**Author:** Harald Uhlig
**Year**: 1999
**Name**: A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily

Fernández-Villaverde, Rubio-Ramírez, Schorfheide (2016) Handbook of Macroeconomics

# Syllabus

### Goal of Paper:

Use perturbation method to solve DSGE

## Theoretical Part

### Modelling principles

Modern macroeconomic theory is applied dynamic general equilibrium analysis. To spell out such a theory, one needs to explicitly specify the

***Environment***

- **T** technologies
- **I** information
- **P** preferences
- **E** endowments

***Object of Study***

- The *social planners problem* In that case, one needs to specify the planners objective function.
- The *competitive equilibrium* In that case, one needs to specify the markets and provide a definition of an equilibrium. In particular, one needs to spell out the precise extent of market powers.
- The *game* In that case, one needs to specify the rules and to provide a definition of an equilibrium

## Algorithm Part

✓

**The General Procedure:**

1. **Algebraic Equation System:** Find the necessary equations characterizing the equilibrium
   - First-order conditions
   - Constraints
   - Exogenous Shocks
2. **Steady-State Calculation:** Pick parameters and find the steady states (s).
3. **Log-Linearization:** Log-linearize the necessary equations characterizing the equilibrium of the system to make the equations approximately linear in the log-deviations from the steady state
4. **Recursive Law of Motion Solving:** Solve for the recursive equilibrium law of motion via the method of undetermined coefficients
5. **Impulse-Response Analysis:** Analyze the solution via impulse-response analysis

## Mathematical Preliminary:

### *Log-linearization*

The principle of log-linearization is to use a Taylor approximation around the steady state to replace all equations by approximations

$$x_t = \log X_t - \log \overline{X}$$

$$\Rightarrow \begin{cases} 1 \equiv f(0,0) = f(x_t, x_{t-1}) \\ 1 \equiv g(0,0) = \mathbb{E}_t[g(x_{t+1}, x_t)] \end{cases}$$

$$\Downarrow$$

$$\begin{cases} 0 \approx f_1 x_t + f_2 x_{t-1} \\ 0 \approx \mathbb{E}_t\left[g_1 x_{t+1} + g_2 x_t)\right] \end{cases}$$

### *Infinitesimal Approximation*

Apply the infinitesimal approximation, eliminate the constant (the steady-state value)

$$\begin{aligned} e^{x_t + a y_t} &\approx 1 + x_t + a y_t \\ x_t y_y &\approx 0 \\ \mathbb{E}_t[a e^{x_{t+1}}] &\approx \mathbb{E}_t[a x_{t+1}] \quad \text{up to a constant} \end{aligned}$$

Example:

$$\begin{aligned} e^{x_t} &\approx 1 + x_t \\ aX_t &\approx a\bar{X}x_t \text{ up to a constant} \\ (X_t + a)Y_t &\approx \bar{X}\bar{Y}x_t + (\bar{X} + a)\bar{Y}y_t \text{ up to a constant} \end{aligned}$$

# Economic Model Examples

## Stochastic Neoclassical Growth Model

- *The environment:*
  1. *Preferences*: The representative agent experiences utility according to

$$U = E\left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta} - 1}{1-\eta}\right]$$

  2. *Technologies*: We assume a Cobb-Douglas production function

$$C_t + K_t = Z_t K_{t-1}^{\rho} N_t^{1-\rho} + (1-\delta)K_{t-1}$$

$$\log Z_t = (1-\psi)\log \overline{Z} + \psi \log Z_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d.\mathcal{N}(0, \sigma^2)$$

  3. *Endowment*: Each period,
     1. the representative agent is endowed with one unit of time $N_t = 1$
     2. he is endowed with capital $K_{-1}$ before $t = 0$
  4. *Information*: $C_t$, $N_t$ and $K_t$ need to be chosen based on all information $\mathcal{I}_t$ up to time $t$
- *The social planners problem:*

