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Year: 2007
Name: Incomplete Markets, Heterogeneity and Macroeconomic Dynamics
Journal: NBER Working Paper 13260

Key Points:

- Select and construct the right state variables to approximate the aggregate distribution
- Additional restrictions from equivalence from first- or second-order approximations of aggregation law

KS Model Review

Model Settings

Households

- There are a continuum of agents with unit measure indexed $i \in [0,1]$
- Preferences

$$\mathbb{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \frac{c_{it}^{1-\gamma} - 1}{1-\gamma}$$

- Constraints:

- Budget constraint for capitals

$$a_{i,t+1} = (1-\delta)a_{i,t} + y_{i,t} - c_{i,t}$$
$$\downarrow$$
$$y_{i,t} = r_t a_{i,t} + w_t l_{i,t} = r_t a_{i,t} + w_t e_{i,t}\bar{l}, \begin{cases} e_{i,t} = (1-\rho_e)\mu_e + \rho_e e_{i,t} + \epsilon^e_{i,t+1} \\ \epsilon^e_{i,t+1} \sim N(0, \sigma^e) \end{cases}$$

- Borrowing constraint:

$$a_{i,t} \geq -b$$

Firms :

- Competitive firms produce output

$$y_t = z_t k_t^{\alpha} l_t^{1-\alpha}$$

- z_t: Aggregate TFP follows the AR(1) process $\begin{cases} z_{t+1} = (1-\rho_z)\mu_z + \rho_z z_t + \epsilon^z_{i,t+1} \\ \epsilon^z_{i,t+1} \sim N(0, \sigma^z) \end{cases}$
- k_t: aggregate capital stock $k_t = \int_0^1 a_{i,t} di$
- l_t: aggregate labor supply $l_t = \int_0^1 l_{i,t} di = \mu_e \bar{l}$

Solution

- **Firm optimization and market clearing:**

$$r(k_t, l_t, z_t) = \alpha z_t (k_t/l_t)^{\alpha-1}$$
$$w(k_t, l_t, z_t) = (1-\alpha) z_t (k_t/l_t)^{\alpha}$$

- **Evolution of distribution**

$$\Gamma_{t+1} = H(\Gamma_t, z_t)$$

- **Household optimization**
  - To impose the restriction, define the interior function

$$I(a_{i,t+1}) = \frac{1}{(a_{i,t+1}+b)^2}$$

> **Interior Methods for optimization problems subject to inequality constraints**
>
> The idea adopted here is to replace the problem of maximizing an objective function subject to this inequality constraint with an unconstrained maximization problem.
>
> 1. For small ϕ, the maximization problem satis…es the constraint $a_{i,t+1} \geq b$
> 2. When $a_{i,t+1}$ approaches b the interior function tends to dominate the value function, leading to large negative values.

  - Optimization (Dynamic Programming) Problem of HH

$$\begin{aligned} v(a_{i,t}, e_{i,t}; \Gamma_t, z_t) = & \max_{c_{i,t}, a_{i,t+1}} u(c_{i,t}) + \beta E_t v(a_{i,t+1}, e_{i,t+1}; \Gamma_{t+1}, z_{t+1}) \\ & - \phi I(a_{i,t+1}) \\ s.t. \quad & a_{i,t+1} = (1-\delta)a_{i,t} + r_t a_{i,t} + w_t e_{i,t} \bar{l} - c_{i,t} \end{aligned}$$

  - The FOC of households w.r.t asset levels

$$u_c(c_{i,t}) = E_t\Big[u_c(c_{i,t+1})(r(k_{t+1}, l_{t+1}, z_{t+1}) + 1 - \delta) - \phi I_a(a_{i,t+1})\Big]$$

Perturbation Method

The Representative Agent Model

To generate a representative agent model, assume that there are no idiosyncratic labor employment shocks and that each household inelastically supplies a unit of labor.

