrow a^2-2a-8=0$

$\rightarrow (a-4)(a+2)=0$

$$\rightarrow a=4 \text{ or } a=-2$$

Knowing this, we can find the two values of b, on for each of the two values of a.

If $a=4 \rightarrow b=2$ and if $a=-2 \rightarrow b=-4$.

Since $-k=3a-2b$, we can now find two values of k.

Case 1 $(a=4, b=2)$:
$-k=12-8$

$k=-4$
Case 2 $(a=-2, b=-4)$:
$-k=-6+8$

$k=-2$

Quadratics Part 1 (Passport to Advanced Math)

1.
$y=x^2-b$
In the equation above, $b$is a positive constant and the graph of the equation in the xyplane is a parabola. Which of the following is an equivalent form of the equation?

A. $y=(x+b)(x-b)$

B. $y=(x+\sqrt{b})(x-\sqrt{b})$

C. $y=(x+\frac{b}{2})(x-\frac{b}{2})$

D. $y=(x-b)^2$

2.

$y=x^2-c^2$

In the equation above, $c<0$and the graph of the equation in the xy-plane is a parabola. Which of the following is an equivalent form of the equation?

A. $y=(x-c)(x+c)$

B. $y=(x-\sqrt{c})(x+\sqrt{c})$

C. $y=(x+ic)(x-ic)$

D. $y=(x+c)^2$

3.

In the $xy-$¿plane, the point $(4,2)$ lies on the graph of the function g. If $g(x)=2a-x^2$ where a is a constant, what is the value of a?

4.

$9x^2-64=(ux-v)(ux+v)$

In the equation above, u and v are *positive constants*. Which of the following can be the value of $u+v$?

A. -5 B. 5 C. 11 D. -11

5. What are the solutions to $3x^2+8x+5=0$?

6.

$2x^2+8x=n$

In the equation above, n is a constant. If the equation has one *real solutions*, which of the following can be the value of n?

A. -8
B. 4
C. 2
D. 8

7.
$3x^2+6x=t$
In the equation above, t is a constant. If the equation has *two real solutions*, which of the following can be the value of t?

A. 1 B. -3 C. -2 D. -1

8. Which of the following is equivalent to?
$\left(u+\frac{v}{3}\right)^2$

A. $u^2+\frac{v^2}{9}$
B. $u^2+\frac{v^2}{9}$
C. $u^2+\frac{uv}{3}+\frac{v^2}{3}$
D. $u^2+\frac{2uv}{3}+\frac{v^2}{9}$

9. Which of the following is equivalent to?
$\left(2a-\frac{b}{4}\right)^2$

A. $4a^2+\frac{b^2}{16}$
B. $4a^2-\frac{b^2}{16}$
C. $4a^2-\frac{1ab}{2}+\frac{b^2}{16}$
D. $a^2-ab+\frac{b^2}{16}$

Solutions to *Quadratics Part 1* (Passport to Advanced Math)

1. The correct answer is B.

   This is a must know trick in algebra called *Difference of squares*. The *Difference of Squares* states that $u^2-v^2=(u-v)(u+u)$.

   We can write $y=x^2-b$ as $y=x^2-(\sqrt{b})^2$. This way, $u=x$ and $v=\sqrt{b}$. As such, $x^2-(\sqrt{b})^2=(x-\sqrt{b})(x+\sqrt{b})$.

2. The correct answer is A.

   To solve this problem, we will once again use the *Difference of Squares*. c is less than 0, but since every real number squared is positive, $c^2>0$. We can also make a substitution to make the problem easier. We will take $c=-a(a>0)$. Now we can write:

   $y=x^2-(-a)^2 \rightarrow y=x^2-a^2$

   Now is where we will use the *Difference of Squares*, we have:
   $y=(x+a)(x-a)$

   Now, substituting back, we have:
   $y=(x+(-c))(x-(-c))$

   $y=(x-c)(x+c)$

3. The correct answer is 9.

   We are given the function $g(x)=2a-x^2$ and we know that the point $(4,2)$ lies on its graph. This means that when we input the number 4 into the function, we get back the number 2. Knowing this, we can write the following equation:
   $g(4)=2a-4^2=2$

   $\rightarrow 2a-16=2 \rightarrow 2a=18$

   $\rightarrow a=9$

4. The correct answer is C. 11.

   We are given the equation $9x^2-64=(ux-v)(ux+v)$. We can write the right-hand side as $(3x)^2-8^2$. Now, to get something looking like the left-hand side which allows us to apply the *Difference of Squares*.

   $(3x-8)(3x+8)=(ux-v)(ux+v)$

   Now, comparing both sides of the equation, we conclude that $u=3$ and $v=8$. Hence, $u+v=3+8=11$. Notice that $u=-3$ and $v=-8$ also work. However, we are told that u and v are positive constants, as such, we have reached the correct conclusion.

5. The correct answer is $x=1, x=\frac{-5}{3}$

To find the solutions of the equation $3x^2+8x+5=0$, we can use the *quadratic* formula which states that when we have a *quadratic in standard form*, $ax^2+bx+c=0$, *both solutions of the equation* can be obtained by plugging in the corresponding values of a, b, *and* c in the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

In the equation $3x^2+8x+5=0$, with $a=3$, $b=8$, and $c=5$. Plugging in the *quadratic formula*, we have: $x=\frac{-8\pm\sqrt{8^2-4(3)(5)}}{2(3)}$

$$x=\frac{-8\pm 2}{6}\rightarrow x=-1, x=\frac{-5}{3}$$

6. The correct answer is A. -8

This question is about the *Discriminant* of a *quadratic*. A *quadratic* in *Standard Form* is $ax^2+bx+c=0$. A *quadratic* written in this form has *one real solution* if the *Discriminant* $b^2-4ac=0$.

Now, the given that the *quadratic* $2x^2+8x=n$ is not exactly in *Standard Form*, subtracting n from both sides, we can write it in *Standard Form* as $2x^2+8x-n=0$.

Now, we can correctly calculate the *Discriminant* as $8^2-4(2)(-n)=64+8n$. For *One Real Solution*, we need to set this *Discriminant* equal to 0.

$\rightarrow 64+8n=0$

$\rightarrow 8n=-64 \rightarrow n=-8$

7. The correct answer is A. 1

This question is about the *Discriminant* of a *quadratic*. A *quadratic* in *Standard Form* is $ax^2+bx+c=0$. A *quadratic* written in this form has *two real solutions* if the *Discriminant* $b^2-4ac>0$.

Now, , subtracting t from both sides, we can first write the given *quadratic* in *Standard Form* as $3x+6x-t=0$. Now, we can correctly calculate the *Discriminant* as $6^2-4(3)(-t)=36+12t$. And for *two real solutions*, we need to set this *Discriminant* greater than 0.

$\rightarrow 36+12t>0$

$\rightarrow 12t>-36 \rightarrow t>-3$

The only answer choice with an appropriate value of t is A. 1.

8. The correct answer is D. $u^2+\frac{2uv}{3}+\frac{v^2}{9}$

This problem is about the *Binomial Square*. We are given the binomial $\left(u+\frac{v}{3}\right)^2$ and so we can use the following formula:

$$(a+b)^2=a^2+2ab+b^2$$

Here, $a=u$ and $b=\frac{v}{3}$.

Therefore, we have $\left(u+\frac{v}{3}\right)^2=u^2+\frac{2uv}{3}+\left(\frac{v}{3}\right)^2=u^2+\frac{2uv}{3}+\frac{v^2}{9}$.

9. The correct answer is D. $4a^2-ab+\frac{b^2}{16}$

Just like the solution to the previous problem, this problem is about the *Binomial Square*.

We are given the binomial $\left(2a-\frac{b}{4}\right)^2$ and so we can use the following formula:

$$(x-y)^2=x^2-2xy+y^2$$

Here, $x=2a$ and y¿$\frac{b}{4}$.

Plugging in, we have $\left(2a-\frac{b}{4}\right)^2=(2a)^2-\frac{2(2a)(b)}{4}+\left(\frac{b}{4}\right)^2=4a^2-ab+\frac{b^2}{16}$.

Equations Quadratic in Form

An equation is considered *quadratic in form* if it can be turned into a *quadratic equation* via a *substitution*. In this section, we will talk about the ways to identify if an *equation is quadratic* in form.

Remember, a *quadratic in Standard Form* is: ax^2+bx+c. This quadratic is a "*quadratic* in x." That makes sense. It is a *quadratic,* and it uses the variable x.

So, a *quadratic* in u is au^2+bu+c.

Now, consider the equation $(x-2)^2+5(x-2)+4=0$.

We can quickly see that this equation is *quadratic in form*. Here is how:

- It has *3 terms*, "technically more" if we expand.
- Considering it as *3 termed*, the first term is the square of the middle term (setting aside the coefficient of 5.)
- The last term is a *constant* just like a *quadratic*.

So then, we make a *substitution*. Always for the middle term.

Letting $u=x-2$, we have $(x-2)^2+5(x-2)+4=0$

$\rightarrow u^2+5u+4=0$

$(u+4)(u+1)=0$

$\rightarrow u+4=0$ *or* $u+1=0$

$u=-4$ *or* $u=-1$

Now, we go back to the *substitution* we had made ($u=x-2$¿ but in reverse.

$\rightarrow x-2=-4$ *or* $x-2=-1$

$\rightarrow x=-4+2=-2$ *or* $x=-1+2=1$

There is nothing more to say about *quadratics*.

Quadratics and Cubics (Passport to Advanced Math)

1. What is the sum of the solutions to $(x-2.7)(x+1.3)=0$

   A. 4 B. -1.4 C. 1.4 D. -4

x^2-8x+3

2. Which of the following is equivalent to the expression above?
   A. $(x-4)^2+13$
   B. $(x+4)^2-13$
   C. $(x-4)^2-13$

D. $(x+4)^2+13$

x^2+4x+2

3. Which of the following is equivalent to the expression above?
   A. $(x-2)^2+2$
   B. $(x-2)^2-2$
   C. $(x+2)^2+2$
   D. $(x+2)^2-2$

$ax^3+bx^2+cx+d=0$

4. In the equation above, a, b, c, and d are constants. If the equation has *roots* $x=-4$, $x=-2$, and $x=-1$, which of the following is a factor of ax^3+bx^2+cx+d.

   A. $x-4$ B. $x-2$ C. $x-1$ D. $x+1$

$f(x)=ax^3+bx^2+cx+24$

5. In the cubic *Polynomial* f above, a, b, and c are constants. If $x-4$, $x-2$, and $x-k$ are all factors of $f(x)$, which of the following could be the value of k?
   A. 8
   B. -8
   C. 3
   D. -3

6. *We hate this number cause we love Jesus, the only God. We're all about the 7s and Jesus.*

$ax^3+bx^2+cx+15=0$

7. In the equation above, a, b, c, and d are constants. If the equation has the three *roots* $x=-3$, $x=-5$, and $x=d$, where d is a constant, which of the following could be the value of d?

A. 3 B. 5 C. -1 D. 1

Solutions to Quadratics and Cubics (Passport to Adv. Math)

1. The correct answer is C. 1.4

   Since we are already given the equation in a *factored form*, we can get a *solution* from *each factor* by setting each factor *equal to zero*. Doing this, we have:

$$x-2.7=0 \textit{ or } x+1.3=0$$

$$x=2.7 \text{ or } x=-1.3$$

$-1.3+2.7=1.4$

2. The correct answer is C.

This is a typical question in *Completing the Square*.

We are given the *quadratic* x^2-8x+3, which is *in Standard Form* ax^2+bx+c. To *Complete the Square*, we must take the first two terms x^2-8x and figure out what we must add to these two terms to get a *Binomial Square*. You should know from lesson prior what we must add is:

$$\left(\frac{b}{2}\right)^2=\left(\frac{-8}{2}\right)^2=16$$

So, we will have $x^2-8x+16=(x-4)^2$

But we must not forget that we started with x^2-8x+3. So $x^2-8x+16+3$ feels wrong. And it is. To make up for <u>adding a</u> 16, we must <u>subtract</u> 16.

That is,

we should write, $x^2-8x+3=x^2-8x+16+3-16=(x-4)^2+3-16$

$\rightarrow x^2-8x+3=(x-4)^2-13$

3. The correct answer is D. $(x+2)^2-2$

This is another exercise in *Completing the Square*. We are given the *quadratic* x^2+4x+2, which is *in Standard Form* ax^2+bx+c. To *Complete the Square*, we must take the first two terms x^2+4x and figure out what we must add to these two terms to get a *Binomial Square*. You should know from the previous solution or lessons earlier in this book that what we must add is

$$\left(\frac{b}{2}\right)^2=\left(\frac{4}{2}\right)^2=4$$

So, we will have $x^2+4x+4=(x-4)^2$

To this end, we proceed as follows:

We should write $x^2+4x+2=x^2+4x+4+2-4=(x+2)^2+2-4$

$\rightarrow x^2+4x+2=(x+2)^2-2$

4. The correct answer is D. $(x+1)$

This is so because, for any polynomial if $x=c$ is a *root* then $x-c$ is *a factor*. From this, since we know that $ax^3+bx^2+cx+d=0$ has the *roots* $x=-4$, $x=-2$, and $x=-1$ and therefore $(x+4)$,$(x+2)$, and $(x+1)$ are *factors*. That is,

$ax^3+bx^2+cx+15=(x+4)(x+2)(x+1)$

Looking at the answer choices, we see that the only *factor* that shoes up is $(x+1)$.

5. The correct answer is D. -3

    For any *polynomial*, if $x=c$ is a *root*, then, $x-c$ is a *factor*. From here since we know that $a x^3+b x^2+cx+24=0$ has as f*actors* $(x-4),(x-2)$, and $(x-k)$,

    $$a x^3+b x^2+cx+24=(x-4)(x-2)(x-k)$$

    You can multiply out the three factors, but it is a simple algebra fact that the last term of a cubic is the product of the three constants from the *factors* (*when it is factorable over the Reals*.) Meaning, we already know without hard labor that:

    $$24=(-4)\cdot(-2)\cdot(-k)\rightarrow 24=-8k$$

    $$\rightarrow -3=k$$

6. *This number is imaginary, so the solution doesn't exist cause the problem doesn't exist.*

7. The correct answer is C. -1

    For any *polynomial*, if $x=c$ is a *root* then $x-c$ is a *factor*. From this, since we know that $a x^3+b x^2+cx+15=0$ has the three *roots* $x=-3$, $x=-5$, and $x=d$, then $(x+3),(x+5)$ and $(x-d)$ are *factors*. That is,

    $$a x^3+b x^2+cx+15=(x+3)(x+5)(x-d)$$

    From the previous solution or simple mathematical logic, you should know without the hard labor that

    $$15=3\cdot 5\cdot(-d)$$

    $$\rightarrow 15=-15d$$

    $$\rightarrow -1=d$$

−−−−−−−¿

Composition of Functions (Other)

There are *many ways to make new functions from familiar functions*. One of the ways to make new functions is *Composition Of Functions*, the subject here.

But first, let’s review the more basic ways of making new functions from the familiar ones.

Adding, *Subtracting*, *Multiplying*, and *Dividing* polynomials

Suppose $f(x)=x^2-4$ and $g(x)=-7x+3$

Then, we can *add* these two functions as follows.

$$f(x)+g(x)=x^2-4+(-7x+3)$$

Before we can say that we are done with the sum of these two functions, we need to *combine like-terms*. As we have already discussed how to do this and because you should be familiar with the basics, without much elaboration, you should understand that the job is done by writing the following:

$$f(x)+g(x)=x^2-7x-1$$

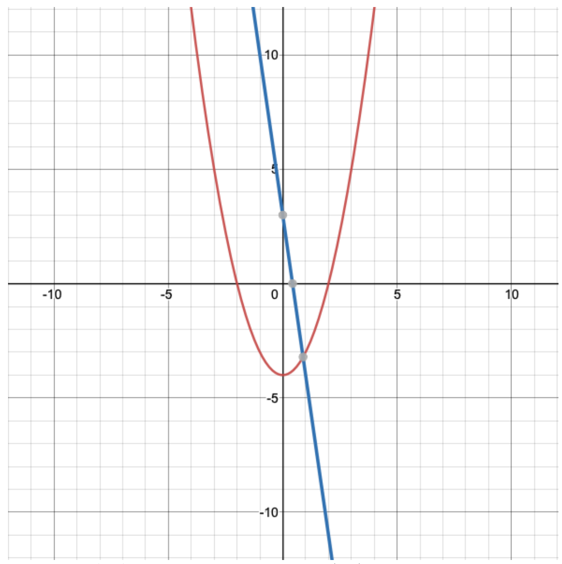
We can also *subtract* the two functions similarly. With extra caution for any negative signs in front of parenthesis.

As we have preached in places in this book, *the best way to learn math is to make connections between the Algebra and Geometry* of “things.” Whether “things” concerns functions or otherwise.

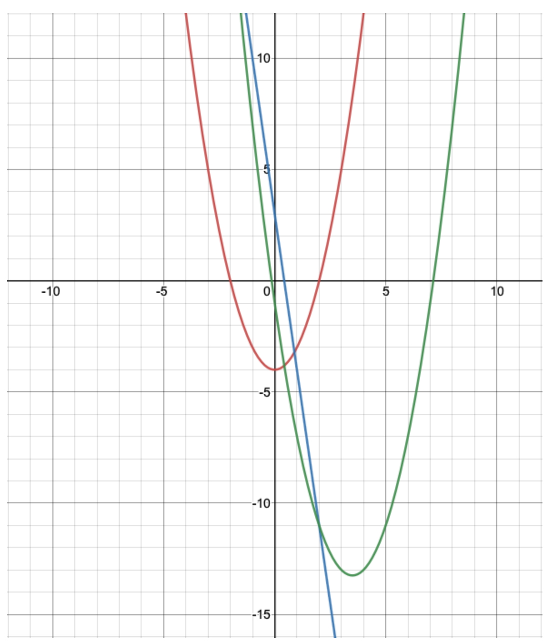
In what is above, here are pictures from a great online graphing tool (great math tool overall) called *Desmos*. In order, we have *graphed*

- f and g
- $f+g$ *added* through *Desmos*
- $f$and g *added* algebraically.

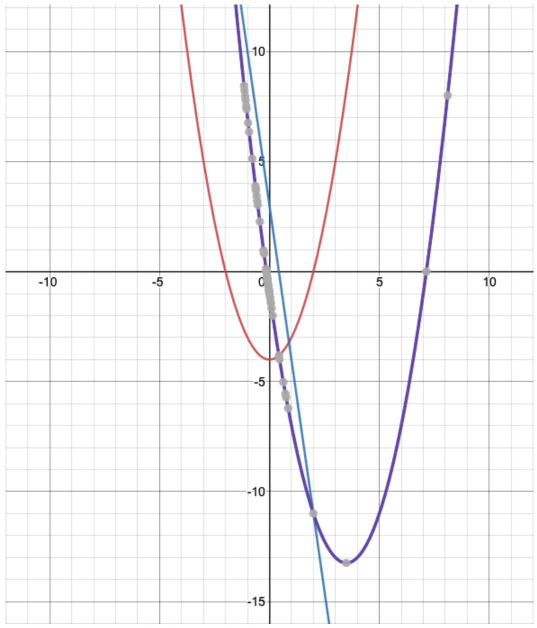
1. $f(x) = x^2 - 4$
2. $g(x) = -7x + 3$
3. $f(x) + g(x)$
4. $h(x) = x^2 - 7x - 1$



$f(x) = x^2 - 4$ and $g(x) = -7x + 3$



$f(x)=x^2-4$, $g(x)=-7x+3$, and $f(x)+g(x)$



$f(x)+g(x)=x^2-7x-1$

Notice whether we instruct *Desmos* to show us the sum function $f(x)+g(x)$ or *add* them algebraically and *combine-like-terms* (the third and fourth visuals above respectively) we get the same result☺

While we will leave it up to you to dig deep, this is what is happening with the visuals.

If we consider say, $x=0$, then the *quadratic* will spit out, $f(x)=0^2-4=-4$ and the *linear* will spit out $g(0)=-7(0)+3=3$. So then, naturally, the sum $f(0)+g(0)=-1$ and this claim is visually corroborated.

As the visual is the same for *differences*, *products*, and *quotients of functions*, the presentation above should take care of you for the visual understanding that may be required in what is to follow.

$f(x)-g(x)=x^2-4-(7x+3)$

Here, the extra care we must exercise is in distributing the negative sign in front of $g(x)$.

$\rightarrow f(x)-g(x)=x^2-4-7x-3$

Of course, once again, we must *combine-like-terms* before we say we are done. Thus, our final answer on the *difference* of f and g is the following:

$$\rightarrow f(x)-g(x)=x^2-7x-6$$

As with numbers, the *subtraction* of two functions doesn't give the same result (not commutative.) That is, we know that $4-2=2$ but $2-4=-2$. Similarly, $f-g\neq g-f$. As such, pay attention to which of the two you're tasked with.

Now, *multiplying* the two functions would look as follows.

$$f(x)g(x)=(x^2-4)(-7x+3)$$

Here, you have to "**FOIL**" to get the job done. We leave it to the student to finish that simple exercise.

The *quotient* (*division*) of the two functions would look as follows:

$$\frac{f(x)}{g(x)}=\frac{x^2-4}{-7x+3}$$

As with *subtraction, division is not commutative* $\left(\frac{f}{g}\neq\frac{g}{f}\right)$. Whereas, unsurprisingly *addition* and *multiplication* are *commutative*.

Notice how in each of the operations we performed above, we took the two original functions and made new ones.

And finally, to the main event.

Composition

A great way to *make even more exotic functions* is *composition*. Just like the four operations above, *composition* <u>is just an operation</u>, albeit less familiar as far as operations go.

The composition of two functions is donated by " $\circ$.

So, when we intend to communicate the *composition of functions*, <u>we write</u> the "*little* O" <u>between the two functions</u>. For example,

$$f\circ g$$

Or more typically

$$(f\circ g)(x)$$

is intended to mean:

$$f(g(x))$$

In other words, all three of the above pairings of two functions symbolize "*Composition*."

That is, $f \circ g = (f \circ g)(x) = f(g(x))$

As the last of the three is most practical in execution, we appeal to it when we want to *compose two functions*. When we see $f(g(x))$, what we are saying is, *plug in the function* $g(x)$ *into the function* $f(x)$. As with *division* and *subtraction* of functions, *composition* is not *commutative* (with rare exceptions.)

So, how do we use the *composition operation*?

Suppose you were asked for $f(2)$. The task should be straight forward, in the function f, you must substitute 2 everywhere you see an x.

That is, you do the following.

$f(2) = 2^2 - 4 = 4 - 4 = 0$

<u>*Composition* is not very different.</u> But let us give a more bizarre example to emphasize the point.

Suppose you were tasked with $f(\ominus)$. Now you will have to write:

$f(2) = \ominus^2 - 4$

So, the point is, everywhere we see an x, we substitute $\ominus$

We do the same thing in *Composition* but instead of the number 2 or $\ominus$, we are substituting the function inside of the function.

That is, $f(g(x)) = g(x)^2 - 4$

But since we know the function $g(x)$, we can find out what $g(x)^2 - 4$ is.

$\rightarrow f(g(x)) = g(x)^2 - 4 = (-7x + 3)^2 - 4$

While the far right of what is just above maybe sufficient, we can do a little better by expanding $(-7x+3)^2$ using the binomial square.

$(-7x+3)^2 = 49x^2 + 2(-7x)(3) + 3^2$

¿ $49x^2 - 42x + 9$

So, using this, we can write that $f(g(x)) = 49x^2 - 42x + 9 - 4$

$\rightarrow f(g(x)) = 49x^2 - 42x + 5$

If instead, we were looking at $g(f(x))$, this time, we are substituting $f(x)$ in place of x in the function $g(x)$.

So, we have

$g(f(x)) = -7(g(x)) + 3$

$\rightarrow g(f(x)) = -7(x^2 - 4) + 3$

$\rightarrow g(f(x)) = -7x^2 + 28 + 3$

$\rightarrow g(f(x)) = -7x^2 + 31$

As we said earlier, $f(g(x)) \neq g(f(x))$, with rare exceptions.

Now, a slightly different task is evaluating a *composition of two functions*.

Let us use the same two functions f and g.

- Suppose you were asked to find the value of $f(g(2))$. This time, you have two options.

- One option is, since we already know the function $f(g(x))$ from our work earlier, we can just plug in 2 everywhere we see an x in this function. That is, we have

$\rightarrow f(g(x)) = 49x^2 - 42x + 5$

$\rightarrow f(g(2)) = 49(2)^2 - 42(2) + 5$

$\rightarrow f(g(2)) = 49(4) - 42(2) + 5 = 117$

- Alternatively, we can first figure out the value of $g(2)$ and move forward as follows.

$g(2) = -7(2) + 3 = -14 + 3 = -11$

Now, since we know $g(2)=-11$, to find $f(g(2))$, we can just write $f(-11)$

So, we evaluate f for $x=-11$,

$\rightarrow f(-11)=(-11)^2-4=121-4=117$

Composition of Functions practice problems **(Other)**

1.

$f(x)=\sqrt{2x+1}$ and $g(x)=\frac{3}{x}$

Which of the following is equal to $g(f(4))$

A. 1

B. $\frac{3}{4}$

C. $\frac{3}{9}$

D. $\frac{3}{2}$

Problems 2 through 4 are based on the table below.

x	$f(x)$	$g(x)$

7	2	2
2	7	7
−2	4	2
4	42	777

2. For what value of x is $f(g(x))=x$?
   A. 2   B. 7   C. −2   D. 9

3. For what value of x is $f(g(x))=2$?
   A. 2
   B. 4
   C. −2
   D. The value cannot be determined from the information given.

4. $h(x)=f(x)+g(f(x))$, what is the value of $h(-2)$
   A. 881
   B. 781
   C. 791
   D. The value cannot be determined from the information given.

5. $f(x)=\frac{4}{(2-x)}$ and $g(x)=\sqrt{x}$

   What is the value of $f(g(0))$?
   A. 4
   B. 2
   C. −2
   D. −4

Solutions to *Composition of Functions Practice Problems* **(Other)**

1. The correct answer is A. 1.

   Since we are asked to find the value of $g(f(4))$, we should find the value of $f(4)$ first, and then evaluate g on the result as follows:

$$f(x)=\sqrt{2x+1}$$

$$f(4)=\sqrt{2(4)+1}=\sqrt{9}=3$$

Now, since

$$g(x)=\frac{3}{x}$$

$$g(3)=\frac{3}{3}=1$$

2. The correct answer is A. 2.

   Since we don't have the actual functions f and g, we cannot solve an equation to find the value of x. In this kind of problem what we must do is simply look through the table and try values of x until we find one that works. When there are many inputs in the table, it's a good idea to skip values that are bound to fail based on prior observation. But here, we only have *three* values to test.

   We want to find the value of x that satisfies $f(g(x))=x$. We will have to begin by first evaluating g.

   First row:

   We start with an x value of 7. For this value, we have that $g(7)=2$. Since we are evaluating f on $g(7)=2$, we now must find $f(2)$. We see that $f(2)=2$. Since $7\neq 2$, this isn't it.

   Second row:

   For the second row, we have that $g(2)=7$. Now, we need to find the value of $f(7)$. According to the table, $f(7)=2$. So, we have $f(g(2))=f(7)=2$, that is, we have $f(g(2))=2$.

3. The correct answer is B. 2.

   The key idea to solve this kind of problem is to work inside out. First, we will look for some number a such that $f(a)=2$. We do this because at the end we will have to evaluate f on some number and get 2 as the output. According to the table, $a=7$, because $f(7)=2$. Now, we will look for some number b such that $g(b)=7$, because we need to evaluate g on some number and get 7 as the output. We can see that this number b is equal to 2, because $g(2)=7$. Hence, this is the final answer $x=2$. We can verify this as follows:

$$f(g(x))=2$$

$$x=2$$

$$f(g(2))=2$$

$$f(7)=2$$

4. The correct answer is B. 781.

   To find the value of $h(-2)$ we need to replace every x with a -2:

$$h(-2)=f(-2)+g(f(-2))$$

Now we will substitute the values we know from the table.

$$h(-2)=4+g(4)$$

Now, $g(4)=777$, so we have:

$$h(-2)=4+777=781$$

5. The correct answer is B. 2.

We will apply the same trick we have applied in the previous problems. First, we will find the value of $g(0)$.

Since $g(x)=\sqrt{x}$ and $g(0)=\sqrt{0}=0 \quad g(0)=0$.

Now we must find $f(0)$. Since we are given:

$$f(x)=\frac{4}{2-x}$$

$$f(0)=\frac{4}{2-0}=\frac{4}{2}=2$$

Hence, the correct answer is B. 2.

−−−−−−−¿

Exponents

Exponent rules are one of the first topics you must have covered as foundation to Algebra. In fact, you should have had exposure to *exponents* even before you started your studies in Algebra.

And so, as we assume exposure, we will go through this topic efficiently.

Let's just quickly list all the important *exponent Rules* you need to know. These are all the important ones you should have already owned.

$1 \rightarrow a^b a^c = a^{b+c}$

$2 \rightarrow (a^b)^c = a^{bc}$

$3 \rightarrow (ab)^c = a^c b^c$

$4 \rightarrow \dfrac{a^b}{a^c} = a^{b-c}$

$5 \rightarrow a^0 = 1$

$6 \rightarrow \dfrac{1}{a^b} = a^{-b}$

$7 \rightarrow a^{\frac{b}{c}} = \sqrt[c]{a^b}$

Assuming the basics, we can just dive right into practice. We remind you that the practice presented here comes with solutions. Therefore, if you put in the time, you should have all your bases covered. Some problems will require that you use the basic *exponent rules* stated above together with other items in Algebra like, "*the difference of squares*," "*the binomial square*," etc. Ok, enough said!

As a first example,

Consider

$$\frac{2^{-3} x^4 (xy)^{-2}}{y^{-3}}$$

Solution

First, by using the 3rd *exponent rule* in the list above, isolating the quantity inside the parenthesis, we can write:

$$(xy)^{-2} = x^{-2} y^{-2}$$

So, the *quotient* becomes:

$$\frac{2^{-3}x^4x^{-2}y^{-2}}{y^{-3}}$$

$$\rightarrow \frac{x^4y^3}{2^3x^2y^2}$$

Doing a little bit of cancelation (using the 4^{th} *exponent rule* above), we have:

$$\frac{x^2y}{2^3}=\frac{x^2y}{8}$$

A second example

should help you dive into (with confidence) the 7 more advanced practice problems to follow.

If $12x-4y=12$, what is the value of:

$$\frac{27^x}{3^y}?$$

Solution

First, note that 27 can be expressed as a power of 3. Specifically, $27=3^3$.

Therefore, we have:

$$\frac{27^x}{3^y}\rightarrow\frac{(3^3)^x}{3^y}=\frac{3^{3x}}{3^y}$$

Notice, from the middle step to the far right, we have used the second *exponent rule*.

Now, using the fourth *exponent rule*, we have:

$$\frac{3^{3x}}{3^y}=3^{3x-y}$$

Now, if you are paying attention, you would know that $12x-4y=4(3x-y)$

So, all we must do is use what we are given to find out the value of $3x-y$.

Specifically, $12x-4y=12\rightarrow 4(3x-y)=12\rightarrow 3x-y=3$

$$\rightarrow\frac{3^{3x}}{3^y}=3^{3x-y}=3^3=27$$

Let's wrap up this section with a final example.

A third and final example

If $x^{\frac{y}{3}}=27$ for positive integers x and y, what is one possible value of y?

Solution

$x^{\frac{y}{3}}=3^3$

This question is both easy and challenging.
It is easy because, well, look at the equation.

What makes it challenging is the fact that we have too many choices. But the choices are narrowed down so we have a direction.
Since it said that x and y are both positive integers, we can choose x, because what we are asked about is y?

A choice that would greatly simplify our path forward is making $x=3$.

$x^{\frac{y}{3}}=3^3 \rightarrow 3^{\frac{y}{3}}=3^3$

So now, quick comparison shows that

$\frac{y}{3}=3$

$\rightarrow y=9$

As we are asked for one possible value of y, we are done.

A final Example (of the sort you're *about to encounter*☺)

If

$\frac{x^{u^2}}{x^{v^2}}=x^{64}, x>1$

and $u+v=4$, what is the value of $u-v$?

A. -4
B. -8
C. 8
D. 16

Solution

The correct answer is D. 16

We know from *exponent rules* of a *quotient with the same base*, we must *subtract exponents*.

$\rightarrow \frac{x^{u^2}}{x^{v^2}}=x^{64} \rightarrow x^{u^2-v^2}=x^{64}$

Now, you should know the *Difference of Squares* $a^2-b^2=(a+b)(a-b)$ (This is very important must know for the SAT math.)

$\rightarrow x^{(u+v)(u-v)}=x^{64}$

At this stage, comparing exponents

$\rightarrow (u+v)(u-v)=64$

$\rightarrow u-v=\frac{64}{u+v}$

But we know that $u+v=4$

$\rightarrow u-v=\frac{64}{4}=16$

Exponents (Passport to Advanced Mathematics)

1. If

$$\frac{x^{a^2-ab}}{x^{ab-b^2}}=x^{64}, x>1$$

where a and b are constants, what is the value of $a-b$?

A. 4
B. -16
C. 8
D. 16

2. If

$$x^{a^2}x^{-b^2}=x^{25}, x>1$$

where a and b are constants and $a+b=1$, what is the value of $a-b$?

A. -5
B. 5
C. -25
D. 25

3. If

$$\frac{x^{a^2+3ab}}{x^{ab-b^2}}=x^{4}, x>1$$

where a and b are constants, what is the value of $a+b$?
A. -4
B. -2
C. 2
D. 8

4.
If

$$\frac{x^{a^4+3a^2b^2}}{x^{a^2b^2-b^4}}=x^{81}, x>1$$

where a and b are constants, what is the value of a^2+b^2?
A. -3
B. 3
C. -9
D. 9

5.
If

$$\left(x^{a^2}\right)^{b^2}=x^{16}, x>1$$

where a and b are constants, what is the value of a if $b=2$?
A. 4
B. -4 and 4
C. -2 and 2
D. 2

6. There is no problem here! And so, there is no solution!

7.
If

$$\left(x^{b^2}\right)^{c}=x^{16}, x>1$$

where b and c are constants, what is the value of c if $b=-4$?
A. -2
B. 2
C. -1

D. 1

Solutions to *Exponents* (Passport to Advanced Mathematics)

1. The correct answer is B. 8

We know from *exponent rules of a quotient with the same base* that we must subtract the exponents.

$$\rightarrow \frac{x^{a^2-ab}}{x^{ab-b^2}} = x^{64} \rightarrow x^{a^2-ab-ab+b^2} = x^{64}$$

$$\rightarrow x^{a^2-2ab+b^2} = x^{64}$$

Now, you should know from the *Binomial Square* that

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$\rightarrow x^{(a-b)^2} = x^{64}$$

Now, comparing exponents in this last equation, we conclude that:

$$\rightarrow (a-b)^2 = 64$$

$$\rightarrow (a-b)^2 = 8^2$$

So then $a-b=8$.

2. The correct answer is D. 25

We know from *exponent rules of a product with the same base* that we must add exponents.

$\rightarrow x^{a^2} x^{-b^2} = x^{25} \rightarrow x^{a^2-b^2} = x^{25}$

Now, we know the *Difference of Squares* that $x^2 - y^2 = (x+y)(x-y)$

$\rightarrow x^{(a+b)(a-b)} = x^{25}$

Now, *comparing exponents*

$\rightarrow (a+b)(a-b) = 25$

$\rightarrow a-b = \frac{25}{a+b}$

But we know that $a+b=1$

$\rightarrow a-b = \frac{25}{1} = 25$

3. The correct answer is C. 2

We know from *exponent rules of a quotient with the same base*, we must subtract exponents.

$\rightarrow \frac{x^{a^2+3ab}}{x^{ab-b^2}} = x^4 \rightarrow x^{a^2+3ab-ab+b^2} = x^4$

$\rightarrow x^{a^2+2ab+b^2} = x^4$

Now, you should know from the *Binomial Square* that $(a+b)^2 = a^2 + 2ab + b^2$

$\rightarrow x^{(a+b)^2} = x^4$

Now, *comparing exponents*

$\rightarrow (a+b)^2 = 4$

$\rightarrow (a+b)^2 = 2^2$

This time, comparing bases, it is obvious that:

$a+b=2$

4. The correct answer is D. 9

We know from *exponent rules of a quotient with the same base* that we must subtract exponents here:

$\rightarrow \frac{x^{a^4+3a^2b^2}}{x^{a^2b^2-b^4}} = x^{81} \rightarrow x^{a^4+3a^2b^2-a^2b^2+b^4} = x^{81}$

$\rightarrow x^{a^4+2a^2b^2+b^4} = x^{81}$

Now from the *Binomial Square* that $(x+y)^2=x^2+2xy+y^2$

Replacing x with a^2 and y with b^2, we get $(a^2+b^2)^2=a^4+2a^2b^2+b^4$

$x^{a^4+2a^2b^2+b^4}=x^{81} \rightarrow x^{(a^2+b^2)^2}=x^{81}$

Comparing exponents on what is to the right of the arrow just above, we write:

$\rightarrow (a^2+b^2)^2=81$

$\rightarrow (a^2+b^2)^2=9^2$

$\rightarrow a^2+b^2=9$

5. The correct answer is C. -2 and 2

Using the *"back-to-back exponents" rule*, we know that we have to multiply the exponents a^2 and b^2."

$\left(x^{a^2}\right)^{b^2}=x^{16} \rightarrow x^{a^2b^2}=x^{16}$

Now, *comparing exponents since the bases are the same*, we have:

$a^2b^2=16$

We are told that $b=2$.

$\rightarrow 2^2a^2=16 \rightarrow 4a^2=16$

$\rightarrow a^2=4 \rightarrow \sqrt{a^2}=\pm\sqrt{4}$

$\rightarrow a=\pm 2 \rightarrow a=2 \vee a=-2$

6. *No solution!*

7. The correct answer is D. 1

Once again, using the *"back-to-back exponents" rule*, we know that we have to multiply the exponents b^2 and c."

$\left(x^{b^2}\right)^c=x^{16} \rightarrow x^{b^2c}=x^{16}$

Now comparing exponents since the bases are the same, we have:

$b^2c=16$

Because we are told that $b=-4$.

$\rightarrow (-4)^2c=16 \rightarrow 16c=16$

$\rightarrow c=1$

If you go back to the list of *exponent rules*, you'd see that the last of them (#7) is

$$a^{\frac{b}{c}}=\sqrt[c]{a^b}$$

Now, if broke this down, we ***would have to first*** see that

$$a^{\frac{1}{c}}=\sqrt[c]{a}$$

With this,

$$a^{\frac{b}{c}}=\left(a^{\frac{1}{c}}\right)^b=\left(a^b\right)^{\frac{1}{c}}$$

The last equation is a trio of equations. Both expressions to the right of the first equal side are valid interpretations of

$$a^{\frac{b}{c}}$$

It is obvious how we get to them, we use the *second exponent rule*, where we multiply *back-to-back exponents*. But in practice, one of the two is better than the other when doing computation.

Let us consider the following.

- $\sqrt[3]{27^4}=?$

$$27^{\frac{4}{3}}=\left(27^{\frac{1}{3}}\right)^4=\left(27^4\right)^{\frac{1}{3}}$$

Going back to the first equation we have written in this section, we see that:

$$\left(27^{\frac{1}{3}}\right)^4=\left(\sqrt[3]{27}\right)^4=3^4=81$$

You see, this first *interpretation* is a much easier path to get to the correct answer of 81.

Had we chosen the path of the second interpretation, we would have to write the following:

$$\sqrt[3]{27^4}=\sqrt[3]{27^3\cdot 27}=27\sqrt{27}=27\cdot 3=81$$

Obviously, we got to the same place, but the second approach was a lot more work than the first.

Other than the above distinct rule on *rational exponents*, the only thing we need to talk about before we close out this section is a quick mention of *square root rules*. Even though we are calling them *square root rules*, they apply to *cube roots*, etc.

Most importantly, <u>these rules are all *exponent rules*</u>. Let's show why this is true.

We know from the *exponent rules* we previously listed that

$(ab)^c = a^c b^c$

Now, if we let $c = \frac{1}{2}$

$\rightarrow (ab)^{\frac{1}{2}} = a^{\frac{1}{2}} b^{\frac{1}{2}}$

Since we know that

$a^{\frac{1}{n}} = \sqrt[n]{a}$

With this, we see that

$(ab)^{\frac{1}{2}} = a^{\frac{1}{2}} b^{\frac{1}{2}} \rightarrow \sqrt{ac} = \sqrt{a}\sqrt{b}$

So then, it should come as no surprise that

$$\sqrt{\frac{a}{b}} = \left(\frac{a}{b}\right)^{\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} = \frac{\sqrt{a}}{\sqrt{b}}$$

In this manner, these are just the *exponent rules* in disguise. So we are done here!

Try these two on your own

- Which of the following is equivalent to $5^{9/4}$

Solution

First, we will rewrite the expression using *radicals*:

$5^{\frac{9}{4}} = \sqrt[4]{5^9}$

Now, we will write the 5^9 as $5^8 \cdot 5$.

$\sqrt[4]{5^8 \cdot 5}$

Using *radical rules*, we can write the expression as $\sqrt[4]{5^8} \cdot \sqrt[4]{5}$

$\rightarrow 5^{\frac{8}{4}} \cdot \sqrt[4]{5} = 25\sqrt[4]{5}$

$-\sqrt{a-5} = x$

- In the equation above, a is a constant. If $x = -5$, what is the value a?

Replacing the value of x with -5, we get the following equation:

$-\sqrt{a-5}=-5$

Solution

To start, we can divide by negative one and we will square both sides to get rid of the square root on the left.

$a-5=25$

Finally, adding 5 to both sides, we get the $a=30$.

- If $\sqrt{x}+\sqrt{16}=\sqrt{36}$, what is the value of x?

A. 1
B. $\sqrt{2}$
C. 2
D. 4

Solution

First, let's simplify the *square roots* that are easy to compute:

$\sqrt{x}+\sqrt{16}=\sqrt{36}\rightarrow\sqrt{x}+4=6$

$\rightarrow\sqrt{x}=2$

$\sqrt{x}=\sqrt{4}$

$x=4$

Rational Exponents (Passport to Advanced Mathematics)

1. Which of the following is equivalent to $27^{\frac{2}{3}}$?

A. $\sqrt[3]{27}$
B. $\sqrt{27}$
C. $\sqrt[3]{27^2}$
D. 9

2. Which of the following is equivalent to $4^{\frac{3}{4}}$?

A. $\sqrt[4]{32}$

B. $\sqrt{27}$
C. $\sqrt[3]{64}$
D. $2\sqrt[4]{4}$

3. Which of the following is equivalent to $8^{\frac{4}{3}}$?

A. $\sqrt[4]{8^3}$
B. 216
C. $\sqrt[3]{64}$
D. $2\sqrt[4]{4}$

4. Which of the following is equivalent to $9^{\frac{3}{2}}$?

A. $\sqrt[3]{81}$
B. 27
C. $\sqrt[2]{81}$
D. $9\sqrt{9}$

Solutions to *Rational Exponents*

1. The correct answer is D. 9

   Because $27^{\frac{2}{3}}=\sqrt[3]{27^2}=\sqrt[3]{27\times 27}=\sqrt[3]{(9)(3)(9)(3)}=\sqrt[3]{(9)(9)(9)}=\sqrt[3]{9^3}=9$

2. The correct answer is D. $2\sqrt[4]{4}$

   Because $4^{\frac{3}{4}}=\sqrt[4]{4^3}=\sqrt[4]{(4)(4)(4)}=\sqrt[4]{16(4)}=\sqrt[4]{16}\sqrt[4]{4}=2\sqrt[4]{4}$

3. The correct answer is B. 16

   Because $8^{\frac{4}{3}}=\left(2^3\right)^{\frac{4}{3}}=2^{3\left(\frac{4}{3}\right)}=2^4=16$

4. The correct answer is B. 27

   Because $9^{\frac{3}{2}}=\left(3^2\right)^{\frac{3}{2}}=3^{\left(2\left(\frac{3}{2}\right)\right)}=3^3=27$

−−−−−−−¿

Radical Equations

In this section, the most important thing for you to keep in mind is the fact that

$(\sqrt{x})^2 = x$

Having already looked at exponent rules, you should be able to easily see why this must be true.

$\left(x^{\frac{1}{2}}\right)^2 = x^{\frac{2}{2}} = x$

More generally,

$(\sqrt[n]{x})^n = \left(x^{\frac{1}{n}}\right)^n = x^{\frac{n}{n}} = x$

Let us do a couple of examples.

- If $k = 3\sqrt{5}$ and $2k = 3\sqrt{5t}$, what is the value of t?

Solution

This is a simple exercise in substitution.

$2k = 3\sqrt{5t} \rightarrow 2(3\sqrt{5}) = 3\sqrt{5t}$

Cancelling the 3s on both sides of the equation on the right above:

We have $2\sqrt{5} = \sqrt{5t}$.

Now, using *properties of square roots* from the previous section, we have:

$2\sqrt{5} = \sqrt{5}\sqrt{t}$

$\rightarrow 2 = \sqrt{t} \rightarrow 2^2 = 4 = \sqrt{t}^2$

$t = 4$

Exponential Growth and Decay

Some *rates of change* are *not linear* (the growth is *not a constant rate*.) Visually, we are saying some *rates of change* cannot be modeled by lines.

While there are a variety of other *rates of changes*, other than *linear* and *quadratic*, *exponential growth* and *decay* are the only two other types of growth we must be concerned with.

Let us start the discussion with an example.

- Suppose you invest $\$1000$ in a bank account that pays a 7% annual interest. What do you expect to get at the end of the year?

Solution

Clearly, at the end of the year, you expect to retain your $\$1000$ *original investment*, this is called *the principal*, and additionally, you expect to earn 7% on this $\$1000$. That means you have a total of

$$\$1000+\$1000\times 0.0=\$1000(1+0.07)$$

What about at the end of the second year?

Well, assuming you don't withdraw any money, you will have

$$\$1000(1+0.07)+\$1000(1+0.07)\times 0.07$$

Just like before, you would have kept your *original amount* (but the original amount is now what you had at the end of the first year) and you earn 7% on *this new original amount*.

¿ $\$1000(1+0.07)(1+0.07)=\$1000(1+0.07)^2$

In this manner, at the end of year t, you will have ¿ $\$1000(1+0.07)^t$

If we call the *initial investment* P for *Principal* (the name usually given to it) and *the interest* r, notice that at the end of year t, you will have:

$A=P(1+r)^t$

The A is for *Amount* at time t. Now, if instead of *appreciating* (*earning positive interest*), you l*ose a certain percent per year* (there are situations that merit this, for example, car values *depreciate*), then, the formula is adjusted to look as follows.

$A = P(1-r)^t$

The first of the two formulas is for *Exponential Growth* and the second is for *Exponential Decay*.

While the names of the variables might be different in a different setting (*radioactive decay* for example), *all exponential growth* and *decay* questions can be answered by adopting one of these two formulas.

Ratios and Percentages (Other)

On a *ratio* problem, when we write $a:b$, it is the same as writing

$\frac{a}{b}$

For example, when we write $7:2$, it is the same as

$\frac{7}{2}$

Regardless of what the quantities seven and two represent.

As this topic is best learned by doing problems, let's dive in by considering a quintessential example.

Example 1

The *ratio* of boys to girls in a math class is $2:3$. If there are 30 students in the class, how many are boys?

