Complex Fuzzy Sets

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***Abstract*—The objective of this paper is to investigate the innovative concept of complex fuzzy sets. The novelty of the complex fuzzy set lies in the range of values its membership function may attain. In contrast to a traditional fuzzy membership function, this range is not limited to [0, 1], but extended to the unit circle in the complex plane. Thus, the complex fuzzy set provides a mathematical framework for describing membership in a set in terms of a complex number. The inherent difficulty in acquiring intuition for the concept of complex-valued membership presents a significant obstacle to the realization of its full potential. Consequently, a major part of this work is dedicated to a discussion of the intuitive interpretation of complex-valued grades of membership. Examples of possible applications, which demonstrate the new concept, include a complex fuzzy representation of solar activity (via measurements of the sunspot number), and a signal processing application. A comprehensive study of the mathematical properties of the complex fuzzy set is presented. Basic set theoretic operations on complex fuzzy sets, such as complex fuzzy complement, union, and intersection, are discussed at length. Two novel operations, namely set rotation and set reflection, are introduced. Complex fuzzy relations are also considered.**

***Index Terms*—Complex fuzzy intersection, complex fuzzy relations, complex fuzzy sets, complex fuzzy union, complex-valued grades of membership, fuzzy complex numbers.**

I. Introduction

In 1965, the concept of fuzzy sets was introduced by Zadeh [14]. This novel method for representing uncertainty questioned many fundamental beliefs held by researchers at the time. In particular, fuzzy sets posed a challenge to probability theory, suggesting an alternative instrument for modeling uncertainty, and contesting its very foundation—Aristotelian dual-valued logic [9]. For this reason, the new theory came under attack. Today however, fuzzy sets have become an increasingly popular method for representing uncertainty and have proven themselves useful in many applications.

In the past few decades following Zadeh's seminal paper [14], the study of fuzzy sets has been extensive. The merits of utilizing fuzzy sets for modeling uncertainty, representing subjective human knowledge, and as a means of emulating human reasoning processes, have been discussed by innumerable authors. To name but a few—Kandel [8], Klir and Yuan [9], Mendel [10], and Zimmermann [15] have contributed some introductory texts on the subject.

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The extension of crisp sets to fuzzy sets, in terms of membership functions, is mathematically comparable to the extension of the set of integers, $\boldsymbol{I}$, to the set of real numbers, $\boldsymbol{R}$. That is, expanding the range of the membership function, $\mu_S(x)$, from $\{0, 1\}$ to $[0, 1]$ is mathematically analogous to the extension of $\boldsymbol{I}$ to $\boldsymbol{R}$.

Of course, the development of the number set did not end with real numbers. Historically, the introduction of real numbers was followed by their extension to the set of complex numbers, $\boldsymbol{C}$. Thus, it may be suggested that a further development of fuzzy set theory should be based on this extension. The result of such an extension, in the context of set theory, is the *complex fuzzy set*, i.e., a fuzzy set characterized by a complex-valued membership function.

In this paper, the innovative complex fuzzy set is introduced. The complex fuzzy set, S, is characterized by a membership function, $\mu_S(x)$, whose range is not limited to [0, 1] but extended to the unit circle in the complex plane. Hence, $\mu_S(x)$ is a complex-valued function that assigns a grade of membership of the form $r_S(x) \cdot e^{j\omega_S(x)}$ $(j = \sqrt{-1})$ to any element x in the universe of discourse. The value of $\mu_S(x)$ is defined by the two variables, $r_S(x)$ and $\omega_S(x)$, both real-valued, with $r_S(x) \in [0, 1]$.

Thus, complex fuzzy set theory modifies the original concept of fuzzy membership by asserting that, at least in some instances, it is necessary to add a second dimension to the expression of membership. However, this added dimension does not alter the basic concept of fuzziness. Membership in a complex fuzzy set remains "as fuzzy" as membership in a traditional fuzzy set. The fuzziness of membership, i.e., the representation of membership as a value in the range [0, 1], is retained in complex fuzzy sets through the *amplitude* of the grade of membership, $r_S(x)$. The novelty of complex fuzzy sets is manifested in the additional dimension of membership: the *phase* of the grade of membership, $\omega_S(x)$. The properties of *membership phase* are discussed at length in the following sections.

Unfortunately, there is inherent difficulty in acquiring intuition for the concept of complex-valued membership, which presents a significant obstacle to the realization of its full potential. Consequently, a major part of this work is dedicated to a discussion of the intuitive interpretation of complex-valued grades of membership. Several examples, which are designed to provide intuition for the merit and potential of complex-valued membership functions, are also presented. In particular, it is shown that complex fuzzy sets are not simply a linear extension of conventional fuzzy sets. Rather, complex fuzzy sets allow a natural extension of fuzzy set theory to problems that are either very difficult or impossible to address with one-dimensional grades of membership. In addition, a comprehensive study of the mathematical properties of this original concept and its potential for various applications is presented.

It should be noted that there is, in fact, an abundance of research on the subject of combining complex numbers with fuzzy sets. However, in most cases, the aspects of applying complex numbers to fuzzy sets, which are addressed in the literature, are profoundly different from the ones at the focus of this paper. Consider, for example, the *fuzzy complex number*. This concept was introduced by J. J. Buckley in 1987 [3]–[6] and has become a prominent research topic in fuzzy set theory. The fuzzy complex number is indeed concerned with complex numbers and fuzzy sets, but in a completely different manner than the one discussed in this work. Buckley's definition incorporates complex numbers into the *support* of the fuzzy set, thus creating the fuzzy complex number. The fuzzy set representing the fuzzy complex number is an ordinary fuzzy set, with grades of membership in the range [0, 1]. This is entirely different from the complex fuzzy set, which is a new type of fuzzy set—one with *complex-valued grades of membership*. The support of the complex fuzzy set is unrestricted, and may include any kind of element, such as real numbers, cars, people, and of course complex numbers.

An original approach, similar to the one outlined in this paper, can be found in Nguyen, Kreinovich, and Shekhter [11]. Nguyen, *et al.* employ complex numbers to represent "truth values," extending the range of truth values from the interval [0, 1] to $\boldsymbol{C}$. The motivation for this extension is the utility of complex-valued degrees of truth in representing inconsistencies or paradoxes. The argument in favor of using truth values that include complex numbers, is that such an extension improves the applicability of fuzzy logic to problems encountered in the humanities (chiefly psychology and philosophy). In contrast, the research presented below emphasizes engineering and real-life applications of complex-valued grades of membership, which make use of the formal framework of *complex fuzzy sets*. This focus on engineering and real-life applications results in the construction of a mathematical framework, and the derivation of a physical/intuitive interpretation, that are markedly different from those suggested by Nguyen, *et al.* [11]. In particular, the significance associated with the "phase quality" of complex-valued membership functions distinguishes the two approaches (see Section III).

The paper is organized as follows. A formal definition of the complex fuzzy set is provided in Section III. An intuitive interpretation of complex-valued membership functions and some examples of possible applications of complex fuzzy sets are also presented in this section.

Set theoretic operations on complex fuzzy sets are discussed in Section IV. The operations of complex fuzzy complement, union and intersection are defined. Two novel operations are introduced, namely set rotation and reflection. These operations allow the manipulation of the phase term of complex-valued membership functions.

Section V is concerned with complex fuzzy relations. Concluding remarks and suggestions for further research are given in Section VI.

II. Introducing Complex Fuzzy Sets

In this section, complex fuzzy sets are introduced. First, a formal definition of the complex fuzzy set is provided, followed by a discussion of the interpretation of this novel concept. The discussion is completed by several examples, which illustrate the potential of complex fuzzy sets.

A. Definition of a Complex Fuzzy Set

Definition 1: A *Complex Fuzzy Set* S, defined on a universe of discourse U, is characterized by a membership function $\mu_S(x)$ that assigns any element $x \in U$ a complex-valued grade of membership in S. By definition, the values $\mu_S(x)$ may receive all lie within the unit circle in the complex plane, and are thus of the form $r_S(x) \cdot e^{j\omega_S(x)}$, where $j = \sqrt{-1}, r_S(x)$ and $\omega_S(x)$ are both real-valued, and $r_S(x) \in [0, 1]$.

The complex fuzzy set S may be represented as the set of ordered pairs

$$S = \{(x, \mu_S(x)) \,|\, x \in U\}. \tag{1}$$

Throughout this paper, the term complex fuzzy Sset refers to a fuzzy set with complex-valued membership function, while the term fuzzy set refers to a traditional fuzzy set with real-valued membership function.

B. Interpretation of the Complex Fuzzy Set

Unfortunately, obtaining intuition for the concept of a complex-valued grade of membership is not a simple task. In contrast, a real-valued grade of membership in the range [0, 1] is a highly intuitive notion, which is one of its powerful attributes. It is quite simple to grasp the idea of a grade of membership of 0.5 in practically any set (for example, the set of tall people). However, it may be quite confusing to consider a grade of membership $0.5 \cdot e^{(j\pi)/2}$, or $0.5 \cdot j$. This is partly because complex numbers themselves are not particularly intuitive.

Much important insight into the complex fuzzy set can be obtained directly from the form of its membership function as it appears in Definition 1: $\mu_S(x) = r_S(x) \cdot e^{j\omega_S(x)}$. Evidently, each complex grade of membership is defined by an amplitude term $r_S(x)$ and a phase term $\omega_S(x)$. Note that the amplitude term $r_S(x)$ is equal to $|\mu_S(x)|$, the amplitude of $\mu_S(x)$.

As complex fuzzy sets are generalizations of ordinary fuzzy sets, it should be possible to represent any ordinary fuzzy set in terms of a complex fuzzy set. Assume the ordinary fuzzy set S is characterized by the real-valued membership function $\lambda_S(x)$. Transforming S into a complex fuzzy set is easily achieved by setting the amplitude term $r_S(x)$ equal to $\lambda_S(x)$, and the phase term equal to zero for all x. From this observation, it is possible to deduce that the amplitude term is essentially equivalent to the traditional real-valued grade of membership, while the phase term is the distinguishing factor between ordinary and complex fuzzy sets. In other words, without a phase term, the complex fuzzy set effectively reduces to a conventional fuzzy set. This interpretation is supported by the fact that the range of $r_S(x)$ is [0, 1], as for an ordinary real-valued grade of membership.

Thus, it is the introduction of a phase term, which makes the complex fuzzy set a distinctive and novel concept. This suggests that the use of a complex-valued membership function to characterize a set S is appropriate only when membership in S has some "phase quality" to it, the existence of which is manifested in a nonzero phase term. For if $\omega_S(x)$ equals zero for all elements in the support of S, the phase term is irrelevant and S is a conventional fuzzy set with real-valued membership function.

