ELSEVIER

Signal Processing 81 (2001) 1101–1107

SIGNAL PROCESSING

www.elsevier.nl/locate/sigpro

Short communication

Recursive implementation of constrained LMS *L*-filters for image restoration

Tao Chen[1], Hong Ren Wu*

School of Computer Science and Software Engineering, Monash University, Clayton Campus, CL(63)G24, Victoria 3168, Australia

Received 10 April 2000; received in revised form 21 October 2000

Abstract

This paper addresses the problem of designing recursive *L*-filters optimized by the least mean square (LMS) algorithm. The sorting operation brings in nonlinear behaviour into the *L*-filter structure, and thus it hampers the derivation of a closed-form solution for optimizing the weights suited to recursive LMS *L*-filters. An alternative training strategy is proposed to iteratively derive the weighting coefficients for recursive *L*-filters. Simulations conducted show the advantage of the recursive *L*-filter over its nonrecursive counterpart in suppressing image noise. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: *L*-filters; Recursive filtering; LMS optimization

1. Introduction

It is well known that linear filters in general introduce undesirable blurring effect when applied to noise suppression in images. To combat this, a wealth of nonlinear filtering techniques have been proposed and applied successfully to image restoration [10]. One of the important classes of nonlinear filters is based on the order statistics, which has shown excellent robustness [11]. Generally, order statistic based filters have good behaviour in the presence of additive white noise or impulse noise, if they are designed properly [11]. Among the rich family of filters based on order statistics, the median filter is the best known and widely used. An important generalization of the median is the *L*-filter [2], whose output is formed as a linear combination of the order statistics of input observation samples. The *L*-filters have been applied extensively in digital signal and image processing (e.g. [2,8,4]), since they have a well-defined design methodology as the estimators which minimize the mean-squared error (MSE) between the filter output and the original noise-free signal.

However, those algorithms have been derived based on the assumption of a constant signal being corrupted by zero-mean additive white noise. In general, an image is a nonstationary process, whose statistics vary in different regions, and the noise attributes change as well. As a result, in practice, we may have no a priori knowledge about the characteristics of the signal and noise at all. Recently, Kotropoulos and Pitas [5] developed a framework of adaptive *L*-filters based on the well-known least

* Corresponding author. Tel: + 61-3-9905-3255; fax: + 61-3-9905-5146.

E-mail address: hrw@csse.monash.edu.au (H.R. Wu).

[1] Now at Visual Information Systems, Sarnoff Corporation, Princeton, NJ 08543-5300, USA.

0165-1684/01/$ - see front matter
PII: S0165-1684(01)00010-X

mean square (LMS) algorithm [12,3] for image restoration. The updating formula for the weighting coefficients was derived without invoking the assumption of signal stationarity [5].

It is noticed that the weighting coefficients of adaptive LMS L-filters [5] were derived for nonrecursive implementation. In many cases, improved performance in noise suppression can be obtained if the filter is implemented recursively. However, the weighting coefficients derived for nonrecursive filtering are not optimal for recursive implementation, where the estimate of current pixel is dependent on the past filter outputs. In this work, an iterative training strategy for deriving the weighting vector used in the recursive filtering is proposed. The performance of recursive filtering design is evaluated by simulations of reducing image noise, and it is shown to be superior to that of nonrecursive implementation.

The remainder of the paper is organized as follows. For completeness, Section 2 reviews the formulation of LMS L-filters and the constrained LMS optimization algorithm. The new training process is then developed in Section 3. Experimental results are presented in Section 4. Finally, Section 5 draws conclusions.

2. Constrained LMS L-filters

Let $C = \{\boldsymbol{c} = (c_1, c_2) \mid 1 \leqslant c_1 \leqslant H, 1 \leqslant c_2 \leqslant W\}$ denote the pixel coordinates of an image, where H and W represent its height and width, respectively. At each position $\boldsymbol{c} \in C$, a filter window is defined in terms of the image coordinates symmetrically surrounding the current pixel, where the window size is $N = 2n + 1$ (n is a nonnegative integer). We assume the filter window slides across the image pixel by pixel in a raster scanning fashion. Let $\boldsymbol{x}(\boldsymbol{c})$ be the vector of observation samples obtained via the filter window, which is given by

$$\boldsymbol{x}(\boldsymbol{c}) = [x_1(\boldsymbol{c}), x_2(\boldsymbol{c}), \ldots, x_N(\boldsymbol{c})]^{\mathrm{T}}. \tag{1}$$

