MATH 115 Fall 2018: Assignment 6 **Due at 8:30am on Friday, November 16, 2018**

It is important that you read the assignment submission instructions and suggestions available on LEARN.

1. (**7 marks**) Let $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. Let $\vec{p} = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 5 \end{bmatrix}$ and $\vec{q} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

   (a) Determine whether $\vec{p} \in \operatorname{Span}\{\vec{x}, \vec{y}, \vec{z}\}$, and whether $\vec{q} \in \operatorname{Span}\{\vec{x}, \vec{y}, \vec{z}\}$. If applicable, write $\vec{p}$ and $\vec{q}$ as explicit linear combinations of $\vec{x}$, $\vec{y}$ and $\vec{z}$ (no variables or parameters in the linear combination).

   (b) Is the set $\{\vec{x}, \vec{y}, \vec{z}, \vec{p}\}$ linearly independent? Explain.

2. (**6 marks**) Let $A = \begin{bmatrix} 1 & 3 & -3 & 2 & 9 \\ -1 & -2 & 1 & -1 & -6 \\ 1 & 1 & 1 & 1 & 4 \\ 2 & 7 & -8 & 3 & 19 \end{bmatrix}$.

   (a) Find a basis for $\operatorname{null} A$.

   (b) Find a basis for $\operatorname{col} A$.

   (c) Find a basis for $\operatorname{row} A$.

   (d) Compute $\dim(\operatorname{col} A)$ and $\dim(\operatorname{row} A)$.

3. (**6 marks**) Let $U = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \text{ and } a = 2b + 3c \right\} \subseteq \mathbb{R}^3$.

   (a) Show that U is a subspace of $\mathbb{R}^3$.

   (b) Find a basis for U.

   (c) Find $\dim U$.

4. (**8 marks**) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \in \mathbb{R}^n$ be such that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly independent. For each of the following, either show that the given set is linearly independent or linearly dependent.

   (a) $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_3, \vec{v}_1 + \vec{v}_4\}$

   (b) $\{\vec{v}_1 - \vec{v}_2, \vec{v}_2 - \vec{v}_3, \vec{v}_3 - \vec{v}_4, \vec{v}_4 - \vec{v}_1\}$

   (c) $\{A\vec{v}_1, A\vec{v}_2, A\vec{v}_3, A\vec{v}_4\}$ where A is an invertible $n \times n$ matrix.

5. (**8 marks**) Let U and W be subspaces of a finite dimensional vector space $\mathbb{V}$. Define their *intersection* $U \cap W$, their *union* $U \cup W$ and their *sum* $U + W$, as follows:

$$\begin{aligned} U \cap W &= \{\vec{x} \in \mathbb{V} \mid \vec{x} \in U \text{ and } \vec{x} \in W\} \\ U \cup W &= \{\vec{x} \in \mathbb{V} \mid \vec{x} \in U \text{ or } \vec{x} \in W\} \\ U + W &= \{\vec{x} \in \mathbb{V} \mid \vec{x} = \vec{x}_1 + \vec{x}_2 \text{ with } \vec{x}_1 \in U \text{ and } \vec{x}_2 \in W\} \end{aligned}$$

For each of the following statements, either show they are true, or provide a counterexample which shows they are false.

(a) $U \cap W$ is a subspace of $\mathbb{V}$.

(b) $U \cup W$ is a subspace of $\mathbb{V}$.

(c) $U + W$ is a subspace of $\mathbb{V}$.

6. (**7 marks**) The functions

$$\begin{aligned} C_0(x) &= 1 \\ C_1(x) &= x \\ C_2(x) &= 2x^2 - 1 \\ C_3(x) &= 4x^3 - 3x \\ C_4(x) &= 8x^4 - 8x^2 + 1 \end{aligned}$$

are known as the first five Chebyshev polynomials of the first kind.

(a) Show that the set $\mathcal{B} = \{C_0, C_1, C_2, C_3, C_4\}$ is a basis for the vector space of polynomials of degree at most 4.

(b) Write the polynomial $p(x) = x^4 + x^3 + x^2 + x + 1$ in this basis, that is, find $[p(x)]_{\mathcal{B}}$.

Total: 42 marks.