The association of the amplitude term with an ordinary real-valued grade of membership should have clarified its role to some extent. The amplitude term, much like an ordinary grade of membership, may be regarded as representing the *degree* to which x is a member of the complex fuzzy set S. The phase term, however, remains an enigma. What indeed, is the *meaning* of a "phase quality" of membership? Moreover, if the phase term is to be seen as representing the *phase* associated with the membership of x in S, what is the significance of this *membership phase*?

A few answers to these questions may be obtained by borrowing some terminology from the branch of physics known as quantum mechanics. In prequantum (classical) mechanics, a basic element was regarded as either a particle or a wave. In a (purposely over-simplified) sense, quantum mechanics may be considered a generalization of classical mechanics, which allows objects to exhibit both wave-like and particle-like properties, without viewing this duality as a paradox. This generalization is formalized mathematically by the use of a complex-valued function, known as a "wave function," to describe the state of the object. An analogy to this aspect of quantum mechanics leads to the following interesting interpretation: The use of a complex value associates "wave-like" qualities with membership functions (which are now similar to quantum "wave functions"). These wave-like properties generalize the "particle-like" properties, if you will, of traditional fuzzy membership functions. That is, membership functions may now interact in a manner that is generally reserved for waves. In particular, they may interfere, constructively or destructively, with other membership functions, revolutionizing the manner in which membership functions interact. For example, grades of membership with large values (i.e., large amplitudes) can no longer guarantee a large result when they interact. The result of their interaction is now also dependent on their various phase terms, as is demonstrated in detail by Example 2 below. (Note: an analogy to quantum mechanics is also used in Nguyen, *et al.* [11] in order to offer motivation for the concept of complex-valued "truth values." However, again their approach differs significantly from the one detailed above).

The examples in the rest of this section and throughout the remainder of the paper attempt to provide still more intuition for the concept of membership phase. It is hoped that with the aid of these examples, intuition for this original concept will be developed.

1) A Noteworthy Comment on Membership Phase: Since the amplitude term is equivalent to a traditional real-valued grade of membership, an amplitude term of 0.9 is perceived as a high degree of membership, while an amplitude term of 0.1 is considered a low degree of membership. This interpretation holds regardless of whether the elements of the set are students, cars, real numbers, etc., i.e., the interpretation of the value of the amplitude term is the same for all applications of complex fuzzy sets.

In contrast, this interpretation is generally inapplicable to the phase term, $\omega_S(x)$. Due to the relative nature of phase, the absolute value of the phase term is often of little significance. In fact, the absolute value of $\omega_S(x)$ is usually determined according to some arbitrary reference, which may vary from one application to another. Therefore, neither a value of π nor zero may be considered as a high or low value of membership phase. Instead, the important parameter is dominantly the relative phase between x and other elements in the support of S. Thus, it is the interpretation of *relative phase* that remains the same for all applications of complex fuzzy sets; e.g., a phase difference of π between two elements in a set is always considered a large phase difference. Furthermore, if all elements in S have equal phase (of any value), the phase term may be set arbitrarily to zero, and a real-valued membership function obtained. (Note: this is true only when S is considered independently; if S interacts with other complex fuzzy sets, relative phase between sets is of particular importance, and must be maintained).

C. Some Examples

Some illustration of the concept of membership phase is provided by the following examples. In order to develop intuition for this new concept, these examples have been kept relatively simple. Nonetheless, even simple examples are often sufficient to demonstrate the potential of membership phase and the kind of real-world problems in which it may be of use.

Example 1: Every eleven years the sun undergoes a period of activity called the "solar maximum," followed by a period of quiet called the "solar minimum." During the solar maximum, there are many sunspots, solar flares, and coronal mass ejections, all of which can affect communications and weather on Earth. During the solar minimum, there are few sunspots. One way solar activity may be tracked is by observing sunspots. Sunspots are relatively cool areas that appear as dark blemishes on the face of the sun, and are sites where solar flares are observed to occur [16]. Fig. 1 shows the monthly average of the number of sunspots observed since 1749.

Let S denote the ordinary fuzzy set *high solar activity*. Assume the grade of membership of a particular month in S is derived from the average number of sunspots observed during this month. Clearly, an average sunspot number of 200 is associated with a large grade of membership, while an average sunspot number of 2 is associated with a small grade of membership.

Consider a month with an average sunspot number of 50. Judging by the data in Fig. 1, it seems reasonable to assign this month a grade of membership of 0.25 in S. However, the interpretation of an average sunspot number of 50 can vary considerably if the solar *cycle* is also considered. For example, in 1805 the sunspot number of 50 was the solar maximum, while in 1956 it was barely a quarter of the way "up" the solar cycle. Thus, for applications in which solar activity is a significant parameter, the representation of a sunspot number of 50 by the single grade of membership 0.25 may be insufficient. This may be particularly relevant to applications that require long-term planning, e.g., a mission to space expected to last several years.

There are several methods for incorporating the additional required information into the fuzzy representation of solar activity. One method is to define a second fuzzy set, *close to solar maximum*, so that the solar activity in any particular month is characterized by its grade of membership in two fuzzy sets. Alternatively, it is possible to determine the grade of membership of a particular month in S by considering not only its absolute sunspot number, but also its phase in the solar cycle. Thus, a month that is a solar maximum would receive a large grade of membership in S even if its average sunspot number were only 50.

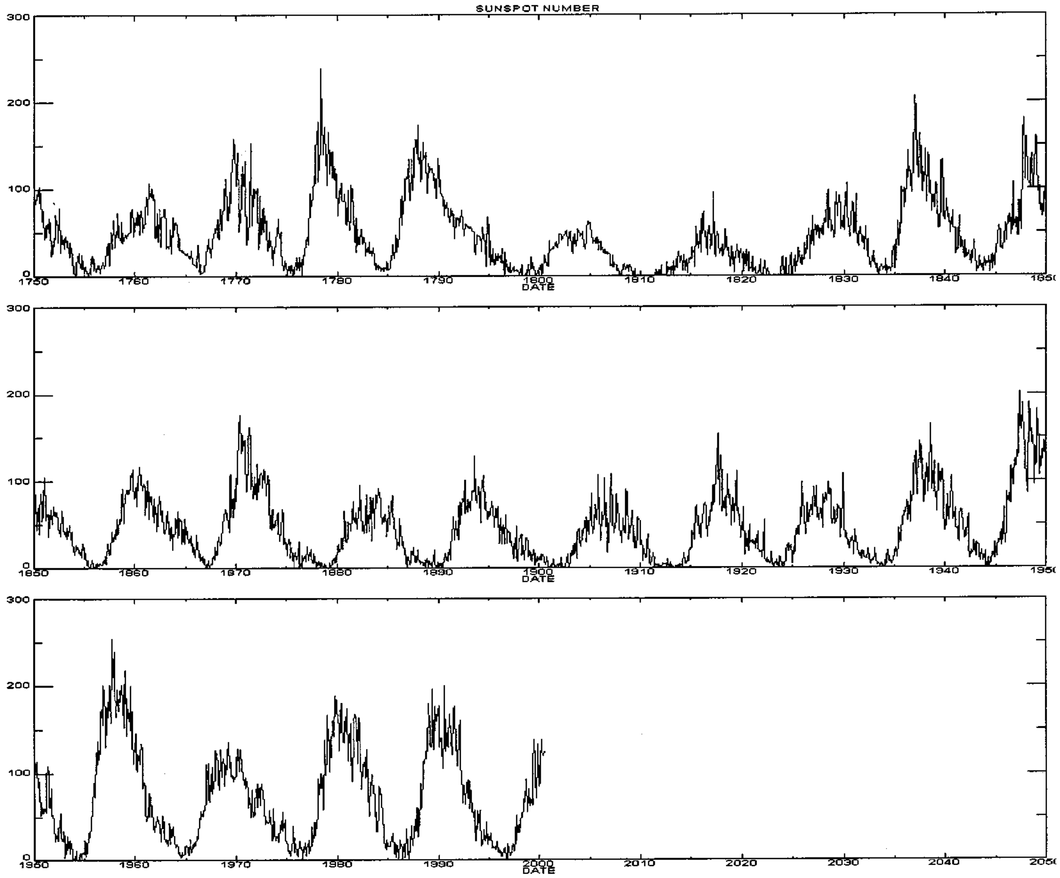

Fig. 1. Monthly average of sunspots observed since 1749 [16].

However, a more elegant and comprehensive manner of representing all the necessary information pertaining to the solar activity in any particular month, is to define S as a complex fuzzy set. When S is a complex fuzzy set, each month is associated with a complex-valued grade of membership consisting of an amplitude term and a phase term.

The role of the amplitude term is simple: it signifies the degree to which a particular month x is a member of S. The amplitude term is derived from the sunspot number in the same manner the real-valued grade of membership was determined when S was considered an ordinary fuzzy set.

The phase term contains information regarding the position of the month under consideration in the solar cycle. Thus, if the month in question, x, is the solar minimum, it is attributed a membership phase of zero. If however, x is the solar maximum, it is given a membership phase of π. Accordingly, any position between the solar minimum and the solar maximum is associated with a membership phase in the range $(0, \pi)$ if solar activity is on the rise, or a value in the range $(\pi, 2\pi)$ if solar activity is decreasing.

The complex fuzzy representation of S is succinct, incorporating all required information in a single grade of membership. It is also more complete than the two traditional fuzzy representations previously suggested.

- The complex fuzzy representation conveys the precise phase of x in the solar cycle, rather than just its proximity to the solar maximum. The fuzzy set *close to solar maximum* does not consider the phase of x relative to the maximum, i.e., whether the maximum is past or whether it is approaching.
- The traditional fuzzy representation that uses a single fuzzy set to represent both the absolute sunspot number and its phase in the solar cycle is often ambiguous. For example, a grade of membership of 0.9 may imply a sunspot number of 150, a sunspot number of 50 that is close to the solar maximum, or several other combinations of sunspot number and "solar phase." In contrast, using the complex representation it is possible to describe both the absolute sunspot number and its phase in the solar cycle without risk of ambiguity.

Note: all information and data on sunspots presented in this example was obtained from http://www.sunspotcycle.com/, [16].

Example 2: In this example, a signal processing application of complex fuzzy sets is presented. The example demonstrates the use of complex fuzzy set theory in an application that identifies a particular signal of interest out of a large number of signals received by a digital receiver.

Assume L different electromagnetic or speech signals, $S_1(t), S_2(t), \ldots S_L(t)$, have been detected and sampled by a digital receiver. Each received signal is sampled N times. Let $S_l(k)$ denote the kth sample $(1 \leq k \leq N)$ of the lth signal $(1 \leq l \leq L)$.

