Sparse Tree Search Optimality Guarantees in POMDPs with Continuous Observation Spaces

Tianci Li

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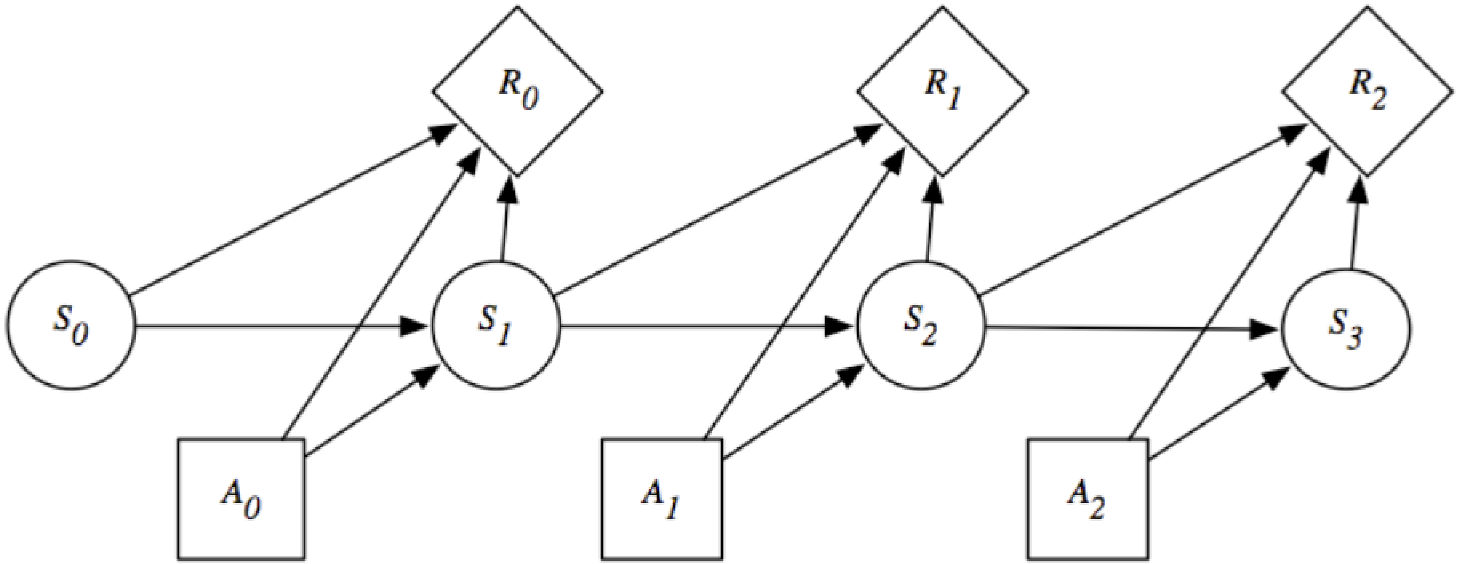
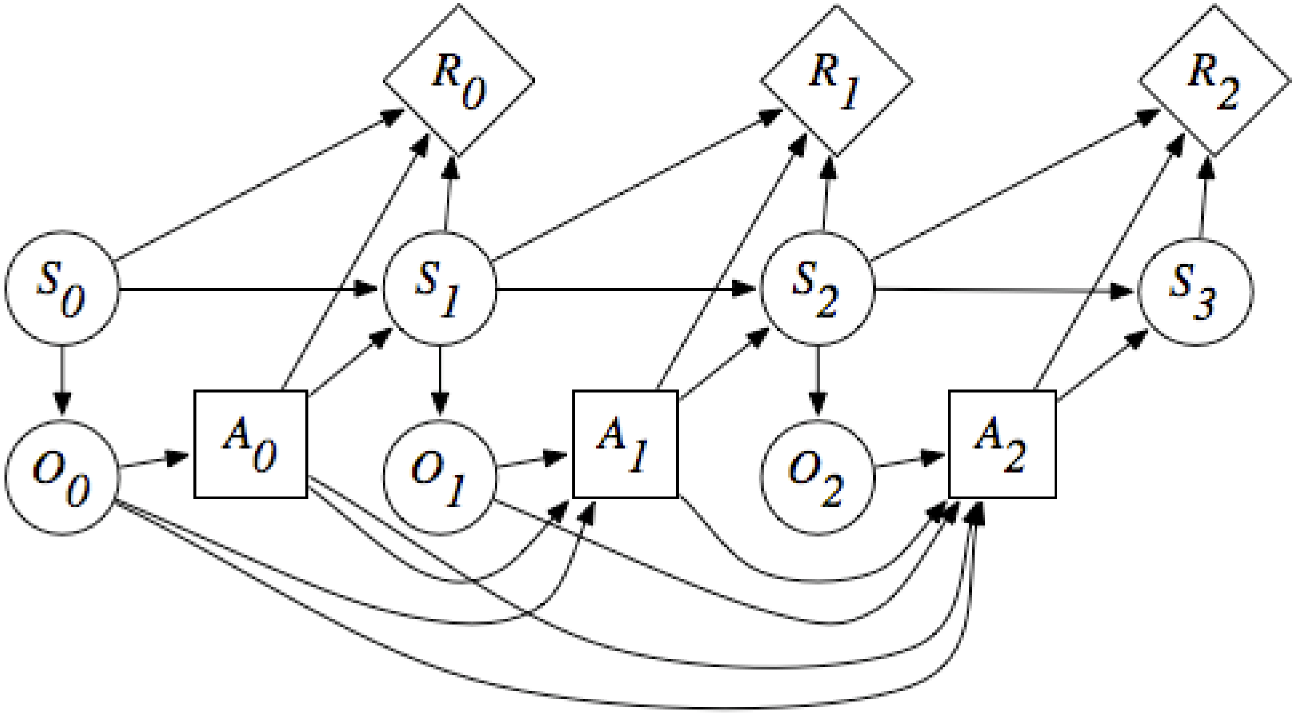
Outline

- Introduction
  - Definition of POMDP
  - Sparse Tree Search Algorithms in POMDP
    - Monte-Carlo Tree Search
    - Particle Filtering
    - Progressive widening
    - Importance Sampling
- Sparse Tree Search Optimality Guarantees

Introduction



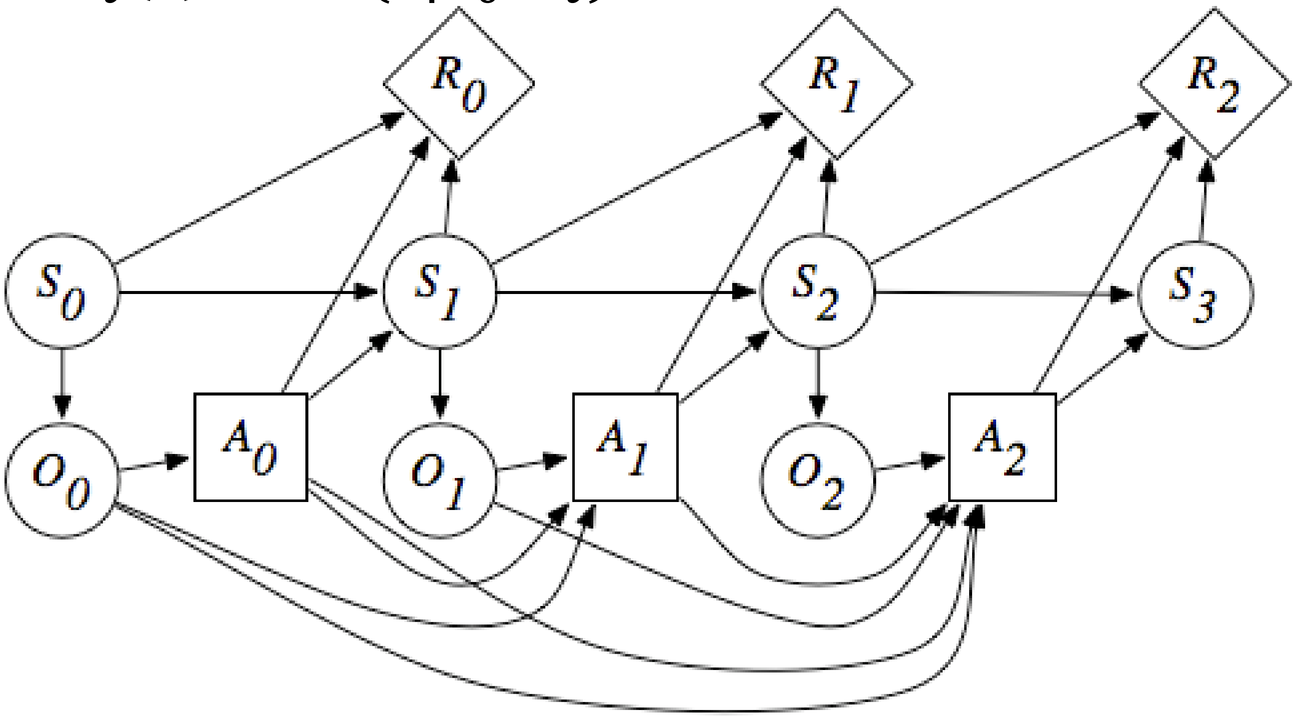
Introduction



Introduction

历史信息：$h_t = \{a_0, o_1, a_2, \ldots, a_{t-1}, o_t\}$

信念状态：$b_t(s) = Pr(s|b_0, h_t)$



POMDP → Belief MDP

Definition of POMDP

A POMDP can formally be described as a 7-tuple $\mathrm{P} = (\mathcal{S}, \mathcal{A}, \mathrm{T}, \mathrm{R}, \Omega, \mathrm{O}, \gamma)$,

- $\mathcal{S} = \{\mathrm{s}_1, \mathrm{s}_2, \dots, \mathrm{s}_\mathrm{n}\}$ is a set of states,
- $\mathcal{A} = \{\mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_\mathrm{m}\}$ is a set of actions,
- T is a set of conditional transition probabilities $T(s' \mid s, a)$ for the state transition $s \rightarrow s'$.
- $R: S \times A \rightarrow R$ is the reward function,
- $\Omega = \{\mathrm{o}_1, \mathrm{o}_2, \dots, \mathrm{o}_\mathrm{k}\}$ is a set of observations,
- O is a set of observation probabilities $\mathrm{O}(\mathrm{o}|\mathrm{s}', \mathrm{a})$
- $\gamma \in [0,1]$ is the discount factor.

Some concepts and equations

历史信息：$h_t = \{a_0, o_1, a_2, \dots, a_{t-1}, o_t\}$

信念状态：$b_t(s) = Pr(s|b_0, h_t)$

多变量贝叶斯公式：

$$\Pr(A|B,C) = \frac{\Pr(B|A,C)\Pr(A|C)}{\Pr(B|C)}$$

贝叶斯更新：$b_t(s') = Pr(s'|a_{t-1}, o_t, b_{t-1})$

$$= \frac{\Pr(o_t|s', a_{t-1}, b_{t-1})\Pr(s'|a_{t-1}, b_{t-1})}{\Pr(o_t|a_{t-1}, b_{t-1})}$$

$$= \frac{\mathrm{O}(o_t|s', a_{t-1})\sum_{s\in S} T(s'|s, a_{t-1}) b_{t-1}(s)}{\sum_{s'\in S} O(o_t|s', a_{t-1})\sum_{s\in S} T(s'|s, a_{t-1}) b_{t-1}(s)}$$

$O(|S|^2)$

计算成本高！使用粒子滤波

Some concepts and equations

累计奖励： $G_t = R_t + \gamma R_{t+1} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$

策略：$\forall s \in S, \forall a \in A, \pi(a|s) = \Pr(a|s)$

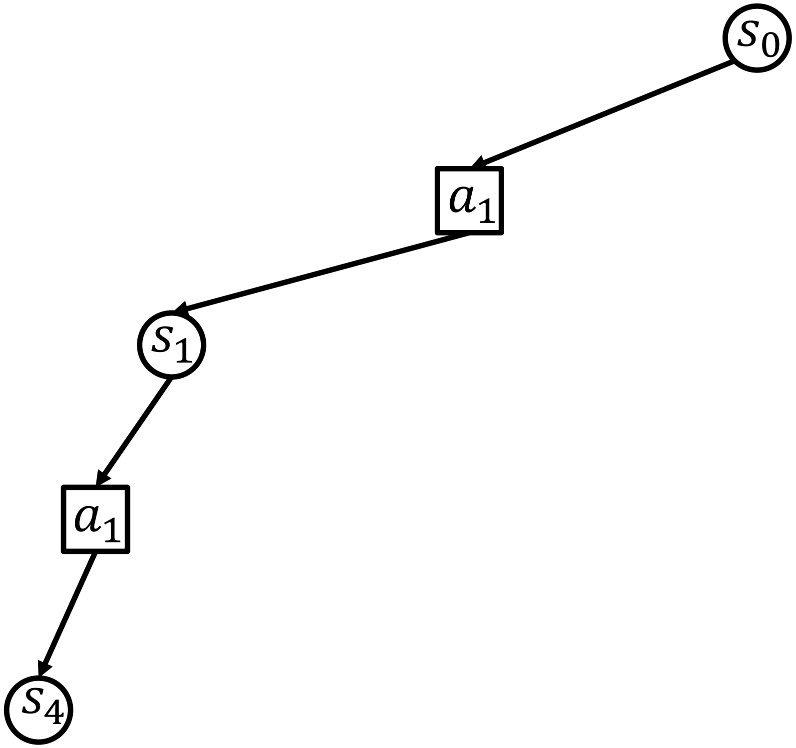
状态值函数：$V_\pi(s) = E[G_t|s_t = s] = E[R_t + \gamma V_\pi(s_{t+1})|s_t = s]$(贝尔曼方程)