- *Optimization Problem:*

$$
\begin{aligned}
&\max_{C_t,K_t} E\left[\sum_{t=0}^{\infty}\beta^t\frac{C_t^{1-\eta}-1}{1-\eta}\right]\\
&s.t.\ K_{-1}, Z_0\\
&\qquad C_t+K_t=Z_tK_{t-1}^{\rho}+(1-\delta)K_{t-1}\\
&\qquad \log Z_t=(1-\psi)\log\overline{Z}+\psi\log Z_{t-1}+\epsilon_t\\
&\qquad \epsilon_t\sim i.i.d.\mathcal{N}(0,\sigma^2)
\end{aligned}
$$

- *The competitive equilibrium:*
  - *Optimization Problem:*
    - Households:

$$
\begin{aligned}
&\max_{C_t,K_t} E\left[\sum_{t=0}^{\infty}\beta^t\frac{C_t^{1-\eta}-1}{1-\eta}\right]\\
&s.t.\ N_t=1\\
&\qquad C_t+K_t=W_tN_t+R_tK_{t-1}\ \text{(Budget Constraint)}\\
&\qquad 0=\lim E_0\Pi_{s=1}^{t}R_t^{-1}K_t\ \text{(Non-Ponzi Condition)}
\end{aligned}
$$

    - Firms:

$$
\begin{aligned}
&\max_{N_t,K_{t-1}} Z_tK_{t-1}^{\rho}N_t^{1-\rho}+(1-\delta)K_{t-1}-W_tN_t-R_tK_{t-1}\\
&\quad s.t.\ \log Z_t=(1-\psi)\log\overline{Z}+\psi\log Z_{t-1}+\epsilon_t\\
&\qquad\quad \epsilon_t\sim i.i.d.\mathcal{N}(0,\sigma^2)
\end{aligned}
$$

    - Markets Clear:
      - [Capital Market Clear:] $K_{t-1}^{(s)}=K_{t-1}^{(d)}$
      - [Labor Market Clear:] $N_t^{(s)}=N_t^{(d)}$

- [Goods Market Clear:] $C_t + K_t = Z_t K_{t-1}^{\rho} + (1-\delta)K_{t-1}$

- *Algebraic Equation:*

$$
\text{F.O.Cs: } \begin{cases} 1 = E_t\left[\beta\left(\frac{C_t}{C_{t+1}}\right)^{\eta} R_{t+1}\right] \Leftarrow \text{ Euler Equation} \\ 0 = \lim_{T\to\infty} E_0[\beta^T C_T^{-\eta} K_T] \Leftarrow \text{ Transversality} \\ R_t = \rho Z_t K_{t-1}^{\rho-1} + (1-\delta) \end{cases}
$$

$$
\text{Budget Constraint: } C_t = Z_t K_{t-1}^{\rho} + (1-\delta)K_{t-1} - K_t
$$

$$
\text{Exogenous Shocks: } \begin{cases} \log Z_t = (1-\psi)\log \overline{Z} + \psi \log Z_{t-1} + \epsilon_t \\ \epsilon_t \sim i.i.d. \mathcal{N}(0, \sigma^2) \end{cases}
$$

# Perturbation Algorithm

## Steady-State

$$
\begin{aligned}
\overline{C} &= \overline{ZK}^{\rho} + (1-\delta)\overline{K} - \overline{K} \\
\overline{R} &= \rho \overline{ZK}^{\rho-1} + (1-\delta) \\
1 &= \beta \overline{R}
\end{aligned}
$$

## Log-Linearization

*Budget Constraint*

$$
C_t = Z_t K_{t-1}^{\rho} + (1-\delta)K_{t-1} - K_t
$$

$$\Downarrow$$

$$c_t \approx \frac{\overline{Y}}{\overline{C}} z_t + \frac{\overline{K}}{\overline{C}} \overline{R} k_{t-1} - \frac{\overline{K}}{\overline{C}} k_t$$