Now, the (discrete) Fourier transforms of the received signals may be obtained, and each signal represented as the sum of its Fourier components

$$S_l(k) = \frac{1}{N} \cdot \sum_{n=1}^{N} C_{l,n} \cdot e^{\frac{j2\pi(n-1)(k-1)}{N}} \tag{2}$$

where $C_{l,n}$ $(1 \le n \le N)$ are the complex Fourier coefficients of S_l.

The aforementioned sum may be rewritten in the form

$$S_l(k) = \frac{1}{N} \cdot \sum_{n=1}^{N} A_{l,n} \cdot e^{\frac{j(2\pi(n-1)(k-1)+\alpha_{l,n})}{N}} \tag{3}$$

where $C_{l,n} = A_{l,n} \cdot e^{j\alpha_{l,n}}$, with $A_{l,n}, \alpha_{l,n}$ real-valued and $A_{l,n} \ge 0$ for all n $(1 \le n \le N)$.

The purpose of the application is to determine which, if any, of the L signals received can be identified as the reference signal, R. The reference signal R has been similarly sampled N times, and its discrete Fourier transform is known. Let the Fourier coefficients of R be $C_{R,n}$, where $1 \le n \le N$. Thus

$$R(k) = \frac{1}{N} \cdot \sum_{n=1}^{N} A_{R,n} \cdot e^{\frac{j(2\pi(n-1)(k-1)+\alpha_{R,n})}{N}} \tag{4}$$

where $C_{R,n} = A_{R,n} \cdot e^{j\alpha_{R,n}}$, with $A_{R,n}, \alpha_{R,n}$ real-valued and $A_{R,n} \ge 0$ for all n.

Calculating a measure of the similarity between two signals is possible by comparing their Fourier transforms. Apply the following method to compare the different signals $S_1, \dots S_L$ to R.

Step 1) Normalize the amplitudes of all Fourier coefficients. Consider $S_l, (1 \le l \le L)$. Denote as $\boldsymbol{A_l}$ the (N-dimensional) vector of amplitudes of S_l's Fourier coefficients: $(A_{l,1}, A_{l,2}, \dots A_{l,N})$, and let $\boldsymbol{A_R}$ denote the vector of amplitudes of R's Fourier coefficients: $(A_{R,1}, A_{R,2}, \dots A_{R,N})$. Let $\boldsymbol{B_l}$ be the normalized vector $1/(\mathrm{norm}(\boldsymbol{A_l})) \cdot \boldsymbol{A_l}$, where $\mathrm{norm}(\boldsymbol{A_l}) = \sqrt{\sum_{n=1}^{N} (A_{l,n})^2}$, and let $\boldsymbol{B_R}$ equal the normalized vector $1/(\mathrm{norm}(\boldsymbol{A_R})) \cdot \boldsymbol{A_R}$. Thus, $\boldsymbol{B_l} = (B_{l,1}, B_{l,2}, \dots B_{l,N})$ is the vector of normalized amplitudes of S_l's Fourier coefficients. Similarly, $\boldsymbol{B_R} = (B_{R,1}, B_{R,2}, \dots B_{R,N})$ is the vector of normalized amplitudes of R's Fourier coefficients.

Step 2) For each Fourier coefficient of S_l, calculate a complex grade of similarity to the relevant coefficient in R, as shown below. The grade of similarity of $C_{l,n}$ to $C_{R,n}$, denoted by $\mu_{S_l,R}(n)$, is defined as

$$\mu_{S_l,R}(n) = r_{S_l,R}(n) \cdot e^{j\omega_{S_l,R}(n)} \tag{5}$$

where

$$r_{S_l,R}(n) = e^{\frac{-(B_{R,n}-B_{l,n})^2}{B_{R,n} \cdot B_{l,n}}} \qquad \omega_{S_l,R}(n) = \alpha_{R,n} - \alpha_{l,n} \tag{6}$$

$\mu_{S_l,R}(n)$ is a complex grade of membership which follows Definition 1, and includes a phase term and an amplitude term. The phase term contains information of the relative phase between $C_{l,n}$ and $C_{R,n}$. The amplitude term $r_{S_l,R}(n)$ is a normalized exponential measure of the distance between the normalized amplitudes of $C_{l,n}$ and $C_{R,n}$, with values in the range [0, 1]. Normalized amplitudes, $B_{l,n}$ and $B_{R,n}$, are used in order to eliminate the effect of outside factors such as path loss, distance of transmission source from digital receiver, etc. In this way, the relative amplitude of $C_{l,n}$ in S_l is compared to the relative amplitude of $C_{R,n}$ in R, so that the same results are obtained for strong and weak signals.

Step 3) Calculate the total complex grade of similarity of S_l to $R, \mu_{S_l,R}$. This is done by summing the grades of similarity of each Fourier coefficient, $\mu_{S_l,R}(n)$, over all n $(1 \le n \le N)$ for which $B_{l,n}$ or $B_{R,n}$ is larger than a threshold, $B_{\mathrm{Threshold}}$. The threshold is used to prevent $\mu_{S_l,R}$ from being affected by insignificant Fourier coefficients, i.e., Fourier coefficients whose amplitudes are small in both S_l and R. The sum is then divided by m, the number of coefficients for which $B_{l,n}$ or $B_{R,n}$ is larger than the threshold, mapping the amplitude of $\mu_{S_l,R}$ onto the range [0, 1]

$$\mu_{S_l,R} = \frac{\sum_M \mu_{S_l,R}(n)}{m} \tag{7}$$

where $M = \{n \,|\, B_{l,n} \text{ or } B_{R,n} > B_{\mathrm{Threshold}}\}$, and m is the number of elements in M.

Hence, $\mu_{S_l,R}$ is the grade of similarity of S_l to R, and it clearly complies with Definition 1. Note that the sum given in (7) is a vector sum that is strongly dependent upon the various phase terms of $\mu_{S_l,R}(n)$. The phase terms determine whether the grades of similarity of each Fourier coefficient, $\mu_{S_l,R}(n)$, sum up (or "interfere") constructively or destructively. It is simple to deduce that if the relative phase of coefficients $C_{l,n}$ and $C_{R,n}$ is identical for all n $(1 \le n \le N)$, then all $\mu_{S_l,R}(n)$ will have equal phase, and therefore, the amplitude of their sum, $\mu_{S_l,R}$, will be maximized. If however, the phase difference between $C_{l,n}$ and $C_{R,n}$ varies for every n, the amplitude of $\mu_{S_l,R}$ may turn out to be very small.

Thus, the amplitude of $\mu_{S_l,R}$, which serves to determine if S_l may be identified as R (see (d) below), will be close to 1 if the following conditions are met:

1) the normalized amplitudes of the Fourier coefficients of S_l and R are similar;
2) the relative phases of the Fourier coefficients of S_l are similar to the relative phases of the Fourier coefficients of R.

Step 4) In order to conclude if S_l may be identified as R, compare: $|\mu_{S_l,R}|$ to a threshold, $\mu_{\mathrm{Threshold}}$. If $|\mu_{S_l,R}|$ exceeds the threshold, identify S_l as R.

Thus, a device for measuring the similarity between two signals is provided. The method may be of use for any signal analysis application, in which the relative phase between the Fourier components of the signals under consideration is important.

III. Set Theoretic Operations on Complex Fuzzy Sets

In this section, set theoretic operations on complex fuzzy sets are discussed. As will be shown below, the extension of fuzzy sets to complex fuzzy sets presents a need, and an opportunity, to extend the definition of the basic set theoretic operations, such as fuzzy complement, union, and intersection.

This section begins with a discussion of the complex fuzzy complement, followed by a consideration of complex fuzzy union and intersection. Next, several novel set theoretic operations are introduced. These original operations provide means for benefiting from the full scope of possibilities presented by complex fuzzy sets. Specifically, the set theoretic operations of *rotation* and *reflection* are defined.

A. Complex Fuzzy Complement

1) A Review of Crisp and Traditional Fuzzy Complement: The following axiomatic-based definition for the complement of a traditional fuzzy set can be found in Klir and Yuan [9], and is supported by many others (e.g., [15], [1], and [13]).

Definition 2: Let S be a fuzzy set on U, and $\mu_S(x)$ the degree to which x is a member of S. Let $c(S)$ denote the *fuzzy complement* of S of type c, defined by the function c: $[0,1] \rightarrow [0,1]$, which assigns a value $c(\mu_S(x))$ to all x in U. $c(\mu_S(x))$ may be interpreted both as the degree to which x is a member of $c(S)$, and as the degree to which x is not a member of S. The fuzzy complement function, c, must satisfy at least the following two axiomatic requirements:

- axiom 1 (boundary conditions): $c(0)=1$, and $c(1)=0$;
- axiom 2 (monotonicity): $\forall x, y \in [0,1]$, if $x \leq y$, then $c(x) \geq c(y)$.

In addition, it is desirable in many practical cases that c also satisfy the following requirements:

- axiom 3 (continuity): c is a continuous function;
- axiom 4 (involutivity): c is *involutive*, namely, $\forall x, y \in [0,1], c(c(x))=x$.

Several different complement functions c, which satisfy the axiomatic requirements of Definition 2, have appeared in the literature. Some examples are as follows.

One) The standard complement:

$$c(\mu_S(x)) = 1 - \mu_S(x) \quad \mu_S(x) \in [0,1]. \tag{8}$$

This form of complement, which satisfies all four axiomatic requirements [9], has been used in the majority of fuzzy applications.

Two) Step-threshold complement:

$$c(\mu_S(x)) = \begin{cases} 1 & \text{for } \mu_S(x) \leq t \\ 0 & \text{for } \mu_S(x) > t \end{cases} \quad \mu_S(x) \in [0,1], \quad t \in [0,1]. \tag{9}$$

This form of complement satisfies only the axiomatic skeleton (Axioms 1 and 2) [9].

Three) Sugeno class complement [12]:

$$c_\lambda(\mu_S(x)) = \frac{1-\mu_S(x)}{1+\lambda \cdot \mu_S(x)}, \quad \mu_S(x) \in [0,1], \quad \lambda \in (-1, \infty) \tag{10}$$

which satisfies all four axiomatic requirements [9].

An illustration of examples is given in Fig. 2(a)–(c). Other examples of fuzzy complements may be found in Ferri and Kandel [7, pp. 303–304].

2) Complex Fuzzy Complement: It would seem only natural to make use of the same axiomatic definition, and the complement functions derived from it, for the complement of a complex fuzzy set. Unfortunately, this approach encounters several difficulties.