Here, the current pixel is denoted by $x_{n+1}(\boldsymbol{c})$. Fig. 1(a) shows an example of applying a 3×3 window (i.e., $N = 9$) which is centered at pixel $x_5(\boldsymbol{c})$. The ordered sample vector is given by

$$\boldsymbol{x}_{\mathrm{r}}(\boldsymbol{c}) = [x_{(1)}(\boldsymbol{c}), x_{(2)}(\boldsymbol{c}), \ldots, x_{(N)}(\boldsymbol{c})]^{\mathrm{T}}, \tag{2}$$

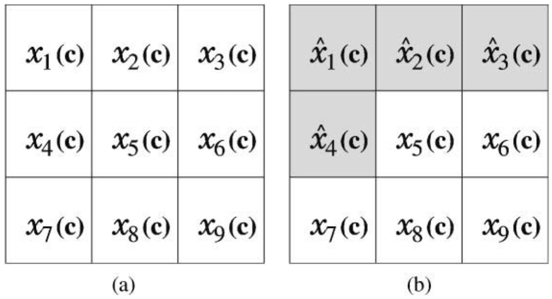


Fig. 1. Observation samples obtained via a 3×3 window for the (a) nonrecursive and (b) recursive filtering, respectively.

where $x_{(i)}(\boldsymbol{c}) \leqslant x_{(i+1)}(\boldsymbol{c})$ for $i = 1, \ldots, N-1$. It is clear that the median corresponds to $x_{(n+1)}(\boldsymbol{c})$.

The output of L-filters is formed as [2]

$$\hat{x}(\boldsymbol{c}) = \boldsymbol{w}^{\mathrm{T}} \boldsymbol{x}_{\mathrm{r}}(\boldsymbol{c}), \tag{3}$$

where the weighting vector $\boldsymbol{w} = [w_1, w_2, \ldots, w_N]^{\mathrm{T}}$ satisfies the location invariance constraint, i.e.,

$$\boldsymbol{w}^{\mathrm{T}} \boldsymbol{e}_N = 1. \tag{4}$$

Here, $\boldsymbol{e}_N = [1,1,\ldots,1]^{\mathrm{T}}$ is an $N \times 1$ vector. This restriction can also be viewed as a sufficient condition for an unbiased estimate as defined for a constant signal with additive noise [2,4].

The optimal set of weights is sought to minimize the MSE $\mathscr{E}$, which is given by [5]

$$\mathscr{E} = E\{[s(\boldsymbol{c}) - \hat{x}(\boldsymbol{c})]^2\}, \tag{5}$$

where $s(\boldsymbol{c})$ and $\hat{x}(\boldsymbol{c})$ denote the values of the original noise-free image pixel and the filter output, respectively, at location $\boldsymbol{c}$. Here, $E\{\cdot\}$ is statistical expectation.

The normalization constraint imposed by Eq. (4) leads to the use of the constrained LMS algorithm for deriving the optimal weights. The reduced weighting vector can be written as follows [5]: $\tilde{\boldsymbol{w}} = [\boldsymbol{w}_1^{\mathrm{T}}, \boldsymbol{w}_2^{\mathrm{T}}]^{\mathrm{T}}$, where $\boldsymbol{w}_1 = [w_1, w_2, \ldots, w_n]^{\mathrm{T}}$ and $\boldsymbol{w}_2 = [w_{n+2}, w_{n+3}, \ldots, w_N]^{\mathrm{T}}$. Correspondingly, the reduced ordered input vector is given by $\tilde{\boldsymbol{x}}_{\mathrm{r}}(\boldsymbol{c}) = [\boldsymbol{x}_{\mathrm{r}1}(\boldsymbol{c})^{\mathrm{T}}, \boldsymbol{x}_{\mathrm{r}2}(\boldsymbol{c})^{\mathrm{T}}]^{\mathrm{T}}$, where $\boldsymbol{x}_{\mathrm{r}1}(\boldsymbol{c}) = [x_{(1)}(\boldsymbol{c}), x_{(2)}(\boldsymbol{c}), \ldots, x_{(n)}(\boldsymbol{c})]^{\mathrm{T}}$ and $\boldsymbol{x}_{\mathrm{r}2}(\boldsymbol{c}) = [x_{(n+2)}(\boldsymbol{c}), x_{(n+3)}(\boldsymbol{c}), \ldots, x_{(N)}(\boldsymbol{c})]^{\mathrm{T}}$.