动作值函数：$Q_\pi(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_\pi(s')$

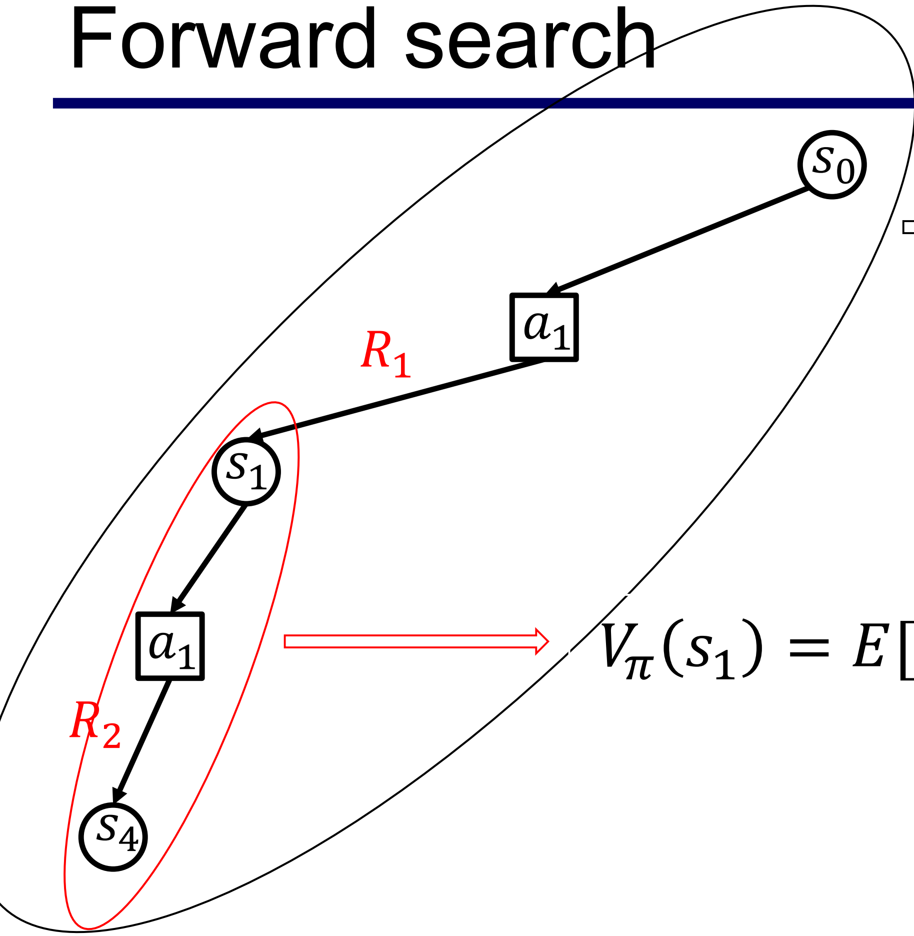
贝尔曼最优方程：$V^*(s) = \max_a Q(s,a)$

最优策略：$\pi^*(a|s) = \begin{cases} 1, & if\ a \in arg\max_a Q(s,a) \\ 0, & otherwise \end{cases}$

Forward search



Forward search

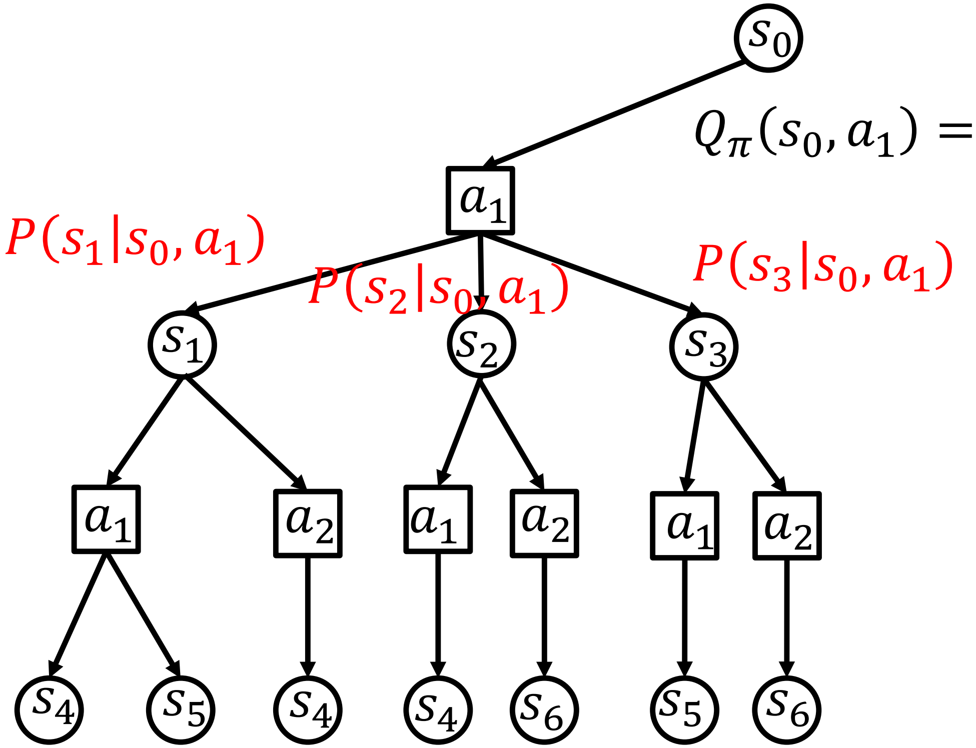


$$V_\pi(s_0) = E[G_0]$$
$$= E[R_1 + \gamma G_1]$$
$$= E[R_1 + \gamma V_\pi(s_1)]$$

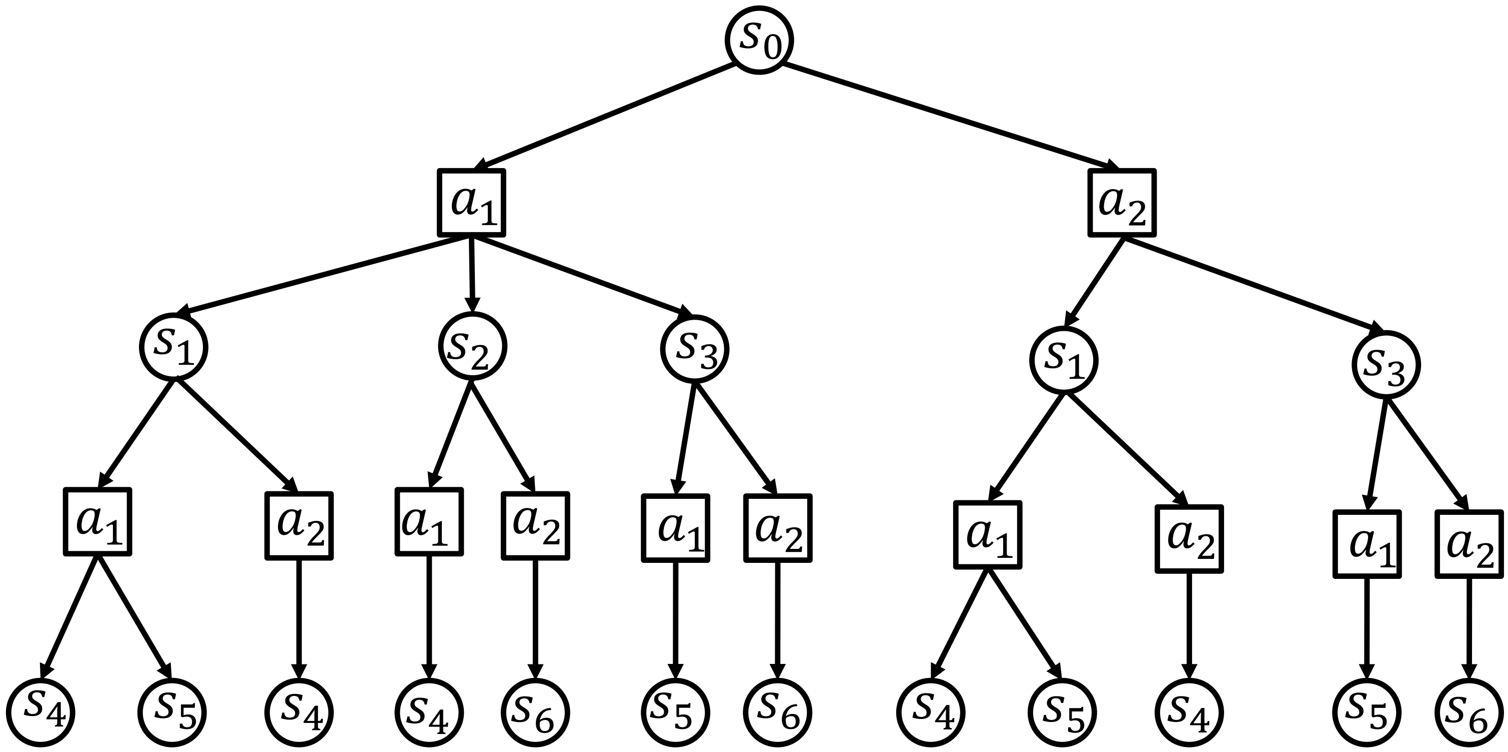
$$V_\pi(s_1) = E[G_1]$$

Forward search

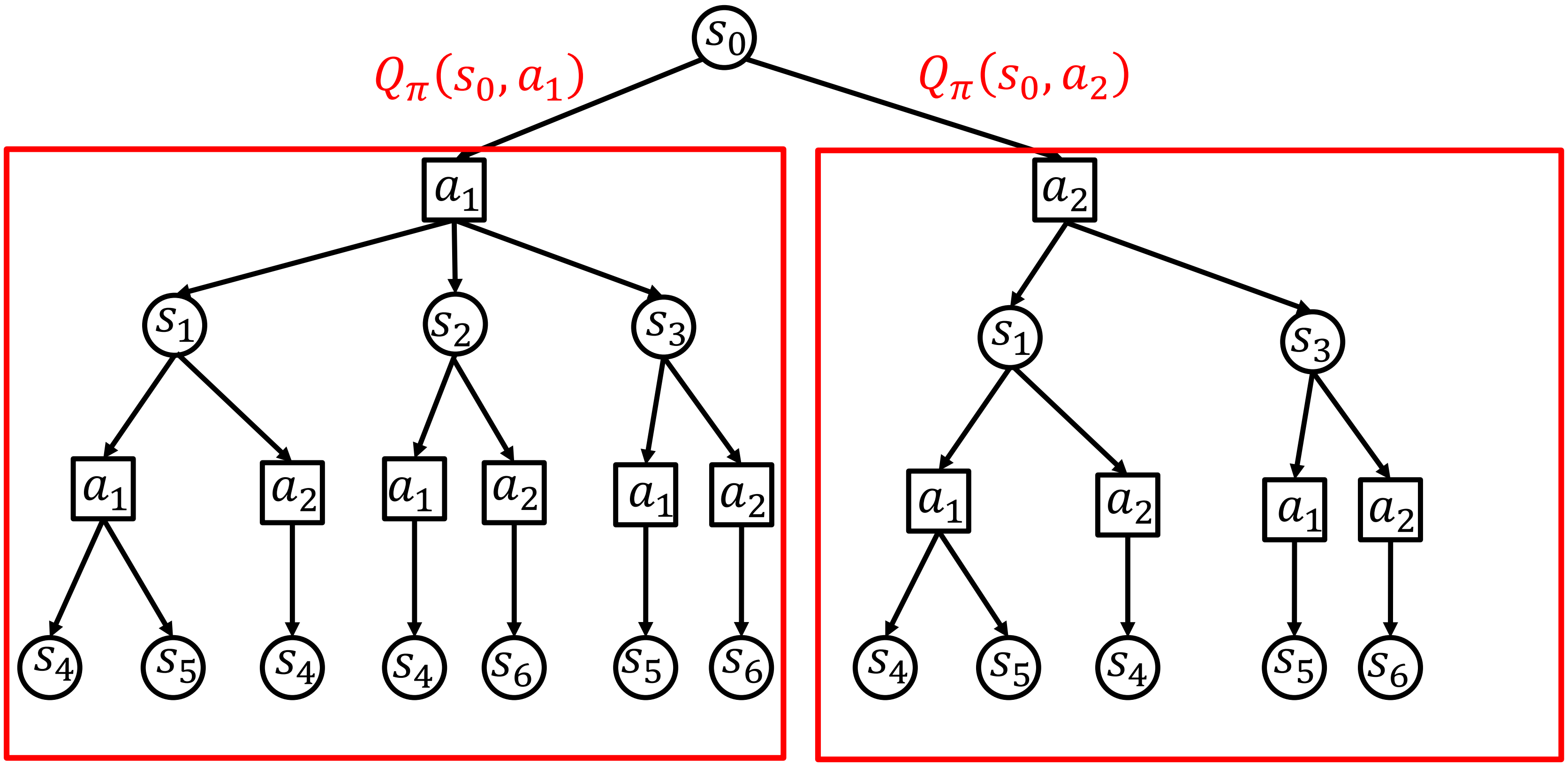
$$Q_\pi(s_0, a_1) = R(s_0, a_1) + \gamma \sum_{s' \in S} P(s'|s_0, a_1) V_\pi(s')$$



Forward search

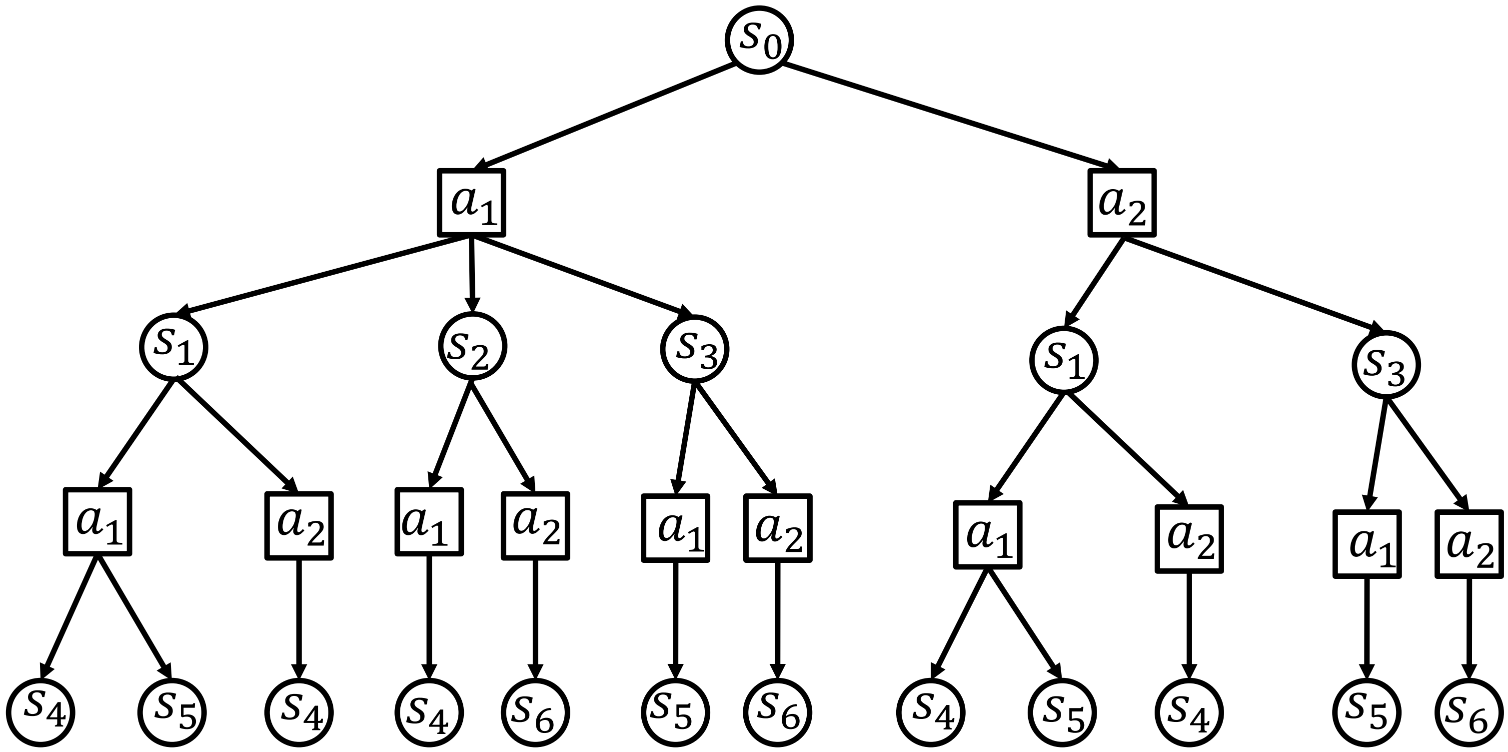


Forward search



$$V_\pi(s_0) = \sum_{a \in A} \pi(a|s_0)\, Q_\pi(s_0, a) \le \max_a Q(s_0, a) \qquad V^*(s_0) = \max_a Q^*(s_0, a)$$