- Closure: Following Definition 1, a complex grade of membership $\mu_S(x)$ is restricted to the unit circle in the complex plane, i.e., $|\mu_S(x)|$ is limited to [0, 1]. The complement functions derived from Definition 2 do not comply with this requirement. Consider, for example, the standard fuzzy complement, $1-\mu_S(x)$. When $\mu_S(x)$ is complex, the expression $1-\mu_S(x)$ is not closed, i.e., $1-\mu_S(x)$ is often outside the unit circle in the complex plane. For example, $\mu_S(x) = -1\ (r=1, \omega=\pi) \Rightarrow \mu_{\bar{S}}(x)=2$. It can be easily shown that the Sugeno class and other forms of known fuzzy complements suffer from the same problem when applied to complex fuzzy sets.
- Physical interpretation: The boundary conditions specified in Axiom 1 of Definition 2 do not take into account the issue of membership phase. For example, by Axiom 1 $\mu_{\bar{S}}(x)=1$ for any x with $\mu_S(x)=0$, a perfectly logical result for real-valued fuzzy membership functions. However, for complex-valued membership functions, where membership phase must be taken into consideration, this result does not hold.

  Recall the relation $\mu_S(x) = r_S(x) \cdot e^{j\omega_S(x)}$. Clearly, for any x with $\mu_S(x)=0, r_S(x)$ must also be equal to 0, but $\omega_S(x)$ may be arbitrary. Nevertheless, from the boundary conditions of Axiom 1, it follows that $\mu_{\bar{S}}(x) = 1$; i.e., the membership phase of x in $\bar{S}$ is 0, regardless of its membership phase in S.

  It may be argued that the membership phase of any element x in S with $\mu_S(x)=0$, is of no significance. Such a claim may indeed have some merit, for what is the importance of phase when the amplitude is zero? However, this does not rectify the problem presented above. Namely, that any complement function which follows the axioms of Definition 2, including those illustrated in Fig. 2, arbitrarily assigns a membership phase of 0 in $\bar{S}$ to all elements which have zero amplitude in S.
- Properties of complex numbers: Complex numbers are not linearly ordered. For this reason, relations such as "x is greater than y" are generally undefined for x and y which are complex. This attribute of complex numbers renders the monotonicity requirement, as specified in axiom 2 of Definition 2, inapplicable to complex fuzzy sets. It also means that complement functions such as

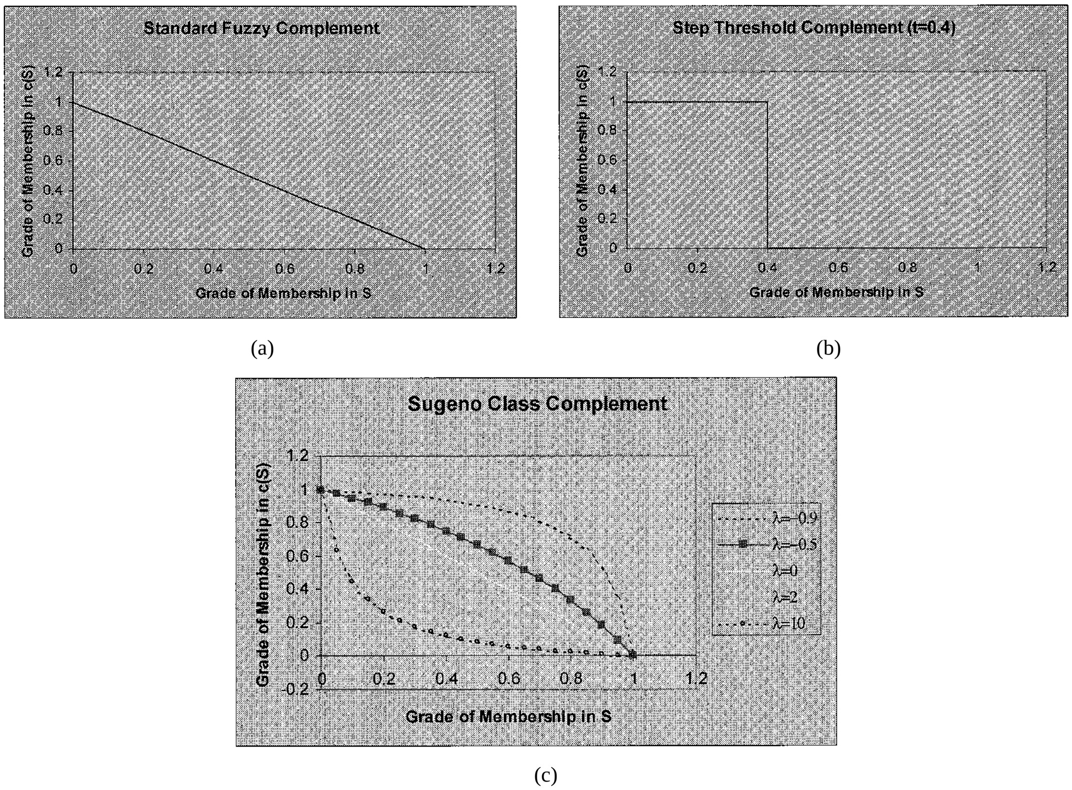

Fig. 2. Examples of traditional fuzzy complements.

the step-threshold complement are irrelevant to complex fuzzy sets.

Perhaps the difficulties detailed above arise from the fact that the axioms of Definition 2 were applied simultaneously to the real and imaginary components of the complex grade of membership. Possibly, if each part, real and imaginary, is considered individually, more favorable results will be achieved. Consider the form of complex fuzzy complement suggested as follows:

$$\begin{aligned}\mu_S(x) &= r_S(x)\cdot e^{j\omega_S(x)}\\ &= r_S(x)\cdot\cos(\omega_S(x)) + j\cdot r_S(x)\cdot\sin(\omega_S(x))\\ \Rightarrow c(\mu_S(x)) &= \mu_{\bar{S}}(x) = r_{\bar{S}}(x)\cdot e^{j\omega_{\bar{S}}(x)}\\ &= c(r_S(x)\cdot\cos(\omega_S(x)))\\ &\quad + j\cdot c(r_S(x)\cdot\sin(\omega_S(x))). \end{aligned}\tag{11}$$

Using the standard fuzzy complement, the following expression is obtained:

$$\begin{aligned}\mu_{\bar{S}}(x) &= [1 - r_S(x)\cdot\cos(\omega_S(x))]\\ &\quad + j\cdot[1 - r_S(x)\cdot\sin(\omega_S(x))]\end{aligned}\tag{12}$$

leading to (13), shown at the bottom of the page.

Unfortunately, this form of complex fuzzy complement does not reduce to its traditional counterpart when a real-valued membership function, i.e., one with $\omega_S(x) = 0$ for all x, is used. In fact, for any x with $\omega_S(x) = 0$, the complement function defined in (12) is equal to $c(r_S(x)) + j$. In addition, the complex fuzzy complement presented above does not maintain closure, e.g., if $\mu_S(x) = 0$ then $\mu_{\bar{S}}(x) = 1 + j$, a value that is not within the unit circle in the complex plane. Note that this result is a direct consequence of Axiom 1 of Definition 2, and is obtained no matter which particular form of complement function c is chosen.

Thus, the axioms of Definition 2, and complement functions derived from these axioms, cannot be applied directly to complex fuzzy sets. This is the case whether the real and imaginary components of the complex grade of membership are considered individually or concurrently. Hence, it is necessary to develop a novel definition for the complex fuzzy complement.

The basis for the definition of the complex fuzzy complement given in this paper is that the axioms of Definition 2, although not directly applicable to $\mu_S(x)$, do apply to the *amplitude* of $\mu_S(x)$, i.e., to $|\mu_S(x)|$. Of course, $|\mu_S(x)|$ equals the amplitude term $r_S(x)$. Note that this is entirely consistent with the interpretation suggested in the previous section (see Section III-B) for the amplitude term of a complex grade of membership. The amplitude term was said to be similar in form and function to a traditional real-valued grade of membership. Therefore, ap-

$$\begin{aligned}r_{\bar{S}}(x) &= \sqrt{2 - 2\cdot r_S(x)\cdot[\cos(w_S(x)) + \sin(w_S(x))] + r_S(x)^2}\\ \omega_{\bar{S}}(x) &= \arctan\left(\frac{1 - r_S(x)\cdot\sin(\omega_S(x))}{1 - r_S(x)\cdot\cos(\omega_S(x))}\right).\end{aligned}\tag{13}$$

plying the traditional fuzzy complement function c to the *amplitude term* of $\mu_S(x)$ is a natural consequence of this interpretation:

$$\mu_{\bar{S}}(x) = c(r_S(x)) \cdot e^{j\omega_{\bar{S}}(x)}$$

or:

$$r_{\bar{S}}(x) = c(r_S(x)). \tag{14}$$

In axiomatic terminology, the function c: $[0, 1] \to [0, 1]$, mapping the amplitude of $\mu_S(x)$ to the amplitude of $\mu_{\bar{S}}(x)$, i.e., $|\mu_{\bar{S}}(x)| = c(|\mu_S(x)|)$, is required to satisfy the axioms of Definition 2.

The primary advantage of this definition of complex fuzzy complement is that it maintains closure: As $r_S(x) \in [0, 1]$, $r_{\bar{S}}(x)$ must also be in the range [0, 1], generating values of $\mu_{\bar{S}}(x)$ which are within the unit circle in the complex plane for any $\omega_{\bar{S}}(x)$.

In addition, considering the interpretation associated with the amplitude term $r_S(x)$, applying the axioms of Definition 2 to $r_S(x)$ and obtaining the relation $r_{\bar{S}}(x) = c(r_S(x))$ is easily justified intuitively. Recall that these axioms, which define the fuzzy complement function c, were designed to furnish it with properties that would make it intuitively acceptable as a fuzzy complement function. These properties, e.g., the boundary conditions: $c(0) = 1$ and $c(1) = 0$, and the requirement of monotonicity, are certainly intuitively relevant to any complement function that is applied to the amplitude of $\mu_S(x)$ (whether it is expressed in the form of $|\mu_S(x)|$ or $r_S(x)$).

Thus, the only issue that remains unresolved is the definition of the *membership phase* of the complex fuzzy complement, $\omega_{\bar{S}}(x)$. This is no simple matter, as illustrated by the following discussion.

From entirely intuitive considerations, it is possible to propose a number of different forms for $\omega_{\bar{S}}(x)$. For example, it would seem reasonable to preserve the original membership phase of x in the complement set $\bar{S}$, i.e.,

$$\omega_{\bar{S}}(x) = \omega_S(x). \tag{15}$$

Alternatively, following the standard fuzzy complement, $\omega_{\bar{S}}(x)$ may be defined by the relation

$$\omega_{\bar{S}}(x) = 2\pi - \omega_S(x) = -\omega_S(x). \tag{16}$$

The rotation of $\omega_S(x)$ by π radians (180°), may also be considered a suitable method for determining $\omega_{\bar{S}}(x)$

$$\omega_{\bar{S}}(x) = \omega_S(x) + \pi. \tag{17}$$

Thus, there are several possible (and arguably, equally viable) methods for determining the membership phase of complex fuzzy complements. This should not be surprising—the existence of several operators for the same set-theoretic operation is common in fuzzy set theory.

