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The graphs with exactly two distance eigenvalues different from −1 and −3

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Abstract In this paper, we completely characterize the graphs with third largest distance eigenvalue at most -1 and smallest distance eigenvalue at least -3. In particular, we determine all graphs whose distance matrices have exactly two eigenvalues (counting multiplicity) different from -1 and -3. It turns out that such graphs consist of three infinite classes, and all of them are determined by their distance spectra. We also show that the friendship graph is determined by its distance spectrum.

Keywords Distance eigenvalue · Distance equitable partition · Friendship graph · Distance spectral characterization

Mathematics Subject Classification 05C50

1 Introduction

Let G be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, edge set $E(G)$ and adjacency matrix $A = A(G)$. Denote by $d(v_i, v_j)$ the *distance* (i.e., the length of a shortest path) between the vertices v_i and v_j of G. Then, the *diameter* $d(G)$ and *distance matrix* $D(G)$ are defined as $d(G) = \max\{d(v_i, v_j) \mid v_i, v_j \in V(G)\}$ and $D(G) = (d(v_i, v_j))_{n\times n}$, respectively.

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Since $D(G)$ is a real symmetric matrix, all its eigenvalues are real and can be denoted and arranged as $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$. These eigenvalues are also called the *distance eigenvalues* of G, and the largest one ∂_1 is called the *distance spectral radius* of G. The *distance spectrum* of the graph G, denoted by $\mathrm{Spec}_D(G)$, consists of the distance eigenvalues (together with their multiplicities). The graph G is said to be *determined by its distance spectrum* (DDS for short) if, for any graph H, $\mathrm{Spec}_D(H) = \mathrm{Spec}_D(G)$ implies that $H \cong G$. The notions of *adjacency eigenvalue*, *adjacency spectrum* (denoted by $\mathrm{Spec}_A(G)$) and *determined by its adjacency spectrum* (DAS for short) can be similarly defined if we consider the adjacency matrix $A(G)$.

Throughout this paper, we denote by $J_{i\times j}$ the $i \times j$ all-ones matrix, I_p the identity matrix of order p and $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ the *neighbor set* of $v \in V(G)$. The *join* of two vertex disjoint graphs G and H is the graph $G \vee H$ obtained by adding all edges with one end vertex in G and the other end vertex in H. In addition, all the symbols and notions not mentioned here are standard and can be found in [3].

The famous question "Which graphs are determined by their spectra?" has been raised by Günthard and Primas [5] over half a century and is far from being solved up to now. For surveys of this question see [9,10]. With regard to the distance spectrum, it is believed that a mass of graphs are DDS since the distance matrix $D(G)$ contains more information than $A(G)$. However, just a few DDS-graphs have been characterized up to now. Recently, Jin and Zhang [6] proved that the complete k-partite graph $K_{n_1,\ldots,n_k}$ is DDS; Lin et al. [7] proved that the graph $K^r_{s,t} = K_r \vee (K_s \cup K_t)$ with $r \geq 1$ is DDS.

The famous friendship graph F_k consists of k edge-disjoint triangles all meeting at one vertex. In 2010, Wang et al. [11] put forward the conjecture that F_k is DAS. This conjecture aroused several activities [1,4] and finally was affirmed by Cioabă et al. [2] for $k \neq 16$ (if $k = 16$, they also showed that there is exactly one graph H satisfying $\mathrm{Spec}_A(H) = \mathrm{Spec}_A(F_k)$ but $H \ncong F_k$). Actually, Cioabă et al. characterized all graphs with all but two adjacency eigenvalues equal to ± 1 and F_k is just contained in this class.

In this paper, we first introduce the notion of a distance equitable partition and give some basic results about it in Sect. 2. Motivated by the work of Cioabă et al., we completely characterize those graphs satisfying $-3 \leq \partial_n$ and $\partial_3 \leq -1$ in Sect. 3. In particular, we determine all graphs with exactly two distance eigenvalues (counting multiplicity) different from -1 and -3 in Sect. 4. In the process, we show that all these graphs are DDS, and particularly, F_k is DDS.

2 The distance equitable partition

Given a graph G, the vertex partition Π: $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ is said to be an *equitable partition* if, for any $u \in V_i$, $|V_j \cap N(u)| = b_{ij}$ is a constant depending only on i, j $(1 \leq i, j \leq r)$, and the matrix $B_\Pi = (b_{ij})_{k\times k}$ is the so-called *divisor matrix* derived from Π. Moreover, B_Π can be regarded as the adjacency matrix of a directed multigraph G/Π, which is called the *divisor* of G with respect to Π. Let A be the adjacency matrix of G, and let C be the *characteristic matrix* with respect to Π whose i-th column C_i is the characteristic function of V_i. Then $AC = CB_\Pi$, and so the columns of C generate an invariant subspace of A, which produces a nice property:

$\det(xI - B_\Pi) | \det(xI - A)$ (see [3], Theorem 3.9.5). Naturally, we ask if there exists analogous "equitable partition" for the distance matrix of G? If it exists, what confuses us is that how it reveals the relation between the eigenvalues and the structure of a graph. In this section, we will introduce the notion of "equitable partition" for the distance matrix of a graph.