According to the analysis shown in [4,5], it can be obtained that the LMS recursive relation for updating the reduced weighting vector is given as follows:

$$\tilde{\boldsymbol{w}}' = \tilde{\boldsymbol{w}} + \eta[s(\boldsymbol{c}) - \hat{x}(\boldsymbol{c})]\tilde{\boldsymbol{x}}_{\mathrm{r}}'(\boldsymbol{c}), \tag{6}$$

where η represents the learning-rate factor, and $\tilde{\boldsymbol{w}}'$ denotes the reduced weighting vector that is updated on the basis of $\tilde{\boldsymbol{w}}$ at location $\boldsymbol{c}$. In Eq. (6), $\tilde{\boldsymbol{x}}_{\mathrm{r}}'(\boldsymbol{c})$ is the $(N-1)\times 1$ vector given by [5]

$$\tilde{\boldsymbol{x}}_{\mathrm{r}}'(\boldsymbol{c}) = \tilde{\boldsymbol{x}}_{\mathrm{r}}(\boldsymbol{c}) - \boldsymbol{e}_{N-1}x_{(n+1)}(\boldsymbol{c}) \tag{7}$$

Obviously, the coefficient for the median sample is evaluated by

$$w_{n+1}' = 1 - \boldsymbol{e}_{N-1}^{\mathrm{T}}\tilde{\boldsymbol{w}}'. \tag{8}$$

It is clear that the update at the next location for the weighting vector is conducted based on $\tilde{\boldsymbol{w}}'$.

On the basis of the conventional assumptions for the derivation of LMS convergence [3,5], it can be shown that the following condition is sufficient for the convergence of the algorithm in the mean square as well as in the mean: $0 < \eta < 2/\lambda$, where λ denotes the total input power given by

$$\lambda = E\left\{\sum_{i=1}^{N}[x_{(i)}(\boldsymbol{c})]^2\right\}. \tag{9}$$

3. Recursive filtering

In recursive filtering, the estimate of current pixel is dependent on the past filter outputs. In the case where the filter window slides from pixel to pixel in a raster scanning fashion, the observation sample vector obtained at location $\boldsymbol{c}$ is then given by

$$\boldsymbol{x}'(\boldsymbol{c}) = [\hat{x}_1(\boldsymbol{c}), \hat{x}_2(\boldsymbol{c}), \ldots, \hat{x}_n(\boldsymbol{c}), x_{n+1}(\boldsymbol{c}), x_{n+2}(\boldsymbol{c}), \ldots, x_{2n+1}(\boldsymbol{c})]^{\mathrm{T}}. \tag{10}$$

When a 3×3 filter window is applied, the observation samples obtained in recursive filtering are shown in Fig. 1(b).

The weighting coefficients obtained by the LMS training process described in the preceding section are therefore not optimal for the recursive implementation. The recursive design of L-filters provides better restoration performance as will be shown in the simulations, however it precludes a closed-form expression for the derivation of optimal weighting coefficients. This consequence transpires due to the feedback of past filter outputs and the nonlinear nature of the L-filter structure.

The nonrecursive filtering operation given in Eq. (3) can be thought of as a modification of linear finite impulse response (FIR) filters [6], and it is next extended to its recursive form which may be viewed as analogous to linear infinite impulse response (IIR) filters. The general structure of linear IIR filters is defined by the difference equation as follows:

$$\hat{x}(\boldsymbol{c}) = \sum_{l=1}^{\mathscr{U}} A_l\hat{x}_{n+1-l}(\boldsymbol{c}) + \sum_{k=-\mathscr{V}_1}^{\mathscr{V}_2} B_k x_{n+1-k}(\boldsymbol{c}), \tag{11}$$

where the output is calculated not only from the input but also from previous filter outputs. Clearly, the weights consist of two sets: the feedback coefficients $\{A_l\}$ and the feedforward coefficients $\{B_k\}$. In total, $\mathscr{U} + \mathscr{V}_1 + \mathscr{V}_2 + 1$ coefficients are required in order to define the recursive difference equation in Eq. (11). For a causal IIR filter implementation, $\mathscr{V}_1 = 0$. An updating formula based on the LMS algorithm is given in [12,7] to derive the weights, $\{A_l\}$ and $\{B_k\}$, for recursive linear filters.