The existence of several complex fuzzy complement functions could be accepted if all of these functions provided a unanimous *interpretation* of the operation they represented, and shared a common basic structure (such as the axioms of Definition 2). However, each function presented above suggests a very different method for calculating $\omega_{\bar{S}}(x)$. This is demonstrated in Figs. 3–5. Compare these figures to those of

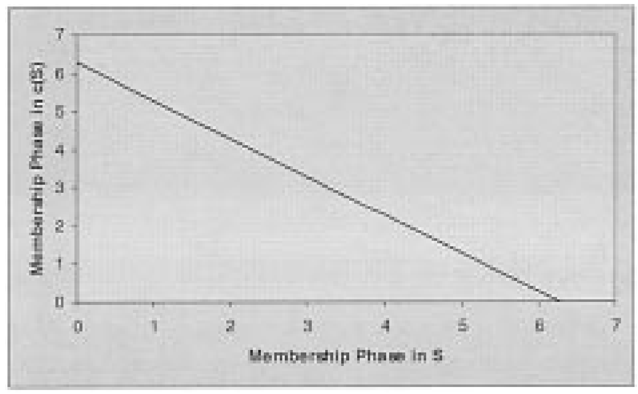

Fig. 3. Illustration of complex complement operator in (15): $\omega_{\bar{S}}(x) = \omega_S(x)$.

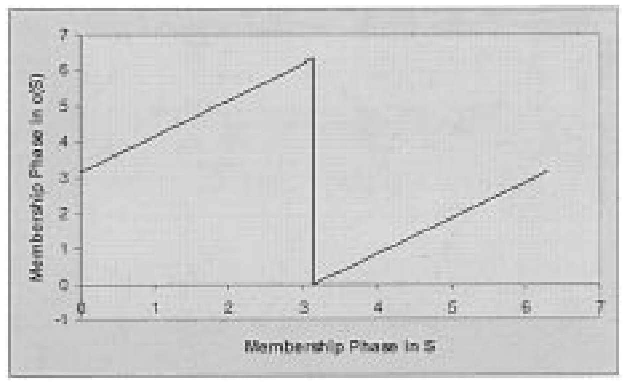

Fig. 4. Illustration of complex complement operator in (16): $\omega_{\bar{S}}(x) = -\omega_S(x)$.

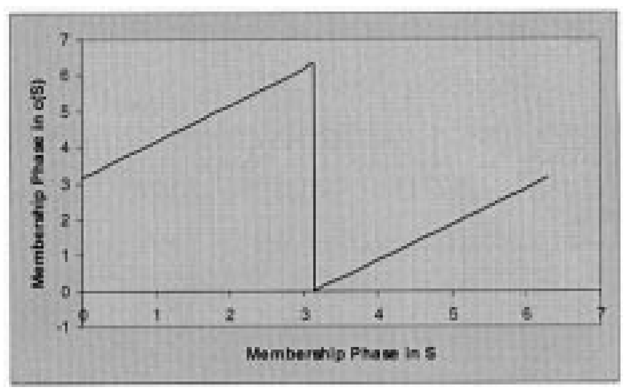

Fig. 5. Illustration of complex complement operator in (17): $\omega_{\bar{S}}(x) = \omega_S(x) + \pi$.

Fig. 2, which illustrate various traditional fuzzy complement operators. Considering the direction of the slope, the boundary conditions, and the general form, it is apparent that the operators of Fig. 2 are similar in nature, as would be expected of operators with a common axiomatic basis. The operators illustrated in Figs. 3–5, however, disagree in every aspect. In particular, there is disparity in the direction of the slope and the boundary conditions.

This significant variation may be attributed to the lack of an axiomatic basis for determining the membership phase of complex fuzzy complements. Yet, in order to develop such an axiomatic basis, an established and accepted approach is required. The axioms of Definition 2 represent an intuitive consensus on the issue of which properties fuzzy complement functions must satisfy. However, no such consensus exists on the issue of complement and membership phase. Each of the operators presented in (15)–(17) suggests a completely different approach to membership phase in complex fuzzy complements, all of which seem intuitively valid.

To decide which complement function and which approach is "correct," the *meaning* of membership phase in the context of complement must first be determined. However, this is precisely the source of the difficulties that have been encountered thus far. For the fact is that the original definition of complement, in crisp set theory and as it appears in the axioms of Definition 2, has no bearing on membership phase. The set-theoretic operation of complement has never been applied to a parameter of the type of membership phase—a parameter whose nature is *periodic*, e.g., a phase of 2π is equal to a phase of 0. The entire discussion above may be regarded as an attempt to apply a familiar concept, complement, to a new type of parameter whose characteristics are very different from those of ordinary grades of membership. It is therefore understandable that relying on "intuition" has led to three very different definitions for the same operator.

How then, is the membership phase of complex fuzzy complements to be defined? Turning to the dictionary for assistance, we obtain the following.

> "A complement is one of two parts that make up a whole; counterpart; the quantity or amount that completes anything; (*logic*) the class of all the things that are not members of a given class; (*math*) the set of all the elements of a universal set not included in a given set" [7].

This definition may point at the expression given in (16) as the appropriate method for calculating $\omega_{\bar{S}}(x)$; i.e., $\omega_{\bar{S}}(x)$ should be defined as $2\pi - \omega_S(x)$, producing two parts, $\omega_S(x)$ and $\omega_{\bar{S}}(x)$, which make up a whole, 2π. However, due to the periodic nature of membership phase, this also means that $\omega_S(x)$ and $\omega_{\bar{S}}(x)$ make up a phase of zero, which can hardly be considered a whole.

Thus, it appears that not even the dictionary definition of complement can provide an outright, clear solution to the dilemma—what is the correct way of applying the operation of complement to membership phase?

It is our conclusion that the set theoretic operation of complex fuzzy complement is intuitively acceptable only if it is limited to the amplitude of complex grades of membership. We have seen complement applied with ease and intuition to the amplitude of $\mu_S(x)$, but despite the lengthy discussion, have found no clear-cut method of applying complement to membership phase. From an intuitive perspective, there seems to be no satisfactory interpretation of complement in the context of membership phase.

We therefore assert that the concept of complement is not applicable to membership phase. To illustrate this point, consider the following example.

Example 3: Recall the complex fuzzy set '*High Solar Activity*' described in Example 1 and denoted S. The set S is characterized by a complex-valued membership function, whose amplitude term specifies the degree to which the average number of sunspots in a particular month, x, is considered high. The phase term represents the location, or phase, of x in the solar cycle.

Now consider the complement of $S, c(S)$, which denotes the set of months with low solar activity. Clearly the degree (or amplitude) of membership of a particular month x in $c(S)$, denoted $r_{\bar{S}}(x)$, must be very different than its degree of membership in $S, r_S(x)$. The relation of $r_{\bar{S}}(x)$ to $r_S(x)$ is given by

$$r_{\bar{S}}(x) = c(r_S(x)) \tag{18}$$

where c is a complement function that satisfies the axioms of Definition 2.

However, the membership phase of x in $c(S)$ cannot be any different than its membership phase in S. For the membership phase is an expression of the phase of the month x in the solar cycle, which has not changed as a result of the application of complement to S. In fact, the application of the complement function c to the set S should have no effect on the membership phase of any element in S.

Thus, the axioms that appear in the definition of complex fuzzy complement (see below) apply solely to the amplitude terms of complex-valued membership functions.

Subsequently in this section, the set theoretic operations of *rotation* and *reflection* are introduced. These operations are applicable solely to membership phase, and may be combined with the complex fuzzy complement to create a compound operation, which affects both phase and amplitude of complex grades of membership.

Definition 3: Let S be a complex fuzzy set on U, the universe of discourse, and let $\mu_S(x)$ be x's complex grade of membership in S. Let $c(S)$ denote the *complex fuzzy complement* of S of type c, defined by the function c: $\{a \mid a \in \boldsymbol{C}, |a| \le 1\} \rightarrow \{b \mid b \in \boldsymbol{C}, |b| \le 1\}$, which assigns a value $c(\mu_S(x))$ to all x in U. The complex fuzzy complement function, c, must satisfy at least the following axiomatic requirements:

- axiom 1 (amplitude boundary conditions):

$$|a| = 0 \Rightarrow |c(a)| = 1, \quad \text{and} \quad |a| = 1 \Rightarrow |c(a)| = 0;$$

- axiom 2 (amplitude monotonicity):

$$\begin{aligned} \forall a, b \in \boldsymbol{C}, \quad & \text{with } |a|, |b| \in [0, 1], \\ & \text{if } |a| \le |b|, \quad \text{then } |c(a)| \ge |c(b)| \end{aligned}$$

In addition, it is desirable in some cases that c should satisfy also the following requirements:

- axiom 3 (continuity): c is a continuous function;
- axiom 4 (involutivity): $\forall a \in \boldsymbol{C}$, with $|a| \in [0, 1], c(c(a)) = a$.

It would be mathematically correct to say that the option of preserving the membership phase of x in the complement set, represented by (15), was chosen as the appropriate method for calculating $\omega_{\bar{S}}(x)$. This, however, would be an incorrect interpretation of the conclusion, for it assumes that the set-theoretic operation of complement has some relevance to membership phase. It is important to stress that in fact, complementation does not apply to membership phase at all, but rather is restricted exclusively to the amplitude term of complex grades of membership.

B. Complex Fuzzy Union

1) A Short Primer on Traditional Fuzzy Union: Much like the traditional fuzzy complement, traditional fuzzy union may also be defined by a set of axioms. These axioms characterize the properties any fuzzy union function is required to satisfy in order to be considered intuitively sound. The axiomatic definition given below follows Klir and Yuan [9].

Definition 4: The fuzzy union of two fuzzy sets A and B on U, denoted $A \cup B$, is specified by a function u:

$$u : [0,1] \times [0,1] \rightarrow [0,1]. \tag{19}$$

The fuzzy union function u must satisfy at least the following axiomatic requirements, for any $a, b, d \in \{x \mid x \in \boldsymbol{C}, |x| \leq 1\}$:

- axiom 1 (boundary conditions): $u(a, 0) = a$;
- axiom 2 (monotonicity): $b \leq d$ implies $u(a, b) \leq u(a, d)$;
- axiom 3 (commutativity): $u(a, b) = u(b, a)$;
- axiom 4 (associativity): $u(a, u(b, d)) = u(u(a, b), d)$.