Denote by $d(v, S) = \sum_{u \in S} d(u, v)$, where $v \in V(G)$ and S is a non-empty subset of $V(G)$. In terms of $d(v, S)$, we give the following definition of distance equitable partition.

Definition 2.1 Given a connected graph G, the vertex partition $\Pi : V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ is called a *distance equitable partition* if, for any $v \in V_i$, $d(v, V_j) = b_{ij}$ is a constant depending only on i, j $(1 \leq i, j \leq k)$. Here the matrix $B^*_\Pi = (b_{ij})_{k \times k}$ is called the *distance divisor matrix* of G with respect to Π.

Let Γ be the automorphism group of G, and let $O_1, O_2, \ldots, O_k$ be all orbits of Γ. Then Π: $V(G) = O_1 \cup O_2 \cup \cdots \cup O_k$ is a partition of $V(G)$ which is generally called the *orbit partition* of G. It is well known that the orbit partition is an equitable partition and, fortunately, we have the following result.

Lemma 2.1 *Let G be a connected graph. Then its orbit partition $\Pi : V(G) = O_1 \cup O_2 \cup \cdots \cup O_k$ is a distance equitable partition.*

Proof Notice that $d(u, v) = d(\sigma(u), \sigma(v))$ for any automorphism σ of G. For $v \in O_i$, we have

$$\begin{aligned} d(\sigma(v), O_j) = d(\sigma(v), \sigma(O_j)) &= \sum_{u \in O_j} d(\sigma(v), \sigma(u)) \\ &= \sum_{u \in O_j} d(v, u) = d(v, O_j). \end{aligned} \tag{1}$$

Since the automorphism group Γ acts transitively on each orbit, $d(v, O_j)$ is a constant independent of the choice of $v \in O_i$. The result follows. □

Now suppose that Π: $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ is a distance equitable partition of G, and C is the characteristic matrix with respect to Π. Then the distance divisor matrix $B^*_\Pi = (b_{ij})$ (with respect to Π) leads to a *distance divisor* of G, also denoted by G/Π, which is the directed multigraph with vertices $V_1, V_2, \ldots, V_k$ and b_{ij} arcs from V_i to V_j. To compare with (adjacency) equitable partition, we will give some parallel results for distance equitable partition.

Lemma 2.2 *Let G be a connected graph with a distance equitable partition $\Pi : V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$. Then $DC = CB^*_\Pi$, where D is the distance matrix of G, C and B^*_Π are the characteristic matrix and distance divisor matrix with respect to Π, respectively.*

Proof Consider the (v, j)-entry of DC and CB^*_Π where v is in V_i. On the one hand,

$$(DC)_{vj} = \sum_{u \in V_j} d(v, u) = d(v, V_j) = b_{ij}.$$

On the other hand, $(CB^*_\Pi)_{vj} = b_{ij}$. Thus, our result follows. □

Theorem 2.3 *Let G be a connected graph with distance matrix D, and let $\Pi : V = V_1 \cup V_2 \cup \cdots \cup V_k$ be a distance equitable partition of G with distance divisor matrix B^*_Π. Then* $\det(xI - B^*_\Pi) \mid \det(xI - D)$.

Proof Let C be the characteristic matrix with respect to Π. Clearly, the matrix C has rank k. Now, we choose a matrix C^* of order $n \times (n-k)$ such that $(C \mid C^*)$ is an invertible matrix of order $n \times n$, where $n = |V(G)|$. Then there exist two matrices X and Y such that

$$DC^* = CX + C^*Y. \tag{2}$$

From Lemma 2.2 and (2), we obtain

$$D(C \mid C^*) = (C \mid C^*)\begin{pmatrix} B^*_\Pi & X \\ 0 & Y \end{pmatrix}.$$

It follows that $\det(xI - D) = \det(xI - B^*_\Pi)\det(xI - Y)$ since $(C \mid C^*)$ is invertible. □

Corollary 2.4 *Let G be a connected graph of order n with distance equitable partition Π, and B^*_Π the distance divisor matrix of G with respect to Π. Then, the largest eigenvalue of B^*_Π is just the distance spectral radius of G.*