- **The Algebraic System:**

$$\star\ E_t F(c_{t+1}, c_t, x_{t+1}, x_t) = E_t \begin{bmatrix} c_t^{-\gamma} - \beta c_{t+1}^{-\gamma}(r_{t+1} + 1 - \delta) - \frac{2\phi}{(k_{t+1}+b)^3} \\ k_{t+1} - (1-\delta)k_t - r_t k_t - w_t \bar{l} + c_t \\ z_{t+1} - (1-\rho_z)\mu_z - \rho_z z_t - \epsilon^z_{i,t+1} \end{bmatrix} = 0$$

where $x_t = (k_t, z_t)'$

- **The Assumed Solution is set as**

$$\begin{cases} c_t & = g(x_t, \sigma) \\ x_{t+1} & = h(x_t, \sigma) + \eta\sigma\epsilon_{t+1} \end{cases}$$

where

- g, h: (1×1) and (2×1) dimensional functional

- $\sigma > 0$: scales the degree of uncertainty in ϵ_{t+1}, itself a (2×1) vector
- η is a (2×2) selection matrix, designating how primitive shocks enter the state equations.

- **The second order approximation of the functions g and h around the steady state $(x_t, \sigma) = (\overline{x}, 0)$:**

$$
\begin{aligned}
g(x,\sigma) = g(\overline{x},0) &+ \sum_m g_{x_m}(\overline{x},0)(x_m - \overline{x_m}) + g_\sigma(\overline{x},0)\sigma \\
&+ \frac{1}{2}\sum_{mn} g_{x_m x_n}(\overline{x},0)(x_m - \overline{x_m})(x_n - \overline{x_n}) \\
&+ \frac{1}{2}\sum_{m} g_{x_m \sigma}(\overline{x},0)(x_m - \overline{x_m})\sigma \\
&+ \frac{1}{2}\sum_{m} g_{\sigma\sigma}(\overline{x},0)\sigma^2
\end{aligned}
$$

and

$$
\begin{aligned}
h(x,\sigma)^j = h(\overline{x},0)^j &+ \sum_m h_{x_m}(\overline{x},0)^j(x_m - \overline{x_m}) + h_\sigma(\overline{x},0)^j\sigma \\
&+ \frac{1}{2}\sum_{mn} h_{x_m x_n}(\overline{x},0)^j(x_m - \overline{x_m})(x_n - \overline{x_n}) \\
&+ \frac{1}{2}\sum_{m} h_{x_m \sigma}(\overline{x},0)^j(x_m - \overline{x_m})\sigma + \frac{1}{2}\sum_{m} h_{\sigma x_m}(\overline{x},0)^j(x_m - \overline{x_m})\sigma \\
&+ \frac{1}{2}\sum_{m} h_{\sigma\sigma}(\overline{x},0)^j\sigma^2
\end{aligned}
$$

where $j, m, n = 1, 2$:

- j indexes the law of motion of the predetermined variable under consideration — either the capital stock or the technology shock
- m and n index the same two state variables in the construction of the approximation.

✓

Algorithm Steps:

1. Taking derivatives of $(\star)$ with respect to x and σ yields:

$$\begin{cases} F_{x_m} = 0 \Rightarrow (g_{x_m}, h^j_{x_m}) \quad \text{6 Equations, 6 Unkown} \\ F_\sigma = 0 \Rightarrow (g_\sigma, h^j_\sigma) \quad \text{3 Equations, 3 Unkown} \end{cases}, \quad j, m = 1, 2$$

2. Taking second-order derivatives of $(\star)$ yields:

$$\begin{cases} F_{x_m x_n} = 0 \Rightarrow (g_{x_m x_n}, h^j_{x_m x_n}) \\ F_{\sigma\sigma} = 0 \Rightarrow (g_{\sigma\sigma}, h^j_{\sigma\sigma}) \\ F_{x_m \sigma} = 0 \Rightarrow (g_{x_m \sigma}, h^j_{x_m \sigma}, g_{\sigma x_m}, h^j_{\sigma x_m}) \end{cases}, \text{17 Equations, 17 Unkown} \quad j, m, n = 1, 2$$

Heterogeneous Agent Model

Individual consumption and saving decisions, and therefore the aggregate capital stock, can now depend on an additional set of state variables relevant to describing the evolving distribution of wealth in the economy.

The Steady State

Assume that capital is equally distributed across agents in this steady state.

$\downarrow$

The cross-sectional distribution has unit probability mass on this aggregate quantity of capital

$\downarrow$

What the second order approximation does, is approximate the wealth distribution in the neighborhood of this degenerate wealth distribution.

State Variables Defined

Consider the set of state variables relevant to individual i's decision problem at the first order. $\{a_{i,t}, e_{i,t}, z_t\}$.

