MATH5472 Assignment 4

Nov 4, 2025

Problem 1

Consider a linear model that relates variables $X_1, \dots, X_p$ to the response Y:

$$Y = \sum_{j=1}^{p} X_j \beta_j + \epsilon,$$

where β_js are random effects, and $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$.

Assume the prior $\beta_j \sim \mathcal{N}(0, \sigma_\beta^2), j = 1, \dots, p$. Obtain the posterior of $\boldsymbol{\beta}$ using the mean-field approximation $q(\boldsymbol{\beta}) = \prod_{j=1}^{p} q(\beta_j)$, where the posterior mean and posterior variance denoted as $\boldsymbol{\mu}_{\mathrm{mf}} \in R^p$ and $\mathbf{S}_{\mathrm{mf}}$ (a p-by-p diagonal matrix). Then compare the obtained mean-field approximation with the exact posterior distribution (e.g., obtained by the standard EM algorithm). You should check whether the approximated posterior mean is accurate and whether the posterior variance is underestimated. Please demonstrate your conclusion using simulation.

Problem 2

Consider the Lasso problem

$$\min_{\beta_1, \dots, \beta_p} \frac{1}{2} \sum_{i=1}^{n} (y_i - \sum_{j=1}^{p} x_{ij} \beta_j)^2 + \lambda \sum_{j=1}^{p} |\beta_j|, \tag{1}$$

and denote its solution path as $\hat{\beta}(\lambda)$. Suppose we are also interested in solving the following problem

$$\min_{u_1, \dots, u_p; v_1, \dots, v_p} \sum_{i=1}^{n} (y_i - \sum_{j=1}^{p} x_{ij} u_j v_j)^2 + \lambda \sum_{j=1}^{p} (u_j^2 + v_j^2), \tag{2}$$

and denote its solution path as $(\hat{\mathbf{u}}(\lambda), \hat{\mathbf{v}}(\lambda))$. Let us define $\tilde{\beta}_j(\lambda) = \hat{u}_j(\lambda) \hat{v}_j(\lambda)$ for $j = 1, \dots, p$. Please compare $\tilde{\boldsymbol{\beta}}(\lambda)$ with the well known Lasso solution path $\hat{\boldsymbol{\beta}}(\lambda)$. Make some discussion based on your observation. (Hint: This is another example of “Ridge + ridge = Lasso”. You can obtain the solution paths by developing your own algorithm or calling some existing packages. Please make sure you know algorithm in the package well).

Problem 3

Based on the paper “Don’t Blame the ELBO! A Linear VAE Perspective on Posterior Collapse” `https://openreview.net/forum?id=HJMo0Vrg8B`, verify Lemma 1 and Theorem 1 by conducting experiments to reproduce the results given in Figure 2 of this paper.

Requirement

- You need to submit a report, in which you should clearly describe your method and explain your idea. The code should also be included.
- You can use R or Python for coding.
- Your report should be in the **pdf** or **html** format, which is automatically generated by either R markdown or Jupyter notebook.