CS880: Algorithmic Mechanism Design

Lecture 8: Prior-free Mechanism and Blackbox Reduction **Scribe:** Sihan Liu

8.1 Setting and meaningful benchmark

When priors are unknown, BIC mechanism doesn't make sense (Bayesian Equilibrium is defined with prior distributions). So, we focus on DSIC mechanism.

The most intuitive goal is to design a single mechanism that stays competitive against the optimal truthful mechanism (with oracle access of the valuation tuple in advance) for any given valuation tuples.

Formally, denote $\mathcal{M}$ as the class of truthful mechanisms. We want a single mechanism $m^* \in \mathcal{M}$ such that for any valuations $\mathbf{v} \in \mathcal{V}$, $\mathrm{Rev}(m^*, \mathbf{v})$ is at least some fraction over $\max_{m \in \mathcal{M}} \mathrm{Rev}(m, \mathbf{v})$.

However, as we will see in the next section, the gap is intractable even in some basic setting.

8.1.1 Inapproximability

Suppose we are trying to sell 1 item to 2 buyers with the valuations (v_1, v_2). In such case, the class of DSIC mechanisms is exactly two pricing function

1. Price faced by buyer 1: $t_1(v_2)$.
2. Price faced by buyer 2: $t_2(v_1)$.

Notice that t_1 is a function of v_2 only and t_2 is a function of v_1 only; otherwise, it incentives buyers to fake his/her valuation to minimize the price.

One extreme case is that v_2 is much larger than v_1. Fix any pricing function $t_2 : \mathbb{R} \mapsto \mathbb{R}$. v_2 could be order of magnitude larger than $t_2(v_1)$. Hence, the revenue achieved with the pricing function t_2 could be much smaller than any mechanism that charges a much higher price on v_2 on this specific instance.

We hence need a more meaningful benchmark to compare our mechanism with.

8.1.2 Envy-free mechanism

Intuitively, inapproximability comes from the fact we are discriminating between the buyers. Some buyer may be *envious* of others if they are charged a much higher price. Hence, a meaningful benchmark is to consider the class of *envy-free mechanism*, where no buyer is envious of the outcome of another.

Definition 8.1.1 (Envy Free Mechanism) *Suppose we are selling to n buyers. Denote the allocation of the mechanism to buyer-i as $\mathbf{x}_i$ and price charged from him/her as p_i. Then, we have buyer i prefers $(\mathbf{x}_i, p_i)$ than outcomes received by any other buyer i.e., $\mathit{Utility}^{(i)}(\mathbf{x}_j, p_j) \le \mathit{Utility}^{(i)}(\mathbf{x}_i, p_i)$ for all $j \in [n]$.*

To better illustrate the concept, we will give a mechanism that breaks the envy-free criteria.

Example: Suppose we are selling one item two buyers with the valuation v_1, v_2. Then, the mechanism with the pricing function $t_1(v_2) = \frac{v_2}{2}$ and $t_2(v_1) = 2 \cdot v_1$ violates the envy-free criteria. In particular, buyer-2 will be envious of buyer-1. ■

8.1.3 $EFO_{\geq 2}$ benchmark

Denote $\mathcal{E}$ as the class of envy-free mechanisms satisfying Definition 8.1.1. Given any valuation instance $\mathbf{v}$, we can compute the benchmark $EFO = \max_{m \in \mathcal{E}} \mathrm{Rev}(m, \mathbf{v})$. Sadly, such benchmark is still inapproximable.

Consider the setting where we are selling k identical items to n unit-demand buyers where the i-th buyer has the valuation v_i. The envy-free mechanism will allocate items to $k^{'} \leq k$ buyers such that

- All allocated buyer will be charged the same price.
- All buyer with valuation exceeding the price must be allocated/served.

It is easy to see the price p must sits within the interval $[v_{k'+1}, v_{k'}]$. Then, it follows that $\mathrm{Rev} = k' \cdot v_{k'}$. Therefore, $EFO = \max_{k' \leq k} k' \cdot v_{k'}$.

Unfortunately, this is also inapproximatable due to the existence of outlier. In particular, if the largest valuation is un-bounded, the envy-free mechanism will cater to only this customer and we fall back to a situation similar to the previously discussed 2-player case.

