The Framework:

The set of equilibrium conditions of a wide variety of dynamic general equilibrium models in macroeconomics can be written as

Rational Expectation:

$$\mathbb{E}_t f(y_{t+1}, y_t, x_{t+1}, x_t) = 0 \tag{1}$$

- x_t: predetermined variables is of size $n_x \times 1$, $x = [x_t^1; x_t^2]'$
  - x_t^1: *endogenous* predetermined state variables
  - x_t^2: *exogeneous* predetermined state variables

$$x_{t+1}^2 = \Lambda x_t^2 + \tilde{\eta}\sigma\epsilon_{t+1}, \quad \|\Lambda\| < 1$$

- y_t: non-predetermined variables is of size $n_y \times 1$
- $n = n_x + n_y$

Example: consider the simple neoclassical growth model.

$$\begin{aligned} c_t^{-\gamma} &= \beta \mathbb{E}_t c_{t+1}^{-\gamma}[\alpha A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta] \\ c_t + k_{t+1} &= A_t k_t^{\alpha} + (1-\delta)k_t \\ ln\ A_{t+1} &= \rho ln\ A_t + \sigma\epsilon_{t+1} \end{aligned}$$

Let $y_t = c_t$ and $x_t = [k_t, ln\ A_t]'$. Then

$$\mathbb{E}_t f(y_{t+1}, y_t, x_{t+1}, x_t) = \mathbb{E}_t \begin{bmatrix} y_t^{-\gamma} - \beta y_{t+1}^{-\gamma}[\alpha e^{x_{2,t+1}} x_{1,t+1}^{\alpha-1} - (1-\delta)] \\ y_t + x_{1,t+1} - e^{x_{2,t}} x_{1,t}^{\alpha} - (1-\delta)x_{1,t} \\ x_{2,t+1} - \rho x_{2,t} \end{bmatrix}$$

Solution:

$$y_t = g(x_t, \sigma) \tag{2}$$

$$x_{t+1} = h(x_t, \sigma) + \underbrace{\begin{bmatrix} \emptyset \\ \tilde{\eta} \end{bmatrix}}_{\eta} \sigma \epsilon_{t+1} \tag{3}$$

Non-stochastic Steady-state:

$$f(\bar{y}, \bar{y}, \bar{x}, \bar{x}) = 0$$

- $\bar{y} = g(\bar{x}, 0)$
- $\bar{x} = h(\bar{x}, 0)$

Approximation the Solution

Substituting the proposed solution given by Eqs. (2) and (3) into Eq. (1), we can define

$$\begin{aligned} F(x, \sigma) \equiv \mathbb{E}_t f[&g(h(x_t, \sigma) + \eta\sigma\epsilon_{t+1}, \sigma), \\ &g(x_t, \sigma), \\ &h(x_t, \sigma) + \eta\sigma\epsilon_{t+1}, \\ &x_t] = 0 \end{aligned}$$

Because $F(x, \sigma)$ must be equal to zero for any possible values of x and σ, it must be the case that the derivatives of any order of F must also be equal to zero

$$F_{x^k \sigma^j}(x, \sigma) = 0 \quad \forall x, \sigma, j, k$$

- ***First-order Derivative:***

$$\begin{aligned} g(x,\sigma) &= \overbrace{g(\bar{x},0)}^{\bar{y}} + g_x(\bar{x},0)(x-\bar{x}) + g_\sigma(\bar{x},0) \\ h(x,\sigma) &= \underbrace{h(\bar{x},0)}_{\bar{x}} + h_x(\bar{x},0)(x-\bar{x}) + h_\sigma(\bar{x},0) \end{aligned}$$