1. For the first equation, the feasibility constraint, one obtains:

$$\begin{aligned}
C_t &= Z_t K_{t-1}^{\rho} + (1-\delta) K_{t-1} - K_t \\
\bar{C} e^{c_t} &= \bar{Z} \bar{K}^{\rho} e^{z_t + \rho k_{t-1}} + (1-\delta) \bar{K} e^{k_{t-1}} - \bar{K} e^{k_t} \\
\bar{C} + \bar{C} c_t &\approx \bar{Z} \bar{K}^{\rho} + (1-\delta) \bar{K} - \bar{K} \\
& \quad + \bar{Z} \bar{K}^{\rho} (z_t + \rho k_{t-1}) + (1-\delta) \bar{K} k_{t-1} - \bar{K} k_t
\end{aligned}$$

Use the steady state relationships

$$\begin{aligned}
\bar{Y} &= \bar{Z} \bar{K}^{\rho} \\
\bar{C} &= \bar{Y} - \delta \bar{K}
\end{aligned}$$

*F.O.C of Firms*

$$R_t = \rho Z_t K_{t-1}^{\rho-1} + (1-\delta)$$

$$\Downarrow$$

$$r_t \approx [1 - \beta(1-\delta)](z_t - (1-\rho) k_{t-1})$$

2. For the second equation, the calculation of the return, one gets

$$
\begin{aligned}
R_t &= \rho Z_t K_{t-1}^{\rho-1} + 1 - \delta \\
\bar{R} e^{r_t} &= \rho \bar{Z} \bar{K}^{\rho-1} e^{z_t + (\rho-1)k_{t-1}} + 1 - \delta \\
\bar{R} + \bar{R} r_t &\approx \rho \bar{Z} \bar{K}^{\rho-1} + 1 - \delta \\
& \quad + \rho \bar{Z} \bar{K}^{\rho-1}(z_t + (\rho - 1)k_{t-1})
\end{aligned}
$$

Use the steady state relationship

$$
\frac{1}{\beta} = \bar{R} = \rho \bar{Z} \bar{K}^{\rho} + 1 - \delta
$$

*Euler Equation*

$$
1 = E_t \left[ \beta \left( \frac{C_t}{C_{t+1}} \right)^{\eta} R_{t+1} \right]
$$

$$\Downarrow$$

$$
0 \approx E_t[\eta(c_t - c_{t+1}) + r_{t+1}]
$$

3. For the third equation, the the Lucas asset pricing equation, one gets

$$\begin{aligned}
1 &= E_t\left[\beta\left(\frac{C_t}{C_{t+1}}\right)^{\eta} R_{t+1}\right] \\
1 &= E_t\left[\beta\left(\frac{\bar{C}e^{c_t - c_{t+1}}}{\bar{C}}\right)^{\eta} \bar{R}e^{r_{t+1}}\right] \\
1 &\approx E_t\left[\beta\bar{R} + \beta\bar{R}(\eta(c_t - c_{t+1}) + r_{t+1})\right]
\end{aligned}$$

Use the steady state relationship

$$1 = \beta\bar{R}$$

*Exogenous Shocks*

$$\log Z_t = (1-\psi)\log\overline{Z} + \psi\log Z_{t-1} + \epsilon_t$$

$$\Downarrow$$

$$z_t \approx \psi z_{t-1} + \epsilon_t$$

4. For the fourth equation:

$$\begin{aligned}
\log Z_t &= (1-\psi)\log\bar{Z} + \psi\log Z_{t-1} + \epsilon_t, \\
\log(\bar{Z}e^{z_t}) &= (1-\psi)\log\bar{Z} + \psi\log(\bar{Z}e^{z_{t-1}}) + \epsilon_t, \\
z_t &= \psi z_{t-1} + \epsilon_t,
\end{aligned}$$

holding exactly.

