AN ITERATIVE METHOD FOR RESTORING NOISY IMAGES

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ABSTRACT

A new iterative image restoration method is presented which incorporates a priori knowledge concerning the image and noise statistics directly in the iterative procedure. The iterative algorithm is computationally efficient in that only a small number of computations per pixel are required and appears to exhibit neither high noise sensitivity nor significant loss of resolution. It is demonstrated that for image signal-to-noise ratio, $\mathcal{L}$, greater than some $\mathcal{L}_{min}$, the procedure converges to the best mean-square estimate of the image. The value of $\mathcal{L}_{min}$ is derived and shown to depend on the correlation parameters of the image model. The basic iterative algorithm is then modified so that the modified algorithm converges to the best mean-square estimate of the image for all values of $\mathcal{L}$. An interesting feature of this technique is that the noisy observed image is taken as the initial approximation to the best estimate. In general, an attractive advantage of iterative algorithms for image restoration is that they readily facilitate man-machine interaction.

1.0 IMAGE MODEL

Consider the brightness or gray-level function of an $(m \times m)$-dimensional noisy observed image arranged in lexicographic order as an $(m^2 \times 1)$-dimensional vector $\underline{g}$. Denote the brighness function of the original image and the additive noise as the $(m^2 \times 1)$-dimensional vectors $\underline{f}$ and $\underline{\eta}$, respectively. The observed image is given by,

$$\underline{g} = \underline{f} + \underline{\eta} \tag{1}$$

The problem of image restoration can be stated as that of obtaining an estimate of $\underline{f}$, denoted by $\hat{\underline{f}}$, given $\underline{g}$ and certain a priori knowledge of the statistical nature of the noise $\underline{\eta}$. If the original image is modeled as a sample function from a zero-mean, stationary stochastic process with covariance R_f, the best linear mean-square estimate of $\underline{f}$ based on $\underline{g}$ is obtained by the Wiener filter[1],

$$\begin{aligned}\hat{\underline{f}} &= R_f(R_f + R_\eta)^{-1}\underline{g} \\ &= (R_\eta^{-1} + R_f^{-1})^{-1}R_\eta\,\underline{g}\end{aligned} \tag{2}$$

where R_η is the covariance matrix of $\underline{\eta}$. If the noise is white, then R_η is of the particularly simple form $R_\eta = \sigma_\eta^2 I$, where σ_η^2 is the noise variance. In many practical image models, the autocovariance function of $\underline{f}$ is assumed to be separable first-order Markovian[2,3], i.e.,

$$E\{f_{i+k,j+\ell}\, f_{ij}\} = \sigma^2 \rho_1^{|k|} \rho_2^{|\ell|} \tag{3}$$

where $0 < \rho_1, \rho_2 < 1$ are the one-step correlation parameters in the vertical and horizontal image directions and σ^2 is the variance of the original image. The covariance matrix R_f corresponding to (3) may be written as,

$$R_f = \sigma^2 R_1 \otimes R_2 \tag{4}$$

in which R_1 and R_2 are positive-definite first-order Markovian Toeplitz Matrices having ρ_i^j $j = 0,1,2,\ldots, m-1$; $i = 1,2$, respectively, as the first row of elements and $\otimes$ denotes the Kronecker product. The matrix $R_1 \otimes R_2$ is also positive-definite.

Based on the first expression in (2), the computation for $\hat{\underline{f}}$ may be carried out in two steps:

Step 1. Let $(R_f + R_\eta)\underline{x} = \underline{g}$. Calculate $\underline{x}$ as the solution to this system of linear equations, viz., $\underline{x} = (R_f + R_\eta)^{-1}\underline{g}$.

Step 2. Calculate the matrix-vector product $R_f\underline{x}$.

In the following sections, we concentrate on developing an efficient iterative procedure for *Step 1*.

2.0 RELEVANT ITERATIVE METHODS

Given a linear system $A\underline{x} = \underline{b}$, let the positive-definite coefficient matrix A be written as $A = B + C$ where B is positive-definite. We now have the system,

$$(B + C)\,\underline{x} = \underline{b} \tag{5}$$

We employ the first-order, stationary, linear Jacobi method[4] to solve for $\underline{x}$. Such an iterative procedure has $\underline{x}_o = B^{-1}\underline{b}$ and,

$$\underline{x}_n = -B^{-1}C\,\underline{x}_{n-1} + B^{-1}\underline{b} \qquad n = 1,2,\ldots \tag{6}$$

It is desired that $\underline{x}_j$ approach $A^{-1}\underline{b}$ as $n \to \infty$. The error at the nth iteration is given by,

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$$\underline{e}_n := \underline{x} - \underline{x}_n$$
$$= (-B^{-1}C)\underline{e}_{n-1} = (-B^{-1}C)^n \underline{e}_o \tag{7}$$

The matrix E: $= B^{-1}C$ is called the iteration matrix and may be written as,

$$E = I - B^{-1}A \tag{8}$$

It can be easily shown that necessary and sufficient conditions for the iterative procedure defined by (6) to converge to the solution of $A\underline{x} = \underline{b}$ are:

(a) $E^n \to [0]$ as $n \uparrow \infty$

(b) Every eigenvalue of E be less than unity in absolute value.

