Complex Fuzzy Logic

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***Abstract*—A novel framework for logical reasoning, termed complex fuzzy logic, is presented in this paper. Complex fuzzy logic is a generalization of traditional fuzzy logic, based on complex fuzzy sets. In complex fuzzy logic, inference rules are constructed and "fired" in a manner that closely parallels traditional fuzzy logic. The novelty of complex fuzzy logic is that the sets used in the reasoning process are complex fuzzy sets, characterized by complex-valued membership functions. The range of these membership functions is extended from the traditional fuzzy range of [0,1] to the unit circle in the complex plane, thus providing a method for describing membership in a set in terms of a complex number. Several mathematical properties of complex fuzzy sets, which serve as a basis for the derivation of complex fuzzy logic, are reviewed in this paper. These properties include basic set theoretic operations on complex fuzzy sets—namely complex fuzzy union and intersection, complex fuzzy relations and their composition, and a novel form of set aggregation—vector aggregation. Complex fuzzy logic is designed to maintain the advantages of traditional fuzzy logic, while benefiting from the properties of complex numbers and complex fuzzy sets. The introduction of complex-valued grades of membership to the realm of fuzzy logic generates a framework with unique mathematical properties, and considerable potential for further research and application.**

***Index Terms*—Complex fuzzy logic, complex fuzzy relations, complex fuzzy sets, complex-valued grades of membership.**

I. Introduction

THE framework of fuzzy logic is unique in its ability to represent subjective or linguistic knowledge in terms of a mathematical model. For this reason, fuzzy logic provides a natural method for constructing systems that emulate human decision making processes. Literature on the subject of fuzzy logic systems (FLSs) is extensive and applications, particularly in the field of fuzzy control and fuzzy expert systems, are prevalent. Mendel [12] and Klir and Yuan [7] provide good introductory texts on FLSs, while some examples of applications of FLSs may be found in [14], [10], and [6].

In essence, fuzzy set theory and fuzzy logic are a generalization of crisp set theory and traditional, dual-valued logic. In fuzzy set theory the range of the membership function characterizing a set is extended from {0,1} to [0,1]—a development that is mathematically comparable to the extension of the set of integers, $\boldsymbol{I}$, to the set of real numbers, $\boldsymbol{R}$. Proceeding along this path of mathematical generalizations, the next natural step is the extension of $\boldsymbol{R}$ to $\boldsymbol{C}$, the set of complex numbers. Applied to fuzzy set theory, this extension leads to the definition of the complex fuzzy set, a fuzzy set characterized by a complex-valued membership function. This novel set is the basis for complex fuzzy logic. Complex fuzzy sets were introduced in [13], where their mathematical properties are developed.

The definition of the complex fuzzy set allows membership in a set to be specified by a complex number, effectively transforming "membership in a set" into a two-dimensional concept. Membership of any element x, in a complex fuzzy set S, is given by a complex-valued grade of membership of the general form $r_S(x) \cdot e^{j\omega_S(x)}$ ($j = \sqrt{-1}$, $r_S(x) \in [0, 1]$), which comprises an amplitude term, $r_S(x)$, and a phase term, $\omega_S(x)$. The amplitude term retains the traditional notion of "fuzziness," i.e., the representation of membership in a set as a value in the range [0,1]. The phase term signifies the assertion of complex fuzzy set theory that, at least in some instances, a second dimension of membership is required. The properties of this "membership phase" are discussed at length throughout this paper.

Complex fuzzy logic is a unique framework, designed to maintain the advantages of traditional fuzzy logic, while benefiting from the properties of complex fuzzy sets. It is shown in the following sections that complex fuzzy logic is not merely a linear extension of conventional fuzzy logic. Rather, complex fuzzy sets allow a natural extension of fuzzy logic to problems that are either very difficult or impossible to address with one-dimensional grades of membership.

It may be interesting to note that previous publications discussing aspects of combining complex numbers and fuzzy sets have concentrated on the *fuzzy complex number*, which differs markedly from the complex fuzzy set (as is pointed out in [13]). The fuzzy complex number, introduced by Buckley [1]–[4], is created by incorporating complex numbers into the support of the fuzzy set. The fuzzy set representing the fuzzy complex number remains an ordinary fuzzy set, with real-valued grades of membership in the range [0,1]. This is entirely different from the complex fuzzy set, which is a new type of fuzzy set—one with *complex-valued grades of membership*. The support of the complex fuzzy set is unrestricted, and may include any kind of element, such as real numbers, students, pets, and, of course, complex numbers.

In this paper, Section II serves to review the fundamental concepts of complex fuzzy sets, following the definitions provided in [13]. The formal definition of the complex fuzzy set is presented in this section, supported by an intuitive interpretation of complex-valued membership functions and an example illustrating the potential of complex fuzzy sets. Section II also includes a review of the basic set-theoretic operations on complex

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fuzzy sets, namely complex fuzzy union and intersection. The section is completed by the introduction of vector aggregation, a form of set aggregation, which is particularly useful in applications of complex fuzzy sets. Complex fuzzy relations and their composition are considered in Section III. The development of complex fuzzy relations and compositions serves as a basis for the derivation of complex fuzzy logic—presented and discussed at length in Section IV. A summary and suggestions for future research are given in Section V.

II. Complex Fuzzy Sets—A Review of Basic Concepts and Operations

Complex fuzzy sets, which form the basis for complex fuzzy logic, are reviewed in this section following [13]. First, a formal definition of the complex fuzzy set is provided. A discussion of a possible interpretation for this concept is presented and supported by an example, which illustrates the potential of complex fuzzy sets. In addition, several set theoretic operations on complex fuzzy sets, which are required for the derivation of complex fuzzy logic, are discussed. These set theoretic operations include the basic operations of complex fuzzy union and intersection, developed in [13], and a novel form of set aggregation, termed *vector aggregation*.

A. Definition of a Complex Fuzzy Set [13]

Definition 1: A *Complex Fuzzy Set* S, defined on a universe of discourse U, is characterized by a membership function $\mu_S(x)$ that assigns any element $x \in U$ a complex-valued grade of membership in S. By definition, the values $\mu_S(x)$ may receive all lie within the unit circle in the complex plane, and are thus of the form $r_S(x) \cdot e^{j\omega_S(x)}$, where $j = \sqrt{-1}$, $r_S(x)$ and $\omega_S(x)$ are both real-valued, and $r_S(x) \in [0,1]$.

The complex fuzzy set S may be represented as the set of ordered pairs

$$S = \{(x, \mu_S(x)) \mid x \in U\}. \tag{1}$$

Throughout this paper, the term complex fuzzy set refers to a fuzzy set with complex-valued membership function, whereas the term fuzzy set refers to a traditional fuzzy set with real-valued membership function.

B. Interpretation of the Complex Fuzzy Set [13]

Considering the form $\mu_S(x) = r_S(x) \cdot e^{j\omega_S(x)}$, it is apparent that complex grades of membership are comprised of an amplitude term $r_S(x)$ and a phase term $\omega_S(x)$. Attempting to represent an ordinary fuzzy set in terms of a complex fuzzy set (this should be possible, as complex fuzzy sets are generalizations of ordinary fuzzy sets), provides initial insight into the role of the amplitude and phase terms. Let $\lambda_T(x)$ be the real-valued membership function characterizing the ordinary fuzzy set T. Representing the fuzzy set T in terms of a complex fuzzy set S can be achieved by setting the amplitude term $r_S(x)$ equal to $\lambda_T(x)$, and the phase term equal to zero for all x.

This representation indicates that the amplitude term is commensurate to a traditional real-valued grade of membership, a conclusion that is supported by the fact that the range of $r_S(x)$ is [0,1], as for an ordinary real-valued grade of membership. Such an interpretation of the amplitude term suggests that, much like an ordinary grade of membership, the amplitude term may be regarded as representing the *degree* to which x is a member of the complex fuzzy set S. In contrast, the phase term is a completely novel parameter of membership, and essentially distinguishes between ordinary and complex fuzzy sets. In fact, it is the phase term, which provides the framework of complex fuzzy logic with its unique properties and sets it apart from traditional fuzzy logic. The significance and potential of the phase term, or *membership phase*, is best demonstrated by examples, such as the one provided below. In order to allow the reader to develop intuition for the concept of membership phase, the example has been kept relatively simple. Nonetheless, even a simple example is sufficient to demonstrate the potential of membership phase and the kind of real-world problems in which it may be of use.