Forward search



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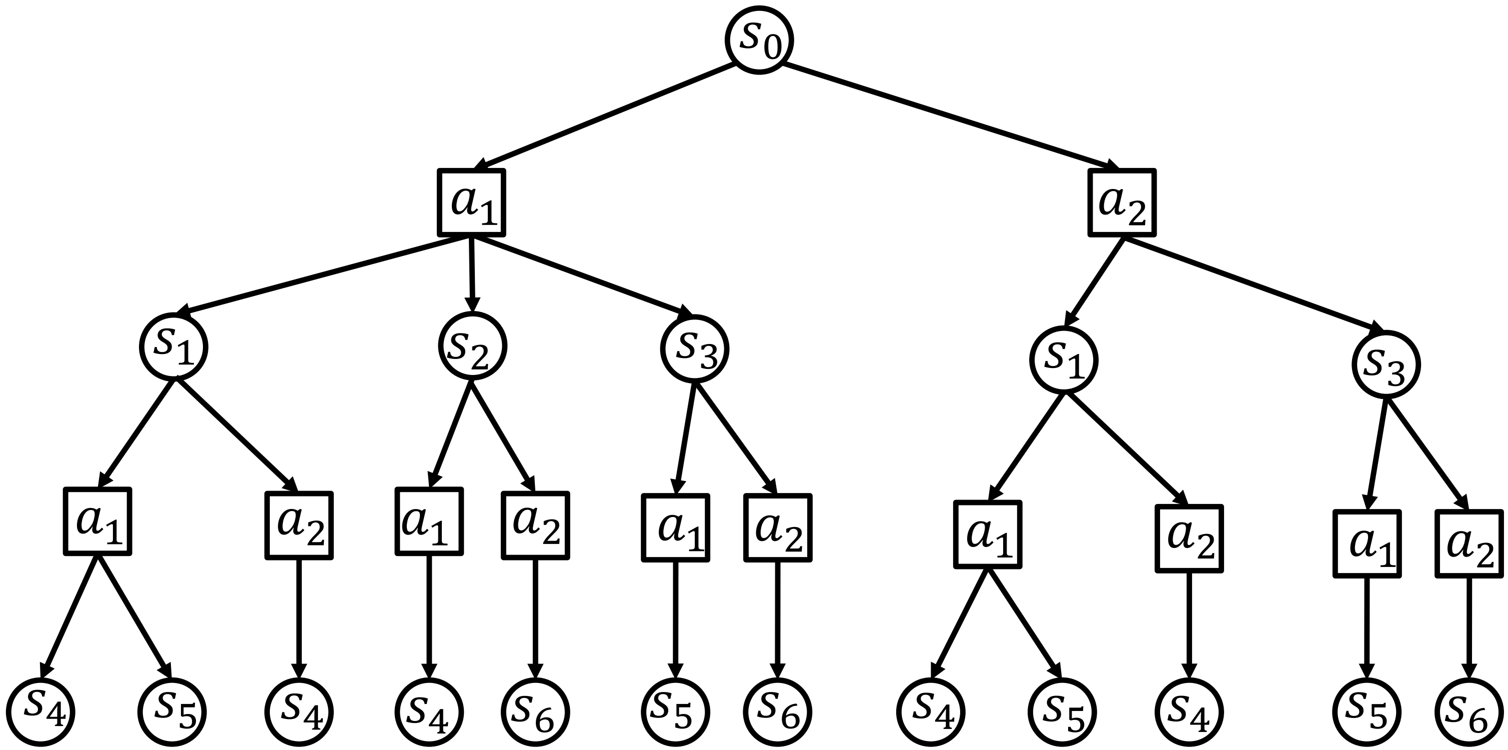
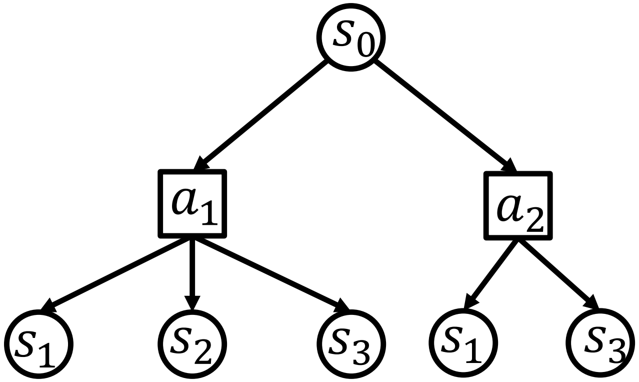


When the search space is large

Forward search ✗ Monte-Carlo Tree Search ✓

Monte-Carlo Tree Search

Each node saves the N value(visited counts) and V value(Node value)



Monte-Carlo Tree Search

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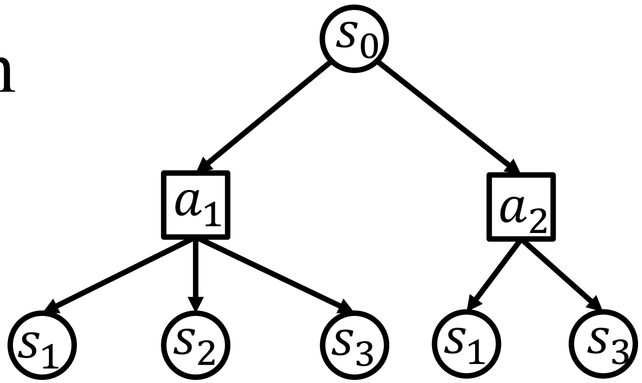
1.Selection



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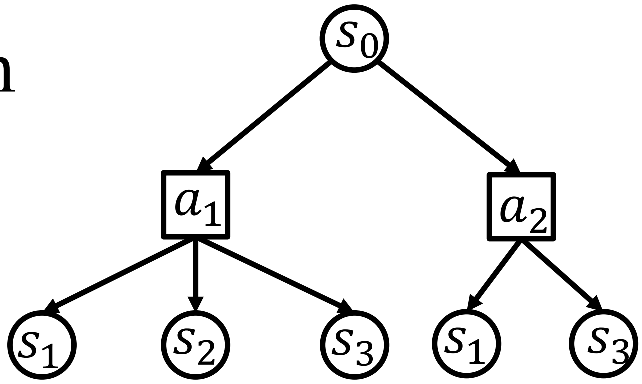

A tree policy that is used while within the search tree. Greedy tree policy ; UCT

UCT: $$A = \underset{a}{argmax}\ [V(sa) + c\sqrt{\frac{\log N(s)}{N(sa)}}\ ]$$

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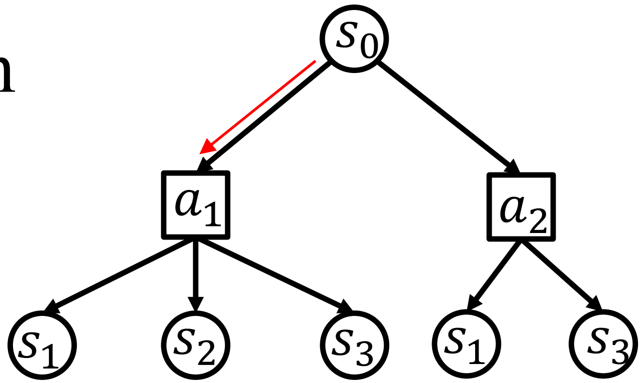

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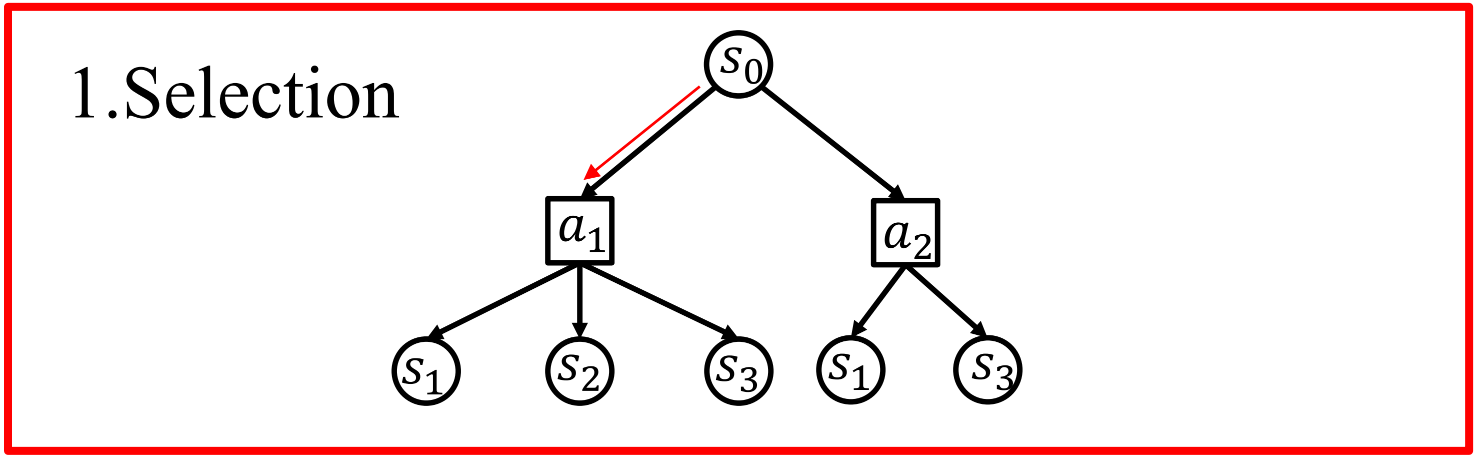

Generative model to generate sequences of states and rewards

$$G(s_t, a_t) = (R_{t+1}, s_{t+1})$$

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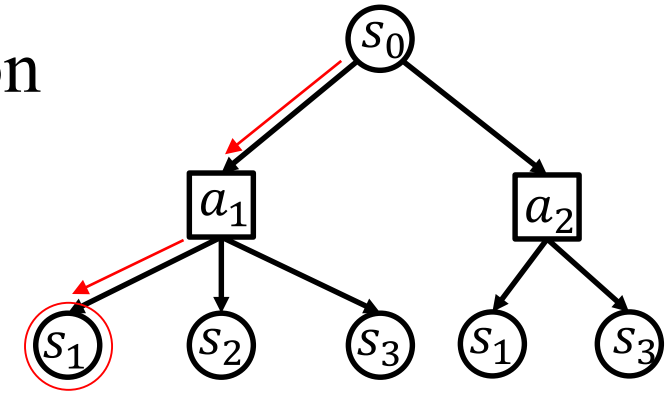

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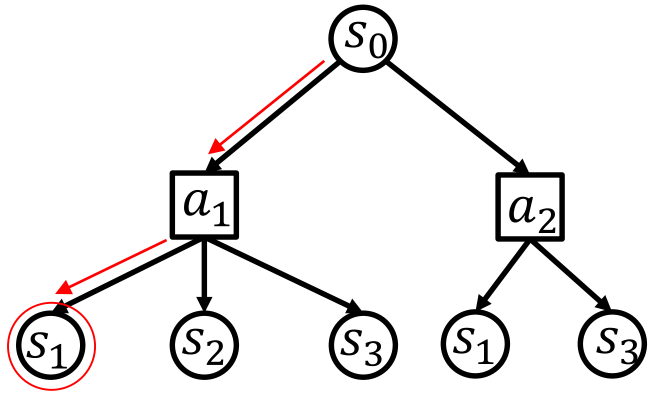

Monte-Carlo Tree Search

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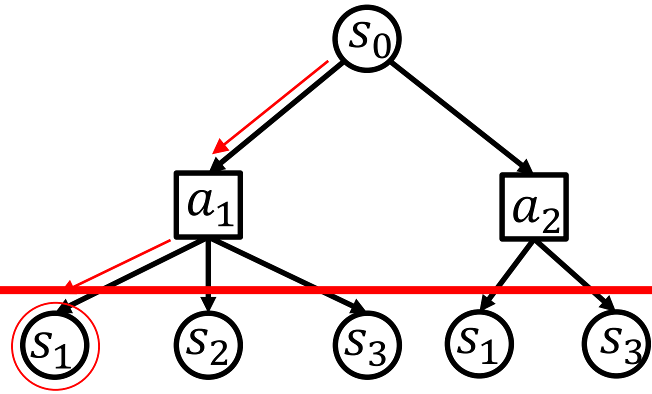
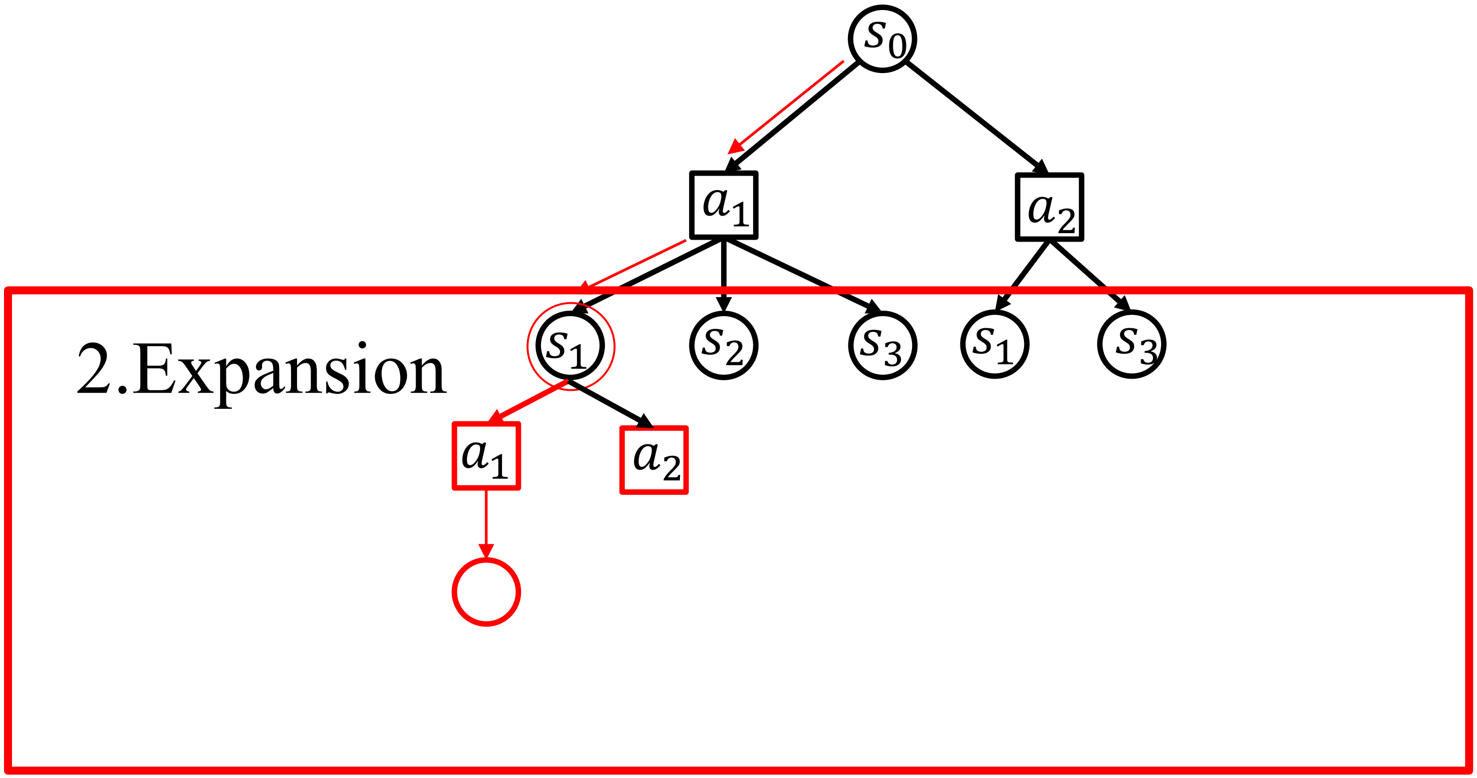