Hence, we slightly modify the benchmark to be

$$EFO_{\geq 2} = \max_{2 \leq k' \leq k} k' \cdot v_{k'}$$

Conceptually, the quantity can be treated as the instance-optimal revenue achieved by any envy-free mechanism that caters to at least two customers. As a result, the benchmark becomes a function of the valuations of at least two buyers, making it less sensitive to outliers.

8.1.4 Alternative interpretation with i.i.d priors

Suppose the valuation of each buyer is drawn independently from the distribution $\mathcal{F}$ (albeit unknown). The optimal mechanism (in expectation) is the vickrey auction with a single reserve price p computed from the virtual function of $\mathcal{F}$. Further, assume that the goods we are selling has unlimited supply (i.e digital goods). Then, the mechanism is just a single pricing rule p (since there is no longer competition among buyers). Now, consider

$$\mathcal{M} = \{\text{all mechanisms that are optimal w.r.t some distribution } \mathcal{F}\}.$$

Essentially, we are considering the class of mechanism of all possible pricing. Given $(v_1, \cdots, v_n)$, it is easy to see that the optimal mechanism from the class is $\max_{p \in M} p \cdot |\{i : v_i \geq p\}| = EFO$, which gives an alternative interpretation of the benchmark in Section 8.1.2.

8.2 Competing against the $EFO_{\geq 2}$ benchmark

In this section, we will see truthful mechanisms (but not necessarily envy-free) that stays competitive against the benchmark $EFO_{\geq 2}$ for unlimited goods supply and unit-demand buyers.

Algorithm 1 Partition and Sell

Input: A set of valuation $V = \{v_1, v_2, \cdots, v_k\}$ where $v_1 \geq v_2 \cdots \geq v_k$.

1: Randomly partition the datas into two groups S and M where $S \cap M = \emptyset$ and $S \cup M = V$. Without loss of generality, assume $v_1 \in M$.

2: Learn the optimal reserved price from the group S. Namely,

$$p_S = \max_p p \cdot |\{i : v_i \geq p\}|.$$

3: Sell to all customers in M with the the price p_S.

Theorem 8.2.1 *Suppose we are selling to unit-demand buyers with valuation $v_1, v_2, \cdots, v_n$ with unlimited goods supply, the mechanism outlined in Algorithm 1 extracts a constant fraction of the revenue $EFO_{\geq 2} = \max_{2 \leq k'} (k' \cdot v_{k'})$.*

We make the following claims. First, the optimal envy-free revenue on a random set is at least a fraction overal the optimal.

Claim 8.2.2 *Given a set of valuations V. Denote EFO_S as the revenue of the optimal envy free mechanism on a set of values $S \subseteq V$. With good probability, the optimal revenue for the sample EFO_S is close to the benchmark $EFO_{\geq 2}$.*

Proof: We will use the following notations. Suppose p^* is the price that achieves $EFO_{\geq 2}$. Let $k = |\{i \in V : v_i \geq p^*\}|$ and $S_k = |\{i \in S : v_i \geq p^*\}|$. Denote $\mathcal{B}$ as the event that $S_k \geq k/2$. It is easy to see that the pricing p^* will have revenue $p^* \cdot S_k$ while $EFO_{\geq 2} = k \cdot p^*$. Hence, conditioning on $\mathcal{B}$, we have $EFO_S \geq S_k \cdot p^* \geq \frac{1}{2} EFO_{\geq 2}$.

It remains to show that the event $\mathcal{B}$ happens with good probability. Since the highest bidder is always put into the market set M, we need to bound the probability that in the remaining $k-1$ customers at least $k/2$ of them is put into the sample set S. This is easy when k is even. Since $k-1$ is odd, in any partition, there is always one subset which has at least $k/2$ items and with probability $1/2$ the set will be put into S. When k is odd $Pr[\mathcal{B}] < 1/2$, one need slightly more complicated arguments to get an accurate lower bound (or we simply get worse constant). We omit the details. ■

Secondly, we claim that the revenue that can be achieved on the two random partitions S and M are comparable for any pricing.