Example 1: Consider an investor intending to invest in a company by buying stock traded on the New York Stock Exchange. Of course, the investor is interested in timing his purchase optimally in order to maximize his profit, and therefore aspires to buy the stock when its price is *low*. The investor consults with an expert, who supplies the following data.

1) The current price of a company's stock in relation to the overall assessment of the company's performance.
2) The current situation of the stock market. The stock market is generally assumed to behave in a periodic manner (for example, a four-year period has been observed in the S&P index since about 1940). Thus, information regarding the current phase of the market in this cycle may be important to the investor.

The investor, being acquainted with fuzzy set theory, attempts to characterize the current situation of the stock by assigning it a grade of membership in the fuzzy set "Low Price." However, in order to accurately and succinctly represent all of the information provided by the expert, the set "Low Price" should be defined as a *complex* fuzzy set. Using a complex fuzzy set, the amplitude term may be set to represent the current stock price with respect to company performance. The smaller the ratio: "stock price to company performance," the higher the amplitude term. The phase term can be used to signify the current phase of the stock market. For example, a phase of 0 would indicate an assessment that the market index is at a low point, while a phase of π would suggest the market had peaked. Hence, a complex fuzzy set allows the full scope of the information provided by the expert to be represented by a single (complex-valued) grade of membership.

It may be worthwhile to consider that in most engineering applications which utilize complex numbers, it is possible to replace the complex representation by ordered pairs (a vector) of real numbers. Despite this, complex numbers are used extensively in such applications. In general, reasons for selecting a complex representation include ease of representation and calculations, physical accuracy of the complex representation (e.g., one of the parameters does indeed represent phase) and utility of complex algebra in the specific application. In the previous example, the investor may also choose to represent his information using two ordinary fuzzy sets. The arguments in favor of employing a single complex fuzzy set are similar to those cited above: elegance of representation and physical accuracy of the description. More on reasons for selecting complex fuzzy representations can be found in the discussion of complex fuzzy logic in Section IV-C and [13, Ex. 1].

C. Complex Fuzzy Union

The definition of complex fuzzy union provided in this section follows [13]. Traditional fuzzy set theoretic operations are usually defined by a commonly accepted set of axioms (see, for example, [7]). Similarly, complex fuzzy union is defined by a set of axioms. These axioms represent properties complex fuzzy union functions must satisfy in order to be intuitively acceptable.

Unfortunately, adopting the axiomatic definition of traditional fuzzy union and applying it directly to complex fuzzy sets results in union functions that are:

a) Not Closed: Consider for example the algebraic sum—a traditional fuzzy union function. This union function does not maintain closure for complex fuzzy sets (e.g., $\mu_A(x) = \mu_B(x) = j \Rightarrow \mu_{A\cup B}(x) = 2j + 1$—a value outside the unit circle in the complex plane, and therefore in violation of Definition 1).

b) Inapplicable to Complex Fuzzy Sets: Complex numbers are not linearly ordered. Therefore, the axiomatic monotonicity requirement [7] imposed on traditional fuzzy union is generally inapplicable to complex-valued membership functions. For the same reason, max and min operators commonly used in traditional fuzzy union functions, are not applicable to complex-valued grades of membership.

Consequently, a definition of complex fuzzy union derived directly from the axioms of traditional fuzzy union would clearly be difficult to justify.

The axioms that are incorporated into the definition of complex fuzzy union follow the intuitive interpretation of the complex fuzzy set presented above. Recall that according to the interpretation given above, the amplitude term of a complex-valued grade of membership is essentially equivalent to a traditional real-valued grade of membership. Thus, any axiomatic requirement imposed upon traditional fuzzy union should also be imposed upon the "amplitude term" of complex fuzzy union, i.e., if the union of two complex fuzzy sets A and B is considered

$$\text{Let,}\quad \mu_A(x) = r_A(x)\cdot e^{j\omega_A(x)}\quad \mu_B(x) = r_B(x)\cdot e^{j\omega_B(x)}. \tag{2}$$

Then, the membership function of $A\cup B$ is given by

$$\mu_{A\cup B}(x) = [r_A(x) \oplus r_B(x)]\cdot e^{j\omega_{A\cup B}(x)} \tag{3}$$

where $\oplus$ represents a t-*conorm* function (t-*conorm* functions satisfy all axiomatic requirements of traditional fuzzy union). Note that the expression above maintains closure, which is an important property.

It remains to determine an expression for $\omega_{A\cup B}$. However, the intuitive requirements that complex fuzzy union must satisfy with regard to membership phase are unclear. Until now, the operation of union has been confined to crisp sets or traditional fuzzy sets. Accordingly, any intuition we may have developed for union operators is limited to sets characterized by real-valued membership functions. The merit of extending this intuition beyond axioms that relate solely to the amplitude term (i.e., the part of complex-valued membership functions that is similar to traditional real-valued grades of membership) is questionable. The properties of membership phase, primarily its periodic nature, make it a unique type of membership parameter, to which traditional union functions are not necessarily applicable.

Following [13], the definition of complex fuzzy union provided below does not specify a unique manner of deriving the phase term of $A\cup B$. In [13], it is suggested that different applications of complex fuzzy union may require significantly different approaches to membership phase, both in the practical sense (i.e., which function to use) and in the intuitive sense. Therefore, the axioms of Definition 2 impose very few restrictions on the method of calculating $\omega_{A\cup B}$, resulting in a definition of complex fuzzy union that is application-dependent with regard to membership phase. The large number of methods available for calculating $\omega_{A\cup B}$ may make planning a system utilizing complex fuzzy sets seem a rather daunting task. However, as is described in [13, Ex. 6], the choice of method for each application may in fact be quite straightforward. Certainly, traditional fuzzy sets have always been quite useful despite the large number of existing t-*conorms*.

Definition 2: Let A and B be two complex fuzzy sets on U, with complex-valued membership functions $\mu_A(x)$ and $\mu_B(x)$. The complex fuzzy union of A and B, denoted $A\cup B$, is specified by a function

$$u\colon \{a \mid a\in \boldsymbol{C},\, |a|\le 1\}\times\{b\mid b\in\boldsymbol{C},\, |b|\le 1\} \rightarrow \{d\mid d\in\boldsymbol{C},\, |d|\le 1\}. \tag{4}$$

u assigns a complex value, $u(\mu_A(x), \mu_B(x)) = \mu_{A\cup B}(x)$ to all x in U.

The complex fuzzy union function u must satisfy at least the following axiomatic requirements, for any a, b, c, $d \in \{x \mid x\in C,\, |x|\le 1\}$

- Axiom 1 (boundary conditions); $u(a, 0) = a$.
- Axiom 2 (monotonicity): $|b| \le |d|$ implies $|u(a, b)| \le |u(a, d)|$.
- Axiom 3 (commutativity): $u(a, b) = u(b, a)$.
- Axiom 4 (associativity): $u(a, u(b, d)) = u(u(a, b), d)$.

In some cases, it is may be desirable that u also satisfy the following requirements.

- Axion 5 (continuity): u is a continuous function.
- Axiom 6 (superidempotency): $|u(a, a)| > |a|$.
- Axiom 7 (strict monotonicity): $|a| \le |c|$ and $|b| \le |d| \Rightarrow |u(a, b)| \le |u(c, d)|$.