To pin down other terms, using $F_x(\bar{x},0)=0$ and $F_\sigma(\bar{x},0)$

$$\Downarrow$$

Einstein Notation

$$\begin{aligned} [F_x(\bar{x},0)]^i_j &= [f_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_x]^\beta_j + [f_y]^i_\alpha[g_x]^\alpha_j + [f_{x'}]^i_\beta[h_x]^\beta_j + [f_x]^i_j \\ &= 0; \quad i=1,\ldots,n; \quad j,\beta=1,\ldots,n_x; \quad \alpha=1,\ldots,n_y \\ [F_\sigma(\bar{x},0)]^i &= E_t\{[f_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_\sigma]^\beta + [f_{y'}]^i_\alpha[g_x]^\alpha_\beta[\eta]^\beta_\phi[\varepsilon']^\phi + [f_{y'}]^i_\alpha[g_\sigma]^\alpha + [f_y]^i_\alpha[g_\sigma]^\alpha \\ &\quad + [f_{x'}]^i_\beta[h_\sigma]^\beta + [f_{x'}]^i_\beta[\eta]^\beta_\phi[\varepsilon']^\phi\} \\ &= [f_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_\sigma]^\beta + [f_{y'}]^i_\alpha[g_\sigma]^\alpha + [f_y]^i_\alpha[g_\sigma]^\alpha + [f_{x'}]^i_\beta[h_\sigma]^\beta \\ &= 0; \quad i=1,\ldots,n; \quad \alpha=1,\ldots,n_y; \quad \beta=1,\ldots,n_x; \quad \phi=1,\ldots,n_\varepsilon. \end{aligned}$$

Notion $[f'_y]^i_\alpha[g_x]^\alpha_\beta[h_x]^\beta_j = \sum_{\alpha=1}\sum_{\beta=1}(\partial f^i/\partial y'^\alpha)(\partial g^\alpha/\partial x^\beta)(\partial h^\beta/\partial x^j)$

- ***Second-order Derivative:***

$$
\begin{aligned}
[g(x,\sigma)]^i = {} & [g(\bar{x},0)]^i + [g_x(\bar{x},0)]^i_\alpha[(x-\bar{x})]_\alpha + [g_\sigma(\bar{x},0)]^i[\sigma] \\
& + \frac{1}{2}[g_{xx}(\bar{x},0)]^i_{\alpha\beta}[(x-\bar{x})]_\alpha[(x-\bar{x})]_\beta \\
& + \frac{1}{2}[g_{\sigma\sigma}(\bar{x},0)]^i[\sigma][\sigma] \\
& + \frac{1}{2}[g_{x\sigma}(\bar{x},0)]^i_\alpha[(x-\bar{x})]_\alpha[\sigma] \\
& + \frac{1}{2}[g_{\sigma x}(\bar{x},0)]^i_\alpha[(x-\bar{x})]_\alpha[\sigma]
\end{aligned}
$$

$$
\begin{aligned}
[h(x,\sigma)]^j = {} & [h(\bar{x},0)]^j + [h_x(\bar{x},0)]^j_\alpha[(x-\bar{x})]_\alpha + [h_\sigma(\bar{x},0)]^j[\sigma] \\
& + \frac{1}{2}[h_{xx}(\bar{x},0)]^j_{\alpha\beta}[(x-\bar{x})]_\alpha[(x-\bar{x})]_\beta \\
& + \frac{1}{2}[h_{\sigma\sigma}(\bar{x},0)]^j[\sigma][\sigma] \\
& + \frac{1}{2}[h_{x\sigma}(\bar{x},0)]^j_\alpha[(x-\bar{x})]_\alpha[\sigma] \\
& + \frac{1}{2}[h_{\sigma x}(\bar{x},0)]^j_\alpha[(x-\bar{x})]_\alpha[\sigma]
\end{aligned}
$$