Solution

To solve this problem, we turn $2:3$ to its equivalent fraction to write the following *proportion*.

$\frac{2}{5} = \frac{x}{30}$

Notice, our denominator on two is five. This is so because if we have a ratio of $2:3$, then $2+3=5$ represents the total.

From here, we can cross multiply and solve or multiply both sides by 30 and go from there. Either is a fine way to finish the problem. If we choose the former, we have the following

$$2\times 30=5x$$

$$\rightarrow 5x=60$$

$$\rightarrow x=\frac{60}{5}=12$$

So, we see that there are 12 boys and therefore, 18 girls. Notice that $12:18$ is the same *ratio* as $2:3$.

It is easy to see why they are the same *ratio* if we write:

$$\frac{2}{3}=\frac{12}{18}=\frac{6(2)}{6(3)}$$

Just as we saw at the end of the last solution, notice that $2:3$ is the same as $12:18$ suggests that if we multiply 2 and 3 by the same positive integer, we will keep getting the same ratio. That is in fact true.
$2:3\rightarrow 4:6\rightarrow 6:9\rightarrow 8:12\rightarrow 10:15\rightarrow 12:18\rightarrow 14:21\rightarrow\cdots$

All the *ratios* above are equal as ratios. This is so again because,

$$\frac{2}{3}=\frac{4}{6}=\frac{6}{9}=\frac{8}{12}=\frac{10}{15}=\frac{12}{18}=\frac{14}{21}=\cdots$$

Now, for the problem we had to solve here, notice, we needed 12 and 18 because $12+18=30$.

There is another approach we could have taken to solve our problem.

Since the key to solving the problem is *finding the positive integer* x so that $2x$ and $3x$ *sum to* 30, we can proceed as follows:

We write

$$2x+3x=30$$

$$5x=30$$

$$\rightarrow x=6$$

Now, to finish the job, we write that $2x=12$ and $3x=18$.

It is up to the student to decide which approach is preferable, but we personally think the second approach is easier to follow and universally efficient.

It is time for a second slightly more challenging example.

Example 2

The *ratio* of peanuts to cashews in a bag is $5:7$ and the *ratio* of cashews to almonds in the same bag is $6:9$. What is the *ratio* of peanuts to almonds in this bag?

Solution

Here, we have a bridge between peanuts and almonds. They are both related to cashews. So, the goal is to turn the 7 and 6 into the same number.

We know from the previous example that $5:7$ is the same as $30:42$. We also know that $6:9$ is the same as $42:63$. Now, we don't just have cashews as a bridge, but we have a direct numerical bridge as follows.

$30:42$

$42:63$

The problem is easily solved now. The *ratio* of peanuts to almonds is $30:63$.

Since both 30 and 63are divisible by 3, you should simplify the *ratio* before you look for the correct answer from the choices.

$30:63$ is the same as $10:21$. So, the correct answer provided should be $10:21$.

The practice required to master this most recent skill will be found in the practice problems to follow.

Example 3

The *ratio* of $x:y$ is $4:7$. What is the *ratio* of $x+4:y+2$?

Solution

We can answer questions like what is the *ratio* of $7x:7y$or $4x:2y$based on what we are given. In the case of $7x:7y$we can say 4:7 and in the case of $4x:2y, 16:14$. And while a very tempting answer to give to $x+4:y+2$ is $4+4:7+2$ or $8:9$ it is incorrect.

From knowing $x:y$ is $4:7$, we can gather that:

$$4x=7y \rightarrow y=\frac{4}{7}x$$

From this, the best we can do to answer a question like the ratio of $x+4:y+2$ is to say:

$x+4:\frac{4}{7}x+2$

But without knowing the value of x or y, we cannot say more.

Percentages

Percentages are intimately linked to *ratios* (the last topic we discussed.) For example, it should now be clear that $7:10$ can be understood to mean

$\frac{7}{10}$

You should know from elementary fundamentals that

$\frac{7}{10}=0.7$

and 0.7 is the same as 70%. Where the *ratios* are different from *percentages* is in that, in the denominator of the quotient above, 10 doesn't represent, *the whole* in a *ratio*. Whereas when we are talking about a *percentage*, in writing

$\frac{7}{10}$

We are saying, 7 *is the part* and 10 *is the whole*. To elaborate, when we write $7:10$, we may mean, for every 7 *of something* (ex: number of boys in class, etc.), there are 10 *of this other thing* (ex: number of girls in a class, etc.)

But, when we are talking about percentages, we must be saying, for every 7 *of something* (ex: number of boys in a class, number of girls in a class, etc.) *there are a total of* 10 *of this something and other things altogether*.

Of course, even as a *ratio*, we may mean $7:10$ as the number of boys to the total number of students in the class, but this is usually not the case. With the comparison over, let us focus on the topic at hand.

The most important thing to understand about *percentages* is the fact that 0.7 means 70%. If we move by analogy, it must mean that

0.77 represents 77%

0.777 represents 77.7%

1.777 represents 177.7%

And 7.777 represents 1777.7%

As such, it should be easily understood that "one" or 1.0 represents 100%.

Let's give a few examples.

Example 1

Blanca went to dinner with *Roxanna*, and they paid $\$112$. If their meal cost $\$97$ and it is polite to tip 15%. Were the two girls polite?

Solution

The difference between 112 and 97 is 16. So, we write the following:

$$\frac{16}{97}=0.165$$

So, we see now that they left $\$16.50$ in tips. So, they were polite although not by much!

Example 2

What *percent* of 112 is 142.

Solution

Here, we can just write

$$\frac{142}{112}=1.268$$

So, we can quickly answer, 142 is 126.8% of 112. This is so because we correctly realize 112 to be *the whole* and 142 to be *the part*, 142 is *what percent **of*** 112 is another wording of the question. From there, we know the translation that $1.268\rightarrow 126.8\%$ so we are done.

Percent Change

When you want to calculate the percent change in a certain quantity, you use the following formula.

$$\frac{New\,Value-Old\,Value}{Old\,Value}\times 100\%$$

Let's see a final example concerning percent change before we close this section.

Example 3

A 5.11 hoodie (yes, they make good product so we will advertise for them for free) costs $\$47.00$. During the Christmas season, the hoodie was marked down to $\$42$. What *percent change* does the holiday discount represent.

Solution

This is just a matter of executing the formula we just mentioned above.

$$\frac{42-47}{47}\times 100\%$$

$$\rightarrow \frac{-5}{47}\times 100\%$$

$$\rightarrow -0.1064\times 100\% = -10.64\%$$

Of course, if you keep in mind our earlier notes on the equivalence between 0.1064 and 10.64%, you should not need to multiply by 100%. Since the question is asking the *percent change*, we give the answer 10.64 percent. The negative sign in front of this number tells us it is a change downward but we ignore the negative sign because all we are after, is the "*percent change*."

Ratios and Percentages (Other)

During a basketball game, Iverson took 42 shots in the second half.

The following three problems depend on the information given above.

1. If the ratio of $2-$¿point shots to $3-$¿point shots Iverson took in the second half was $5:2$, how many $3-$¿point shots did Iverson take in the second half?

   A. 6 B. 30 C. 12 D. 9

2. Given that the number of total shots Iverson took in the first half was a 142.86% *increase* from the number of total shots he took in the second half, which of the following closely approximates the *percent decrease* in the number of shots Iverson took when comparing his first half performance to his second half performance?

   A. 5 B. 108.33 C. 52 D. 48

3. If Iverson made 42% of all shots he took in the game (first half and second half combined), how many $3-$¿point shots did he make during the entire game if he took a total of 25 three pointers during the game? (*Round your answer to the nearest integer.*)
   A. 15.54
   B. 11
   C. 16
   D. It cannot be determined from the given information!

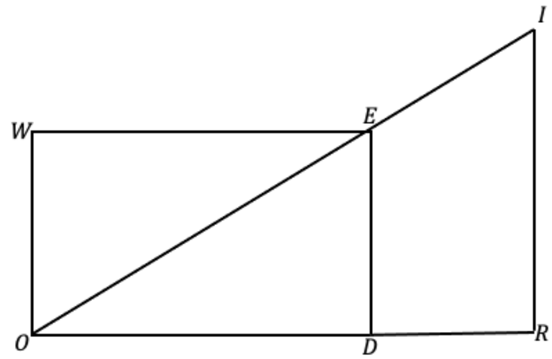
4. The *ratio* of the three angles in a triangle is $2:3:4$, what is the <u>difference between the largest and the smallest</u> angles in this triangle?

   A. 20 B. 60 C. 80 D. 40

5. A bag contains *Red*, *Black*, and *yellow* marbles. There are a total of 112 marbles. If the *ratio* of *Red* to *Black* marbles is $5:3$ and the *ratio* of *Red* to *yellow* marbles is $7:4$, what is the *ratio* of *Black* to *yellow* marbles?

   A. $3:4$ B. $7:4$ C. $21:20$ D. $4:7$

6. In the figure below, WEDO is a quadrilateral. Points O, E, and I are *collinear* and OR and IR form a right angle at R. If the *ratio* of angles $\angle IOR = x°$ and $\angle OIR = y°$ is $2:1$, what is the value of $\sin y°$?



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7. In the *Rational Equation* below the *ratio* of $a:b$ is $4:1$. What is the value of $a+b$?

$$\frac{x^2+ax+12}{x^2-b^2}=\frac{x+6}{x-2}$$

   A. $\frac{45}{4}$ B. $\frac{4}{45}$ C. $\frac{9}{4}$ D. $\frac{4}{9}$

Solutions to *Ratios and Percentages* (Other)

1. The correct answer is C. 12.

   Since we are told that the *ratio* of $2-$¿point shots to $3-$¿point shots is $5:2$, and that Iverson took 42 shots in the second half, we can set up the following equation and solve the problem:

   $$5x+2x=42$$

   $$\rightarrow 7x=42 \rightarrow x=6$$

   Here, the number of $3-$¿point shots is represented by $2x$, so the answer is 12.

2. The correct answer is C. 52.

   The first thing we should do is to find the number of total shots Iverson took in the first half of the game.

   The number of shots in the first half is equal to 142.86% of the number of shots in Iverson took in the second half.

   So, if x represents the number of shots in the second half, the number of shots in the first half is $1.4286x$. Additionally, we need

   $$1.4286x+x=42$$

   $$\rightarrow 2.4286x=42$$

   $$\rightarrow 2.4286x=\frac{42}{2.4286}=17.3$$

   So, to the nearest integer, Iverson scored $42-17$ or 25 shots in the first half and clearly 17 shots in the second half.

   Now, we can simply use the formula given in the lesson.

   $$\left(\frac{17-25}{25}\right)100\%=\left(\frac{-8}{25}\right)100\%=-32\%$$

   The negative sign is letting us know it is a *percent decrease*, so the *percent decrease* is 32.

3. The correct answer is C.

Since Iverson took 25 three pointers during the game and we are told that he made 42% of all the shots he took, this suggests that he made 42% of the 25 three pointers he took. However, this is not correct as we only know the percent of total shots made and not how many of those total shots made were $2-¿$point as opposed to three pointers.

4. The correct answer is D.

Since the *sum of angles* of a triangle is $180°$, we can set up the following equation based on the given *ratio*:

$$2x+3x+4x=180$$

$$\rightarrow 9x=180 \quad \rightarrow x=20$$

Here, the largest angle is represented by $4x=(4)(20)=80$, and the smallest angle is represented by $2x=(2)(20)=40$. This means that the difference between the largest and smallest angle is $80-40=40$.

5. The correct answer is $D.21:20$.

We will build the following table for the *ratios* (we will use the following order on purpose)

Black	*Red*	*Yellow*
3	5	
	7	4

As you can see from the table, we must "*make equal*" the 5 and the 7. We can do this by finding their LCM (Least Common Multiple), which in this case is 35. So the table will transform to the following *ratios*:

Black	*Red*	*Yellow*
21	35	
	35	20

We can now see that the *ratio* of *Black* to *Yellow* marbles is $21:20$. Notice that even though we are told there are a total of 112 marbles, this is extra information put in place to see if you can separate the meats from the bones.

6. The correct answer is $\frac{1}{2}$.

Since we are told that ORI is a right triangle, $x+y$ must equal $90°$. Hence, we can set up and solve the following equation based on the given *ratio*:

$$2k+k=90 \rightarrow 3k=90$$

$$\rightarrow k=30$$

Here, angle y is represented by $k$which is equal to 30. This is a well-known $\sin(x)$ value, which is equal to $\frac{1}{2}$. In fact, the triangle with the two acute angles should be very familiar, it is the $30°-60°-90°$.

7. The correct answer is D. $\frac{45}{4}$

There are a few different approaches but here is a neat one. Let us first cross multiply and get *Cubic Polynomials* on both sides of the qual sign so we can hope to compare *coefficients* thereafter

$$\frac{x^2+ax+12}{x^2-b^2}=\frac{x+7}{x-2}$$

$$\rightarrow (x\text{¿¿}2+ax+12)(x-2)=(x^2-b^2)(x+7)\text{¿}$$

$$\rightarrow x^3+ax^2+12x-2x^2-2ax-24=x^3-b^2x+7x^2-7b^2$$

$$\rightarrow (a-2)x^2+(12-2a)x-24=7x^2-b^2x-7b^2$$

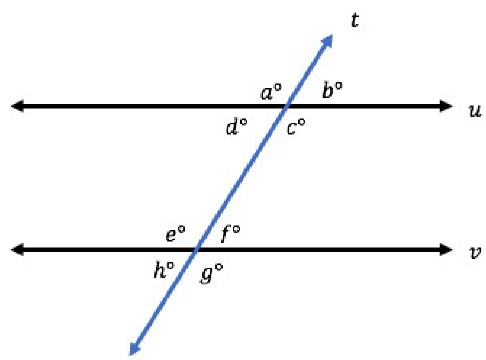
Comparing the *coefficients* of x^2on both sides of the equation, we can see that $-2=7 \rightarrow a=9$. We can deduct from the ratio that $a=4b$, which means that $b=\frac{9}{4}$. Hence, $a+b=9+\frac{9}{4}=\frac{45}{4}$.

$$-------\text{¿}$$

Geometry

Parallel lines & the transversal.

You should already know what *parallel lines* are. A *transversal* is a line that cuts across two or more *parallel lines*. The visual below shows that 8 angles are formed when one *transversal* cuts across two *parallel lines*.



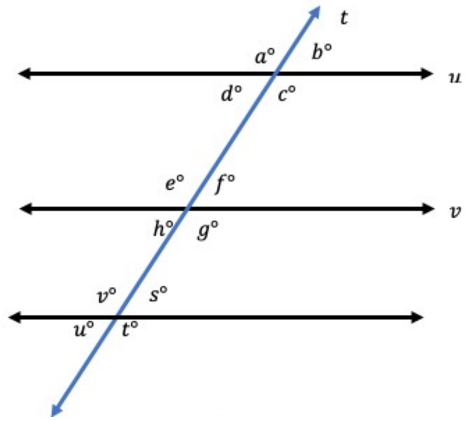
Notice that of the eight angles formed, four are *acute* $(b°, d°, f°, h°)$ and four are *obtuse* $(a°, c°, e°, g°)$.

Now, in geometry, you must have learned specific names for each pair of angles. For example, $c°$ and $f°$ are called *consecutive interior angles* and all *consecutive interior angles* formed by *two parallel lines* and a *transversal* are *supplementary*.

Since the goal is efficiency and knowing these names is not required (you can always look it up unless you must memorize for your test) we will just state the angle relationships using the following **Golden Rule**.

The Golden Rule – *all the small angles (acute angles)* are equal, and all the *big angles (obtuse angles)* are equal. *One small* and *one big* angle together *add to 180* degrees (are supplementary.)

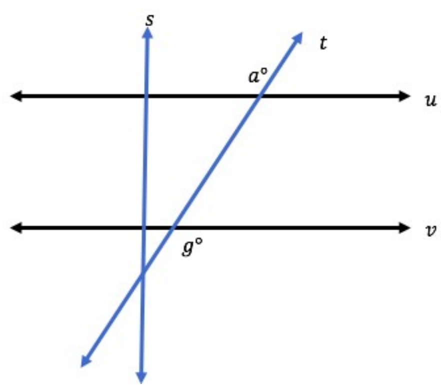
Now, questions on this topic might be presented in the context of algebra or from a slightly different angle (no pun intended) than that which we have presented above. Below, we show you what happens if it were three *parallel lines* that are cut by a *transversal*.



The **Golden Rule** remains the same!

In the following exercise problems, sufficient practice of different varieties is provided so that you are amply practiced in this topic.

Parallel lines and *transversals* (Geometry?)



1. In the figure above, lines u and v are both *perpendicular* to line s. If the measure of angle a is $115°$ ($m\angle a = 115°$), then $m\angle g = ?$

A. $75°$
B. $115°$
C. $105°$
D. $85°$

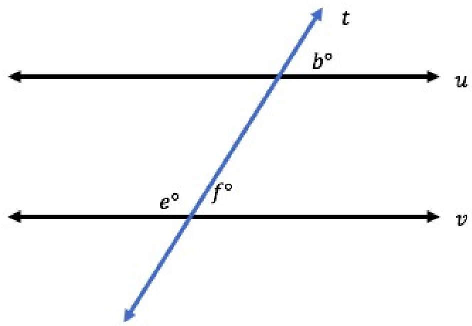
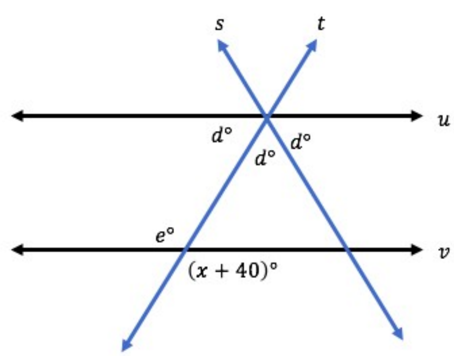


Figure not drawn to scale.

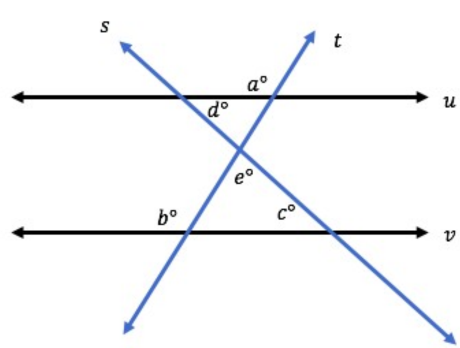
2. In the figure above, lines u and v are *parallel* and line t is a *transversal*. If the measure of angle f is $95°$, $m\angle e = (2x-3)°$, and $m\angle b = (2x+7)°$, what is the value of x?

   A. $75°$
   B. $115°$
   C. $105°$
   D. $44°$



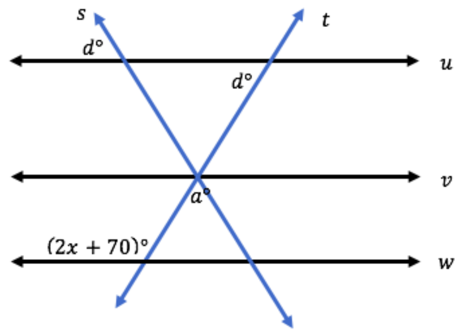
3. In the figure above, lines u and v are both *parallel*. What is the value of x?

   A. $75°$ B. $115°$ C. $105°$ D. $80°$



4. In the figure above, lines u and v are *parallel* and lines s and t are *transversals*. If the measure of angle a is $110°$ and the measure of angle d is $70°$, what is the measure of angle $e°$?

A. $40°$ B. $60°$ C. $70°$ D. $30°$

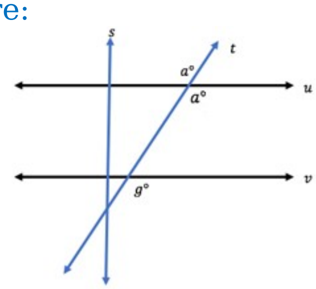


5. In the figure above, line v is parallel to lines u and w. Lines s and t are transversals and they are concurrent (as in the figure above) with line v. If the measure of angle a is $60°$, what is the value of x?

A. $50°$ B. $25°$ C. $60°$ D. $40°$

Answers *Parallel lines* and *transversals* (Geometry)

1. The correct answer is B. $115°$.
To find the measure of angle g, we would first apply the following property. When two lines cross (intersect), two pairs of vertical angles are formed. And we know that opposite angles (*vertical angles*) are equal. So then, now we have the following picture:



From here, notice that because lines u and v are parallel (because line s is *perpendicular* to both lines u and v), we see that angle a is equal to angle g. which means that the correct answer is $115°$, which is answer choice B. There are other ways to reach the same conclusion.

2. The correct answer is D. $44°$.
Since angle f is $95°$, angle e must measure $85°$. But we are told that angle e is also equal to $(2x-3)°$, which means we can set up the following equation to solve for x.

$2x-3=85$

$\rightarrow 2x = 88$

$\rightarrow x = 44$

Which is answer choice D.

A different route would be to write:

$2x + 7 = 95$

3. The correct answer is D. $80°$.

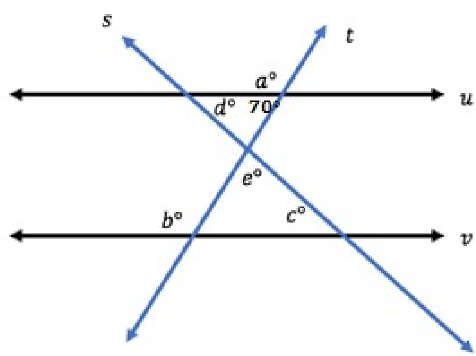
First, we can easily solve for $d°$. We see that 3 $d°$s makes a half turn, which implies that $3d° = 180 \rightarrow d° = 60$. Now, because $d°$ and $e°$ are *consecutive interior* (or one small one big with lines u and v are parallel) we can see that $e° = 120°$. This in turn means that $(x + 40)° = e°$

$\rightarrow x + 40 = 120 \rightarrow x = 80$

Which is answer choice D.

4. The correct answer is A. $40°$

The first thing we should do is notice that since angle a is $110°$, the angle next to it should be $70°$ because together they (*linear pairs*) must add up to $180°$ as they are on the same line (line t). Now we have the following scenario:



Since angle d is $70°$, the other angle inside the small triangle in the center of the figure must be $40°$. Since this angle whose value we just found and angle e make a pair of *vertical angles* (as they are formed by the intersection of two lines) they must be equal, the value of e is $40°$.

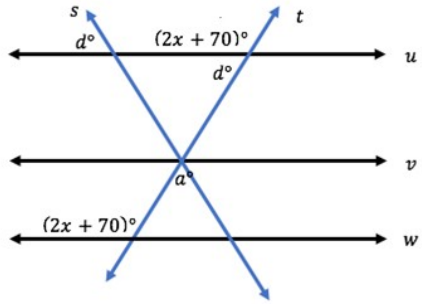
5. The correct answer is B. $25°$

First, we see that since angle a is formed by the intersection of lines s and t, the angle opposite to it must have the same value. Another application of *vertical angles* will bring the angle d in the exterior (top left) to the triangle above line u. So, this way, we can set up the following equation and solve for d.

$d° + d° + 60° = 180°$

$\rightarrow 2d° = 120° \rightarrow d° = 60°$

Now, we know that lines $u$and w are parallel because they are both parallel to a third angle, angle v. Now, we can look at the visual below for some guidance on how to proceed.



We observe that angle d and $(2x+70)°$ add up to $180°$ (are supplementary) because they form a linear pair. Finally, because we know the value of d, we can solve for x.

$$2x+70+d=180 \rightarrow 2x+70+60=180$$

$$\rightarrow 2x=50 \rightarrow x=25°$$

Which is answer choice B.

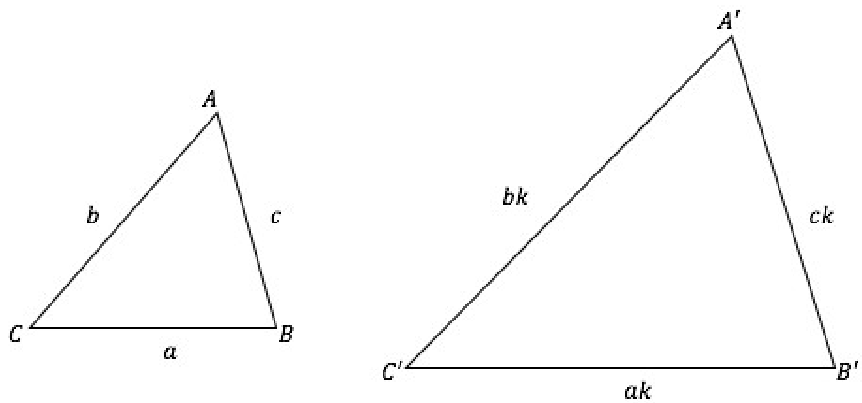
−−−−−−−¿

Similar Triangles

Two triangles are said to be *similar if they are scaled versions of one another*. That is, *the three angles in the two triangles are identical*.

The theorem we use to gauge if two triangles are *similar* is called the *AA- Theorem*. That is, the *Angle-Angle Theorem*.

The name comes from the fact that we are looking for two pairs of angles on the two triangles to be the same. Of course, *if two pairs of angles in two triangles are the same, the third pair is automatically the same*.



$\Delta ABC \quad \Delta A'B'C'$

Since the two triangles are *scaled versions of one another*, their *corresponding sides are proportional*. The symbol for *similarity* is whether we are talking about triangles or other *polygons*.

Notice that the *side proportions* in the figures above would tell us:

$$\frac{AB}{BC}=\frac{c}{a}=\frac{ck}{ak}=\frac{A'B'}{B'C'}$$

We can also write

$$\frac{BC}{AC}=\frac{a}{b}=\frac{ak}{bk}=\frac{B'C'}{A'C'}$$

Using similar logic (no pun intended), we can finally also write:

$$\frac{AC}{AB}=\frac{b}{c}=\frac{bk}{ck}=\frac{A'C'}{A'B'}$$

Similarity extends to *polygons* with more sides. For example, two rectangles, pentagons, etc. can be similar if the angles in them are equal. That is, again, if one is a scaled version of the other.

Before we say we are done here, let us consider one example.

Example

A square (a special rectangle) has a side length of a. If a scaled version of this square has a side length of ka, where $k>1$. What is the *ratio* (small: big) of the areas of these two squares?

A. k^2
B. k
C. $\frac{1}{k^2}$
D. $\frac{1}{k}$

Solution

We can do this in one of two ways. We can either calculate the areas and then figure out the *ratio* of the areas (the laborious way) or recall a theorem from geometry (the efficient way.)

If we start with the longer path, we know that the area of the smaller squares is a^2 and the area of the bigger square is $(ka)(ka)=k^2a^2$.

So, when we take the *ratios*, we have:

$$\frac{a^2}{k^2 a^2} = \frac{1}{k^2}$$

The more efficient path is to remember that if the sides of two similar *polygons* is in a *ratio* r, their areas will have to be in a *ratio* r^2.

Since we know the *ratio* of the sides is $r = \frac{1}{a}$, the *ratio* of their areas is thus

$$r^2 = \left(\frac{1}{a}\right)^2 = \frac{1}{a^2}.$$

Answers to *Similarity and miscellaneous problems* (Geometry + Other)

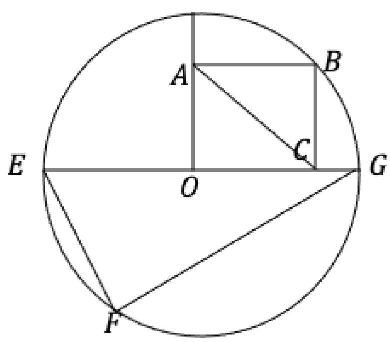


Figure not drawn to scale.

1. In the figure above, ABCO is a square and EFG is a triangle. Additionally, EG is the *diameter,* and the *center* of *circle* is O. If $FG=8$ and one side of the *square* measures 4, what is the length of EF?

2.

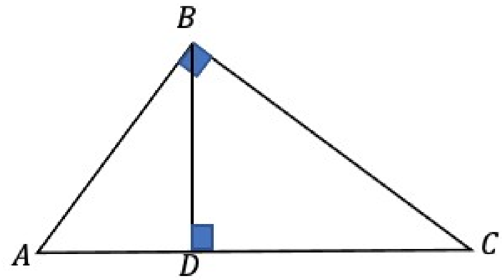


Figure not drawn to scale.

ABC is a *right triangle*. BD is drawn to the *altitude* AC of triangle ABC. If $AD=7$, $DC=10$, and $BD=x$, what is the value of x?

3. In the triangle below, $m\angle ACB=x=m\angle CBD$. If $\tan x=4$, $DC=8$, what is the length of AC?

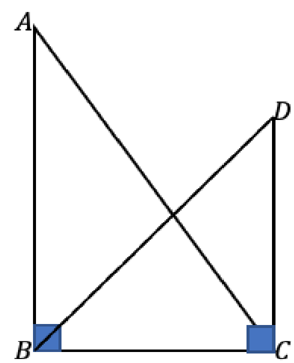
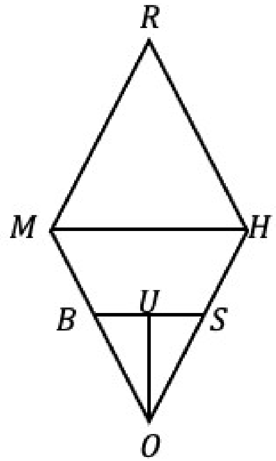


Figure not drawn to scale.

4. In the figure below, $RHOM$ is a rhombus and MH is the *shorter diagonal* of this *rhombus*. If we know that $MH \parallel BS$, $UO \perp US$ with one side of the *rhombus* measuring 10, MH equaling 6 with *midpoint* U while $BS=3$, what is the length of OU?



For the following problem, basic understanding of ratios is assumed. This should be a fair assumption, but should you need to brush up, you can find lessons on ratios in future lessons.

5. In the figure below, BO is *parallel* to EI and EI is *parallel* to ST. If $BW = WI = 6$, $BO = 8$ and $WT : TI$ is $4:2$, then, what is the length of ST?

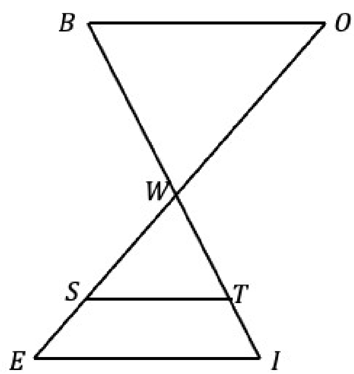


Figure not drawn to scale.

Solutions to Similarity and miscellaneous geometry problems (Other)

1.

The answer is 8.

Since the *square* has a side measuring 4, we know that the *diagonal* of the *square*, AC must be equal to $4\sqrt{2}$. As such $\overline{OB} = r = 4\sqrt{2}$. Therefore then, $8\sqrt{2}$ is the length of the *diameter* $\overline{EG}$. If we say $\overline{EG} = x$, then we can do the *Pythagorean Theorem* on triangle GFE because we know angle $F = 90°$. We know angle F is a *right angle* because it is *an inscribed angle* where the arc it intercepts has measure $180°$, thus the *inscribed angle* is one half of the measure of the arc it intercepts.

Doing *Pythagorean Theorem* on this triangle GFE, we have

$(FG)^2 + x^2 = (8\sqrt{2})^2 \rightarrow 8^2 + x^2 = (8\sqrt{2})^2$

$\rightarrow 64 + x^2 = 64 * 2 = 128$

$\rightarrow x^2 = 128 - 64$

$\rightarrow x^2 = 64$

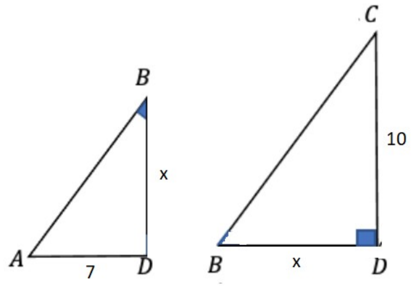
$\rightarrow x = 8$

We could have avoided the *Pythagorean Theorem* because we know two sides of triangle GFE have measure 8 and $8\sqrt{2}$ so it is immediate that it is a $45° - 45° - 90°$ and thus, the other side must be 8 as the sides of an *Isosceles Right Triangle* are of the form $x, x, x\sqrt{2}$.

2.

The answer is $x = \sqrt{70}$.

Since BD is the *altitude of the hypotenuse* of a *right triangle*, it divides the triangle into *two similar triangles* which are in turn similar to the biggest right triangle that they is comprised of them. This means that the angles in these triangles will measure the same, and the corresponding sides will be in proportion. Because of this, we can solve for x as follows. The two *similar triangles* of interest will be the following (in order of corresponding angles): ADB and BDC. We now have the following visuals.



Because of *similarity* and subsequent *proportionality of corresponding sides*, we can write:

$$\frac{x}{7} = \frac{10}{x}$$

Cross multiplying to solve for x, we have:

$$x^2 = 70 \rightarrow x = \sqrt{70}$$

This is a well-known theorem (the *Altitude to the Hypotenuse*) and as such, students familiar with it can get to the solution without drawing the triangles separately as we have showed above.

3.

The correct answer is $\sqrt{68}$.

The main observation required to solve this problem is to notice that triangles ABC and BCD are *congruent*. This is so because they are both *right triangles* and share the side BC and angle x (ASA congruence.) Therefore, the task is reduced to finding the *hypotenuse* of either of the two triangles (this is so because $AC=BD$ by CPCTC.) Because we know that side DC measures 8 and that $\tan x=4$, we can write the following equation.

$$\tan x = ¿\frac{8}{BC}=4 \rightarrow BC=2¿$$

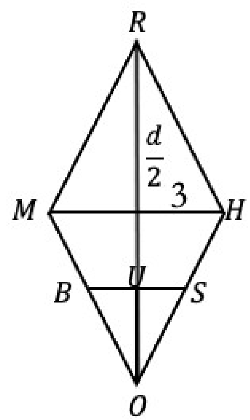
Now, letting $BD=y$, we can solve for $y$as follows:

$$2^2+8^2=y^2 \rightarrow 68=y^2 \rightarrow y=\sqrt{68}$$

4.

The correct answer is $\frac{\sqrt{91}}{2}$.

We should first draw the other diagonal (RO) of the *rhombus* (see visual below.) Doing so, we can see that four *right triangles* are formed by the sides of the *rhombus* and the two *diagonals*. This we know because the *diagonals* of a *rhombus* are *perpendicular*. We can now use this to find the length of the *longer diagonal*. Using the *Pythagorean Theorem* on the *right triangle* with hypotenuse RH, we write the following.



$$\left(\frac{d}{2}\right)^2+3^2=10^2$$

$$\rightarrow \frac{d^2}{2^2}+9=100 \rightarrow \frac{d^2}{4}=91$$

$$\rightarrow d^2=(4)(91) \rightarrow d=\sqrt{(4)(91)}=2\sqrt{91}$$

Now that we have the value of d, we can use similarity to find the required value. Since MH and BS are *parallel*, we see that triangle BUO is *similar to* all six *right triangles* in the previous visual. From what we have so far, we can see that the *ratio* of interest is $2:1$ (MH is 6 while BS is 3.) This means that OU is half of the length of the segment from point O to the *midpoint* of MH.

Now, we also know that if two triangles are similar, then their corresponding *altitudes* are in the same *proportion* as their corresponding sides. In addition to the similarity of all the right triangles we alluded to, we also know that triangles MHO and BSO are *similar*. This is because BS is a *side-splitter*, in fact, a specific *side-splitter* (the *midsegment* of MHO.)

So, then, assigning the midpoint of MH the letter P, we can write the *proportion*

$$\frac{3}{6}=\frac{1}{2}=\frac{OU}{OP}=\frac{OU}{\frac{d}{2}}=\frac{OU}{\sqrt{91}}$$

$$\rightarrow\frac{1}{2}=\frac{OU}{\sqrt{91}}\rightarrow\frac{\sqrt{91}}{2}=OU$$

5.

The correct answer is $\frac{16}{3}$.

We should first notice that the two big triangles are *congruent* to each other (SAS: we have angle B equal to angle I, $BW=¿\ WI$ and *vertical angles* formed at W.) We also know that the smaller triangle WST is *similar to* the two big triangles (WOB and WEI.) The similarity is established because we know that ST is parallel to EI since a line parallel to two lines dictates that those two lines be parallel (we were given: BO is *parallel* to EI and EI is *parallel* to ST.¿ Subsequently, as we already said, we can use the *AA-similarity* to conclude that the smaller triangle is *similar to* both *congruent* triangles. We will label the side we want to find (ST) as x. Because of the *ratio* we are given, since side WI measures 6, the side should be split into two sides of length 4 and 2 for WT and TI respectively. Now, using *similarity* and the fact that $BO=EI$, we can set up the following equation to solve for x.

$$\frac{ST}{WT}=\frac{EI}{WI}\rightarrow\frac{x}{4}=\frac{8}{6}\rightarrow x=\frac{16}{3}$$

$$-------¿$$

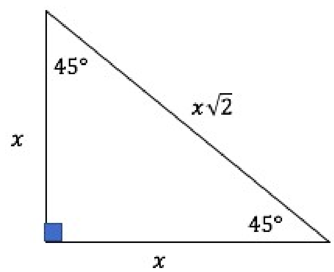
*Tri*angles

The *area of a triangle* is not even worth mentioning. You should have known about it since your middle school years.

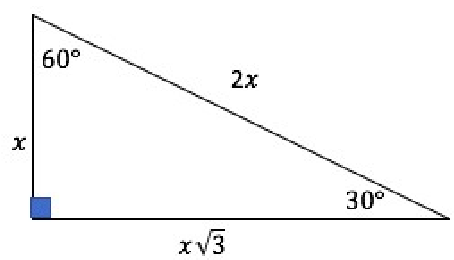
The *Two* Special Right *Tri*angles

The two *special Right Triangles* are worth a brief look. These are such staples in foundational geometry that you had to have already learned about them. If you’re unfamiliar, you will need to know them like the back of your hands for further studies in mathematics.

The $45°-45°-90°$

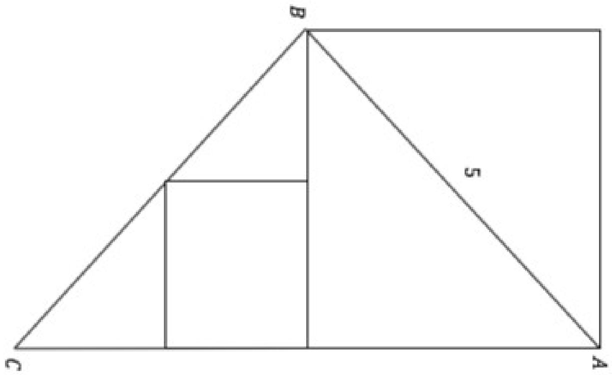


The $30°-60°-90°$



Let us do a few quick examples to practice basic question on these two special right triangles.

- Note: All corners in the figure are *right angles*, including $\angle ABC$. The big square is in a *ratio* of $2{:}1$ with the small square. What is the *diagonal* of the smaller square?



Solution

Notice that half of a square is always a $45° - 45° - 90°$ triangle. The formal name for this triangle if you don't know is the *Isosceles Right Triangle* (Makes sense, it is a *right triangle* and it is *isosceles*.)

Now, by the side relations we stated earlier in this section, we know that whenever the hypotenuse of a right isosceles triangle is $x\sqrt{2}$, one leg is x.

So, allowing $x\sqrt{2}=5$, we see that

$$x=\frac{5}{\sqrt{2}}$$

Now, usually, we would *rationalize* the denominator. But in this case, no need since we aren't done.

Since x is a side of the big square, we can see that a side of the small square is

$$\frac{1}{2}\left(\frac{5}{\sqrt{2}}\right)=\frac{5}{2\sqrt{2}}=y$$

Because we are given the *ratio* of the sides of the two squares. Because y is a side of the smaller square and all we are after is the *diagonal* of the smaller square, we are almost there.

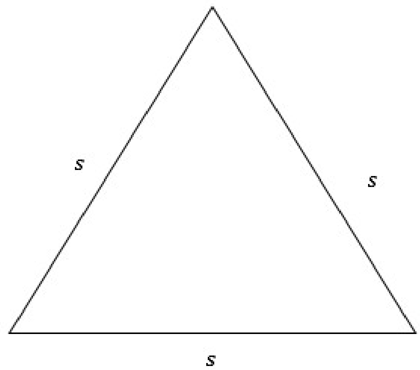
Once again, since we are dealing with a *right-isosceles* triangle where this time, we have a hypotenuse of length $y\sqrt{2}$.

$$y\sqrt{2} \rightarrow \left(\frac{5}{2\sqrt{2}}\right)\sqrt{2}=\frac{5\sqrt{2}}{2\sqrt{2}}=\frac{5}{2}$$

Now, we are done!

It is trivial to present an example on a $30° - 60° - 90°$. But we will so that the triangle doesn't feel left out.

- The triangle below is an *equilateral triangle*. Find the *area of the triangle* in terms of one side s.

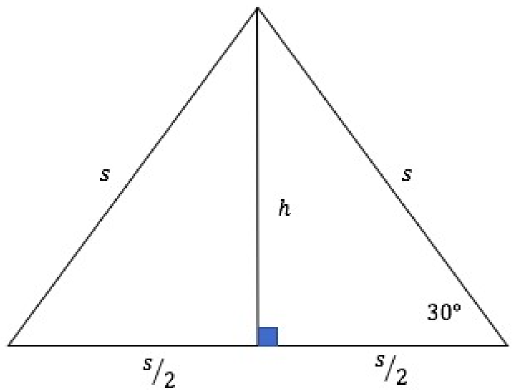


Solution

We know, the *area of a triangle* is

$$A=\left(\frac{1}{2}\right)bh=\left(\frac{1}{2}\right)sh$$

Let's draw *an altitude* (h¿ and take a closer look.



Notice that *one half of an equilateral is always* a $30°-60°-90°$. Now, if you look back at the side relationships we presented for the $30°-60$, you'd know that the two legs are in the *ratio* of $1:\sqrt{3}$.

So, this means that

$$h=\frac{s}{2}\sqrt{3}$$

With this, our task is practically complete.

$$A=\left(\frac{1}{2}\right)sh=\left(\frac{1}{2}\right)s\left(\frac{s}{2}\sqrt{3}\right)$$

$$\rightarrow A=\frac{s^2\sqrt{3}}{4}$$

So, we are done!

The two special right triangles – (Passport to Advanced Mathematics)

1. In the figure below, $ABEF$ is a rectangle. AE and BF intersect at point G and $CD \parallel BE$. The length of ¿$= ED = 2$ and $CD : ED : \geq$¿ are in *ratio* $2:1:1$. What is the length of FC ?

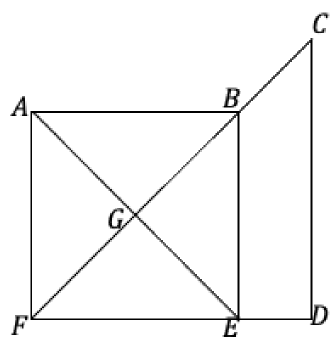


Figure not drawn to scale.

2. In the figure below, the measure of angle $\angle ABC$ is $45°$ while the measure of angle $\angle BDA$ is $60°$. If $AB = AE = ED = 4$, what is the length of FE?

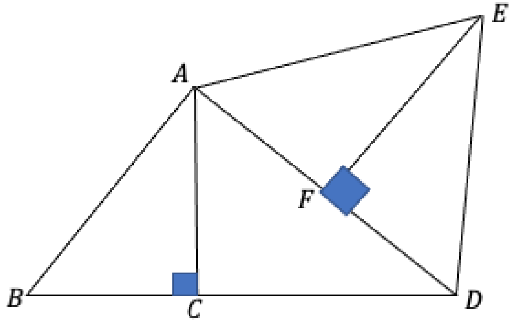


Figure not drawn to scale.

3. In the figure below, $JCLE$ is a square and rectangle $CRUL$ shares side CL with the square with $CL \parallel RU$. If the measure of angle $s = 30°$, and the square has a side length of 7, how long is LU?

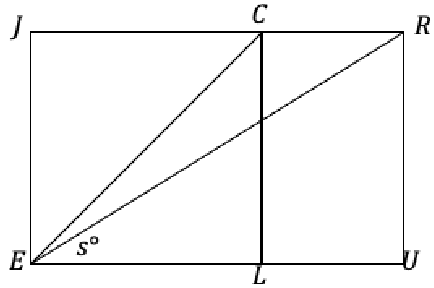


Figure not drawn to scale.

A. $7\sqrt{2}$ B. $7\sqrt{3}$ C. $7\left(1-\sqrt{2}\right)$ D. $7\left(\sqrt{3}-1\right)$

4. An *equilateral triangle* and *Isosceles right triangle* both have the same area. If the *equilateral triangle* has an area equal to 7, what is the sum of one *leg* of the *isosceles right triangle* and one side of the *equilateral triangle*?

5. In the figure below, $AE = AB$, $BE = BC$, and $CE = CD$. If the length of CE is $7\sqrt{2}$, what is the length of AE or AB?*

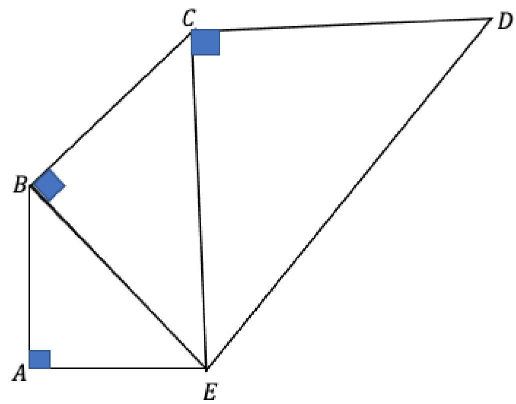


Figure not drawn to scale.

*All questions in this book are original written by Reus but this question credit to a fun contest day at TJHSST.

Solutions to the two special right triangles

(Passport to Adv. Mathematics)

1. Let us start with the fact that we are given that $CD:BE:\geq$¿ are in ratio $2:1:1$ which means that the lengths of these segments must be 4, 2, and 2 (this is so because the length of GE and ED are assigned a fixed value equaling 2.) Thus, of $CD=4$.

Now, we know that BE and CD are parallel and $ABEF$ is a rectangle. So that means angle D is a right angle and therefore, this in turn means that FCD is a right triangle.

Looking at triangle FGE, we have a *Right Isosceles* triangle. Because one of the legs of this triangle is 2, we know that FE is equal to $2\sqrt{2}$. As the task is to find FC, we can use the *Pythagorean Theorem* and write the following about triangle FCD.

$$FD^2+CD^2=FC^2$$

Since we know that $FD=FE+ED$, we have $FD=2\sqrt{2}+2=2\left(\sqrt{2}+1\right)$

$$\therefore FD^2+CD^2=FC^2\rightarrow\left(2\left(\sqrt{2}+1\right)\right)^2+4^2=FC^2$$

$$2^2\left(\sqrt{2}+1\right)^2+16=FC^2$$

You should all know by the *Binomial Square* that $\left(\sqrt{2}+1\right)^2=\sqrt{2}^2+2\sqrt{2}+1^2$

$$\rightarrow 2+1+2\sqrt{2}=3+2\sqrt{2}$$

$$4\left(3+2\sqrt{2}\right)+16=FC^2$$

$$\rightarrow 12+8\sqrt{2}+16=FC^2\rightarrow 28+8\sqrt{2}=FC^2$$

$$\rightarrow\sqrt{28+8\sqrt{2}}=FC$$

2. Notice that triangles ACB and $ACD$are both Special right triangles. If the solution here doesn't make complete sense to you, you ought to consult the appropriate section in this book on these two triangles. Now, since we know $AB=4$, if we let

$$BC=x\rightarrow x\sqrt{2}=4$$

$$\rightarrow x=\frac{4}{\sqrt{2}}=\frac{4\sqrt{2}}{2}=2\sqrt{2}$$

Now because the *legs* of an *Isosceles right triangle* are equal, we know $BC=AC=¿\,2\sqrt{2}$.