The following are several possibilities for calculating $\omega_{A\cup B}$, which, if combined with an appropriate function for determining $r_{A\cup B}$, satisfy the axiomatic requirements of the definition above (proof of this is trivial):

$$\text{a) Sum:}\qquad \omega_{A\cup B} = \omega_A + \omega_B \tag{5}$$

$$\text{b) Max:}\qquad \omega_{A\cup B} = \max(\omega_A, \omega_B) \tag{6}$$

$$\text{c) Min:}\qquad \omega_{A\cup B} = \min(\omega_A, \omega_B) \tag{7}$$

$$\text{d) ``Winner Take All'':}\quad \omega_{A\cup B} = \begin{cases}\omega_A & r_A > r_B\\ \omega_B & r_B > r_A.\end{cases} \tag{8}$$

D. Complex Fuzzy Intersection

Complex fuzzy intersection is defined in a manner analogous to the definition of complex fuzzy union [13]. Again, the traditional fuzzy axioms are applied directly to the amplitude terms, while the derivation of the phase term is considered application-dependent. The reasons for this approach are detailed beforehand.

Definition 3: Let A and B be two complex fuzzy sets on U, with complex-valued membership functions $\mu_A(x)$ and $\mu_B(x)$. The complex fuzzy intersection of A and B, denoted $A \cap B$, is specified by a function

$$i\colon \{a \mid a \in \boldsymbol{C},\, |a| \le 1\} \times \{b \mid b \in \boldsymbol{C},\, |b| \le 1\} \to \{d \mid d \in \boldsymbol{C},\, |d| \le 1\}. \quad (9)$$

The complex fuzzy intersection function i, assigns a complex value, $i(\mu_A(x),\, \mu_B(x)) = \mu_{A\cap B}(x)$, to all x in U, and must satisfy at least the following axiomatic requirements for any a, b, c, $d \in \{x \mid x \in \boldsymbol{C},\, |x| \le 1\}$ [13].

- Axiom 1 (boundary conditions): if $|b| = 1$, $|i(a, b)| = |a|$.
- Axiom 2 (monotonicity): $|b| \le |d|$ implies $|i(a, b)| \le |i(a, d)|$.
- Axiom 3 (commutativity): $i(a, b) = i(b, a)$.
- Axiom 4 (associativity): $i(a, i(b, d)) = i(i(a, b), d)$.

In some cases, it is may be desirable that i also satisfy the following requirements.

- Axiom 5 (continuity): i is a continuous function.
- Axiom 6 (subidempotency): $|i(a, a)| < |a|$.
- Axiom 7 (strict monotonicity): $|a| \le |c|$ and $|b| \le |d| \Rightarrow |i(a, b)| \le |i(c, d)|$.

Thus, the operation of complex fuzzy intersection may be represented as

$$\mu_{A\cap B}(x) = [r_A(x) \star r_B(x)] \cdot e^{j\omega_{A\cap B}(x)} \quad (10)$$

where $\star$ signifies any t-norm function (functions which satisfy all axiomatic requirements of traditional fuzzy intersection), such as the minimum or algebraic product. The precise form of $\omega_{A\cap B}$ is dependent upon the application under consideration. Possible methods of calculating $\omega_{A\cap B}$ are given in (5)–(8). Note that in theory, $\omega_{A\cup B}$ and $\omega_{A\cap B}$ may be simultaneously defined by the same function (e.g., min). However, in practice, this is unlikely to be a useful method for calculating these terms.

E. Complex Fuzzy Aggregation

Aggregation operations on fuzzy sets provide a means for combining several sets in order to produce a single fuzzy set. An example of fuzzy aggregation is an operation that combines the scores awarded by referees in a gymnastics competition. Suppose three referees award the scores: *very high*, *high*, and *average*. Each linguistic term is represented by an appropriate fuzzy set, defined on the interval [0,10]. An aggregation operation on these three sets produces a single fuzzy set, which summarizes the overall assessment of a competitor's performance.

The aggregation operation on complex fuzzy sets defined in this section, although not consistent with the axiomatic approach used for the definition of complex fuzzy union and intersection, provides a particularly useful method for manipulating complex fuzzy sets. The merit of defining this form of aggregation operation, termed *vector aggregation*, will become evident with the development of complex fuzzy logic.

Definition 4: Let $A_1, A_2, \ldots, A_n$ be complex fuzzy sets defined on the universe of discourse U. *vector aggregation* on A_1, $A_2, \ldots, A_n$ is defined by a function v

$$v\colon \{a \mid a \in \boldsymbol{C},\, |a| \le 1\}^n \to \{b \mid b \in \boldsymbol{C},\, |b| \le 1\}. \quad (11)$$

The function v produces an aggregate fuzzy set A by operating on the membership grades of $A_1, A_2, \ldots, A_n$ for each $x \in U$. For all $x \in U$, v is given by

$$\mu_A(x) = v(\mu_{A_1}(x), \mu_{A_2}(x), \ldots, \mu_{A_n}(x)) = \sum_{i=1}^{n} w_i \mu_{A_i}(x) \quad (12)$$

with $w_i \in \{a \mid a \in \boldsymbol{C},\, |a| \le 1\}$ for all i, and $\sum_{i=1}^{n} |w_i| = 1$.

(Note: The complex weights are incorporated into Definition 4 for the purpose of maintaining a definition that is as general as possible. A method for choosing the complex weights remains to be developed, and is beyond the scope of this work. As a rule, we select all weights to be equal and real-valued, i.e., $w_i = 1/n$).

For any x in U, vector aggregation produces a weighted sum of the grades of membership of x in sets $A_1, A_2, \ldots, A_n$. If, for example, $w_i = 1/n$ for all i, the result of vector aggregation is the arithmetic average of all $\mu_{A_i}(x)$

$$\mu_A(x) = \frac{1}{n} \cdot \sum_{i=1}^{n} \mu_{A_i}(x). \quad (13)$$

The significance of vector aggregation is that all $\mu_{A_i}(x)$ are complex valued. Thus, the sums in (12) and (13) are in fact vector sums. If, for example, all the phases of the various grades of membership, $\mu_{A_i}(x)$, are equal, the amplitude of the sum will be maximized. If, however, the $\mu_{A_i}(x)$ are not "aligned," or coherent, "destructive interference" may occur. In this instance, the amplitude of the sum may turn out to be much smaller than the individual amplitudes of its arguments.

Hence, vector aggregation allows the aggregation of complex fuzzy sets in a manner that incorporates phase considerations. As mentioned before, vector aggregation is a major part of complex fuzzy logic.

III. Complex Fuzzy Relations

In this section, complex fuzzy relations are reviewed following [13], and compositions of these relations are introduced. The development of complex fuzzy relations and compositions is consequential as it affords the basis for the derivation of complex fuzzy logic. For the sake of simplicity, the discussion is limited to relations between two sets, although the extension of any result presented in this section to relations between any number of sets is straightforward. The section begins with a brief reminder of traditional fuzzy relations and their composition.

A. Traditional Fuzzy Relations

1) Defining Fuzzy Relations: "*Fuzzy Relations* represent a *degree* of presence or absence of association, interaction, or interconnectedness" between the elements of two or more crisp sets, [12, p. 352].

Let U and V be two crisp sets. A fuzzy relation $R(U, V)$ is a fuzzy subset of the product space $U \times V$, and is characterized by the membership function $\mu_R(x, y)$, where $x \in U$ and $y \in V$. $\mu_R(x, y)$ assigns to each pair (x, y) a grade of membership in the range [0,1], which represents the degree to which the relation R holds for elements x and y.

Like any fuzzy set, $R(U, V)$ may be represented as the set of ordered pairs

$$R(U, V) = \{((x, y), \mu_R(x, y)) \,|\, (x, y) \in U \times V\}. \quad (14)$$

2) Compositions of Traditional Fuzzy Relations on the Same Product Spaces: As traditional fuzzy relations are in fact fuzzy sets in product space, set theoretic operations such as union, intersection and complement may be applied to them. Thus, compositions of fuzzy relations on the same product space may be performed using standard functions for fuzzy union and intersection.