The modification of Eq. (11) to a recursive L-filter structure is straightforward. A noncausal implementation is assumed in the following, where $\mathscr{V}_1 = n$ and $\mathscr{V}_2 = 0$. With $\mathscr{U} = n$, it leads to

$$\hat{x}(\boldsymbol{c}) = \boldsymbol{w}^{\mathrm{T}}\boldsymbol{x}_{\mathrm{r}}'(\boldsymbol{c}), \tag{12}$$

where $\boldsymbol{x}_{\mathrm{r}}'(\boldsymbol{c})$ denotes the ordered vector of $\boldsymbol{x}'(\boldsymbol{c})$ in Eq. (10). Obviously, the well-known recursive median filter is subsumed under the framework of recursive L-filters. It has been shown that recursive median filters have higher immunity to impulses than nonrecursive median filters [1]. Also, recursive median filters possess some nice deterministic properties [9]. The output of recursive median filters tends to be much more correlated than that of nonrecursive versions, however, the recursive implementation also introduces an increase in blurring [9]. Since the weights of adaptive LMS L-filters are designed by minimizing a certain criterion, i.e., MSE, such blurring effect can be alleviated.

On the other hand, the recursive LMS algorithm derived for adaptive linear filter [12,7] is not applied to this case due to the sorting operation, which makes the problem nonlinear. As an alternative, we propose an approximate optimization means to design the weighting vector for recursive L-filters, where the weights are derived in an iterative way based on the constrained LMS algorithm.

The iterative training process is realized with the use of the associated filtering procedure. Let $\boldsymbol{w}^t$ denote the weighting vector obtained after the tth iteration of training, i.e.,

$$\boldsymbol{w}^t = [w_1^t, w_2^t, \ldots, w_N^t]^{\mathrm{T}}, \tag{13}$$

where $t = 1,2,\ldots$. After the first iteration of training, the weighting vector $\boldsymbol{w}^1$ is derived by applying Eqs. (3), (6), and (8) based on the input observation sample vector $\boldsymbol{x}(\boldsymbol{c})$ given by Eq. (1). Clearly, $\boldsymbol{w}^1$ corresponds to the weighting vector used in nonrecursive filtering. In the subsequent training iterations, $\boldsymbol{w}^t$ ($t \geqslant 2$) is updated according to the above three equations while the input observation samples are given in the form of $\boldsymbol{x}'(\boldsymbol{c})$ defined in Eq. (10). Here, the new values of the previously processed pixels are computed based on the weighting vector $\boldsymbol{w}^{t-1}$ that results from the last iteration. In other words, the filtering result obtained by applying $\boldsymbol{w}^{t-1}$ ($t \geqslant 2$) is utilized in the tth iteration of training. It is obvious that $\boldsymbol{w}^t$ ($t \geqslant 2$) is used as the weighting vector for recursive filtering. As will be shown in the experiments, satisfactory filtering results can be obtained when $t = 2$.

The condition for convergence still holds, since the proposed scheme is based on the LMS algorithm at each iteration of training. The learning factor $\eta(\lambda)$ then satisfies Eq. (9), where the input observation vector given by Eq. (10) is taken into account when $t > 1$.

A flowchart illustrated in Fig. 2 depicts the iterative training process for deriving the weights used in the recursive implementation of LMS L-filters. In spite of being intractable in analysis, the proposed training strategy is observed, from extensive simulations conducted on real image data, to yield solutions which produce better performance in noise suppression than its nonrecursive design.

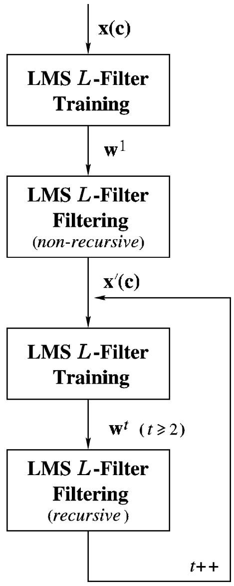


Fig. 2. Iterative training design of the recursive LMS L-filters.

4. Performance evaluation

The efficacy of recursive filtering scheme is evaluated in reducing mixed Gaussian and impulse noise for a variety of test images. In our simulations, corrupted images are obtained by adding zero-mean additive Gaussian noise with standard deviation σ and impulse noise with probability p together.