But then, triangle ACD is the $30°-60°-90°$ Special right triangle with CAD is equal to $30°$, meaning CD is the shortest side of this triangle. As such, if we let

$$CD=y\rightarrow y\sqrt{3}=AC=2\sqrt{2}$$

$$y=\frac{2\sqrt{2}}{\sqrt{3}}$$

$y=2\sqrt{\frac{2}{3}}$

Now, we know on the $30°-60°-90°$, the *hypotenuse* is twice the shortest side. This means that

$AD=4\sqrt{\frac{2}{3}}$

Now, $AED$is an *Isosceles triangle* (because $AE=ED=4$) with a base AD. The *altitude* EF bisects the base AD because that is true of all *Isosceles* triangles. Hence, we can get to EF using the *Pythagorean theorem* on triangle AEF with legs EF and AF. This means we must write:

$AF^2+FE^2=AE^2$

$\rightarrow \left(2\sqrt{\frac{2}{3}}\right)^2+FE^2=4^2$

$\rightarrow 2\left(\frac{2}{3}\right)+FE^2=16 \rightarrow FE^2=\frac{16}{3}-\frac{4}{3}=\frac{12}{3}=4$

$\rightarrow FE=2.$

3. The correct answer is D $7\left(\sqrt{3}-1\right)$

   The key to solving this problem is to notice the special $30°-60°-90°$ *right triangle*. Since angle s measures $30°$, angle ERU must measure 60 degrees because we already have a right angle at U in triangle EUR (we are told $CL \parallel RU$ and therefore, by corresponding angles CLE and RUL are both $90°$. Thus, triangle EUR is the $30°-60°-90°$.

   Now, in this special right triangle, we know that if the *leg* opposite the $30°$, that is, $RU=x$, then the *hypotenuse* of the triangle must be equal to $2x$ and the longer *leg* must be equal to $x\sqrt{3}$.

   Since the leg opposite the $30°$ measures 7 as RU=CL (the details are cumbersome but the fact easy to see.) Subsequently *hypotenuse* ER measures 14 and side EU measures $7\sqrt{3}$. Since $EU=EL+LU$, we can write $7\sqrt{3}=7+LU$.

$$7\sqrt{3}-7=LU \rightarrow LU=7\left(\sqrt{3}-1\right)$$

4. The correct answer is $\sqrt{14}+\frac{2\sqrt{7}\sqrt[4]{3}}{3}$

   We should recall that in an *Isosceles right triangle*, the area is given by $\frac{1}{2}x^2$, where x is the length of the *legs*. To find the value of x, we can write the following equation.

$$\frac{x^2}{2}=7 \rightarrow x^2=14 \rightarrow x=\sqrt{14}$$

You should also know from the section on the Special Right Triangles that the area of an *equilateral triangle* with side s is given by:

$$\frac{\sqrt{3}}{4}s^2 \rightarrow \frac{\sqrt{3}}{4}s^2=7 \rightarrow \sqrt{3}s^2=(7)(4)$$

$$\rightarrow s^2=\frac{(7)(4)}{\sqrt{3}} \rightarrow s^2=\frac{(7)(4)\sqrt{3}}{3}$$

$$\rightarrow s=\frac{2\sqrt{7}\sqrt[4]{3}}{3}$$

With this, we have that: $x+s=\sqrt{14}+\frac{2\sqrt{7}\sqrt[4]{3}}{3}$

5. The correct answer is $\frac{7\sqrt{2}}{2}$

   Since we see that each triangle has two legs of equal length therefore all three right triangles are the $45°-45°-90°$. This means that if the length of the *leg* on any of them is x, then the *hypotenuses* must measure $x\sqrt{2}$. Thus, we can work backwards, starting with $x, x, x\sqrt{2} \rightarrow x, x, 7\sqrt{2} \rightarrow 7,7,7\sqrt{2}$ on the triangle in the middle. Now if we let AE = AB $=y$, and the *hypotenuse* on this triangle, that is, BE = $7=y\sqrt{2}$, we have that

$$7=y\sqrt{2} \rightarrow y=\frac{7}{\sqrt{2}}=\frac{7\sqrt{2}}{2}$$

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Circles

The *Equation* of a Circle

The *Equation of a Circle* is:

$$(x-h)^2+(y-k)^2=r^2$$

The derivation of this equation is a path very well traversed. It is just a consequence of applying the *distance formula* to a point ¿) *called the center* and any arbitrary point ¿) and fixing the distance between these two points to be r.

As sufficient practice is provided in two sets on diverse problems on the *Equation of a Circle*, providing sophisticated examples is unwarranted. Especially since the practice problems are accompanied by solutions with sufficient details.

Perhaps highlighting the usual pitfalls is worthwhile.

What is the *Radius* and the *Center* of the circle with the following equation?

$x^2+(y+2)^2=2$

A task like this only requires mere comparison of what is given to the *Equation of a Circle* that we wrote atop this section.

To clearly decipher the *Center* and *Radius*, let us write the equation just above exactly in the form of the equation atop.

$\rightarrow (x-0)^2+(y-(-2))^2=(\sqrt{2})^2$

So, the *Center* is now much easier to see.

$(h,k)=(0,-2)$

$r=\sqrt{2}$

So, if you are unable to see the pitfalls

*students forget that on the right side is r^2 so the typical rushed bad answer in this example is $r=2$.

*Additionally, notice in picking out the $y-$¿coordinate of the *center*, we are looking at $(y+2)^2$. But notice that we must take the opposite sign, $k=-2$ because the equation says $(y-k)^2$.

Before we conclude, notice that for a *Circle* with equation

$(x-1)^2+(y+3)^2=5$

We can expand the two *binomials* on the left side of the equal sign and get what we call, "a scrambled *Equation of a Circle*."

Doing so, we have the following

$x^2-2x+1+y^2+6x+9=5$

Notice that we can scramble this a bit more and combine constants to write an equivalent equation which turns into

$\rightarrow x^2-2x+y^2+6x=-5$

In this last version, we have combined the 1 and $9$and moved them to the right side of the equal sign. In short, the two equations are the same.

The question is, how do we go back?

That is a simple exercise in *completing the square*.

Let us look at the following example.

$x^2+y^2-10x+8y=50$

- The equation above defines a circle in the $xy-¿$plane. What are the coordinates of the *Center* of the circle?

Solution

First, we need to manipulate the equation so that it is in the form $(x-h)^2+(y-k)^2=r^2$, where again (h,k) is the *Center*. For this, we will need to *Complete the Square* both on x and y.

$x^2+y^2-10x+8y=50$

$\rightarrow x^2-10x+25+y^2+8y+16=50+25+16$

$(x-5)^2+(y+4)^2=\sqrt{91}^2$

The *center* is $(5,-4)$.

This next example is still about the equation of a circle but a completely different task than that which we just faced. And as such, it might be just what we need as a cap to this section.

- What is the *Equation of a Circle* in the $xy-¿$plane with *Center* $(0,2)$ and a *Radius* with endpoints at $\left(\frac{1}{2},3\right)$?

Solution

Let us start by laying down
$(x-h)^2+(y-k)^2=r^2$

So, we already have the *Center*, $(h,k)=(0,2)$.

This means, so far, we have
$\rightarrow (x-0)^2+(y-2)^2=r^2$

$\rightarrow x^2+(y-2)^2=r^2$

Obviously, the *Radius* is, the *distance between the center and a point on the circle*. Since we have both, we can get going. We just need the *Distance Formula*.
$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$\rightarrow r=\sqrt{\left(\frac{1}{2}-0\right)^2+(3-2)^2}$

Since what we want is actually r^2, we don't need the square root. We can just simplify what is inside the square root and we are done.

$$r^2 = \frac{1}{4} + 1$$

$$\rightarrow r^2 = \frac{1}{4} + \frac{4}{4} \rightarrow \frac{5}{4}$$

So now, we have everything to claim the correct answer.

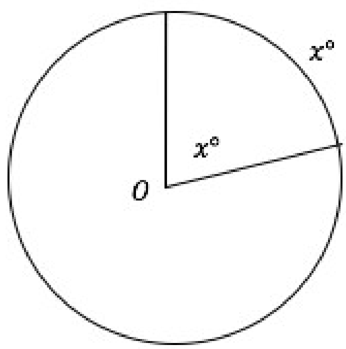
$$x^2 + (y-2)^2 = \frac{5}{4}$$

Area of a Circle & the *Circumference* of a Circle.

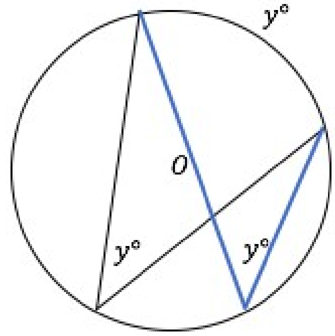
There is a formula sheet that is provided with every SAT test that reminds you of these two formulas (for the two items in the title above.) Thus, no further discussion of either of these two items is required here.

Angles in and *on a Circle*

When we talk about *angles in a circle*, we are only talking about *a central angle*.



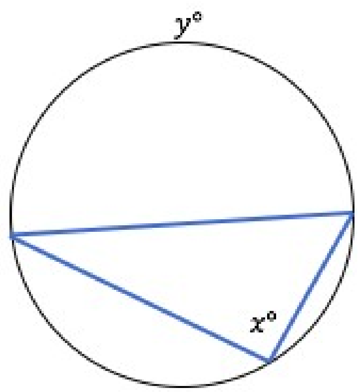
In the figure above, since the *Center* of the circle is O, x is what we call a *Central Angle*. The degree measure of the *Central Angle* is the <u>same as the degree measure of the arc it intercepts</u>.



In the second figure above, notice that the angles formed are formed *on* the circle. An angle formed on the circle like this while intercepting an arc across from it, is called, an *Inscribed Angle*.

An *Inscribed Angle* is <u>half of the measure of the arc in intercepts</u> and therefore, an *Inscribed Angle* is also half of the measure of a *Central Angle* intercepting the same arc.

Notice, a consequence of the *Inscribed Angle theorem* is the following.



It should be clear that here, the value of $x = \frac{1}{2}(y)$.

What we take away from the visual and the equation just above is the fact that any triangle inscribed in a circle with one side as the diameter must be right triangle. That is, $x° = 90°$!

Since *Trig*onometry is just advanced Geometry, we will cap this unit with *Trig* but first, the promised exercises on circles and some discussion of *Volume* thereafter.

Equation of a Circle (Problem Solving and Data Analysis)

$x^2+y^2-4x+6y=17$

1. © The equation above defines a *circle* in the $xy-¿$plane. What are the coordinates of the *center* of the *circle*?
   A. $(-2,3)$
   B. $(2,-3)$
   C. $(2,3)$
   D. $(-2,3)$

$x^2-8x+y^2=9$

2. © The equation above defines a *circle* in the $xy-¿$ plane. What are the coordinates of the *center* of the *circle*?
   A. $(4,0)$
   B. $(16,0)$
   C. $(-4,4)$
   D. $(-4,0)$

$x^2-4x+8y+y^2=5$

3. © The equation above defines a *circle* in the $xy-¿$plane. What are the coordinates of the *center* of the *circle*?
   A. $(-2,4)$
   B. $(2,-4)$
   C. $(-2,-4)$
   D. $(-4,4)$

$x^2-2y+4x+y^2=9$

4. © The equation above defines a *circle* in the $xy-¿$plane. What are the coordinates of the *center* of the *circle*?
   A. $(-2,4)$
   B. $(2,4)$
   C. $(4,2)$
   D. $(-2,1)$

Polar Pi

Solutions to *Equation of a Circle*

(Problem Solving and Data Analysis)

1. The correct answer is B. $(2,-3)$

We call the given equation $x^2+y^2-4x+6y=17$, *"the Scrambled Equation of a Circle."* So, we *complete the square* both on x and y to turn it into "*the Standard Equation of a Circle*." To start completing the square, we add a 4 to the "$x-$¿terms" and a 9 to the "$y-$¿terms." Of course, we must do the same to the other side of the equal sign to "balance the equation" so to speak.

Doing what we just said, we have

$$x^2-4x+4+y^2+6y+9=17+4+9$$

$$\rightarrow (x-2)^2+(y+3)^2=30$$

Comparing to the *Standard Equation of a Circle,*

$(x-h)^2+(y-k)^2=r^2$ where we know the *center is* (h,k)

We see that the *Center is* $(2,-3)$.

2. The correct answer is A. $(4,0)$

This solution is much like the solution to the problem we just solved before. At the heart of what we need to do is completing the square. To this end, we add a 16 to the "$x-$¿terms". Of course, we must add 16 on the right side of the equal sign to "balance the equation."

So, then we will have:

$$x^2-8x+16+y^2=9+16$$

$$\rightarrow (x-4)^2+(y-0)^2=25$$

Comparing what we just wrote to the *Standard Equation of a Circle* $(x-h)^2+(y-k)^2=r^2$ where we know the *Center is* (h,k)

We see that the *Center* of the *Circle* we are deal with here must be $(4,0)$.

3. The correct answer is B.$(2,-4)$

If you need more details, look at the solution to the first problem in this set. But we know that what we must do is complete the square on what we are given. To start, we were given

$$x^2-4x+8y+y^2=5$$

So we proceed as follows:

$x^2-4x+4+y^2+8y+16=5+4+16$

$\rightarrow (x-2)^2+(y+4)^2=25$

Comparing to the *Standard Equation of a Circle,* $(x-h)^2+(y-k)^2=r^2$ where we know the *center is* (h,k)

We see that the *Center is* $(2,-4)$.

4. The correct answer is D. $(-2,1)$

At this point, you should be familiar with the procedure. If you need more support than presented here, you can see more detailed solution to problem 1 above.

A succinct solution goes like:

We have

$x^2+4x+4+y^2-2y+1=9+4+1$

$\rightarrow (x+2)^2+(y-1)^2=14$

Comparing this last equation to the *Standard Equation of a Circle,*

$(x-h)^2+(y-k)^2=r^2$ where we know the *Center is* (h,k)

We see that the *Center is* $(-2,1)$.

Equation of a circle (Part 2) (Passport to Advanced Mathematics)

1. Which of the following is an equation of a *circle* in the $xy-$¿plane with *center* $(0,3)$and *radius* with the other endpoint at $(4,-2)$?
   A. $x^2+(y-3)^2=\sqrt{5}$
   B. $x^2+(y-3)^2=25$
   C. $x^2+(y-3)^2=5$
   D. $x^2+(y-3)^2=10$

2. Which of the following is an equation of a *circle* in the $xy-$¿plane with *center* $(-4,2)$and a *radius* of length 5?
   A. $(x+4)^2+(y-2)^2=\sqrt{5}$

B. $(x-4)^2+(y+2)^2=25$
C. $(x+4)^2+(y-2)^2=25$
D. $(x+4)^2+(y-2)^2=5$

3. Which of the following is an equation of a *circle* in the $xy-$¿plane with *center* $(-1,2)$ and a *radius* having another endpoint at $(2,-3)$?
   A. $(x-1)^2+(y+2)^2=\sqrt{34}$
   B. $(x+1)^2+(y-2)^2=34$
   C. $(x+1)^2+(y-2)^2=\sqrt{34}$
   D. $(x-1)^2+(y+2)^2=34$

4. Which of the following is an equation of a *circle* in the $xy-$¿plane with *center* $(0,-4)$ and a *radius* of length $\sqrt{2}$?
   A. $x^2+(y-4)^2=\sqrt{2}$
   B. $x^2+(y+4)^2=\sqrt{2}$
   C. $(x+4)^2+y^2=2$
   D. $x^2+(y+4)^2=2$

Solution to *Equation of a circle* (Part 2)
(Passport to Advanced Mathematics)

1. The correct answer is C.
   We know that the *Standard Equation of a circle* is
   $(x-h)^2+(y-k)^2=r^2$
   with *center* (h,k) and *radius* r.

we are given $(h,k)=(0,3)$.

The radius involves a bit of work. The *radius* is *the distance between the center and a point on the circle*. So, either by drawing a right triangle with vertices $(0,3)$, $(0,-2)$, and $(4,-2)$ and subsequently applying the Pythagorean theorem with the radius as the hypotenuse or by using the *distance formula* on the points $(0,3)$ and $(4,-2)$, we can figure out that the length of the *radius* of the *circle* is $r=\sqrt{5}$.

Now that we have everything, substituting these values into the *Standard Equation* as follows, we can get the job done!

$\rightarrow (x-h)^2+(y-k)^2=r^2$

$\rightarrow (x-0)^2+(y-3)^2=(\sqrt{5})^2$

$\rightarrow x^2+(y-3)^2=5$

2. The correct answer is C.

We know the *Standard Equation of a Circle* is

$(x-h)^2+(y-k)^2=r^2$

with *center* (h,k) and *radius* r.

Since we know $(h,k)=(-4,2)$ and $r=5$, the only work we have left to do is plugging in these values into the standard equation above.

$\rightarrow (x-h)^2+(y-k)^2=r^2$

$\rightarrow (x-(-4))^2+(y-2)^2=5^2$

$\rightarrow (x+4)^2+(y-2)^2=25$

3. The correct answer is B.

We know the *Standard Equation of a Circle* is

$(x-h)^2+(y-k)^2=r^2$

with *Center* (h,k) and *radius* r.

Since (h,k) is given as $(-1,2)$, no work is involved in finding the coordinates of the *center*.

For the *radius*, just as we did for the solution to problem one in this set, either by drawing a right triangle with vertices $(-1,2)$, $(-1,-3)$, and $(2,-3)$ and subsequently applying the *Pythagorean theorem* with the radius as the *hypotenuse* or by applying the *Distance Formula* on the points $(-1,2)$ and $(2,-3)$, we can figure out that the length of the *radius* of the *circle* is $r=\sqrt{34}$. Now, we have everything. So, plugging in, we have

$\rightarrow (x-h)^2+(y-k)^2=r^2$

$\rightarrow (x-(-1))^2+(y-2)^2=(\sqrt{34})^2$

$\rightarrow (x+1)^2+(y-2)^2=34$

4. The correct answer is D.
   This is because the *Equation of a Circle* is

$(x-h)^2+(y-k)^2=r^2$

with center (h,k) and radius r.

Since we know $(h,k)=(0,-4)$ and $r=\sqrt{2}$, the only work we must do is to plug in the values correctly:

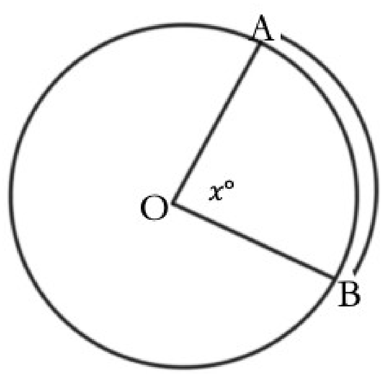
$\rightarrow (x-h)^2+(y-k)^2=r^2$

$\rightarrow (x-0)^2+(y-(-4))^2=\sqrt{2}^2$

$\rightarrow x^2+(y+4)^2=2$

Arcs, Sectors, and Central Angles of a Circle
Add a lesson here.

The most important thing to relating arcs, sectors and *central angles* is the captured in the following visual and the symmetric equation that follows it.



$$\frac{\alpha AOB}{\pi r^2}=\frac{x°}{360°}=\frac{Arc\ AB}{2\pi r}$$

Where $\alpha\ AOB$ refers to the area of *sector* AOB and $Arc\ AB$ is referring to the length of *minor arc* AB with the *central angle* measuring $x°$. Remember, there is a difference between the *length* of an *arc* and the *degree measure* of an *arc*. The *degree measure* of an *arc* as you should know from earlier lessons is the <u>same as the degree measure of the *central angle*</u>.

In the symmetric equation we just stated, we rarely want to pair up the first and last quotients because they both require the same information, the radius of the circle (or the *diameter*.)

So, the pairing we want is:

$$\frac{\alpha AOB}{\pi r^2} = \frac{x°}{360°}$$

or

$$\frac{x°}{360°} = \frac{Arc\,AB}{2\pi r}$$

Obviously, there are exceptions, that is, there are times that we want the pairing:

$$\frac{\alpha AOB}{\pi r^2} = \frac{Arc\,AB}{2\pi r}$$

As heavy practice is not necessary here, in the set of questions to follow, you will find the necessary and sufficient practice backed with solutions.

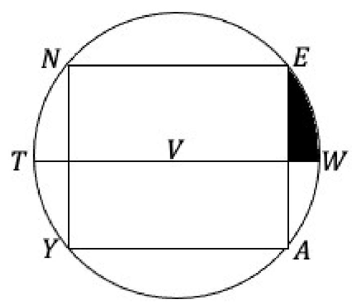
More on Circles – Arcs, sectors, and Area (Part 3) –

(Passport to Advanced Mathematics)

1. In the figure below, the bigger circle has *center* at O and CD is the *diameter* of the smaller circle. $m\angle x = 60°$ and the length of *radius* OB of the bigger circle is 7. If the area of *sector* AOB is equal to the area of either *semicircle* that make up the smaller circle, what is the *radius* of the smaller circle?

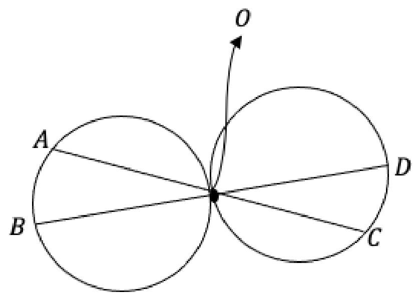
A. $\frac{7}{3}$ B. $\frac{14}{\sqrt{3}}$ C. $\frac{7\sqrt{6}}{3}$ D. $\frac{7\sqrt{2}}{3}$

2. The *circle* below has center at V. *Minor arcs* NE, NY, YA, and AE are all equal. If *diameter* TW has length 10, what is the area of the shaded region?



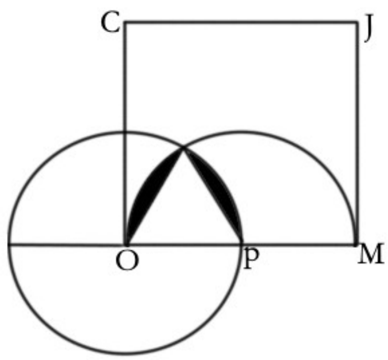
A. $\frac{5(5\pi-1)}{8}$ B. $\frac{(25\pi-2)}{8}$ C. $\frac{5(5\pi-2)}{8}$ D. $\frac{5(5\pi-1)}{4}$

3. In the figure below, AC and BD are *concurrent* at point O. Point O is also a *point of tangency* for the two circles. If OB and OD are both *diameters* for the *circle* on the left and right respectively, and the measure of $\angle AOB$ is $30°$, what is the measure of *minor arc* AO?



A. $30°$ B. $60°$ C. $120°$ D. $80°$

4.



In the figure above, OMJC is a square with a side length of 10. OP is a *radius* both to the *circle* and the *semicircle*. What is the area of the *shaded region*?

5. If a *semicircle* has *perimeter* $14\pi + 14$. What is the area of a *circle* with the same *radius*?

A. $\frac{7\pi}{2}$ B. $\frac{49\pi}{2}$ C. $\frac{5\pi}{2}$ D. 49π

Answers to More on Circles – Arcs, sectors, and Area (Part 3) – (Passport to Advanced Mathematics)

1. The correct answer is C. $\frac{7\sqrt{6}}{3}$

We first need to find the area of *sector* AOB. First, notice that angle APB is an *inscribed angle* and we know it measures $60°$. Now, because APB intercepts the same minor *arc* AB that the *central angle* at O (angle AOB) intercepts, it follows that angle AOB has measure equal to $2x = ¿\ 120°$. Now to find the area of the *sector* formed by AOB, that is, *sector* AOB, we can set up the following proportion.

$$\frac{\theta°}{360°} = \frac{x}{\pi r^2}$$

where x is the *sector* area.

Sine we know that $\theta = 120$ (the *central angle*) and $r = 7$, substituting these values the equation above turns to:

$$\frac{120°}{360°} = \frac{x}{\pi 7^2} \rightarrow \frac{1}{3} = \frac{x}{49\pi}$$

$$\rightarrow x = \frac{49\pi}{3}$$

Now, we are told that the area of this *sector* is the same as the area of either *semicircle* in the smaller square.

Since we know the area of a *semicircle* is

$$\frac{\pi r^2}{2}$$

We must equate this to the area of the *sector* we found a little bit ago.

$$\rightarrow \frac{\pi r^2}{2} = \frac{49\pi}{3}$$

We do this because recall that the end goal is to find the *radius* r of the smaller circle and the equation we just wrote will allow us to do just that.

$$\rightarrow \pi r^2 = \frac{98\pi}{3} \rightarrow r^2 = \frac{98}{3} \rightarrow r = \sqrt{\frac{49(2)}{3}} = \sqrt{49}\sqrt{\frac{2}{3}} = 7\sqrt{\frac{2}{3}}$$

$$\rightarrow r = 7\frac{\sqrt{2}}{\sqrt{3}} = 7\frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = 7\frac{\sqrt{6}}{3}$$

2. The correct answer is B. $\frac{25(\pi-2)}{8}$

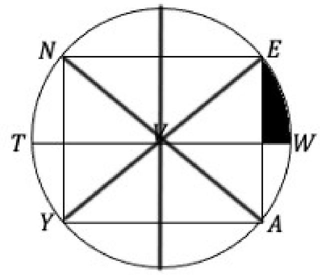
Since we are told that all *minor arcs* are equal, then AENY must be a square. Because the diameter of the circle is 10, the area of the circle is $\pi 5^2 = 25\pi$. To find the area of the square, we will use the properties of the special $45-45-90$ right triangle. <u>The diameter of the circle is also the *diagonal* of the square</u>. Now, to find the length of the sides of the square we must divide the *diagonal* of the square by $\sqrt{2}$. Thus, one sides of the square measure $\frac{10}{\sqrt{2}}$. To find the area of the square, we can use the formula $A = l^2$.

$$A = \left(\frac{10}{\sqrt{2}}\right)^2 = \frac{100}{2} = 50$$

Now, if we look at the visual below, first, it becomes clear why we said earlier that the *diagonal* of the square is also a *diameter* for the *circle* (just look at YE or NA.) More importantly, we can see that the circle is divided up into 8 equal parts as are the sectors (slices in a pizza pie so to speak.) So, we see that the area of interest is $\frac{1}{8}$ circle -

$\frac{1}{8}$ square. We can write this more succinctly as: $\frac{1}{8}(A_c - A_S)$ where A_c stands for the area of the circle and A_S stands for the area of the squre.

We already have both $A_S = 50$ and $A_c = 25\pi$. And so then, we write

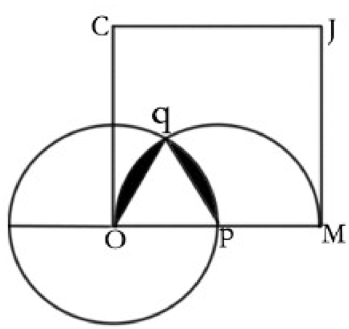


$$\frac{1}{8}(A_c - A_S) = \frac{1}{8}(25\pi - 50) = \frac{25(\pi - 2)}{8}$$

3. The correct answer is C. $120°$.

This is so because, while there is a lot of extra information given to us here, BO is a diameter and as such, the measure of *arc* OB or BO is 180 degrees. Since $m\angle AOB = 60°$, we see that the measure of *arc* AO must be $120°$.

4. The correct answer is $\frac{25}{12}(2\pi - 3\sqrt{3})$.



Consider the visual above. It is the same visual as what was given in the question but with a point q added at the intersection of the *semicircle* and the *circle*.

Now, first, triangle OPQ is an *equilateral* triangle. We know this because OP and PQ are radii for the *circle* and the *semicircle* respectively.

Since we were told that these two (the *semicircle* and the circle share OP as common *radius*, it must mean that $OP = r = PQ = OQ$, hence triangle OPQ is *equilateral*.

Moving forward, consider the *sector* made by *minor arc* OQ and the *center* P of the *semicircle*. Notice that this *sector* is of the same area as the *sector* formed by *minor arc* QP and center O of the *circle*. If we can find the area of either *sector*, we can double the value to finish the given task. To find one of these two *shaded areas*, all we must do is subtract the *area of equilateral triangle* OPQ from the area of either sector. This is a straightforward task.

Let's go with the *sector* made by the *minor arc* QP and the *center* O of the *circle*.

Since triangle OPQ is *equilateral*, each angle is $60°$. As such, we quickly see that the *central angle* that intercepts minor *arc* QP measures $60°$. So, we return to the proportion we used in the solution to the first problem and stated on the lesson titled *Arcs, Sectors, and Central angles*. Letting $x = ¿$ the area of the sector in discussion, we write:

$$\frac{\theta°}{360°} = \frac{x}{\pi r^2}$$

$$\rightarrow \frac{60°}{360°} = \frac{x}{\pi 5^2} \rightarrow \frac{1}{6} = \frac{x}{25\pi}$$

$$\rightarrow x = \frac{25\pi}{6}$$

Now, from the section on *Special Right Triangles*, we know that the area of an *equilateral* triangle with a side length of s is given by:

$$\frac{s^2\sqrt{3}}{4} = \frac{25\sqrt{3}}{4}$$

We have used the value 5 for the side of the *equilateral triangle* because one side of the square measures 10 and P is the *midpoint* of OM.

The only thing we have left to do is, as we already said, subtract the two quantities.

$$\rightarrow \frac{50\pi}{12} - \frac{75\sqrt{3}}{12} = \frac{25}{12}\left(2\pi - 3\sqrt{3}\right)$$

5. The correct answer is C. $\pi \frac{196(\pi+1)^2}{(\pi+2)^2}$.

The *perimeter* of a *semicircle* is equal to half the *circumference* of the circle with the same *radius* plus two times the *radius*. We can write this as:

$$P=\frac{2\pi r}{2}+2r=\pi r+2r=r(\pi+2)$$

Now, we can solve the following equation to solve for the radius of the semicircle.

$$r(\pi+2)=14\pi+14$$

$$\rightarrow r=\frac{14\pi+14}{(\pi+2)}=\frac{14(\pi+1)}{(\pi+2)}$$

Now, since we know the *radius*, we can get to the area with good old πr^2.

$$A=\pi\left(\frac{14(\pi+1)}{(\pi+2)}\right)^2=\frac{196(\pi+1)^2}{(\pi+2)^2}$$

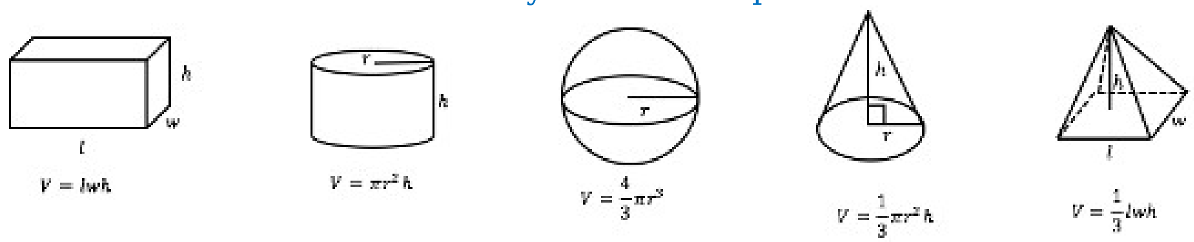
Volume

While *Volume* is an important idea to master, to avoid tedious repetition, we haven't represented it with any real visibility in the *practice tests* and *min tests* that follow the lessons in this book.

In other words, the discussion and practice in this section should be considered adequate.

Remember, any standardized exam you take will provide you the formulas for *Volume* etc. But some familiarity can save you time and knowing these formulas will come in handy away from *standardized exams*. To this end, you will find the following quick couple of pointers helpful!

Here is the formula sheet you would be provided on the SAT.



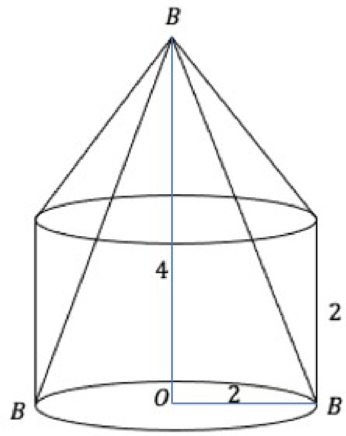
- Build familiarity with this formula sheet and get to know it well ahead of the test. That way, you don't have to flip back and forth to it.

- Notice, the *Volume of a Cone* is exactly *a third of the Volume of a Cylinder*.

- Also notice, similarly, *the Volume of a Pyramid* is exactly *a third of the Volume of a Box* (A *Rectangular Prism*.)

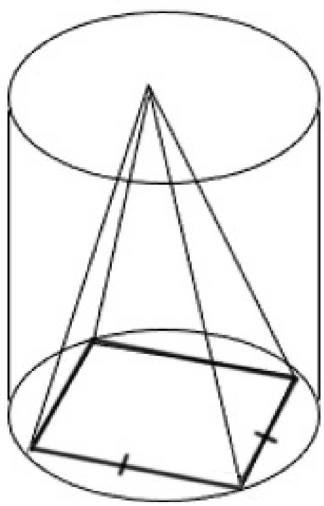
- Lastly, notice that *the Volume of a shape that has uniform cross sections* (if you slice it parallel to the base, you will get slices that are identical to the base) *is always* ***Base Area*** × ***height***.

In what follows, you will find a couple of neat practice questions…



1. A *hut-like* figure is made of a *cone* on top of a *cylinder*. The *cone* and they *cylinder* share the same base radius. A thinner taller *cone* is inscribed in the *hut-like* figure. What is the *volume* of the shape made inside the *hut-like* figure and outside of the thinner cone with height 4? All necessary values to solve this problem are given on the figure.

2. In the figure below, a square based *pyramid* is *inscribed* inside of a *cylinder* so that the *diagonal* of the *square* is also the *diameter* of the base of the *cylinder*. If the height of the *cylinder* is 4 and one side of the square is 2, what is the volume of the space outside of the *pyramid* but inside of the *cylinder*?



Solution

1. The correct answer is $\frac{16\pi}{3}$

   Notice that the volume of the solid (*Snake, Snake, Snaaaaakkeee*?) we are asked for is equal to the volume of the entire solid (the *cone* +¿ the *cylinder*) minus the volume of the thinner taller *cone*. To find the volume of the thinner *cone*, we just apply the volume formula a *cone*. We will denote the volume of the entire figure as V_E, the volume of the *cylinder* with height 4 as V_4.

   $$V_E=\pi 2^2(2)+\frac{1}{3}\left(\pi 2^2(4-2¿\right)=8\pi+\frac{8\pi}{3}=\frac{32\pi}{3}$$

   $$V_4=\frac{1}{3}\pi(2^2)(4)=\frac{16\pi}{3}$$

   So then, the volume of the figure we are asked to find is equal to $V_E-V_4=\frac{32\pi}{3}-\frac{16\pi}{3}=\frac{16\pi}{3}$

2. The correct answer is $\frac{8\pi}{3}$

   Once again, as in the solution to problem 1, we will apply the same idea of subtracting volumes. In this case, the volume we want to find is equal to the volume of the *cylinder* minus the volume of the *pyramid*. We will denote the volume of the cylinder by V_c, and the volume of the pyramid as V_p.

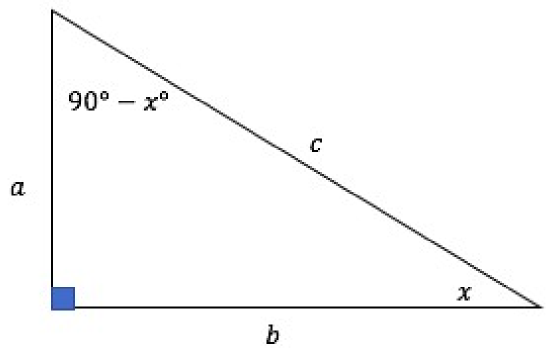
   First, we will find the volume of the *pyramid*. The base is a square, so its area is equal to l^2, and since $l=2$, the are of the base will be $2^2=4$. The formula for the volume of a pyramid is $\frac{1}{3}$ ***Base Area*** × ***height***. We are told that the height is 4, so $V_p=\frac{(4)(4)}{3}=\frac{16}{3}$.

   Now, we will find the volume of the *cylinder*. For this, we need to find the *radius* of its base. Since we are told that the *diagonal* of the square is the *diameter* of the circular base of the *cylinder*, we can use the *Pythagorean theorem* to find the *diagonal* of the square. The *diagonal* of the square is the *hypotenuse* of the *right isosceles triangle* that is formed after cutting the square in half. Thus, the *diagonal* will be equal to $2\sqrt{2}$ using the side relations of the Right Isosceles triangle (which we have abusively represented prior to this problem.) Dividing this value by 2, we get $r=\sqrt{2}$. Now, we are ready to apply the formula for the volume of a *cylinder*.

   $$V_c=\pi\sqrt{2}^2(4)=8\pi$$

----

*Trig*onometry



Now, we assume that we can safely skip the basic definitions. If not, SOHCAHTOA is really all you need to know.

$$\sin(x°) = \cos(90° - x°) = \frac{a}{c}$$

Feel free to slow down and check that the equation above against SOHCAHTOA.

This equation is better known as the *Complement-Angle Rule* in Trigonometry.

In this section, one example should suffice.

Example

In a *right triangle*, one angle measures x°, with $\cos x^{\circ} = \frac{3}{5}$, what is the value of

$\sin(90^{\circ} - x^{\circ})$¿ ?

Solution

Simple! Since we just learned the *Complement-Angle Rule*, we know that

$\sin(x°) = \cos(90° - x°)$ *but it is also true that* $\cos(90° - x°) = \sin(x°)$

Now, we can easily solve this problem.

$\rightarrow \cos(90^{\circ} - x^{\circ}) = \sin x^{\circ} = \frac{3}{5}$

Trigonometry (Other)

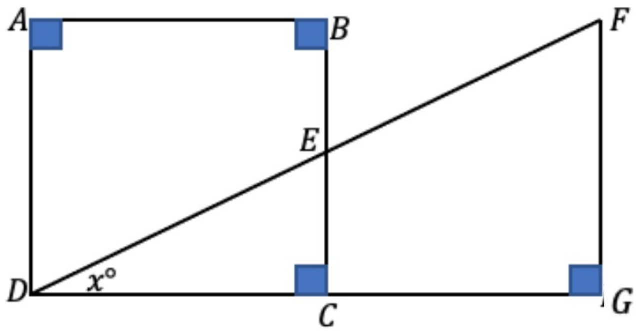


Figure not drawn to scale and all marked corners are right angles.

1. In the figure above, $ABCD$ is a square and $AD = FE = 4$. In addition, E is the *midpoint* of BC. What is the value of $\sin x°$?

2.

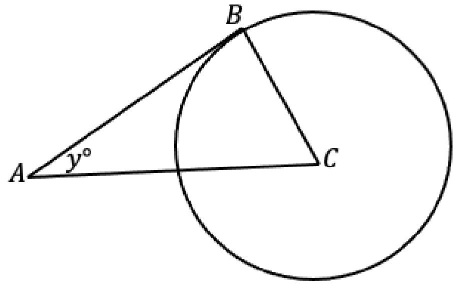


Figure not drawn to scale.

In the figure above, *circle* C has *radius* BC. B is a *point of tangency*. BC intersects AB at point B. If the length of AB is 7and the *radius* of the *circle* is 4, what is $\tan y°$?

3. In the figure below, line LR is tangent to the *circle* at point A. $PR=14$ and $LR=8$ with A as the *midpoint* of LR. If IA is concurrent with the *center* of the circle at I, $m\angle AIR=z°$, and $\angle L=90°$ what is $\tan z°$?

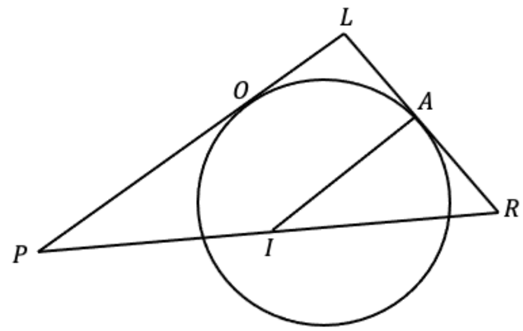


Figure not drawn to scale.

4. In the figure below, ME intersects AB at point Y. If $\sin E° = \frac{3}{4}$, what is $\cos Y°$ in triangle MAY?

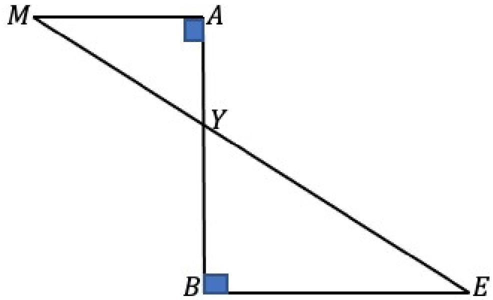


Figure not drawn to scale.

5. In the figure below, $MAGI$ is a square. $AG=8$ and the *ratio* of EI to IG is 4 to 2. If the *ratio* of MN to ¿ is 1to 3, what is the value of $\sin v°$?

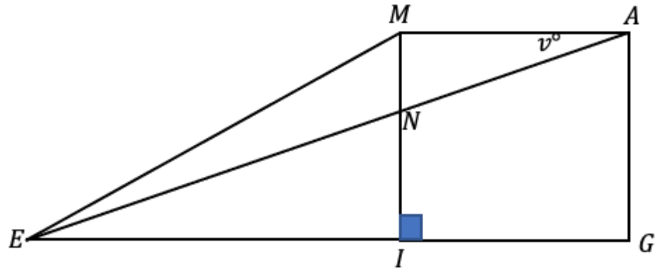


Figure not drawn to scale.

Solution to *Trigonometry* (Other)

1. The correct answer is $\sin x = \frac{\sqrt{5}}{5}$.

We will first label some lengths and find $\sin x°$. We will label side DE as b, and side FG as a. Notice that after these labels, the value of $\sin x$ will be $\frac{a}{b+4}$. This is so because sine is equal to the opposite side of the angle divided by the *hypotenuse*. The opposite side is a and we are given that FE measures 4. Which means the *hypotenuse* will be $b+4$. Now, all we have left to do is solve for $a$and b. Let us begin by solving for b.

Notice that side DC measures 4, because it is a side of the square and we were given that another side, AD measures 4. Since we also know that E is the midpoint of side BC, this means that EC measures 2. Therefore, we can use the *Pythagorean theorem* to solve for b as follows!

$$4^2+2^2=b^2 \rightarrow b^2=20$$

$$\rightarrow b=\sqrt{20}$$

Now that we have the value of b, we can use it to solve for a. We will use similarity of triangles because triangles CDE and GDF are similar by the *AA theorem*. We know they are similar because the two triangles share angle FDG and $\angle C=\angle G=90°$.

As we must, to solve for a, we write:

$$\frac{2}{b}=\frac{a}{b+4}\rightarrow a=\frac{2}{b}(b+4)$$

Recall that we said earlier, the value that we are after, that is $\sin x$ is $\frac{a}{b+4}$. We will substitute the value of a, and simplify. Let us keep the variable b for now as we can substitute at the end and avoid cumbersome writing in the meantime!

$$\sin x=\frac{a}{b+4}$$

$$\sin x=\frac{\frac{2}{b}(b+4)}{b+4}$$

Here, we can simplify the $b+4$ that appears in the numerator and denominator of this last equation just above, leaving us with:

$$\sin x=\frac{2}{b}$$

To conclude, substituting the value of b and rationalizing the denominator, we write:

$$\sin x=\frac{2}{\sqrt{20}}=\frac{2}{2\sqrt{5}}=\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}$$

2. The correct answer is $\frac{4}{7}$.

   Since we are told that point B is a *point of tangency* of line AB and AB intersects the circle's *radius* at the *point of tangency*, angle ABC must measure $90°$. As such, this is simple, we merely had to ensure that we were working with a right triangle before we jump to trigonometry. Tangent is the ratio of the opposite side of a *right triangle* divided by the adjacent side.

$\tan y°=\frac{r}{AB}\rightarrow$

$$\tan y°=\frac{4}{7}$$

3. The correct answer is $\frac{4\sqrt{5}}{15}$

Notice, because the *center* of the circle is on I, the *radius* of the *circle* IA intersects LR at the point A, the *point of tangency* of LR to the *circle*. Therefore, we know that angle RAI is a *right angle*. Since we also know that angle L is a *right angle*, we have IA parallel to PL (corresponding angles mean lines are parallel, with transversal RL.)

Now, because of the pair of right angles we just mentioned, and the fact that angle R is shared, triangles RAI and RLP are *similar triangles* (by the *AA Theorem*.)

This means that the final pair of angles in these two triangles are the same. That is, the measure of angle P is also $z°$.

As such, $\tan z°$ can be found using the bigger triangle RLP. Knowing the opposite side of this triangle is LR and the adjacent side of this triangle is LP, we only need the length of LP to execute. Thus, so far, we have:

$$\tan z° = \frac{LR}{LP} = \frac{8}{LP}$$

Now, to find LP, we can apply the *Pythagorean theorem* on triangle RLP.

It would look as follows:

$$LP^2 + LR^2 = PR^2$$

$$\rightarrow LP^2 + 8^2 = 14^2 \rightarrow LP^2 + 16 = 196$$

$$\rightarrow LP^2 = 196 - 16 = 180 \rightarrow LP = \sqrt{180} = 6\sqrt{5}$$

So now, we have the final answer as

$$\tan z° = \frac{RL}{PL} = \frac{8}{6\sqrt{5}} = \frac{4}{3\sqrt{5}}$$

Since we need to rationalize the denominator, we write:

$$\tan z° = \frac{4\sqrt{5}}{(3\sqrt{5})\sqrt{5}} = \frac{4\sqrt{5}}{(3)5} = \frac{4\sqrt{5}}{15}$$

4. The correct answer is $\frac{3}{4}$.

Since lines ME and AB intersect at point Y, the measure of angle $Y°$ must be the same on both sides of the intersection (*vertical angles*.) Since we are told that $\sin E° = \frac{3}{4}$ but we don't know if the opposite side and the hypotenuse are 3 and 4 respectively, we can only assert that the possible set of values must be of the form, $3x$ and $4x$.

Now, remember from the section on Trigonometry, the *Complement Angle Rule*. Which stated, if x and y are the acute angles in a right triangle, $\sin x° = \cos y°$.

So, we know using this formula that $\cos Y° = \sin E° = ¿ \frac{3}{4} ¿$.

Now, we already said that angles EYB and MYA are equal because of *vertical angles*. Since we additionally know that angle A and angle B are equal in the smaller and bigger right triangles respectively, using the *AA theorem*, we know that the two triangles are *similar*. Meaning, all corresponding angles in the two triangles are equal. As such, we know that *sine*, *cosine*, and *tangent* values in the two triangles are the same as the size of the triangle doesn't matter, the input in these trigonometric ratios (alternatively functions) is only the angle.

Therefore, the *cosine* of angle $Y°$ in the bigger right triangle is the same as the *cosine* of angle $Y°$ in the smaller triangle.