3) Compositions of Traditional Fuzzy Relations on Different Product Spaces: The composition of fuzzy relations on different product spaces is possible if the relations share a common set, for example: $R(U, V)$ and $S(V, W)$. The composition of these two relations is denoted

$$C(U, W) = R(U, V) \circ S(V, W). \quad (15)$$

The relation $C(U, W)$ is a subset of the product space $U \times W$, and has membership function $\mu_C(x, z)$, where $x \in U$ and $z \in W$. The following definition of the membership function for the composition of two fuzzy relations on different product spaces can be found in [12], and is supported by many others (e.g., [7] and [16]).

Definition 5: The membership function for the composition of fuzzy relations $R(U, V)$ and $S(V, W)$ is given by the *sup-star composition* of R and S

$$\mu_{R \circ S}(x, z) = \sup_{y \in V} [\mu_R(x, y) \star \mu_S(y, z)] \quad (16)$$

where $\star$ signifies any t-norm (fuzzy intersection) function.

An important type of composition is the composition of R and S when the fuzzy relation R is just a fuzzy set. In this case $\mu_R(x, y)$ becomes simply $\mu_R(x)$. An example of this special case is "x is large and x is greater than z."

If R is a fuzzy set rather than a fuzzy relation, then $V = U$ and $\sup_{y \in V} [\mu_R(x, y) \star \mu_S(y, z)]$ becomes $\sup_{x \in U} [\mu_R(x) \star \mu_S(x, z)]$, which is only a function of the output variable z. Hence, the notation of $\mu_{R \circ S}(x, z)$ can be simplified to $\mu_{R \circ S}(z)$, so that in the case of R being just a fuzzy set,

$$\mu_{R \circ S}(z) = \sup_{x \in U} [\mu_R(x) \star \mu_S(x, z)]. \quad (17)$$

This result completes the discussion of fuzzy relations and their composition. It is now possible to consider complex fuzzy relations.

B. Complex Fuzzy Relations

1) Defining Complex Fuzzy Relations: "*Complex Fuzzy Relations* represent both the *degree* of presence or absence of association, interaction, or interconnectedness, and the *phase* of association, interaction, or interconnectedness between the elements of two or more crisp sets," [13].

Let U and V be two crisp sets. A complex fuzzy relation $R(U, V)$ is a complex fuzzy subset of the product space $U \times V$. The relation $R(U, V)$ is characterized by the complex membership function $\mu_R(x, y)$, where $x \in U$ and $y \in V$ and $\mu_R(x, y)$ assigns each pair (x, y) a complex-valued grade of membership to the set $R(U, V)$. $R(U, V)$ may be represented as the set of ordered pairs:

$$R(U, V) = \{((x, y), \mu_R(x, y)) \,|\, (x, y) \in U \times V\}. \quad (18)$$

The complex-valued membership function $\mu_R(x) = r_R(x) \cdot e^{j\omega_R(x)}$ is interpreted in the following manner [13].

a) $r_R(x)$ represents a *degree* of presence or absence of association, interaction, or interconnectedness between the elements of U and V.
b) $\omega_R(x)$ represents the *phase* of association, interaction, or interconnectedness between the elements of U and V.

Note that this interpretation is consistent with the approach presented in the previous section, whereby the amplitude term of a complex grade of membership is equivalent to a traditional fuzzy grade of membership.

What does "phase of association" mean? Ramot *et al.* [13, Ex. 5 and 6] provides a good illustration of this original concept.

2) Compositions of Complex Fuzzy Relations on the Same Product Spaces: Complex fuzzy relations are complex fuzzy sets in product space. Thus, compositions of complex fuzzy relations on the same product space may be performed using complex fuzzy set theoretic operations, such as the ones defined in the previous section.

Let $R(U, V)$ and $S(U, V)$ be two complex fuzzy relations. Two possible compositions of these relations are.

a) The complex fuzzy intersection of $R(U, V)$ and $S(U, V)$, denoted by $R \cap S(U, V)$, with membership function

$$\mu_{R \cap S}(x, y) = \mu_R(x, y) \blacklozenge \mu_S(x, y). \quad (19)$$

b) The complex fuzzy union of $R(U, V)$ and $S(U, V)$, denoted by $R \cup S(U, V)$, with membership function

$$\mu_{R \cup S}(x, y) = \mu_R(x, y) \otimes \mu_S(x, y). \quad (20)$$

Where $\blacklozenge$ and $\otimes$ are used here to represent the set theoretic operations of complex fuzzy intersection and union, as defined in Definitions 3 and 2, respectively.

3) Compositions of Complex Fuzzy Relations on Different Product Spaces: The composition of complex fuzzy relations on different product spaces is defined later. The approach used for deriving this definition is consistent with the one utilized in the development of complex fuzzy set-theoretic operations, reviewed in the previous section. The amplitude term of the membership function for the composition of complex fuzzy relations is calculated using the traditional fuzzy definition, i.e., the sup-star composition. The phase term of this membership function is calculated separately, and the precise method of composition is determined individually for each application.

Definition 6: Let the membership functions of relations $R(U, V)$, $S(V, W)$ and the composition of these relations $R \circ S(U, W)$, be

$$\mu_R(x, y) = r_R(x, y) \cdot e^{j\omega_R(x, y)} \quad (21)$$

$$\mu_S(y, z) = r_S(y, z) \cdot e^{j\omega_S(y, z)} \tag{22}$$
$$\mu_{R\circ S}(x, z) = r_{R\circ S}(x, z) \cdot e^{j\omega_{R\circ S}(x, z)}. \tag{23}$$

The amplitude term of the membership function for the composition of complex fuzzy relations $R(U, V)$ and $S(V, W)$ is calculated using the sup-star composition

$$r_{R\circ S}(x, z) = \sup_{y\in V} \left[r_R(x, y) \star r_S(y, z)\right]. \tag{24}$$

The phase term of $\mu_{R\circ S}(x, z)$ is determined in the following manner:

$$\omega_{R\circ S}(x, z) = f\left[g(\omega_R(x, y), \omega_S(y, z))\right] \tag{25}$$

where g is the membership phase equivalent of the t-norm ("star") function, i.e., g is any function used to calculate the intersection of two membership phases. Examples of an appropriate form for g, such as functions (5)–(8), are discussed in the previous section. Generally, the choice of g is application-dependent.

The function f is also selected according to the application at hand, and is the membership phase equivalent of the sup operation. Possible forms of f are

$$\omega_{R\circ S}(x, z) = \sup_{y\in V} \left[g\left(\omega_R(x, y), \omega_S(y, z)\right)\right] \tag{26}$$
$$\omega_{R\circ S}(x, z) = \inf_{y\in V} \left[g\left(\omega_R(x, y), \omega_S(y, z)\right)\right] \tag{27}$$
$$\omega_{R\circ S}(x, z) = g\left(\omega_R(x, y'), \omega_S(y', z)\right) \tag{28}$$

where y' in (28) is the value of y for which the supremum, $\sup_{y\in V}\left[r_R(x, y) \star r_S(y, z)\right]$ is obtained.

IV. Complex Fuzzy Logic

In this section, the framework termed complex fuzzy logic is developed. Complex fuzzy logic is a unique framework designed to maintain the advantages of traditional fuzzy logic, while benefiting from the properties of complex fuzzy sets. The most notable property of complex fuzzy logic is that rules constructed using this framework are strongly related; a relation manifested in the phase term associated with complex fuzzy implications. This relation leads to a unique interaction, or dependence, between rules, which allows the construction of a novel system termed the complex fuzzy logic system (CFLS).

Much like the traditional FLS, the complex FLS is based on inference rules. However, the output of each of these rules is a complex fuzzy set, so that phase considerations play a major part when several rules are combined to determine the final output of the CFLS. In contrast, in a traditional FLS the only consideration is that of amplitude, or degree of membership. Note that this characteristic of complex FLSs is a direct consequence of the properties of complex numbers, and cannot be paralleled by traditional fuzzy sets.