Claim 8.2.3 *Denote $Rev(S, p)$ as the revenue one can achieve on a subset of customers S with the pricing p. With high constant probability, it holds that $Rev(M, p) \geq \frac{1}{3} \cdot Rev(S, p)$.*

Proof: First, assume all valuations are sorted in descending order i.e., $v_1 \geq v_2 \cdots v_n$. We can think of the partition process as a stochastic process where we iterate throught the valuation in order and randomly decide whether we put it in the set S or M. Denote $X_t = |M_t| - \frac{1}{3}|S_t|$, where M_t and S_t

are the sets obtained after we have made the decision for v_t. Denote $\mathcal{E}$ as the event $X_t \geq 0$ for all $t \in [n]$. For all p, it holds that $Rev(M,p) - \frac{1}{3}Rev(S,p) = X_t \cdot p$ for some t. Then, conditioning on $\mathcal{E}$, it is easy to see that $Rev(M,p) \geq \frac{1}{3} \cdot \text{Rev}(S,p)$. It remains to bound the probability of $\mathcal{E}$. This type of calculation is known as the probability of ruin in analogy to a gambler's fate when playing a game with such a payoff structure and we summarize it in Lemma 8.2.4. ■

Lemma 8.2.4 (Balanced Sampling, Lemma 6.4 in [Har12]) *For $X_1 = 0$, X_i for $i \geq 1$ an indicator variable for a independent fair coin flipping to heads, and sum $S_i = \sum_{j \leq i} X_i$, it holds*

$$Pr\left[\forall i, (i - S_i) \geq S_i/3\right] \geq 0.9.$$

With the above two claims, conditioning on the event $\mathcal{E}$ and $\mathcal{B}$, it holds the partition based mechanism is at least 1/6 over the benchmark $EFO_{\geq 2}$. By unioin bound, $Pr[\mathcal{E} \cap \mathcal{B}] \geq 0.4$. Overall, we obtain a competitive ratio of 15. With more careful analysis, one can imrpove the bound to about 4.68 (See in [AMS09]).

Remark 8.2.5 *To see why it is crucial to compare against $EFO_{\geq 2}$, recall that we need the market to contain the highest bidder. Without it, Lemma 8.2.4 will hold for a much lower probability (less than 1/2). Hence, the sample set S must contain the highest bidder. It is therefore impossible for EFO_S to stay competitive against EFO due to the existence of outlier.*

8.3 Random Sampling Profit Extraction Mechanism

An alternative mechanism that also stays competitive with the $EFO_{\geq 2}$ benchmark is the *Random Sampling Profit Extraction Mechanism.* The procedures are similar but are much easier to analyze.

1. Randomly partition the datas into two groups S and M.
2. Learn the optimal revenue R_S that can be achieved by pricing on S. If $EFO_M \leq EFO_S$, simply abort. Try to extract the same revenue on M with the lowest price (See the routine below).

Algorithm 2 Profit-Extraction

Input: A set of valuation $M = \{v_1, v_2, \cdots, v_k\}$ where $v_1 \geq v_2 \cdots \geq v_k$ and a target revenue B

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i = k
while v_i ≤ B/i do
    i = i-1
end while
return B/i
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3. Repeat step (2) by switching the roles of S and M.

Claim 8.3.1 *The revenue of the mechanism, conditioning on the partition (M,S), is exactly $\min(EFO_S, EFO_M)$.*

Proof: If $EFO_M \geq EFO_S$, the mechanism extracts a revenue of EFO_S in step (2). If $EFO_M < EFO_S$, the mechanism extracts a revenue of EFO_M in step (3). ■

Claim 8.3.2 *The mechanism is D.S.I.C*

Proof: Though it seems like the pricing rule applied to the set M depends on the actual values inside M, we argue the dependency does not break the incentive compatibility. Suppose a customer reports a lower value. Then, the while loop in Algorithm 2 will only run more iterations. Hence, we will end up with a higher price, which is against all customers' wills. Suppose the customer i reports a higher value $\tilde{v}_i \geq v_i$. It will change the outcome if and only if $v_i \leq B/i \leq \tilde{v}_i$. Hence, the price will become B/i, which results in negative utility of customer i. Thus, the mechanism is truthful/DSIC. ■

Claim 8.3.3 $\mathbf{E}_{(S,M)\sim\mathcal{P}}\left[\min\left(EFO_S, EFO_M\right)\right] \geq \frac{1}{4} \cdot EFO_{\geq 2}$..