$\rightarrow \cos Y° = \frac{3}{4}$

5. The correct answer is $\frac{\sqrt{17}}{17}$.

First, notice that since side MI is one of the sides of the square, it must be of length 8. Because the *ratio* of MN to $¿$ is $1$to 3, side MNmust measure 2, and side $¿$ is therefore of length 6.

Now, to find the value of $\sin v°$, we first need to know the opposite side and *hypotenuse* of triangle MAN from the perspective of angle $v°$. Now, we know the opposite side MN is 2, but we need the *hypotenuse* NA. The *Pythagorean theorem* comes in handy in what we seek to accomplish so that we right the following:

$$MN^2 + MA^2 = NA^2 \rightarrow$$

$$2^2 + 8^2 = NA^2 \rightarrow NA = \sqrt{68}$$

Hence, $\sin v° = ¿ \frac{2}{\sqrt{68}} = \frac{2\sqrt{68}}{\sqrt{68}\sqrt{68}} = \frac{2\sqrt{68}}{68} = \frac{2\sqrt{4}\sqrt{17}}{68} = \frac{4\sqrt{17}}{68} = \frac{\sqrt{17}}{17} ¿$.

Notice that we can also use angle AEG to answer this question. This is so because angle $v°$ and angle AEG are *alternate interior angles*. Subsequently, they have the same measure and their *sine* values are the same. We will leave it to the student to figure out how to use this path to get to the same solution.

The Warm up before the Mini Tests

Mixed Review before Min

1. Find an expression is equivalent to:

$(x^2 y+3y^2-x^2+5y^2 x)-(-x^2+2y^2 x^2-5x y^2-3y^2)$

Solution

This is a simple exercise in *combining like-terms*. The only thing you need to worry about is *distributing the negative sign in front of the second pair of parentheses*.

To get rid of both parentheses, *we must multiply everything in the second pair of parentheses by a negative one*. Thus, we first write the following (if you are adept at this, you can skip a step or two.)

$x^2 y+3y^2-x^2+5y^2 x+x^2-2y^2 x^2+5x y^2+3y^2$

Now, we put terms that are identical (except for differing coefficients) next to each other and combine. This, we do as follows:

Notice, $5y^2 x$ and $5xy^2$ are *like-terms* but

$x^2 y+3y^2+3y^2-x^2+x^2+5y^2 x+5xy^2-2y^2 x^2$

$\rightarrow x^2 y+6y^2+10y^2 x-2y^2 x^2$

And we are done!

Cleary, some terms *might not have a like-term to combined to* (oh so lonely and single☺)

2. © For what value of m is $|m-3|+3$ equal to 0?

Solution

Should you need more support for this problem than provided in this solution, you ought to consult a prior section dedicated to precisely problems like these.

$|m-3|+3=0 \rightarrow |m-3|=-3$

Now, we all should all know that the *absolute value* of an expression or a quantity <u>cannot equal a negative number</u>. So, we are done. Our answer would be, no solution.

3. Give two equivalent expressions that equal $b^{\frac{5}{3}}$

Solution

If necessary, again, consult the appropriate section. But, using the *back-to-back exponent rule*, we can write the following:

$$b^{\frac{5}{3}}=\left(b^{\frac{1}{3}}\right)^5=\left(\sqrt[3]{b}\right)^5$$

Or

$$b^{\frac{5}{3}}=\left(b^{\frac{1}{3}}\right)^5=\left(b^5\right)^{\frac{1}{3}}=\sqrt[3]{b^5}$$

Important note:

While there are two ways to interpret *rational exponents*, notice that in execution, one maybe preferable over another. Here is an example.

$$27^{\frac{5}{3}}$$

On the one hand, this is the same as:

$$\left(\sqrt[3]{27}\right)^5$$

On the other hand, it is the same as:

$$\sqrt[3]{27^5}$$

The former is clearly much preferable to the latter. Because, in the first interpretation, we must figure out the cube root of 27 which we know to be 3 and then do 27 times 9. In the latter, we must first figure out what 27 to the 5^{th} is, a monstrous number, and thereafter try to find out the cube root, a much harder task. The result is the same so take the easier road.

4. Given $x>0$, if

$$\frac{3}{x}=\frac{x-5}{2}$$

What is the value of:

$$\frac{x}{3}?$$

Solution

First, we will solve for x in the equation. We'll start by multiplying both sides b $2x$.

$$\left(\frac{3}{x}=\frac{x-5}{2}\right)2x$$

$$\rightarrow 6=x(x-5)$$

$$\rightarrow 6=x^2-5x$$

$$\rightarrow x^2-5x-6=0$$

Now we have a *quadratic equation* which we can solve by factoring. The correct factorization will be

$$\rightarrow (x-6)(x+1)=0$$

$$\rightarrow x=6 \vee x=-1$$

We have two values for x that solve the equation. But we are told that $x>0$, which means that the only value of x we should keep is $x=6$. Now, to answer the question, we need to find $\frac{x}{3}$.

$$\frac{x}{3}=\frac{6}{3}=2$$

5. In the xy- plane, *the parabola* with equation $y=2(x-5)^2$ intersects the line with equation $y=8$ at two points, A and B. What is the length of $\overline{AB}$?

Solution

First, we will find the coordinates of the points A and B and then we will apply the *distance formula*.

To find the intersections, we need to set the lines equal.

$$2(x-5)^2=8$$

Next, we have solve this *quadratic* just above as follows:

$$2(x^2-10x+25)=8$$

$$x^2-10x+25=4$$

$$x^2-10x+21=0$$

$$(x-7)(x-3)=0$$

$$\rightarrow x=7 \vee x=3$$

Now, we have the $x-¿$values of the two points that we are after. To find the $y-¿$values, we need to plug these two $x-¿$coordinates into the the original equations. If done correctly, you should find the following two points:

$A=(7,8), B=(3,8)$

Now we will use *the distance formula*:

$D=\sqrt{(7-3)^2+(8-8)^2}=\sqrt{4^2}=4.$

6.

$y=c(x+5)(x-3)$

In the *quadratic equation* above, c is a *nonzero constant*. The graph of the equation in the $xy-¿$ plane is a parabola with *vertex* (h,k). What is the value of k?

First, we will expand the equation:

$y=c(x+5)(x-3) \rightarrow y=cx^2+2cx-15c$

The formula for the $x-¿$coordinate of the *vertex* is $\frac{-b}{2a}$. Here, $a=c$ and $b=2c$. Then the $x-¿$coordinate of the *vertex* is $\frac{-2c}{2c}=-1$

Now, we must plug-in this $x-¿$coordinate into the original equation to find the $y-¿$value.

$y=c(-1+5)(-1-3)=c(4)(-4)=-16c$

The *vertex* is $(-1,-16c)$. Hence, the value of k is $-16c$.f

7. What are the solutions to $5x^2+13x+5=0$?

You should with reasonable efficiency see that this is not factorable. Therefore, it must be a simple exercise in using the *quadratic formula*:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Using the appropriate values of $a=5, b=13$ and $c=5$, we write:

$$x=\frac{-13\pm\sqrt{13^2-4\cdot5\cdot5}}{2\cdot5}=\frac{-13\pm\sqrt{69}}{10}$$

As so, the *solutions* to the given *quadratic* are

$$x=\frac{-13+\sqrt{69}}{10}, \vee x=\frac{-13-\sqrt{69}}{10}$$

8.

A *triangle* has two sides measuring 4 and 9. Given that the third side has a length that is also an *integer*, the *Triangle Inequality Theorem* tells us that the third side can be found by writing the inequality $9-4<x<9+4$. In addition, *Heron's* (aka *Hero's*) formula for the area of a triangle tells us that the area of a triangle with three sides a, b, and c can be found using the following formula:

$$A=\sqrt{s(s-a)(s-b)(s-c)}$$

Where s is the *semi-perimeter* (half of the triangle's *perimeter*.) If this triangle with the given two sides has an area of $2\sqrt{77}$, which of the following *integer* values can be the third side does?

A. 7 B. 8 C. 9 D. 12

Solution

The correct answer is C.

This approach which is an efficient approach is akin to solving an equation, but more like the *back-solving* strategy we talked about at the start of this book (last resort strategies☺) We only have four answer choices and as you will see, testing each doesn't require reinventing the wheel for each. To start, let us push the 2 in front of the square root inside of the square root with the 77 as follows:

$$A=2\sqrt{77}=\sqrt{4}\sqrt{77}=\sqrt{308}$$

Now, notice that the following equation must be satisfied:

$$A=\sqrt{308}=\sqrt{s(s-4)(s-9)(s-c)}$$

$$\rightarrow 308=s(s-4)(s-9)(s-c)$$

Now, for each answer choice, we will replace the corresponding values of s and c, and if we get a value of 308, and the side is in the valid range, then we have a solution:

Option A:

$$c=7, s=\frac{4+9+7}{2}=10$$

$$10(10-4)(10-9)(10-7)=10\cdot 6\cdot 1\cdot 3=180$$

We have found that the first answer choice is not correct.

Option B:

$c=8, s=\frac{4+9+8}{2}=10.5$

$10.5(10.5-4)(10.5-9)(10.5-8)=10.5\cdot 6.5\cdot 1.5\cdot 2.5=255.94$

Clearly, this answer choice also fails.
Option C:

$c=9, s=\frac{4+9+9}{2}=11$

$11(11-4)(11-9)(11-9)=11\cdot 7\cdot 2\cdot 2=308$

Aha. This answer choice is correct because 9 is in the valid range (the range given by the *Triangle Inequality Theorem*) of $(5,13)$.

9. *Nahom* earned the following scores on three *Chemistry* exams: $80,90,100$. On three *Math* exams, he scored: $89,90,91$. Which of the following conclusions below is correct when comparing the *Standard Deviation* on his *Math* scores to the *Standard Deviation* of his *Chemistry* scores?

A. The *Standard Deviation* is the same for his score on both subjects.
B. The *Standard Deviation* is greater for *Math* than it is for *Chemistry*.
C. The *Arithmetic Mean* (*Average*) is the same for his score on both subjects.
D. The *Standard Deviation* is greater for *Chemistry* than it is for *Math*.

Solution

The correct answer is D.

The *Standard Deviation* is a metric that gives us an idea of how close a set of statistic is to the *mean* (another way to say this is that *SD* measures spread of data.) The more spread values are, the bigger the *Standard Deviation*. We see that the values in *Chemistry* are more spread than the values in *Math*, which means that *Chemistry* will have a greater *Standard Deviation*.

10. *Elastic energy* is the *Mechanical Potential Energy* stored in the configuration of a material or physical system as it is subjected to *elastic* deformation by work performed upon it. *Elastic Energy* occurs when objects are impermanently compressed, stretched or generally deformed in any manner. The following formula relates *Elastic Energy* (U) and a change in *position* (Δx) with spring constant k.

$$U=\frac{1}{2}k(\Delta x)^2$$

U is also sometimes called *Spring Potential Energy*.

While this energy is measured in *Jewels* (*J*), ignoring units, if a spring has a *Spring Potential Energy* of 8 for a change in *position* equal to 2, what is the change in *position* for this same spring when it has a *Spring Potential Energy* of 4.5?

A. 3 B. $\frac{3\sqrt{2}}{2}$ C. $\frac{3}{2}$ D. $\frac{-3}{2}$

Solution

The correct answer is C.

We are told that a spring has *Potential Energy* of 8 for a change in *position* equal to 2. Substituting the values of $U=8$and $\Delta x=2$, the given equation will look as follows:

$$8=\frac{1}{2}k(2)^2$$

Now, we can solve for k in this equation writing:

$$\rightarrow 16=4k \rightarrow k=4$$

With the value of k handy, the main equation becomes:

$$U=\frac{1}{2}\cdot 4\cdot(\Delta x)^2=2(\Delta x)^2$$

Since the real task is to find the change in position $(\Delta x)$when $U=4.5$, we can now proceed as follows:

$$4.5=2(\Delta x)^2 \rightarrow \frac{4.5}{2}=(\Delta x)^2$$

$$\rightarrow \Delta x=\sqrt{\frac{4.5}{2}}=\sqrt{\frac{\frac{9}{2}}{2\cdot}}=\sqrt{\frac{9}{4}}=\frac{3}{2}$$

Practice Min Test 1

Calculators are allowed on all parts of this exam.

1. In the 90s in America, a standard cellphone plan required \$42 per month for the first 1000 *minutes* and 42 *cents* for each additional *minute* thereafter. If the number of *minutes* used is represented by t, which of the following equations below gives the cost C (in dollars) of a monthly cellphone plan?

A. $C=\$42t+\$42$
B. $C=\$0.42t+\$42$
C. $C=\$42(t-1000)+\$42$
D. $C=\$0.42[max(0,t-1000)]+\$42$

2. On *planet X*, there are two types of animals, *XaXas* and *XiXis*. *Xaxas* have 7legs and *XiXis* have 3legs. If *Reus* counted a total of 52 legs of both animals and there were 4 *XaXas*, how many *XiXis* were there?

A. 4 B. 10 C. 16 D. 8

3. In store *NunyaBizness*, seven *MacIntosh apples* cost \$42 and *oranges* cost \$3 apiece. If *Reus* bought a total of 21 *apples* and *oranges* for \$111, how many *apples* were there?

A. 9 B. 6 C. 12 D. 16

4. *Una* bought a *PlayStation* that was *marked up* by 50 percent. If the original price of the *PlayStation* was x dollars and she gave a tip of y dollars to the cashier, which of the following represents the total amount of money that *Una* paid for her *PlayStation*?

A. $1.5xy$ B. $150xy$ C. $1.5x+y$ D. $150x+y$

5. In a survey, 300 people were asked about what they prefer, *La Liga* or the *Premier League*. The following table shows the results of this survey. The table also shows what team each surveyed person prefers, *Manchester United* or *Real Madrid*. Based on the results displayed, what is the probability that a randomly selected person who prefers *La Liga* also prefers *Manchester United* or prefers the *Premier League* and *Real Madrid*.

	La Liga	*Premier League*	*Total*
Real Madrid	120	10	130
Manchester United	20	150	170
Total	140	160	300

The following three questions depend on the information provided in the chart below.

Annual budget for different programs (in dollars) for New York State (2018 to 2021)

Program	*Year*			
	2018	2019	2020	2021
Parks and Rec	77,742	42,777	74,747	77,422
Health Services	4,274,242	4,427,777	4,274,277	7,774,242
Education	74,727	47,272	74,277	74,272
Infrastructure	4,242,777	7,427,742	7,774,242	4,724,742
Law Enforcement	774,242	424,277	427,427	422,777

6. Of the following, which Program's *ratio* of its 2018 budget to its 2021 budget is closest to the *ratio* of the 2018 budget to 2021 budget for *Education*?
   A. *Health Services* B. *Infrastructure* C. *Law Enforcement* D. *Parks and Rec*

7. Which of the following best approximates the *Average Rate of Change* in the annual budget of Health Services in New York state from 2019to 2021?
   A. 167323 B. 16732325 C. 1.7 million D. 1673235

8. What is the approximate (to the nearest integer) *ratio* of the total 2018 budget to the total budget for *Parks and Rec* from 2018 to 2021?
   A. $35:1$ B. $1:35$ C. $34:63$ D. $42:1$

The following two questions rely on the information provided below.

Water from a hole in a wall rises and falls according to the model given by the following equation

$$h(t) = -9.8t^2 + 42t + 12$$

where h is the *height* of the water measured from the ground and t is *time* in *minutes* ($t=0$ represents the instant the water is turned on.)

9. Which of the following best interprets the meaning of the constant 12 in the equation?
   A. It is the *time* when the water first started flowing.
   B. It is the *time* when the water is at its highest elevation.
   C. It is the *height* of the water at its maximum.
   D. It is the *height* of the water when it first started flowing from the wall.

10. Approximately at what *time* t does the water hit the ground?
A. 4.55 *minutes* B. 5 *minutes* C. 3 *minutes* D. 2.68 *minutes*

11. It takes *Luis* 7 to 17 *minutes* to fold 10 shirts. On Tuesday, *Luis* folded shirts at his slowest pace. If he folded shirts for 102 *minutes* at this slowest pace, how many shirts did he fold?

12. An elevator has a weight capacity [maximum weight allowed for safe operation] of $(1200)10^4$ *grams*. There were 198 people in the elevator (it's a new type of elevator on a different planet called *Planet X*) with an *average* weight of 60 Kilograms [kg] and *Hannah* and her mom were the last two people to enter the elevator to bring it to full capacity. If the elevator is to operate safely and *Hannah's* weight to her mom's weight is in a *ratio* of 4 *to* 2, what must be *Hannah's* weight? (1 gram is 1000 kilograms [kg].)

The following four questions use the information provided below.

The *Alchemist* invested $\$42$ in a bank account that pays 7% *interest quarterly*. *Jayceon* invested $\$16$ dollars in an account that pays 12% *interest annually*. The following two equations tell us how much money the *Alchemist* and *Jayceon* (respectively) will have t years after they made their initial investments.

$$A(t)=\$42\left(1+\frac{x}{n}\right)^{4t} \quad \text{and} \quad J(t)=\$a(1-y)^t$$

13. What is the value of $x+n+a+y$?

14. How many *years* will it take for the *Alchemist's* initial investment to *quadruple*?

15. How many *years* will it take for *Jayceon's* investment to be more than $\$100$?

16. In the figure below, a *cone* of *height* 4 has *diameter* d with *volume* equaling 12π. What is the value of r^3 where r is the *radius*?

The volume of a cone is given by $V=\frac{1}{3}\pi h\left(\frac{d}{2}\right)^2$

The following three questions rely on the information provided below

Ernesto was given a budget of $\$42$ dollars per week for a 4 week summer camp away from home. The equation below allows us to find out how much money *Ernesto* has left d days after the summer camp begins.

$$M = x - 8d$$

17. What is the value of x in the equation and what is the meaning of this value in the context of the information provided?

18. What is the best interpretation of the *coefficient* on d? That is, what is the meaning of the number 8 in the equation?

19. Does *Ernesto* have enough money to last him through the four-week program? Show the work that leads to your conclusion.

20. It takes *Monica* m *hours* to bake n *tortillas*. It takes *Hannah* p hours to bake q *tortillas*. Which of the following represents the total number of *tortillas Monica* and *Hannah* can bake in 10 *hours*?

A. $\frac{10n}{m} + \frac{10q}{p}$

B. $\frac{10nq}{mp}$
C. $\frac{10m}{n}+\frac{10q}{p}$
D. $\frac{10m}{n}+\frac{10p}{q}$

21. In *planet Pi*, there are two types of animals, *Pipies* and *Pups*. If *Pipies* have 7 legs and *Pups* have 9legs, which of the following represents the number of legs that x *Pipies* and y *Pups* have?
A. $9x+7y$ B. $7x+9y$ C. $63xy$ D. $63(x+y)$

22. In an assembly line, a quality control person checks shoe soles for defectiveness. She finds that 15% of *men's shoes* are defective and 4 out of every 200 *women's shoes* are defective. If the ratio of *men's shoes* to *women's shoes* is $6:9$ and 150 shoes are inspected, to the nearest integer, how many of the 150shoes are defective?

23. A *zero-sum game* is a game where a point gained for one team is a point lost for the other. For example, when one soccer team scores 1 goal, the other team must score a goal just to get back to *parity* (equal scores.) If *Real Madrid* (a soccer team) scores x goals on *Manchester United* (another soccer team, the best in the world) but then *Manchester united* scores y goals to be 10 goals ahead, which of the following is the correct relationship between x and y?
A. $x+y=10$
B. $x-y=10$
C. $y-x=10$
D. $x+10=y$

24. *Salmina* has x quarters (25 cents in the US) and y dimes (10 cents in the US.) If the total of the quarters and dimes she has makes 42 dollars, which of the following does not give the correct equation for *Salmina's* balance in quarters and dimes.

A. $25x+10y=42$
B. $0.25x+0.10y=\$42$
C. $2.5x+1.0y=\$420$
D. $25x+10y=\$4200$

25. A researcher asked a sample of 142 people on 145th street in *Harlem* who they are going to vote for in the 2021 Mayoral race. His more experienced partner told the sampling researcher that there is something wrong with his sampling. Which of the following is the best explanation of what she could be talking about?

A. *The sample size*.
B. *Lurking variables*.
C. *Population size*.
D. *Geographic bias*.

The following two questions involve the use of the information provided below.

Silly goofy *Tutu* decided to measure *Titi*'s height in *nanometers*. He also measured his sister *Tata*'s height in *nanometers*. He used the table provided below to assist him in his unit conversion. He found that *Titi* is 4 *feet* and 10 *inches* tall and *Tata* is 5 *feet* and 1 *inch* tall.

Tutu's Conversion Table	
1 *foot*	0.305 *meters*
1 *meter*	1000000000

26. What best approximates the *ratio* of *Titi*'s height to *Tata's* height in *nanometers*?

27. Using *scientific notation*, what is the sum of *Titi*'s and *Tata*'s height in *nanometers*?

Survey results on alien *Reus*

	Stay	*Go*
Ethiopian Girl	42	100
Hispanic Girl	4	2

28. The table above shows a survey taken on an alien named *Reus*. The numbers displayed represent whether he should go *back to Africa* or *not* and whether he should try to *date* a *Hispanic girl* or an *Ethiopian girl*. Based on the table, what is the probability that *Reus* should stay and date a *Hispanic girl* or *go back to Africa* and date an *Ethiopian girl*?

29. The Price of *PlayStation X* is $\$777$. The *mean* price of *PlayStations* in the past has stayed around $\$400$ and there have been *PlayStation* 1 through *PlayStation* 5. Which of the following most likely happens by the unusual price of *PlayStation* X?

A. The *median* is affected more than the *mean*.
B. The *mean* is more affected by the latest *PlayStation* price than the *median*.
C. Neither the *mean* nor the *median* is affected much.
D. The *mean* and the *median* are affected equally.

30. *Luis* spent 42 *minutes* more than twice as many minutes as *Reus* on programming. If *Luis* spent 4.2 *hours* programming, how many minutes did *Reus* spend on programming?

The following three questions depend on the information provided below.

Davor measured the height of all 44 players in 2 soccer teams, *Team A* and *Team B*. The heights of the players were rounded to the nearest integer and measured in *feet*. The *frequency table* below shows the data he collected.

Height measured in feet (*rounded to the nearest integer*)	*Team A*	*Team B*

4	2	4
5	7	9
7	11	7
10	2	2

31. What is the *positive* *difference* in the *arithmetic mean* (*average*) height of players in the two teams?

32. What is the *positive* *difference* in the *median* height of players in the two teams?

33. What is the *positive* *difference* in the *range* of the height of players in the two teams?

34. To calculate how much money M it would take to ship 4200 boxes of *Mangoes*, *Luis* used the following formula.

$$M = 7xt$$

If x represents the *number of boxes* a truck can carry and t represents the *number of trucks*, given that each truck is used to its maximum capacity, what is the best interpretation of the number 7 in the equation above?

A. The price per face of each box.
B. The price per box of the *Mangoes*.
C. The price for shipping one box of *Mangoes*.
D. The price of one *Mangoes*.

The following two questions depend on the information provided below

It is estimated that a new car loses 15 percent of its value as soon as it is driven off the lot. Thereafter, a car dealer determined that on *average*, cars lose value according to the formula given below. In the formula, V represents the value of the car, P represents the original price of the car and t represents the number of years after the 15 percent decline off the lot.

$$V = aP(.81)^t$$

35. What is the value of a?

A. 0.19 B. 19 C. 0.15 D. 0.85

36. Because Jesus is God, the only God, the only Messiah, we don't like this number. We are about the 7s. So, this question is skipped. 777☺

37. What percent of its value is the car losing per year?

A. 81% B. 15% C. 85% D. 19%

38. An elevator has a carrying capacity of 4200 *kgs* or 77 persons. The *average* weight of *a*dults that use this elevator is 72 *kgs* and the *average* weight of *k*ids who use this elevator is 42 *kgs*. If a represents the *number of* *a*dults who use the elevator while k represents the *number of* *k*ids, which of the following system of inequalities best represents this relationship?

A. $\{42a+72k \le 4200\ a+k \le 77$

B. $\{\frac{a}{42}+\frac{k}{72} \le 4200\ a+k \le 77$

C. $\{72a+42k \le 4200\ a+k \le 77$

D. $\{a+k \le 4200\ 42a+72k \le 77$

Problems 41 and 42 use the information provided below.

Today, *Eteti* has 42 black candies and 7 yellow candies. Every week, she loses 4 black candies and gains 2 yellow candies.

39. x weeks from today, how many candies will *Eteti* have?

A. $42-4x+7+2y$

B. $2\left(21-\frac{2x}{7}\right)+7+2x$

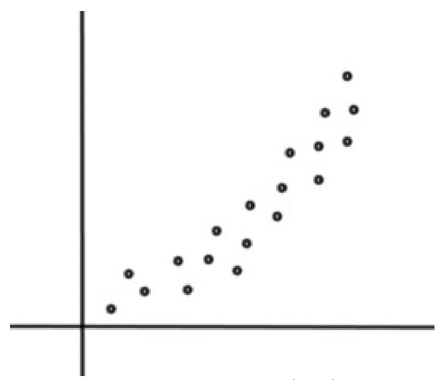
C. $49-2x$

D. $49-\frac{2x}{7}$

40. After several weeks, *Eteti* will clearly only have yellow candies. How many candies will she have at the time when all of them are yellow?

A. 147 B. 49 C. 21 D. 42

41. Which of the *models* provided in the answer choices NOT a good fit for the *Scatter Plot* given below? ($a.b>0$¿



A. $y=ab^x$ B. $y=ax^2$ C. $y=ax+b$ D. $y=a\ln(bx)$

42. Of a *random sample* of 42 students, 42% said that *Saturday* is their favorite day of the week. 7% of the same sample said *Monday* is their favorite. If there are 442 students in total, based on the sample, how many of the total student population chose a day different from *Saturday* or *Monday* as their favorite day of the week? (All surveyed had to choose a day of the week.)

A. 31 B. 217 C. 186 D. 225

Solutions to Practice Min Test 1

1. The correct answer is D. $C=0.42[max(0,t-1000)]+\$42$
The correct answer is D. $C=0.42[max(0,t-1000)]+\$42$

Since every month we must pay \$42no matter how many *minutes* we use, we need to include this as a constant in our equation. The next thing we must take care of is the 42 *cents* we have to pay for each additional *minute* after the first 1000 *minutes* that come with the \$42.
We can represent this quantity as $0.42 \cdot max(0,t-1000)$.

This is so because when t is *less than or equal* to 1000, $0.42[max(0,t-1000)]$ will be equal to 0, so we won't have to pay for those *minutes*, but for every $t>1000$, it will represent the amount we have to pay for the additional *minutes*. Another reason we need the max function is because if we just write as it is tempting to do, the equation:
$C=\$0.42(t-1000)+\$42$

Notice that if t is not restricted to be greater or equal to 1000, we would pay negative dollars which would be sweet but **TANSTAAFL** (**T**here **A**int **N**o **S**uch **T**hing **A**s **A** **F**ree **L**unch.)

2. The correct answer choice is D.

We will denote the number of *XaXas* with x_a and the number of *XiXis* with x_i. We are told that the value of x_a is 4, and we need to find the value of x_i. Since we are told that in total *Reus* counted 52 legs, we can set up the following equation to solve for x_i.

$$7x_a+3x_i=52$$

This is so because the total number of legs is equal to the sum of the legs of every animal, and we know that each *XaXas* has 7 legs and each *XiXis* has 3 legs.

Now, we will substitute the value of x_a .

$$(7)(4)+3x_i=52$$

$$\rightarrow 3x_i=52-28=24$$

$$\rightarrow x_i=8$$

3. The correct answer is D.

Since we know that in total, we have 21 *apples* and *oranges*, if we have x apples then we should have $21-x$ *oranges* so that their sum is equal to 21. Using this, we can set up an equation that we can solve for x. Since we are given that 7 apples cost 42 dollars, this means that 1 apple costs:

$$\rightarrow \frac{42}{7}=6$$

$$\to 6x+(21-x)3=111$$

$$\to 6x+63-3x=111 \to 3x=111-63=48$$

$$\to x=16$$

4. The correct answer is C $1.5x+y$

Since the original price of the *PlayStation* was x and we are told it was *marked up* by 50 percent, the new price will be $1.5x$. Since we are told that *Una* is giving a tip of y dollars to the cashier, then we must add y dollars to the total price.

5. The correct answer is 0.1.

We want to find the probability that someone who likes *La Liga* also likes *Manchester United* or someone who likes the *Premier League* also likes *Real Madrid*. First, we reckon that since this is a probability question involving "or," we must add. Looking at the chart, we see that 20 people like *Manchester United* and *La Liga* (*Event A*) or 10people like the *Premier League* and *Real Madrid* (*Event B*.) Adding these two, we have 30people. Since in total we have 300 people, we will have a probability of

$$P(A \vee B)=\frac{20}{300}+\frac{10}{300}=\frac{30}{300}=\frac{1}{10}=0.1$$

6. The correct answer is D. *Parks and Rec*.

We will first fill the following chart and see which *ratio* is closer to the 2018 budget to the 2021 budget for *Education*.

Program	*Year*		*Ratio*
	2018	2021	
Parks and Rec	77,742	77,422	$\frac{77{,}742}{77{,}422}=1.004$
Health Services	4,274,242	7,774,242	$\frac{4{,}274{,}242}{7{,}774{,}242}=.55$
Education	74,727	74,272	$\frac{74{,}727}{74{,}272}=1.006$
Infrastructure	4,242,777	4,724,742	$\frac{4{,}242{,}777}{4{,}724{,}742}=0.898$
Law Enforcement	774,242	422,777	$\frac{774{,}242}{422{,}777}=1.831$

Looking at the calculations in the table, we can see that the *ratio* closest to the ratio of the 2018 budget to the 2021 budget for *Education* is the corresponding ratio for *Parks and Rec*. A faster way of solving this problem is to notice that the *ratios* of these two categories over the appropriate years are very close to one whereas the others are not nearly as close [to one.]

7. The correct answer is D. 1673235.

We can interpret the *Average Rate of Change* as the change in the $y-$¿coordinate divided by the change in the $x-$¿coordinate of a function.

In this case, our function would be the budget for *Health Services*. The $x-$¿coordinate will represent the *year*, and the $y-$¿coordinate will represent the *allocation in the budget* to Health Services.

Since again, we want to know the *Average Rate of Change* in a period of two years, this will be the difference of the corresponding values to 2021 and 2019 respectively (in the Health Services *budge*t row in the table) divided by 2.

$$\frac{7774242-4427777}{2021-2019}=\frac{73346465}{2}=1673235$$

8. The correct answer is A.

The total 2018 budget is equal to $77742+4274242+74727+4242777+774242=9443730$.

The budget for *Parks and Rec* from 2018 to 2021 is equal to $77742+42777+74747+77422=272688$. The quotient of these two numbers is approximately equal to 34.63, which is closest to $35:1$. You either want to use a Calculator in reducing the number $9443730:272688$ or you want to round off both numbers and that should be good to give you a clear idea. For example, you could work with $9500000:270000$.

9. The correct answer is D.

We observe that the constant 12 is the $y-$¿intercept of the function. In other words, when $t=0, h(0)=12$. This is telling us that at the instant the water is turned on, it flows from the wall at a *height* of 12. The units also tell us that we are correct since 12 is a value of $h(t)$ which is *height*.

10. The correct answer is A.

Since we want to find the *time* when the water hits the ground, we want to find the moment at which the *height* ($h(t)$) is equal to 0. So, we must write:

$$-9.8t^2+42t+12=0$$

From here, to solve for t, we can use the *quadratic formula*.

$$t=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Plugging in the corresponding values, we have:

$$t=\frac{-42\pm\sqrt{42^2-4(-9.8)(12)}}{2(-9.8)}$$

After doing these computations (clearly, one needs a calculator to simplify the quantity just above), we get the following two values of t.

$$t=-0.27, t=4.55$$

Since negative time doesn't make sense in the context of this problem, we will only accept the positive value.

11. The correct answer is 60.

Since we are told that *Luis* folded shirts at his slowest pace, this means that he took 17 *minutes* to fold 10 shirts. This means that he took $\frac{17}{10}=1.7$ *minutes* to fold each shirt. With this in mind, we can set up the following equation to solve for x, the number of shirts he folded in 102 *minutes*.

$$1.7x=102$$

$$x=\frac{102}{1.7}=60$$

12. The correct answer is 80 *kg*.

First we will find out how much more weight we can add to the elevator so that it operates safely. To do this, we will subtract the current weight in the elevator with the maximum allowed. The current weight in the elevator is equal to $(60)(198)=11880$ (because the *average* weight persons in the elevator is 60 *kg*).

So the additional weight we can add into the elevator while keeping it safe is therefore $12000-11880=120kg$. We have divided $(1200)10^4$ by 1000 because we are working in *kilograms* not *grams*.

Now, we haven't used the information about the *ratios* of the weights so we will do that. Since we are told that *Hannah's* weight *to* her mom's weight is in a *ratio* of 4 *to* 2, we will assign *Hannah's* weight the value $4x$, and her mom's weight $2x$ Since we know the sum of their weights [120 *kg*], we can find their individual weight by writing

$$4x+2x=120$$

$$\rightarrow 6x=120$$

$$x=20$$

Hannah's weight is equal to $(4)(20)=80\,kg$.

13. The correct answer is 19.95

The *compound interest* formula is written correctly in what is given us in the question where we saw:

$$A=P\left(1+\frac{r}{n}\right)^{nt}$$

And we know that A is the *Amount* after time t, P is the *Principal* (initial) investment, r is the *annual interest rate* as a decimal, and n is the number of times the interest is compounded per unit t (so monthly means $n=12$.) Plugin in the corresponding values in the appropriate equation, we get the following for each person:

$$A(t)=\$42\left(1+\frac{0.07}{4}\right)^{4t}$$

$$J(t)=\$16(1+0.12)^{t}=\$16(1+0.12)^{t}=\text{¿}\$16(1-(-0.12))^{t}$$

Thus, we have:

$x=0.07,\ n=4,\ a=16,\ y=-0.12$

$x+n+a+y=0.07+4+16-0.12=19.95$

14. The correct answer is 20 *years*.

To solve this question, we should set up an equation and solve for the *time* t. Since we want the initial investment to *quadruple*, we will set the equation involving the *Alchemist* to equal $(4)(\$42)=\$168$ because \$42 is his initial investment and multiplying it by 4 is the same as *quadrupling*.

$$\$42\left(1+\frac{0.07}{4}\right)^{4t}=\$168$$

Dividing both sides by \$42, we have:

$$(1.0175)^{4t}=4$$

Now, we can take the *Natural Logarithm* of both sides:

$$\ln(1.0175)^{4t}=\ln 4=1.386$$

In addition, using the *Log Power Rule*, we will bring the exponent to the front of the *Logarithm*.

$$4t\ln(1.0175)=\ln 4=1.386$$

$$\rightarrow t=\frac{1.386}{4\ln(1.0175)}\approx 19.977\approx 20$$

There is a small precision error margin, since we have used decimal values throughout the computations, but the answer rounded to the closest integer will still be 20 years.

15.The correct answer is 17 *years*.

To start, we use the given equation that concerns *Jayceon's* investment output and set up an inequality as follows:

$$\$16(1+0.12)^t > \$100$$

[*Notice* that the investment equation for *Jayceon* would have us write $1-0.12$ inside of the parenthesis. The equation was given with a minus sign to make question 13 more interesting and trickier. But, because *Jayceon's* investment is growing account that pays 12% *interest annually*, in answering this question, we must write $1+0.12$.]

$$\rightarrow (1+0.12)^t > \frac{\$100}{\$16} = 6.25$$

$$\rightarrow \ln(1+0.12)^t > \ln(6.25)$$

$$\rightarrow t\ln(1.12) > \ln(6.25) = 1.83258$$

$$\rightarrow t > \frac{\ln(1.83258)}{\ln(1.12)} = 5.345$$

Thus, after 6or more years, *Jayceon's* investment will be more than $\$100$. Another way to solve this problem is to manually apply the famous Binary Search algorithm, which you may know if you're interested in *Computer Science*, or *Competitive Programming*.

16.The correct answer is 27.

As given, the formula for the *volume* of a *cone* is $\frac{1}{3}\pi r^2 h$. We know that the *height* of the *cone* is 4 and that the *volume* of the *cone* is 12π. We can substitute these values in the formula for the *volume* and solve for r, and then easily compute r^3.

$$V = \frac{1}{3}\pi r^2 h \rightarrow$$

$$\frac{4\pi r^2}{3} = 12\pi$$

$$\rightarrow r^2 = \frac{(12)(3)}{4} = 9$$

$$\rightarrow r = 3 \rightarrow r^3 = 27$$

17. The value of x is 168. This is so because x represents the total money *Ernesto* has, which is 42 dollars per week, for a 4 week camp. This amount of money is $(\$42)(4)=\$168$.

18. The meaning of the *coefficient* on d is the rate per day of Ernesto's expenditure. This is so because we are told that d represents the number of days after the beginning of the camp, so the best interpretation of the number 8 is the amount he spends every day.

19. Ernesto does not have enough money to last him through the four-week program. To last him, we need to get to 28 days otherwise, the answer is "no." We figure this out by setting $M=0$ and solve the day in which *Ernesto* runs out of money.

$$M=0=168-8d$$

$$\rightarrow 8d=168$$

$$\rightarrow d=\frac{168}{8}=21$$

We see that after 21days, *Ernesto* runs out of money.

20. The correct answer is A.

Since *Monica* takes m *hours* to bake n *tortillas*, she can bake $\frac{n}{m}$ *tortillas* in one *hour*. On the other hand, because *Hannah* takes p *hours* to bake q *tortillas*, she can bake $\frac{q}{p}$ *tortillas* in one *hour*. Therefore, each of them can bake $\frac{10m}{n}$ and $\frac{10q}{p}$ tortillas in 10 *hours* respectively. As such, together they can bake $\frac{10n}{m}+\frac{10q}{p}$ *tortillas.*

21. The correct answer is B.

Since we are told that each *Pipies* has 7 legs, x *Pipies* must have $7x$ legs. The same applies to *Pups*, since *Pups* have 9 legs, y *Pups* must have $9y$ legs. Adding these two amounts together, we get that the total number of legs is $7x+9y$.

22. The correct answer is 11.

First, we will find what percentage of *women's shoelaces* are defective. Since we are told that 4 out of every 200 *women's shoes* are defective, $\frac{4}{200}\cdot 100\%=2\%$ is the percentage of defective *women's shoes*.

In addition, since we know that the ratio of *men's shoes* to *women's shoes* is $6:9$ and 150 shoes are inspected, we can solve for x in the following equation. $6x$ will represent the number of *men's shoelaces*, and $9x$ will represent the number of *women's shoelaces*.

$$6x+9x=150$$

$$15x=150 \rightarrow x=10$$

Thus, there are 60 men's shoelaces and 90 women's shoelaces. Out of the 60 men's shoes, 15% $\left[60\cdot\frac{15}{100}=9\right]$of them will be defective [the 15% was given.] Out of the 90*women's shoes*, 2% $\left[60\cdot\frac{2}{100}=1.8\right]$of them will be defective. Thus, in total, there will be $9+1.8=10.8$ defective shoes, rounded to the nearest integer, 11.

23.The correct answer is C.

An easy way to keep track of the relationship between x and y is to consider the difference between them $(y-x)$. A difference of 0 means equal scores, while a difference of positive 10means y is 10 goals ahead, and a difference of -5 means x is 5goals ahead of y. Since we are told that *Manchester United* is 10 goals ahead, then the correct relationship between x and y is $y-x=10$.

24.The correct answer is A.

First, we will find the main correct equation representing *Salmina's* balance. Once we have done so, any other equation that is a "*multiple*" of it (multiplied by the constant on both sides of the equal sign) will also be correct. The correct equation is $0.25x+.10y=42$. This is so because a quarter is worth 0.25 dollars, a dime is worth 0.10 dollars, and we know that the total is $\$42$.

We see that this is answer choice B, the first correct equation B. Now we will analyze the other choices. We see that answer choice A is not a multiple of it [the main correct equation] since the terms on the left-hand side have been multiplied by 100, but the term on the right-hand side is not multiplied by 100. In answer choice C, we see that every term has been multiplied by 10, so C is also correct. Finally, we see that every term on the answer choice D has been multiplied by 1000, so it is also correct. Thus, the only incorrect answer is answer choice A.

25. This is one of those questions where more than one answer choice may be appropriate. But there is a best answer. The best answer here is *Geographic bias*. *Harlem* tends to have a cluster of a particular demographic and as such, D is the correct answer. But

notice that if your sampling is not airtight, then, “lurking variables” is always a one size fits all answer for all biased sampling.

26. The correct answer is $210:221$

To avoid working with large numbers, we will work using the *feet* units, since the *ratio* of *Titi’s* height to *Tata’s* height will be the same, regardless of the units. We should recall that 12 *inches* ¿1 *feet*. With this conversion in mind, we will represent each person’s height in *feet*. We know that *Titi* is 4 *feet* and 10 *inches* tall and *Tata* is 5 *feet* and 1 *inch* tall. Converting *inches* to *feet*, we get the following heights:

Titi	*Tata*
$4+\frac{10}{12}=4.83$ *feet*.	$5+\frac{1}{12}=5.083$ *feet*.

The *ratio* of *Titi’s* height to *Tata’s* height is $4.83:5.083$. To remove the decimal point, we can multiply both of number by 1000 to write $4830:5083$. We can simplify a little bit more to write as our final answer, the *ratio*: $210:221$.

27. The correct answer is $3.023(10^9)$ *nanometers*.

Using the information from the previous problem, we can tell that the sum of *Titi’s* and *Tata’s* height is equal to $4.83+5.083=9.813$ *feet*. We will now use the conversion table to convert this value into *meters* and then to *nanometers*. According to the table, 9.913 *feet* is equal to $(9.913)(0.305)=3.023$ *meters*.

Now, to get this value into *nanometers*, we just need to multiply it by a billion (by 10^9.) So we will have $3.023(10^9)$ *nanometers*.

This is our final answer, because it meets the criteria for the *scientific notation* (<u>the absolute value of the coefficient should be between 1 and 10, and the exponent is a non-zero integer</u>:)

28. The correct answer is: $\frac{1252}{1173}$

Here, we must find the table *Total* which should (if you do it correctly) look as follows:

	Stay	*Go*	*Totals*
Ethiopian Girl	42	100	142
Hispanic Girl	4	2	6
Totals	46	102	148

Notice that if you add the columns under *Total* (the *Total* of the *Totals*) you get the same 148. I.e., $46+102=142+6=148$.

With this, to see the probability that he (*Reus*) should stay and date a *Hispanic* girl, we need him to both *Stay* and date a *Hispanic* girl, meaning: $\frac{4}{46}=\frac{2}{23}$. Similarly, we need to consider the case where he should *Go* and date an *Ethiopian* girl, which should come out to be: $\frac{100}{102}=\frac{50}{51}$. Since we want one <u>or</u> the other and *we know "<u>or</u>" means <u>add</u> in Probability*, we will have to <u>add</u> the two fractions to answer the question.

$$\frac{2}{23}+\frac{50}{51}=\frac{1252}{1173}$$

29. The correct answer is B.

An efficient way to answer this question is the understanding that *outliers* (unusual points in a statistic) always affect the *mean* (the *average*) more than they affect the median. But for more details, let's say more.

We know that the *mean* price of *PlayStations* in the past has stayed around 400 and there have been *PlayStation* 1 through *PlayStation* 5. This means (no pun) that every *PlayStation* has price around 400, so the distribution of prices will look something like the following:

$$400,\ 400,\ 400,\ 400,\ 400$$

Here, 400 means a number around 400. Since the *median* is the middle element (when the number of elements is odd) in a sorted order of the data, we see that the original *median* is also a number around $400\,(\ 400)$. Adding a price of $\$777$ would make the distribution look like the following:

$$400,\ 400,\ 400,\ 400,\ 400, 777$$

Now, the new *median* will be the *average* of the two central elements, which is the *average* of two numbers around $400\,(\ 400)$ and thus, it will remain around 400 [showing little movement or change.] On the other hand, the arithmetic mean (average) was around $\$400$ to start. But the latest *PlayStation* (*X*) will push up (increase) the overall *arithmetic mean*. If we must do some calculations, we would have to write something like the following:

Without *PlayStation X*:

$$\frac{(\ 400+\ 400+\ 400+\ 400+\ 400)}{5}=\frac{(\ 2000)}{5}=\ 400$$

Including *PlayStation X*:

$$\frac{400+\ 400+\ 400+\ 400+\ 400+777}{5}=\frac{2777}{5}=\ 462.83$$

We see that the change in the *average price* is considerable.

30. The correct answer is 105 *minutes*.

We know that *Luis* spent 4.2 hours programming. Since we are asked about the number of *minutes Reus* spent programming, we will convert this value to *minutes*. To convert hours to *minutes*, we need to multiply the *hours* by 60. Doing so, we see that *Luis* spend $(4.2)(60)=252$ *minutes* programming.

The next step will be to set up an equation using the other piece of information. We are also told that *Luis* spent 42 more than twice as many *minutes* as *Reus* on programming. Now, we need take all the above and transform it to an algebraic expression.

To do so, we will represent the number of *minutes Reus* spent programming by x so that we can write the following equation:

$$252=42+2x$$

$$\frac{210}{2}=x \rightarrow x=105$$

31. The correct answer is 0.451

First, we will find the *average* height for each team, and then find their difference. We know there are a total of 22 players in each team. Now, to find the average, we need to add up the values for each team, and then divide this sum by 22. A more efficient way of doing this is to instead of adding number by number, to multiply each value by its count, because this is the number of times this number will appear in the sum. We will do this for each team:

Team A (*Average*)

$$¿\frac{4\cdot 2+5\cdot 7+7\cdot 11+10\cdot 2}{22}=\frac{140}{22}=6.36$$

Team B (*Average*)

$$¿\frac{4\cdot 4+5\cdot 9+7\cdot 7+10\cdot 2}{22}=\frac{130}{22}=5.909$$

The difference of the averages is equal to $6.36-5.454=0.586$.

32. The correct answer is 2.

To find the median, this time, we do need to list all the values for each team. The complete list of values for both teams is the following:

	Team A	*Team B*
Heights	4, 4, 5, 5, 5, 5, 5, 5, 5, 7, 7, 7, 7, 7, 7, 7, 7, 7,7, 7, 10, 10	4, 4, 4, 4, 5, 5, 5, 5, 5, 5, 5, 5, 7, 7, 7, 7, 7, 7, 7, 10, 10

The *median* of each team is equal to the *average* of the two central *heights* (since we have an even number of players). We see that in the first team, the two central elements are equal to 7, and in the second team the central elements are equal to 5, so the *medians* will be 6 and 5, respectively. The *positive difference* in the medians is equal to $|7-5|=2$.

33. The correct answer is 0.

The *range* of a dataset is simply the difference between the largest value and the smallest value. So, a quick way to answer this question is to notice that both teams have the same smallest and largest numbers and therefore, the *range* for each team will be the same. Meaning, the difference in the ranges will be 0. If you are paranoid, here is a more meticulous approach!

$$Team\ A\ (Range)\ ¿7-4=3$$

$$Team\ A\ (Range)\ ¿7-4=3$$

As such, the difference will be equal to $3-3=0$.

34. The correct answer is C. The price for shipping one box of *mangoes*.

Since we are told that x represents the capacity (*number of boxes*) of each truck, and t represents the *number of trucks* used, then xt is the total number of boxes we are shipping. This is so because for every one of the t trucks, we will have x boxes. Since the whole formula represents the money, we need and there is no other mention of price per box or per truck, the coefficient seven must be the price per box (each box will cost $\$7$ as this number is multiplying the total *number of boxes*. Frankly, it wouldn't have mattered if $\$7$ was multiplying t instead of x because multiplication is firstly commutative but more importantly, xt represents the total number of boxes.)