Noting that optimal decisions will be linear in these state variables in approximation

$\downarrow$

Look for an equilibrium solution to the model in which decisions are linear functions(?), at the first order, of the terms $\{a_{i,t}, e_{i,t}, z_t, k_t\}$

$\downarrow$

Which second order terms are relevant to the household's saving decision?

In principle, decisions could depend on all pair-wise combinations of $\{a_{i,t}, e_{i,t}, z_t, k_t\}$ appearing in a second order polynomial of these first-order state variables.

$$(a_{i,t}-\overline{a})(e_{i,t}-\overline{e}),(a_{i,t}-\overline{a})(k_t-\overline{k}),(a_{i,t}-\overline{a})(z_t-\overline{z}),$$
$$(a_{i,t}-\overline{a})^2,(e_{i,t}-\overline{e})^2,(e_{i,t}-\overline{e})(k_t-\overline{k}),(e_{i,t}-\overline{e})(z_t-\overline{z}),$$
$$(k_t-\overline{k})^2,(k_t-\overline{k})(z_t-\overline{z}),(z_t-\overline{z})^2$$

$$\Downarrow$$

$$\frac{\text{Individual saving decisions is linear in these state variables}\quad\text{\& Individual decisions must satisfy the aggregation constraint}}{}$$

$$\Downarrow$$

$$\text{Aggregate State Variables } = \begin{cases} \int_0^1 (a_{i,t}-\overline{a})(e_{i,t}-\overline{e})di \\ \int_0^1 (a_{i,t}-\overline{a})^2 di \\ \int_0^1 (a_{i,t}-\overline{a})(k_t-\overline{k})di = (k_t-\overline{k})^2 \\ \int_0^1 (a_{i,t}-\overline{a})(z_t-\overline{z})di = (k_t-\overline{k})(z_t-\overline{z}) \\ \int_0^1 (e_{i,t}-\overline{e})^2 di \\ \int_0^1 (z_t-\overline{z})^2 di \\ \int_0^1 (e_{i,t}-\overline{e})(k_t-\overline{k})di \equiv 0 \\ \int_0^1 (e_{i,t}-\overline{e})(z_t-\overline{z})di \equiv 0 \end{cases}$$

$$\Downarrow \qquad .$$

The second-order State Variables

$$
\begin{aligned}
\Psi_t &= \int_0^1 (a_{i,t} - \overline{a})(e_{i,t} - \overline{e})di \\
\Phi_t &= \int_0^1 (a_{i,t} - \overline{a})^2 di \\
\hat{k}_t^2 &= (k_t - \overline{k})^2 \\
\hat{z}_t^2 &= (z_t - \overline{z})^2 \\
\hat{k}_t \hat{z}_t &= (k_t - \overline{k})(z_t - \overline{z})
\end{aligned}
$$

while $\int_0^1 (e_{i,t} - \overline{e})^2 di$ is variance of e, a constant

$$\Downarrow \qquad .$$

The State Variable Set: $\{a_{i,t}, e_{i,t}, z_t, k_t, \Psi_t, \Phi_t\}$

Algebraic System

- **The Algebraic System:**

$$
E_t F(c_{t+1}, c_t, x_{t+1}, x_t) = \begin{bmatrix} F^c \\ F^a \\ F^k \\ F^\Phi \\ F^\Psi \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{aligned}
\S \\
=& E_t \left[\begin{array}{lc}
u_c(c_{i,t}) - \beta u_c(c_{i,t+1})(r_{t+1} + 1 - \delta) - \frac{2\phi}{(a_{i,t+1}+b)^3} & \text{Euler Equation} \\
a_{i,t+1} - (1-\delta)a_{i,t} - r_t a_{i,t} - w_t e_{i,t}\bar{l} + c_{i,t} & \text{Constraints} \\
\left. \begin{array}{l}
(h^k \equiv) k_{t+1} - \int_0^1 a_{i,t+1} di \\
(h^\Psi \equiv) \Psi_{t+1} - \int_0^1 (a_{i,t+1} - \bar{a})(e_{i,t+1} - \bar{e}) di \\
(h^\Phi \equiv) \Phi_{t+1} - \int_0^1 (a_{i,t+1} - \bar{a})^2 di
\end{array} \right\} & \text{Aggregate Law} \\
\left. \begin{array}{l}
z_{t+1} - (1-\rho_z)\mu_z - \rho_z z_t - \epsilon^z_{i,t+1} \\
e_{i,t+1} - (1-\rho_e)\mu_e - \rho_e e_{i,t} - \epsilon^e_{i,t+1}
\end{array} \right\} & \text{Exogenous Shock}
\end{array} \right] \\
=& 0
\end{aligned}
$$

