Author: Paul Klein
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Model

The problem

$$A\mathbb{E}[x_{t+1}|\mathscr{F}_t] = Bx_t + Cz_t$$

- *Stochastic Space:* $(\Omega, \mathscr{F}, P)$
- *Filtration:* $\tilde{\mathscr{F}} = \{\mathscr{F}_t; t = 0, 1, 2, \cdots\}$
- $\tilde{\mathscr{F}}$*-adapted stochastic process:* $z = \{z_t; t = 0, 1, \cdots\}$, exogenously given, with n_z dimensional

Mathematic Background

Generalized Eigenvalues: Let $P : \mathbb{C} \to \mathbb{C}^{n\times n}$ be a matrix-valued function of a complex variable (a matrix pencil). Then the set of its generalized eigenvalues is defined via

$$\lambda(P) = \{z \in \mathbb{C} : |P(z)| = 0\}$$

- When $P(z) = Az - B$, then the set of generalized eigenvalues $\lambda(A, B)$
- When exists $x \in \mathbb{C}^n$, such that $Bx = \lambda Ax$, then $\lambda \in \mathbb{C}$ is a generalized eigenvalue

Regular: Let $P(z)$ be a matrix pencil. Then P is said to be regular if there is $z \in \mathbb{C}$ such that $|P(z)| \neq 0$

Complex Generalized Schur Form:

Let A and B be $n \times n$ matrices of complex numbers such that $P(z) = Az - B$ is a regular matrix pencil. Then there exist unitary $n \times n$ matrices of complex numbers Q and Z such that

- There exist following decompositions, here both T and S are **upper triangular**

$$QAZ = S, \quad QBZ = T$$

- The corresponding diagonal elements are
  - For each i, s_{ii} and t_{ii} are not both zero
  - The pairs (s_{ii}, t_{ii}), $i = 1, 2, \cdots, n$ can be arranged in any order.
- The generalized eigenvalues are

$$\lambda(A, B) = \left\{ \frac{t_{ii}}{s_{ii}} : s_{ii} \neq 0 \right\}$$

- $\text{Unstable eigenvalues} \begin{cases} \text{Finite Unstable} \\ \text{Infinite} \end{cases}$, means $|\lambda| > 1$
- $\text{Stable eigenvalues}$ means $|\lambda| < 1$

Algorithm

Assumption

- **Stability:** Let x be a stochastic process with values in $\mathbb{R}^n$, x is stable if there is an M such that

$$\|x_t\|_{max} \leq M$$
$$\downarrow$$
$$\|x_t\|_{max} = \max_i \sqrt{E[|x_i|]}$$

> The unconditionally expected values of the moduli of the elements of x_t do not blow up as t increases beyond all bounds.

- **Backward-looking:** Let $(\Omega, \mathscr{F}, P, \tilde{\mathscr{F}})$ be a filtered probability space. A process k is called backward-looking if
  1. The prediction error ξ defined via $\xi_{t+1} \equiv k_{t+1} - E[k_{t+1}|\mathscr{F}_t]$ is an exogenous martingale difference process, and

     $(P, \tilde{\mathscr{F}})$***-Martingale Difference Process:*** Let $(\Omega, \mathscr{F}, P, \tilde{\mathscr{F}})$ be a filtered probability space. A vector process ξ is called a martingale difference process if

     1. ξ is adapted to $\tilde{\mathscr{F}}$
     2. $E[\xi_{t+1}|\tilde{\mathscr{F}}_t] = 0$ for each $t = 0, 1, \cdots$
  2. $k_0 \in \mathscr{F}_0$ is exogenously given

Assumptions:

- **Assumption 1:** The exogenous n_z-dimensional process z is stable and adapted to the given filtration $\tilde{\mathscr{F}}$.

$$z_t = \phi(z_{t-1}) + \epsilon_t$$

- **Assumption 2:** k_0 is an exogenously given $\tilde{\mathscr{F}}_0$-measurable random variable and

$$k_{t+1} - \mathbb{E}[k_{t+1}|\mathscr{F}_t] = \xi_{t+1}$$

Here ξ is a Martingale Difference Process

- **Assumption 3:** There exists a $z \in \mathbb{C}$ such that $|Az - B| \neq 0$
- **Assumption 4:** There is no $z \in \mathbb{C}$ such that $|Az - B| = 0$ and $|z| = 1$
- **Assumption 5:** Z_{11} is square and invertible.