One of the basic requirements set for the development of complex fuzzy logic was to maintain the advantages inherent in traditional fuzzy logic. The framework would thus benefit from the properties of both complex numbers and fuzzy logic. Specifically:

- the framework must retain the unique ability of fuzzy logic to handle both numerical data and linguistic knowledge;
- constructing systems using complex fuzzy logic must remain relatively simple and intuitive (despite the unintuitive nature of complex grades of membership);
- CFLSs will be based on many rules which "fire" in parallel, thus achieving computational efficiency.

Consequently, complex fuzzy logic is based upon methods similar to those of conventional fuzzy logic. A complex FLS involves fuzzification of the problem at hand (including the definition of fuzzy rules), fuzzy inference, and a defuzzification stage used to obtain a crisp solution. Of course, all fuzzy sets used for the construction of complex FLSs are characterized by complex valued membership functions.

The suggested framework is simply one of many possible alternatives. It does have several distinct properties, as discussed above, which make it an appealing choice. However, there is undoubtedly much potential for further research into different frameworks for the application of complex fuzzy sets to the realm of fuzzy logic.

A. Review of Fuzzy Logic

Fuzzy logic is based on Rules. These rules are expressed in the form of IF–THEN statements, e.g., "IF p, THEN q," where p and q are unconditional (and unqualified) fuzzy *propositions*.

An unconditional fuzzy proposition is expressed by the phrase: "X is A," where X is a variable that receives values x from a universal set U, and A is a fuzzy set on U that represents a fuzzy predicate. Examples of a fuzzy predicate A are low, short, big, etc., and examples of a relevant variable X are temperature, height, and size. Each fuzzy proposition p is associated with a degree of truth $T(p)$. In general, for any value x of X, $T(p)$ is equal to $\mu_A(x)$, the grade of membership of x in A.

Assume the following fuzzy propositions are given:

$$p\text{: X is } A \tag{29}$$
$$q\text{: Y is } B \tag{30}$$

where X receives values x from a universal set U, A is a fuzzy set on U, Y is a variable that receives values y from a universal set V, and B is a fuzzy set on V.

The rule "IF X is A, THEN Y is B" combines the fuzzy propositions (p, q) into a logical *implication* (denoted $p \rightarrow q$). The implication $p \rightarrow q$ signifies a fuzzy relation between p and q, and has a membership function denoted $\mu_{A\rightarrow B}(x, y)$. The function $\mu_{A\rightarrow B}(x, y)$ represents the degree of truth of the implication $p \rightarrow q$, given that X is equal to x and Y is equal to y. The IF part of an implication is called *antecedent*, whereas the THEN part is known as *consequent*. Note that a rule is also a (conditional and unqualified) fuzzy proposition.

Implication relations are used in crisp propositional logic to construct *inference rules*. One important type of inference rule is *Modus Ponens*, which has the following structure.

Premise 1: "X is A"; *Premise 2*: "IF X is A, THEN Y is B"; *Consequence*: "Y is B."

Modus Ponens is associated with the implication "A implies B" ($A \rightarrow B$), and is the basic inference rule used in engineering applications of logic. In terms of propositions p and q, Modus Ponens is expressed as $(p \wedge (p \rightarrow q)) \rightarrow q$.

In fuzzy logic, Modus Ponens are extended to *Generalized Modus Ponens* [16, pp. 56].

Premise 1: "X is A^*"; *Premise 2*: "IF X is A, THEN y is B"; *Consequence*: "Y is B^*."

The differences between Modus Ponens and Generalized Modus Ponens are subtle, but of great significance. Namely, the fuzzy set A^* is not necessarily the same as rule antecedent fuzzy set A, and as a result, the fuzzy set B^* is not necessarily the same as rule consequent fuzzy set B.

Note that Generalized Modus Ponens is, in fact, a fuzzy composition of two fuzzy relations. The first relation is simply the fuzzy set A^*. The second relation is the implication relation, or *Premise 2* of the Generalized Modus Ponens. As the two relations share a common set (the universal set U), it is possible to obtain their composition using the sup-star composition function of Definition 5. Using the so-called compositional rule of inference, the consequent of the Generalized Modus Ponens, $\mu_{B^*}(y)$, may thus be calculated in the following manner:

$$\mu_{B*}(y) = \sup_{x \in U} \left[\mu_{A^*}(x) \star \mu_{A \to B(x,\, y)}\right]. \tag{31}$$

The most widely used forms of implication in applications of fuzzy logic are probably the *minimum implication* (for example, see [11]) and *product implication* (see [9]) which are given as

$$\mu_{A \to B}(x,\, y) = \min\left[\mu_A(x),\, \mu_B(y)\right] \tag{32}$$

$$\mu_{A \to B}(x,\, y) = \mu_A(x) \cdot \mu_B(y). \tag{33}$$

Using these implications for fuzzy inference, the consequent of the generalized Modus Ponens $\mu_{B^*}(y)$ may be obtained

Minimum:

$$\mu_{B^*}(y) = \sup_{x \in A} \left[\mu_{A*}(x) \star (\min\left[\mu_A(x),\, \mu_B(y)\right])\right] \tag{34}$$

Product:

$$\mu_{B^*}(y) = \sup_{x \in A} \left[\mu_{A*}(x) \star (\mu_A(x) \cdot \mu_B(y))\right]. \tag{35}$$

B. Review of FLSs

An FLS is a nonlinear mapping of an input data vector into an output scalar (the vector output case is considered equivalent to a collection of multiple-input–single-output systems). At the heart of the FLS is a fuzzy rule base, which contains fuzzy rules expressed in the form of IF–THEN statements. The mapping of the input data to the desired output is generally performed in three stages. These are *fuzzification* of the input data (assuming this data is crisp), *fuzzy inference* (using *fuzzy rules*) which is discussed above, and a *defuzzification* stage used for producing a crisp scalar output. Generally, if the rule base of the FLS includes several rules, their individual outputs are combined in the inference stage to produce a single fuzzy output set. The method of combining the outputs of the separate rules is usually disjunctive. Geometrical and mathematical interpretations of the mapping performed by a FLS may be found in [12] and [7]. For insights on various methods of defuzzification, review [5] or [15].

C. Complex Fuzzy Logic

1) Complex Fuzzy Implication and Inference: Complex fuzzy logic, as presented in this paper, employs rules constructed with complex fuzzy sets to create a CFLS. These rules are expressed in the form of IF–THEN statements—"IF X is A, THEN Y is B," where X is a variable that receives values x from a universal set U, A is a complex fuzzy set on U, Y receives values y from a universal set V, and B is a complex fuzzy set on V. A rule represents a complex fuzzy implication relation between unconditional complex fuzzy propositions p and q, where proposition p is described by the phrase "X is A," and q by the phrase "Y is B."

The complex fuzzy implication relation is characterized by a complex-valued membership function, and denoted $\mu_{A \to B}(x,\, y)$. As usual, this membership function may be split into two separate terms, an amplitude term and a phase term. The amplitude term, $r_{A \to B}(x,\, y)$, is equivalent to a real-valued grade of membership. As such, it specifies the degree of truth of the implication relation, i.e., the degree to which the rule is fired.

The phase term, $\omega_{A \to B}(x,\, y)$, signifies the phase associated with the implication. In itself, the phase of the implication is of little consequence; however, it becomes a significant parameter when several implication relations are considered together, as occurs in CFLSs. CFLSs combine implications using vector aggregation (see below). This use of vector aggregation, a form of aggregation whose result is highly dependent on the relative phase of its arguments (Section II-E), is responsible for the potent effect $\omega_{A \to B}(x,\, y)$ has on the final output of the CFLS. The unique role of the phase term of complex fuzzy implication is illustrated in Example 2. Note that in effect, this method of combining implications creates a form of dependence, or interaction, between rules.