## Undetermined Coefficients

Write the system with undetermined coefficient with state variables $k_{t-1}, z_t$

$$\left.\begin{aligned}
&c_t \approx \frac{\overline{Y}}{\overline{C}} z_t + \frac{\overline{K}}{\overline{C}} \overline{R} k_{t-1} - \frac{\overline{K}}{\overline{C}} k_t \\
&r_t \approx [1 - \beta(1-\delta)](z_t - (1-\rho) k_{t-1}) \\
&0 \approx E_t[\eta(c_t - c_{t+1}) + r_{t+1}] \\
&z_t \approx \psi z_{t-1} + \epsilon_t
\end{aligned}\right\} \iff \begin{cases}
k_t = \nu_{kk} k_{t-1} + \nu_{kz} z_t \\
r_t = \nu_{rk} k_{t-1} + \nu_{rz} z_t \\
c_t = \nu_{ck} k_{t-1} + \nu_{cz} z_t
\end{cases}$$

1. for the first equation ("feasibility"):

$$\begin{aligned} c_t &= \left(1+\delta\frac{\bar{K}}{\bar{C}}\right)z_t + \frac{\bar{K}}{\beta\bar{C}}k_{t-1} - \frac{\bar{K}}{\bar{C}}k_t \\ \nu_{ck}k_{t-1}+\nu_{cz}z_t &= \frac{\bar{Y}}{\bar{C}}z_t + \left(\frac{1}{\beta}-\nu_{kk}\right)\frac{\bar{K}}{\bar{C}}k_{t-1} - \frac{\bar{K}}{\bar{C}}\nu_{kz}z_t \end{aligned}$$

2. For the second equation ("calculation of the return"),

$$\begin{aligned} r_t &= (1-\beta(1-\delta))(z_t-(1-\rho)k_{t-1}) \\ \nu_{rk}k_{t-1}+\nu_{rz}z_t &= (1-\beta(1-\delta))(z_t-(1-\rho)k_{t-1}) \end{aligned}$$

3. For the third equation ("asset pricing"),

$$\begin{aligned} 0 &= E_t\left[\eta(c_t-c_{t+1})+r_{t+1}\right] \\ 0 &= E_t[\eta((\nu_{ck}k_{t-1}+\nu_{cz}z_t)-(\nu_{ck}k_t+\nu_{cz}z_{t+1})) \\ &\quad +\nu_{rk}k_t+\nu_{rz}z_{t+1}] \\ &= (\nu_{rk}-\eta\nu_{ck})k_t+\eta\nu_{ck}k_{t-1}+((\nu_{rz}-\eta\nu_{cz})\psi+\eta\nu_{cz})z_t \\ &= ((\nu_{rk}-\eta\nu_{ck})\nu_{kk}+\eta\nu_{ck})k_{t-1} \\ &\quad +((\nu_{rk}-\eta\nu_{ck})\nu_{kz}+(\nu_{rz}-\eta\nu_{cz})\psi+\eta\nu_{cz})z_t \end{aligned}$$

# Framework

## Situation I: Endo State + Exg Shock

$x_t$ is endogenous state variables, $z_t$ is exogenous shocks, all formula below is in Matrix Form

- **Algebraic Formula:**

$$\begin{cases} 0 = E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Lz_{t+1} + Mz_t] \\ z_{t+1} = Nz_t + \epsilon_{t+1}; \quad E_t[\epsilon_{t+1}] = 0 \end{cases} \tag{1}$$

- **Recursive Equilibrium Law of Motion:**

$$x_t = Px_{t-1} + Qz_t \tag{2}$$

**Theorem of $P$ and $Q$:**

If there is a recursive equilibrium law of motion solving equations (1) then the following must be true

1. $P$ satisfies the (matrix) quadratic equation

$$FP^2 + GP + H = 0$$

   The equilibrium described by the recursive equilibrium law of motion is stable iff all eigenvalues of $P$ are smaller than unity in absolute value
2. Given $P$, $Q$ is solved by the following formula

$$[N' \otimes F + I_k \otimes (FP + G)]Q = -vec(LN + M)$$

**Proof** Plugging the recursive equilibrium law of motion (2) into equation (1) twice and The coefficient matrices on $x_{t-1}$ and $z_t$ need to be zero.