35. The correct answer is D.

Since the formula is supposed to represent the value of a car t years after the 15 percent decline in the value of the car going off the lot, then it must take into consideration this 15 percent decline. If a car loses 15 percent of its value, then its new value V will be equal to $(1-0.15)V=0.85V$. This coefficient of 0.85 must be included in the formula, and since a is the only part of the formula whose value is not assigned, this must be its purpose. That is, a must be this missing coefficient of0.85.

36. We skip 36 like the 13th floor. We are all about the 7s (777) and first and foremost, all about Jesus, the Christ, the only God, the only Messiah.

37. The correct answer is D

Every year, the new value of the car V is equal to $0.81\,V_{old}$, where V_{old} represents the previous value. In other words, every year the car loses $(100-81)\%=19\%$ of its value.

38. The correct answer is C.

This is so because we must check two things to solve this problem. We need that the total *weight* is *less than or equal to* 4200 and that the total *number of persons* is *less than or equal to* 77.

The total *number of persons* is equal to $a+k$, so one condition we need to satisfy is that $a+k\leq 77$. The total *weight* of the people who use the elevator is equal to $72a+42k$ and we need this quantity to be *less than or equal* 4200. So, the other condition that must be true is $72a+42k\leq 4200$. Answer choice C correctly has both conditions in place.

39. The correct answer is C. $49-2x$

At the beginning, *Eteti* has 42 black candies and 7 yellow candies, which means that she has a total of $42+7=49$ candies. Every week, she loses 4balck candies and gains2 yellow candies, which means that every week the net change in the number of candies is $-4+2=-2$, so every week *Eteti* will lose 2 candies. As such, in x weeks she will lose a total of $2x$ candies ¿) from her starting total of 49 and so *Eteti* will have $49-2x$ candies. Technically, this cannot go on indefinitely because the number of candies she has will be negative after some time. But the question hasn't tasked us with this detail albeit it is worth pointing out.

40. The correct answer is

As we can tell from reading the question, we want to find the number of candies when all of them are yellow. Which means, at that time, that the number of black candies will have to be 0. Since every week we lose 4 black candies, the number of weeks required to get to 0black candies is equal to $\frac{42}{4}=10.5$. Now, we will find out how many yellow candies she will have gained (in addition to the original 7) in these 10.5 weeks. Since she gains 2 yellow candies per week, we will gain $2\cdot 10.5=21$ yellow candies in 10.5 weeks. And so, in total, when all the black candies are gone, she will have $21+7=28$.

41. Looking through the answer choice, we see that answer choice D is a *Logarithmic* function. We know that these functions grow very slowly and are the inverse of exponential functions. On the *Scatter Plot*, we see a *direct relationship* between the variables that is a more rapid growth than the slow growth exhibited by a *Logarithmic model*. If you use any graphing device and take a quick look at the parent log function, that is, $y=\ln(x)$, it will be easy to see that the pattern above doesn't fit the graph of the natural log [$y=\ln(x)$.¿

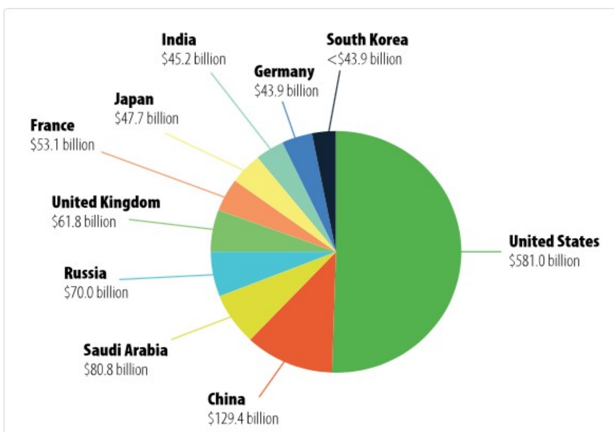
42.The correct answer is D. 225

Here, we know that a total of $42\%+7\%=49\%$ of the surveyed said *Saturday* or *Monday* is their favorite day of the week. We are going to use the surveyed to *approximate* the choices that would be made by the 442 in the total population of students. Since $100\%-49\%=51\%$of the surveyed must have chosen a different day of the week, we estimate that $(0.51)(442)=225.42$ and therefore approximately 225 people chose away from the two days (*Saturday* and *Monday*.)

Practice Mini Test 2

Calculators are allowed on all parts of this mini test.

1. *John F Kennedy* high school enrolls only Freshmen, Sophomores, Juniors, and Seniors. There are 250 Freshmen and a total of 400 Juniors and Seniors. If 8% of the Freshmen and 16% of the Sophomores play sports, how many Sophomores play sports?
   A. Cannot be determined from the information given
   B. 44
   C. 144
   D. 88
2.



2014 Military Spending by Country (Army University Press – armyupress.army.mil)

According to the *Pie Chart* above, the 2014Military budget of which three countries is approximately the same percentage as the combined Military budget of the United Kingdom and Saudi Arabia?
A. India, South Korea, and Germany
B. China and Japan
C. France, Germany, and Korea
D. Germany, India, and France

The following two questions depend on the information provided below.

Ludwig bikes at a rate of 16*miles* per hour for the first 20 *miles* of a trip and he can run at a rate of 12 *miles per hour* the rest of the way.

3. If *Ludwig* sets on a 42−¿*mile* trip on Saturday, how long did he spend traveling?

4. If *Ludwig* travels for 42*minutes*, how many *miles* is he able to cover (travel?)

The following two questions depend on the information provided in the table below.

The table below shows the distribution of weight for dogs or cats

	Weight (*in lbs.*)		**Total**
Pet	*Under 7lbs*	*Over 7lbs*	
Dog	14	4	18
Cat	7	17	24
Total	21	21	42

5. What is the probability that a randomly selected pet is a Cat is under 7 lbs?

6. N/A We are all about Jesus (God, the only God) and the 7s. Thus, there is no question six.

7.

		Subject			
		Math	Physics	Chemistry	**Total**
Religion	Christian	77	27	38	142
	Muslim	45	55	100	200
	Total	122	82	138	342

A group of high school students were asked about their creed (with two choices, *Christian* or *Muslim*) as well as what subject they preferred (*Math*, *Physics*, or *Chemistry*.) The table above shows the result of this set of *categorical data*. Based on the table, which of the following categories accounts for approximately 40% of the survey respondents?

A. The total number of *Physics* students together with the number of *Christian Math* students
B. total number of *Math* students and the *Christian Physics* students
C. The total number of *Christian* students
D. The total number of *Chemistry* students

The following two questions depend on the information provided below.

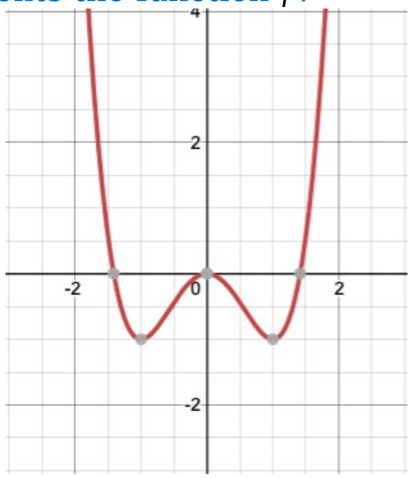
A sneaker collector bought *Saucony Kinvara* for $\$110$ dollars on January $1,2022$. The equation below models the value of this shoe over time where t represents the number of years since January $1,2022$ and P represents the value of the shoe after t years.

$$P=\$42\left(\frac{t}{4}\right)+\$110$$

8. What is the meaning of $\$42$ in the equation?
A. The change the price of the shoes from month to month.
B. The amount of dollars the shoe gains after four years.
C. The amount of dollars the shoe gains after one year.
D. The amount of dollars the shoe gains after three months.

9. Approximately how many years (rounded to the nearest month) will it take for the *Saucony Kinvara* shoes to *quadruple* (four times the original price) in value.

10. The function f shown below has three turning points (*local maxima and minima*.) Which of the equations given in the answer choices best represents the function f?



A. $f(x)=x(x^2-2)^2$

B. $f(x)=x\left(x+\frac{3}{2}\right)\left(x+\frac{3}{2}\right)$

C. $f(x)=x^2(x^2-2)$

D. $f(x)=x(x^2-2)$

The following two questions depend on the information and table provided below.

From a sample of 42 students in a Catholic school in Mexico City, the height of all the sampled students was between 4.8 *feet* to 7.0 *feet*. The *frequency table* below gives the height of all the sampled students rounded to the first number after the decimal.

Student Height	4.8	4.9	5.0	5.1	5.2	5.3	5.4	5.5	7.0
Number of Students	4	2	7	7	7	10	2	2	1

11. *Reus*, the tallest *Mexican* is an *outlier* as you can easily tell looking at the table above. If he was to be removed from the sample, which of the following is least affected?

A. The *median* B. The *mode* C. The *mean*

D. The *range*

12. What is the difference between the *average* of all 42 students and the *average* when *Reus* is removed from the group?

A. 5.2 B. 5.1 C. 5.15 D. 2.2

The following four questions depend on the information provided below.

Idan Raichel is a musical act who had 7 number one songs in the charts in Israel. The rating network rates songs on a scale from 0 to 7. For 42 of his songs, *Idan Raichel*'s portfolio has a rating over time that can be modeled using the following equation where R represents the *average rating* he received and t represents the number of years starting from his 42^{nd} output.

$$R(t)=\begin{cases} 4.2+0.42(t+5) & 0\le t\le \frac{5}{3} \\ 7 & t>\frac{5}{3} \end{cases}$$

13. What is the meaning of the number $\frac{5}{3}$ in the context of the information provided above?

A. It tells us the *time* when *Idan Raichel* achieved his *minimum average* rating.
B. It tells us the *time* when *Idan Raichel's* music rating began.
C. It tells us the *time* when the *limiting value* of $R(t)$ is achieved.
D. It signifies the *time* when *Idan Raichel's* portfolio *average* rating is concluded.

14. If time $t=0$ (the time *Idan* completed his portfolio) was February 2007, on what month of what year does he achieve a perfect rating of 7 for his portfolio of songs (according to the model provided above)?

A. January 2009
B. February 2009
C. March 2009
D. October 2008

15. Looking back at the model provided above, what is the significance of the number 5?

A. The number five tells us that the *average* rating of *Idan's* songs starts five years after 2007

B. The number five tells us that the *average* rating of *Idan's* songs ends five years after 2007

C. The number five tells us that the *minimum* value of $R(t)$ is 6.3

D. The number five tells us that the *minimum* value of $R(t)$ is 5

16. Looking back at the model provided above, what is the significance of the number 0.42?

A. The *decrease* per year in the *average* rating of *Idan's* 42songs.

B. The *increase* over $\frac{5}{3}$ years in the *average* rating of *Idan's* 42 songs.

C. The *increase* per year in the *average* rating of *Idan's* 42 songs.

D. The *decrease* over $\frac{5}{3}$ years in the *average* rating of *Idan's* 42 songs.

17. The cost of a hotel rental for a week in *Addis Ababa*, Ethiopia is \$$42$ with an additional internet fee of x dollars per day. If there is a VAT (Ethiopian service tax) of y dollars per day for all services rendered in the hotel, which of the following expressions tells us the total cost of a stay at said hotel?

A. $\$xy+\$42$

B. $(xy+42)\$7$

C. $\$7(x+y)+\$42$

D. $(42+7x)\$+\$y$

18. The price of a *Mercedes G Wagon* depreciates according to the formula $G(t)$ whereas the price of a *Mercedes 300SL Gullwing* appreciates according to the formula $g(t)$ (both are given below.) If $t=0$ represents the year 2000, what year will the two *Mercedes* cars have the same value?

$$G(t)=\$142{,}000-\$4200t$$

$$g(t)=\$42{,}000+\$4200t$$

A. The year 2012

B. The year 2022

C. The year 2021

D. The year 2011

19. The middle of the world is where the *prime meridian* intersects with the *equator*. The earth, which is a *sphere* is divided into two

equal halves by the *prime meridian*, east and west. Earth is similarly divided into two equal halves by the *equator*, north and south. One of the two <u>middles of the earth</u> is in Ecuador. Assuming the earth is a perfect sphere, if *Reus* walks diagonally (at a $45°$ angle) starting from the <u>middle of the world</u> to a point halfway around the world, then heads north bound from that point to a point halfway around the world while *Luis* starts from the same point and makes one complete loop around the world, who traveled more?

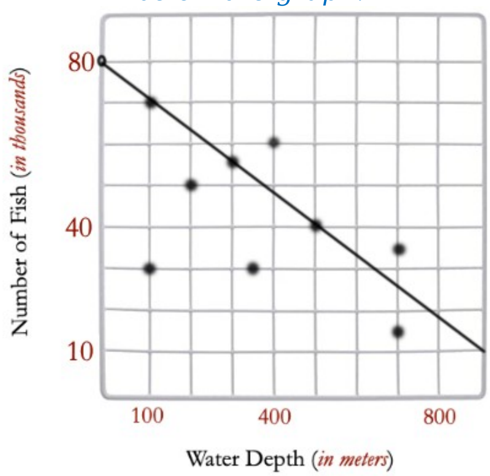
A. *Reus*

B. Luis

C. They traveled the *same* distance

D. None of the above

The next four questions use the graph provided below as well as the information below the graph.



The graph above shows a *Scatter Plot* along with a line fitted to it. The graph gives *water depth* on a lake called Lake vs *number of fish*. On the grid above, all points that appear as lattice points in fact are.

20. If the *line of best fit* was to be removed (only the *Scatter Plot* considered), then, which of the following most closely estimates the *correlation coefficient* for Water Depth vs Number of Fish?

A. $r=0.7$

B. $r=0.777$

C. $r=-0.5$

D. $r=-0.7$

21. *Joshua* used a calculator to learn that the *line of best* fit has the equation:

$$y=-75x+80{,}000$$

where x is the water depth in meters. What is the meaning of the *coefficient* on x ?

A. It tells us how many fish die per one unit change in water depth.
B. It tells us about the decline in the fish population per one *meter* change in water depth.
C. It tells us about the decline in the fish population per 100-*meter* change in water depth.
D. It tells us how many fish are found in the first *meter* of water depth.

22. *Luis* graphed (on the same *Scatter Plot* above) a line with equation $y=c$ where c is a constant and he found two points that contained this line. He labeled one of the two points x_1 and the other x_2. Which of the following is a possible difference between these two x values $(x_1-x_2$¿?

A. 350 B. 290 C. 40 D. -242

23. Using the equation of the *line of best fit* that *Joshua* came up with,

$$y=-75x+80{,}000$$

approximately how many fish can be found at a water depth of 1000 meters?

A. 500 B. 50,000 C. 5,000 D. 500,000

24. As of January 2022, of 13.1 million millionaires, 56% are *Christians*. Additionally, more than half of the billionaires in the world are *Christians*. If there are 2,755 billionaires, then which of the following could be the total number of *Christians* who are either billionaire or millionaires?

25. The number n is the 150% of the *positive difference* between two numbers x and y with $x<y$. If $x-y=-42$, what is the value of $x+y+n$ given $x=7$?

A. 112 B. -112 C. -98 D. 119

26. Which of the following situations cannot be modeled using a *linear relationship*?

A. A bank account that pays a monthly interest of 0.777 percent on an initial deposit of $\$200$
B. A piggy bank where Sammy's mom first deposited $\$42$ and $\$7$ every month thereafter
C. A telephone service provider that charges a fixed fee of $\$82$ per month
D. A taxi meter that displays $\$4.20$ cents as soon as you enter and adds to that balance, $\$0.777$ per quarter mile that you travel in the taxi.

27. The number $x=5n+4$ where n is an integer. But it is also true that $x=7m$ where m is an integer (notice this latter fact means x is a multiple of 7.) If the first three positive values of x are a, b, and c, with $a<b<c$and $c-b=b-a$, what is $c-a$?

A. 21 B. 35 C. 70 D. 77

28. The *volume* of a *cylinder* is $\pi r^2 h$. If the height of the *cylinder* is increased by 50%, how must the *base radius* of the *cylinder* change so that the *volume* remains the same?

A. An *increase* of 18.3%
B. An *increase* of 25%
C. A *decrease* of 25%
D. A *decrease* of 18.3%

29. A bacteria colony had an initial population of p bacteria. If this population doubles every two years, which of the following equations can be used to find the bacteria population t years from the initial time?

A. $p(2)^t$ B. $p(2)^{\frac{t}{2}}$ C. $p(2)^{2t}$ D. $p(2t)$

30. At store *Reus*, notebooks cost $\$4$ and pens cost $\$2$. If *Luis* has a budget of between $\$22$ and $\$42$ and he must buy at least 7 pens, which of the following cannot be the number of notebooks he buys?

A. 4 B. 2 C. 8 D. 0

31. *Mr. Barack* gives grades in his math class so that a grade of A is given to any student who scores between a 95 and 100. If *Obama*, a student in *Mr. Barack's* class has an *average* of 93 on 2 math tests, which of the following could be his score on his final exam to get an A *average* from *Mr. Barack*? (Assume fairly that the tests and the exams have the same weight.)

I. 97
II. 98
III. 99
IV. 100

A. I only B. I and II C. III only D. III and IV

32. In the $xy-¿$plane, the point (p,q) lies in the solution set of the *system of inequalities* below. If we considered the *maximum value* of p and the *minimum value* of q, which answer choice below gives us their difference?

$y \geq 10x - 142$

$y \leq -5x + 38$

A. 54 B. 154 C. -154 D. 42

Use the information below to solve the following two equations.

x	$g(x)$
4	0
2	2
-4	-2

5	0

33. Given the table of values above for the *polynomial* function $g(x)$, which of the following conclusions can be made.

A. $g(0)=4$
B. $x-5$ is a *factor* of $g(x)$
C. $g(-4)=2$
D. None of the above

34. Once again, referring to the table given above, which of the following conclusions can be made?

A. The *remainder* when $g(x)$ is divided by 2 is 2
B. The *remainder* when $g(x)$ is divided by $x+2$ is 2
C. The *remainder* when $g(x)$ is divided by $x-2$ is 2
D. None of the above

35. *Meri* wrote the following *proportion* on a piece of paper:

$$\frac{x-y}{c}=\frac{k}{u+v}$$

Of the following, which is NOT equivalent to the equation *Meri* wrote?

A. $\frac{c}{(x-y)}=\frac{(u+v)}{k}$

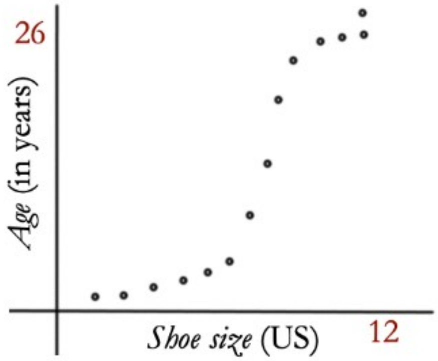
B. $\frac{x}{c}-\frac{y}{c}=\frac{k}{(u+v)}$

C. $(x-y)(u+v)=kc$

D. $\frac{x}{c}-\frac{y}{c}=\frac{k}{u}+\frac{k}{v}$

36. There is no question here cause we don’t like you know what. We are about the 777s and Jesus (aka God aka the Holy Trinity☺)

37. Apparently, science concludes that a person grows (in height etc.) until the age of 26, although feet and nose continue to grow beyond the age of 26. The *Scatter Plot* below is an accurate graph for a sample of the US population. The graph displays age (*vertical axis*) vs shoesize (*horizontal axis*.)



If the line $y = x$ was plotted along with the *Scatter Plot* above, what is the approximate *ratio* of points that would lie above this line to below it?

A. $1:2$ B. $3:7$ C. $1:6$ D. $3:4$

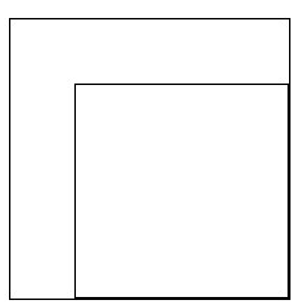
38. If $-\sqrt{x} < |x|$, which of the following conclusions can be made about all numbers x that satisfy this inequality?

A. $x > 0$
B. $x \geq 0$
C. $x < 0$
D. There is no x value that satisfies the given inequality

39. If $f(x) = x^2(4x - 2) + 42(4 - 8x) + h(x)$ and $h(x) = (8 - 16x)^2$, which of the following divides $f(x)$?

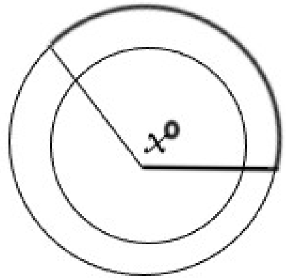
A. $4 - 8x$ B. $8 - 16x$ C. $4x - 2$ D. $2 - 4x$

40. In the figure below of two overlapping squares, the bigger square has a side length of 42. If the *ratio* of a side of the smaller square to the bigger square is $1:3$, what is the ratio of the areas?



A. $1:3$ B. $3:1$ C. $1:\sqrt{3}$ D. $1:9$

41. The figure below shows two *concentric circles* that share a *central angle* x. By definition, a *radian* measure of a *circle* is the same as if the *radius* of the very circle was made untaught and perfectly laid down on the *circumference* (along the circle's border) of the *circle*. With this definition, if the darkened arc on the bigger circle is equal to 2 *radians*, and the conversion $\pi = 3.14 = 180°$ *radians* can be counted on, what is the approximate measure of $x°$ given that the two *circles* have *radii* that are in a *ratio* of $1:2$?



A. 84 B. 120 C. 114 D. 142

42. To convert a *Fahrenheit degree* measure to a *Kelvin measure* for temperature, the following formula is used

$$K = \frac{5}{9}(F - 32) + 273$$

According to this formula, what temperature in *Fahrenheit* is equivalent to the freezing temperature of water in *Kelvin* (water Freezes at approximately 273.1 *Kelvin*)?

A. 273 B. 0 C. 32 D. $\frac{5}{9}$

Solutions to Practice Min Test 2

1. The correct answer is A. Cannot be determined from the information given
   To know how many Sophomores played sports, in this case, we would need to know how many Sophomores there are in total. We could find this amount if we knew some extra information, for example, the total number of students. Although we know the number of Freshmen as well as the total number of Juniors and Seniors, this information is not sufficient to we determine how many Sophomores played sports. We are feeling that this solution is clear enough but leaves begging for more. So, another way to think about it like a system of equations where you have too many variables and not enough equations.

2. The correct answer is C. *Germany*, *India*, and *France*.

We see that the *Pie Chart* is labeled in sorted order. This is helpful because what we are tasked with is basically balancing the Military Budget of 2 countries against the answer choices. The first thing to do is find the total of Military Budget of the *UK* and *Saudi*. Done correctly, you should find that this sum is 142.6 billion. Notice that we didn't need to compute the percentages of every country, since dividing by the total budget among all of them (this is what we would do for finding the percentage for each country) would not affect the position of the countries in the sorted order.

Having said this, the question is essentially a number sense question where a bit of rounding is helpful. There is no more science to this question. If you know that $3 \cdot 45$ is 135, simple multiplication, the countries at the bottom should attract your attention. With this guess in mind, *Germany* +¿ *India* +¿ *France* ¿$43.9+45.2+53.1=142.2$achieves the intended goal.

3. The correct answer is3.08 hours.

Ludwig will bike the first 20 *miles* of the trip at a rate of 16 *miles per hour*. This means that he will run the remaining 22 *miles*, at a rate of 12 miles per hour. To find the total duration of the trip, we need to find the time *Ludwig* takes on each of these two parts, and then add them. Since *Ludwig* bikes at a rate of 16 *miles per hour*, he takes $\frac{1}{16}$ *hours* to bike 1 *mile*. Thus, to bike 20 *miles*, it will take him $\frac{20}{16}=1.25$ *hours*. Since he runs at a rate of 12 *miles per hour*, it takes him $\frac{1}{12}$ *hours* to run 1 *mile*, which means that he takes $\frac{22}{12}=1.83$ *hours* to run 22 *miles*. Adding these two results up, we get a total time of $1.25+1.83=3.08$ *hours*.

Alternate solution: This is what is called a rate question. For rate questions, you should automatically write this *DIRT* formula abbreviated $D=R \cdot T$ where the D stands for Distance, the R stands for Rate and the T stands for Time.

Now, in every question, you will want to write as many of these equations as you have rates involved. For example, if there are two different rates, you want to write two Dirt formulas. Frankly, while this is a great methodical approach, the solution provided above is not only sufficient but more efficient.

Here, we are told a biking rate, and a running rate. Let's distinguish these two with R_1 and R_2 respectively. Then, it means: $R_1=16$ and $R_2=12$, now, correspondingly, we have D_1, D_2, and T_1, T_2. Now, as already explained in the first solution, $D_1=20$ and $D_2=22$. It is T_1 and T_2

that we are after. For each rate question, there is only one unknown, so we can write the following to DIRT equations and solve for the two times.

$$D_1 = R_1 \cdot T_1 \text{ and } D_2 = R_2 \cdot T_2$$

$$\rightarrow 20 = 16 \cdot T_1 \text{ and } 22 = 12 \cdot T_2$$

$$\rightarrow \frac{20}{16} = T_1 \text{ and } \frac{22}{12} = T_2$$

So, the total time is $T_1 + T_2 = \frac{20}{16} + \frac{22}{12}$

Feel free to reduce and turn into decimal to check that this is the same as the answer provided at the end of the first solution above. As a matter of fact, the two approaches are the same, one is more methodical and detailed which some students appreciate, that is the only difference.

4. The correct answer is 11.34 *miles*

    Since 42 *minutes* is less than an hour, and we know that *Ludwig* can bike the first 20 *miles* of a trip at a rate of 16 *miles per hour*, we know in an *hour* he can bike 16 *miles*. Therefore, in 42 minutes, he will travel a shorter distance. Which means that he will only be on his bike and will not need to run. Hence, we only need to find out how many *miles* he can bike in 42 *minutes*. Because he can bike 16 *miles* in 60 minutes (1 *hour*), he can bike $\frac{16}{60} = 0.27$ *miles* in 1 *minute*. As such, he will be able to bike $(42)(0.27) = 11.34$ *miles* in 42 *minutes*.

    Once again, we can solve this question using the *DIRT* formula once as we did above, we figure out that only a bike is needed. As the execution of the *DIRT* formula is much the same as the previous problem, we will consider it sufficient guidance to have provided you the *DIRT* formula just once as we did in the previous solution.

5. The correct answer is $\frac{1}{6}$.

    There are in total 7 cats under 7*lbs*. In total, we have 42 pets, so the probability that one of them is a cat under 7lbs is equal to $\frac{7}{42} = \frac{1}{6}$.

6. We skip six like the 13^{th} floor on most buildings. We are all about the 7s. So there is no question here and therefore no solution.

7. The correct answer is D. The total number of *Chemistry* students.

We will fill the following chart, and then use it to get the answer.

Group	*Amount*	*Percentage*
The total number of *Physics* students and the *Christian Math* students	$82+77=159$	$\frac{159}{342}=46.5\%$
The total number of Math students and the *Christian Physics* students	$122+27=149$	$\frac{149}{342}=43.5\%$
The total number of *Christian* students	142	$\frac{142}{342}=42.5\%$
The total number of *Chemistry* students	138	$\frac{138}{342}=40.4\%$

We see that the percentage of *Chemistry* students is 40.4%, which is the closest to the approximate percentage we were given.

8. The correct answer is the amount of dollars the shoe gains after four years.

We see that in the equation, the number of years is being divided by 4. This means that when t increases by 1, the value of $\frac{t}{4}$ increases by $\frac{1}{4}$. This also means that when t increases by 4, $\frac{t}{4}$ increases by 1. Thus, every 4 years we get an extra $\$42$ in the price of the sneaker, which is what the answer choice B tells us.

9. The correct answer is 31 years and 5 months.

We need to set the price equal to 4 times the original price (that is $4\cdot\$110$) and solve for the corresponding value of t.

$$p=440=42\left(\frac{t}{4}\right)+110$$

$$\rightarrow 330=\frac{42t}{4} \rightarrow t=\frac{(4)(330)}{42}=31.43$$

Now, we just need to convert 0.43 into months. We can do so by multiplying by 12, obtaining $(0.43)(12)=5.04$ months.

10. The correct answer is C because the function is a *quartic* or of higher degree. This is because we know by the *Fundamental Theorem of Algebra* that "an nth degree polynomial has at most n *zeroes* and $n-1$ turning points (local *maxima* and *minima*.)

Now, from the graph, the function has three roots, $x=0$, and two other roots symmetric about the $y-¿$axis with one between $x=1$ and $x=2$ the other between the respective *additive inverses*.

As it is not clear where these two *roots* lie, a quick glance at the answer choices gives us some ideas. Three of the answer choices have the *factor* x^2-2. The answer choice that doesn't have this *factor* is not a *quartic*, it is a *cubic* so we can eliminate it because a *cubic* can only have at most 2 turning points and our graph has 3. So, we conclude that we should write the following to locate the two other *roots* that we are after.

$$x^2-2=0 \rightarrow x^2=2$$
$$\rightarrow x=\pm\sqrt{2}$$

Now, as may have been already mentioned, the following three numbers are very useful to memorize. Their approximate value anyway.

$$\sqrt{2}\approx 1.41 \rightarrow \sqrt{3}\approx 1.73$$

$$\sqrt{5}\approx 2.23$$

So, we see that the two other *roots* are $x\approx\pm 1.41$ Now then, we see that these are the two other $x-¿$*intercepts* (another name for *roots*) we were seeking along with $x=0$. $x=0$ is a repeated *root* because the function doesn't cut through the graph at $x=0$, it bounces on the $x-¿$axis. This means $x=0$ is a double *root*.

11. The correct answer is the B. *mode*.

We see that the most common number (entry) is 5.3. After removing the only occurrence of 7, 5.3 will still be the most common element, which means that the mode will not be affected at all. Notice that it was not necessary to compute all the metrics involved. Usually, *outliers* pull the *mean* (*average*) and clearly the range, and the *median* and *mode* are the least affected.

12. The correct answer is 0.05.

The *average* of all 42 students is equal to the following expression:

$$av_1=\frac{4.8\cdot 4+4.9\cdot 2+5.0\cdot 7+5.1\cdot 7+5.2\cdot 7+5.3\cdot 10+5.4\cdot 2+5.5\cdot 2+7\cdot 1}{42}=\frac{217.9}{42}=5.19$$

Which is equivalent to the sum of the heights divided by the number of numbers. After removing *Reus*, we will have 41 students instead of 42, so the expression will be the same, but instead of dividing by 42, we will divide by 41, and in the numerator we will no longer include the last 7 which corresponds to *Reus's* height.

$$av_2=\frac{4.8\cdot 4+4.9\cdot 2+5.0\cdot 7+5.1\cdot 7+5.2\cdot 7+5.3\cdot 10+5.4\cdot 2+5.5\cdot 2}{41}=\frac{210.9}{41}=5.14$$

The difference between these averages is *approximately* (round of error) equal to $5.19-5.14=0.05$.

Albeit obvious to some, the additional tip here is that to notice that before we divide by 42 and 41 respectively, the total for each of the two *averages* is given by the numerators of the two quotients above. The only difference between the totals is 7, so don't do too much work. Save the total from the first average and just subtract 7 when you need the total for the second average calculation.

13. The correct answer is C. It tells us the time when the *limiting value* of $R(t)$ is achieved

We are told that the system only assigns values from 0 to 7. After $\frac{5}{3}$ years have passed, the values of $R(t)$ becomes more than 7. This would not make sense in the rating system, as it (7) is the maximum possible rating *Idan* can achieve.

14. The correct answer is D. October, 2008

We know that the cutoff for the value of t is $\frac{5}{3}$ as more than this number of years would have $R(t)$ exceed the *maximal* possible rating (note that we have already said this in the solution to problem 13), to find the month and the year, we need to convert $\frac{5}{3}$ years to months and add it on to February 2007 because Feb, 2007 is time zero in this case (based on what we are given in this question.) To do the conversion, we simply multiplying $\frac{5}{3}$ years by 12 (as there are, you should know if you live on earth, 12 months in a year.) Doing so, we write that $\frac{(12)(5)}{3}=20$ months. Adding 12 months onto Feb 2007 would have us land in October 2008.

15. The correct answer is C. The number five tells us that the *minimum* value of $R(t)$ is 6.3

The *minimum* possible value of t is 0. R(t) is an always increasing function. This means that the smaller the value of t, the smaller the *average* rating will be. Since we have the number 5 acompaining t, everyting will be "shifted" by 5, which means that even though the minimum value of t is 0, when we input $t=0$, the 0.42 coefficient won't disappeared. The number 5 ensures that the minimum value of the *average* rating is bigger than if it weren't there. So, if we plug in $t=0$, we will get a value of 6.3, which is as small as we can make $R(t)$.

16. The correct answer is C. The *increase* per year in the *average* rating of *Idan's* 42 songs.

    We observe that the number 0.42 is the multiplying the term which contains the variable t. This means that when t increases by 1, the average rating will increase by 0.42. If this is not a satisfying example, there is a discussion on the meaning of the slope of a linear equation much earlier in the book and so, you should feel free to consult that section. Although the function provided here doesn't appear to be in one of the three forms of the equation of a line, upon distributing the 0.42, you will come to realize that it is in fact a linear equation.

17. The correct answer is C. $\$7(x+y)+\$42$

    For a week, we will spend $\$42$ only renting the hotel. This means that we need to include a fixed cost of $\$42$ in our expression. We are told that for every service, there is a VAT of y dollars per day. Since the internet fee is daily, we will have to pay $\$7x$ to have internet for a week, and similarly, we will have to pay $\$7y$ for the VAT. Thus, the total the service fee including VAT will be $\$7x+\$7y=\$7(x+y)$. Adding the fixed fee of $\$42$to this total service charge, we get $\$42+\$7(x+y)$.

18. The correct answer is A. The year 2012.

    To find the year t when the two Mercedes cars will have the same value, we will need to set the two given equations equal to each other's.

$$G(t)=g(t)$$

$$\$142{,}000-\$4200t=\$42{,}000+\$4200t$$

$$\rightarrow 142{,}000-42{,}000=4200t+4200t$$

$$\rightarrow 100{,}000=8400t$$

$$\rightarrow t=\frac{100{,}000}{8400}=11.9$$

This means that the cars will have the same value in the year 2011.9. This is in the middle of the 11^{th} year and since it said after how many years, 11 years are not adequate, as such, we need to round to the next year.

19. The correct answer is C. They traveled the same distance.

Since we assume that the Earth is a perfect sphere, every straight path (with straight meaning that it maintains its trajectory while surrounding the Earth) from one point to a point at the opposite side of the *sphere* will have the same distance, no matter what the angle is. This is so because if we change the angle, we can "rotate" the sphere by that angle, so that relatively to us, it is now at an angle of $0°$. Since *Reus* walked two halves of the *circumference* Earth, and *Luis* walked one complete *circumference* of the Earth, they traveled the same distance. Another way to say this is that two halfway around the earths equal a full way around the earth because they both make one *great circle* on a *sphere*. On a *sphere*, all *great circles* are equal.

20. The correct answer is D. -0.7

The first thing we should do is to determine the sign of the *correlation coefficient*. Since we see that when the depth increases, most of the time the number of fish decreases, therefore we have a *negative correlation*. This discards two answer choices. Now, to get the final answer, we should estimate how strong the relation is between the two variables. Looking at the plot, we observe that the points tend to go down in a similar direction, indicating that the variables are close to being *highly correlated*. This means that we should pick the larger value between the two possible options.

21. The correct answer is C.

It tells us about the decline in the fish population per one meter change in water depth. The coefficient of the x term in the equation represents the slope of the line. This representees the rate of change, and, based on the current values, means that when x increases by 1, y must decrease by 75. In other words, the -75 represents the decline in the fish population per one meter change in water depth, which is what the answer choice C sustains.

22. The correct answer is D. -242

A line with equation $y=c$, where c is a constant is a horizontal line. Looking at the plot, we see that there are only two points lying on the same y-value. One of them has an $x-$¿ coordinate of $100$and the other one about an $x-$¿coordinate of 350 (in these kinds of questions, it is very important to pay attention to the demarcation along the x and y axes.) Because the question doesn't demand that either be x_1, we need to consider both $100-350$ and $350-100$. Either way, the absolute difference of between these $x-$¿ coordinates is $350-100=250$.

Looking at the answer choices, the only reasonable answer is -242 because we are picking from either $100-350=-250$ or $350-100=250$. The small differential between our calculation and the best answer choices stems only from the approximation involved in the latter of the two $x-$¿ coordinates.

23. The correct answer is C. $5{,}000$.

This is very straight forward, plugging in the value of $x=1{,}000$ in the equation, we have:

$$y=-75(1000)+80{,}000 \rightarrow y=5{,}000$$

24. The correct answer is

Since we are told that of 13.1 million millionaires, 56% are *Christians*, the number of *Christians* that are millionaires is $13.1\cdot 56\%=7.336$ million. Since we are told that more than half of the billionaires in the world are *Christians* and we are told that there are $2{,}755$ billionaires, then the number of *Christians* who are billionaires is $\geq\frac{2{,}755}{2}=1377.5$ Adding those number up, we get that the total number of *Christians* who are either billionaires or millionaires is greater than or equal to $7{,}336{,}000+1377.5=7337377.5$. Any number smaller than this can't be the number of *Christians* who are either millionaires or billionaires.

25. The correct answer is D. 119.

Since we are told that $x=7$, and we are given $x-y=-42$, we have $7-y=-42$.

$$\rightarrow 7+42=y$$

$$\rightarrow y=49$$

Now we need to get to n. The *positive difference* of x and y is equal to $49-7=42$ (notice that we were also given the value of the difference between x and y; the absolute value of that difference is the positive difference. Alternatively, since we know $x<y$, we know the positive difference must be $y-x$ because y is bigger than x.)

Finally, because n is 150% of this *positive difference* we just realized and we know from the section on percentages that $150\%=1.5$ times, The value of n will be $1.5\cdot 42=63$. Putting all of this together, $x+y+n=7+49+63=119$.

26. The correct answer is a bank account. This is so because when you deposit money into a bank you are paid compounded interest. This means the model would have to be exponential.

In a piggy bank, you are the one who will put in a certain amount of money periodically. As such, you are not paying yourself interest. In fact, a simple linear equation of the form $y=mx+b$ would do the job. For example, in answer choice B, the model would be $y=\$7x+\$42$.

A Taxi fare works the same way as the piggy bank. You will incur a fee of something like $\$3.50$ as soon as you sit in the cab and then you are charged a fixed amount per mile you travel.

A telephone bill also works similarly as the last two models and thus, a linear model would suffice.

If you need another solution here it is! Once again, the correct answer is A. A bank account that pays a compounded monthly interest of 0.777 percent on an initial deposit of $\$200$

This is so because the situation we have is compound interest which requires that we use the following formula.

$A=P(1+r)^t$

Where here t is the number of months after the initial deposit. You should know from lessons in this book in the appropriate section that $P=\$200$ and r is the decimal equivalent to the percentage of interest paid. But the short end of the story is that this is an *exponential equation*. So, it cannot be modeled by a line in the form $y=mx+b$.

27. Even though the first piece of information from the question told us (perhaps not to give too much away) only that x is a positive number, because we should first note that x is an integer because both m and n are integers and they are both multiplied and added to or just multiplied by integers (and we all know the integers are closed both under addition and multiplication, that is the sum and product of two integers stays an integer.)

At least as important is to see that x is both 4 more than a multiple of 5 $x=5n+4$ and a multiple of 7 ($x=7m$.)

So the name of the game is to find the numbers x that are both of these things. We can do this by just trying out some values of n and m and eventually observing some patterns and or using the additional information given to the end of efficiency.

Her is a table!

n	4 more than a multiple of 5 $x=5n+4$	Is it a multiple of 7? Yes or No
1	$x=5(1)+4=9$	No
2	$x=5(2)+4=14$	Yes
3	$x=5(3)+4=19$	No
4	$x=5(4)+4=24$	No

5	$x=5(5)+4=29$	No
6	$x=5(6)+4=34$	No
7	$x=5(7)+4=39$	No
8	$x=5(8)+4=44$	No
9	$x=5(9)+4=49$	Yes

Now on the rest of the task, we can use the fact that a b and c are ordered as we are told that $a<b<c$ and equally importantly, we know that $c-b=b-a$. The fact that they are ordered is telling us that $c-b$ and $b-a$ are both positive integers. But the fact that these two quantities are equal is telling us that $b-a=49-14=35$. With this then, we know that $c-b=35 \rightarrow c-49=35$

$\rightarrow c=35+49=84$

So, without continuing the table, we have all three of the numbers we are after. Therefore, $c-a=2(35)=70$.

28. The correct answer is D. A decrease of 18.3%

Here is a never fail approach for these kinds of problems. So, we need the new volume to remain $\pi r^2 h=V_n$ where V_n denotes the new volume. Now, we know the new height is $1.5h$, a 50% increase on the height h the original cylinder had. So, $V_n=\pi r_n^2(1.5h)$ but again, we need this to remain $16\pi h$.

Even though it looks cumbersome, r_n^2 is not that confusing. The $n$is because it is the new radius that we need to relate to the old radius r and it is squared because the volume of a cylinder is $\pi r^2 h$ and so the radius is always squared when writing the volume formula of a cylinder. With all of this, we have the following equation in which we want to isolate r_n^2.

$$\rightarrow \pi r^2 h=\pi r_n^2(1.5h)$$

$$\rightarrow r^2=r_n^2(1.5)$$

Clearly, we have used the cancellation law from the first step to the second in what we just wrote above.

$$\rightarrow \frac{r^2}{1.5}=r_n^2$$

$$\rightarrow r_n = \sqrt{\frac{r^2}{1.5}} = \frac{r}{\sqrt{1.5}} = 0.8165r$$

$$\rightarrow r_n = 0.8165r$$

What this last equation is telling us is that the new radius should be 81.65 percent of the original radius r, or in other words, a $1-0.816492=.183503$ or 18.3 percent decrease.

29. The correct answer is B. $p(2)^{\frac{t}{2}}$

Initially there are p bacteria, and they double every two years. This means that every two years the population doubles (multiplied by powers of 2¿ but since it is after 2 years that this happens, every one year the population grows by $2^{\frac{1}{2}}$. Hence, after t years, the population will be $p\left((2)^{\frac{1}{2}}\right)^t = p(2)^{\frac{t}{2}}$. Another way to think about this is that doubling means times 2^{α}. Clearly, because we start with a population p, we figure we need something like $p2^{\alpha}$ such that when $\alpha=1$after two years. If $\alpha=\frac{t}{2}$,notice that we would have $\alpha=\frac{2}{2}=1$. Therefore, we now see that we need it to read $p2^{\frac{t}{2}}$.

30. The correct answer is C.8.

Luis must buy at least 7 pens. Since every pen cost \$2, Luis will spend at least $2\cdot 7=14$ dollars on pens. Now, we will use an inequality to model his budget. We know that his budget is between \$22 and \$42. We will denote his budget as b.

$$22 \leq b \leq 42$$

Since we must spend at least $\$14$ on pens, the bounds on the budget will be reduced. (Since we are being asked about the number of notebooks, he may be able to buy, we will assume that he spends as little as possible on pens).

$$22-14 \leq b \leq 42-14$$

$$8 \leq b \leq 28$$

Since every notebook cost \$4, we will divide these numbers by 4, to get the number of notebooks we can get. We will denote the number of notebooks by n.

$$\rightarrow \frac{8}{4} \leq n \leq \frac{28}{4} \rightarrow 2 \leq n \leq 7$$

Thus, we can buy at most 7 notebooks, which means that the correct answer is C. 8.

31. The correct answer is D. III and IV

We will find the *lower bound* for the grade in the final exam using an inequality. Initially, Obama's average of two math tests is 93, which can be obtained in many ways, but one of them (any will work) is $\frac{93+93}{2}$. After the final exam, the average will be $\frac{93+93+x}{3}$, where x is the grade Obama got. Since Obama wants to get an A, his *average* on all three exams (tests) must be greater than or equal to 95. This information can be modeled by the following inequality, which we will use to solve for x.

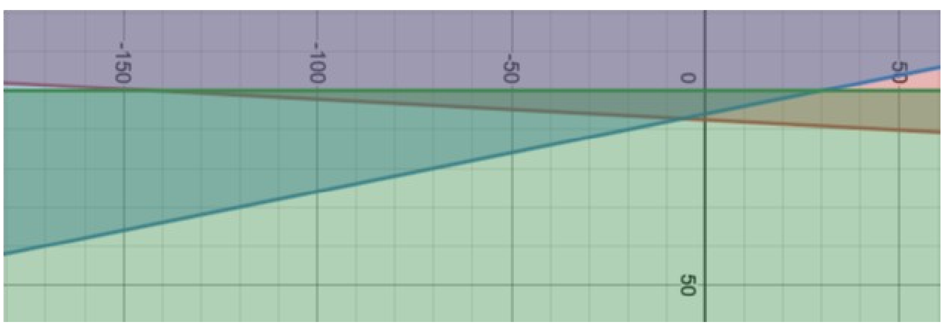
$$\frac{93+93+x}{3} \geq 95$$

$$186+x \geq 285$$

$$x \geq 285-186=99$$

Obama's grade in the final exam must be greater than or equal to 99. This means that he can either get a 99 or a 100, which is III and IV, which is answer choice D.

32. The correct answer is B. 154



If you graphed the *inequalities* correctly and you twist your book sideways, the triangular region with verities at $(0,-142)$, $(12,-22)$ and $(0,38)$ is the *solution set*. Since we are not dealing with strict inequalities, the edges of this triangle are included in this *solution set*. The only laborious work is done in figuring out where how to get to $(12,-22)$.

This is the intersection of the lines with equation $y=10x-142$ and $y=38-5x$. Now, as such, since we want the maximal value of p and minimal value of q, we must look as far to the right along the $x-$¿axis to find the desired value of $p$and as far down as we can along the $y-$¿axis to find the required q. This means $p=12$ and $q=-142$. So, the *positive difference* between these two numbers is $12-(-142)=154$.

33. The correct answer is B.

We see that $x=5$ is a *root* of the polynomial $g(x)$, because for $x=5 \rightarrow g(5)=0$. If a is *a root* of a polynomial, then $x-a$ is a factor of the polynomial (this is called the Factor Theorem.) Here $a=5$, which means that $x-5$ is a factor if $g(x)$.

34. The correct answer is C. The remainder when $g(x)$ is divided by $x-2$ is 2.

Using the *Remainder theorem*, we know that if a polynomial $P(x)$ is divided by $x-a$, the remainder is $P(a)$. Looking at the answer choices and the table, we can quickly eliminate the incorrect answers, or find the correct one.

In answer choice A, we do not see the form of $x-a$ so we quickly rule it out.

The answer choice B does have the form we are looking for. It states that the *reminder* when $g(x)$ is divided by $x+2=x-(-2)$ is 2. Here, $a=-2$, but looking at the table, we see that the value of $g(-2)$ is not given to us, so we cannot make this conclusion.

Answer choice C also has the form we desire, and it states that the *remainder* when $g(x)$ is divided by $x-2$ is 2. Looking at the table, we see that $g(2)=2$, so eureka!