2.Expansion

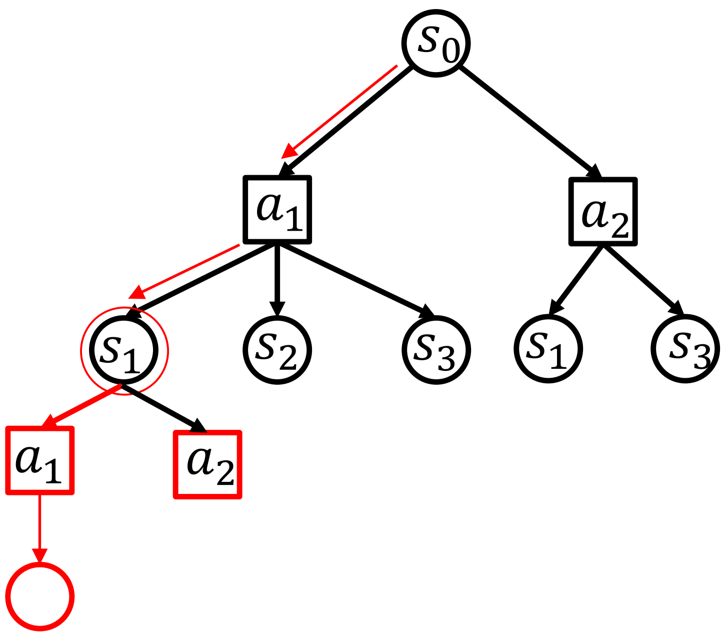
Monte-Carlo Tree Search

Each node saves the N value(visited counts) and V value(Node value)



Monte-Carlo Tree Search

Each node saves the N value(visited counts) and V value(Node value)



Monte-Carlo Tree Search

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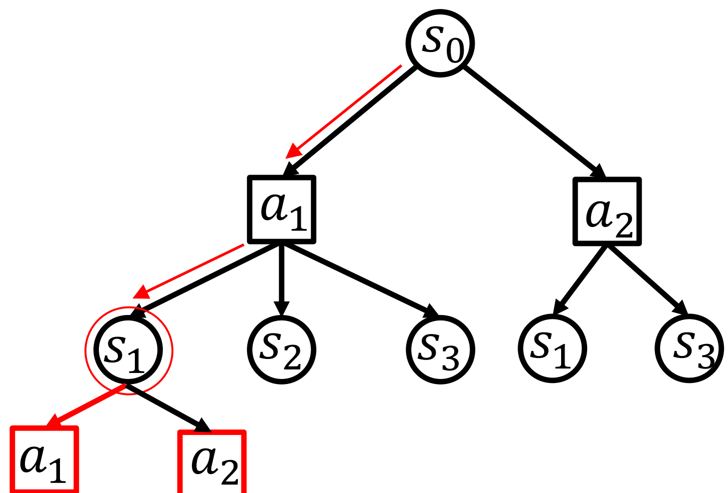
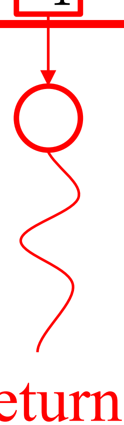


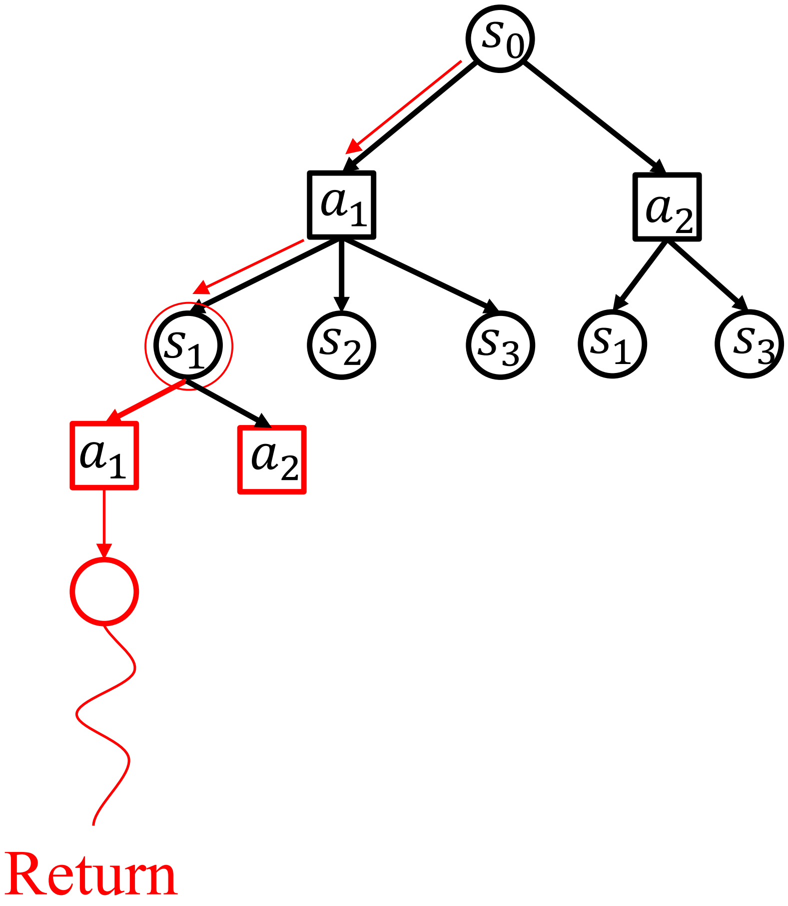
3.Simulation



A rollout policy that is used once simulations leave the scope of the search tree

Monte-Carlo Tree Search

Each node saves the N value(visited counts) and V value(Node value)



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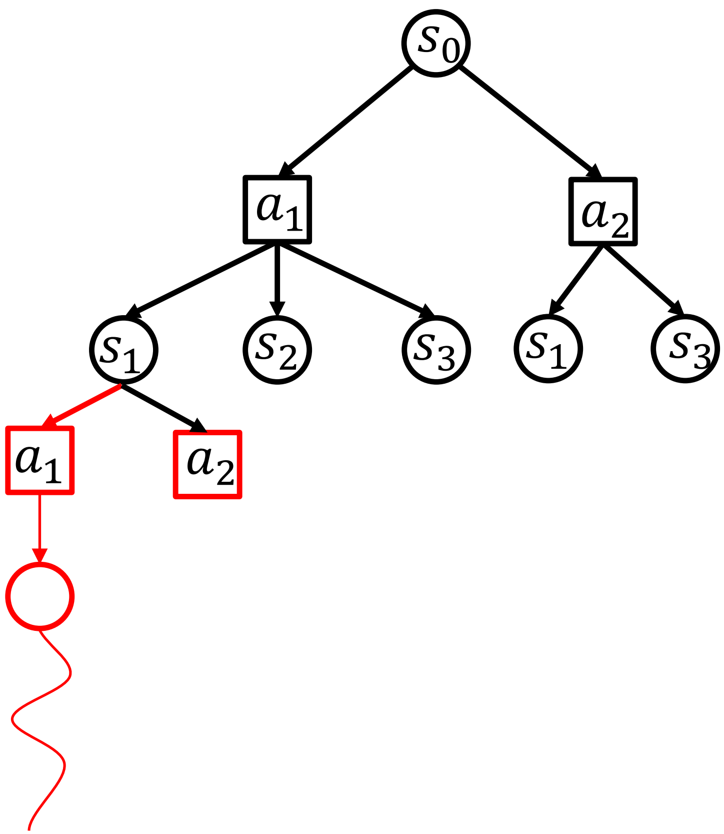
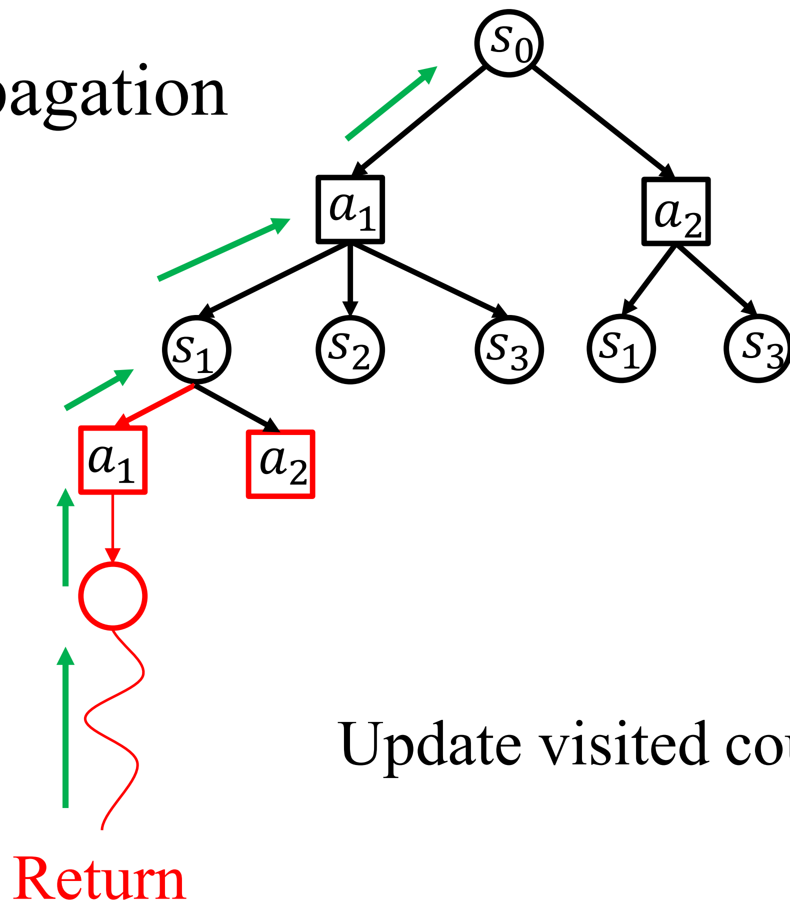


Monte-Carlo Tree Search

Each node saves the N value(visited counts) and V value(Node value)

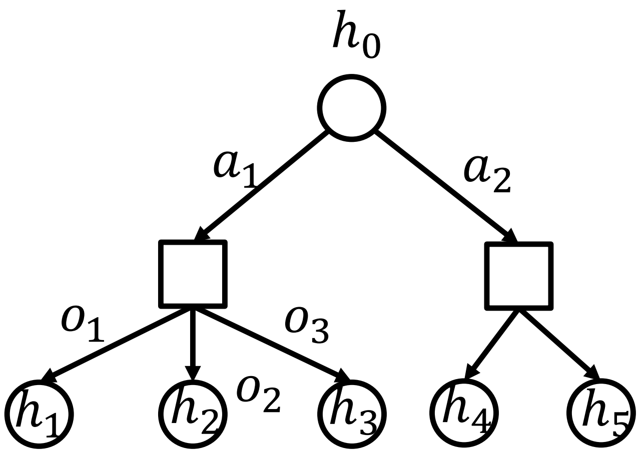
4.Backpropagation



Update visited counts and node value

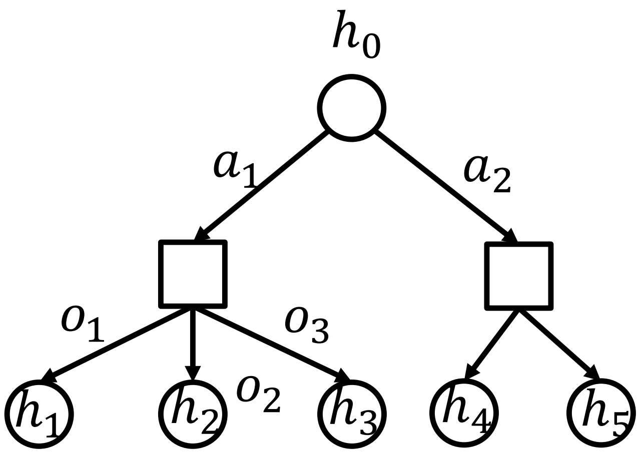
Monte-Carlo Tree Search

In POMCP:



Monte-Carlo Tree Search

In POMCP:

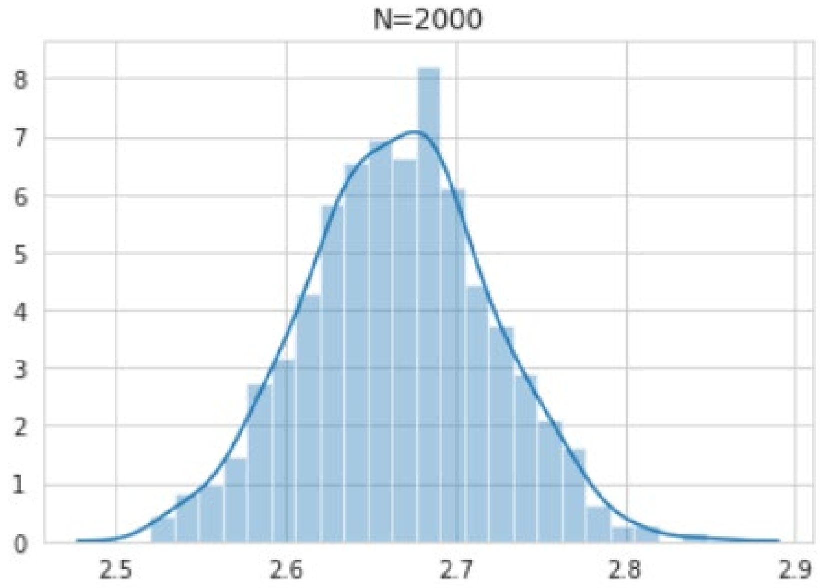


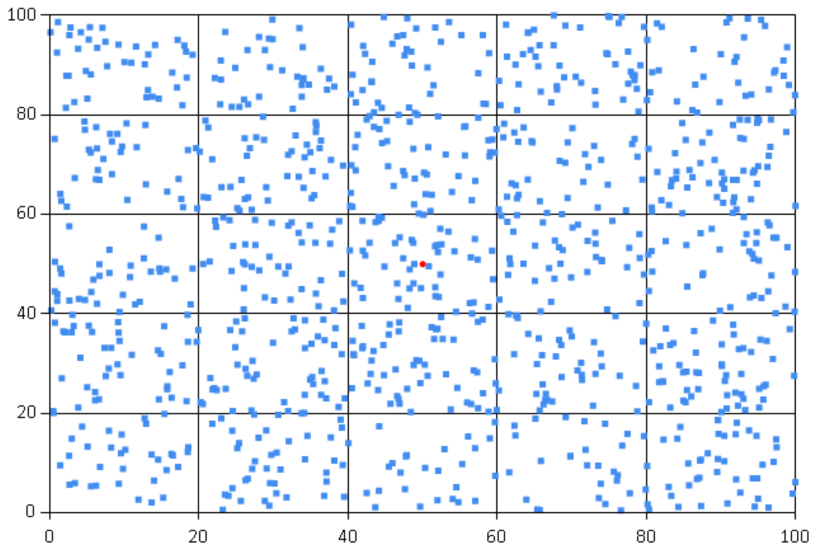
PO-UCT: $$A = \underset{a}{argmax}\ [V(ha) + c\sqrt{\frac{\log N(h)}{N(ha)}}]$$

Generative model: $$G(s_t, a_t) = (R_{t+1}, o_{t+1}, s_{t+1})$$

Particle Filtering

Using a set of particles to represent the posterior distribution.