Proof: We will use the following notations. Assume p^* is the optimal pricing realizing $EFO_{\geq 2}$. Then denote i^* as the number of total values exceeding p^*. Denote i^*_M as the number of values in M exceeding p^*. Denote i^*_S as the number of values in S exceeding p^*. It holds

$$\mathbf{E}_{(S,M)\sim\mathcal{P}}\left[\min\left(EFO_S, EFO_M\right)\right] \geq \mathbf{E}_{(S,M)\sim\mathcal{P}}\left[\min\left(i^*_M, i^*_S\right)\right] \cdot p^* \geq \frac{1}{4} i^* \cdot p^* = \frac{1}{4} EFO_{\geq 2}.$$

■

8.4 Blackbox Reduction

Social welfare maximization (S.W) for single parameter buyers $\mathbf{v}_1, \cdots, \mathbf{v}_n$ can be expressed as the program

$$\max \sum_i \mathbf{x}_i \mathbf{v}_i, \text{ subject to feasibility constrain}$$

Given some algorithm $\mathcal{A}$ that solves the S.W program approximately, we want to study whether we can convert $\mathcal{A}$ into an I.C mechanism in a black-box manner. The following theorem explores the direction under Bayesian setting.

Theorem 8.4.1 *Given an algorithm $\mathcal{A}$ which solves the S.W problem, we can construct a mechanism $\mathcal{M}$ with black-box access to $\mathcal{A}$ satisfying*

1. *The mechanism is B.I.C up to accuracy ϵ (limited by the number of probes allowed for $\mathcal{A}$).*
2. $\mathbf{E}_{\mathbf{v}\sim\mathcal{D}}\left[SW(\mathcal{M}(\mathbf{v}))\right] \geq \mathbf{E}_{\mathbf{v}\sim\mathcal{D}}\left[SW(\mathcal{A}(\mathbf{v}))\right](1-\epsilon)$. *Namely, the welfare achieved by mechanism M approaches to the objective $\sum_i \mathbf{x}_i \mathbf{v}_i$ achieved by $\mathcal{A}$.*
3. *The mechanism invokes* $\text{poly}(n, 1/\epsilon)$ *calls to $\mathcal{A}$.*
4. *The output (allocation rule) $\mathbf{x}$ of $\mathcal{M}$ is the output of $\mathcal{A}$ under some input. Hence, $\mathcal{M}$ satisfies the feasibility constrain.*

Now, we outline the proof for Theorem 8.4.1. For the mechanism to be incentive compatible, for customer i, we require the allocation rule to be monotonically increasing in $\mathbf{v}_i$ after taking expectation over the rest of the customers $\mathbf{v}_{-i}$. Namely, $\mathbf{x}_i(\mathbf{v}_i) := E_{\mathbf{v}_{-i}\sim\mathcal{D}}\left[\mathbf{x}_i(\mathbf{v}_i, \mathbf{v}_{-i})\right]$ is monotonically increasing. The main idea is to probe the allocation function $\mathbf{x}_i(\mathbf{v}_i)$ on an buyer i and applies ironing. The remaining is to show:

1. There is a proper way of ironing such that the feasibility constrain is maintained and it has no influence to the expected allocation function of other customers.

2. After ironing, the revenue won't decrease too much. (Actually, as we will see, the revenue won't decrease at all.)

In the rest of the section, we will focus on just the allocation of a single customer. Hence, we will omit the subscript i.

First, we state the ideal allocation function. To do that, we first look at the allocation rule in the quantile space. Formally, denote $x(q)$ as the probability of selling to a customer with valuation at the q-th percentile (from low values to high values). Furthermore, define

$$X(q) = \int_0^q x(q)dq.$$

Notice that $X(q)$ is convex if and only if $x(q)$ is non-decreasing. Hence, we will compute $\bar{X}$ as the convex lower envelop of X. The ironed allocation function will be $\bar{x}(q) = \frac{d}{dq}X(q)$. Notice that whenever $X \neq \bar{X}$, $\bar{x}$ is linear. Essentially, if $X(q) \neq \bar{X}(q)$ on interval $[a,b]$, we need to make $x(q)$ constant on the interval by "flattening" the allocation in the interval (See Figure 8.4.1).