In some cases, it is may be desirable that u also satisfy the following requirements:

- axiom 5 (continuity): u is a continuous function;
- axiom 6 (superidempotency): $u(a, a) > a$;
- axiom 7 (strict monotonicity): $a \leq c$ and $b \leq d$ implies $u(a, b) \leq u(c, d)$.

Below are some examples of commonly used fuzzy union functions, expressed in the form of membership functions:

$$\text{Maximum (Standard Union): } \mu_{A\cup B} = \max[\mu_A, \mu_B] \tag{20}$$

$$\text{Algebraic Sum: } \mu_{A\cup B} = \mu_A + \mu_B - \mu_A \cdot \mu_B \tag{21}$$

$$\text{Bounded Sum: } \mu_{A\cup B} = \min[1, \mu_A + \mu_B]. \tag{22}$$

The properties fuzzy union functions must satisfy, as specified by Definition 4, are equivalent to properties of functions known in the literature as *t-conorms*. For this reason, fuzzy union functions are also known as t-conorms, and are generally denoted $\oplus$.

2) Complex Fuzzy Union: The most natural method for defining complex fuzzy union is to adopt the axiomatic definition given above and apply it to complex fuzzy sets. However, as shown below, this approach is not particularly useful.

- Closure: Consider the algebraic sum. This union function does not maintain closure for complex fuzzy sets (e.g., $\mu_A(x) = \mu_B(x) = j \Rightarrow \mu_{A\cup B}(x) = 2j + 1$).
- Properties of complex numbers: Complex numbers are not linearly ordered. Therefore the monotonicity requirement, as specified in axiom 2 of Definition 4, is generally undefined for complex-valued membership functions. For the same reason, the max and min operators used in (20) and (22) are not applicable to complex-valued grades of membership.

Thus, the attempt to apply the traditional fuzzy definition of union to its complex counterpart presents a number of difficulties. Recall that these difficulties were also encountered in the derivation of the complex fuzzy complement, which established that the intuitive requirements of traditional fuzzy complement could not be applied directly to complex fuzzy sets. Instead, it was maintained that these axiomatic requirements should be applied to the *amplitude terms* of the complex-valued membership functions that characterize complex fuzzy sets. This approach was based on the interpretation of the amplitude term as a traditional real-valued grade of membership.

The same concept is applied as follows to a complex fuzzy union:

$$\text{Let:} \quad \mu_A(x) = r_A(x) \cdot e^{j\omega_A(x)} \quad \mu_B(x) = r_B(x) \cdot e^{j\omega_B(x)}.$$

The membership function of $A \cup B$ is given by

$$\mu_{A\cup B}(x) = [r_A(x) \oplus r_B(x)] \cdot e^{j\omega_{A\cup B}(x)}. \tag{23}$$

One immediate advantage of the relation above is closure. Since r_A and r_B are real-valued, operators such as max and min can be applied to them. From an intuitive perspective, the amplitude term of a complex-valued grade of membership is equivalent to a traditional, real-valued grade of membership. Hence, all axiomatic (intuitive) requirements imposed upon traditional fuzzy union should also be relevant to union functions that are applied to the amplitude terms of complex-valued grades of membership. Thus, the complex fuzzy union function given in (23) appears to be a valid choice.

However, the true dilemma is the expression for $\omega_{A\cup B}$. The main obstacle in defining $\omega_{A\cup B}$ is that, thus far, intuition for the operation of union has been limited to union of crisp sets or fuzzy sets with real-valued grades of membership. Membership in these sets does not include a phase term, and therefore familiar union functions do not incorporate phase considerations.

In both crisp and fuzzy set theory, an element x is said to be a member of the set $A \cup B$ *to the degree* that it is a member of sets A or B. Of course, the notion of degree of membership varies significantly between crisp and fuzzy sets. However, this does not alter the fact that the traditional concept of union is concerned solely with the *degree* of membership in a set. As expected, the traditional definition of union includes no reference to membership phase. For this reason, traditional union functions are directly applicable only to the amplitude terms of complex grades of membership.

What about the phase terms? Can the same union functions also be applied to the phase terms of an element's grades of membership in A and B? In fact, there appears to be no intuitive justification for applying the axioms of Definition 4 (in particular, the requirement of monotonicity) to membership phase. The properties of membership phase, primarily its periodic nature, make it a unique type of membership measure, to which the traditional union functions are not necessarily applicable. For example, consider the *Max* union function. Suppose this function is used to calculate $\omega_{A\cup B}(x)$. Assume $\omega_A(x) = \pi/2$ and $\omega_B(x) = \pi$. There seems to be no particular justification for setting $\omega_{A\cup B}(x) = \omega_B(x)$ for the sole reason that $\omega_B(x)$ is larger than $\omega_A(x)$. Especially, since a membership phase of $\pi/2$ also implies a membership phase of $5\pi/2$, which is larger than π.

In the derivation of the complex fuzzy complement, the solution to this dilemma was straightforward. The set theoretic operation of complement was simply determined to have no relevance to membership phase. Unfortunately, in the case of complex fuzzy union such an approach is not practicable. Clearly all elements in $A \cup B$ need to be given a full grade of membership, which includes a phase term. If however, the approach of the complex fuzzy complement were adopted, the phase term of $A \cup B$ would remain undefined. Therefore, some method of determining the membership phase of elements in $A \cup B$ must be prescribed. Naturally, this membership phase should be derived in some manner from the relevant membership phases in A and B.

The manner of deriving the phase term of $A \cup B$ is not unique. Different applications of complex fuzzy union may require very different approaches to membership phase, both practical (i.e., which function to use) and intuitive. Possibly, in some applications of complex fuzzy union, a function derived from the axioms of Definition 4 may be the appropriate manner of determining $\omega_{A\cup B}(x)$. In other applications, an operator such as *Minimum*, i.e., $\omega_{A\cup B} = \min(\omega_A, \omega_B)$, may be more pertinent. In short, there is no clear intuitive requirement that complex fuzzy union must satisfy with regard to membership phase. For this reason, the axiomatic based definition of complex fuzzy union provided below imposes very few restrictions on the application of union to membership phase. In effect, this definition states that as far as membership phase is concerned, complex fuzzy union is almost entirely *application-dependent*.

Definition 5: Let A and B be two complex fuzzy sets on U, with complex-valued membership functions $\mu_A(x)$ and $\mu_B(x)$. The complex fuzzy union of A and B, denoted $A \cup B$, is specified by a function

$$u : \{a \mid a \in \boldsymbol{C}, |a| \le 1\} \times \{b \mid b \in \boldsymbol{C}, |b| \le 1\} \to \{d \mid d \in \boldsymbol{C}, |d| \le 1\}. \quad (24)$$

u assigns a complex value, $u(\mu_A(x), \mu_B(x)) = \mu_{A\cup B}(x)$ to all x in U.

The complex fuzzy union function, u, must satisfy at least the following axiomatic requirements, for any $a, b, c, d \in \{x \mid x \in \boldsymbol{C}, |x| \le 1\}$:

- axiom 1 (boundary conditions): $u(a, 0) = a$;
- axiom 2 (monotonicity): $|b| \le |d|$ implies $|u(a,b)| \le |u(a,d)|$;
- axiom 3 (commutativity): $u(a,b) = u(b,a)$;
- axiom 4 (associativity): $u(a, u(b,d)) = u(u(a,b), d)$.

In some cases, it is may be desirable that u also satisfy the following requirements:

- axiom 5 (continuity): u is a continuous function;
- axiom 6 (superidempotency): $|u(a,a)| > |a|$;
- axiom 7 (strict monotonicity): $|a| \le |c|$ and $|b| \le |d| \Rightarrow |u(a,b)| \le |u(c,d)|$.

The following are several possibilities for calculating $\omega_{A\cup B}$, which, if combined with an appropriate function for determining $r_{A\cup B}$, satisfy the axiomatic requirements of the definition above (proof of this is trivial)

One) Sum: $$\omega_{A\cup B} = \omega_A + \omega_B \quad (25)$$

Two) Max: $$\omega_{A\cup B} = \max(\omega_A, \omega_B) \quad (26)$$

Three) Min: $$\omega_{A\cup B} = \min(\omega_A, \omega_B) \quad (27)$$

Four) ``Winner Take All'': $$\omega_{A\cup B} = \begin{cases} \omega_A & r_A > r_B \\ \omega_B & r_B > r_A \end{cases} \quad (28)$$

In addition, the functions given below are also intuitively acceptable possibilities, however, they do not satisfy the axiomatic requirements of commutativity and/or associativity

One) Weighted Average: $$\omega_{A\cup B} = \frac{r_A \cdot \omega_A + r_B \cdot \omega_B}{r_A + r_B} \quad (29)$$

Two) Average: $$\omega_{A\cup B} = \frac{\omega_A + \omega_B}{2} \quad (30)$$

Three) Difference: $$\omega_{A\cup B} = \omega_A - \omega_B. \quad (31)$$

As stated above, each of these options, and others which have been omitted, may be argued as a suitable way for determining $\omega_{A\cup B}$. Eventually, the "appropriate" function must be selected according to the application at hand.

Example 4: Consider two complex fuzzy sets A and B, defined on a universe of discourse U. Assume the sets A and B are given by

$$A = 1 \cdot e^{j\cdot 0}/x + 0.4 \cdot e^{j\cdot\pi}/y + 0.8 \cdot e^{j\cdot\frac{\pi}{2}} \Big/ z \quad (32)$$

$$B = 0.2 \cdot e^{j\cdot\frac{3\pi}{4}} \Big/ x + 0.3 \cdot e^{j\cdot 2}/y + 1 \cdot e^{j\cdot\frac{\pi}{5}} \Big/ z. \quad (33)$$

The representation of complex fuzzy sets A, B used above is common in fuzzy set theory. The expression $A = 1 \cdot e^{j\cdot 0}/x$ for example, should be interpreted as $\mu_A(x) = 1 \cdot e^{j\cdot 0}$.

Using the max union function for calculating $r_{A\cup B}$ and the "winner take all" method in (28) for determining $\omega_{A\cup B}$, the following results are obtained for $A \cup B$:

$$A \cup B = 1 \cdot e^{j\cdot 0}/x + 0.4 \cdot e^{j\cdot\pi}/y + 1 \cdot e^{j\cdot\frac{\pi}{5}} \Big/ z. \quad (34)$$

C. Complex Fuzzy Intersection

The derivation of complex fuzzy intersection closely parallels that of complex fuzzy union.

In general, the intersection of two fuzzy sets A and B on U is denoted $A \cap B$. An axiomatic definition of traditional fuzzy intersection can be found in Klir and Yuan [9]. The properties which fuzzy intersection functions must satisfy are the same as properties of functions known in the literature as *t-norms*. Hence, in traditional fuzzy set theory, set theoretic operators for fuzzy intersection are known as *t-norms*, and are denoted $\star$.