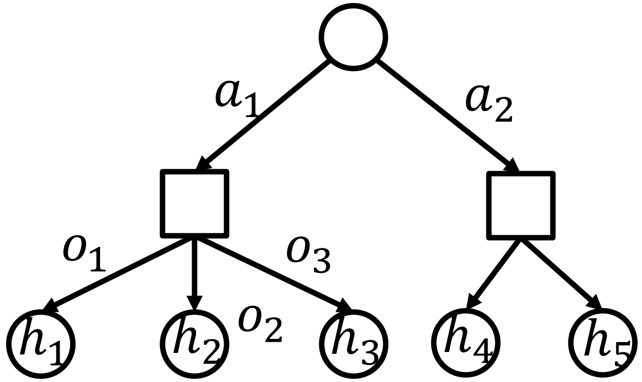




Particle Filtering

In POMCP:

h_0 $\mathrm{B}(h_0) = [s_0, s_0, s_1, s_2]$



$\mathrm{B}(h_1)$

Particle Filtering

In POMCP:

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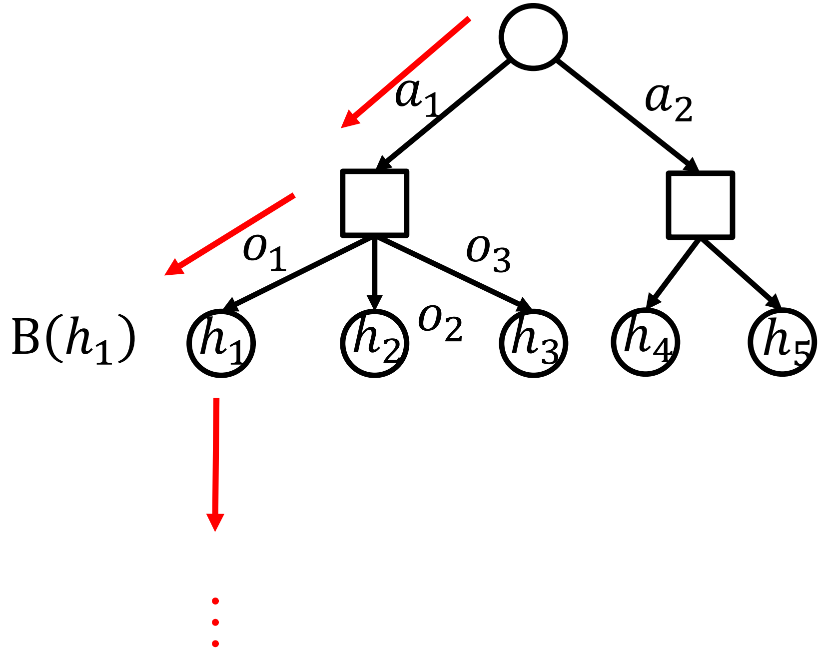


$s \sim B(h_0)$

↓

$G(s, a_0) = (R_1, o_1, s_1)$

↓

$B(h_0 a_0 o_1) \cup \{s_1\}$

↓

$s \sim B(h_0 a_0 o_1)$

⋮

Partially Observable Monte-Carlo

Algorithm 1 Partially Observable Monte-Carlo Planning

```
procedure SEARCH(h)
    repeat
        if h = empty then
            s ~ I
        else
            s ~ B(h)
        end if
        SIMULATE(s, h, 0)
    until TIMEOUT()
    return argmax_b V(hb)
end procedure

procedure ROLLOUT(s, h, depth)
    if γ^depth < ε then
        return 0
    end if
    a ~ π_rollout(h, ·)
    (s', o, r) ~ G(s, a)
    return r + γ.ROLLOUT(s', hao, depth+1)
end procedure

procedure SIMULATE(s, h, depth)
    if γ^depth < ε then
        return 0
    end if
    if h ∉ T then
        for all a ∈ A do
            T(ha) ← (N_init(ha), V_init(ha), ∅)
        end for
        return ROLLOUT(s, h, depth)
    end if
    a ← argmax_b V(hb) + c sqrt(log N(h) / N(hb))
    (s', o, r) ~ G(s, a)
    R ← r + γ.SIMULATE(s', hao, depth + 1)
    B(h) ← B(h) ∪ {s}
    N(h) ← N(h) + 1
    N(ha) ← N(ha) + 1
    V(ha) ← V(ha) + (R − V(ha)) / N(ha)
    return R
end procedure
```

POMCP in Continues Space

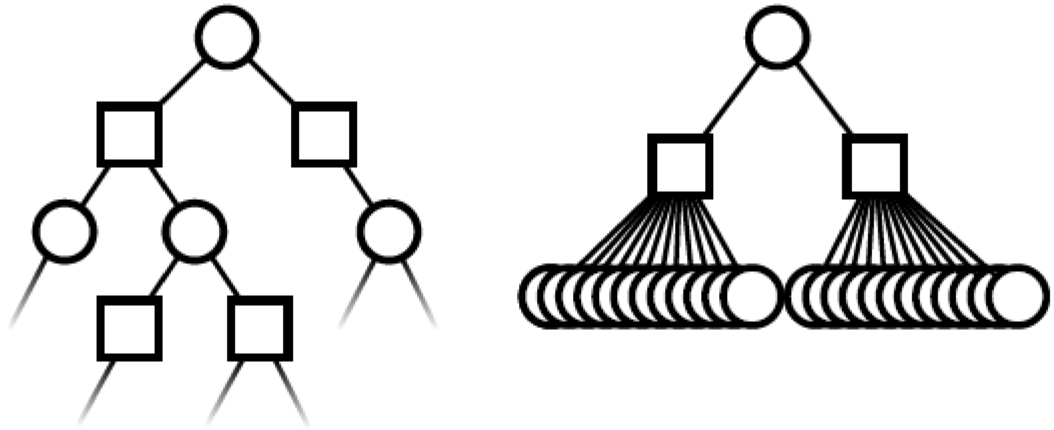
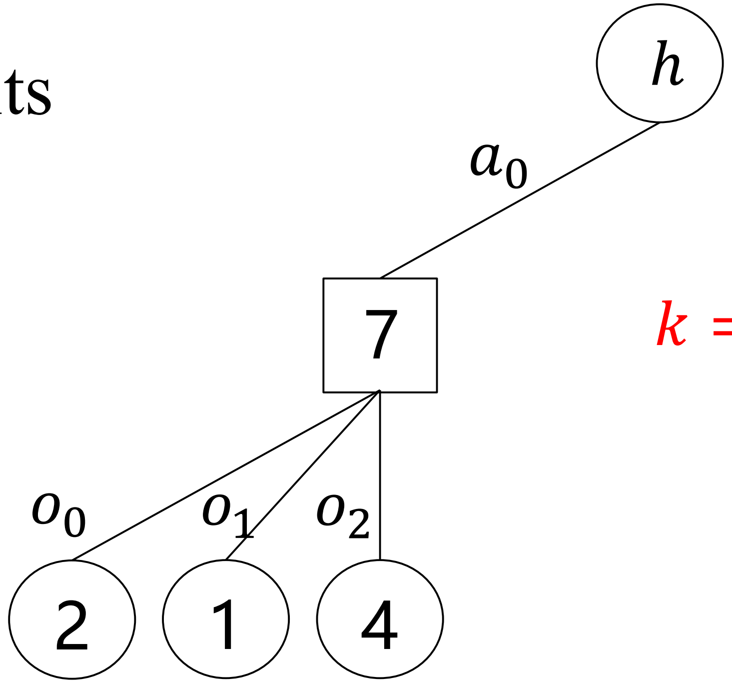


Figure 1: POMCP tree for a discrete POMDP (left), and for a POMDP with a continuous observation space (right). Because the observation space is continuous, each simulation creates a new observation node and the tree cannot extend deeper.