Let $\lambda(\cdot)$ denote the spectral radius of the indicated matrix. The asymptotic rate of convergence of the iterative procedure is given by,

$$R(E^n) := -(1/n)\log[\lambda(E^n)]$$
$$= -\log[\lambda(E)] = R(E) \tag{9}$$

The rate R(E) can be employed as an indication of the "effectiveness" of each iteration.

The rate of convergence of the above procedure can be improved by using an extrapolated Jacobi method. In this case $\underline{x}_o = \omega B^{-1}\underline{b}$, where ω is a scalar parameter, and,

$$\underline{x}_n = (I - \omega B^{-1}A)\underline{x}_{n-1} + \omega B^{-1}\underline{b} \qquad n = 1,2,\ldots \tag{10}$$

Denote $\lambda(I - B^{-1}A)$ by $\tilde{\mu}$ and $\lambda(I - \omega B^{-1}A)$ by μ; clearly, we have $(1-\mu) = \omega(1-\tilde{\mu})$. Suppose that all elements of the eigenspectrum of $B^{-1}A$ are strictly positive. Let λ be an eigenvalue of $B^{-1}A$; then $\tilde{\mu} = 1-\lambda, \mu = 1-\omega\lambda$. From the above, we must have $|1-\lambda| < 1$ for the Jacobi method to converge and $|1-\omega\lambda| < 1$ for the extrapolated method to converge. Using the maximum eigenvalue, λ_{max}, the conditions are $\lambda_{max} < 2$ and $\omega\lambda_{max} < 2$, respectively. The best choice for ω is given by,

$$\omega = 2/(\lambda_{max} + \lambda_{min}) \tag{11}$$

in which case,

$$\mu = |1-\omega\lambda| \leq \left|\frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}}\right| < 1 \tag{12}$$

From (12), we see that the extrapolated method is most effective when the spread $\lambda_{max}/\lambda_{min}$ is close to unity.

3.0 ITERATIVE ALGORITHM FOR RESTORATION - COMPLEXITY

The iterative approach to solving the system defined in *Step 1* of Section 1.0 may now be immediately obtained from (6). The recurrence with $\underline{x}_o = R_f^{-1}\underline{g}$ is,

$$\underline{x}_n = -R_f^{-1}R_\eta \underline{x}_{n-1} + R_f^{-1}\underline{g} \qquad n = 1,2,\ldots \tag{13}$$

Note that the initial step uses the observed noisy image $\underline{g}$. The iterative procedure may be terminated when $\|\underline{x}_n - \underline{x}_{n-1}\| < \xi$, where the tolerance ξ is a predetermined small quantity. We first examine the computational complexity of (13).

The formation of each iterate may be carried out in two steps.

Step 1a. Compute $\underline{y}_{n-1} = -R_f^{-1}R_\eta \underline{x}_{n-1}$

Step 1b. Compute $\underline{x}_n = \underline{y}_{n-1} + \underline{x}_o$

From (4) and the fact that $R_\eta = \sigma_\eta^2 I$, the computation of $\underline{y}_{n-1}$ is given by,

$$\underline{y}_{n-1} = -\frac{\sigma_\eta^2}{\sigma^2}(R_1^{-1} \otimes R_2^{-1})\underline{x}_{n-1} \tag{14}$$

If the $(m^2 \times 1)$-dimensional vectors $\underline{x}_n$, $\underline{y}_n$ are arranged as matrices X_n and Y_n, respectively, each of order m, we can write (14) equivalently as,

$$Y_{n-1} = -\frac{\sigma_\eta^2}{\sigma^2} R_1^{-1} X_{n-1} R_2^{-1} \tag{15}$$

using the fact that R_i $i = 1,2$ are symmetric. Using methods similar to those found in [5], it is easy to show that $2m^2$ multiplications and $2m(m-1)$ additions are required to compute $(\sigma_\eta^2/\sigma^2)R_1^{-1} X_{n-1}$. Right multiplication by R_1^{-1} requires an equal number of operations; thus, the total complexity per iteration of (13) is $4m^2$ multiplications and $m(5m-4)$ additions.