Examples of commonly used fuzzy intersection functions, expressed in the form of membership functions, are

Minimum (Standard Intersection): $$\mu_{A\cap B} = \min[\mu_A, \mu_B] \quad (35)$$

Algebraic Product: $$\mu_{A\cap B} = \mu_A \cdot \mu_B \quad (36)$$

Bounded Difference: $$\mu_{A\cap B} = \max[0, \mu_A + \mu_B - 1]. \quad (37)$$

As in the case of complex fuzzy union, the axiomatic definition of traditional fuzzy intersection cannot be applied directly to complex fuzzy sets. The same problems encountered in the discussion of complex fuzzy union, such as lack of closure and the fact that complex numbers are not linearly ordered, arise here.

Instead, as previously done, the axiomatic (and intuitive) requirements of traditional fuzzy intersection are imposed solely on the amplitude terms of the complex membership functions. No specific axiomatic requirements are given for the method of determining the membership phase of an element in $A \cap B$.

Thus, the membership phase of any element in $A \cap B$ is derived from the membership phase of that element in complex fuzzy sets A and B, using any method that is most suitable to the application at hand. The arguments in favor of this approach may be found in the discussion of complex fuzzy union above.

Definition 6: Let A and B be two complex fuzzy sets on U, with complex-valued membership functions $\mu_A(x)$ and $\mu_B(x)$. The complex fuzzy intersection of A and B, denoted $A \cap B$, is specified by a function

$$i: \{a \mid a \in \boldsymbol{C}, |a| \leq 1\} \times \{b \mid b \in \boldsymbol{C}, |b| \leq 1\} \rightarrow \{d \mid d \in \boldsymbol{C}, |d| \leq 1\}. \quad (38)$$

The complex fuzzy intersection function i, assigns a complex value, $i(\mu_A(x), \mu_B(x)) = \mu_{A\cap B}(x)$, to all x in U, and must satisfy at least the following axiomatic requirements for any $a, b, c, d \in \{x \mid x \in C, |x| \leq 1\}$::

- axiom 1 (boundary conditions): if $|b| = 1, |i(a, b)| = |a|$;
- axiom 2 (monotonicity): $|b| \leq |d|$ implies $|i(a, b)| \leq |i(a, d)|$;
- axiom 3 (commutativity): $i(a, b) = i(b, a)$;
- axiom 4 (associativity): $i(a, u(b, d)) = i(i(a, b), d)$.

In some cases, it is may be desirable that i also satisfy the following requirements:

- axiom 5 (continuity): i is a continuous function;
- axiom 6 (subidempotency): $|i(a, a)| < |a|$;
- axiom 7 (strict monotonicity): $|a| \leq |c|$ and $|b| \leq |d| \Rightarrow |i(a, b)| \leq |i(c, d)|$.

The consequence of this axiomatic definition is that the operation of complex fuzzy intersection may be represented in the following manner:

$$\mu_{A\cap B}(x) = [r_A(x) \star r_B(x)] \cdot e^{j\omega_{A\cap B}(x)} \quad (39)$$

where $\star$ represents any t-norm function, such as the minimum or algebraic product. As specified above, the form of $\omega_{A\cap B}$ is selected according to the application at hand. Possible choices are given in (25)–(31).

D. Properties of Complex Fuzzy Union and Intersection

1) De Morgan's Laws: De Morgan's laws for crisp sets are: $\overline{A \cup B} = \bar{A} \cap \bar{B}$ and $\overline{A \cap B} = \bar{A} \cup \bar{B}$. These laws are generally useful for proving properties of more complicated operations on sets. As noted in Zimmerman [15], if a corresponding pair of t-conorm and t-norm functions is used along with the standard fuzzy complement function, then De Morgan's laws also hold for fuzzy union and intersection. For example, De Morgan's laws are satisfied by fuzzy sets when the max and min functions are used for fuzzy union and intersection. Thus, matching pairs of t-conorms and t-norms satisfy the following generalization of De Morgan's laws [2]:

$$c(\mu_A(x) \oplus \mu_B(x)) = c(\mu_A(x)) \star c(\mu_B(x)) \quad (40)$$
$$c(\mu_A(x) \star \mu_B(x)) = c(\mu_A(x)) \oplus c(\mu_B(x)). \quad (41)$$

Where, $\oplus$ and $\star$ represent a corresponding pair of t-conorm and t-norm functions, respectively, and $c(A)$ denotes the standard fuzzy complement of A.

Complex fuzzy sets satisfy the same generalization of De Morgan's laws under one condition—that the same function is used for determining both $\omega_{A\cup B}$ and $\omega_{A\cap B}$.

Proof: Recall that following Definition 3, $\omega_{\bar{A}}(x) = \omega_A(x)$ for all x. Thus

$$\begin{aligned}\mu_{\overline{A\cup B}}(x) &= c\left([r_A(x) \oplus r_B(x)] \cdot e^{j\omega_{A\cup B}(x)}\right)\\ &= c(r_A(x) \oplus r_B(x)) \cdot e^{j\omega_{\overline{A\cup B}}(x)}\\ &= c(r_A(x) \oplus r_B(x)) \cdot e^{j\omega_{A\cup B}(x)} \quad (42)\\ \mu_{\bar{A}\cap\bar{B}}(x) &= [c(r_A(x)) \star c(r_B(x))] \cdot e^{j\omega_{\bar{A}\cap\bar{B}}(x)}\\ &= [c(r_A(x)) \star c(r_B(x))] \cdot e^{j\omega_{A\cap B}(x)}. \quad (43)\end{aligned}$$

As $r_A(x)$ and $r_B(x)$ are real-valued in the range [0, 1], they satisfy the generalized De Morgan's laws for traditional fuzzy sets. Consequently, $\mu_{\overline{A\cup B}}(x)$ is equal to $\mu_{\bar{A}\cap\bar{B}}(x)$ if and only if $\omega_{A\cup B} = \omega_{A\cap B}$. The second De Morgan's law is similarly proved.

2) Laws of Contradiction and Excluded Middle: The two fundamental (Aristotelian) laws of crisp set theory are as follows.

1) Law of Contradiction: $A \cap \bar{A} = U$, i.e., a set and its complement comprise the entire universe of discourse.
2) Law of Excluded Middle: $A \cap \bar{A} = \emptyset$, i.e., an element can be either in a set or in its complement, but not in both.

Of course, in traditional fuzzy set theory these laws are broken (for an example, see Mendel [10, p. 351]). This is in fact one of the major distinctions between crisp and fuzzy set theory. In this respect, complex fuzzy set theory is in complete agreement with its traditional counterpart. The complement, union and intersection functions applied to the amplitude terms of complex fuzzy sets are the same as those used for traditional fuzzy sets. Consequently, the laws of contradiction and excluded middle do not hold in complex fuzzy set theory either.

E. Set Rotation and Reflection

The set theoretic operations of *rotation* and *reflection* (see below) are novel operations, designed to allow manipulation of the phase component of complex-valued membership functions. The effect of these operations on a complex fuzzy set is restricted to the membership phase of elements of that set.

Definition 7: Let S be a complex fuzzy set on U, and $\mu_S(x) = r_S(x) \cdot e^{j\omega_S(x)} x$'s grade of membership in S. The *Rotation* (anticlockwise) of S by θ radians is defined as

$$\mathrm{Rot}_\theta(\mu_S(x)) = r_S(x) \cdot e^{j(\omega_S(x)+\theta)}. \quad (44)$$

Thus, if x has a membership phase of $\omega_S(x) = a$ in S, then x's membership phase in $\mathrm{Rot}_\theta(S)$ is given by $\omega_{\mathrm{Rot}_\theta(S)}(x) = a + \theta$.

Definition 8: Let S be a complex fuzzy set on U, and $\mu_S(x) = r_S(x) \cdot e^{j\omega_S(x)} x$'s grade of membership to S. The *reflection* of S, is defined as

$$\mathrm{Ref}(\mu_S(x)) = r_S(x) \cdot e^{-j\omega_S(x)}. \quad (45)$$

Thus, if x has a membership phase of $\omega_S(x) = a$ in S, then x's membership phase in $\mathrm{Ref}(S)$ is given by $\omega_{\mathrm{Ref}(S)}(x) = -a$.

Essentially, the set theoretic operation of reflection, as defined above, is a reflection of $\mu_S(x)$ about the real axis in the complex plane. It is possible to define other forms of reflection, such as the reflection of $\mu_S(x)$ about the imaginary axis. However, any such reflection operation may be performed by a combination of reflection, as it is defined above, and rotation by the appropriate θ. For example, reflection about the imaginary axis may be obtained by combining the reflection operation of Definition 8 and a rotation by π radians.

Both operations of rotation and reflection may be combined with the complex fuzzy complement, whose effect is restricted to amplitude terms. In this manner, it is possible to create a compound operation, which involves both phase and amplitude components of a complex fuzzy set.

The operation created by combining rotation and complement is termed the *directional domplex (DC) fuzzy complement*, and is defined below. The DC fuzzy complement is similar to the complex fuzzy complement (refer to Definition 3), with the addition of a "phase term," θ, which gives it its directional quality.

Definition 9: Let S be a complex fuzzy set on U, with membership function μ_S. The *DC fuzzy complement* of S, denoted by $\bar{S}^\theta$, is the complex fuzzy set whose membership function is given by

$$\mu_S(x) = r_S(x) \cdot e^{j\omega_S(x)} \Rightarrow \mu_{\bar{S}^\theta}(x) = c(r_S(x)) \cdot e^{j(\omega_S(x)+\theta)} \tag{46}$$

where c is any complement function, c: $[0, 1] \to [0, 1]$, which satisfies the axioms of Definition 2.

IV. Complex Fuzzy Relations

In this section, complex fuzzy relations are considered. For the sake of simplicity, the discussion is limited to relations between two sets, although the extension of any result presented in this section to relations between any number of sets is straightforward.

A. Review of Traditional Fuzzy Relations

"*Fuzzy relations* represent a *degree* of presence or absence of association, interaction, or interconnectedness" between the elements of two or more crisp sets, Mendel [10, p. 352].

Let U and V be two crisp sets. A fuzzy relation $R(U, V)$ is a fuzzy subset of the product space $U \times V$. The relation $R(U, V)$ is characterized by the membership function $\mu_R(x, y)$, where $x \in U$ and $y \in V$. $\mu_R(x, y)$ assigns to each pair (x, y) a grade of membership in the range [0, 1], which represents the degree to which the relation R holds for elements x and y.