The implication function utilized in complex fuzzy logic is the product implication

$$\mu_{A \to B}(x,\, y) = \mu_A(x) \cdot \mu_B(y). \tag{36}$$

Thus, the expression for complex fuzzy implication is in fact quite similar to the expression for ordinary fuzzy implication. The difference is, of course, that the grades of membership in (36) are complex-valued. Considering the amplitude term and the phase term of $\mu_{A \to B}(x,\, y)$ separately, complex fuzzy implication is given by

$$r_{A \to B}(x,\, y) = r_A(x) \cdot r_B(y) \tag{37}$$

$$\omega_{A \to B}(x,\, y) = \omega_A(x) + \omega_B(y). \tag{38}$$

Note that the effect of product implication on the amplitude terms of A and B is identical to its effect on traditional real-valued grades of membership. Thus, product implication is consistent with the interpretation of the amplitude term presented in this paper, whereby the amplitude term is equivalent to a traditional real-valued grade of membership. This is one of the reasons that, of all possible forms of implication functions, product implication is selected for complex fuzzy logic.

An effective method of obtaining intuition for the application of product implication to complex fuzzy sets is to view it as a transformation of the grade of membership of the consequent set $\mu_B(y)$. Following this approach, it is easy to see that the effect of product implication on $\mu_B(y)$ is its rotation by a phase of $\omega_A(x)$, and a scaling of its amplitude by a factor $r_A(x)$. The unique rotation effect of product implication on the consequent set B is another reason for selecting product implication as the implication function of choice in complex fuzzy logic.

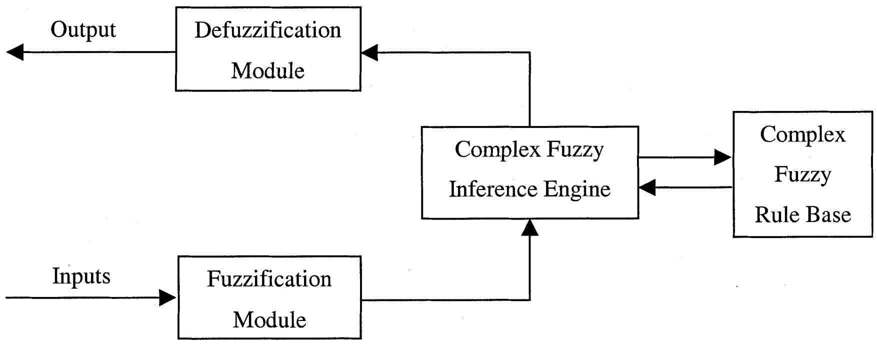

Fig. 1. General scheme of a CFLS.

Complex fuzzy implication may be utilized for the construction of complex fuzzy inference rules, in the form of Generalized Modus Ponens.

Premise 1: "X is A^*"; *Premise 2*: "IF X is A, THEN Y is B"; *Consequence*: "Y is B^*."

Of course, the sets A, B, A^* and B^* are all complex fuzzy sets.

Following the compositional rule of inference, the Generalized Modus Ponens may be regarded as a composition of two fuzzy relations. The first relation is simply the complex fuzzy set A^*, or *Premise 1* of the Generalized Modus Ponens. The second relation is the complex fuzzy implication, which appears in *Premise 2* of the Generalized Modus Ponens. These relations share a common set (the universal set U), and therefore, their composition is given by Definition 6.

Thus, using the implication membership function $\mu_{A\to B}(x, y)$ defined above, the membership function of B^* may be calculated in the following manner. Denote the membership function of B^* by $\mu_{B^*}(y) = r_{B^*}(y) \cdot e^{j\cdot\omega_{B^*}(y)}$. Let the membership functions of complex fuzzy sets A^*, A and B, and that of the complex fuzzy implication relation $A \to B$ be similarly denoted. Following Definition 6

$$\begin{aligned} r_{B^*}(y) &= \sup_{x\in U} \left[r_{A^*}(x) \star r_{A\to B}(x, y)\right] \\ &= \sup_{x\in U} \left[r_{A*}(x) \star (r_A(x) \cdot r_B(y))\right] \end{aligned} \tag{39}$$

$$\begin{aligned} \omega_{B^*}(y) &= f\left[g\left(\omega_{A*}(x),\, \omega_{A\to B}(x, y)\right)\right] \\ &= f\left[g\left(\omega_{A^*}(x),\, (\omega_A(x) + \omega_B(y))\right)\right] \end{aligned} \tag{40}$$

where g is any function used to calculate the intersection of two membership phases (see 3.4). Examples of an appropriate form of g are given in (5)–(8). Generally, the choice of g is application-dependent. The function f is also application-dependent, and is the membership phase equivalent of the sup operation. Possible forms of f are given in (26)–(28), Section III-B3.

2) CFLSs: Much like the traditional FLS (see Section IV-B), the CFLS is a nonlinear mapping of an input data vector into an output scalar. A CFLS is characterized by a complex fuzzy rule base containing a set of rules, which are in fact complex fuzzy implications expressed in the form of IF–THEN statements. In essence, a CFLS is a generalization of its traditional counterpart, obtained by replacing the fuzzy sets and fuzzy implications found in a traditional FLS with a their complex equivalents.

The output of a CFLS is determined in three stages, as illustrated in Fig. 1. The first stage is *Fuzzification*, used to map crisp inputs into fuzzy input sets. These fuzzy sets may or may not be complex, depending on the application.

The second stage, *Fuzzy Inference*, utilizes a complex fuzzy rule base to map the fuzzy input sets into fuzzy output sets. The method of inference follows the outline described in the previous subsection. Each rule is combined with the relevant fuzzy input sets (specifically, those sets that appear in its antecedent), to generate a Generalized Modus Ponens. The outputs of these Generalized Modus Ponens are given by (39) and (40). Next, using the operation of *Vector Aggregation* (Definition 4), the complex fuzzy outputs of the separate rules are combined to produce a single complex fuzzy output set, denoted B^*.

Defuzzification, as usual, is the final stage of the mapping performed by the CFLS. This stage involves defuzzification of the complex fuzzy output set to produce a crisp output. One possible approach to defuzzification of the complex fuzzy output is to neglect all phase terms and consider only the amplitude term of the output set B^*. Any defuzzification method used in a traditional FLS may be utilized for this purpose. This is the approach selected in Example 2. Alternatively, it is possible to consider forms of defuzzification that make use of the phase information in this final stage of the mapping. These forms may prove quite useful in applications of CFLS, and although beyond the scope of this paper, certainly merit further research.

The operation of vector aggregation is the chosen method of rule aggregation in the second stage (fuzzy inference), because it stresses the aspect of relative phase between rules. Vector aggregation is in essence a vector sum of its arguments. Therefore, the result of vector aggregation is strongly dependent on the relative phase of its arguments. For example, if all of the arguments are in phase, the amplitude of the sum is maximized. If, however, the arguments are not in phase, the result of the sum may be a grade of membership whose amplitude is smaller than that of its separate arguments. As amplitude is a parameter of consequence in the defuzzification stage, relative phase between inference rules is of great importance. Thus, a form of interaction between rules in a CFLS is defined, requiring that they maintain some sort of phase correlation, or coherence, for a large output grade of membership to be obtained. This interaction between rules is a direct result of the combination of vector aggregation and complex fuzzy implication.

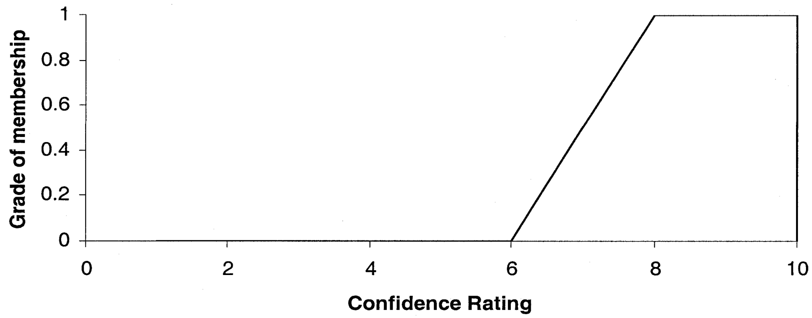

Fig. 2. Membership function of complex fuzzy set "*High Confidence in Democracy*."