## Situation II: Endo State + Endo Other Variables + Exg Shock

$x_t$ is endogenous state variables, $y_t$ is endogenous other variables, $z_t$ is exogenous shocks, all formula below is in Matrix Form

- **Algebraic Formula:**

$$\begin{cases} 0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t \\ 0 = E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Lz_{t+1} + Mz_t] \\ z_{t+1} = Nz_t + \epsilon_{t+1}; \quad E_t[\epsilon_{t+1}] = 0 \end{cases} \tag{3}$$

- **Recursive Equilibrium Law of Motion:**

$$\begin{cases} x_t = Px_{t-1} + Qz_t \\ y_t = Rx_{t-1} + Sz_t \end{cases} \tag{4}$$

**Theorem of $P$, $Q$ and $R$, $S$:**

If there is a recursive equilibrium law of motion solving equations (3) then the coefficient matrices can be found as follows

- $C^+$: the pseudoinverse of $C_{l\times l}$
- $C^0$: an $(l-n)\times l$ matrix, whose rows form a basis of the null space of $C'$

---

1. Given $P$ and $R$, $Q$ and $S$ satisfy the following Equations

$$V\begin{bmatrix} vec(Q) \\ vec(S) \end{bmatrix} = -\begin{bmatrix} vec(D) \\ vec(LN+M) \end{bmatrix}$$
$$\downarrow$$
$$V = \begin{bmatrix} I_k \otimes A & I_k \otimes C \\ N' \otimes F + I_k \otimes (FP + JR + G) & N' \otimes J + I_k \otimes K \end{bmatrix}$$

2. $R$ is given by

$$R = -C^+(AP+B)$$

3. $P$ satisfies the (matrix) quadratic equations

$$
\begin{aligned}
0 &= C^0AP + C^0B \\
0 &= (F - JC^+A)P^2 - (JC^+B - G + KC^+A)P - KC^+B + H
\end{aligned}
$$

⚠ 这里可以把$C^+$ 和 $C^0$ 理解为投影矩阵和残差矩阵

## Solving the matrix quadratic equation

To generally solve the matrix quadratic equations

$$\Psi P^2 - \Gamma P - \Theta = 0$$

### Solution

**General Eigenvalue and eigenvector**

Recall that a generalized eigenvalue $\lambda$ and eigenvector $s$ of a matrix $\Xi$ with respect to a matrix $\Delta$ are defined by

$$\lambda \Delta s = \Xi s$$

**LQ Problem $\sim$ General Eignvector Problem**

Define the $2m \times 2m$ matrices $\Xi$ and $\Delta$ via

$$\Xi = \begin{bmatrix} \Gamma & \Theta \\ I_m & 0_{m\times m} \end{bmatrix}$$

and

$$\Delta = \begin{bmatrix} \Psi & 0_{m\times m} \\ 0_{m\times m} & I_m \end{bmatrix}$$

1. $s$ can be written as

$$s = \begin{bmatrix} \lambda x \\ x \end{bmatrix}$$

2. If there are m generalized eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$ and corresponding m generalized eigenvectors $s_1, s_2, \cdots, s_m$, written as $s_i' = [\lambda_i x_i', x_i']$ for some $x_i \in \mathbb{R}^m$, and if $(x_1, x_2, \cdots, x_m)$ is linearly independent, then

$$P = \overbrace{\Omega}^{\Omega=[x_1,x_2,\cdots,x_m]} \underbrace{\Lambda}_{\Lambda=[\lambda,\lambda_2,\cdots,\lambda_m]} \Omega^{-1}$$