Like any fuzzy set, $R(U, V)$ may be represented as the set of ordered pairs

$$R(U, V) = \{((x, y), \mu_R(x, y)) \,|\, (x, y) \in U \times V\}. \tag{47}$$

An example of a fuzzy relation R is the relation of *Proximity* between two sets U and V, i.e., the relation "x is close to y," where $x \in U$ and $y \in V$. The membership function $\mu_R(x, y)$ specifies the degree of proximity between any two elements of U and V.

B. Complex Fuzzy Relations

Complex fuzzy relations represent both the *degree* of presence or absence of association, interaction, or interconnectedness, and the *phase* of association, interaction, or interconnectedness between the elements of two or more crisp sets.

Let U and V be two crisp sets. A complex fuzzy relation $R(U, V)$ is a complex fuzzy subset of the product space $U \times V$. The relation $R(U, V)$ is characterized by the complex membership function $\mu_R(x, y)$, where $x \in U$ and $y \in V$ and $\mu_R(x, y)$ assigns each pair (x, y) a complex-valued grade of membership to the set $R(U, V)$. As always, $R(U, V)$ may be represented as the set of ordered pairs

$$R(U, V) = \{((x, y), \mu_R(x, y)) \,|\, (x, y) \in U \times V\}. \tag{48}$$

The values $\mu_R(x, y)$ may receive lie within the unit circle in the complex plane, and are of the form: $r(x) \cdot e^{j\omega(x)} (j = \sqrt{-1}), r(x)$ and $\omega(x)$ both real-valued, with $r(x) \in [0, 1]$.

The complex membership function $\mu_R(x, y)$ is to be interpreted in the following manner:

One) $r(x)$ represents a *degree* of presence or absence of association, interaction, or interconnectedness between the elements of U and V;

Two) $\omega(x)$ represents the *phase* of association, interaction, or interconnectedness between the elements of U and V.

Note that this interpretation is consistent with the approach presented throughout this paper, whereby the amplitude term of a complex grade of membership is equivalent to a traditional fuzzy grade of membership.

What is the significance of the phase term of a complex fuzzy relation? Consider the following examples.

Example 5: Let U be the set of financial indicators or indexes of the American economy. Possible elements of this set are unemployment rate, inflation, interest rates, growth rate, GDP, Dow-Jones industrial average, etc. Let V be the set of financial indicators of the Japanese Economy. Let the complex fuzzy relation $R(U, V)$ represent the relation of influence of American financial indexes on Japanese financial indexes: "y is influenced by x," where $x \in U$ and $y \in V$.

The membership function for the relation $R(U, V)$, $\mu_R(x, y)$, is complex valued, with an amplitude term and a phase term. The amplitude term indicates the degree of influence of an American financial index on a Japanese financial index. An amplitude term with a value close to one implies a large degree of influence, while a value close to zero suggests small to no influence. The phase term indicates the "phase" of influence, or time lag that characterizes the influence of an American index on a Japanese index. Thus, the phase term represents the time that elapses before the influence of a certain occurrence in an American financial indicator is evident in a Japanese counterpart.

Consider, for example, μ_R $(\mathrm{Growth\ Rate,\ Export})$, i.e., the grade of membership associated with the statement "American growth rate influences Japanese Export." The value of μ_R $(\mathrm{Growth\ Rate,\ Export})$ may be calculated from available economic statistics using a variety of methods (genetic algorithms, neural nets, etc.), or obtained from an expert. In this example, the latter of the two options is considered.

Suppose an expert were to state that "The influence of American growth rate on Japanese export is large, and the effect of a decline or increase in American growth is evident in Japanese export in three–five months." If $R(U,V)$ were a traditional fuzzy relation, a value of about 0.8 would be selected for the grade of membership μ_R (Growth Rate, Export), and all information regarding the time frame of the interaction between these two economic parameters would be lost. However, $R(U,V)$ is a complex fuzzy relation, and thus μ_R (Growth Rate, Export) can be assigned a complex value which incorporates all of the information provided by the expert.

Assume $R(U,V)$ measures interactions between American and Japanese financial indicators in the limited time frame of 12 months. The following value may, therefore, be attributed to μ_R (Growth Rate, Export)::

$$\mu_R\,(\text{Growth Rate, Export}) = 0.8 \cdot e^{\frac{4}{12}2\pi\cdot j}. \tag{49}$$

Note that the amplitude term of μ_R (Growth Rate, Export) was selected to be 0.8, similar to the grade of membership of a traditional fuzzy set. The phase term was chosen to be $(4/12)\cdot 2\pi$; 4 as an average of "three–five months," normalized by 12 months—the maximum time frame the relation was designed to take into account.

Example 6: As in Example 5, let U be the set of financial indexes of the American economy, and let V be the set of financial indexes of the Japanese Economy. In addition, let W be the set of indicators of Japanese public opinion—such as job approval ratings of the Japanese Prime Minister, confidence in the Japanese economy, etc. Now, consider the following two complex fuzzy relations.

1) The relation $R(U,V)$, discussed in detail in the previous example, representing the relation of influence of American financial indexes on Japanese financial indexes.
2) The relation $S(V,W)$, representing the relation of influence of Japanese financial indexes on Japanese public opinion.

Suppose the following information is available from an expert.

1) The influence of American consumption of imported goods on Japanese unemployment rate is medium, and its effect is evident in six–ten months. Thus, as in Example 5, μ_R (Import, Unemployment Rate) is given a value of $0.6\cdot e^{(8/12)2\pi\cdot j}$.
2) The influence of an increase or decrease in Japanese unemployment on the Prime Minister's job approval rating is very large, and becomes apparent within a month. Thus, μ_S (Unemployment Rate, Prime Minister Approval Rating) $= 0.9\cdot e^{(1/12)2\pi\cdot j}$.

The two relations defined above may be combined in order to produce a third relation, $T(U,W)$, the relation of influence of American financial indexes on Japanese public opinion. The relation $T(U,W)$ is obtained through the composition of relations $R(U,V)$ and $S(V,W)$. It is possible to provide a general and rigorous definition for the composition of complex fuzzy relations. However, for the sake of illustration, it will suffice in this example to consider the composition of the two grades of membership derived above: μ_R (Import, Unemployment Rate) and μ_S(Unemployment Rate, Prime Minister Approval Rating).

The result of this composition is the grade of membership μ_T(American Import, Japanese Prime Minister Approval\ Rating). From intuitive considerations, we suggest that the value of μ_T (Import, Approval Rating) should equal the product of μ_R (Import, Unemployment Rate) and μ_S(Unemployment Rate, Prime Minister Approval Rating).

$$\begin{aligned}&\mu_T\,(\text{Import, Approval Rating})\\ &\quad = 0.6\cdot e^{\frac{8}{12}2\pi\cdot j}\cdot 0.9\cdot e^{\frac{1}{12}2\pi\cdot j} = 0.54\cdot e^{\frac{8+1}{12}2\pi\cdot j}\\ &\quad = 0.54\cdot e^{\frac{9}{12}2\pi\cdot j}.\end{aligned} \tag{50}$$

Thus, the amplitude term of μ_T(Import, Approval Rating) is derived by intersecting the amplitudes of μ_R(Import, Unemployment Rate) and μ_S(Unemployment Rate, Approval Rating), with product used as the intersection function of choice. This is in agreement with accepted methods of composition of traditional fuzzy sets (again, the amplitude term is equated with a traditional real-valued grade of membership). Intuitively, the use of intersection for determining the resultant degree of influence when two influence relations are combined appears a suitable choice.

The phase term of μ_T(Import, Approval Rating) is obtained through the summation of the phase terms of μ_R(Import, Unemployment Rate) and μ_S(Unemployment Rate, Approval Rating). This is easily understood by recalling that for a given relation, the phase term represents the time frame of the interaction expressed by that relation. It is only natural to deduce that the total time frame of both interactions must equal the sum of "time frames" of each interaction. Thus, a time frame of nine months is calculated for the influence of American import on Japanese Prime Minister approval rating.

Hence, the use of multiplication in this example makes good intuitive sense. Note that the product operation emphasizes a unique property of complex fuzzy sets—the complex algebra of its grades of membership. It is a feature of complex fuzzy sets that is difficult to reproduce using traditional fuzzy sets.

The examples presented above demonstrate the potential of complex fuzzy sets in representing relations between parameters. This sort of representation may prove to be extremely useful in applications that utilize fuzzy cognitive maps (FCMs). FCMs describe interactions between various parameters in terms of fuzzy relations. Introducing complex fuzzy relations to FCM applications may provide an effective method for describing the temporal dependence between the parameters of the FCM, an aspect of interaction that is not covered by FCMs today.

V. Summary and Conclusion

A new type of set, the complex fuzzy set, was presented in this paper. The complex fuzzy set represents a novel approach to the concept of membership by allowing it to be described in terms of a complex number. The introduction of a complex form of membership signifies a paradigm shift, whereby membership is transformed into a two-dimensional parameter.

Acquiring intuition for the concept of complex-valued membership is a difficult task. However, in order to benefit from the full potential of complex fuzzy sets, an in depth understanding of their properties is required. Consequently, a major section of this paper was dedicated to a discussion of the intuitive interpretation of complex-valued grades of membership, as well as to examples of possible applications of complex fuzzy set theory. These examples included a complex fuzzy representation of the sunspot number, and a signal processing application.

A comprehensive study of the mathematical properties of complex fuzzy sets was presented. The study began by considering the basic set theoretic operations of complement, union, and intersection, and their application to complex fuzzy sets. Following this, novel set theoretic operations were introduced, namely set rotation and set reflection.

Complex fuzzy relations were also considered, and examples of applications of complex fuzzy relations provided. The use of a complex fuzzy relation in these applications was advantageous because it provided a method for describing the time frame of the relation. In addition, the singular algebra of complex numbers was utilized—by multiplying complex-valued grades of membership in order to calculate the composition of two complex fuzzy relations. Thus, another unique property of complex fuzzy sets was demonstrated. Furthermore, it was suggested that the introduction of complex fuzzy relations to applications of FCMs could prove an effective method for describing the temporal dependence between parameters of an FCM. This aspect of interaction is not covered by FCMs today.

This paper should be considered an introduction to complex fuzzy sets. Indeed, much research of this novel concept is still needed to fully comprehend its properties and potential. While signal processing applications and FCMs appear to be likely areas for the successful application of complex fuzzy sets, the concepts presented in this paper are entirely general and are not limited to a specific application. Thus in all, the complex fuzzy set seems to be a promising new concept, paving the way to numerous possibilities for future research.

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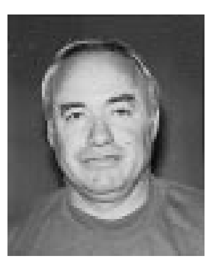

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