As the *Remainder theorem* has neat mechanics, for the curious, here is a short proof:

It is a fact that *Dividend* ¿*Divisor* · *Quotient* + *Remainder*

In this case, the *dividend* is $P(x)$, and the *divisor* is $x-a$. We will denote the *quotient* as $q(x)$, and the *remainder* as r. The remainder is not r(x), it must be a constant because the *divisor* is a linear and so the *remainder* must be a degree less (a constant.) Otherwise, it means the *remainder* offers us more division. With all this, we have:

$$P(x)=(x-a)q(x)+r$$

At this stage, notice that if we evaluate $P(x)$ at $x=a$, we must write:

$$P(a)=(a-a)q(a)+r$$

$$\rightarrow P(a)=0\cdot q(a)+r \rightarrow P(a)=r$$

35. The correct answer is D.

If you review the appropriate section (in this book) on *proportions* and you can efficiently solve this problem. Answer choice B shows that answer choice D is the correct answer because, in general:

$$\frac{a}{b+c}\neq\frac{a}{b}+\frac{a}{c}$$

If you need more details, here is another approach:

To solve this kind of problems is akin to solving for a variable in each equation and see if the result is the same. In each instance, we will solve for k.

$$\frac{x-y}{c}=\frac{k}{u+v}$$

In the original equation (written just above), we can multiply both sides by $u+v$, to write:

$$\frac{(x-y)(u+v)}{c}=k$$

Now, we will look at the first option:

$$\frac{c}{(x-y)}=\frac{(u+v)}{k}$$

Inverting both sides, we have:

$$\frac{(x-y)}{c}=\frac{k}{(u+v)} \rightarrow \frac{(x-y)(u+v)}{c}=k$$

Continuing to *option B*:

$$\frac{x}{c}-\frac{y}{c}=\frac{k}{(u+v)} \rightarrow \frac{(x-y)}{c}=\frac{k}{(u+v)}$$

$$\rightarrow \frac{(x-y)(u+v)}{c}=k$$

Option C:

$$(x-y)(u+v)=kc$$

Dividing both sides by c, we have

$$\frac{(x-y)(u+v)}{c}=k$$

Finally, we know that answer choice D must be the odd one out.

$$\frac{x}{c}-\frac{y}{c}=\frac{k}{u}+\frac{k}{v}$$

$$\rightarrow \frac{(x-y)}{c}=\frac{kv+ku}{uv}=\frac{k(v+u)}{uv}$$

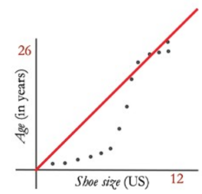
$$\rightarrow \frac{(x-y)uv}{c(u+v)}=k$$

We see that the other side of k here is not the same as the others!

36. There is no question and therefore no solution. We are about the 777s and Jesus (aka the Holy Trinity, aka God.)

37. The correct answer is C. $1:6$

We should remember that the line $y=x$ is a line which forms $45°$ with the x-axis and has a positive slope. With this in mind, we can draw n imaginary line $y=x$ (you can also use something straight to represent it, like a pencil or a ruler), and you will notice that out of the total of 14 points, about only 2 lie above the line and thus 12 lie below it. This is a ratio of $2:12$, which reduces to $1:6$.



38. The correct answer is A. $x>0$.

First, notice that since we are dealing with a square root, <u>we must be sure the number inside the square root is non-negative</u>, as if it is not, we are necessarily working with *c*omplex (*i*maginary) numbers. From this observation, we are automatically forced to consider only x values such that $x \geq 0$.

Now, notice that since $x \geq 0$, $-\sqrt{x} \leq 0$. This is so because <u>the result of the square root is always positive (non-negative)</u>, but when $x=0$, we see that the condition is not met, as $-\sqrt{0}=|0|=0$. We get equality not the strictly less than that comes with the problem. Hence, we discard all values of x which are less than or equal to 0. This leads to the conclusion that $x>0$. For any positive value of x, the condition will be met. Since the absolute value will return a positive number and the square root is always positive. But because there is a negative sign in front of the square root, the positive number returned by the square root is then made negative

39. All answers are correct.

$$f(x)=x^2(4x-2)+42(4-8x)+h(x) \text{ and } h(x)=(8-16x)^2$$

If a polynomial leaves a remainder of 0 when divided by another polynomial, this second polynomial must be a factor of the first one. Hence, we will try to find which of the answer choice is a factor of $f(x)$. An easier and faster way than to factor $f(x)$, is to simply try the answer choices since we only have 4. If a polynomial has a factor $(ax-b)$, then the value of x when this factor is set to 0 must be one of the roots of the polynomial. In other words, the polynomial evaluated at this value of x must be 0. Using this fact, we will start trying the answer choices.

Option A:

We have the factor $4-8x$, and we will set it equal to 0 and solve for x:

$$4-8x=0 \to 4=8x \to x=\frac{1}{2}$$

Polar Pi

Now, we will find $f\left(\frac{1}{2}\right)$:

$$f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^2\left(4\left(\frac{1}{2}\right)-2\right)+42\left(4-8\left(\frac{1}{2}\right)\right)+\left(8-16\left(\frac{1}{2}\right)\right)^2$$
$$\rightarrow f\left(\frac{1}{2}\right)=\left(\frac{1}{4}\right)0+42(0)+(0)^2=0$$

This means that $x=\frac{1}{2}$ is a root of $f(x)$, and hence $x-\frac{1}{2}$ is a factor of $f(x)$. Notice that we can “scale” this factor by a number, and the remainder will still be $0*$¿, which means that $8\left(x-\frac{1}{2}\right)=8x-4$ divides $f(x)$. Any “scalation” of this factor divides $f(x)$, and since all the answer choices are “scalations” of this factor, all of them divide $f(x)$.

*Say $x-a=0$, then surely, $b(x-a)=0$ because $b(0)=0$.

40. The correct answer is A. $1:9$.

Knowing the length of a side of the bigger square is not the efficient way to answer this question. You should know (feel free to experiment) that if we have two similar polygons (two squares are clearly similar as one can be magnified to overlap on the other), and we know that the ratio of corresponding sides is a, the ratio of their areas must be a^2.

Now, since the ratio of the sides is $\frac{1}{3}$, it follows naturally from what we just said that the ratio of the areas must be

$$\left(\frac{1}{3}\right)^2=\frac{1}{3^2}=\frac{1}{9}.$$

41. The correct answer is C.

It is for the ambitious or intellectually curious that the definition of a radian is precisely given in the question. Otherwise, it has nothing to do with answering the question posed. In fact, we can get to the correct solution either by setting up a proportion or the following set of equations which are immediate upon the proportion we set up. Since $x°$ must correspond to measure 2 *radians*, all we must do is to convert 2 *radians* to *degrees*. Using the proposed relation that is given, we write:

$$3.14\,radians\text{ ¿}180°$$

$$\rightarrow 2\cdot 3.14\,radians\text{¿}(2)(180°)$$

$$\rightarrow 2\,radians\text{¿}\frac{(2)180}{3.14}\approx 114°$$

42. The correct answer is C.

We were given the following conversion equation which goes in the direction *Kelvin* → *Fahrenheit*:

$$K=\frac{5}{9}(F-32)+273$$

To turn it into the direction *Fahrenheit* →*Kelvin*, which is one way to solve this problem, we can first solve for F as follows:

$$K=\frac{5}{9}(F-32)+273$$

$$\rightarrow K-273=\frac{5}{9}(F-32)$$

$$\rightarrow \frac{9}{5}(K-273)=F-32$$

$$\rightarrow F=\frac{9}{5}(K-273)+32$$

Now, we can plug in the value $K=273.1$, and get the answer:

$$F=\frac{9}{5}(273.1-273)+32=\frac{9}{5}(0.1)+32=\frac{9}{50}+32=32.18$$

This means that the correct answer is C. (the value is a little bit bigger because of rounding to the first decimal place.)

Alternatively, from the get-go to the same conclusion (end), we could have solved for F after writing the following equation:

$$273.1=\frac{5}{9}(F-32)+273$$

Math Test – No Calculator

25 MINUTES, 20 QUESTIONS

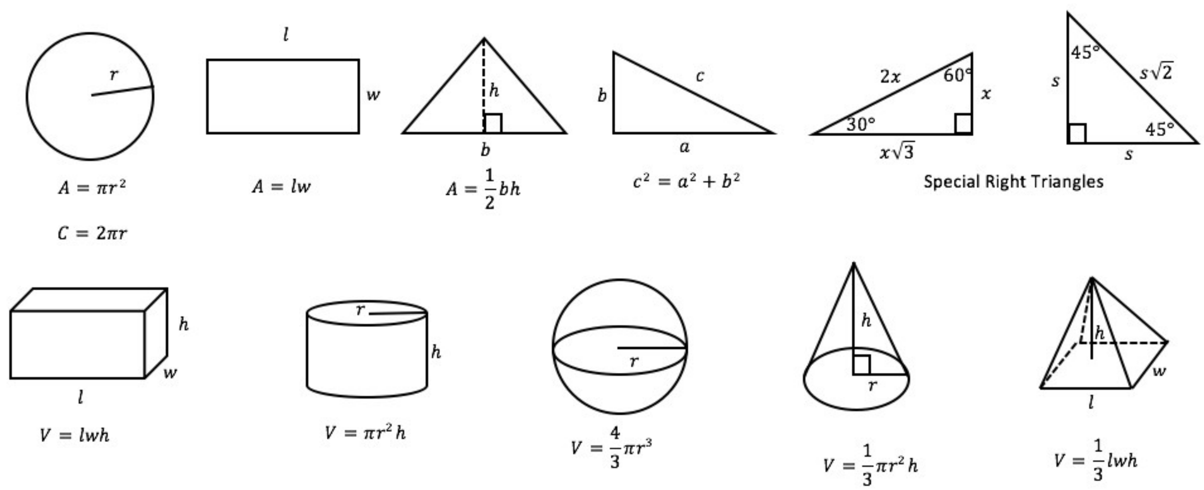
DIRECTIONS

For questions 1-30, solve each problem, choose the best answer from the choices provided, and fill in the corresponding circle in your answer sheet. **For questions 31-38,** solve the problem and enter your answer in the grid on the answer sheet. Please refer to the directions before question 31 on how to enter your answers in the grid. You may use any available space in your test booklet for scratch work.

NOTES

1. The use of a calculator is permitted.
2. All variables and expressions used represent real numbers unless otherwise indicated-
3. Figures provided in this test are drawn to scale unless otherwise indicated.
4. All figures lie in a plane unless otherwise indicated.
5. Unless otherwise indicated, the domain of a given function *f* is the set of all real numbers *x* for which *f(x)* is a real number.

REFERENCE



The *number of degrees* of arc in a circle is 360.
The *number of radians* of arc in a circle is 2π.
The *sum of the measures* in degrees of the angles of a triangle is 180.

1

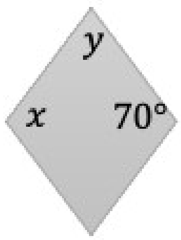
If $5a+2b=4$, then what is the value of $-10a-4b$?

A. Cannot be determined from the information given.
B. -16
C. 16
D.

2

D. -8

The figure below shows a *rhombus*, what is the value of $x+y=?$



A. 110
B. 70
C. 180
D. 170

$$ax^4-bx^3+2x-\left(4x^4-2x^3+2x\right)=0$$

In the equation above, a and b are constants with $b=4a$. If the only two *solutions* to the equation are $x=0$ and $x=7$, what is the value of a?

A. -10
B. 10
C. 30
D. -30

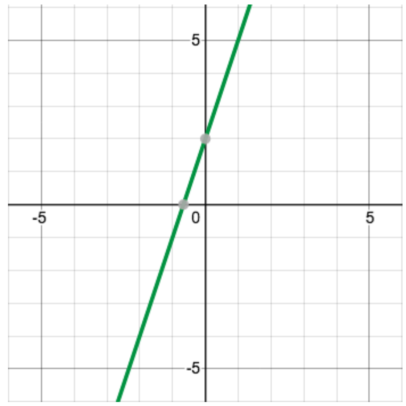
4

$$\frac{(x-2)^2}{4}+1=0$$

The equation above has two *solutions* of the form $2(a\pm bi)$ where a and b are real constants. What is the value of $a+b$?

A. 2
B. -2
C. 0
D. -1

5



Which of the following is an equation of the line graphed in the $xy-¿$plane above?

A. $x=2y+2$

6

B. $y=2x+3$
C. $y=3x+2$
D. $y=3x+1$

What is the *solution set* to the equation $-\sqrt{x+2}=x$?

A. $x=-1$ or $x=2$
B. $x=1$ or $x=-2$
C. $x=-1$ or $x=-2$
D. $x=-1$

7

What are the *solutions* to the *quadratic* equation $2x^2-10x+12=0$?

A. $x=6$ and $x=5$
B. $x=2$ and $x=3$
C. $x=-2$ and $x=-3$
D. $x=3$ and $x=-2$

8

If $u^2 - v^2 = x$ and $y = v$, then which of the following is equivalent to $25x + (4y)^2$?

A. $(5u - v)(u + 3v)$
B. $(5u - 3v)^2$
C. $(25u - 9v)(u + v)$
D. $(5u - 3v)(5u + 3v)$

9

n students paid t dollars for a lunch. If two students joined the lunch group and no contribution was required of them, what was the cost per meal for one student?

A. $\frac{t}{(n+2)}$
B. $\frac{(t+2)}{(n+2)}$
C. $\frac{(t+2)}{n}$

D.

10

D.

$2 + \frac{(t+2)}{n}$

$$\frac{1}{x^2 - 1} = \frac{a}{x - 1} + \frac{b}{x + 1}$$

where a and b are constants. Which of the following is the value of $a + b$?

A. 1
B. -1
C. 2
D. 0

11

$$\frac{2}{i} - \frac{4}{1+i}$$

which of the following is equivalent to the above difference of quotients, where $i = \sqrt{-1}$,?

A. $-4i - 2$
B. $4i - 2$
C. 0
D. -2

12

If the *system of equations* below has *infinitely many solutions*, what is the value of a?

$-x + ay = 5$

$2x + y = -10$

A. -1
B. -2
C. $\frac{-1}{2}$
D. $\frac{1}{2}$

13

If $(27^4)^a = 81$, what is the value of a?

A. $\frac{1}{2}$
B. 3
C. 2
D. $\frac{1}{3}$

A student incorrectly wrote the *quadratic formula* as follows:

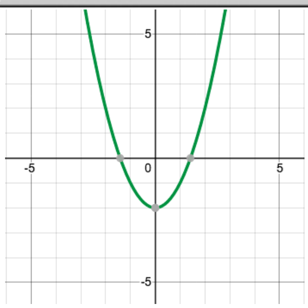
$x = -b + \sqrt{c^2 - 4a}$

Shortly after, this student solved for a in terms of the other constants and variables. Which of the following should be this student correctly solving for a in the equation above?

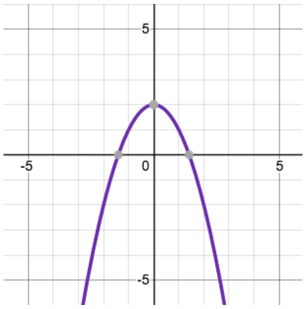
A. $\frac{c^2 - (x - b)^2}{4} = a$
B. $\frac{c^2 + (x + b)^2}{4} = a$

C. $\dfrac{c^2+(x-b)^2}{4}=a$

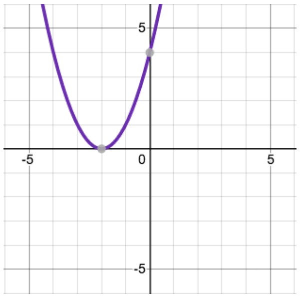
D. $\dfrac{c^2-(x+b)^2}{4}=a$



C.



D.

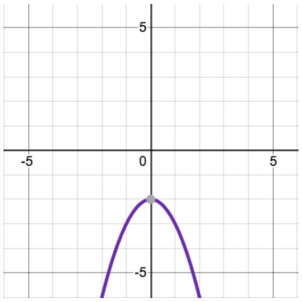


15

$$f(x)=x^2-2$$

The function f is defined above, which of the following is the graph of $-f(x)$?

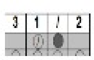
A.

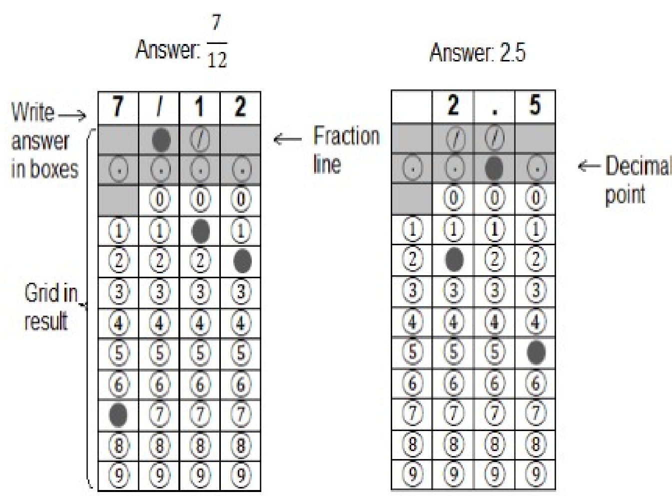


B.

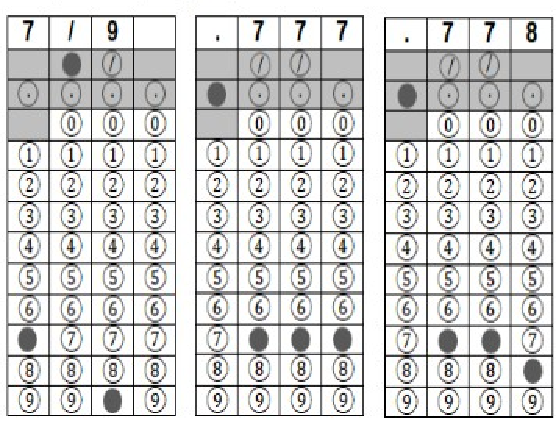
DIRECTIONS

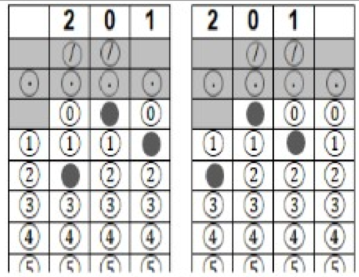
For questions 31-38, solve the problem and enter your answer in the grid, as described below, on the answer sheet.

1. Although not required, it is suggested that you write your answer in the boxes at the top of the columns to help you fill in the circles accurately. You will receive credit only if the circles are filled in correctly.
2. Mark no more than one circle in any column.
3. No question has a negative answer.
4. Some problems may have more than one correct answer. In such cases, grid only one answer.
5. **Mixed numbers** such as must be gridded as 3.5 or 7/2. If is  entered into the grid, it will be interpreted as , not .
6. **Decimal answers**: If you obtain a decimal answer with more digits than the grid can accommodate, it may be either rounded or truncated, but it must fill the entire grid.



Acceptable ways to grid $\frac{7}{9}$ are:





NOTE: You may start your answers in any column, space permitting. Colums you don't need to use should be left blank.

16

If $4x^2 = 5x + t$, what is one possible value of t so that the *quadratic* has no real *solutions*?

17

t	$a(t)$	$b(t)$
-2	4	2
-1	5	1
0	-4	-2
1	2	4
2	-2	-4

In the table above, if $2a(t) - b(t) = 0$, then what is the value of t?

18

If

$2(x-1)^2 + 5(x-1) + 3 = 0$, what is one possible value of x?

19

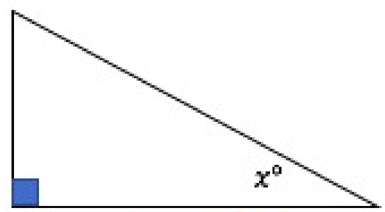
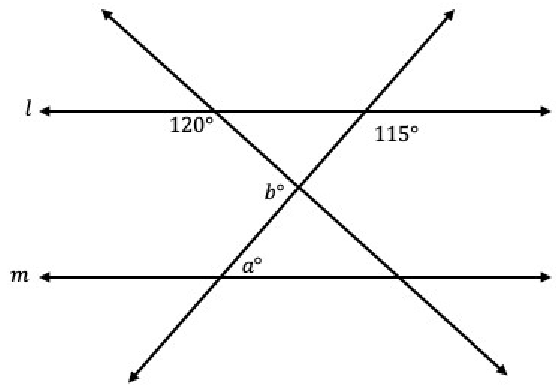


Figure not drawn to scale

In the figure above, if $\tan x° = \sqrt{3}$, what is the value of $\sin x°$?

20



In the figure above, lines *l* and *m* are *parallel*. As marked, given that two obtuse angles formed by line l and two *transversals* measure $120°$ and $115°$, what is the value of $a° + b°$?

Math Test – Calculator

55 MINUTES, 38 QUESTIONS

DIRECTIONS

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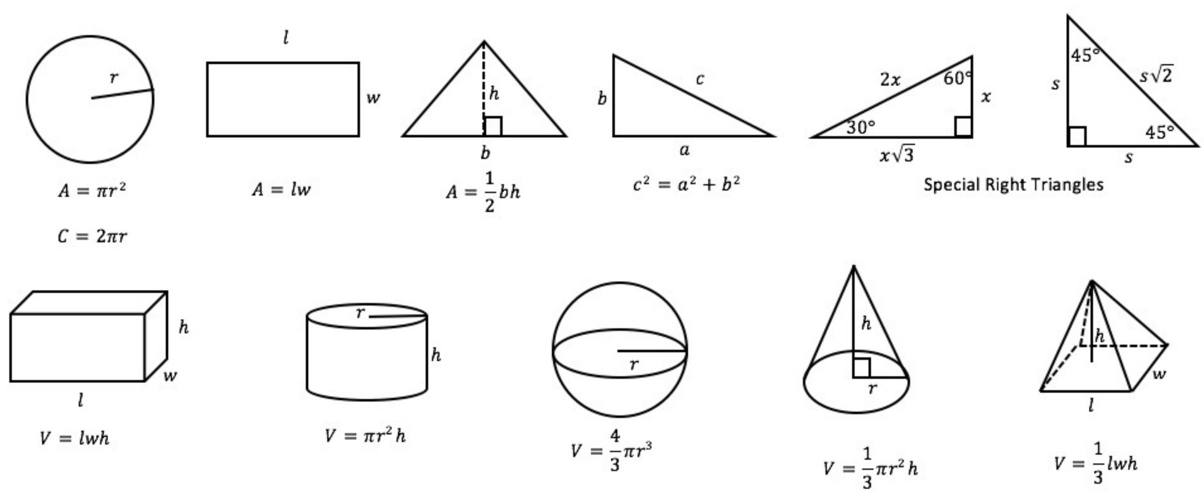


The *number of degrees* of arc in a circle is 360.
The *number of radians* of arc in a circle is 2π.
The *sum of the measures* in degrees of the angles of a triangle is 180.

1

	La Liga	*Prem*	
Christiano	25	55	
Messi	45	35	
	70		

The partially completed table above shows the preference of people for *La Liga* or the *Premier League* (*Prem*.) It also shows the same people's preference for *Christiano* or *Messi*. Of the total people in this survey, what is probability that a person who prefers the *Premier League* also prefers *Messi*?

A. $\frac{7}{32}$

B. $\frac{1}{2}$

C. $\frac{11}{18}$

2

$D.\frac{7}{18}$

Luis solved the *quadratic* equation $x^2-2x+5=0$. *Luis* correctly solved this *quadratic* and found the *difference* between the *two roots*. He could have found *the difference* in two different ways. Which of the following could be *the difference* between the two *solutions*?

A. $-1-2i$

B. $-1+2i$

C. $-4i$

D. -4

3

On his math exams, *Luis* scored between 50 and 70 in the first semester of his senior year. Which of the following equations covers the range of *Luis'* math scores (s)?

A. $|L-60|=10$

B. $|L-60|\leq 10$

4

C. $|L-60|\geq 10$

D. $|L+60|\leq 1$

$y=-2x+3$

$y=x+3$

If the system of equations above is graphed and (a,b) is the point of intersection, what is the value of $a-b$?

A. 0

B. 3

C. -3

D. 2

5

$y\leq 2x-5$

Which of the following is equivalent to the inequality above?

A. $2x+y\geq 5$

B. $2x+y\geq -5$

C. $2x-y\geq 5$

D. $2x-y\leq 5$

6

A cylinder of *height* h and *radius* r has a *volume* V. If the *volume* is to be kept the same and the *radius* is increased by 10%, how should the *height* of the *cylinder* change?

A. Lowered by 82.6%

B. Increased by 8.26%

C. Lowered by 1.73%

D. Increased by 17.4%

7

n	-1	0	1	2
$f(n)$	$\frac{7}{2}$	2	-1	-7

$$\frac{3(x-1)^2+5(x-1)+2}{x-1}=3x+a+\frac{2}{x-1}$$

In the equation above, if a is a constant, what is the value of a?

A. -3
B. 2
C. 5
D. -2

The table above shows selected values for a function $f(n)$. Which of the following defines f?

A. $f(n)=-3(2^n)-5$
B. $f(n)=2(-3)^n+5$
C. $f(n)=3(2^n)+5$
D. $f(n)=-3(2^n)+5$

9

Lumber Loggers Co. chops logs of wood in 17 *feet* pieces of all equal size. If a world record breaking $5100-$¿*yard* tree was to be chopped into these pieces, how many pieces could be counted at the end? (1 *yard* = 3 *feet*:)

A. 300
B. 600
C. 900
D. 1200

10

If $11x^2-8x+12-2(x^2-4x+7)=$¿ $(ax+b)(ax-b)$, what is the value of $a+b$?

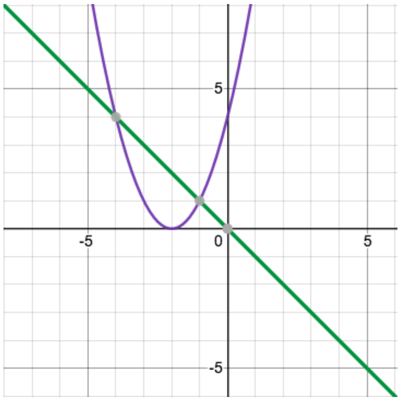
A) $3-\sqrt{2}$
B) 5
C) 1
D) $3+\sqrt{2}$

11

If the *circle* with equation $x^2+y^2=4$ and the line with equation $y=x-1$ intersect at $x=a$ and $x=b$. What is the value of $a+b$?

A. $\frac{4+2\sqrt{30}}{2}$
B. -4
C. -2
D. 2

12



The graphs above are that of $f(x)=(x+2)^2$ and $g(x)=-x$. *Einstein* claimed $x=a$ and $x=b$ are the two correct solutions to the equation

$$f(x)=g(x).$$

If he is right, what is the value of $a+b$?

A. -4
B. -3
C. -5
D. 4

13

$$f(x)=ax^2+b$$

If $f(-1)=7$ and $f(4)=2$, what is the value of a?

A. $\frac{1}{3}$
B. $\frac{-1}{3}$
C. $\frac{-1}{2}$
D. $\frac{1}{2}$

14

$$\frac{2x^2+5x+3}{x-1}$$

Reus performed *Polynomial Long Division* on the *quotient* above and he correctly found the expression above is the same as

$$2x+7-\frac{a}{x-1}$$

where a is a constant. What is the value of a?

A. 5
B. -5
C. 10
D. -10

15

If $f(x)=x^2-x+2$, what are the two *solutions* to the equation $f(-3x)=0$.

A. $\frac{1}{3}$
B. $\frac{-2}{3}$

16

C. $\frac{(-1\pm i\sqrt{7})}{6}$
D. $\frac{(-1\pm\sqrt{7})}{3}$

A plot of land in *Ethiopia* costs 42 *birr* for 2 *square centimeters*. If *Reus* bought a plot of land that is 40 by 20 *meters*, how much did *Reus* spend? (Note: 1 meter is 100 *centimeters*.)

A. 442,000,000 *birr*
B. 442,000 *birr*
C. 168,000,000 *birr*

D. 168,000*birr*

17

The graph above shows the five-day stock performance of *Ocugen*, a *Pennsylvania* based pharmaceutical company. According to the graph, what is the *approximate percent change* from its *minimum* value to its *peak value* over the course of the five days?

A. 2.3%
B. 230%
C. 23%
D. 0.23%

18

If $\frac{\frac{a}{2}}{a}=\frac{b}{a}$ and $b=42$, what is the value of $\frac{a}{b}$?

A. $\frac{1}{2}$
B. 2
C. 4
D. $\frac{1}{2}$

19

Which of the following *complex numbers* is equal to $(-4-2i^3)-(-4i^2+2i)$ for $i=\sqrt{-1}$?

A. $-4i$
B. $4i$
C. -8
D. 0

20

$$5x+3y=-10$$
$$bx-4.5y=13.5$$

In the *systems of equations* above, b is a constant and x and y are variables. For what value of b will the *system of equations* have no solution?

A. -10
B. -7.5
C. 2.5
D. 4

21

Other than zero, I am the only number a such that one gets the same result whether you take the sum of me with myself or the product of me with myself.

A. 4
B. 1
C. -1
D. 2

22

$$4c-5<4c+3$$

Which of the following best describes the *solutions* to the *inequality* shown above?

A. All real numbers
B. $c<\frac{1}{2}$
C. $c>\frac{1}{4}$
D. No Solution

23

Osario correctly completed the square on $y=-x^2+4x+5$ to find the equivalent

Vertex Form $y=a(x-h)^2+k$. What is the value of $a+h+k$?

A. -1
B. -10
C. 9
D. 10

24

$f(x)=ax^2+bx+4$ and $g(x)=ax+2$. If $f(-1)=7=g(1)$, what is the value of $a+b$?

A. 2
B. 5
C. 7
D. -7

25

$$-4(x+1)^2+1=ax^2+bx+c$$

In the equation above, a, b, and c are constants. What is the value of $a+b+c$?

A. -11
B. 13
C. -15
D. 17

26

If x is the length of the *hypotenuse* of *an isosceles right triangle*, which of the following is the length of a *leg* of the triangle?

A. $\frac{x\sqrt{2}}{2}$
B. $x\sqrt{2}$
C. $x^2\sqrt{2}$
D. $\frac{x\sqrt{2}}{2}$

27

If $g(x)=-(x-4)^2-4$ and $f(x)=x^2$, which set of transformations below accurately describes the change from f to g?

A. A *horizontal shift* of 4 to the right, a *reflection* over the $y-$¿axis followed by a *vertical shift* of down 4.
B. A *horizontal shift* of 4 to the left, a *reflection* over the $x-$¿axis and then a *vertical shift* of down 4.
C. A *horizontal shift* of 4 to the right, a *reflection* over the $x-$¿axis and then a *vertical shift* of down 4.
D. A *horizontal shift* of 4 to the right, a *reflection* over the $x-$¿axis and then a *vertical shift* of up 4.

28

What is the bigger *ratio* of *the roots* to the equation $-(x-5.1)(x+1.7)=0$?

A. $\frac{-1}{3}$
B. -3
C. 3
D. $\frac{1}{3}$

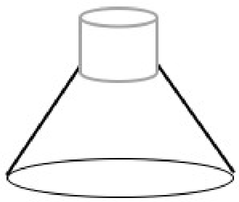
29

If $\sqrt{75}+\sqrt{12}=a\sqrt{b}$, what is the value of $\frac{a}{b}$?

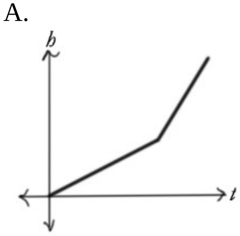
A. $\frac{3}{4}$
B. $\frac{3}{7}$
C. $\frac{7}{3}$
D. $\frac{4}{3}$

30

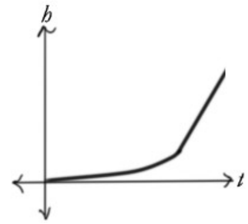
An interesting vase is shaped like a *partial cone* at the bottom and a *cylinder* at the top. *Blanket* started pouring water into the vase at a constant rate. Which of the following could be the graph of *time* vs *height of water* being poured into the vase?



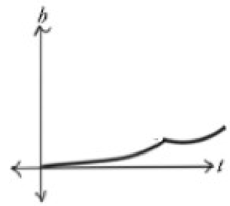
A.



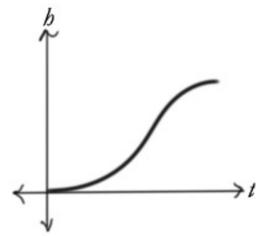
B.



C.

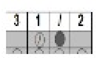


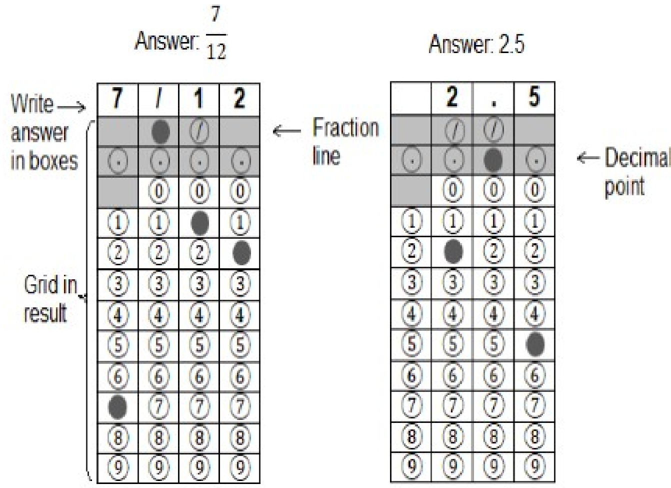
D.



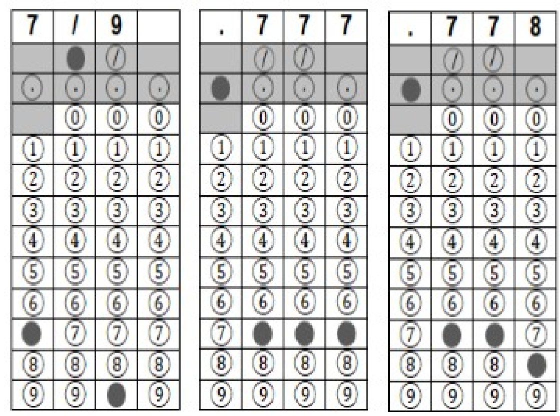
DIRECTIONS

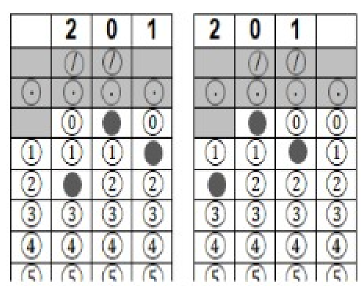
For questions 31-38, solve the problem and enter your answer in the grid, as described below, on the answer sheet.

1. Although not required, it is suggested that you write your answer in the boxes at the top of the columns to help you fill in the circles accurately. You will receive credit only if the circles are filled in correctly.
2. Mark no more than one circle in any column.
3. No question has a negative answer.
4. Some problems may have more than one correct answer. In such cases, grid only one answer.
5. **Mixed numbers** such as must be gridded as 3.5 or 7/2. If is  entered into the grid, it will be interpreted as , not .
6. **Decimal answers**: If you obtain a decimal answer with more digits than the grid can accommodate, it may be either rounded or truncated, but it must fill the entire grid.



Acceptable ways to grid $\frac{7}{9}$ are:





NOTE: You may start your answers in any column, space permitting. Colums you don't need to use should be left blank.

31

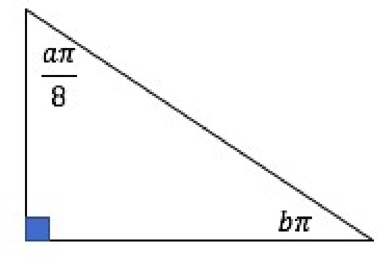


Figure not drawn to scale.

In the right triangle above, what is the value of b if $a=2$?

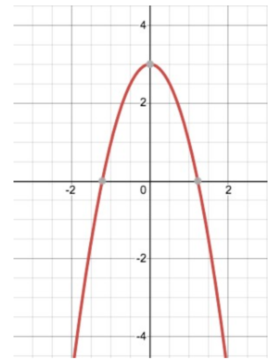
32

Approval voting is a system where a voter can vote for any candidate he or she approves of. In other words, a voter is allowed to vote for more than one candidate. Out of 42 votes cast, 70% went to candidate A and 7% went to candidate C. If candidate B got 42% of the same voters as candidate A in addition to 7 other voters who didn't vote for either of the other two candidates, rounded to the nearest integer, how many people voted for candidate B?

33

At a Fruit stand in New York City, the sign says 4 *Apples* for 2 dollars and 7 *Oranges* for 3 dollars. Individually, *Apples* are priced at 55 cents and *Oranges* at 42 cents. If *Rosario* buys 8 *Apples* and 21 *Oranges*, how many dollars has she saved by buying in bulk instead of the individual prices?

34

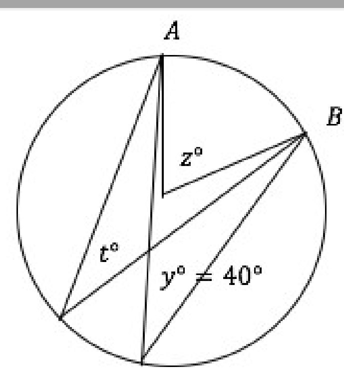


The graph in the $xy-$¿plane above has an equation of the form $y=ax^2+bx+c$. What is the value of $a+b+c$?

35

An elevator had x people in it. y people walk in where y is double x. If z people then joined to fill the elevator to capacity and z is half of x, which equation below can be used to solve for y if the elevator fills once 42people are in it?

36



In the figure above, z is a *Central Angle*. All three angles [the two *Inscribed Angles* $t°$ and $y°$ as well as the central angle $z°$] intercept *minor arc* AB. What is the value of $t°+z°$?

What is the absolute value of $x-$¿

oor
dinate of the *center* of the circle with equation:

$$x^2+y^2-4y+8x=1$$

$$v+v+v+v-5+1=v^2$$

In the equation above, what is the value of v?

Solutions to Polar Pi SAT Practice Test 1

Section 3

While detailed solutions are provided for every problem, should you need more support, you should consult the appropriate section. For example, problem 23 in section 4 is entirely about *"Completing the Square,"* and there is a section in this book with lessons and practice strictly on *Completing the Square*.

1. Notice as we are given one equation in two unknowns, we cannot solve for either unknown. But, multiplying both sides of the given equation by -2, we have:

   $-2(5a+2b)=-2(4)$

   $\rightarrow -10a-4b=-8$

   Hence, the correct answer is D.

2. A *Rhombus* is quadrilateral with four equal sides. But a *Rhombus* is more globally categorized as a *parallelogram* in the realm of *quadrilaterals*. As such, it [a *Rhombus*] inherits all the properties of *parallelograms*. As such, to the end of our goal here, we will recall that all *parallelograms* have equal opposite angles and supplementary consecutive interior angles. With this, we see that $x° = 70°$ and $y° + 70° = 180°$.

   $\rightarrow y° + 70° = 180° - 70° = 110°$. So then, what we are asked for, that is $x° + y° = 70° + 110° = 180°$ which matches answer choice C.

3. Her is one approach, we can told the equation $ax^4 - bx^3 + 2x - (4x^4 - 2x^3 + 2x) = 0$, has solutions $x=0$ and $x=7$. We are additionally told that $b=4a$. Let us first simplify and factor the equation to make it easier to work with.

   $ax^4 - bx^3 + 2x - (4x^4 - 2x^3 + 2x) = 0$

   $\rightarrow ax^4 - bx^3 + 2x - 4x^4 + 2x^3 - 2x = 0$

   $\rightarrow (a-4)x^4 + (2-b)x^3 = 0$

   Since we are given $b=4a$, we will use it in the last step of our equation:

$\rightarrow (a-4)x^4+(2-4a)x^3=0$

Now, we will use the information about the solutions. With $x=0$, we will not get any closer to find the value of a, because that would make every term equal to 0. However, using the value of $x=7$, we can get a linear equation that we can solve for a. Let's substitute $x=7$.

$\rightarrow (a-4)7^4+(2-4a)7^3=0$

$\rightarrow 7^4a-4\cdot7^4+2\cdot7^3-4\cdot7^3a=0$

Now, combining like terms and moving the constant terms to the right-hand side, we have:

$\rightarrow (7^4-4\cdot7^3)a=4\cdot7^4-2\cdot7^3$

$$\rightarrow a=\frac{4\cdot7^4-2\cdot7^3}{7^4-4\cdot7^3}=\frac{26}{3}$$

4. The correct answer is A. 2

The given equation reads,

$$\frac{(x-2)^2}{4}+1=0$$

Subtracting 1 from both sides, we have

$$\frac{(x-2)^2}{4}=-1$$

$\rightarrow (x-2)^2=-4$

$\rightarrow x-2=\sqrt{-4}=\pm\sqrt{-1}\sqrt{4}=\pm i\sqrt{4}=\pm i2$

$\rightarrow x-2=\pm2i$

$\rightarrow x=2\pm2i=2(1\pm i)=2(1\pm1i)=2(a\pm bi)$

It is clear form this last step that $a=b=1$. Meaning, $a+b=1+1=2$.

5. To find the equation of the line in the *Slope-Intercept* form $y=mx+b$, we need to find the slope m and the $y-¿$intersect point (the $y-¿$coordinate of the point where the graph crosses the y-axis.) is b.

First, we can see the line crosses the $y-¿$axis at the point $(0,2)$, which means the $y-¿$intercept is at $y=2$. So far, we have $y=mx+2$. Now we just need to find the *slope* of the line. To this end, we can just pick two lattice points (points with integer coordinates) that we are sure lie on our graph. While other choices exist for the pair of points that we need, the points $(0,2)$ and $(-1,-1)$ were good choices for us. Now then, using the slope formula

$$m=\frac{y_1-y_2}{x_1-x_2}=\frac{y_2-y_1}{x_2-x_1}$$

$\rightarrow m=\frac{2-(-1)}{0-(-1)}=\frac{3}{1}=3$

Hence, the equation of the line is $y=3x+2$, which matches answer choice C.

6. Our task is to solve the equation $-\sqrt{x+2}=x$. We'll start by squaring both sides:

$(-\sqrt{x+2})^2=x^2$

In the left-hand side, the negative sign and the square root will disappear because we are squaring so, we have.

$\rightarrow x+2=x^2$

To solve the quadratic, we get all the terms on one side and once again employ the quadratic formula:

$x^2-x-2=0$

Using the *quadratic formula*, we have

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\rightarrow x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-2)}}{2(1)}=\frac{1\pm\sqrt{9}}{2}=\frac{1\pm 3}{2}$

So, the solutions are $x=2$and $x=-1$. These solutions satisfy the quadratic equation, but we must make sure that they also satisfy the original equation. Substituting both in the original equation, we observe that the only valid solution is $x=-1$ as the other solution is extraneous. Hence the correct answer is D.

7. To solve the equation, first, we can divide every term by $2$to simplify a little bit.

$2x^2-10x+12=0$

$\rightarrow x^2-5x+6=0$

Now, we can factor this quadratic as:

$\rightarrow (x-3)(x-2)=0$

From here, we get $x=3$and $x=2$, which is answer choice B.

8. We are given that $u^2-v^2=x$ and $y=v$, and we need to find and equivalent expression to $25x+(4y)^2$ in terms of $u$and v. This is mere substitution.

$25x+(4y)^2 \rightarrow 25(u^2-v^2)+(4v)^2$

Now, we can simplify by distributing the and combining like terms.

$\rightarrow 25u^2-25v^2+16v^2 \rightarrow 25u^2-9v^2$

Using the "*difference of squares*," we can write that

$$25u^2-9v^2=(5u-3v)(5u+3v)$$

Which is answer choice D.

9. This is exactly the type of the type of question where "*Picking numbers*" is appropriate.

"*Picking numbers*" means we pick numbers for the variables. We then use the "picked numbers" to test out the answer choices. "*Pick numbers*" that are friendly with each other. For example, here $n=10$ and $t=20$. Using these numbers, we can gather that 10 students paid 20 dollars to start, but now, 2 students have joined so it will be 20 dollars for 12 students. Thus, each student must pay

$$\frac{20}{12}=\frac{10}{6}=\frac{5}{3}$$

But point is, we have $\frac{t}{(n+2)}=\frac{20}{12}$ which matches answer choice A.

10. First, we will combine the fractions on the right-hand side:

$$\frac{1}{x^2-1}=\frac{(x+1)a+(x-1)b}{x^2-1}$$

Now, distributing the a and b, and canceling the denominators, we have

$$1=ax+a+bx-b$$

Now, we will factor the right-hand side and compare coefficients.

$$\rightarrow 1=(a+b)x+a-b$$

Since there is no x in the left-hand side, it means that its coefficient is 0. In the right-hand side, the coefficient of x is $a+b$, which means that $a+b=0$. The correct answer is D.

11. For this kind of problems, we should try to get rid of the i's in the denominator. For this, we can multiply and divide by a convenient number. In this case, it's a good idea to multiply and divide the first by i, and the second fraction by $(1-i)$.

$$\frac{2}{i}\cdot\frac{(-i)}{(-i)}-\frac{4}{(1+i)}\cdot\frac{(1-i)}{(1-i)}$$

We have exploited conjugation in what we just wrote. See section on Complex Numbers [in this book] if you need more support.
Keeping in mind that $i^2=-1$ and simplifying more, we write:

$$\rightarrow \frac{-2i}{1}-\frac{4(1-i)}{1-(-1)}=-2i-2(1-i)$$

$$¿-2i-2+2i=-2$$

Which is answer choice D.

12. The given system of equations is the following:

$$-x+ay=5$$
$$2x+y=-10$$

To have infinitely many solutions, we need to be able to turn one of the two equations into the other by simply multiplying both sides of the equation by a constant. Here, the easiest approach is to try and make the equation with the constant a turn into second equation.
In the second equation given, since the coefficient of x is -1, let's divide both sides by -2 to make the two equations identical (*this is what you want to do on all problems that task you with this same challenge*.)

$$2x+y=-10$$
$$\rightarrow -x-\frac{1}{2}y=5$$

Comparing this last equation above (*a scaled version of the original second equation*) to the equation with a, it is abundantly clear that $a=\frac{1}{2}$, which is answer choice C.

13. To find the value of a, we will manipulate the given equation as follows. First, we will apply rules of exponents.

$$(27^4)^a=81 \rightarrow (27^a)^4=81$$

Now, we know that $3^4=81$, so let's use this together with the fact that $\sqrt[3]{27}=3=27^{\frac{1}{3}}$.

$$\left(27^{\frac{1}{3}}\right)^4=81$$

$$\rightarrow (27^4)^a=(27^a)^4=81=3^4=\left(27^{\frac{1}{3}}\right)^4$$

$$\rightarrow (27^4)^a=\left(27^{\frac{1}{3}}\right)^4$$

Now then, the correct answer is clearly D.

14. We need to solve for a in the following equation:

$$x=-b+\sqrt{c^2-4a}$$

First, we add b to both sides

$x+b=\sqrt{c^2-4a}$

Now, we will square both sides to get rid of the square root on the right side:

$\rightarrow (x+b)^2=c^2-4a$

The next step is to subtract c^2 from both sides.

$\rightarrow (x+b)^2-c^2=-4a$

Finally, we divide both sides by -4 to isolate a, we have:

$$\frac{(x+b)^2-c^2}{-4}=a$$

We can move the negative side in the denominator as follows. This amounts to multiplying the left side by negative one top and bottom.

So now we have:

$$\rightarrow \frac{c^2-(x+b)^2}{4}=a$$

which is answer choice D.