Progressive widening

- Limit the number of child nodes: $\leq kN^{\alpha}$

  N is the number of visits

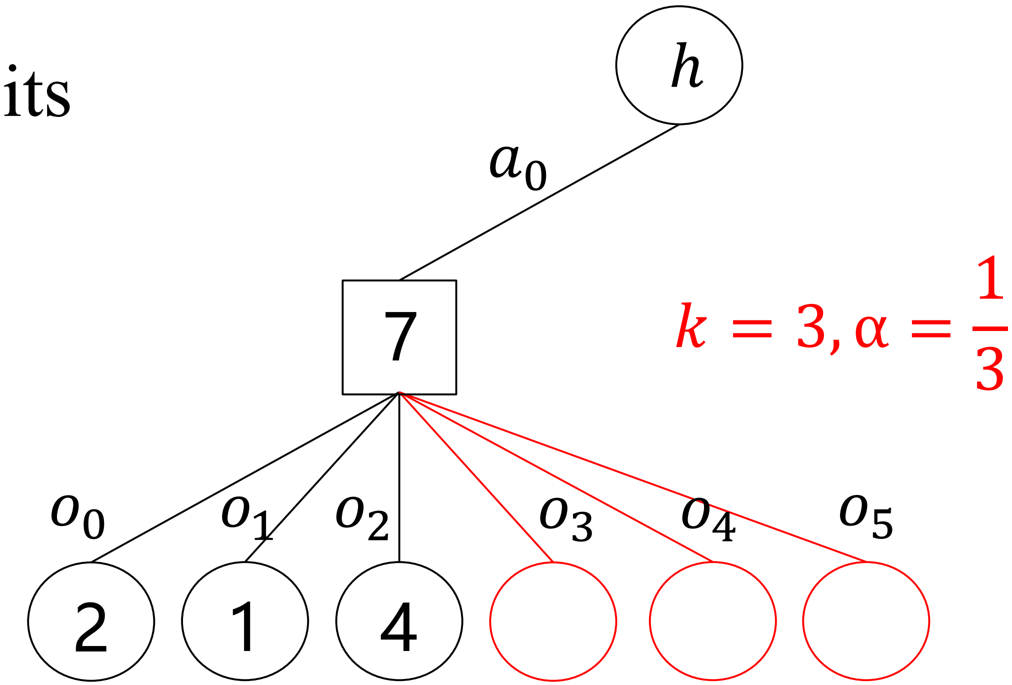


$k = 3, \alpha = \frac{1}{3}$

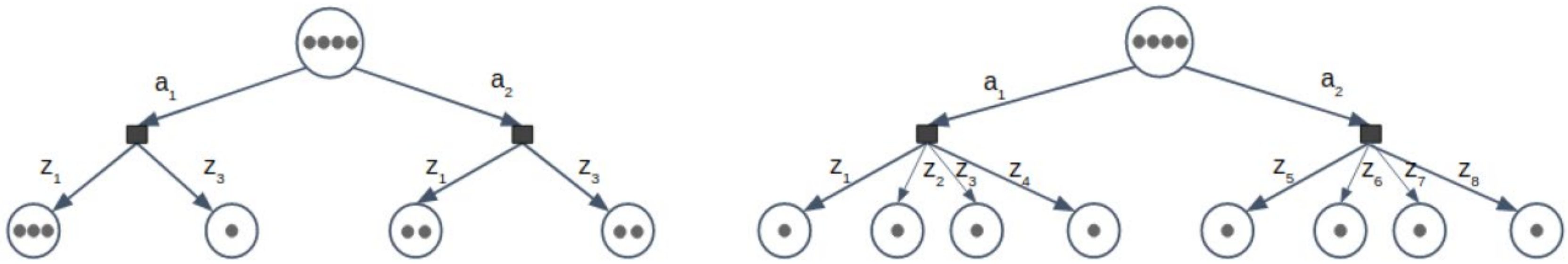
Progressive widening

- Limit the number of child nodes: $\leq kN^{\alpha}$

  N is the number of visits



Belief Update



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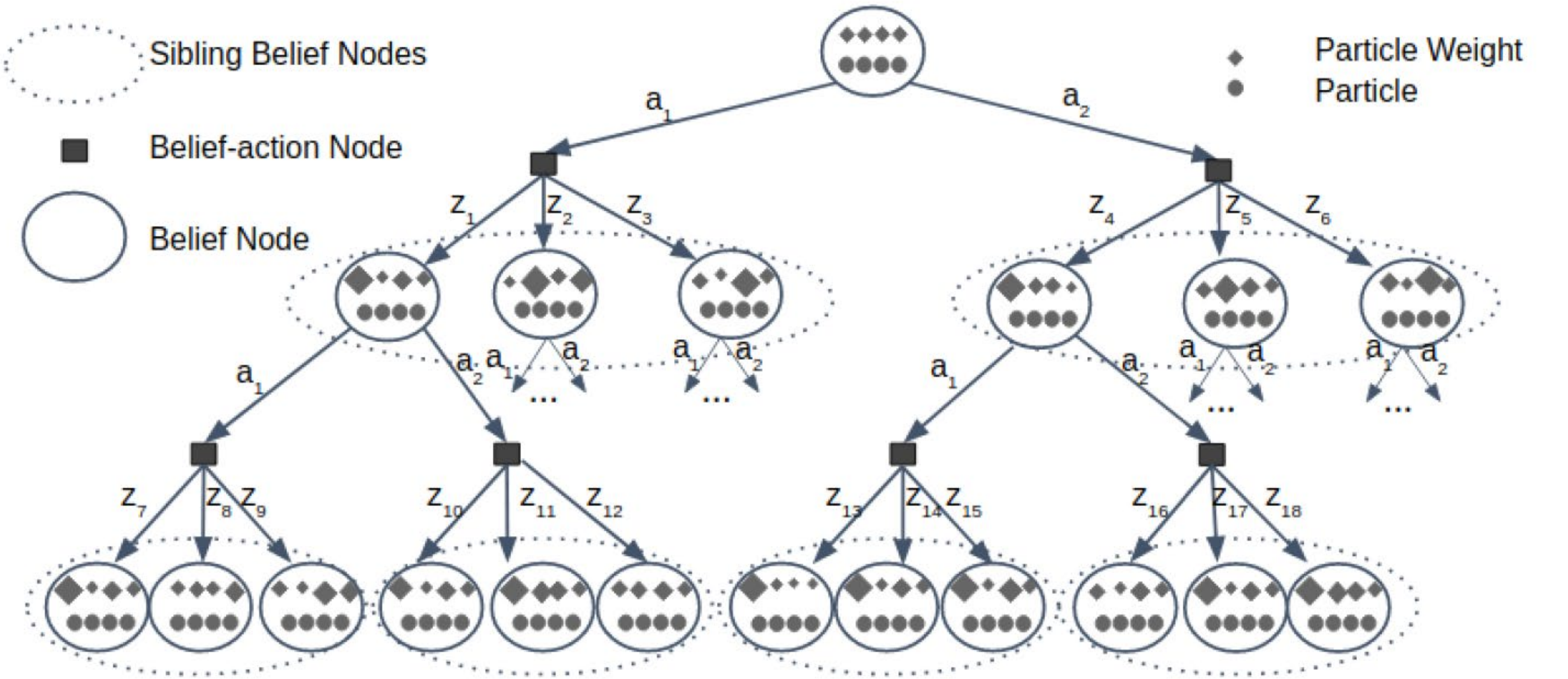
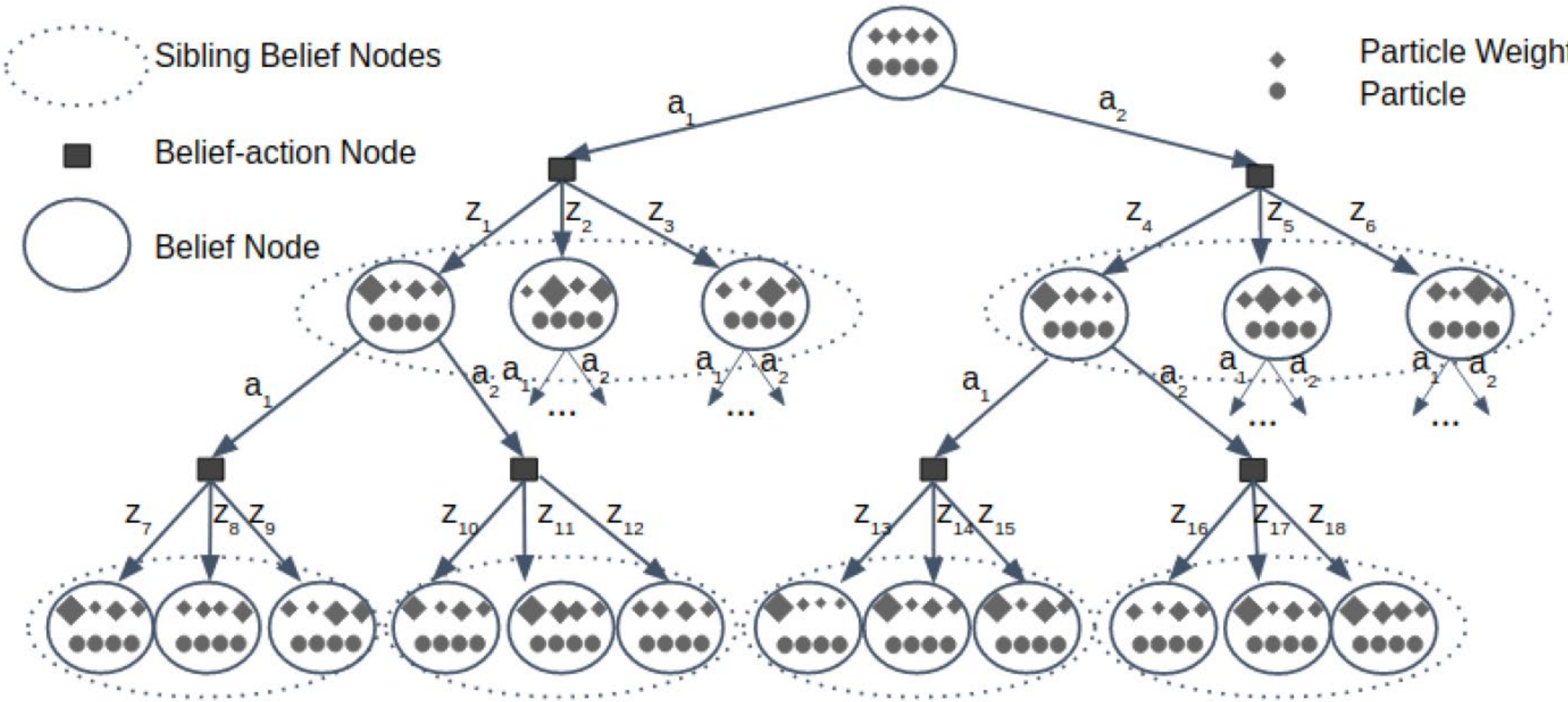


Belief Update



State particles are shared among all observation branches and importance sampling weights are given

Importance Sampling

Constructs an approximation using samples from a proposal distribution, and corrects for the discrepancy between the target and proposal using (importance) weights.

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$$E_p[f(x)] = \int f(x)p(x)dx = \int \frac{f(x)p(x)}{g(x)} g(x)dx = \frac{E_g\left[\frac{f(x)p(x)}{g(x)}\right]}{E_g[\frac{p(x)}{g(x)}]}$$

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$$= \frac{E_g\left[\frac{f(x)p(x)}{g(x)}\right]}{E_g[\frac{p(x)}{g(x)}]} = \frac{\frac{1}{N}\sum_{i=1}^{N} f(x)W(x_i)}{\frac{1}{N}\sum_{j=1}^{N} W(x_j)} \approx \sum_{i=1} \widehat{W}(\mathrm{x_i})f(x_i)$$

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$$W(x_i) = \frac{p(x_i)}{q(x_i)} \qquad \widehat{W}(x_i) = \frac{W(x_i)}{\sum_{j=0}^{N} W(x_j)}$$

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In POMCPOW:

Target Distribution $\Pr(s_{t+1} \mid b_t, a_t, o_{t+1})$

Proposal Distribution $\Pr(s_{t+1} \mid b_t, a_t)$

$$W(s_{t+1}) = \frac{p(s_{t+1})}{q(s_{t+1})} = \frac{\Pr(o_t \mid s', a_{t-1}, b_{t-1})}{\Pr(o_t \mid a_{t-1}, b_{t-1})}$$

Importance Sampling

Constructs an approximation using samples from a proposal distribution, and corrects for the discrepancy between the target and proposal using (importance) weights.

In POMCPOW:

$$b_t(s') = Pr(s'|a_{t-1}, o_t, b_{t-1})$$

Target Distribution

$$= \frac{\Pr(o_t|s', a_{t-1}, b_{t-1}) \Pr(s'|a_{t-1}, b_{t-1})}{\Pr(o_t|a_{t-1}, b_{t-1})}$$

Proposal Distribution $\Pr(s_{t+1}\,|b_t, a_t)$

$$W(s_{t+1}) = \frac{p(s_{t+1})}{q(s_{t+1})} = \frac{\Pr(o_t|s', a_{t-1}, b_{t-1})}{\Pr(o_t|a_{t-1}, b_{t-1})}$$

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Proposal Distribution $\Pr(s_{t+1} \mid b_t, a_t)$

$$W(s_{t+1}) = \frac{p(s_{t+1})}{q(s_{t+1})} = \frac{\Pr(o_t | s', a_{t-1}, b_{t-1})}{\Pr(o_t | a_{t-1}, b_{t-1})}$$

Importance Sampling

Constructs an approximation using samples from a proposal distribution, and corrects for the discrepancy between the target and proposal using (importance) weights.

$$\tilde{w}_{\mathcal{P}/\mathcal{Q}}(x) \equiv \frac{w_{\mathcal{P}/\mathcal{Q}}(x)}{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)} \qquad \text{(SN Importance Weight)}$$

$$d_\alpha(\mathcal{P}||\mathcal{Q}) \equiv \mathbb{E}_{x\sim\mathcal{Q}}[w_{\mathcal{P}/\mathcal{Q}}(x)^\alpha] \qquad \text{(Rényi Divergence)}$$

$$\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \equiv \sum_{i=1}^{N} \tilde{w}_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i) \qquad \text{(SN Estimator)}$$

Importance Sampling

Constructs an approximation using samples from a proposal distribution, and corrects for the discrepancy between the target and proposal using (importance) weights.

Theorem 1

Theorem 1 (SN d_∞-Concentration Bound). *Let $\mathcal{P}$ and $\mathcal{Q}$ be two probability measures on the measurable space $(\mathcal{X}, \mathcal{F})$ with $\mathcal{P} \ll \mathcal{Q}$ and $d_\infty(\mathcal{P}||\mathcal{Q}) < +\infty$. Let $x_1, \cdots, x_N$ be i.i.d.r.v. sampled from $\mathcal{Q}$, and $f : \mathcal{X} \to \mathbb{R}$ be a bounded Borel function ($\|f\|_\infty < +\infty$). Then, for any $\lambda > 0$ and N large enough such that $\lambda > \|f\|_\infty d_\infty(\mathcal{P}||\mathcal{Q})/\sqrt{N}$, the following bound holds with probability at least $1 - 3\exp(-N \cdot t^2(\lambda, N))$:*

$$|\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}| \leq \lambda$$

where $t(\lambda, N)$ is defined as:

$$t(\lambda, N) \equiv \frac{\lambda}{\|f\|_\infty d_\infty(\mathcal{P}||\mathcal{Q})} - \frac{1}{\sqrt{N}}$$

Proof 1

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Lemma1:

$$
\begin{aligned}
\mathbb{P}\left(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim P}[f(x)] \geq \lambda\right) &= \mathbb{P}\left(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim Q}\left[W_{\mathcal{P}/\mathcal{Q}}(x) f(x)\right] \geq \lambda\right) \\
&\leq \exp\left(-\frac{2N^2\lambda^2}{\sum_{i=1}^{N} 2(d_\infty(\mathcal{P} \parallel \mathcal{Q}) \parallel f \parallel_\infty)^2}\right) \\
&\leq exp\left(-\frac{N\lambda^2}{d_\infty^2(\mathcal{P} \parallel \mathcal{Q}) \parallel f \parallel_\infty^2}\right) \equiv \delta \\
\mathbb{P}\left(\left|\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim P}[f(x)]\right| \geq \lambda\right) &\leq 2\exp\left(-\frac{N\lambda^2}{d_\infty^2(\mathcal{P} \parallel \mathcal{Q}) \parallel f \parallel_\infty^2}\right) = 2\delta
\end{aligned}
$$

Proof 1

Lemma1:

$$\mathbb{P}\left(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim P}[f(x)] \geq \lambda\right) = \mathbb{P}\left(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim Q}\left[W_{\mathcal{P}/\mathcal{Q}}(x)f(x)\right] \geq \lambda\right) \quad (1)$$

$$\leq \exp\left(-\frac{2N^2\lambda^2}{\sum_{i=1}^{N} 2\left(d_\infty(\mathcal{P} \parallel \mathcal{Q}) \parallel f \parallel_\infty\right)^2}\right)$$

$$\leq exp\left(-\frac{N\lambda^2}{d_\infty^2(\mathcal{P} \parallel \mathcal{Q}) \parallel f \parallel_\infty^2}\right) \equiv \delta$$

$$\mathbb{P}\left(\left|\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim P}[f(x)]\right| \geq \lambda\right) \leq 2\exp\left(-\frac{N\lambda^2}{d_\infty^2(\mathcal{P} \parallel \mathcal{Q}) \parallel f \parallel_\infty^2}\right) = 2\delta$$

(1) $\mathbb{E}_{x\sim P}[f(x)] = \int f(x)p(x)dx = \int \frac{p(x)f(x)}{q(x)} q(x) = \mathbb{E}_{x\sim Q}\left[W_{\mathcal{P}/\mathcal{Q}}(x)f(x)\right]$

Proof 1

Lemma1:

$$\mathbb{P}\left(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim P}[f(x)] \geq \lambda\right) = \mathbb{P}\left(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim Q}\left[W_{\mathcal{P}/\mathcal{Q}}(x)f(x)\right] \geq \lambda\right)$$

$$\leq \exp\left(-\frac{2N^2\lambda^2}{\sum_{i=1}^{N} 2(d_\infty(\mathcal{P} \parallel \mathcal{Q}) \parallel f \parallel_\infty)^2}\right)$$

$$\leq exp\left(-\frac{N\lambda^2}{d_\infty^2(\mathcal{P} \parallel \mathcal{Q}) \parallel f \parallel_\infty^2}\right) \equiv \delta$$

$$\mathbb{P}\left(\left|\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim P}[f(x)]\right| \geq \lambda\right) \leq 2\exp\left(-\frac{N\lambda^2}{d_\infty^2(\mathcal{P} \parallel \mathcal{Q}) \parallel f \parallel_\infty^2}\right) = 2\delta$$

Proof 1

Lemma1:

$$\begin{aligned}
\mathbb{P}\left(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim P}[f(x)] \geq \lambda\right) &= \mathbb{P}\left(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim Q}\left[W_{\mathcal{P}/\mathcal{Q}}(x) f(x)\right] \geq \lambda\right) \\
&\leq \exp\left(-\frac{2N^2\lambda^2}{\sum_{i=1}^{N} 2(d_\infty(\mathcal{P} \parallel \mathcal{Q}) \parallel f \parallel_\infty)^2}\right) \quad ② \\
&\leq exp\left(-\frac{N\lambda^2}{d_\infty^2(\mathcal{P} \parallel \mathcal{Q}) \parallel f \parallel_\infty^2}\right) \equiv \delta \\
\mathbb{P}\left(\left|\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim P}[f(x)]\right| \geq \lambda\right) &\leq 2\exp\left(-\frac{N\lambda^2}{d_\infty^2(\mathcal{P} \parallel \mathcal{Q}) \parallel f \parallel_\infty^2}\right) = 2\delta
\end{aligned}$$

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Proof 1

2 $$\hat{\mu}_{\mathcal{P}/\mathcal{Q}} = \frac{1}{N}\sum_{i=1}^{N} W_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i) = \frac{1}{N}\sum_{i=1}^{N} g(x_i)$$ $$g(x) = W_{\mathcal{P}/\mathcal{Q}}(x) f(x)$$

Proof 1

② $\hat{\mu}_{\mathcal{P}/\mathcal{Q}} = \frac{1}{N}\sum_{i=1}^{N} W_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i) = \frac{1}{N}\sum_{i=1}^{N} g(x_i)$ $\quad g(x) = W_{\mathcal{P}/\mathcal{Q}}(x) f(x)$

$\hat{\mu}_{\mathcal{P}/\mathcal{Q}}$ is unbiased , which means $E[\hat{\mu}_{\mathcal{P}/\mathcal{Q}}] = E[g(x)]$, then:

Proof 1

② $$\hat{\mu}_{\mathcal{P}/\mathcal{Q}} = \frac{1}{N}\sum_{i=1}^{N} W_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i) = \frac{1}{N}\sum_{i=1}^{N} g(x_i) \qquad g(x) = W_{\mathcal{P}/\mathcal{Q}}(x) f(x)$$

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$$\mathbb{P}\left(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim Q}\left[W_{\mathcal{P}/\mathcal{Q}}(x) f(x)\right] \geq \lambda\right) = \mathbb{P}\left(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim Q}\left[\hat{\mu}_{\mathcal{P}/\mathcal{Q}}\right] \geq \lambda\right)$$

Proof 1

② $$\hat{\mu}_{\mathcal{P}/\mathcal{Q}} = \frac{1}{N}\sum_{i=1}^{N} W_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i) = \frac{1}{N}\sum_{i=1}^{N} g(x_i) \qquad g(x) = W_{\mathcal{P}/\mathcal{Q}}(x) f(x)$$

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$$\mathbb{P}\left(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim Q}\left[W_{\mathcal{P}/\mathcal{Q}}(x) f(x)\right] \geq \lambda\right) = \mathbb{P}\left(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim Q}\left[\hat{\mu}_{\mathcal{P}/\mathcal{Q}}\right] \geq \lambda\right)$$

Hoeffding's inequality:

Let $\{X_1, X_2, \dots, X_N\}$ be independent random variables bounded by the interval [a, b], and $\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$, then:

Proof 1

② $$\hat{\mu}_{\mathcal{P}/\mathcal{Q}} = \frac{1}{N}\sum_{i=1}^{N} W_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i) = \frac{1}{N}\sum_{i=1}^{N} g(x_i) \qquad g(x) = W_{\mathcal{P}/\mathcal{Q}}(x) f(x)$$

$\hat{\mu}_{\mathcal{P}/\mathcal{Q}}$ is unbiased , which means $E[\hat{\mu}_{\mathcal{P}/\mathcal{Q}}] = E[g(x)]$, then:

$$\mathbb{P}\left(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim Q}\left[W_{\mathcal{P}/\mathcal{Q}}(x) f(x)\right] \geq \lambda\right) = \mathbb{P}\left(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim Q}\left[\hat{\mu}_{\mathcal{P}/\mathcal{Q}}\right] \geq \lambda\right)$$

Hoeffding's inequality:

Let $\{X_1, X_2, \dots, X_N\}$ be independent random variables bounded by the interval [a, b], and $\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$, then:

$$\forall\, \lambda > 0, \qquad \mathbb{P}\left(\overline{X} - \mathbb{E}_{x\sim Q}\left[\overline{X}\right] \geq \lambda\right) \leq \exp\left(-\frac{2N\lambda^2}{\sum_{i=1}^{N}(b_i - a_i)^2}\right)$$

Proof 1

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2 $$\hat{\mu}_{\mathcal{P}/\mathcal{Q}} = \frac{1}{N}\sum_{i=1}^{N} W_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i) = \frac{1}{N}\sum_{i=1}^{N} g(x_i) \qquad g(x) = W_{\mathcal{P}/\mathcal{Q}}(x) f(x)$$

$\hat{\mu}_{\mathcal{P}/\mathcal{Q}}$ is unbiased , which means $E[\hat{\mu}_{\mathcal{P}/\mathcal{Q}}] = E[g(x)]$, then:

$$\mathbb{P}\big(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim Q}\big[W_{\mathcal{P}/\mathcal{Q}}(x) f(x)\big] \geq \lambda\big) = \mathbb{P}\big(\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim Q}\big[\hat{\mu}_{\mathcal{P}/\mathcal{Q}}\big] \geq \lambda\big)$$

$$0 \leq g(x_i) \leq \big\|g(x)\big\|_{\infty} = d_{\infty}(\mathcal{P}\|\mathcal{Q})\big\|f(x)\big\|_{\infty}$$

$$\forall\, \lambda > 0, \qquad \mathbb{P}\big(\overline{X} - \mathbb{E}_{x\sim Q}\big[\overline{X}\big] \geq \lambda\big) \leq \exp\Big(-\frac{2N\lambda^2}{\sum_{i=1}^{N}(b_i - a_i)^2}\Big)$$

Proof 1

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Further:

$$
\begin{aligned}
&\mathbb{P}(|\,\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}| \geq \lambda) \\
&\quad \leq \mathbb{P}(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim\mathcal{P}}[f(x)] \geq \lambda) + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda) \\
&\quad \leq \mathbb{P}(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] \geq \tilde{\lambda}) + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \tilde{\lambda}) \\
&\quad \leq \tilde{\delta} + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda)
\end{aligned}
$$

Proof 1

Further:

$$\begin{aligned}
&\mathbb{P}(|\,\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}| \geq \lambda) \\
&\quad \leq \mathbb{P}(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim\mathcal{P}}[f(x)] \geq \lambda) + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda) \qquad ① \\
&\quad \leq \mathbb{P}(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] \geq \tilde{\lambda}) + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \tilde{\lambda}) \\
&\quad \leq \tilde{\delta} + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda)
\end{aligned}$$

① $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$

Proof 1

Further:

$$
\begin{aligned}
&\mathbb{P}(|\,\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}| \geq \lambda) \\
&\quad \leq \mathbb{P}(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim\mathcal{P}}[f(x)] \geq \lambda) + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda) && 1 \\
&\quad \leq \mathbb{P}(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] \geq \tilde{\lambda}) + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \tilde{\lambda}) && 2 \\
&\quad \leq \tilde{\delta} + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda)
\end{aligned}
$$

2 $$\mathbb{P}\left(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim P}[f(x)] \geq \lambda\right)$$

$$= \mathbb{P}\left(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim P}[f(x)] - \left|\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \mathbb{E}_{x\sim\mathcal{Q}}\left[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}\right]\right| \geq \tilde{\lambda}\right)$$

$$\leq \mathbb{P}\left(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim P}[f(x)] - (\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \mathbb{E}_{x\sim\mathcal{Q}}\left[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}\right]) \geq \tilde{\lambda}\right)$$

Proof 1

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Further:

$$
\begin{aligned}
&\mathbb{P}(|\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}| \geq \lambda) \\
&\quad \leq \mathbb{P}(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim\mathcal{P}}[f(x)] \geq \lambda) + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda) && ① \\
&\quad \leq \mathbb{P}(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] \geq \tilde{\lambda}) + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \tilde{\lambda}) && ② \\
&\quad \leq \tilde{\delta} + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda) && ③
\end{aligned}
$$

③ Using Lemma1

$$\tilde{\lambda} = \lambda - \left|\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}]\right|$$

$$\tilde{\delta} = \exp\left(-\frac{N\tilde{\lambda}^2}{d_\infty^2(\mathcal{P}\parallel\mathcal{Q})\parallel f\parallel_\infty^2}\right)$$

Proof 1

Further:

$$
\begin{aligned}
&\mathbb{P}(|\,\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}| \geq \lambda) \\
&\quad \leq \mathbb{P}(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim\mathcal{P}}[f(x)] \geq \lambda) + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda) && \text{1} \\
&\quad \leq \mathbb{P}(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] \geq \tilde{\lambda}) + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \tilde{\lambda}) && \text{2} \\
&\quad \leq \tilde{\delta} + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda) && \text{3}
\end{aligned}
$$

Proof 1

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Further:

$$\begin{aligned}
&\mathbb{P}(|\,\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}| \geq \lambda) \\
&\quad \leq \mathbb{P}(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim\mathcal{P}}[f(x)] \geq \lambda) + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda) && 1 \\
&\quad \leq \mathbb{P}(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] \geq \tilde{\lambda}) + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \tilde{\lambda}) && 2 \\
&\quad \leq \tilde{\delta} + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda) && 3
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda) &\leq \mathbb{P}(\mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \tilde{\lambda}) \\
&\leq \mathbb{P}(|\,\mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}| \geq \tilde{\lambda}) \leq 2\tilde{\delta}
\end{aligned}$$

Proof 1

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Further:

$$
\begin{aligned}
&\mathbb{P}(|\,\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}| \geq \lambda) \\
&\quad \leq \mathbb{P}(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim\mathcal{P}}[f(x)] \geq \lambda) + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda) && (1) \\
&\quad \leq \mathbb{P}(\tilde{\mu}_{\mathcal{P}/\mathcal{Q}} - \mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] \geq \tilde{\lambda}) + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \tilde{\lambda}) && (2) \\
&\quad \leq \tilde{\delta} + \mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda) && (3)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \lambda) &\leq \mathbb{P}(\mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}} \geq \tilde{\lambda}) \\
&\leq \mathbb{P}(|\,\mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}| \geq \tilde{\lambda}) \leq 2\tilde{\delta}
\end{aligned}
$$

$$
\mathbb{P}\left(|\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}| \geq \lambda\right) \leq 3\tilde{\delta}
$$

Proof 1

$$
\begin{aligned}
|\,\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}]| &= \left|\mathbb{E}_{x\sim\mathcal{Q}}[\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}]\right| \leq \mathbb{E}_{x\sim\mathcal{Q}}[|\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}|] \\
&\leq \mathbb{E}_{x\sim\mathcal{Q}}\left|\frac{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)f(x_i)}{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)} - \frac{1}{N}\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)f(x_i)\right| \\
&= \mathbb{E}_{x\sim\mathcal{Q}}\left[\left|\frac{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)f(x_i)}{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)}\right|\left|1 - \frac{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)}{N}\right|\right] \\
&\leq \mathbb{E}_{x\sim\mathcal{Q}}\left[\left(\frac{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)f(x_i)}{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)}\right)^2\right]^{1/2} \mathbb{E}_{x\sim\mathcal{Q}}\left[\left(1 - \frac{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)}{N}\right)^2\right]^{1/2} \\
&\leq \|f\|_\infty \sqrt{\frac{d_2(\mathcal{P}||\mathcal{Q}) - 1}{N}} \leq \|f\|_\infty \frac{d_\infty(\mathcal{P}||\mathcal{Q})}{\sqrt{N}}
\end{aligned}
$$

Proof 1

$$
\begin{aligned}
|\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}]| &= \left|\mathbb{E}_{x\sim\mathcal{Q}}[\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}]\right| \leq \mathbb{E}_{x\sim\mathcal{Q}}[|\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}|] \\
&\leq \mathbb{E}_{x\sim\mathcal{Q}}\left|\frac{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i)}{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)} - \frac{1}{N}\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i)\right| \\
&= \mathbb{E}_{x\sim\mathcal{Q}}\left[\left|\frac{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i)}{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)}\right|\left|1 - \frac{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)}{N}\right|\right] \qquad 1 \\
&\leq \mathbb{E}_{x\sim\mathcal{Q}}\left[\left(\frac{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i)}{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)}\right)^2\right]^{1/2} \mathbb{E}_{x\sim\mathcal{Q}}\left[\left(1 - \frac{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)}{N}\right)^2\right]^{1/2} \\
&\leq \|f\|_\infty \sqrt{\frac{d_2(\mathcal{P}||\mathcal{Q}) - 1}{N}} \leq \|f\|_\infty \frac{d_\infty(\mathcal{P}||\mathcal{Q})}{\sqrt{N}}
\end{aligned}
$$

1 Cauchy-Schwarz inequality

$$
\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)
$$

Proof 1

$$\left| \mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] \right| = \left| \mathbb{E}_{x\sim\mathcal{Q}}[\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}] \right| \leq \mathbb{E}_{x\sim\mathcal{Q}}[|\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}|]$$

$$\leq \mathbb{E}_{x\sim\mathcal{Q}} \left| \frac{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i)}{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)} - \frac{1}{N}\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i) \right|$$

$$= \mathbb{E}_{x\sim\mathcal{Q}} \left[\left| \frac{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i)}{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)} \right| \left| 1 - \frac{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)}{N} \right| \right] \qquad 1$$

$$\leq \mathbb{E}_{x\sim\mathcal{Q}} \left[\left(\frac{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i)}{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)} \right)^2 \right]^{1/2} \mathbb{E}_{x\sim\mathcal{Q}} \left[\left(1 - \frac{\sum_{i=1}^{N} w_{\mathcal{P}/\mathcal{Q}}(x_i)}{N} \right)^2 \right]^{1/2} \qquad 2$$

$$\leq \|f\|_\infty \sqrt{\frac{d_2(\mathcal{P}||\mathcal{Q}) - 1}{N}} \leq \|f\|_\infty \frac{d_\infty(\mathcal{P}||\mathcal{Q})}{\sqrt{N}}$$

Proof 1

$$
\begin{aligned}
|\,\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}]| &= \left|\mathbb{E}_{x\sim\mathcal{Q}}[\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}]\right| \leq \mathbb{E}_{x\sim\mathcal{Q}}[|\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}|] \\
&\leq \mathbb{E}_{x\sim\mathcal{Q}}\left|\frac{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i)}{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i)} - \frac{1}{N}\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i)\right| \\
&= \mathbb{E}_{x\sim\mathcal{Q}}\left[\left|\frac{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i)}{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i)}\right|\left|1 - \frac{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i)}{N}\right|\right] \qquad (1) \\
&\leq \mathbb{E}_{x\sim\mathcal{Q}}\left[\left(\frac{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i) f(x_i)}{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i)}\right)^2\right]^{1/2} \mathbb{E}_{x\sim\mathcal{Q}}\left[\left(1 - \frac{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i)}{N}\right)^2\right]^{1/2} \qquad (2) \\
&\leq \|f\|_\infty \sqrt{\frac{d_2(\mathcal{P}||\mathcal{Q}) - 1}{N}} \leq \|f\|_\infty \frac{d_\infty(\mathcal{P}||\mathcal{Q})}{\sqrt{N}} \qquad (3)
\end{aligned}
$$

Proof 1

$$|\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \mathbb{E}_{x\sim\mathcal{Q}}[\tilde{\mu}_{\mathcal{P}/\mathcal{Q}}]| = \left|\mathbb{E}_{x\sim\mathcal{Q}}[\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}]\right| \leq \mathbb{E}_{x\sim\mathcal{Q}}[|\hat{\mu}_{\mathcal{P}/\mathcal{Q}} - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}|]$$

$$\leq \mathbb{E}_{x\sim\mathcal{Q}}\left|\frac{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i)f(x_i)}{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i)} - \frac{1}{N}\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i)f(x_i)\right|$$

$$= \mathbb{E}_{x\sim\mathcal{Q}}\left[\left|\frac{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i)f(x_i)}{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i)}\right|\left|1 - \frac{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i)}{N}\right|\right] \quad ①$$

$$\leq \mathbb{E}_{x\sim\mathcal{Q}}\left[\left(\frac{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i)f(x_i)}{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i)}\right)^2\right]^{1/2}\mathbb{E}_{x\sim\mathcal{Q}}\left[\left(1 - \frac{\sum_{i=1}^N w_{\mathcal{P}/\mathcal{Q}}(x_i)}{N}\right)^2\right]^{1/2} \quad ②$$

$$\leq \|f\|_\infty\sqrt{\frac{d_2(\mathcal{P}||\mathcal{Q})-1}{N}} \leq \|f\|_\infty\frac{d_\infty(\mathcal{P}||\mathcal{Q})}{\sqrt{N}} \quad ③$$