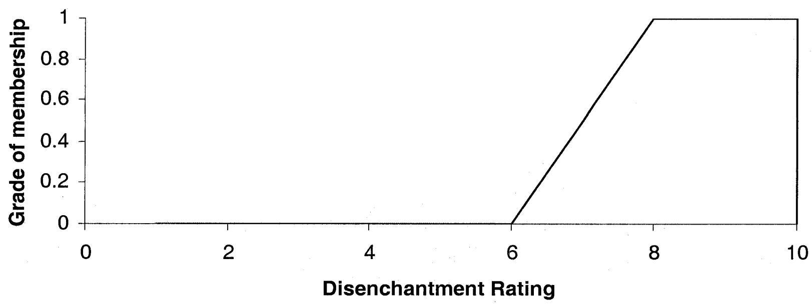

Fig. 3. Membership function of complex fuzzy set "*High Disenchantment with Leaders*."

The example that follows illustrates the significance of the phase term, the effect of product implication and the importance of the operation of *Vector Aggregation* in a CFLS. Again, a relatively simple example has been chosen, in an attempt to facilitate the development of intuition for complex fuzzy logic.

Example 2: Consider a CFLS constructed using the pair of inference rules

$$\begin{aligned} \text{Rule 1:}\quad & \text{IF ``Confidence in Democracy''; is ``High''} \\ & \text{THEN ``Voter Turnout'' is ``High''} \\ \text{Rule 2:}\quad & \text{IF ``Disenchantment with Leaders'' is ``High''} \\ & \text{THEN ``Voter Turnout'' is ``Very High''} \end{aligned}$$

where "*High Confidence in Democracy*" and "*High Disenchantment with Leaders*" are complex fuzzy sets defined on the set $U = \{x \,|\, x \in [1,\, 10]\}$. "*High Voter Turnout*" and "*Very High Voter Turnout*" are complex fuzzy sets defined on the set $V = \{x \,|\, x \in [0,\, 100]\}$. The inputs to this CFLS are the variables "*Confidence in Democracy*" and "*Disenchantment with Leaders*," which receive values from the set U, i.e., are measured on a scale of 1 to 10. The output is the variable "*Voter Turnout*," which receives values from the set V.

The goal of this CFLS is to predict voter turnout for an election (as a percentage of the adult population who are eligible to vote), given fuzzy measures of the current public sentiment toward the democratic process and its leaders. The rules that make up the CFLS signify that both a strong belief in democracy and a lack of faith in contemporary leaders are factors that contribute to high voter turnout.

The CFLS will attempt to model the following well-known phenomenon: Although individually the two rules above make good sense, when considered together their combination may actually produce a contrary result. That is to say, if confidence in the democratic process is high *and* there exists a high degree of dissatisfaction with the present political leadership, there is likely to be relatively low voter turnout on Election Day. In short, the combination of disillusionment with current leaders and complacency toward the state of democracy may lead to certain electoral apathy, which is the result the CFLS will attempt to reproduce. As is shown below, such interaction between two fuzzy rules is difficult to model using traditional fuzzy logic. The CFLS, however, can provide an elegant and relatively straightforward solution to this problem.

Let the membership phase of all elements in the four complex fuzzy sets introduced above equal zero. This is a reasonable choice, as there does not appear to be any intuitive justification for associating particular phase values with membership in these sets. In other words, as these sets can be well represented by traditional fuzzy sets, selecting any membership phase other than zero will be difficult to account for. Hence, the system described thus far is no different from an ordinary FLS with real-valued membership functions.

Figs. 2–4 illustrate the four fuzzy sets, which make up the rule base of the CFLS.

Now, assume the inputs to the system are expressed by the premises

$$\text{```Confidence in Democracy' is } A_1^*\text{''}$$
$$\text{```Disenchantment with Leaders' is } A_2^*\text{''}$$

The output of the system will be calculated in the following two cases: 1) the system is limited to the confines of an ordinary FLS; and 2) complex fuzzy logic is employed. This comparison will serve to delineate the potential of systems of complex fuzzy logic.

Case 1: An Ordinary FLS: Assume A_1^* and A_2^* are both ordinary fuzzy singletons, selected to signify that "Confidence

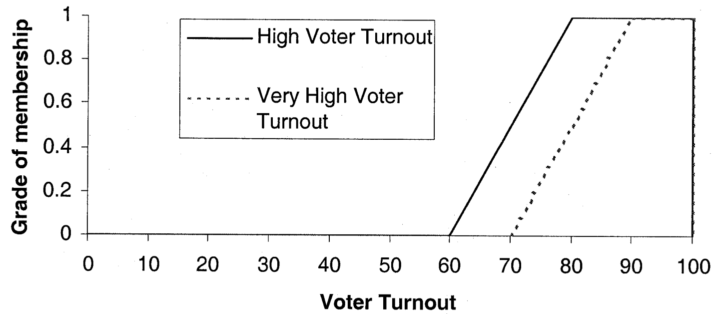

Fig. 4. Membership functions of complex fuzzy sets "*High Voter Turnout*" and "*Very High Voter Turnout*."

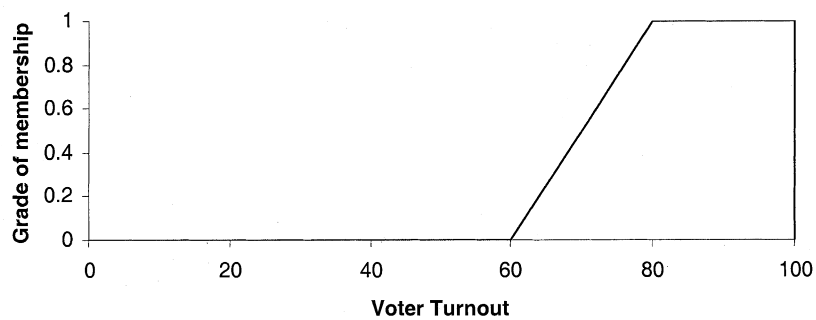

Fig. 5. Membership function of output set of ordinary FLS.

in Democracy" and "Disenchantment with Leaders" are indeed relatively high. (Note: fuzzy singletons are considered for simplicity of computation and illustration—results obtained using fuzzy inputs that are not fuzzy singletons are very similar).

$\mu_{A_1^*}(x) = 1$ for $x = 9$, and $\mu_{A_1^*}(x) = 0$ for all other $x \in U$.

$\mu_{A_2^*}(x) = 1$ for $x = 8$, and $\mu_{A_2^*}(x) = 0$ for all other $x \in U$.

Using (35) for calculating the output fuzzy set of each rule, choosing Product as the conjunction operator ($\star$) and a disjunctive method (Maximum) for combining output sets, the fuzzy set depicted in Fig. 5 is obtained. Defuzzification of this output set using *Mean of Maxima* defuzzification (see [12] for a discussion of defuzzification methods) produces a prediction of 90% turnout on Election Day. Additional selections of ordinary fuzzy sets A_1^* and A_2^* (as long as choices remain faithful to the assumption that "Confidence in Democracy" and "Disenchantment with Leaders" are relatively high) result in similar predictions. It is in fact quite easily demonstrated that for an ordinary FLS, if either or both of the premises that make up the FLS rule base are true, then the resulting prediction is for high voter turnout (over 90%).

Case 2: CFLS: Using a CFLS, it is possible to model the phenomenon previously suggested, i.e., relatively low voter turnout when both premises hold true. Assume A_1^* and A_2^* are *complex* fuzzy singletons. As in the ordinary FLS case, the amplitudes of these singletons will be selected to represent a high degree of "Confidence in Democracy" and "Disenchantment with Leaders," respectively. The membership phases of these complex singletons will be chosen to reflect the contrary effect the two inputs have on one another. Namely, that their individual "high" outputs interact, or interfere, resulting in a final output that is lower than expected. Again, it may be worthwhile to stress that it is the introduction of membership phase, which sets the CFLS apart from its ordinary fuzzy forebear.