Using the Generalized Schur Form

Triangularizing the system: to find an upper triangular system of expectational difference equations in the auxiliary variables y_t defined via

$$y_t \equiv Z^H x_t = Z^H \begin{bmatrix} k_t \\ d_t \end{bmatrix} = \begin{bmatrix} s_t \\ u_t \end{bmatrix}$$

- k_t: backward looking variables
- s_t: stable variables
- u_t: unstable variables

$$\Downarrow$$

$$\star \quad S\mathbb{E}[y_{t+1}|\mathscr{F}_t] = Ty_t + QCz_t$$
$$\Rightarrow \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \mathbb{E}\left\{ \begin{bmatrix} s_{t+1} \\ u_{t+1} \end{bmatrix} \middle| \mathscr{F}_t \right\} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} Cz_t$$

Solving the triangular system:

Solving for $\mathbf{u}_t$:

1. When n_u is not large, Hint: $u_\infty = 0$ to get a solution

Forward Iteration for Unstable Variables

$$u_t = -T_{22}^{-1}\sum_{k=0}^{\infty}[T_{22}^{-1}S_{22}]^k Q_2 C\mathbb{E}[z_{t+k}|\mathscr{F}_t]$$

When z is a stationary VAR process with autocorrelation matrix Φ

$$\begin{aligned} u_t &= M z_t \\ &\downarrow \\ vec(M) &= \left[\Phi^T \otimes S_{22} - I_{n_z} \otimes T_{22}\right]^{-1} vec[Q_2 C] \end{aligned}$$

Hints:

$$\left.\begin{aligned} &S = \textstyle\sum_{k=0}^{\infty} \Phi^k A \Psi^k = A + \sum_{k=1}^{\infty} \Phi^k A \Psi^k \\ &\Phi S \Psi = \textstyle\sum_{k=1}^{\infty} \Phi^k A \Psi^k \end{aligned}\right\} \Rightarrow S - \Phi S \Psi = A$$

$$\Rightarrow vec(S) - (\Psi^T \otimes \Phi) vec(S) = vec(A)$$
$$\Rightarrow vec(S) = \left[I - \Psi^T \otimes \Phi\right]^{-1} vec(A)$$

and

$$\mathbb{E}_t z_{t+k} = \Phi^k z_t$$

2. When n_u is large, calculating row by row.

Solving for s_t:

$$
\begin{aligned}
&\star \\
&\Downarrow \\
\mathbb{E}[s_{t+1}|\mathscr{F}_t] &= S_{11}^{-1}T_{11}s_t + S_{11}^{-1}\{T_{12}u_t - S_{12}\mathbb{E}[u_{t+1}|\mathscr{F}_t] + Q_1Cz_t\} \\
&\Downarrow (u_t = Mz_t, \mathbb{E}z_{t+1} = \Phi z_t) \\
\mathbb{E}[s_{t+1}|\mathscr{F}_t] &= S_{11}^{-1}T_{11}s_t + S_{11}^{-1}\{T_{12}M - S_{12}M\Phi + Q_1C\}z_t
\end{aligned}
$$

$$
\underbrace{\begin{bmatrix} k_{t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} s_{t+1} \\ u_{t+1} \end{bmatrix} \qquad k_{t+1} - \mathbb{E}[k_{t+1}|\mathscr{F}_t] = \xi_{t+1}}
$$

$$
\Downarrow
$$

$$
Z_{11}(s_{t+1} - \mathbb{E}[s_{t+1}|\mathscr{F}_t]) + Z_{12}(u_{t+1} - \mathbb{E}[u_{t+1}|\mathscr{F}_t]) = \xi_{t+1}
$$

$$
\Downarrow (z_t = \phi(z_{t-1}) + \epsilon_t)
$$

$$
s_{t+1} - \mathbb{E}[s_{t+1}|\mathscr{F}_t] = Z_{11}^{-1}\xi_{t+1} - Z_{11}^{-1}Z_{12}M\epsilon_{t+1}
$$

$$
\begin{aligned}
s_{t+1} = {} & S_{11}^{-1}T_{11}s_t + S_{11}^{-1}\{T_{12}M - S_{12}M\Phi + Q_1C\}z_t \\
& + Z_{11}^{-1}\xi_{t+1} - Z_{11}^{-1}Z_{12}M\epsilon_{t+1}
\end{aligned}
$$

➤ Eliminating the auxiliary process $\mathbf{y}_t$