Due to the lack of original noise-free images in practice, the LMS L-filters are trained over an image other than those to be filtered in order to represent a realistic scenario. Specifically, the filters are trained over the *Couple* image, and the weighting vector is then used in filtering other test images, namely *Boats*, *Bridge*, *Goldhill*, *Lake*, and *Lena*. The peak signal-to-noise ratio (PSNR) is employed to quantitatively measure the image quality, which is defined as

$$\mathrm{PSNR} = 10\log\left(\frac{\sum_{c\in C} 255^2}{\sum_{c\in C}[s(\boldsymbol{c}) - \hat{x}(\boldsymbol{c})]^2}\right)\mathrm{dB}. \tag{14}$$

In order to exploit local correlation of pixels, the window size is set to 3×3. It is noticed that, when

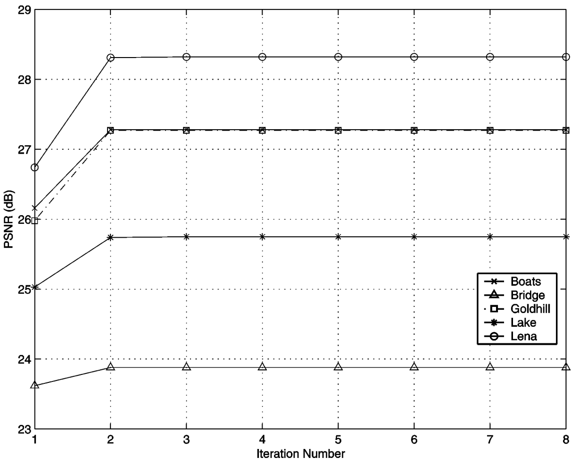
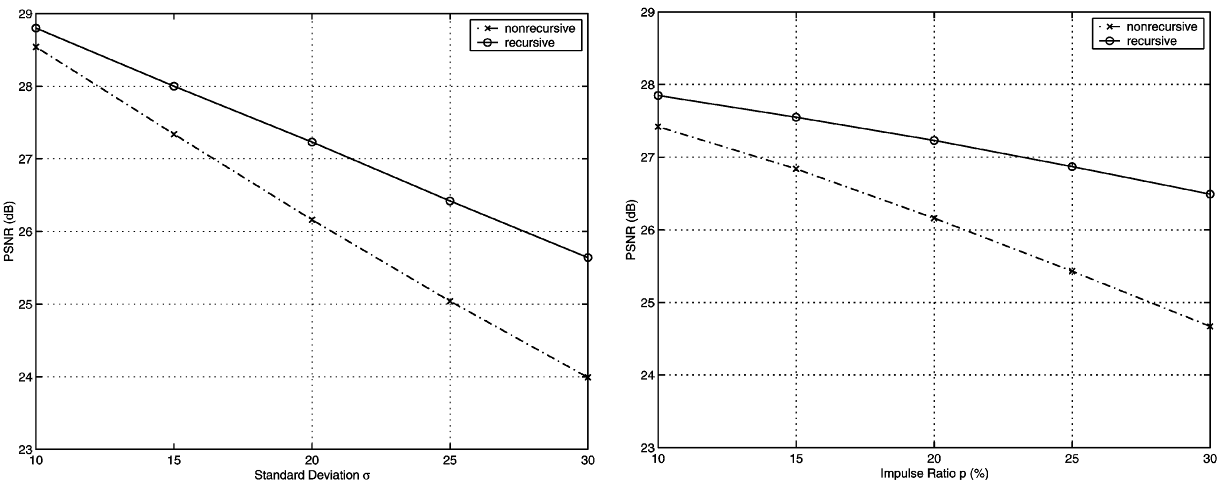


Fig. 3. Performance of recursive filtering in reducing mixed Gaussian (= 20) and impulse ($p = 20\%$) noise for different images.

a larger window is used, the improvement diminishes. This could be explained by that, roughly speaking, FIR filters with a larger window tend more to be IIR filters if they are designed properly. In addition, pixels distant from each other in a large window generally demonstrate less correlation.

