

# Logistic Regression

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## 1 Model

Let  $\mathbf{x} \in \mathbb{R}^{s \times n}$  be the inputs, where  $n$  is the number of samples; and  $s$  is the dimension of each feature vector. The logistic regression model is:

$$\hat{\mathbf{y}} = \sigma(\mathbf{x}^\top \mathbf{w} + b) \quad (1)$$

$$= \frac{1}{1 + e^{-(\mathbf{x}^\top \mathbf{w} + b)}}. \quad (2)$$

Here,  $\mathbf{w} \in \mathbb{R}^s$  is the weight.  $b \in \mathbb{R}$  is the bias. For an input sample  $\mathbf{x}_i$ , let  $y_i$  be the ground-truth for  $\mathbf{x}_i$ . The prediction  $0 < \hat{y}_i < 1$  is the probability  $p(y_i = 1 | \mathbf{x}_i)$ .

## 2 Objective Function

Typically, the logistic regression is a binary classification. Therefore, we use a binary cross-entropy loss as the objective function. For an input sample  $\mathbf{x}_i$ , the binary cross-entropy loss  $L(\mathbf{x}_i, y_i | \mathbf{w})$  is:

$$L(\mathbf{x}_i, y_i | \mathbf{w}) = -(y_i \log \hat{y}_i + (1 - y_i) \log (1 - \hat{y}_i)) \quad (3)$$

$$\begin{aligned} &= - \left( y_i \log \left( \frac{1}{1 + e^{-(\mathbf{x}_i^\top \mathbf{w} + b)}} \right) + (1 - y_i) \log \left( \frac{e^{-(\mathbf{x}_i^\top \mathbf{w} + b)}}{1 + e^{-(\mathbf{x}_i^\top \mathbf{w} + b)}} \right) \right) \\ &= \log \left( 1 + e^{-(\mathbf{x}_i^\top \mathbf{w} + b)} \right) + (\mathbf{x}_i^\top \mathbf{w} + b) (1 - y_i). \end{aligned} \quad (4)$$

For all input samples  $\mathbf{x}$ , the total loss is an average of the losses for all samples:

$$\begin{aligned} L(\mathbf{x}, \mathbf{y} | \mathbf{w}) &= \frac{1}{n} \sum_{i=1}^n L(\mathbf{x}_i, y_i | \mathbf{w}) \\ &= \frac{1}{n} \sum_{i=1}^n \left( \log \left( 1 + e^{-(\mathbf{x}_i^\top \mathbf{w} + b)} \right) + (\mathbf{x}_i^\top \mathbf{w} + b) (1 - y_i) \right). \end{aligned} \quad (5)$$

## 3 Back-propagation

$\mathbf{w}$  and  $b$  are the only variable in  $L$ . Therefore, we only need to calculate the gradient for  $\mathbf{w}$  and  $b$ :

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{w}} &= \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \mathbf{w}} \log \left( 1 + e^{-(\mathbf{x}_i^\top \mathbf{w} + b)} \right) + \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \mathbf{w}} (\mathbf{x}_i^\top \mathbf{w} + b) (1 - y_i) \\ &= \frac{1}{n} \sum_{i=1}^n \frac{-\mathbf{x}_i e^{-(\mathbf{x}_i^\top \mathbf{w} + b)}}{1 + e^{-(\mathbf{x}_i^\top \mathbf{w} + b)}} + \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i (1 - y_i) \\ &= \frac{1}{n} \sum_{i=1}^n -\mathbf{x}_i (1 - \hat{y}_i) + \frac{1}{N} \sum_{i=1}^n \mathbf{x}_i^\top (1 - y_i) \\ &= \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i (\hat{y}_i - y_i), \end{aligned} \quad (6)$$

$$\frac{\partial L}{\partial b} = \frac{1}{n} \sum_{i=1}^n (\hat{y}_i - y_i). \quad (7)$$