One way for doing that is to re-sample v from $[a,b]$ uniformly if originally $v \in [a,b]$. That way, we have

1. The allocation rules $x(v)$ are flattened to be the same in expectation for $v \in [a,b]$.

2. The feasibility constrain is preserved since the output $\mathcal{M}(\mathbf{v})$ always corresponds to the output of $\mathcal{A}(\tilde{\mathbf{v}})$ for some $\tilde{\mathbf{v}}$ after re-sampling.

3. The distribution for $\mathbf{v}_i$ does not change since the re-sampling is uniform.

Thus, uniform re-sampling on all intervals $[a,b]$ where $X(q) \neq \bar{X}(q)$ when $q \in [a,b]$ performs the ideal ironing.

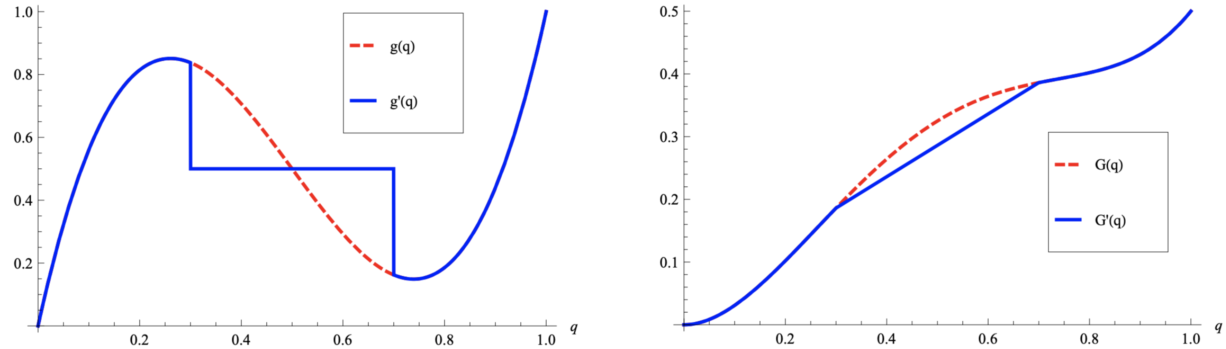
It remains to show that the ironing won't decrease the expected revenue.

Claim 8.4.2 *Let $v(q)$ be the valuation of some buyer at the q-th percentile and $x(q)$ be the allocation function of the buyer at the q-th percentile (taking expectation over the other buyers). Then, it holds*

$$\int_0^1 v(q) \cdot x(q)dq \leq \int_0^1 v(q) \cdot \bar{x}(q)dq$$

Proof: The inequality is true by applying integration by parts

$$\begin{aligned}\int_0^1 v(q)(x(q) - \bar{x}(q))dq &= \left[v(q)\left(X(q) - \bar{X}(q)\right)\right]_0^1 - \int_0^1 v'(q)\left(X(q) - \bar{X}(q)\right)dq \\ &= -\int_0^1 v'(q)\left(X(q) - \bar{X}(q)\right)dq\end{aligned}$$



(a) (b)

Figure 8.4.1: (a) A non-monotone ironing g' (solid) of curve g (dashed). (b) The corresponding integral curves G (solid) and G (dashed) in probability space. Figures from [HL10].

Since $v(q)$ is always increasing (recall that in the qunatile space values are sorted from low to high), $v'(q)$ is positive. Besides, $X(q) - \bar{X}(q) \geq 0$ by the definition of lower convex envelop. Overall, we have

$$\int_0^1 v(q)(x(q) - \bar{x}(q))dq = -\int_0^1 v'(q)\left(X(q) - \bar{X}(q)\right) dq \leq 0.$$

Hence, we conclude the ironing will only increase the expected revenue. ∎

Remark 8.4.3 *The ϵ in Theorem 8.4.1 comes from the fact that we are unable to accurately obtain the function $x(q)$. Hence, we may miss iron some region with small probability mass (makes the mechanism only approximately B.I.C). Besides, the re-sampling area may not correspond exactly to regions where the lower envelop $\bar{X}(q)$ is below $X(q)$. As a result, the approximate lower convex envelop may be slightly above $X(q)$, resulting slight decrease in the expected revenue.*

On the lower bound side, we have the claim

Claim 8.4.4 *It is impossible to achieve (1) and (2) in Theorem 8.4.1 in the ex-post setting (D.S.I.C and instance level approximation) with only polynomial number of queries to $\mathcal{A}$.*

References

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