where $x_t = (a_{i,t}, e_{i,t}, z_t, k_t, \Psi_t, \Phi_t)'$

- **The Assumed Solution is set as**

$$
\begin{cases}
c_t = g(a_{i,t}, e_{i,t}, z_t, k_t, \Psi_t, \Phi_t, \sigma) \\
x_{t+1} = \begin{bmatrix} a_{i,t+1} \\ e_{i,t+1} \\ z_{t+1} \\ k_{t+1} \\ \Psi_{t+1} \\ \Phi_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} h^a(a_{i,t}, e_{i,t}, z_t, k_t, \Psi_t, \Phi_t, \sigma) \\ h^e(e_{i,t}) \\ h^z(z_t) \\ h^k(z_t, k_t, \Psi_t, \Phi_t, \sigma) \\ h^\Psi(z_t, k_t, \Psi_t, \Phi_t, \sigma) \\ h^\Phi(z_t, k_t, \Psi_t, \Phi_t, \sigma) \end{bmatrix}}_{h(x_t,\sigma)} + \eta\sigma\epsilon_{t+1}
\end{cases}
$$

where

- g, h: (1×1) and (6×1) dimensional functional
- $\sigma > 0$: scales the degree of uncertainty in ϵ_{t+1}, itself a (6×1) vector
- η is a (6×6) selection matrix, designating how primitive shocks enter the state equations.

- **The second order approximation of the functions g and h around the steady state $(x_t, \sigma) = (\overline{x}, 0)$:**

$$
\begin{aligned}
g(x, \sigma) = {} & g(\overline{x}, 0) + \sum_m g_{x_m}(\overline{x}, 0)(x_m - \overline{x_m}) + g_\sigma(\overline{x}, 0)\sigma \\
& + \frac{1}{2} \sum_{mn} g_{x_m x_n}(\overline{x}, 0)(x_m - \overline{x_m})(x_n - \overline{x_n}) \\
& + \frac{1}{2} \sum_m g_{x_m \sigma}(\overline{x}, 0)(x_m - \overline{x_m})\sigma \\
& + \frac{1}{2} \sum_m g_{\sigma\sigma}(\overline{x}, 0)\sigma^2 + g_\Phi(\overline{x}, 0)(\Phi - \overline{\Phi}) + g_\Psi(\overline{x}, 0)(\Psi - \overline{\Psi})
\end{aligned}
$$

and

$$
\begin{aligned}
h(x,\sigma)^j = h(\overline{x},0)^j &+ \sum_m h_{x_m}(\overline{x},0)^j(x_m - \overline{x_m}) + h_\sigma(\overline{x},0)^j\sigma \\
&+ \frac{1}{2}\sum_{mn} h_{x_m x_n}(\overline{x},0)^j(x_m - \overline{x_m})(x_n - \overline{x_n}) \\
&+ \frac{1}{2}\sum_m h_{x_m\sigma}(\overline{x},0)^j(x_m - \overline{x_m})\sigma + \frac{1}{2}\sum_m h_{\sigma x_m}(\overline{x},0)^j(x_m - \overline{x_m})\sigma \\
&+ \frac{1}{2}\sum_m h_{\sigma\sigma}(\overline{x},0)^j\sigma^2 + h_\Phi(\overline{x},0)^j(\Phi - \overline{\Phi}) + h_\Psi(\overline{x},0)^j(\Psi - \overline{\Psi})
\end{aligned}
$$

Solution

✓

Algorithm Steps to Solve First-order Terms:

1. Taking derivatives of the first two equations of (§) with respect to $\{a, k, z, e\}$ yields:

$$
\left.
\begin{aligned}
dF^c &\begin{cases}
F^c_a = \beta u_{cc} g_a h^a_a (r+1-\delta) - u_{cc} g_a - \frac{6\phi}{(a+b)^4} h^a_a \\
F^c_k = \beta u_{cc}(g_a h^a_k + g_k h^k_k)(r+1-\delta) + \beta u_c r_k h^k_k - u_{cc} g_k - \frac{6\phi}{(a+b)^4} h^a_k \\
F^c_z = \beta u_{cc}(g_a h^a_z + g_k h^k_z + g_z \rho_z)(r+1-\delta) + \beta u_c r_k (h^k_z + \rho_z) - u_{cc} g_z - \frac{6\phi}{(a+b)^4} h^a_z \\
F^c_e = \beta u_{cc} g_e \rho_e (r+1-\delta) - u_{cc} g_e - \frac{6\phi}{(a+b)^4} h^a_e
\end{cases} \\
dF^a &\begin{cases}
F^a_a = (r+1-\delta) - h^a_a - g_a \\
F^a_e = w - g_e - h^a_e \\
F^a_k = r_k a_{i,t} + w_k e_{i,t} - g_k - h^a_k \\
F^a_z = r_z a_{i,t} + w_z e_{i,t} - g_z - h^a_z
\end{cases}
\end{aligned}
\right\} = 0
$$

$$\frac{\downarrow}{\text{8 Equations, but 12 Unkown: } g_a, g_e, g_k, g_z, h_a^a, h_k^a, h_e^a, h_k^a, h_a^k, h_e^k, h_k^k, h_z^k}$$

2. Make up the 4 Equations need to solve the 12 unknowns above:

$$\begin{cases} F^k : h^k & \equiv k_{t+1} \\ & = \int_0^1 a_{i,t+1} di \overset{FOC}{=} \int_0^1 [h_a^a(a_{i,t} - \overline{a}) + h_e^a(e_{i,t} - \overline{e}) + h_k^a(k_t - \overline{k}) + h_z^a(z_t - \overline{z})] \\ & = (h_a^a + h_k^a)(k_t - \overline{k}) + h_z^a(z_t - \overline{z}) \\ h^k : h^k & = h_k^k(k_t - \overline{k}) + h_z^k(z_t - \overline{z}) \end{cases}$$

$$\frac{\downarrow}{h_a^k = h_e^k = 0, \; h_k^k = h_a^a + h_k^a, \; h_z^k = h_z^a}$$

3. Solving the last three first-order derivative $g_\sigma, h_\sigma^a, h_\sigma^k$

$$\left. \begin{array}{l} F_\sigma^c = \beta u_{cc} g_\sigma (r + 1 - \delta) - u_{cc} g_\sigma + \beta u_c r_k h_\sigma^k - 6\phi(a+b)^{-4} h_\sigma^a \\ F_\sigma^a = -g_\sigma - h_\sigma^a \\ h^k - \int_0^1 h^a di (\text{if } h_\sigma^a \neq h_\sigma^k = 0, \text{ then this equation can not stand forever}) \end{array} \right\} = 0$$

$$\frac{\downarrow}{g_\sigma = h_\sigma^a = h_\sigma^k = 0}$$

✓

Algorithm Steps to Solve Second-order Terms:

1. Taking derivatives of the first two equations of (§) with respect to pairs of $\{a, k, z, e\}$ yields:

$$d^2F^c \left\{ \begin{gathered} F^c_{aa} = -u_{ccc}(g_a)^2 - u_{cc}g_{aa} + \beta[u_{ccc}(g_a h^a_a)^2 + u_{cc}(g_{aa}(h^a_a)^2 \\ +u_{cc}g_a h^a_{aa}](r+1-\delta) - \langle I_{aa}\rangle \\ F^c_{ae} = -u_{ccc}g_e g_a - u_{cc}g_{ae} + \beta[u_{ccc}(g_a h^a_e + g_e\rho_e)g_a h^a_a + u_{cc}(g_{aa}h^a_e + g_{ae}\rho_e)h^a_a \\ +u_{cc}g_a h^a_{ae}](r+1-\delta) - \langle I_{ae}\rangle \\ F^c_{ak} = -u_{ccc}g_k g_a - u_{cc}g_{ak} + \beta[u_{ccc}(g_a h^a_k + g_k h^k_k)g_a h^a_a + u_{cc}(g_{aa}h^a_k + g_{ak}h^k_k)h^a_a \\ +u_{cc}g_a h^a_{ak}](r+1-\delta) + \beta u_{cc}g_a h^a_a r_k h^k_k - \langle I_{ak}\rangle \\ F^c_{az} = -u_{ccc}g_z g_a - u_{cc}g_{az} + \beta[u_{ccc}(g_a h^a_z + g_z\rho_z + g_k h^k_z)g_a h^a_a + u_{cc}(g_{aa}h^a_z + g_{az}h^k_z)h^a_a \\ +u_{cc}g_a h^a_{az}](r+1-\delta) + \beta u_{cc}g_a h^a_a(r_k h^k_z + r_z\rho_z) - \langle I_{ak}\rangle \\ F^c_{ka} = \beta u_{ccc}g_a h^a_a(g_a h^a_k + g_k h^k_k)(r+1-\delta) + \beta u_{cc}[g_{aa}(h^a_k)^2 + g_a h^a_{ka} + g_{ka}h^a_a h^k_k \\ +g_k h^k_{ka}](r+1-\delta) + \beta u_c g_a r_k h^k_k - u_{ccc}g_a g_k - u_{cc}g_{ka} + \langle I_{ka}\rangle \\ F^c_{ke} = \beta u_{ccc}(g_a h^a_e + g_e\rho_e)(g_a h^a_k + g_k h^k_k)(r+1-\delta) + \beta u_{cc}[h^a_k(g_{aa}h^a_e + g_{ae}\rho_e) \\ +g_a h^a_{ke} + h^k_k(g_{ka}h^a_e + g_{ke}\rho_e)](r+1-\delta) + \beta r_k h^k_k u_{cc}(g_a h^a_e + g_e\rho_e) - u_{ccc}g_e g_k \\ -u_{cc}g_{ke} + \langle I_{ke}\rangle \\ F^c_{kk} = \beta u_{ccc}(g_a h^a_k + g_k h^k_k)^2(r+1-\delta) + \beta u_{cc}[h^a_k(g_{aa}h^a_k + g_{ak}h^k_k) + g_a h^a_{kk} + \\ h^k_k(g_{ka}h^a_k + g_{kk}h^k_k) + g_k h^k_{kk}](r+1-\delta) + \beta u_{cc}(g_a h^a_k + g_k h^k_k)r_k h^k_k \\ +\beta u_{cc}(g_a h^a_k + g_k h^k_k)r_k h^k_k + \beta u_c r_{kk}(h^k_k)^2 + \beta u_c r_k h^k_{kk} - u_{ccc}(g_k)^2 - u_{cc}g_{kk} + \langle I_{kk}\rangle \\ F^c_{kz} = \beta u_{ccc}(g_a h^a_z + g_z\rho_z + g_k h^k_z)(g_a h^a_k + g_k h^k_k)(r+1-\delta) + \beta u_{cc}[h^a_k(g_{aa}h^a_z + \\ g_{az}\rho_z + g_{ak}h^a_{kz}) + g_a h^a_{kz} + h^k_k(g_{ka}h^a_z + g_{kz}\rho_z + g_{kk}h^k_z) + g_k h^k_{kz}](r+1-\delta) \\ +\beta u_{cc}(g_a h^a_k + g_k h^k_k)(r_k h^k_z + r_z\rho_z) + \beta u_{cc}(g_a h^a_z + g_z\rho_z + g_k h^k_z)r_k h^k_k \\ +\beta u_{cc}(r_{kk}h^k_z + r_{kz}\rho_z)h^k_k - u_{ccc}g_z g_k - u_{cc}g_{kz} + \langle I_{kz}\rangle \end{gathered} \right\} = 0$$

$$\vdots$$

2. Taking derivatives of the first two equations of (§) with respect to $\{\Psi, \Phi\}$

$$F^c_\Phi = u_{cc}(g_a h^a_\Phi + g_k h^k_\Phi + g_\Phi h^\Phi_\Phi + g_\Psi h^\Psi_\Phi)(r+1-\delta) + \beta u_c r_k h^k_\Phi - u_{cc}g_\Phi - \langle I_\Phi\rangle$$
$$F^c_\Psi = u_{cc}(g_a h^a_\Psi + g_k h^k_\Psi + g_\Phi h^\Phi_\Psi + g_\Psi h^\Psi_\Psi(r+1-\delta) + \beta u_c r_k h^k_\Psi - u_{cc}g_\Psi - \langle I_\Psi\rangle$$

$\vdots$

3. The leaving restrictions come from second-order expansion on both sides of Aggregate law
4. Finally, including derivative w.r.t $\{\sigma\sigma, a\sigma, e\sigma, k\sigma, z\sigma\}$