Say $\|\underline{x}_N - \underline{x}_{N-1}\| < \xi$, then as per *Step 2* of Section 1,

$$\hat{\underline{f}} = R_f \underline{x}_N \quad \text{or} \quad \hat{\underline{f}} = R_1 X_n R_2 \tag{16}$$

From (13) we observe that the product indicated in the first form of (16) is unnecessary. Write (13) in the following manner at termination,

$$R_f \underline{x}_N = -R_\eta \underline{x}_{N-1} + \underline{g} \tag{17}$$

Therefore, we have,

$$\hat{\underline{f}} = -R_\eta \underline{x}_{N-1} + \underline{g} \tag{18}$$

and do not require explicit evaluation of $\underline{x}_N$. Once $\underline{x}_{N-1}$ is known, the estimate $\hat{\underline{f}}$ can be obtained in m^2 multiplications and additions.

Steps 1 and *2*, performed in the above-described manner require $5m^2$ multiplications and $m(6m-4)$ additions. Since the image has m^2 pixels, we have 5 multiplications and less than 6 additions per pixel per iteration. The computational complexity is seen to be independent of image dimensions.

4.0 ITERATIVE ALGORITHM FOR RESTORATION - CONVERGENCE

Using (6), (8), and (14), the iteration matrix for the proposed technique is given by,

$$E = -\frac{\sigma_\eta^2}{\sigma^2}\,(R_1^{-1} \otimes R_2^{-1}) \tag{19}$$

Since R_1 and R_2 are positive-definite, $R_1^{-1} \otimes R_2^{-1}$ is also positive-definite. Denote by λ_{min} and λ_{max}, the smallest and largest eigenvalues (in absolute value) of E; then, it is straight-forward to show that,

$$\lambda_{min} \geq \frac{\sigma_\eta^2}{\sigma^2}\left(\frac{1-\rho_1}{1+\rho_1}\right)\left(\frac{1-\rho_2}{1+\rho_2}\right)$$

and

$$\lambda_{max} \leq \frac{\sigma_\eta^2}{\sigma^2}\left(\frac{1+\rho_1}{1-\rho_1}\right)\left(\frac{1+\rho_2}{1-\rho_2}\right) \tag{20}$$

From the results of Section 2.0, the iterates $\underline{x}_n$ $n = 0,1,2,\ldots$, converge if and only if,

$$\left(\frac{1+\rho_1}{1-\rho_1}\right)\left(\frac{1+\rho_2}{1-\rho_2}\right) < \sigma^2/\sigma_\eta^2 := \mathcal{L} \tag{21}$$

where $\mathcal{L}$ is the average image signal-to-noise ratio. For example, if $\rho_1 = \rho_2 = 0.7$, then (21) implies that the iterates converge if $\mathcal{L} > 15.1$ dB. The asymptotic rate of convergence of the iterative method is given, from (9) and (20), by

$$R(E) = \log\left[\mathcal{L}\left(\frac{1-\rho_1}{1+\rho_1}\right)\left(\frac{1-\rho_2}{1+\rho_2}\right)\right] \tag{22}$$

We note from (2) that we can interchange the rôles of R_f and R_η in (13) to obtain an equivalent recursion having $\underline{x}_o = R_\eta^{-1}\underline{g}$, i.e.,

$$\underline{x}_n = -R_\eta^{-1} R_f\, \underline{x}_{n-1} + R_\eta^{-1}\,\underline{g} \qquad n = 1,2,\ldots \tag{23}$$

An analysis similar to that above establishes the computational complexity of the iterative procedure using (23) to be of the same order. The approach using (23), however, differs in that convergence is assured if and only if,

$$\mathcal{L} < \left(\frac{1-\rho_1}{1+\rho_1}\right)\left(\frac{1-\rho_2}{1+\rho_2}\right) \tag{24}$$

For example, if $\rho_1 = \rho_2 = 0.7$, then (24) implies that the iterates converge if $\mathcal{L} < -15.1$ dB.

Clearly, the rate of convergence in both cases depends on $\mathcal{L}$ and the correlation parameters (ρ_1, ρ_2). Since the number of iterations depends on the rate of convergence, the total computational complexity also depends on $\mathcal{L}$ and (ρ_1,ρ_2). Such a result is highly desirable for images having $\mathcal{L}$ above (or below) a certain threshold. Since this may not always be the case, we investigate a modification to the iterative scheme for moderate $\mathcal{L}$.

