Author: Olivier Jean Blanchard and Charles M. Kahn
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Fernández-Villaverde, Rubio-Ramírez, Schorfheide (2016) Handbook of Macroeconomics

The Model

The model is given by

The structural model.

$$\begin{bmatrix} X_{t+1} \\ P_{t+1}^{(t)} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_t \tag{1.a}$$

- X_t: is an $n \times 1$ vector of variables predetermined at t
- P_t: is an $m \times 1$ vector of variables non-predetermined at t
- Z_t: is an $m \times 1$ vector of exogenous variables at t

Linear Rational Expectation

$$P_{t+1}^{(t)} = \mathbb{E}_t P_{t+1} | \Omega_t \tag{1.b}$$

- Ω_t: the information set at time t

$$\forall t, \exists \overline{Z}_t \in \mathbb{R}^k,\ \theta \in \mathbb{R}$$

such that

$$-(1+i)^{\theta_t}\overline{Z}_t \leq E(Z_{t+i}|\Omega_t) \leq (1+i)^{\theta_t}\overline{Z}_t \tag{1.c}$$

Predetermined or backward looking variables

- Functions only of variables known at time t time t, that is of variables in Ω_t. i.e.,

$$X_{t+1} = \mathbb{E}_t X_{t+1}|\Omega_t \quad \text{or} \quad X_{t+1} = f(X_t) + \epsilon$$

- Forward iteration:

$$X_{t+1} = f^{(t)} \circ X_0 + \sum_{i=1}^{t} f^{(i)} \circ \epsilon_{t-i}$$

Forward looking variables | Non-predetermined variables

- Variables following such formula

$$\mathbb{E}_t P_{t+1} = g(P_t) + \epsilon_t$$

- Forward iteration:

$$P_t = (g^{-1})^{(n)} \circ \mathbb{E}_t P_{t+n} - \sum_{i=0}^{n-1} (g^{-1})^{(i+1)} \circ \mathbb{E}_t \epsilon_{t+i}$$

Solution

BK Condition

Perspective of Jordan Canonical Form

- Simplify the notations:

$$Y_{t+1} = AY_t + \gamma Z_t$$

$\Downarrow$

- The ***Jordan Decomposition:***

$$A = C^{-1}JC$$

$\Downarrow$

- Ordering by magnitudes of eigenvalues

$$J = \begin{bmatrix} J_{1\,(\tilde{n}\times\tilde{n})} & 0 \\ 0 & J_{2\,(\tilde{m}\times\tilde{m})} \end{bmatrix} \quad C = \begin{bmatrix} C_{11\,(\tilde{n}\times n)} & C_{12\,(\tilde{n}\times m)} \\ C_{21\,(\tilde{m}\times n)} & C_{22\,(\tilde{m}\times m)} \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} B_{11\,(n\times\tilde{n})} & B_{12\,(m\times\tilde{m})} \\ B_{21\,(n\times\tilde{m})} & B_{22\,(m\times\tilde{m})} \end{bmatrix} \quad \gamma = \begin{bmatrix} \gamma_{1\,(\tilde{n}\times k)} \\ \gamma_{2\,(\tilde{m}\times k)} \end{bmatrix}$$

J_1: all eigenvalues are on or inside the unit circle
J_2: all eigenvalues are outside the unit circle

Blanchard-Kahn Condition:

- if the number of eigenvalues of A outside the unit circle is equal to the number of non-predetermined variables $\tilde{m} = m$, then there exists a unique solution

$$
\begin{bmatrix} X_{t+1} \\ P_{t+1}^{(t)} \end{bmatrix} = \begin{bmatrix} B_{11\,(n\times\tilde{n})} & B_{12\,(m\times\tilde{m})} \\ B_{21\,(n\times\tilde{m})} & B_{22\,(m\times\tilde{m})} \end{bmatrix} \begin{bmatrix} J_{1\,(\tilde{n}\times\tilde{n})} & 0 \\ 0 & J_{2\,(\tilde{m}\times\tilde{m})} \end{bmatrix} \begin{bmatrix} C_{11\,(\tilde{n}\times n)} & C_{12\,(\tilde{n}\times m)} \\ C_{21\,(\tilde{m}\times n)} & C_{22\,(\tilde{m}\times m)} \end{bmatrix} \begin{bmatrix} X_t \\ P_t \end{bmatrix}
+ \begin{bmatrix} \gamma_{1\,(\tilde{n}\times k)} \\ \gamma_{2\,(\tilde{m}\times k)} \end{bmatrix} Z_t
$$

$$\Downarrow$$

$$
C \begin{bmatrix} X_{t+1} \\ P_{t+1}^{(t)} \end{bmatrix} = \begin{bmatrix} J_{1\,(\tilde{n}\times\tilde{n})} & 0 \\ 0 & J_{2\,(\tilde{m}\times\tilde{m})} \end{bmatrix} C \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \begin{bmatrix} \gamma_{1\,(\tilde{n}\times k)} \\ \gamma_{2\,(\tilde{m}\times k)} \end{bmatrix} Z_t
$$

$$\Downarrow$$

$$
\begin{cases} \tilde{X}_{t+1} = J_1 \tilde{X}_t + \gamma_1 Z_t \\ \tilde{P}_{t+1} = J_2 \tilde{P}_t + \gamma_2 Z_t \end{cases}
$$

- if the number of eigenvalues outside the unit circle exceeds the number of non-predetermined variables $\tilde{m} > m$, then there exists a no solution
- if the number of eigenvalues outside the unit circle is lower than the number of non-predetermined variables $\tilde{m} < m$, then there exists a infinite solution

Perspective of Schur Decomposition

See Klein (2000)