Consider the following choices for A_1^* and A_2^*.

$\mu_{A_1^*}(x) = 1 \cdot e^{0 \cdot j} = 1$ for $x = 9$, and $\mu_{A_1^*}(x) = 0$ for all other $x \in U$.

$\mu_{A_2^*}(x) = 1 \cdot e^{\pi \cdot j} = -1$ for $x = 8$, and $\mu_{A_2^*}(x) = 0$ for all other $x \in U$.

The amplitudes of membership are identical to the real-valued grades of membership considered in the ordinary FLS case, thus facilitating the comparison between the two cases. The membership phase of A_1^* has been arbitrarily selected as 0, while the membership phase of A_2^* has been selected as π radians. Note that these choices produce the desired effect of "destructive interference" between the output sets of the two rules.

Consider *Rule 2*. Let the result of firing this rule be a complex fuzzy set denoted B_2^*. The set B_2^* is characterized by a complex-valued membership function, whose amplitude and phase terms are calculated using the compositional rule of inference

$$\begin{aligned} r_{B_2^*}(y) &= \sup_{x \in U}[r_{A_2^*}(x) \star (r_{\text{High Confidence in Democracy}}(x) \\ &\qquad \cdot r_{\text{Very High Voter Turnout}}(y))] \\ &= 1 \star (r_{\text{High Confidence in Democracy}}(8) \\ &\qquad \cdot r_{\text{Very High Voter Turnout}}(y)) \\ &= r_{\text{Very High Voter Turnout}}(y) \end{aligned} \tag{41}$$

$$\begin{aligned} \omega_{B_2^*}(y) &= f[g(\omega_{A_2^*}(x), (\omega_{\text{High Confidence in Democracy}}(x) \\ &\qquad + \omega_{\text{Very High Voter Turnout}}(y)))] \\ &= f[g(\omega_{A_2^*}(x), 0)] = \pi \end{aligned} \tag{42}$$

where g is chosen as the *Sum* function (5) discussed in Sections II-C and II-D, and f is chosen as the function in (28) (Section III-B3): $f[\omega_{A_2*}(x) + \omega_B(y)] = \omega_{A_2*}(s) + \omega_B(y)$,

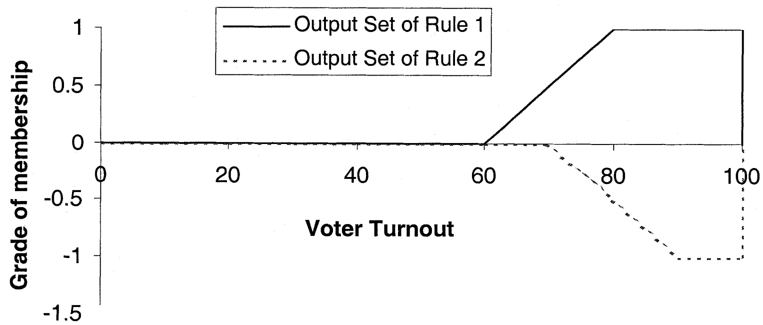

Fig. 6. Membership functions of output sets of CFLS.

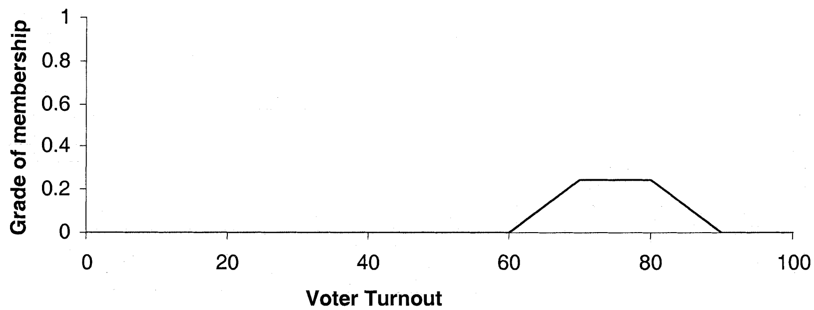

Fig. 7. Membership function of final output set of CFLS (Obtained by applying vector aggregation to individual output sets.)

with s representing the value of x for which the supremum in (41) is obtained (in this example $s = 8$).

Thus, the actual result of firing *Rule 2* is a rotation of the original consequent of the rule ("*Very High Voter Turnout*") by a constant phase (π radians). Note the amplitude term of the original consequent is unchanged in this case, as it was multiplied ("scaled") by a factor of one. Similarly, the result of firing the first rule B_1^* is given by

$$r_{B_1^*}(y) = r_{\text{Very High Voter Turnout}}(y) \tag{43}$$

$$w_{B_1^*}(y) = 0. \tag{44}$$

The membership functions $\mu_{B_1^*}(y)$ and $\mu_{B_2^*}(y)$ are plotted in Fig. 6.

Next, the complex fuzzy set B^*, the complex fuzzy output of the CFLS, can be calculated by applying vector aggregation to the sets B_1^* and B_2^*:

$$\mu_{B^*} = \sum_{i=1}^{2} w_i \cdot \mu_{B_i^*}(y). \tag{45}$$

The elements in the support of B_1^* and B_2^* have different membership phases. Therefore, the grades of membership of B_1^* and B_2^*, if considered as vectors in the complex plane, are not "aligned," i.e., do not point in the same direction. Due to this lack of alignment, the effect of vector aggregation on complex fuzzy sets is very different than that of a simple averaging operation used in ordinary additive FLS [8]. Assume $w_1 = w_2 = 0.5$. The result of the vector aggregation operation, B^*, is depicted in Fig. 7.

Now, all that remains is to defuzzify the complex fuzzy output set B^* in order to obtain a crisp output. In this example, all phase terms (which are zero anyway) are neglected in the defuzzification stage, and only the amplitude terms of the output set B^* are considered. Defuzzifying B^* using the *Mean of Maxima* method [12], the output of the CFLS described previously is found to be 75%. This result is considerably lower than the one obtained using ordinary fuzzy logic (90% or above), and is a direct consequence of the use of complex fuzzy sets in the reasoning process. In many democratic countries, 75% turnout would be considered relatively poor, which is precisely the result that the example was designed to recreate.

To conclude, the use of complex fuzzy sets, product implication and vector aggregation defines an interaction between fuzzy rules. In the example, this interaction is significant despite the choice of relatively simple inputs (singletons) and complex fuzzy sets. As the example demonstrates, a large grade of membership in the output set of a specific rule B_1^* provides no guarantee that an element y will have a large grade of membership in B^*, the final output set of the CFLS. It is not even enough for y to have large amplitudes of membership in all output sets B_i^*. In order for an element to have a large grade of membership in B^*, its phase terms in all B_i^*s which have a relatively large $\mu_{B_i^*}(y)$ must be in phase. As stated previously, this requirement effectively defines a relation that the rules of the CFLS must satisfy if a large output grade of membership is to be obtained.

V. Conclusion

The novel framework of complex fuzzy logic was presented in this paper. The most notable property of complex fuzzy logic is that rules constructed using this framework are strongly related—a relation manifested in the phase term associated with complex fuzzy implication. This relation leads to a unique interaction, or dependence, between rules, which is enhanced by the use of vector aggregation in the inference stage of CFLSs. Thus, the novel framework may be utilized to solve problems, in which rules are related to one another with the nature of the relation varying as a function of the input to the system. These problems may be very difficult or impossible to solve using traditional methods of fuzzy logic.

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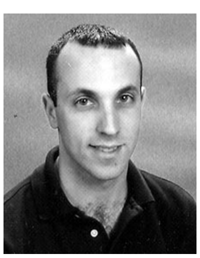

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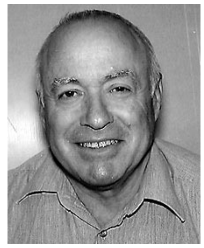

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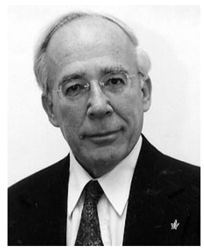

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