5.0 NEW ALGORITHM FOR MODERATE $\mathcal{L}$

There are several approaches possible to insure convergence of (13) when $\mathcal{L}$ is between the two extremes previously mentioned. We investigate an approach which involves writing the coefficient matrix as,

$$R_f + R_\eta = R_f(1+\alpha) + (R_\eta - \alpha R_f) \tag{25}$$

where the bias $\alpha > 0$. We then define the recurrence for $\underline{x}_o = (1+\alpha)^{-1} R_f^{-1}\underline{g}$, by,

$$\underline{x}_n = -(1+\alpha)^{-1}R_f^{-1}(R_\eta - \alpha R_f)\underline{x}_{n-1} + (1+\alpha)^{-1} R_f^{-1}\,\underline{g} \qquad n = 1,2,\ldots \tag{26}$$

The iteration matrix associated with (26) is given as,

$$E = -(1+\alpha)^{-1} R_f^{-1}(R_\eta - \alpha R_f) = -(1+\alpha)^{-1}(R_f^{-1}R_\eta - \alpha\, I) \tag{27}$$

The smallest and largest eigenvalues (in absolute value) of E are,

$$\lambda_{min} = (1+\alpha)^{-1}\left[\frac{\sigma_\eta^2}{\sigma^2}\left(\frac{1-\rho_1}{1+\rho_1}\right)\left(\frac{1-\rho_2}{1+\rho_2}\right) - \alpha\right]$$

and

$$\lambda_{min} = (1+\alpha)^{-1}\left[\frac{\sigma_\eta^2}{\sigma^2}\left(\frac{1+\rho_1}{1-\rho_1}\right)\left(\frac{1+\rho_2}{1-\rho_2}\right) - \alpha\right] \tag{28}$$

We choose the value of α such that λ_{max} and λ_{min} are equal and opposite in magnitude (cf(11)), i.e.,

$$\alpha = \frac{\sigma_\eta^2}{\sigma^2}\left[\left(\frac{1+\rho_1}{1-\rho_1}\right)\left(\frac{1+\rho_2}{1-\rho_2}\right) + \left(\frac{1-\rho_1}{1+\rho_1}\right)\left(\frac{1-\rho_2}{1+\rho_2}\right)\right] \tag{29}$$

Now, the condition for convergence,

$$(1+\alpha)^{-1}\,\frac{\sigma_\eta^2}{2\sigma^2}\left[\left(\frac{1+\rho_1}{1-\rho_1}\right)\left(\frac{1+\rho_2}{1-\rho_2}\right) - \left(\frac{1-\rho_1}{1+\rho_1}\right)\left(\frac{1-\rho_2}{1+\rho_2}\right)\right] < 1 \tag{30}$$

is always satisfied if (29) is inserted in (30). The rate of convergence depends on $\mathcal{L}$ and (ρ_1,ρ_2) as before. However, we note that if $\rho_1 = \rho_2 = 0.7$ and $\mathcal{L} = 10$ (10dB), then $\alpha = 1.61$ from (29) and the recurrence defined in (25) converges with a rate of convergence, $R(E) = -\log_e(0.615) = 0.486$. This indicates a very small number of iterations are required to achieve excellent restoration quality at moderate signal-to-noise ratio.

The multiplicative complexity for the new, biased algorithm is $5m^2$ multiplications and $m(6m-4)$ additions per iteration. To compute the initial value, $\underline{x}_o$, an additional $4m^2$ multiplications and $4m(m-1)$ additions are required per iteration. The total count is 9 multiplications and less than 10 additions per pixel per iteration.

6.0 CONCLUSIONS

An iterative approach to image restoration based on the first-order Jacobi method has been presented. Assuming a widely used first-order Markovian image correlation model and image corruption by additive white noise, explicit relations are provided linking the convergence of the method with the average signal-to-noise ratio, $\mathcal{L}$, and the horizontal and vertical one-step correlation parameters (ρ_1,ρ_2). It is found that the unmodified iterative approach works well for high $\mathcal{L}$ and, therefore, may be considered a "touching-up" procedure. A new algorithm is presented by introducing a bias parameter in the Jacobi method. The bias parameter is based on a priori knowledge (or estimates) of $\mathcal{L}$ and (ρ_1,ρ_2) and effectively insures convergence of the iterative method for low-to-moderate values of $\mathcal{L}$. Preliminary experiments indicate that the approach exhibits neither the typical high noise sensitivity nor significant loss of image resolution. Either iterative method can be implemented using digital hardware that operates on the image on a column-by-column and row-by-row basis. We also mention that it is possible to extend the results presented here to the case where each pixel is a vector in order to include restoration in color imagery.

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