15. C is the correct answer.

This is so because multiplying a function by -1 is equivalent to reflexing across the $x-$¿axis. Since the graph of y ¿ x^2-2 is a parabola pointing up 2 units under the x-axis, se we need to look for a parabola pointing down with the vertex at $y=2$, which is answer choice C.

16. For a quadratic equation to have *no real solutions*, the *discriminant* needs to be negative. The *discriminant* of a quadratic in *Standard Form* [ax^2+bx+c] is b^2-4ac. We are given the quadratic $ax^2=5x+t$. As such, first task is to turn the given quadratic into the *Standard Form*. [Δ is *the discriminant*.]

$4x^2-5-t=0$

Here, the discriminant is $25-4(4)(-t)=25+16t=\Delta$.

To make sure this is always negative, we need to solve the following inequality:

$\Delta=25+16t<0$

$\rightarrow 25<-16t$

$$\rightarrow \frac{25}{-16}>t$$

17. In this problem we have just have to try every row until we find a value of t that meets the given condition. This value of t is 1, because $a(1)=2$and $b(1)=4$, so $2a(t)-b(t)=2(2)-4=4-4=0.$ Notice that $t=2$ also meets the given condition, so $t=2$ is another possible value.

18. The correct answer is $\frac{\sqrt{3}}{2}$. Since $\tan x°=\sqrt{3}$, then, the opposite side will be $k\sqrt{3}$, and the adjacent side is k, where k is a constant. We see that this is the special right triangle of 30-60-90, so the hypotenuse will be $2k$. Therefore, $\sin x°$ is the opposite side over the hypotenuse, $\frac{k\sqrt[3]{3}}{2k}$, which simplifies to $\frac{\sqrt{3}}{2}$.

19. The possible values of x are 0 and $\frac{-1}{2}$, so any of these would be considered correct. To solve the equation, first, we will use the binomial square formula to expand the binomial, and we will distribute the parenthesis:

$$2(x-1)^2+5(x-1)+3=0$$

$$\rightarrow 2(x^2-2x+1)+5x-5+3=0$$

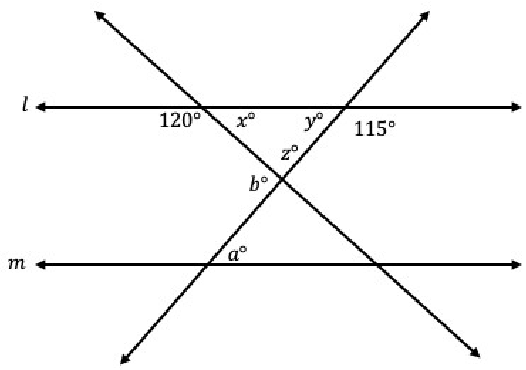
$$\rightarrow 2x^2-4x+2+5x-2=0$$

$$\rightarrow 2x^2+x=0 \rightarrow x(2x+1)=0$$

$$x=0 \text{ or } 2x+1=0$$

$$x=0 \text{ or } x=\frac{-1}{2}$$

20. To start, look at the labeled figure below. It is the same figure as in the question with a few addons.



Now, the angle with measure 120 degrees and angle x are supplementary (linear pair.) From this, we see that $x°=60°$. Using similar reasoning, we know that $y°=65°$. Now, since we have a triangle with angles x, y, and z, we know that $z°=180°-65°-60°=55°$. Since $z°$ and $b°$ are supplementary, we know that $b°=125°$. To find the value of $a°$, considering the two parallel lines and the positively sloped transversal, we know that $a°$ and the angle labeled $115°$ are consecutive interior

angles. As such, they are supplementary. Hence, the value of $a° = 65°$. Now, we are done.
$a° + b° = 65° + 125° = 190°$.

Solutions to Polar Pi SAT Practice Test 1

Section 4

While detailed solutions are provided for every problem, should you need more support, you should consult the appropriate section. For example, problem 23 in section 4 is entirely about *"Completing the Square,"* and there is a section in this book with lessons and practice strictly on *Completing the Square*.

1. The first piece of information we need to find is the number of people who took part in the survey. The number of people who voted for La Liga is $25 + 45 = 70$, and the number of people who voted for the Premier League is $55 + 35 = 90$, so in total there where $70 + 90 = 160$ people in the survey. Of these 160 people, 35 prefer the Premier League and Messi. So, the correct answer is $\frac{35}{160} = \frac{7}{32}$ (A.)

2. First, we need to find the two solutions.

$$x^2 - 2x + 5 = 0$$

Using the quadratic formula, we can write:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\rightarrow x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2}$$

$x = 1 + 2i$ or $x = 1 - 2i$

Now, we can find the two possible values of the difference of the roots. This is like $x - y = 7 - 4 = 3$ vs $y - x = 4 - 7 = -3$, the other one is the same but in reverse order, $(1 - 2i) - (1 + 2i)$.

$$(1 + 2i) - (1 - 2i) = 1 + 2i - 1 + 2i = 4i$$

$$(1 - 2i) - (1 + 2i) = 1 - 2i - 1 - 2i = -4i$$

Thus, the possible values are $4i$ and $-4i$. The only value that appears in the answer choices is $-4i$, C.

3. The correct answer is B) $|s-60|\leq 10$. A good way to quickly solve this kind of problems is to look at the answer choices and to try to anticipate what it involves an inequality after some manipulations. B, is correct because we see that after getting rid of the absolute value, we will produce a -10 on the left-hand side, and a 10 in the right-hand side, with $s-60$ in between. After this, we can add 60 to all sides, to get a 50 on the left, and a 70 on the right, which is what we want. We also need to be careful with the inequality symbol. Let's solve the inequality to check our answer.

$|s-60|\leq 10$
$\rightarrow -10\leq s-60\leq 10$

Now, adding 60 to the whole inequality, we have

$50\leq s\leq 70$

Which is what we needed. The correct answer is B.

4. To find the intersection, we should set the ys equal:

$-2x+3=x+3$

$-2x+3=x+3$

$\rightarrow 0=3x$

$\rightarrow x=0$

Now, we have the value of x, which is 0, so a is also 0. Let's plug in the value of $x=0$ in the second equation:
$y=x+3$
$\rightarrow y=0+3=3$
$\rightarrow y=3$

The point of intersection is $(0,3)$, which means that a is 0 and b is 3, hence,
$a-b=0-3=-3$ (C.)

5. Looking at the answer choices we see that all inequalities have a single number in the right-hand side, so let's start by isolating the number -5 in the given inequality:

$y\leq 2x-5$

Subtracting $2x$ from both sides, we have

$y-2x\leq -5$

Now, looking at the answer choices, again, we see that the only inequality that has a -5 in the right-hand side is different from the inequality we have. The rest of inequalities have a positive 5 in the right-hand side. So, it's clear that we need a positive 5 to appear in our inequality. To achieve this, we can multiply both sides of our last step by -1, but we must remember that we must invert the inequality sign.

$$y-2x\leq-5$$

$$\rightarrow -y+2x\geq5$$

$$\rightarrow 2x-y\geq5$$

The correct answer is C.

6. Initially, the volume V is given by the formula $V=\pi r^2 h$. Now, we want the volume to be the same, but1 we want the radius to be 10% bigger. For this to be possible, the height h needs to be reduced by some percentage. Now, we can set up the following equation:

$$V=\pi(1.1r)^2 x$$

Now, we just need to solve for x.

$$\rightarrow \frac{V}{\pi(1.21r^2)}=hx$$

$$\rightarrow x=\frac{V}{\pi(1.21r^2)}\cdot\frac{1}{h}$$

But we know the original volume $V=\pi r^2 h$.

$$\rightarrow x=\frac{\pi r^2 h}{\pi(1.21r^2)}\cdot\frac{1}{h}=\frac{1}{1.21}$$

7. Here is an approach, we can simplify the equation as much as possible and then compare coefficients.

$$\frac{3(x-1)^2+5(x-1)+2}{x-1}=3x+a+\frac{2}{x-1}$$

First, we will multiply every term by $(x-1)$.

$$\rightarrow 3(x-1)^2+5(x-1)+2=3x(x-1)+a(x-1)+2$$

We don't want to do more work than necessary. First, we expand the *Binomial Squared* and distribute. Thereafter, since the $x-1$ is a common factor, and the only different variable term is $3x$, we will rewrite it in terms of $x-1$ cleverly. Instead of writing x, we will write $(x-1)+1$, which is the same.

$$\rightarrow 3(x-1)^2+5(x-1)=3((x-1)+1)(x-1)+a(x-1)$$

Now, we can make a substitution to make it simpler. We will let $u=x-1$ to write:

$3u^2+5u=3u(u+1)+au$

Now, we will distribute the $3u$.

$\rightarrow 3u^2+5u=3u^2+3u+au$

Finally, we can combine like terms.

$\rightarrow 3u^2+5u=3u^2+(3+a)u$

Comparing coefficients, we see that $5=3+a$, which implies that $a=2$.

8. Since the table shows that when $n=0$, $f(n)=2$ and we know that every number raised to the $0th$ power is 1, this value of n is a good one to test at the start. So, essentially what we are doing is *Back-solving* but *Picking Numbers* at the same time. For testing exponential functions, for the reason we already stated, 0 is a good value to test.

If we first, try:

$f(n)=-3(2^n)-5$

$\rightarrow f(0)=-3(2^0)-5$

$\rightarrow f(0)=-3-5=-8\neq 2$

And therefore, we see that this first answer choice will not work.

Let's try with answer choice B then.

$f(n)=2(-3)^n+5$

$\rightarrow f(0)=2(-3)^0+5$

$\rightarrow f(0)=2+5=7\neq 2$

We can similarly test answer choice C only to learn that it will not work either when $n=0$ is tested.

So then, let's finally try answer choice D.

$f(n)=-3(2^n)+5$

$\rightarrow f(0)=-3(2^0)+5=03(1)+5=-3+5=2$

Ah, since this is the only answer choice that works at all (we say this because we have only tested one value of n), we conclude that this must be the correct answer.

9. The first thing we should do is to convert to the same unit so we will convert *yards* to *feet*. Using the given fact that 1 *yard* ¿ 3 *feet*, we find that 5100 *yards* = 15300 *feet*.
Now, we must know how many "17 *feet* pieces" fit into the 15300 *feet*. So, we write:

$$\frac{15300}{17}=900$$

Which is answer choice C.

10. First, we will distribute the -2 and combine like-terms on the left side of the equation:

$$11x^2-8x+12-2(x^2-4x+7)=(ax+b)(ax-b)$$

$$\rightarrow 11x^2-8x+12-2x^2+8x-14$$

$$\rightarrow 9x^2-2=(ax+b)(ax-b)$$

We can now write the left-hand side as follows:

$$\rightarrow (3x)^2-\sqrt{2}^2$$

Subsequently, using the *Difference of Squares* $[a^2-b^2=(a+b)(a-b)]$, we can factor the left-hand side in the following manner:

$$\rightarrow (3x)^2-\sqrt{2}^2=(3x+\sqrt{2})(3x-\sqrt{2})$$

Comparing this expression with the right -hand side $(ax+b)(ax-b)$, we see that $a=3$ and $b=\sqrt{2}$. Hence, $a+b=3+\sqrt{2}$, which is answer choice D.

Alternatively, we can expand $(ax+b)(ax-b)$ and write via FOIL or using the *Difference of Squares*:

$$a^2x^2-b^2$$

Therefore,

$$9x^2-2=a^2x^2-b^2$$

$$\rightarrow a^2=9 \text{ and } b^2=2$$

$$\rightarrow a=\pm\sqrt{9} \text{ and } b=\pm\sqrt{2}$$

And since

$$a^2x^2-b^2=(ax+b)(ax-b)$$

$$9x^2-2=(3x+\sqrt{2})(3x-\sqrt{2})$$

11. While we can go about the Algebra in many ways, the following is both clever and most efficient!

$$x^2+y^2=4$$
$$y=x-1$$

We want to isolate the y^2 in both equations. In the first equation it is simple.

$$x^2+y^2=4$$
$$\rightarrow y^2=4-x^2$$

Now, for the second equations, we must square both sides and expand the binomial on the right as an easier set up for what is to follow.

$$y=x-1$$

$$\rightarrow y^2=(x-1)^2$$

$$\rightarrow y^2=x^2-2x+1$$

Now, since both equations are equal to y^2, we can set them equal.

$$4-x^2=x^2-2x+1$$
$$\rightarrow 0=2x^2-2x-3$$

Now, we have a quadratic equation of the form ax^2+bx+c. To find the sum of roots, we can use Vieta's formula, which states that the sum of roots is equal to $-\frac{b}{a}$. Here, $a=2$and $b=-2$, so the sum of roots is $-\frac{(-2)}{2}=1$.

12. We can set up a quadratic equation and solve, but a more efficient (clever) approach is to use the graph. It is just easier to find the $x-$¿values of the intersections from the graph. We can see that the curves intersect at $x=-4$ and $x=-1$. As such, either can be used as the value of a and the other the value of b. Hence, $a+b=-4+(-1)=-5$, which is answer choice C.

Should you prefer the Algebra approach, the quadratic equation that you would have to solve is clearly

$$(x+2)^2=-x$$

13. To find the value of a, we should evaluate the function at the given numbers and set up a system of equations.

$$f(x)=ax^2+b$$

We are told that$f(-1)=7$. If we plug in -1 in $f(x)$, we should get back the value of 7:

$$f(-1)=a(-1)^2+b=7$$

$$\rightarrow a+b=7$$

Now, to get the second equation, we will use the other piece of information.

$f(4)=2$

$\rightarrow f(4)=a(4)^2+b=2$

$\rightarrow 16a+b=2$

Now, we have two equations.

$1¿16a+b=2$

$2¿a+b=7$

We can subtract both equations to get rid of the variable b, and find the value of a. Subtracting $2¿$ from $1¿$, we have:

$16a-a+b-b=2-7$

$\rightarrow 15a=-5$

$\rightarrow a=\frac{-1}{3}$

The correct answer is B.

14. For this problem, but it would take longer. There is a simpler method, working backwards from the resulting expression after the long division Reus performed.
To get back to the original expression,

$$2x+7-\frac{a}{x-1}$$

We need to get a common denominator. So, we write the following:

$$2x\frac{(x-1)}{(x-1)}+7\frac{(x-1)}{(x-1)}-\frac{a}{x-1}$$

Now, we can distribute and write everything under a single denominator as follows:

$$\rightarrow \frac{2x^2-2x+7x-7-a}{x-1}$$

We recall that this expression must be equal to the initial expression

$$\frac{2x^2+5x+3}{x-1}=\frac{2x^2+5x-7-a}{x-1}$$

Now, comparing the constant terms, we see that $-7-a$ must be equal to 3.

$\rightarrow 3=-7-a$

$\rightarrow a=-7-3=-10$

Thus, the correct answer is D.

15. The correct answer is C) $\frac{\left(-1\pm i\sqrt{7}\right)}{6}$.

Since $f\left(x\right)=x^2-x+2$, if we plug in $-3x$, we will get the following.

$$f\left(-3x\right)=\left(-3x\right)^2-\left(-3x\right)+2$$

We must next solve the following equation

$$\left(-3x\right)^2-\left(-3x\right)+2=0$$

Let's carefully get rid of the parenthesis.

$$9x^2+3x+2=0$$

Now we can solve this equation with the quadratic formula, as it is not clearly factorable upon immediate inspection.

$$x=\frac{-3\pm\sqrt{3^2-4\left(9\right)\left(2\right)}}{2\left(9\right)}=\frac{-3\pm\sqrt{-63}}{18}=\frac{-3\pm i\sqrt{9\cdot 7}}{18}=\frac{-3\pm 3i\sqrt{7}}{18}$$

$$¿\frac{-1\pm i\sqrt{7}}{6}$$

Which is answer choice C.

16. First, we should convert all the information to the same unit. Since the price is given in relation to square centimeters, we will work with centimeters. The plot of land is 40 by 20 meters. Since 1 meter is 100 centimeters, the plot is 4000 by 2000 centimeters. Now, we need to multiply this value to find the area, of the plot of land.

$$2000\cdot 4000=8{,}000{,}000\,cm^2$$

Now we know that the plot of land measures 8 million square centimeters.

We are told that a plot of land in Ethiopia costs 42 *birr* for 2 square centimeters. We can set up the following equation:

$$42\ \textit{birr}\ ¿\ 2\,cm^2$$

$$42\,2\,cm^2$$

$$\rightarrow 21\,birr=1\,cm^2$$

Since every square centimeter cost 21 *birr*, to find the total amount Reus spent, we must multiply the size of the plot of land by 21.

$$8000000\cdot 21=168000000$$

Reus spent $168{,}000{,}000$ *birr*.

17. To start, we need to find the *approximate values* of the *minimum* and the *maximum* in the graph of the *Ocugen* stock and then apply the *Percent Change Formula*. The formula adopted to this situation would look as follows:

$$\frac{100(v¿¿2-v_1)}{v_1}\%¿$$

Where v_1 is the *minimum* value and v_2 is the *peak value*. This assignment is done with the understanding that we want the *Percent Change* in the order of the *minimum* to the peak, and as such note that the denominator of the quotient we wrote above must be the *minimum* value v_1.

We can pick out that the *minimum* value is close to 7.3 and that the *maximum* value is close to 9. This means, the *approximate Percent Change* will work out as follows:

$$\frac{100(9-7.3)}{7.3}\%$$

Enlisting the help of a calculator you will get *approximately* 23.29%, so the correct answer is C.

18. We have the following equation, with $b=42$.

$$\frac{4}{\frac{2}{a}}=\frac{b}{a}$$

Important note to self is, we must be careful with the order of the fractions. We can rewrite the equation as follows.

$$4\cdot\frac{1}{\frac{2}{a}}=\frac{b}{a}$$

$$\rightarrow\frac{4a}{2}=\frac{b}{a}$$
$$\rightarrow 2a=\frac{b}{a}$$

Now, we can substitute the value of b, and solve the equation.

$$2a=\frac{42}{a}$$
$$\rightarrow a=\frac{21}{a}$$
$$\rightarrow a^2=21$$
$$\rightarrow a=\pm\sqrt{21}$$

Hence, the value of $\frac{a}{b}$ is $\pm\frac{\sqrt{21}}{42}$.

19. We want to simplify the following complex number:

$$\left(-4-2i^3\right)-\left(-4i^2+2i\right)$$

First, we will distribute the negative sign and combine like terms.

$$\rightarrow -4-2i^3+4i^2-2i$$

Now, we know that $i^3=-i$ and that $i^2=-1$. To understand this, recall that $i=\sqrt{-1}$. Making these substitutions,

$$\rightarrow -4-2(-i)+4(-1)-2i$$

$$\rightarrow -4+2i-4-2i$$

$$\rightarrow -8$$

The correct answer is C.

20. Let's isolate y in both equations.

$$5x+3y=-10$$

$$bx-4.5y=13.5$$

$\downarrow$

$$y=\frac{-5x-10}{3}$$

$$y=\frac{bx-13.5}{4.5}$$

To have no solution, we need both equations to have the same slope, and different $y-$¿intercepts. Let us first make the slopes the same. The slope of the first line is $\frac{-5}{3}$, and the slope of the second one is $\frac{b}{4.5}$. We can set up the following equation and solve for b.

$$\frac{-5}{3}=\frac{b}{4.5}$$

$$\rightarrow b=4.5\left(\frac{-5}{3}\right)=-7.5$$

Which is answer choice B.

Before we are done, note the

$$\frac{-7.5}{5}=\frac{-4.5}{3}=-1.5$$

and so, we must insure,

$-10(-1.5) \neq 13.5$

Which is clearly the case. We need to go through this check because, if instead of 13.5 we had 15, the system would have infinitely many solutions.

21. The correct answer is D.2 You can either think about it without pencil and paper or do Algebra to get there. If you choose the former, realize that products get bigger much faster than sums. For example, $9+9=18$ but $9 \cdot 9=81$. So if you choose to "just think about it," you should start small and given your reasoning is good, you can get there relatively efficiently. If you choose to do Algebra, you will need to write something like the following:

$a+a=a \cdot a$

$\rightarrow 2a=a^2$

$\rightarrow a^2-2a=0$

$\rightarrow a(a-2)=0$

$a=0$ or $a=2$.

22. The correct answer is A) All real numbers. We are given the following inequality.

$4c-5<4c+3$

We can subtract $4c$ on both sides, and we will get the following.

$-5<3$

This is always true, regardless of the value of c, which means that the solution set is all real numbers.

Alternate Solution (*an analytic approach*)
We could have answered this question without writing anything down. We can see that both on the left and right of the inequality, we started off with $4c$. From one side, we subtracted 5 while we added 3 on the other side. This means, the inequality is true regardless of the value of c.

23. To *Complete the Square*, we need the coefficient of x^2 to be 1. We can achieve this by factoring out a negative one from every term of:

$y=-x^2+4x+5=-(x^2-4x-5)$

Now, we must add and subtract (inside of the parenthesis) the square of half of the coefficient of x.

So, in this case, the number we are looking for is $\left(\frac{-4}{2}\right)^2=4$.

$-(x^2-4x+4-4-5)$

The first three terms of what we have now will form a perfect square.

$\rightarrow -((x-2)^2-9)$

Distribute back the negative one as follows:

$\rightarrow -(x-2)^2+9$

We are finally ready to compare this expression with the *vertex form* of a *quadratic* $y=a(x-h)^2+k$ to easily see that $a=-1$, $h=2$ and $k=9$. Hence, $a+h+k=-1+2+9=10$ which means that the correct answer is D.

24. We can easily find the value of a, because we know that $g(1)=7$, and since $g(x)=ax+2$, we can solve a linear equation for a, as follows.

$g(1)=a+2=7$

$\rightarrow a=5$

Now, we can find the value of b similarly. We will substitute the value of a in f.

$f(x)=5x^2+bx+4$

Since we know that $f(-1)=7$, we can set up the following equation:

$f(-1)=5(-1)^2+b(-1)+4=7$

$\rightarrow 5-b+4=7$

$\rightarrow b=2$

Finally, the value of $a+b=5+2=7$. The correct answer is C.

25. We need to manipulate the expression until we make it fit into the standard for, and then compare coefficients on both sides.

$-4(x+1)^2+1=-4(x^2+2x+1)+1$

¿ $-4x^2-8x-4+1=-4x^2-8x-3$

Now, we have the quadratic in standard form. Let's compare coefficients.

$-4x^2-8x-3=ax^2+bx+c$

We observe that $a=-4, b=-8$ and $c=-3$. Thus, $a+b+c=-4-8-3=-15$, giving us as the correct answer, C.

26. Since a right triangle can only have 1 *hypotenuse*, and the triangle is *isosceles* which means that both *legs* measure the same. Now, we can use the *Pythagorean Theorem*. Calling the legs y.

$$x^2=y^2+y^2$$

$$\rightarrow x^2=2y^2$$

$$\rightarrow \frac{x^2}{2}=y^2$$

Since a distance is positive, we will only take the positive square root.

$$\frac{x}{\sqrt{2}}=y$$

Now, if we rationalize the denominator as follows, we can get an equivalent expression that does appears in the answer choices.

$$\frac{x}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}}=y$$

$$\rightarrow \frac{x\sqrt{2}}{2}=y$$

We could have also remembered the special isosceles right triangle and solved the problem in 1 step.

27. The correct answer is C. that the graph of g is the graph of f shifted 4 units to the right [the $x-4$ inside of the parenthesis, remember -4 is to the right and $+4$ is to the left for horizontal shifts], reflected over the x-axis and then shifted down 4. It must be done in this order *HSRV* (*H*orizontal, *S*tretch, *R*eflect, and *V*ertical) because *composition of functions* is not commutative, but it is through composition that we build-in shifts in the algebraic transformation of a function.

28. The correct answer is A) $\frac{-1}{3}$. This is so because the *roots* of the equation are $x=5.1$ and $x=-1.7$.

Since one number is positive and the other one is negative, the ratio will be negative, but to make it as big as possible, we need to pick the smaller number to be the nominator, because for negative numbers, the smaller they are in absolute value, the bigger they are.

$$\rightarrow -\frac{1.7}{5.1}=\frac{-1}{3}$$

If we, do it the other way, we would get a ratio of -3, but $\frac{-1}{3}>-3$, so this is not the maximum ratio. The correct answer is $\frac{-1}{3}$.

29. The correct answer is C. $7/3$. This is so because we can rewrite the given equation as follows:
$\sqrt{75}+\sqrt{12}=\sqrt{25\cdot3}+\sqrt{4\cdot3}=a\sqrt{b}$

Now, applying the multiplication rule for radicals

$¿\sqrt{25}\cdot\sqrt{3}+\sqrt{4}\cdot\sqrt{3}=5\sqrt{3}+2\sqrt{3}=7\sqrt{3}=a\sqrt{b}$

Comparing the resulting expression, we see that $a=7$ and $b=3$. Hence, $\frac{a}{b}=\frac{7}{3}$.

30. The correct answer is B.
Because the vase gets narrower all the way to the *cylindrical* top piece, the *height of the water* will increase faster at each instant in time in the partial cone part. Equally importantly, the height of the water doesn't increase at a constant rate in this part and therefore, cannot be linear until the water reaches *cylindrical* top piece. This eliminates answer choices A because in answer choice A, the first piece of the graph is *linear*. Now, we see that on the *cylindrical* top portion of the vase, the water's height increases at a constant rate (*linear*) because each horizontal slice (*horizontal cross-section*) of the *cylinder* gives identical circles. As answer choices C and D are *nonlinear* in considering the *cylindrical* top, we are left with the correct answer choice as B. More than just process of elimination, B is correct because it follows our analysis. Another reason that we can rule out C is that the graph there shows a decreasing *height* on the *cylindrical* top part of the vase, which is wrong because the *height of the water* continues to grow for the entire duration [*time*] of water being poured in.

31. To find the value of b, we should start by finding the value of the other acute angle in the given right triangle as we are told that $a=2$. With this, the angle at the top $\frac{a\pi}{8}$ becomes $\frac{2\pi}{8}=\frac{\pi}{4}$. This is an easy to recognize *Special Right Triangle*, the *Right-Isosceles*. Therefore, the other angle ($b\pi$ ¿must be equal to $\frac{\pi}{4}$.

$\rightarrow b\pi=\frac{\pi}{4}\rightarrow b=\frac{1}{4}$

32. First, let us find 42% of 70%, which is, $0.42\cdot0.7=0.294$. So now, we know that 29.4% of the total 42 voters voted for candidate B. That is, $0.294\cdot42=12.348$. So then, since we know there are 7 additional people who voted for Candidate B, we have a total of $12.348+7=17.348$ or rounded to the nearest integer, 17 people who voted for candidate B.

33. Since we are told that four *apples* cost two dollars [$4\rightarrow\$2$¿ and 7 *oranges* costs three dollars [$7\rightarrow\$3$¿ and we are taking $8=2(4)=2(\$2)=\$4$ worth of *apples* and $21=3(7)=3(\$3)=\$9$ worth of oranges, we would have to pay $\$4+\$9=\$13$. On the other hand, since individually one *apple* costs $\$0.55$ and one orange $\$0.42$ dollars, we would have to pay $8\cdot\$0.55+21\cdot\$0.42=\$13.22$ dollars if we don't buy in bulk. Therefore, the difference is $13.22-13.00=0.22$.

34. While answer choice B is a tempting answer to pick, there is no reasonable way for us to deduce that life expectancy grew more today than 1990. However, it is easy to see that both plots show, as median household income goes up, life expectancy also goes up. Therefore, the correct answer is C.

35.

Since the elevator is at full capacity when x, y and z people enter and the full capacity is 42, we have

$$x+y+z=42$$

From the relationships between x and the two other variables given in the question, it is clear that

$$y=2x \wedge z=\frac{x}{2}$$

$$\rightarrow x+2x+\frac{x}{2}=42 \rightarrow \left(x+2x+\frac{x}{2}=42\right)2$$

$$\rightarrow 2x+4x+x=84 \rightarrow 7x=84$$

36. The correct answer is $120°$. Since the angle $y=40°$ interceps the minor arc AB, and so does t, and they both are *Inscribed Angles* of AB. Therefore, they must be equal [so $t°=y°=40°$], and they must also be equal to half of the measure of the *Central Angle* that intercepts the same arc ($z°$ in this case.) With this, because $t°=y°=40°$, $z°$ must be twice as large and thus, $z=80°$. Hence, $t°+z°=40°+80°=120°$.

37. To solve this problem, we must *Complete the Square* both on x and y. We start with the given equation:

$$x^2+y^2-4y+8x=1$$

$$\rightarrow x^2+8x+16+y^2-4y+4=1+16+4$$

There is sufficient practice [with detailed solutions] on precisely this kind of a problem in the appropriate section on Circles. If you find the solution here too brief, feel free to consult the content there.

$$\rightarrow (x+4)^2+(y-2)^2=21$$

Comparing this equation with *Standard Form* of the equation of a circle:

$$(x-h)^2+(y-k)^2=r^2$$

We see that the $x-¿$coordinate of the *center* is -4. Since what we are asked for is the absolute value of this $x-¿$coordinate, the correct answer is 4.

38. Our task is to solve the following equation:

$$v+v+v+v-5+1=v^2$$

The first thing we should do is combine like terms.

$\rightarrow 4v-4=v^2$

Now, we can move everything to the right-hand side.

$\rightarrow 0=v^2-4v+4$

Now, we could use the quadratic formula to solve the equation, but a faster and easier method is to notice that this quadratic is a perfect square.

$\rightarrow 0=v^2-4v+4=(v-2)^2$

This means the equation only has 1 solution.

$0=(v-2)^2$

$\rightarrow v-2=0$

$v=2$
So, the correct answer is D.

Math Test – No Calculator

25 MINUTES, 20 QUESTIONS

DIRECTIONS

For questions 1-30, solve each problem, choose the best answer from the choices provided, and fill in the corresponding circle in your answer sheet. **For questions 31-38,** solve the problem and enter your answer in the grid on the answer sheet. Please refer to the directions before question 31 on how to enter your answers in the grid. You may use any available space in your test booklet for scratch work.

NOTES

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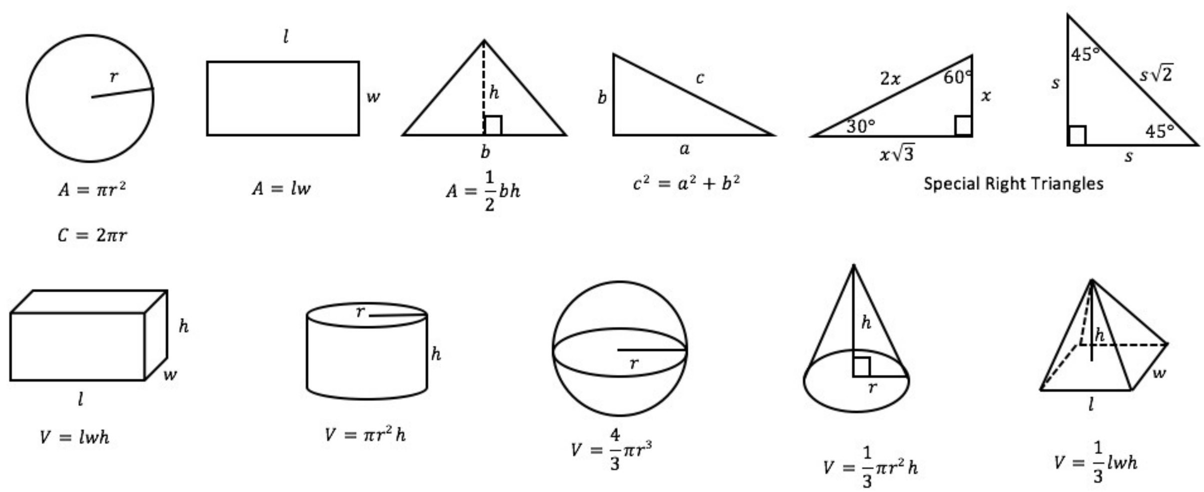
REFERENCE



The *number of degrees* of arc in a circle is 360.
The *number of radians* of arc in a circle is 2π.
The *sum of the measures* in degrees of the angles of a triangle is 180

1

$$2ax+2b+b-7b=4x+8$$

In the equation above, a and b are constants. What is the value of $a+b$?

A. -1

B. 8

C. 2

D. 0

2

5

If $f(x)=-x^3-8$ and $f(-2x)=0$, what is the value of x?

A. -2

B. -1

C. 1

D. 2

3

$f(x)=9x^2-7$

Which of the following is equivalent to $f(x)$?

A. $(9x-7)(x+1)$

B. $(3x-7)(3x+1)$

C. $\left(3x-\frac{7}{2}\right)\left(3x+\frac{7}{2}\right)$

D. $(3x-\sqrt{7})(3x+\sqrt{7})$

4

Which of the following statements includes a function divisible by $x+1$?

I. $f(x)=x^2-1$

II. $g(x)=4(x+1)^2-7(x^2-1)^{77}$

III. $h(x)=x^3-x$

A. I only
B. I and II only
C. I and III only
D. I, II, and III

Which of the following is equal to $27^{\frac{4}{3}}$?

A. 3
B. 9
C. 81
D. 243

6

A 42 ounce *mixture* contains 30 ounces of *water* and 7 ounces of *coffee*. The rest of the *mixture* is *sugar*. How much *sugar* needs to be added to the mixture to have 42% *sugar* by *volume* in the *mixture*?

A. 14.7
B. 21.8
C. 17
D. 20.7

7

A and B have $x-$¿coordinates -1 and 2 respectively on the parabola $y=(x-1)^2+1$. If line l is perpendicular to line $\acute{AB}$ and goes through the vertex of the parabola. Which of the following is the equation of line l?

A. $y-2=(x-1)$
B. $y+2=-1(x+1)$
C. $y-2=-1(x-1)$
D. $y=x$

8

If $b^3 \cdot (b^4)^2 = b^x$, what is the value of x?

A. 9
B. 11
C. 18

9

D. 19

$$(x^2 - b^2)(x^2 + b^2) = x^4 + ax^2 - 16$$

In the equation above, a and b are constants. What is the value of $b - a$?

A. -1
B. 2
C. -2

10

D. 1

$$\frac{i}{1-i}-\frac{i}{1+i}$$

Which of the following is equivalent to the *complex number* shown above?

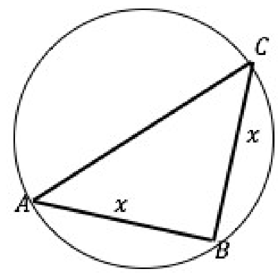
A. i

B. 1

C. $-i$

11

D. -1



14

In the circle above, $\overline{AC}$ is a diameter and $\overline{AB}=x=\overline{BC}$. If the area of the circle is 16π, what is the value of x?

A. $8\sqrt{2}$

B. $16\sqrt{2}$

C. $4/\sqrt{2}$

D. $4\sqrt{2}$

12

Which of the following statements is true about the *parabola* whose equation in the $xy-$¿plane is given by $y=(x-4)(x-2)$?

I. The line $x=-3$ is a *line of symmetry*.

II. The *minimum* value of y is -1.

III. The $y-$¿intercept is -8.

A. I and II

B. I and III

C. II only

D. I, II and III

13

$$k=\sqrt{4+3k}+2$$

What values of k satisfy the equation above?

A. 0 only

B. -7 only

C. 0 and 7

D. 0 and -7

A *polynomial* has *zeroes* at -4, 2 and 0. Which of the following could be the polynomial?

A. $(x+4)(x-2)x^2$

B. $(x-4)(x+2)x$

C. $(x-4)(x-2)x$

D. $(x-4)(x-2)x^2$

Two lines graphed in the $xy-$¿plane have the equations $2x+5y=20$ and $y=kx-3$, where k is a constant. For what value of k will the two lines be *parallel*?

A. $\frac{-2}{5}$

B. $\frac{5}{2}$

C. $\frac{-5}{2}$

D. $\frac{2}{5}$

15

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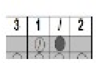
5. **Mixed numbers** such as must be gridded as 3.5 or 7/2. If is  entered into the grid, it will be interpreted as , not .
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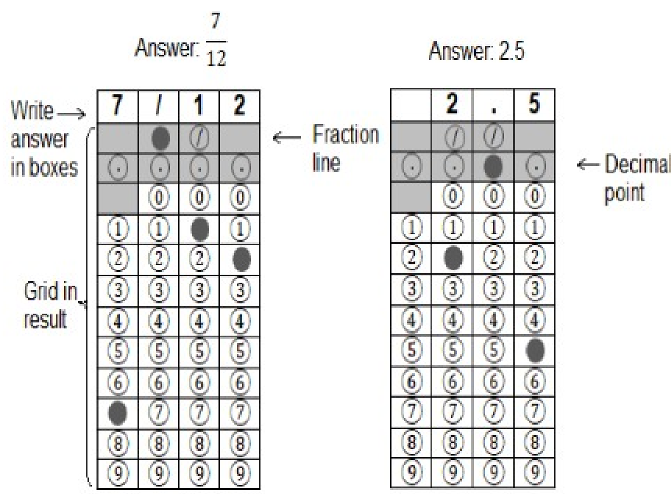

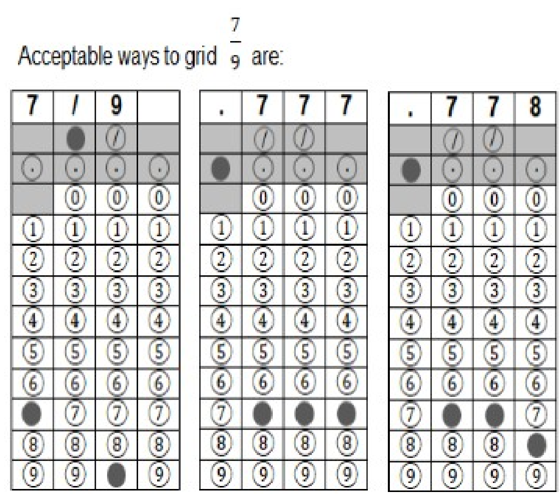

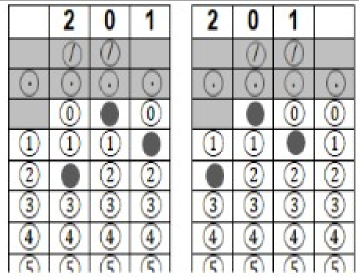

NOTE: You may start your answers in any column, space permitting. Colums you don't need to use should be left blank.

16

$$x^2 - ax + b$$

If one of the *zeros* of the *quadratic* above is $\sqrt{2}$ and the other one is $-\sqrt{2}$, what is the value of b?

17

Two different points on a number line are 7 units from the point with coordinate $x = -2$. If the solution to the equation $|x + a| = b$ gives both coordinates, what is the value of $b - a$?

18

$$\left(\frac{1}{x}\right)^2 - 3\left(\frac{1}{x}\right) + 2 = 0$$

What is one of the two values of x that satisfies the equation above?

19

A rectangle was redesigned by *decreasing* its by $\frac{p}{2}$ percent. If these alterations *decreased* the *area* of the rectangle by 8 percent, what is the value of p?

20

$$-3x + 9y = 12$$
$$x + ky = 2$$

In the system of equations above, what must be the value of k so that there is *no solution* (x, y) that satisfies both equations?

Math Test – Calculator

55 MINUTES, 38 QUESTIONS

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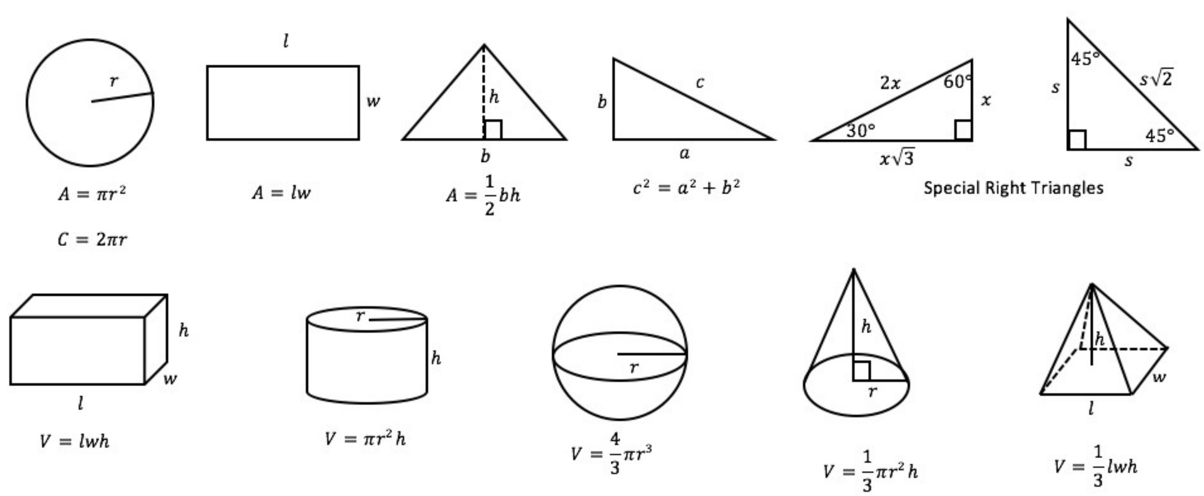


The *number of degrees* of arc in a circle is 360.
The *number of radians* of arc in a circle is 2π.
The *sum of the measures* in degrees of the angles of a triangle is 180.

x	4	2	-4
$g(x)$	2	4	-2

x	-2	-4	4
$h(x)$	-4	-2	7

Values of $g(x)$ and $h(x)$ are listed in the tables above. What is the value of $h(g(2))$?

A. 4
B. 2
C. -4
D. 7

2

$$\frac{1+i}{1-i}+\frac{1-i}{1+i}$$

Which of the following is equivalent to the *complex number* shown above?

A. $4-4i$
B. $4+4i$
C. 0
D. $-2i$

3

$$g(x)=x^2-x \text{ and } f(x)=x+3$$

If *Luis* solved the equation $g(x)-f(x)=0$, which of the following must be a correct value of x?

I. 3
II. -1
III. 1

A. I only
B. I and II
C. II and III
D. I, II, and III

4

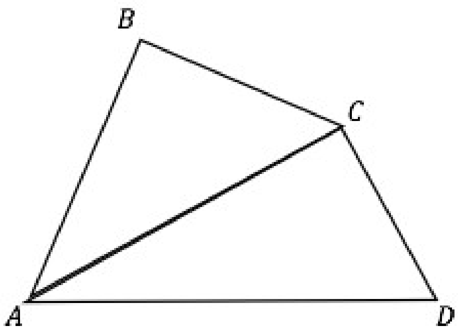
The numbers e and π are two of the most important five numbers in mathematics. The value of e is *approximately* equal to 2.72 and the *ratio* of boys to girls in a class is $e:\pi$. If there are 20 boys in the class, *approximately* how many girls are in the class?

A. 24
B. 30
C. 23
D. 25

C. $4u^2 - \frac{1ab}{2} + \frac{b^2}{16}$

D. $4a^2 - ab + \frac{b^2}{16}$

5



In the triangle above, $m\angle B = m\angle ACD = 90°$. If $\overline{BC} = \overline{CD} = 3$ and $\overline{AB} = 4$, what is the value of $\overline{AD}$?

A. $3\sqrt{2}$
B. $2\sqrt{6}$
C. $6\sqrt{2}$
D. $\sqrt{34}$

6

Which of the following is equivalent to?

$\left(2a - \frac{b}{4}\right)^2$

A. $4a^2 + \frac{b^2}{16}$

B. $4a^2 - \frac{b^2}{16}$

7

Given $(a,b)(c,d) = (a+c, b+d)$
if

$(4,2)(7,77) = (x,y)$

then, what is the value of $\frac{x}{y}(xy + xy^2)$?

A. 11
B. 121
C. 79
D. 9680

8

$$4x^2 + 2x = t$$

In the equation above, t is a constant. If the equation has two real solutions, which of the following can be the value of t?

A. -4
B. -3
C. 1
D. -1

9

The *cost* of cleaning one *cubic inch* of an oil spill is related to the *volume* of the oil spill according to the formula:

$$c = \frac{200}{v}$$

where v is the *volume* of the spill (in *cubic inches*.) Based on this model, which of the following statements below is true?

A. As the *volume* of the spill increases, so does the *cost*.
B. The *cost* is inversely related to the *volume* of the spill.
C. The *cost* decreases as the *volume* of the spill decrease.
D. The *cost* is the opposite of the *volume* of the spill.

10

In the xy-plane, line l and m have slope 2 and -3 respectively. The two lines intersect at point C. If the parabola with equation $y = -(x-1)^2 + 4$ intersects each line at each of its zeroes, what is the sum of the x and y coordinates of C?

A. $\frac{4}{5}$

B. $\frac{22}{5}$

C. $\frac{26}{5}$

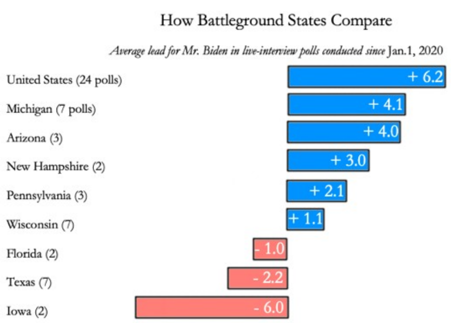
D. $\frac{5}{4}$

11

In a political science class, test scores were determined to be three more than 20times the number of hours, h, the student studied. Which of the following best models this situation?

A. $f(h)=3h+20$
B. $f(h)=20h$
C. $f(h)=60h$
D. $f(h)=20h+3$

12



By how much does the difference between the states of *Michigan* and *Iowa* exceed the difference between the states of *Pennsylvania* and *Texas*?

A. 9.1
B. -1.9
C. 2.9
D. 1.9

13

Let x, y, and z be negative numbers such that $x<y<z$. Which expression is the smallest?

A. $z \cdot z$
B. yz
C. xz
D. yx

14

$$ax-3y=21$$
$$bx+9y=42$$

In the *systems of equations* above,a and b are constants. What must be the relationship between a and b so that the *system of equations* has *no solution*?

A. $b=-3b$
B. $a=3b$
C. $b=-2a$
D. $b=2a$

15

If $-uv=4$ and $u-v=2$, then $u^2v-uv^2=?$

A. 8
B. -8
C. 16
D. -16

16

In the $xy - ¿$plane, the lines $y = 2ax$ and $y = \frac{-b}{2}x$ are *parallel*. What is the relationship between a and b?

A. $b = -4a$
B. $b = -4$
C. $b = 4a$
D. $a = -4b$

17

$$4 - 2\sqrt{x} = 7 - \sqrt{x}$$

Which value of x is the *solution* to the above equation?

A. $3i$
B. $-3i$
C. 9
D. No value of x satisfies the given equation.

18

If
$$\frac{4}{3} = \frac{4x - 2}{2 - 4x}$$

What is the value of $\frac{5x}{2}$?

A. $\frac{1}{2}$
B. $\frac{5}{2}$
C

19

C. $\frac{5}{4}$
D. $\frac{2}{5}$

If
$$x^{a^2} x^{-b^2} = x^{25}, x > 1$$

where a and b are constants and $a + b = 1$, what is the value of $a - b$?

A. -5
B. 5
C. -25
D. 25

20

$$\frac{x}{x-2} + \frac{x}{(x-2)(x-3)} = \frac{4}{x-3}$$

What is (are) the solution(s) to the equation above?

A. 2 only
B. 4 only
C. 2 and 3
D. 2 and 4

In the $xy - ¿$ plane, the point $(6, 0)$ lies on the

21

graph of the function f. If $f(x) = 4c - 2x$ where c is a constant, what is the value of c?