Proof Let λ be the largest eigenvalue of B^*_Π with eigenvector $\mathbf{x}$, then $B^*_\Pi\mathbf{x} = \lambda\mathbf{x}$. By Perron–Frobenius Theorem, we may assume that $\mathbf{x} > 0$. Putting $\mathbf{y} = C\mathbf{x}$, where C is the characteristic matrix with respect to Π. From Lemma 2.2, we have

$$D\mathbf{y} = D(C\mathbf{x}) = (DC)\mathbf{x} = (CB^*_\Pi)\mathbf{x} = C(B^*_\Pi\mathbf{x}) = C(\lambda\mathbf{x}) = \lambda(C\mathbf{x}) = \lambda\mathbf{y}.$$

Thus $\mathbf{y}$ is an eigenvector of D and, by Perron–Frobenius Theorem again, λ is the distance spectral radius of G because $\mathbf{y}$ is positive. □

3 The graphs with distance spectrum $[\partial_n, \partial_3] \subseteq [-3, -1]$

There are some results about the smallest distance eigenvalue ∂_n of a graph. Recently, Yu [12] proved that $\partial_n(G) \leq -2.383$ when G is neither a complete graph nor a complete k-partite graph. In this section, we will characterize those graphs satisfying $-3 \leq \partial_n$ and $\partial_3 \leq -1$.

Lemma 3.1 (Cauchy Interlace Theorem) *Let A be a Hermitian matrix with order n, and let B be a principal submatrix of A with order m. If $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ lists the eigenvalues of A and $\mu_1(B) \geq \mu_2(B) \geq \cdots \geq \mu_m(B)$ the eigenvalues of B, then*

$$\lambda_{n-m+i}(A) \leq \mu_i(B) \leq \lambda_i(A) \quad \text{for } i = 1, \ldots, m.$$

Let H be a connected induced subgraph of the connected graph G. Then $A(H)$ must be a principal submatrix of $A(G)$, while $D(H)$ may not be a principal submatrix of $D(G)$. For example, the path P_4 is an induced subgraph of the cycle C_5, and the distance matrix of P_4 and C_5 are, respectively, given by

$$D(P_4) = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}, \quad D(C_5) = \begin{pmatrix} 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 \end{pmatrix}.$$

Obviously, $D(P_4)$ is not a principal submatrix of $D(C_5)$. Nevertheless, if H is an induced subgraph of G with diameter $d(H) < 3$, the matrix $D(H)$ must be a principal submatrix of $D(G)$. In fact, for any $u, v \in H$, we have $d_H(u, v) \leq d(H) < 3$. This implies that either u and v are adjacent or they are not adjacent but have a common neighbor in H. If u and v are adjacent in H, then they are adjacent in G, and so $d_H(u, v) = d_G(u, v) = 1$. If u and v are not adjacent but have a common neighbor in H, then they are not adjacent in G and still have a common neighbor because H is an induced subgraph of G, and so $d_H(u, v) = d_G(u, v) = 2$. Thus for any $u, v \in H$, we have $d_H(u, v) = d_G(u, v)$, which implies that $D(H)$ is a principal submatrix of $D(G)$. By Lemma 3.1, we have the following result.

Lemma 3.2 *If H is a connected induced subgraph of G with diameter $d(H) < 3$, then the eigenvalues of $D(H)$ interlace those of $D(G)$.*

Corollary 3.3 *Let G_1 be a connected graph with diameter $d(G_1) < 3$. If $G = K_r \vee G_1$, then the eigenvalues of $D(G_1)$ interlace those of $D(G)$.*

The distance eigenvalues are closely linked to the structure of a graph. In fact, some special structure of a graph can lead to some special distance eigenvalues. Conversely, some special distance eigenvalues also can determine some special structure of a graph.

Lemma 3.4 *Let G be a connected graph on n vertices. If $S = \{v_1, \ldots, v_p\}$ $(p \geq 2)$ induces a clique of G with $N(v_i)\backslash S = N(v_j)\backslash S$ for $1 \leq i, j \leq p$. Then -1 is an eigenvalue of $D(G)$ with multiplicity at least $p - 1$.*

Proof According to our assumption, we can suppose that $N = N(v_i)\backslash S = \{u_1, u_2, \ldots, u_q\}$ where $i = 1, 2, \ldots, p$. Set $T = V(G)\backslash(S \cup N) = \{w_1, w_2, \ldots, w_{n-p-q}\}$. Then $V(G) = S \cup N \cup T$ is a vertex partition of G. Since $d(v_i, u_j) = 1$, the submatrix $D(S, N)$ of $D(G)$ induced on the row set S and the column set N equals to $J_{p\times q}$. Similarly, since $d(v_i, v_j) = 1$, we have $D(S, S) = J_{p\times p} - I_p$. Furthermore, for $j \in \{1, 2, \ldots, n - p - q\}$ we see that $d(v_i, w_j) = a_j$ for any $i \in \{1, 2, \ldots, p\}$. Hence the submatrix $D(S, T)$ can be written as

$$D(S, T) = \begin{array}{c} \begin{array}{cccc} w_1 & w_2 & \cdots & w_{n-p-q} \end{array} \\ \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-p-q} \\ a_1 & a_2 & \cdots & a_{n-p-q} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_{n-p-q} \end{pmatrix} \end{array} \begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_p \end{array}$$