To pin down other terms, using $F_{xx}(\bar{x},0) = 0$ and $F_{\sigma\sigma}(\bar{x},0)$

$$\Downarrow$$

$$
\begin{aligned}
[F_{xx}(\bar{x},0)]^i_{jk} = {} & ([f_{y'y'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{y'y}]^i_{\alpha\gamma}[g_x]^\gamma_k \\
& + [f_{y'x'}]^i_{\alpha\delta}[h_x]^\delta_k + [f_{y'x}]^i_{\alpha k})[g_x]^\alpha_\beta[h_x]^\beta_j \\
& + [f_{y'}]^i_\alpha[g_{xx}]^\alpha_{\beta\delta}[h_x]^\delta_k[h_x]^\beta_j \\
& + [f_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_{xx}]^\beta_{jk} \\
& + ([f_{yy'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{yy}]^i_{\alpha\gamma}[g_x]^\gamma_k + [f_{yx'}]^i_{\alpha\delta}[h_x]^\delta_k + [f_{yx}]^i_{\alpha k})[g_x]^\alpha_j \\
& + [f_y]^i_\alpha[g_{xx}]^\alpha_{jk} \\
& + ([f_{x'y'}]^i_{\beta\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{x'y}]^i_{\beta\gamma}[g_x]^\gamma_k + [f_{x'x'}]^i_{\beta\delta}[h_x]^\delta_k + [f_{x'x}]^i_{\beta k})[h_x]^\beta_j \\
& + [f_{x'}]^i_\beta[h_{xx}]^\beta_{jk} \\
& + [f_{xy'}]^i_{j\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{xy}]^i_{j\gamma}[g_x]^\gamma_k + [f_{xx'}]^i_{j\delta}[h_x]^\delta_k + [f_{xx}]^i_{jk} \\
= {} & 0; \quad i = 1,\ldots,n, \quad j,k,\beta,\delta = 1,\ldots,n_x; \quad \alpha,\gamma = 1,\ldots,n_y.
\end{aligned}
$$

$$
\begin{aligned}
[F_{\sigma\sigma}(\bar{x},0)]^i = {} & [f_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_{\sigma\sigma}]^\beta \\
& + [f_{y'y'}]^i_{\alpha\gamma} [g_x]^\gamma_\delta [\eta]^\delta_\xi [g_x]^\alpha_\beta [\eta]^\beta_\phi [I]^\phi_\xi \\
& + [f_{y'x'}]^i_{\alpha\delta} [\eta]^\delta_\xi [g_x]^\alpha_\beta [\eta]^\beta_\phi [I]^\phi_\xi \\
& + [f_{y'}]^i_\alpha [g_{xx}]^\alpha_{\beta\delta} [\eta]^\delta_\xi [\eta]^\beta_\phi [I]^\phi_\xi \\
& + [f_{y'}]^i_\alpha [g_{\sigma\sigma}]^\alpha \\
& + [f_y]^i_\alpha [g_{\sigma\sigma}]^\alpha \\
& + [f_{x'}]^i_\beta [h_{\sigma\sigma}]^\beta
\end{aligned}
$$

$$
\begin{aligned}
[F_{\sigma x}(\bar{x},0)]^i_j = {} & [f_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_{\sigma x}]^\beta_j + [f_{y'}]^i_\alpha [g_{\sigma x}]^\alpha_\gamma [h_x]^\gamma_j + [f_y]^i_\alpha [g_{\sigma x}]^\alpha_j + [f_{x'}]^i_\beta [h_{\sigma x}]^\beta_j \\
= {} & 0; \quad i = 1, \ldots n; \quad \alpha = 1, \ldots, n_y; \quad \beta, \gamma, j = 1, \ldots, n_x.
\end{aligned}
$$

$$
g_\sigma(\bar{x},0) = 0, \quad h_\sigma(\bar{x},0) = 0, \quad g_{x\sigma}(\bar{x},0) = 0, \quad h_{x\sigma}(\bar{x},0) = 0
$$

Examples:

Example 1: The neoclassical growth model:

Parameters:

$$
\beta = 0.95, \delta = 1, \alpha = 0.3, \rho = 0, \gamma = 2
$$

Variables:

$$x_t = \begin{bmatrix} ln\ k_t \\ ln A_t \end{bmatrix}, y_t = \ln c_t$$

Non-Stochastic Steady-state:

$$\bar{x} = \begin{bmatrix} -1.7932 \\ 0 \end{bmatrix}, y_t = -0.8734$$