A. $\frac{9}{4}$
B. $\frac{9}{2}$
C. $\frac{81}{4}$
D. $\frac{4}{9}$

22

Circle P has a *center* $(4, -2)$. If the point $A(7, 3)$ lies on the circle, what is the area of circle P?

A. 24π
B. 34π
C. 54π

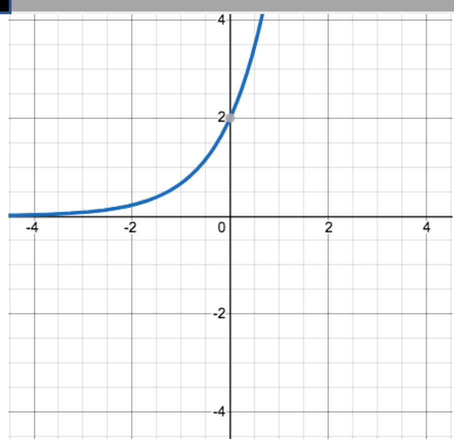
D. 64π

23

I... ction g is defined by $g(x)=x-1$ and
$2g(c$ ...

A. 5
B. 16
C. 15
D. 14

24



A portion of the graph of $y=2(3^x)$ is shown above. If Eteti correctly solved the equation $2(3^x)=a(x-h)^2+k$ to find the only solution of $x=0$, which of the following could be the equation $y=a(x-h)^2+k$?

A. $y=x^2+2$
B. $y=x^2-2$
C. $y=(x-2)^2$
D. $y=(x+2)^2$

Reus needed to lacquer (coat or cover) 42 boxes (*rectangular prisms*) of dimension m by m by n. If the rectangular faces of the boxes are k times biger than the square faces and it takes *Reus* 7 *minutes* to lacquer the square faces but it takes him 28 *minutes* for the rectangular faces, working at the same pace (rate), which of the following gives us the total time it will take *Reus* to finish lacquering all the boxes?

25

A. 12.6 *hours*
B. 42 *hours*
C. 756 *hours*
D. 88.2 *hours*

26

$$ax^3+bx^2+cx+15=0$$

In the equation above, a, b, and c are constants. If the equation has roots -3, -5, and d where d is a constant, which of the following could be the value of d?

A. 3
B. 5
C. -1
D. 1

27

The expression

$$\frac{4x-1}{x+7}$$

is equivalent to which of the following?

A. $\frac{4-1}{7}$

B. $4-\frac{1}{7}$

C. $4-\frac{7}{x+7}$

D. $4-\frac{29}{x+7}$

28

$$-(x+4)(x-2)=0$$

If $x=s$ and $x=t$ are the solutions to the equation above with $s>t$, which of the following is equal to the value of ¿ $s-t \vee$ ¿?

A. -6

B. 5

C. 7

D. 6

29

It takes a group of 4 men 7 hours to do a job, at the same rate, how many hours will it take a group of 7 men to do the same job?

A. 4.2

B. 28

C. 4

D. 7

In modern Physics, one $l$ength contraction formula is written as follows:

$$l=\frac{l_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

In this formula, which of the following correctly solves for c in terms of the other quantities?

A. $c=\frac{lv}{\sqrt{l^2-l_0}}$

B. $c=\frac{lv}{\sqrt{l^2-l_0^2}}$

C. $c=\frac{lv}{\sqrt{l-l_0^2}}$

D. $c=\frac{l^2 v}{\sqrt{l^2-l_0^2}}$

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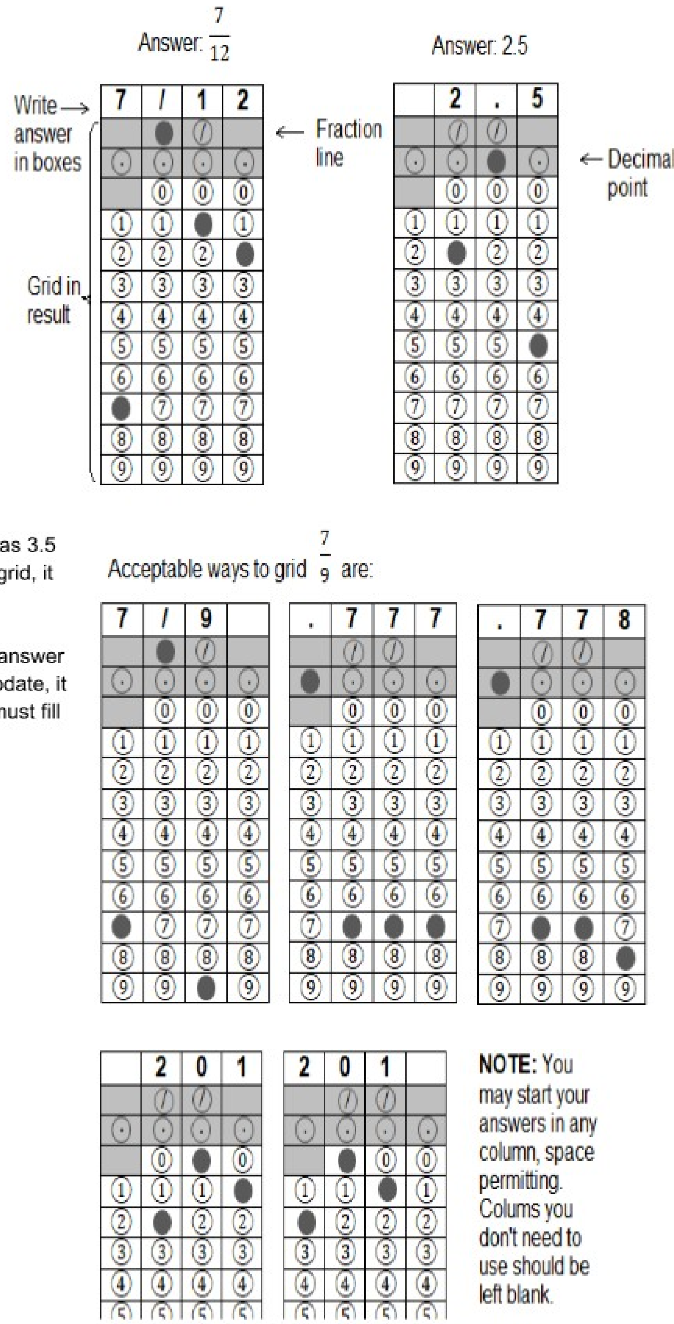
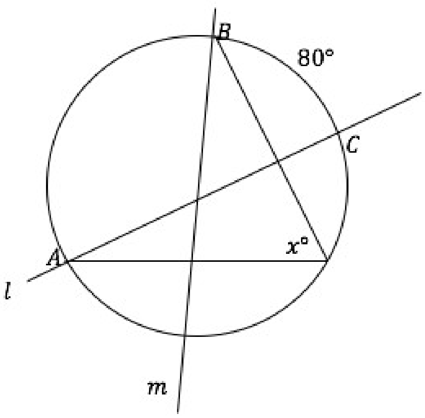


31



Lines l and m intersect at the *center* of the circle in the figure above. In addition, *minor arc* AB contains a point from the two lines as well as two *chords* as in the figure above. If the mearusre of *minor arc* BC is $80°$, what is the meausre of $x°$?

32

number

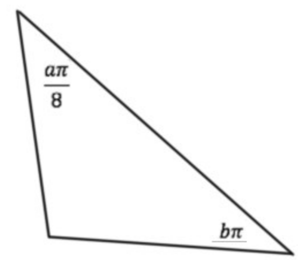
33

$x^2 - 4y^2 = 2$
$x - y = 1$

If $7^k = 100$, what is the value of $7^{\frac{k}{2}+1}$?

34

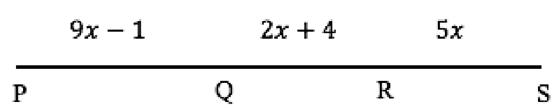
35



Note: Figure not drawn to scale.

In the triangle above, if the ratio of $a:b$ is $4:2$ and $\sin\left(\frac{a\pi}{8}\right) = \cos(b\pi)$, what is the value of $a+b$?

36



Note: Figure not drawn to scale

On $\overline{PS}$ above, $PQ = RS$. What is the length of $\overline{PS}$?

37

In the $xy-¿$plane, the graph of $y=4x^2-2x$intersects the graph of $y=2x$ at the points $(0,0)$ and $(a,2a)$. What is the value of a?

38

If $a^4-b^4=4$, $a-b=4$, and $a+b=2$ what is the value of a^2+b^2?

Solutions to Polar Pi SAT Practice Test 2

Section 3

While detailed solutions are provided for every problem, should you need more support, you should consult the appropriate section. For example, problem 23 in section 4 is entirely about *"Completing the Square,"* and there is a section in this book with lessons and practice strictly on *Completing the Square*.

1. To find the value of $a+b$, we should first simplify the given expression and then compare it with the right-hand side. We will start by combining like terms.

$$2ax+2b+b-7b=4x+8$$

$$\rightarrow 2ax-4b=4x+8$$

Now, comparing coefficients on both sides, we have:

$$2a=4, -4b=8$$

$$a=2, b=-2$$

Hence, $a+b=2-2=0$, which is answer choice D.

2. Since $f(x)=-x^3-8$, $f(-2x)=-(-2x)^3-8=-(-8x^3)-8=8x^3-8$, but we are told that $f(-2x)=0$. To find the value of x, we need to solve the equation $8x^3-8=0$. We'll start by adding 8 to both sides.

$$8x^3=8$$

Now, diving both sides by 8, we write:

$$\rightarrow x^3=1$$

Taking the cube root on both sides, we have:

$$x=1$$

Which is answer choice C.

3. A quick look at the answer choices instructs us to try to factor the given quadratic [$f(x)=9x^2-7$]. Applying the *Difference of Squares*.

$$a^2-b^2=(a+b)(a-b)$$

$$9x^2-7=(3x)^2-\sqrt{7}^2$$

$\rightarrow 3x+\sqrt{7}$¿$(3x-\sqrt{7})$

Which is answer choice D.

4. To find whether or a polynomial is divisible by another polynomial, we can try to factor it and see if any of its factors matches with the *divisor* (the *polynomial we are dividing by*.)

First,

$$f(x)=x^2-1=(x-1)(x+1)$$

We see that one of the factors of f is precisely $x+1$, which means f is divisible by $x+1$.
In II, we can write the following to realize that g is also divisible by $x+1$.

$$\rightarrow g(x)=4(x+1)^2-7(x^2-1)^{77}$$

$$\rightarrow g(x)=4(x+1)^2-7((x+1)(x-1))^{77}$$

$$\rightarrow g(x)=4(x+1)^2-7(x+1)^{77}(x-1)^{77}$$

$$\rightarrow g(x)=4(x+1)\left[x+1-7(x+1)^{76}(x-1)^{77}\right]$$

For III, a bit more work is involved. First a GCF of x then the *Difference of Squares* does the job. We again find that $x+1$ is a factor. So, $h(x)$ [$x^3-x=x(x^2-1)=x(x-1)(x+1)$] is also divisible by $x+1$. Thus, the correct answer is D.

5. A few different approaches are possible, but we should start by noting that $3^3 = 27$.
Substituting this into the expression, we write:

$$27^{\frac{4}{3}}=(3^3)^{\frac{4}{3}}$$

Now, using the *product rule for exponents*, we have:

$$(3^3)^{\frac{4}{3}}=3^{3\cdot\frac{4}{3}}=3^4=81$$

Hence, the correct answer is C.

6. The correct answer is B.
The total *volume* is equal to $w+c+s$, where w is the number of ounces of *water*, c the number of ounces of *coffee* and s the number of ounces of *sugar*. After adding u additional ounces of *Sugar*, the *volume* will be $w+c+s+u$, and we want the total *volume* of *sugar* to be 42%. We can translate this into the following equation:

$$(w+c+s+u)0.42=s+u$$

The task that remains here is to solve for u, and of course, we will need to substitute the given values of w, c and s to get a numerical value at the end (but that is just number crunching so it can wait.)

$$\rightarrow 0.42w+0.42c+0.42s+0.42u=s+u$$

$$\rightarrow 0.42w+0.42c+0.42s-s=u-0.42u$$

$$\rightarrow 0.42(w+c)-0.58s=0.58u$$

$$\rightarrow u=\frac{0.42(w+c)-0.58s}{0.58}$$

Now, because we know the values of $w=30, c=7$, and $s=42-30-7=5$, we have

$$u=\frac{0.42(30+7)-0.58\cdot 5}{0.58}=21.8$$

7. The first step will be to find the $y-$¿coordinates of the points A and B. We are given the equation $y=(x-1)^2+1$, and we know that points A and B have $x-$¿coordinates -1 and 2. To find the $y-$¿ coordinates, we should plug in the $x-$¿coordinates in the equation.

Point A:

$$y=(-1-1)^2+2=5$$

$$\rightarrow A=(-1,5)$$

Point B:

$$y=(2-1)^2+1=2$$

$$\rightarrow B=(2,2)$$

Since we are asked to find the equation of a line that is *perpendicular* to line $\acute{A}B$, we first need to first find the slope of the line $\acute{A}B$. Using the coordinates of points A and B, we write:

$$m=\frac{y_2-y_1}{x_2-x_2}$$

$$\rightarrow m=\frac{5-2}{-1-2}=\frac{3}{-3}=-1$$

We now know the slope of the line $\acute{A}B$, so the slope of any line that is *perpendicular* to $\acute{A}B$ will be 1. This is so because if a line has slope m, then any line *perpendicular* to it will have a slope of $\frac{-1}{m}$. So far, we know that the equation of line l is $y-y_1=(x-x_1)$ in the *Point Slope* form. We can find $(x_1,y)$by using the fact that the line l goes through the *vertex* of the parabola.

The equation of the parabola is given in the form $a(x-h)^2+k$, where the vertex is (h,k). So, we gather that the vertex is $(1,1)$. Substituting this into the *Point Slope form*, we write:

Line l:

$y-1=(x-1)$

$y=x$

8. The first step is to use rules of exponents to simplify the left-hand side of the given equation.

When we have a "power to a power" situation, so we know to multiply the powers.

$b^3 \cdot (b^4)^2=b^x \rightarrow b^3 \cdot b^8=b^x$

$\rightarrow b^{11}=b^x$

$\rightarrow x=11$

The correct answer is B.

9. We should expand the left-hand side and then compare coefficients with the left-hand side. We can apply difference of squares to expand the left-hand side quickly.

$(x^2-b^2)(x^2+b^2)=(x^2)^2-(b^2)^2=x^4-b^4$

$\rightarrow x^4-b^4=x^4+a\,x^2-16$

Now, we will compare coefficients to find the values of a and b. We see that there is no x^2 term in the left-hand side. This means that there should not be an x^2 term in the right-hand side either. The only value of a that makes this possible is $a=0$. Now, the constant terms must be equal in both sides. Hence

$-b^4=-16 \rightarrow b^4=16$

$\rightarrow b=2$

Therefore, the value of $b-a=2-0=2$.

10. We want to find an equivalent expression to the following complex number.

$$\frac{i}{1-i}-\frac{i}{1+i}$$

We do this so by combining the fractions and simplifying. First, to get a common denominator, we will multiply and divide the first fraction by $1+i$, and the second one by $1-i$.

$$\frac{i}{1-i}-\frac{i}{1+i}=\frac{i}{(1-i)}\cdot\frac{(1+i)}{(1+i)}-\frac{i}{(1+i)}\cdot\frac{(1-i)}{(1-i)}$$

We can use *difference of squares* to simplify a few parts as follows,

$$\rightarrow \frac{i+i^2-(i-i^2)}{1-i^2}=\frac{2i^2}{1-i^2}$$

We see that we can further simplify because we know that $i^2=-1$

$$\frac{2i^2}{1-i^2}=\frac{2(-1)}{1-(-1)}=\frac{-2}{2}=-1$$

Which is answer choice D.

11. Since we are told that $\overline{AB}$ is a diameter, the angle ABC must measure $90°$ because the arc AC measures $180°$. Now, we will find the value of the radius r. Since we know that the area of the circle is 16π, we can setup the following equation to solve for r.

$$16\pi=\pi r^2$$

$$\rightarrow r^2=16$$

$$\rightarrow r=4$$

The diameter which we know to also be the hypotenuse of triangle ABC measures 8. Now, we can use the special *Isosceles Right triangle*, or the *Pythagorean Theorem* as follows, to find x, a leg of the triangle.

$$x^2+x^2=8^2$$

$$\rightarrow 2x^2=64$$

$$\rightarrow x^2=32$$

$$\rightarrow x=\sqrt{32}=\sqrt{16\cdot 2}=\sqrt{16}\cdot\sqrt{2}=4\sqrt{2}$$

Hence, the correct answer is D.

12. To make it easier to analyze the function, we will transform the equation into the form $y=ax^2+bx+c$.

$$y=(x-4)(x-2)$$

$\rightarrow y=x^2-6x+8$

Now, we know that the $x-$¿coordinate of the *vertex* is also the *line of symmetry*. This $x-$¿ coordinate is given by $x=\frac{-b}{2a}=\frac{--6}{2}=3$. Thus, the *line of symmetry* occurs for $x=3$, not $x=-3$.

For II, since the coefficient of the $x^2$is positive, the minimum value of y is the $y-$¿coordinate of the vertex. Since we already have the $x-$¿coordinate, we just plug in $x=3$ in the equation to find the $y-$¿coordinate, as follows.

$y=3^2-6(3)+8=-1$

The minimum value of y is -1, so II is true.

Finally, for III, we quickly see that it is false. Because the $y-$¿intercept is always in every quadratic, equal to the last term (the constant at the end.)

Thus, $y=8$ not $y=-8$ is the correct answer. As such, the correct answer is C.

13. Our task is to solve the following equation:

$k=\sqrt{4+3k}+2$

First, we will subtract 2 from both sides.

$\rightarrow k-2=\sqrt{4+3k}$

Now, we can square both sides to get rid of the square root.

$\rightarrow (k-2)^2=4+3k$

$\rightarrow k^2-4k+4=4+3k$

$\rightarrow k^2-7k=0$

Now, we just must solve a quadratic equation. We can solve it by factoring.

$\rightarrow k(k-7)=0$

$k=0 \vee k=7$

Now in conclusion, we have found two solutions. Recall, when solving radical equations, we must always check if the solutions because we might get extraneous solutions. You can do this yourself to find that the only valid solution is $k=7$.

14. To build a *polynomial* with *roots/zeroes* at -4, 2 and 0, we can represent a general *polynomial* in a factored form, and our polynomial must be a special case of this factored form. Since we know that if a polynomial has a root a, it must follow that $x-a$is a factor of the polynomial, we immediately understand that our polynomial of interest must look as follows:

$P(x)=a(x-(-4))(x-2)(x-0)$

$P(x)=a(x+4)(x-2)x$

Now, as a is just a constant, our polynomial $p(x)$ will give rise to the three *zeroes*, there may be other *zeroes* and neither this nor the actual value of a is of concern to us. But at minimum, we require that our polynomial has $p(x)$ as follows with $a=1$.

$P(x)=(x+4)(x-2)x$

A quick scan of the answer choices reveals that A. $(x+4)(x-2)x^2$ is the correct answer. Although it is one degree higher than the minimum that we required, it fits the bill.

15. We have the following lines:

$2x+5y=20$

$y=kx-3$

To have both lines being parallel, we must have the same slope on both lines. Let's put the first line into the *Slope-Intercept form* so we can compare them easier.

$$y=\frac{-2x+20}{5}=\frac{-2}{5}x+4$$

$y=kx-3$

Now it is abundantly clear that

$$\frac{-2}{5}=k$$

Giving us correct answer as A.

16. We know $(x-\sqrt{2})$ and $(x+\sqrt{2})$ are factors from the given roots. Now, using difference of squares, we know

$$(x-\sqrt{2})(x+\sqrt{2})=x^2-(\sqrt{2})^2=x^2-2$$

Finally, because $x^2-2=x^2-ax+b$ we see that $b=-2$.

17. The absolute value equation that would give the coordinates of the two solutions is:
$|x+2|=7$

In fact, we would find the two coordinates by proceeding as follows.

$\rightarrow x+2=7 \vee x+2=-7$

$\rightarrow x=5 \vee x=-9$

But the task was to find out the difference $b-a$ if the solution is:

$|x+2|=7$

Clearly then, $b-a=7-2=5$.

18. We must solve the equation

$$\left(\frac{1}{x}\right)^2-3\frac{1}{x}+2=0$$

$$\rightarrow \frac{1}{x^2}-\frac{3}{x}+2=0$$

Let's multiply every term by x^2.

$\rightarrow 1-3x+2x^2=0$

$\rightarrow 2x^2-3x+1=0$

Now, we can factor the quadratic as follows:

$(2x-1)(x-1)=0$

From here, we get that the solutions are $x=1$ and $x=\frac{1}{2}$, so either of these two values can be given as the correct answer.

19. Let's say the rectangle has $w=1w=100\%w$ and $l=1l=100\%l$ as length. Then, the area of the rectangle is $A=wl$.

Decreasing the length of the rectangle by p percent means we have $\left(1-\frac{p}{100}\right)l$ because $1=100\%$.

Similarly, increasing the width by $p/2$ percent means $\left(1-\frac{p}{200}\right)w$.

With these modifications, the area of the rectangle has become

$$A=\left(1-\frac{p}{100}\right)l\left(1-\frac{p}{200}\right)w=¿$$

$$\left(1-\frac{p}{100}\right)\left(1-\frac{p}{200}\right)wl=0.92\,wl$$

Now on the right side of this last equation, 0.92 represents an 8% decrease in the area (wl) of the rectangle.

20. If there is no solution to a system of linear equations, the two lines must be *parallel* AND have different $y-$¿intercepts. This means that the coefficients of x and y must be multiples of one another (of the same magnitude) while the constants on the other side of the equal signs aren't multiples by the same magnitude. Since we have the coefficients on x on both equations, we can quickly see that the coefficient on x in the second equation is $\frac{-1}{3}$ of the coefficient of x in the first equation. So, to find the value of k, we must write $9\left(\frac{-1}{3}\right)=k=-3$. Now, if the question is well written, we should not have to check the constants, and such is the case because we can see that $12\left(\frac{-1}{3}\right)=-4\neq 2$ as desired.

Solutions to Polar Pi SAT Practice Test 2

Solutions to Section 4

While detailed solutions are provided for every problem, should you need more support, you should consult the appropriate section. For example, problem 23 in section 4 is entirely about *"Completing the Square,"* and there is a section in this book with lessons and practice strictly on *Completing the Square*.

1. $g(2)=4$ reading from the table concerning the function g. So then, $h(g(2))=h(4)=7$, which is answer choice D. Clearly, we are reading off the table concerning h for the value $h(4)$.

2. To simplify the *complex number*, we will combine the fractions. We need to get a common denominator, and then we will expand the numerator. As we proceed, we should keep in mind that $i^2=-1$.

$$\frac{1+i}{1-i}+\frac{1-i}{1+i}=\frac{(1+i)(1+i)}{(1-i)(1+i)}+\frac{(1-i)(1-i)}{(1+i)(1-i)}=\frac{(1+i)^2+(1-i)^2}{1-i^2}=\frac{2(1+i^2)}{2}=0$$

This *complex number* is equal to 0, which is answer choice C.

3. $g(x)-f(x)=0$ means that we must solve $x^2-x-(x+3)=0$.
Simplifying, we can write:
$\rightarrow x^2-2x-3=0$

Now we can factor the equation as follows:

$\rightarrow (x-3)(x+1)=0$

From this factored form, we get two *solutions*: $x=3$and $x=-1$. Therefore, the correct answer is B. I and II.

4. Since the *ratio* of boys to girls is $e:\pi$, we can rewrite this as $ek:\pi k$, where $k$is a constant. Therefore, we set up the following proportion.

$$\frac{b}{g}=\frac{e}{\pi}=\frac{20}{g}\rightarrow$$

Cross multiplying,

$\rightarrow \geq ¿ 20\pi \rightarrow 2.72g=20\pi$

$$\rightarrow g=\frac{20\pi}{2.72}$$

Now using a calculator, we learn that the number of girls is *approximately* equal to 23[C.]

5. The first thing we should do is to find the value of the side AC. We see that since one leg is 4 the other one is 3, by the well-known *Pythagorean triple* (the $3-4-5$), the hypotenuse must be 5. Now, we can apply the *Pythagorean theorem* and find the value of AD. We do so recognizing say x, as the hypotenuse with the two legs measuring 5 and 3.

$$x^2=AD^2=5^2+3^2=34$$

$$AD^2=34\rightarrow AD=\sqrt{34}$$

6. We know by the binomial square that $(x-y)^2=x^2-2xy+y^2$. Therefore, $x=2a$ and $y=\frac{b}{4}$. Plugging in on the right side of the equation we have just written, we find that

$$\left(2a-\frac{b}{4}\right)^2=(2a)^2-2(2a)\left(\frac{b}{4}\right)+\left(\frac{b}{4}\right)^2$$

Simplifying the right side, we then have:

$$\left(2a-\frac{b}{4}\right)^2=4a^2-ab+\frac{b^2}{16}$$

Which is answer choice D.

7. The first thing we should do is factor the expression we are asked for to see if there are some common factors we can cancel and simplify. Otherwise, this is a simple task of applying a given definition (operation.)

   To the end that we seek, we write:

   $$\frac{x}{y}\left(xy+xy^2\right)=\frac{x}{y}\left(xy(1+y)\right)=x^2(1+y)$$

   Now, since we have that:

   $$(a,b)(c,d)=(a+c,b+d)$$

   It follows that

$$(4,2)(7,77)=(x,y)=(4+7,2+77)=(11,79)$$

$$x=11 \text{ and } y=79.$$

$$\rightarrow \frac{x}{y}\left(xy+xy^2\right)=x^2(1+y)=121(80)=9680 \text{ [D]}$$

8. The given equation is $4x^2+2x-t=0$. Now, the number of solutions and the type of solutions to a quadratic can be understood by computing the discriminant b^2-4ac and considering its value. To have *two real solutions*, *the discriminant* must be a *positive number*.

   Here, *the discriminant* is:

   $$(2)^2-4(4)(-t)=4+16t$$

   Now that we have simplified the discriminant to $4+16t$, *any positive value* of t would make *the discriminant* a *positive number*. Therefore, the correct answer is C as it is the only positive number among the answer choices.

9. The correct answer is B. The cost is inversely related to the volume of the spill because as the volume increases, the cost decreases. For example, when $v=1$, $c=200$ and when $v=200$, $c=200/200=1$. Inversely related variables behave this way, when one variable increases, the other decreases.

10. We are told that we have two lines, line l with slope 2 and line m with slope -3. Knowing this, we can write the equations of the lines as follows.

    $$l=2x+k$$

    $$m=-3x+c$$

    We will find the values of k and c soon. First, we need to find the *zeros* of the *quadratic*. We can do this by solving the equation:

$-(x-1)^2+4=0 \rightarrow 4-(x-1)^2=0 \rightarrow 4=(x-1)^2$

We can now take the square root on both sides, not forgetting to write ±.

$$\pm 2=x-1 \rightarrow x=1\pm 2 \rightarrow x=-1 \text{ or } x=3$$

We now have the *zeros* of the *quadratic*. Since we also know that the lines l and m both go through these points (the *zeros* of the *quadratic*) we can use this to get to the values of k and c.

Line l goes through the point $(-1,0)$ on the parabola because, since the parabola is facing down, line m which has a negative slope would cut through the parabola twice. We cannot allow this because we know the lines are *tangents* not *secants*. And therefore, we have:

$0=2(-1)+k \rightarrow 0=-2+k \rightarrow k=2$

Using similar reasoning as above, we see that line m is the one that goes through $(3,0)$. Meaning:

$0=-3(3)+c \rightarrow 0=-9+c \rightarrow c=9$

And so then, the equations of the lines $l$and m are:

$l=2x+2 \wedge m=-3x+9$

To find their intersection, we solve the following linear equation:

$2x+2=-3x+6 \rightarrow 5x=4 \rightarrow x=\frac{4}{5}$

But this is only the $x-$¿coordinate of the point of intersection. To find the $y-$¿ coordinate, we merely do substitution as follows.\:

$y=-2\left(\frac{4}{5}\right)+6 \rightarrow y=\frac{-8}{5}+\frac{30}{5} \rightarrow y=\frac{22}{5}$

Hence, the points $C$is located at $\left(\frac{4}{5},\frac{22}{5}\right)$ and the sum of the coordinates is

$\frac{4}{5}+\frac{22}{5}=\frac{26}{5}$. The correct answer is C.

11. Since the number of hours (h) is the variable, and the score is obtained by multiplying the hours by 20 and adding 3, the equation should be $f(h)=20h+3$, which is answer choice D.

12. Let's calculate the difference for each paired state. For *Michigan* and *Iowa*, the difference is $4.1-(-6)=10.1$On the other hand, for *Pennsylvania* and *Wisconsin*, the difference is $2.1-1.1=1$. Therefore, the difference of the differences is $10.1-1=9.1$, which means that the correct answer is A.
13. In all the answer choices, since we are multiplying pairs of negative numbers, the result will be positive. Hence, to get the smallest product, we should multiply the biggest negative numbers. In other words, we

should pick the number with the smallest absolute value. z is the biggest negative number, so the smallest expression will be $(z)(z)$, which is answer choice A.

14. For additional support, consult previous similar problems and respective solutions or appropriate section in this book. We know the task before us concerns the coefficients.

$$ax-3y=21$$
$$bx+9y=42$$

We need the lines to be parallel but with different y-intercepts. So, we need $b=-3a$ and $42\neq-3(21)$ The latter is easy to see, and the former is the relationship we are seeking.

15. Soon as you face a similar task to the one presented in this problem; your instinct should be to factor.

$$u^2v-uv^2=uv(u-v)$$
¿$(-4)(2)$

¿-8(B)

16. When two *lines are parallel*, they must have the *same slope*. As both equations are given in the *Slope-Intercept* form, we can immediately write:

$$2a=\frac{-b}{2}\rightarrow b=-4a$$

The slope in the *Slope-Intercept* form is always the coefficient on x. Thus, the correct answer is A.

17. Our task is to solve the following equation

$$4-2\sqrt{x}=7-\sqrt{x}$$

First, we will combine like terms. Doing so, we write:

$$\rightarrow 4-7=2\sqrt{x}-\sqrt{x}\rightarrow -3=\sqrt{x}$$

Since our last step gives the equation $\sqrt{x}=-3$, the correct answer must be D. No solution. This is so because the square root is always positive unless there is a "$-$¿" sign in front of the square root. We could have reached this same conclusion differently. If we take our last step and square both sides, we will have:

$$(-3)^2=(\sqrt{x})^2\rightarrow 9=x$$

Now, as the equation is a radical equation, we need to check for extraneous solutions. This requires that we plug in this solution into the original equation. When we do, we would have to write:

$$4-2\sqrt{9}=7-\sqrt{9}\rightarrow 4-2(3)=7-3$$

$$\rightarrow 4-6=-2=7-3=4$$

But $-2 \neq 4$.

18.

$$\frac{4}{3}=\frac{4x-2}{2-4x}$$

Cross multiplying, we have

$$4(2-4x)=3(4x-2)$$

$$\rightarrow 8-16x=12x-6$$

Combining like-terms, we have

$$\rightarrow 8+6=12x+16x$$

$$\rightarrow 14=28x \rightarrow x=\frac{14}{28}=\frac{1}{2}$$

Now, we must answer the question, which asks for the value of

$$\frac{5x}{2}=\frac{1}{2}(5x)=\frac{1}{2}\left(5\left(\frac{1}{2}\right)\right)=\frac{5}{4}$$

Therefore, the correct answer is D.

19. Using exponent rules, we can write the given equation

$$x^{a^2}x^{-b^2}=x^{25}, x>1$$

as

$$x^{a^2-b^2}=x^{25}, x>1$$

Now, since the bases are the same, we only need to set the exponents equal.

Setting exponents equal, we have

$$a^2-b^2=25$$

On the left-side, we have a *difference of squares*. Which means

$$a^2-b^2=(a+b)(a-b)=25$$

Since we are given that $a+b=1$, the equation reduces to

$$1(a-b)=25$$

Therefore, we see that $a-b=25$.

20. Our task is to solve the following equation:

$$\frac{x}{x-2}+\frac{x}{(x-2)(x-3)}=\frac{4}{x-3}$$

Of a variety of ways, we can move forward, the following is both fun and efficient.

The given equation is the same as:

$$\frac{x}{(x-2)(x-3)}=\frac{4}{x-3}-\frac{x}{x-2}$$

Now, common denominators on the right side

$$\frac{x}{(x-2)(x-3)}=\frac{4(x-2)-x(x-3)}{(x-2)(x-3)}$$

$$\frac{x}{(x-2)(x-3)}=\frac{-x^2+7x-8}{(x-2)(x-3)}$$

Keeping in mind the restriction that $x\neq 2 \wedge x\neq 3$, as the denominators are now the same, we just set numerators equal.

$$x=-x^2+7x-8$$

Getting everything on one side, we will be left with a simple quadratic.

$$x^2-6x+8=0$$

$$\rightarrow (x-4)(x-2)=0$$

$$\rightarrow x=4 \text{ or } x=2$$

One of our restrictions was that $x\neq 2$, so the only valid solution is $x=4$ only (B.)

21. We are told that the point $(6,9)$ lies on the graph of the function $f(x)=4c-2x^2$. So, we write:

$$9=4c-2(6)^2$$

$$\rightarrow 9=4c-72$$

$$\rightarrow c=\frac{81}{4}$$

Hence, the correct answer is C.

22. To find the area of the circle, we need to know the length of the radius. Since we know where the center is, and we know a point that lies on the circle, we can use the distance formula to find the length of the radius, all too crucial for our ultimate task, area.

$$r=\sqrt{(4-7)^2+(-2-3)^2}=\sqrt{9+25}=\sqrt{34}$$

$$\rightarrow A=\pi r^2=\pi(\sqrt{34})^2=\pi\left(34^{\frac{1}{2}}\right)^2=34\pi \text{(B.)}$$

23. First, we should find the value of c. We can do it by solving the following equation:

$$2g(c)=8$$

$\rightarrow 2(c-1)=2c-2=8$

$\rightarrow 2c=10 \rightarrow c=5$

Now, we want to find $g(3c)=g(15)$. So we simply write:

$$g(15)=15-1=14 \text{ (D)}$$

If a function g is defined by $g(x)=x-1$ and $2(c)=8$, what is the value of $g(3c)$

24. Essentially what we need to do is find the equation of a *quadratic* that goes through the point $(0,2)$. In other words, a *quadratic* with $y-$¿intercept equal to 2. This is any *quadratic* of the form ax^2+bx+2. If we look though the answer choices, we see that the only equation that satisfies this criterion is $y=x^2+2$. Thus, the correct answer is A.

25. The correct answer is D. 88.2 *hours*

    Since *Reus* has 42 boxes, the total time will be equal to $42 \cdot$ *Time per box*. Since we can easily see that there are two square faces, the time for lacquering these two faces will be $7 \cdot 2=14$ *minutes*. Moreover, we know that a box has four additional rectangular sides, and thus, the total time for covering these rectangular sides will be $4 \cdot 28=112$ *minutes*.

    Combining these two, the total time for all boxes will be $42 \cdot (14+112)=5292$ *minutes*. But we are asked to give an answer in terms of hours and as such, we just have a bit more arithmetic to do. Specifically, we must write: $\frac{5292}{60}=88.2$ *hours*.

26. If the cubic has roots $x=-3,-5$, and d, then, it means the polynomial must be of the form $a(x-(-3))(x-(-5))(x-d)$

    $\rightarrow a(x+3)(x+5)(x-d)$

    Now, we are given that this polynomial is the same as

    $x^3+bx^2+cx+15$

    So, we immediately see that $a=1$ and 15, the constant at the end is equal to $3 \cdot 5 \cdot (-d)$

    $\rightarrow -15d=15 \rightarrow d=-15$

27. We can do *long division* here, but we can avoid doing *long division* through the clever trick below. In the appropriate section in this book, you should have had enough practice on this.

$$\frac{4x-1}{x+7}=\frac{4x+28-28-1}{x+7}$$

$$\rightarrow \frac{4x+28-29}{x+7}=\frac{4x+28}{x+7}-\frac{29}{x+7}$$

$$\rightarrow = \frac{4(x+7)}{x+7} - \frac{29}{x+7} \rightarrow = 4 - \frac{29}{x+7}$$

Therefore, the correct answer is D.

28. We must find the solutions to the following equation.

$$-(x+4)(x-2)=0$$

We can multiply both sides by -1 to get rid of the negative sign the equation leads with.

$$\rightarrow (x+4)(x-2)=0$$

Since the equation is already in a factored form, we can easily find the roots.

$$x=-4 \text{ or } x=2$$

We were also told that s is the bigger number $(s>t)$, so $s=2$ and $t=-4$. Thus, $|s-t|=|2-(-4)|=|6|=6$. The correct answer is D.

29. So, we have a section in this book on inverse and direct relations. We also have a section on Work Problems. While it is tempting to confuse this question for a work problem, it is a problem about inverse relations. Because, the more men do the work, the shorter (timewise) it takes to finish the job. The relationship between the two variables would be intact even if it we changed “job” into the plural, “jobs.”

Having acknowledged an inverse relation, we immediately write:

$$y=\frac{k}{x} \rightarrow j=\frac{k}{h}$$

Now, since it takes 4 men 7 hours, we have

$$\rightarrow 4=\frac{k}{7} \rightarrow k=28$$

$$\rightarrow j=\frac{28}{h}$$

Now, we want to know how long it would take 7 men to do the job.

$$\rightarrow 7=\frac{28}{h} \rightarrow \frac{1}{7}=\frac{h}{28}$$

$$\rightarrow h=\frac{28}{7}=4$$

Meaning, four hours [C.]

30. To solve for the variable c, we proceed as follows:

The given equation is:

$$l=\frac{l_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

First, multiplying both sides by the denominator on the right side, we have:

$$l\sqrt{1-\frac{v^2}{c^2}}=l_0$$

Now, dividing both sides by l, we can write:

$$\sqrt{1-\frac{v^2}{c^2}}=\frac{l_0}{l}$$

Squaring both sides, we have:

$$1-\frac{v^2}{c^2}=\left(\frac{l_0}{l}\right)^2 \rightarrow 1-\left(\frac{l_0}{l}\right)^2=\frac{v^2}{c^2}$$

$$\rightarrow 1-\frac{l_0^2}{l^2}=\frac{v^2}{c^2} \rightarrow \frac{l^2}{l^2}-\frac{l_0^2}{l^2}=\frac{v^2}{c^2}$$

Getting a common denominator on the left side means that we write:

$$\rightarrow \frac{l^2-l_0^2}{l^2}=\frac{v^2}{c^2}$$

Now, flipping both sides, that is, taking the reciprocal of both sides of the equal sign, we have:

$$\rightarrow \frac{l^2}{l^2-l_0^2}=\frac{c^2}{v^2}$$

Next, multiplying both sides by v^2 we write:

$$\rightarrow \frac{v^2 l^2}{l^2-l_0^2}=c^2$$

Finally, taking the square root of both sides, and simplifying looks as follows:

$$\sqrt{\frac{v^2 l^2}{l^2-l_0^2}}=c \rightarrow \frac{vl}{\sqrt{l^2-l_0^2}}=c$$

31. To find the angle $x°$, we will do the following. Since lines l and m meet at the center, they are both diameters of the circle. Thus, arc AC must be $180°$,because half of a circle is $180°$. Now, since BC measures $80°$, thus the rest of the arc AC must measure $100°$. Arc AB is two times the value of x (An *Inscribed Angle* is half of the measure of the arc it intercepts) we can write:

Hence,

$$100°=2x \rightarrow 50°=x$$

32. To find the number we can set up the following equation:

$$0.3x=1 \rightarrow x=\frac{1}{0.3}=3.333$$

33. Substitution is the best approach here:

$$x^2-4y^2=2$$

$$x-y=1 \rightarrow x=y+1$$

We will substitute the value of x in the first equation:

$$(y+1)^2-4y^2=2$$

$$y^2+2y+1-4y^2=2$$

$$-3y^2+2y-1=0$$

We can solve for y using the quadratic formula.

$$x=\frac{-2\pm\sqrt{2^2-4(-3)(-1)}}{2(-3)}=\frac{-2\pm\sqrt{-8}}{-6}=\frac{-2\pm 2i\sqrt{2}}{-6}=\frac{-1\pm i\sqrt{2}}{-3}=\frac{1\pm i\sqrt{2}}{3}$$

Therefore, we see that the real par of y is $\frac{1}{3}$.

34. If $7^k=100$, we can raise both sides of this equation to the power $\frac{1}{2}$ and apply the appropriate rule(s) of exponents to write:

$$(7^k=100)^{\frac{1}{2}}$$

$$\rightarrow 7^{\frac{k}{2}}=10$$

Now, we can multiply both sides by 7.

$$7^{\frac{k}{2}}\cdot 7^1=10\cdot 7^1$$

Adding exponents on the left-side and multiplying on the right, we have:

$$7^{\frac{k}{2}+1}=70$$

35. Although it doesn't appear so, we know that the triangle is a right triangle because it otherwise doesn't make sense to talk about the Sine and Cosine values of the other two angles in the triangle. Since we are given

$\sin\left(\frac{a\pi}{8}\right)=\cos(b\pi)$, we know that $\frac{a\pi}{8}$ and $b\pi$ are the two complementary acute angles in the triangle. As such, we write:

$$\frac{a\pi}{8}+b\pi=\frac{\pi}{2}\rightarrow$$

$$\frac{a}{8}+b=\frac{1}{2}\rightarrow$$

Now, because we are asked simply for $a+b$, no amount of manipulation of this last equation will get us there.

Before we proceed, notice that if we recall the *Complement – Angle Rule* in Trigonometry that told us that if α and β are two acute angles in a right triangle, it must follow that:

$$\sin\alpha=\cos\left(\frac{\pi}{2}-\alpha\right)$$

This means, because of the given $\sin\left(\frac{a\pi}{8}\right)=\cos(b\pi)$

$$b\pi=\frac{\pi}{2}-\frac{a\pi}{8}$$

$$\rightarrow b\pi=\frac{\pi}{2}-\frac{a\pi}{8}\rightarrow b=\frac{1}{2}-\frac{a}{8}$$

$$\frac{a}{8}+b=\frac{1}{2}$$

So now that we are in the same place as the first path, to finish the solution, we need to use the fact that the *ratio* of $a:b$ is $4:2$.

$$\rightarrow\frac{a}{b}=\frac{4}{2}=2$$

$$\rightarrow a=2b$$

$$\frac{a}{8}+b=\frac{1}{2}\rightarrow\frac{2b}{8}+b=\frac{1}{2}$$

$$\rightarrow\frac{b}{4}+b=\frac{5b}{4}=\frac{1}{2}$$

$$5b=\frac{4}{2}=2\rightarrow b=\frac{2}{5}$$

So then,

$$\frac{4}{5}+\frac{2}{5}=\frac{6}{5}$$

36. We need to find the length of the whole segment PS. We are told that $PQ=RS$, so we can set up the following equation and solve for x.

$$9x-1=5x \rightarrow 4x=1 \rightarrow x=\frac{1}{4}$$

Now, the total length of PS is $9x-1+2x+4+5x=16x+3$. Plugging in the value of $x=\frac{1}{4}$, we have $PS=16\left(\frac{1}{4}\right)+3=7$.

37. To find the intersections, we set $y=y \rightarrow 4x^2-2x=2x$.

$$\rightarrow 4x^2-4x=0 \rightarrow 4x(x-1)=0$$

$$4x=0 \text{ or } x-1=0$$

$$x=0 \text{ or } x=1$$

We have found the $x-¿$coordinates. To find the y-values we must plug in in the $x-¿$coordinate values in either of the equations. The easier one to plug into is $y=2x$.

$$y=0(2)=0$$

$$y=1(2)=2$$

We now have all we need: $(0,0)$ and $(1,2)$. Comparing the second points with $(a,2a)$, we see that $a=1$.

38. This is just a giant exercise on the *difference of squares* $\rightarrow$ $x^2-y^2=(x+y)(x-y)$

Here, in what we are given, we start with $x^2=a^4=(a^2)^2$ and $y^2=b^4=(b^2)^2$.

$$a^4-b^4=4$$

$$\rightarrow (a^2-b^2)(a^2+b^2)=4$$

We can further factor the left-hand side:

$$\rightarrow (a-b)(a+b)(a^2+b^2)=4$$

Now, we can substitute the values of $(a-b)$ and $(a+b)$:

$$\rightarrow (4)(2)(a^2+b^2)=4 \rightarrow 8(a^2+b^2)=4$$

$$\rightarrow a^2+b^2=\frac{4}{8}=\frac{1}{2}.$$

Definitions

A

Altitude: The shortest vertical distance from the highest point on a triangle or a quadrilateral to another point on a side across from first point.

Altitude to the hypotenuse

Average rate of change (*average slope*): The length of a secant line on a curve or the quotient of the difference between two y values and the difference between the corresponding x values to the y-values, in the same order.

B

Bisect: to cut in half.

C

Categorical data: Data about say gender (male vs female) etc.
Central Angle: An angle formed by two rays containing two radii of a circle.
Circumference: The length around a
Coefficient: A constant usually in front of (multiplying) a variable. Note sometimes, the constant may not be in front for mathematical grammar reasons ex: $x\sqrt{2}$

Completing the square: The process of taking a quadratic from the standard form $a\,x^2+bx+c$ to a form $t^2 \mp a$, with a variable substitution where a is a constant.
Concentric circles: circles that share the same center.
Correlation coefficient: A measure of the strength of the relationship between two variables, one along a horizontal axis and the other along a vertical axis.

D

Diagonal: A line connecting two nonconsecutive vertices of a polygon.

E

F

G

A Great circle: The biggest circle that can be drawn on a sphere (note that there are infinitely many great circles of which the equator and the prime meridian are two examples.)

H

I

Inscribed Angle: an angle formed by two intersecting but not overlapping chords.

J

K

L

A leg: the two shortest sides on a right triangle.

M

N

Natural logarithm: this is a logarithm of base number e or the inverse of $f(x)=e^x$. This is something you should study more as a follow up to what you've learned in this book.

O

P

Parallel lines: two lines that never intersect.
Perimeter of a semicircle: A half circumference together with the length of the diameter of the semicircle.
A point - that which has no parts
Point slope form
Point of tangency: the point of intersection of a line and a circle where the line intersects the circle only once.

The Pythagorean theorem: A theorem stating that a triangle is a right triangle if and only if the sum of the squares of the length of two smallest sides equals the square of the length of the largest side.

Q

A quadratic in Standard Form: a quadratic (second degree polynomial) written in the form ax^2+bx+c where b and c maybe equal to zero.

R

a *radian* measure of a *circle* is the same as if the *radius* of the very circle was made untaught and perfectly laid down on the *circumference* (along the circle's border) of the *circle*.
Radius: The distance from the center of a circle to a point on the circle.

Ratio: the quotient of a and b, where a and b may represent any quantities (including things like, $a=x+1$ and $b=x^2-5$.)
Rectangular prism: A box
Rhombus: A quadrilateral where all the sides are qual.
A right angle: A corner

S

Semi-perimeter: Half of the length of a perimeter.
Slope Intercept form
Solution set
Standard Deviation: A measure of the spread of numbers representing a particular data.
Standard form of a line

T

Transversal: A line cutting across two or more lines. Notice though transversals come up in the context of two or more parallel lines, the lines don't necessarily have to be parallel, we'd still use the name transversal to describe the line cutting across a bunch of lines.

U

V

W

X

Y

Z