③ $$d_2(\mathcal{P}||\mathcal{Q}) - 1 \leq d_2(\mathcal{P}||\mathcal{Q}) = \mathbb{E}_{x\sim\mathcal{Q}}[W_{\mathcal{P}/\mathcal{Q}}(x)^2]$$

$$\leq W_{\mathcal{P}/\mathcal{Q}}(x)^2_{max} = d_\infty(\mathcal{P}||\mathcal{Q})^2$$

Proof 1

② $$\mathbb{E}_{x\sim\mathcal{Q}}\left[\left(1-\frac{\sum_{i=1}^{N}W_{\mathcal{P}/\mathcal{Q}}(x_i)}{N}\right)^2\right]=\mathbb{E}_{x\sim\mathcal{Q}}\left[\sum_{i=1}^{N}\left(\frac{1-W_{\mathcal{P}/\mathcal{Q}}(x_i)}{N}\right)^2\right]$$

$$\leq\sum_{i=1}^{N}\mathbb{E}_{x\sim\mathcal{Q}}\left[\left(\frac{1-W_{\mathcal{P}/\mathcal{Q}}(x_i)}{N}\right)^2\right]\leq\sum_{i=1}^{N}\mathbb{E}_{x\sim\mathcal{Q}}\left[\left(\frac{W_{\mathcal{P}/\mathcal{Q}}(x_i)}{N}\right)^2-\left(\frac{1}{N}\right)^2\right]$$

$$=\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}_{x\sim\mathcal{Q}}\left[\left(W_{\mathcal{P}/\mathcal{Q}}(x_i)\right)^2\right]-\frac{1}{N}\leq\frac{d_2(\mathcal{P}||\mathcal{Q})-1}{N}$$

Proof 1

Last:

$$\tilde{\delta} \leq \exp\left(-\frac{N(\lambda - \|f\|_\infty d_\infty(\mathcal{P}\|\mathcal{Q})/\sqrt{N})^2}{d_\infty^2(\mathcal{P}\|\mathcal{Q})\|f\|_\infty^2}\right)$$

$$= \exp\left(-N\left(\frac{\lambda - \|f\|_\infty d_\infty(\mathcal{P}\|\mathcal{Q})/\sqrt{N}}{\|f\|_\infty d_\infty(\mathcal{P}\|\mathcal{Q})}\right)^2\right)$$

$$\equiv \exp\left(-N \cdot t^2(\lambda, N)\right)$$

Lemma 1

Lemma 1 (SN Estimator Leaf Node Convergence). *$\hat{Q}^*_{D-1}(\bar{b}_{D-1}, a)$ is an SN estimator of $Q^*_{D-1}(b_{D-1}, a)$, and the following leaf-node concentration bound holds with probability at least* $1 - 3\exp(-C \cdot t^2_{\max}(\lambda, C))$,

$$|Q^*_{D-1}(b_{D-1}, a) - \hat{Q}^*_{D-1}(\bar{b}_{D-1}, a)| \leq \lambda$$

Proof 2

$\hat{Q}^*_{D-1}\left(\overline{b}_{D-1}, a\right)$ is an SN estimator of $Q^*_{D-1}\left(\overline{b}_{D-1}, a\right)$:

$$\hat{Q}^*_{D-1}\left(\overline{b}_{D-1}, a\right) = \sum_{i=1}^{C} \widetilde{w}_{\mathcal{P}^{D-1}/\mathcal{Q}^{D-1}}(\{s_n\}_i) R\left(s_{D-1,i}, a\right)$$

$$Q^*_{D-1}\left(b_{D-1}, a\right) = \int_S R(S_{D-1}, a) dS$$

Proof 2

$\hat{Q}^*_{D-1}(\overline{b}_{D-1}, a)$ is an SN estimator of $Q^*_{D-1}(\overline{b}_{D-1}, a)$:

Using Theorem 1:

first bound R by $3V_{max}$:

$$\mathrm{V}_{max} \equiv \frac{R_{max}}{1-\gamma} \geq R_{max}$$

Proof 2

Theorem 1 (SN d_∞-Concentration Bound). *Let $\mathcal{P}$ and $\mathcal{Q}$ be two probability measures on the measurable space $(\mathcal{X}, \mathcal{F})$ with $\mathcal{P} \ll \mathcal{Q}$ and $d_\infty(\mathcal{P}||\mathcal{Q}) < +\infty$. Let $x_1, \cdots, x_N$ be i.i.d.r.v. sampled from $\mathcal{Q}$, and $f : \mathcal{X} \to \mathbb{R}$ be a bounded Borel function ($\|f\|_\infty < +\infty$). Then, for any $\lambda > 0$ and N large enough such that $\lambda > \|f\|_\infty d_\infty(\mathcal{P}||\mathcal{Q})/\sqrt{N}$, the following bound holds with probability at least $1 - 3\exp(-N \cdot t^2(\lambda, N))$:*

$$|\mathbb{E}_{x\sim\mathcal{P}}[f(x)] - \tilde{\mu}_{\mathcal{P}/\mathcal{Q}}| \leq \lambda \tag{6}$$

where $t(\lambda, N)$ is defined as:

$$t(\lambda, N) \equiv \frac{\lambda}{\|f\|_\infty d_\infty(\mathcal{P}||\mathcal{Q})} - \frac{1}{\sqrt{N}} \tag{7}$$

Proof 2

$\hat{Q}^*_{D-1}(\overline{b}_{D-1}, a)$ is an SN estimator of $Q^*_{D-1}(\overline{b}_{D-1}, a)$:

Using Theorem 1:

first bound R by $3V_{max}$:

$$\mathrm{V}_{max} \equiv \frac{R_{max}}{1-\gamma} \geq R_{max}$$

Proof 2

$\hat{Q}^*_{D-1}(\overline{b}_{D-1}, a)$ is an SN estimator of $Q^*_{D-1}(\overline{b}_{D-1}, a)$:

Using Theorem 1:

first bound R by $3V_{max}$: $\mathrm{V}_{max} \equiv \frac{R_{max}}{1-\gamma} \geq R_{max}$

$$t_{D-1}(\lambda, C) = \frac{\lambda}{3V_{\max} d_\infty(\mathcal{P}^{D-1} \parallel \mathcal{Q}^{D-1})} - \frac{1}{\sqrt{C}} \geq \frac{\lambda}{3V_{\max} d_\infty^{\max}} - \frac{1}{\sqrt{C}} \equiv t_{\max}(\lambda, C)$$

$\forall\, i < D,$ $d_\infty(\mathcal{P}^i \parallel \mathcal{Q}^i) \leq d_\infty^{\max}$

$$t_{\mathrm{i}}(\lambda, C) \geq t_{\max}(\lambda, C)$$

Lemma 2

Lemma 2 (SN Estimator Step-by-Step Convergence). *$\hat{Q}^*_d(\bar{b}_d, a)$ is an SN estimator of $Q^*_d(b_d, a)$, and for all $d = 0, \cdots, D-1$ and a, the following holds with probability at least* $1 - 3|A|(3|A|C)^D \exp(-C \cdot t^2_{\max})$:

$$|Q^*_d(b_d, a) - \hat{Q}^*_d(\bar{b}_d, a)| \leq \alpha_d$$

$$\alpha_d \equiv \lambda + \gamma \alpha_{d+1}; \ \alpha_{D-1} = \lambda$$

Proof 3

$$|Q_d^*(b_d, a) - \hat{Q}_d^*(\bar{b}_d, a)| \leq \underbrace{\left|\mathbb{E}[R(s_d, a)|b_d] - \frac{\sum_{i=1}^{C} w_{d,i} r_{d,i}}{\sum_{i=1}^{C} w_{d,i}}\right|}_{\text{(A)}} + \gamma \underbrace{\left|\mathbb{E}[V_{d+1}^*(bao)|b_d] - \frac{\sum_{i=1}^{C} w_{d,i} \hat{V}_{d+1}^*(\overline{b_d a o_i})}{\sum_{i=1}^{C} w_{d,i}}\right|}_{\text{(B)}}$$

Proof 3

$$|Q_d^*(b_d, a) - \hat{Q}_d^*(\bar{b}_d, a)| \leq \underbrace{\left| \mathbb{E}[R(s_d, a)|b_d] - \frac{\sum_{i=1}^{C} w_{d,i} r_{d,i}}{\sum_{i=1}^{C} w_{d,i}} \right|}_{\text{(A)}} + \gamma \underbrace{\left| \mathbb{E}[V_{d+1}^*(bao)|b_d] - \frac{\sum_{i=1}^{C} w_{d,i} \hat{V}_{d+1}^*(\overline{b_d a o_i})}{\sum_{i=1}^{C} w_{d,i}} \right|}_{\text{(B)}}$$

$$|A_1 + B_1 - A_2 - B_2| \leq |A_1 - A_2| + |B_1 - B_2|$$

$$\hat{Q}_d^*(\bar{b}_d, a) = \frac{\sum_{i=1}^{C} w_{d,i} \left(r_{d,i} + \gamma \overline{V^*}_{d+1}(\overline{b_d a o_i}) \right)}{\sum_{i=1}^{C} w_{d,i}}$$

$$Q_d^*(b_d, a) = E[R(s_d, a) + \gamma V_{d+1}^*(bao)|b_d]$$

Proof 3

$$|Q_d^*(b_d,a) - \hat{Q}_d^*(\bar{b}_d,a)| \le \underbrace{\left| \mathbb{E}[R(s_d,a)|b_d] - \frac{\sum_{i=1}^C w_{d,i} r_{d,i}}{\sum_{i=1}^C w_{d,i}} \right|}_{\text{(A)}} + \gamma \underbrace{\left| \mathbb{E}[V_{d+1}^*(bao)|b_d] - \frac{\sum_{i=1}^C w_{d,i} \hat{V}_{d+1}^*(\overline{b_d a o_i})}{\sum_{i=1}^C w_{d,i}} \right|}_{\text{(B)}}$$

For part A: bound R by R_{max} and let $\lambda = \frac{R_{max}}{3V_{max}}\lambda$, using Thero1:

$$A \le \frac{R_{max}}{3V_{max}}\lambda \qquad \Pr \ge 1 - \exp(-t_{\max}^2)$$

Proof 3

For part B:

$$(B) \le \underbrace{\left| \mathbb{E}[V^*_{d+1}(bao)|b_d] - \frac{\sum_{i=1}^{C} w_{d,i} \mathbf{V}^*_{d+1}(s_{d,i}, b_d, a)}{\sum_{i=1}^{C} w_{d,i}} \right|}_{\text{Importance sampling error}}$$

$$+ \underbrace{\left| \frac{\sum_{i=1}^{C} w_{d,i} \mathbf{V}^*_{d+1}(s_{d,i}, b_d, a)}{\sum_{i=1}^{C} w_{d,i}} - \frac{\sum_{i=1}^{C} w_{d,i} V^*_{d+1}(b_d a o_i)}{\sum_{i=1}^{C} w_{d,i}} \right|}_{\text{MC next-step integral approximation error}}$$

$$+ \underbrace{\left| \frac{\sum_{i=1}^{C} w_{d,i} V^*_{d+1}(b_d a o_i)}{\sum_{i=1}^{C} w_{d,i}} - \frac{\sum_{i=1}^{C} w_{d,i} \hat{V}^*_{d+1}(\overline{b_d a o_i})}{\sum_{i=1}^{C} w_{d,i}} \right|}_{\text{Function estimation error}}$$

$$\le \frac{1}{3}\lambda + \frac{2}{3\gamma}\lambda + \alpha_{d+1}$$

Proof 3

For Important Sampling error ,

$$\mathbf{V}^*_{d+1}(s_{d,i}, b_d, a) \equiv \int_S \int_O V^*_{d+1}(b_d a o)\mathcal{Z}(o|a, s_{d+1})\mathcal{T}(s_{d+1}|s_{d,i}, a) ds_{d+1} do$$

$$\begin{aligned}\mathbb{E}[V^*_{d+1}(bao)|b_d] &= \int_S \int_S \int_O V^*_{d+1}(b_d a o)(\mathcal{Z}_{d+1})(\mathcal{T}_{d,d+1}) b_d \cdot ds_{d:d+1} do \\ &= \int_S \mathbf{V}^*_{d+1}(s_d, b_d, a) b_d \cdot ds_d \\ &= \frac{\int_{S^{d+1}} \mathbf{V}^*_{d+1}(s_d, b_d, a)(\mathcal{Z}_{1:d})(\mathcal{T}_{1:d}) b_0 ds_{0:d}}{\int_{S^{d+1}} (\mathcal{Z}_{1:d})(\mathcal{T}_{1:d}) b_0 ds_{0:d}}\end{aligned}$$

Using SN inequality: $\left\|V^*_{d+1}(s_d, b_d, a)\right\|_\infty = V_{max}$

$$\mathrm{IS} \le \frac{1}{3}\lambda \qquad \Pr \ge 1 - \exp(-\mathrm{t}^2_{\max})$$

Proof 3

For MC error:

the quantity $V^*_{d+1}(b_d a o_i)$ for a given $(s_{d,i}, b_d, a)()$ is an unbiased 1-sample MC estimate of $\mathbf{V}^*_{d+1}(s_{d,i}, b_d, a)$

$$\Delta_{d+1}(s_{d,i}, b_d, a) \equiv \mathbf{V}^*_{d+1}(s_{d,i}, b_d, a) - V^*_{d+1}(b_d a o_i) \qquad E(\Delta_{d+1}) = 0$$

$$\left| \frac{\sum_{i=1}^{C} w_{d,i} \mathbf{V}^*_{d+1}(s_{d,i}, b_d, a)}{\sum_{i=1}^{C} w_{d,i}} - \frac{\sum_{i=1}^{C} w_{d,i} V^*_{d+1}(b_d a o_i)}{\sum_{i=1}^{C} w_{d,i}} \right|$$

$$= \left| \frac{\sum_{i=1}^{C} w_{d,i} \Delta_{d+1}(s_{d,i}, b_d, a)}{\sum_{i=1}^{C} w_{d,i}} - 0 \right| \leq \frac{2}{3}\lambda \leq \frac{2}{3\gamma}\lambda \qquad ||\Delta_{d+1}||_\infty \leq 2V_{max}$$

Proof 3

For Function Estimation Error:

the third term is bounded by the inductive hypothesis, since each i-th absolute difference of the Q-function and its estimate at step $d+1$, and furthermore the value function and its estimate at step $d+1$, are all bounded by α_{d+1}.

Proof 3

For part B:

$$|Q_d^*(b_d, a) - \hat{Q}_d^*(\bar{b}_d, a)| \leq \frac{R_{\max}}{3V_{\max}}\lambda + \gamma[\frac{1}{3}\lambda + \frac{2}{3\gamma}\lambda + \alpha_{d+1}]$$

$$\leq \lambda + \gamma\alpha_{d+1} = \alpha_d$$

Thanks!