In the first experiment, the images are corrupted by mixed Gaussian ($\sigma = 20$) and impulse ($p = 20\%$) noise. Fig. 3 shows the performance comparison of the recursive LMS L-filter and its nonrecursive design. The PSNR values presented at the first iteration correspond to those for the nonrecursive filtering scheme. In particular, the PSNR gain obtained by recursive filtering over its nonrecursive design is 0.26–1.58 dB, which is image dependent. It is interesting to notice that, in general, satisfactory results can be achieved by using the weighting vector obtained after two iterations of training, and experimentally the convergence is observed for the recursive filtering.

Fig. 4 presents the comparative results of filtering images corrupted by noise with different characteristics. In both cases of varying σ for Gaussian noise and varying p for impulses, the weights are obtained by training the filters over the *Couple* image corrupted by mixed Gaussian ($\sigma = 20$) and impulse ($p = 20\%$) noise. The recursive filtering consistently provides superior performance to the nonrecursive design. It is observed that significant improvement in terms of PSNR is obtained when the images are heavily corrupted. Similar results are also obtained for other images.



(a) (b)

Fig. 4. Comparative results in PSNR for the nonrecursive and recursive filters in restoring the *Boats* image corrupted by (a) Gaussian noise with varied standard deviations and impulse noise ($p = 20\%$), and (b) Gaussian noise ($\sigma = 20$) and impulses with varied noise ratio p, respectively.

Fig. 5. Comparison of restoration performance for the *Lena* image. (a) Original image; (b) image corrupted by mixed Gaussian ($\sigma = 20$) and impulse ($p = 20\%$) noise; images filtered by (c) nonrecursive (26.74 dB) and (d) recursive (28.32 dB) LMS L-filters, respectively.

In addition to the achieved gains in PSNR, the recursive design yields visually better quality in respect of noise suppression than its nonrecursive counterpart. The filtered *Lena* images are shown in Fig. 5. The resultant image by nonrecursive filtering manifests insufficient reduction of noise, while the recursive filter is more efficient in noise removal without excessive blurring of image details.

5. Conclusions

In this paper, the problem of designing the weighting coefficients for recursive implementation of LMS L-filters is considered. The nonlinearity of L-filters introduces difficulty in deriving a closed-form solution to the weights for recursive filtering. An alternative training strategy is proposed as an approximation to design the weighting vector for recursive filtering. The recursive design has been shown to provide better performance in noise suppression than its nonrecursive version.

References

[1] G. Arce, N.C. Gallagher, Stochastic analysis for the recursive median filter process, IEEE Trans. Inform. Theory 34 (July 1988) 669–679.

[2] A.C. Bovik, T. Huang, D.C. Munson, A generalization of median filtering using linear combinations of order statistics, IEEE Trans. Acoust., Speech, Signal Processing ASSP-31 (6) (1983) 1342–1350.

[3] S. Haykin, Adaptive Filter Theory, Third Edition, Prentice-Hall, Englewood Cliffs, NJ, 1996.

[4] C. Kotropoulos, I. Pitas, Constrained adaptive LMS *L*-filters, Signal Processing 26 (1992) 335–358.

[5] C. Kotropoulos, I. Pitas, Adaptive LMS *L*-filters for noise suppression in images, IEEE Trans. Image Processing 5 (12) (1996) 1596–1609.

[6] H.G. Longbotham, A.C. Bovik, Theory of order statistic filters and their relationship to linear FIR filters, IEEE Trans. on Acoust., Speech, Signal Processing 37 (2) (1989), pp. 275–287.

[7] O. Macchi, Adaptive Processing: The Least Mean Squares Approach with Applications in Transmission, Wiley, New York, 1995.

[8] L. Naaman, A.C. Bovik, Least-squares order statistic filters for signal restoration, IEEE Trans. Circuits Syst. CAS-38 (3) (1991) 244–257.

[9] T.A. Nodes, N.C. Gallagher Jr., Median filters: Some modifications and their properties, IEEE Trans. Acoust., Speech, Signal Processing ASSP-30 (5) (1982) 739–746.

[10] I. Pitas, A.N. Venetsanopoulos, Nonlinear Digital Filters: Principles and Applications, Kluwer Academic Publishers, Boston, 1990.

[11] I. Pitas, A.N. Venetsanopoulos, Order statistics in digital image processing, Proc. IEEE 80 (12) (1992) 1893–1921.

[12] B. Widrow, S.D. Streams, Adaptive Signal Processing, Prentice-Hall, Englewood Cliffs NJ, 1985.