Now putting $A = D(S, T)$, $X = D(N, N)$, $Y = D(N, T)$ and $Z = D(T, T)$, the distance matrix $D(G)$ can be written as

$$D(G) = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{ccc} S & N & T \\ \end{array}$$

$$D(G) = \left(\begin{array}{ccc} J_{p\times p} - I_p & J_{p\times q} & A \\ J_{q\times p} & X & Y \\ A^T & Y^T & Z \end{array}\right)\begin{array}{c} S \\ N \\ T \end{array}.$$

For $i = 2, \ldots, p$, let $x^{(i)} \in \mathbb{R}^n$ be the vector defined on $V(G)$ with $x^{(i)}_{v_1} = 1$, $x^{(i)}_{v_i} = -1$ and $x^{(i)}_v = 0$ for $v \neq v_1, v_i$. Since $A^T(x^{(i)}_{v_1}, \ldots, x^{(i)}_{v_p})^T = 0$, we have $D(G)x^{(i)} = (-1)x^{(i)}$. Moreover, $x^{(2)}, x^{(3)}, \ldots, x^{(p)}$ are linearly independent. Thus the result follows. □

Lemma 3.5 *Let G be a connected graph on n vertices. If $S = mK_r$ ($m \geq 2$) is an induced subgraph of G with $N(u)\backslash V(S) = N(v)\backslash V(S)$ for any $u, v \in V(S)$, then $-(r+1)$ is an eigenvalue of $D(G)$ with multiplicity at least $m-1$.*

Proof First, we partition the vertices of S as $V(S) = V(S_1) \cup V(S_2) \cup \cdots \cup V(S_m)$ such that each $V(S_i)$ induces a K_r in G, where $i = 1, \ldots, m$. Assume that $V(S_i) = \{v^{(i)}_1, \ldots, v^{(i)}_r\}$, $N = N(v)\backslash V(S) = \{u_1, \ldots, u_q\}$ for any $v \in S$ and $T = V(G)\backslash(N \cup V(S)) = \{w_1, \ldots, w_{n-q-mr}\}$, like the proof of Lemma 3.4, the distance matrix $D(G)$ can be written as

$$D(G) = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} V(S) & N & T \end{array} \\ \left(\begin{array}{ccc} B & J_{mr\times q} & A \\ J_{q\times mr} & X & Y \\ A^T & Y^T & Z \end{array}\right)\begin{array}{c} V(S) \\ N \\ T \end{array} \end{array}$$

where

$$B = \begin{array}{c} \begin{array}{cccc} V(S_1) & V(S_2) & \cdots & V(S_m) \end{array} \\ \left(\begin{array}{cccc} J_{r\times r} - I_r & 2J_{r\times r} & \cdots & 2J_{r\times r} \\ 2J_{r\times r} & J_{r\times r} - I_r & \cdots & 2J_{r\times r} \\ \vdots & \vdots & \ddots & \vdots \\ 2J_{r\times r} & 2J_{r\times r} & \cdots & J_{r\times r} - I_r \end{array}\right)\begin{array}{c} V(S_1) \\ V(S_2) \\ \vdots \\ V(S_m) \end{array} \end{array},$$

$$A = \begin{array}{c} \begin{array}{cccc} w_1 & w_2 & \cdots & w_{n-q-mr} \end{array} \\ \left(\begin{array}{cccc} a_1 & a_2 & \cdots & a_{n-q-mr} \\ a_1 & a_2 & \cdots & a_{n-q-mr} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_{n-q-mr} \end{array}\right)\begin{array}{c} v^{(1)}_1 \\ v^{(1)}_2 \\ \vdots \\ v^{(m)}_r \end{array} \end{array}$$

For $i = 2, \ldots, m$, let $y^{(i)} \in \mathbb{R}^n$ be the vector defined on $V(G)$ with $y^{(i)}_{v^{(1)}_j} = 1$, $y^{(i)}_{v^{(i)}_j} = -1$ for $j = 1, \ldots, r$ and $y^{(i)}_v = 0$ for $v \notin \{v^{(1)}_j, v^{(i)}_j \mid 1 \leq j \leq r\}$. Since

$B(y^{(i)}_{v_1^{(1)}}, \ldots, y^{(i)}_{v_r^{(m)}})^T = -(r+1)(y^{(i)}_{v_1^{(1)}}, \ldots, y^{(i)}_{v_r^{(m)}})^T$ and $A^T(y^{(i)}_{v_1^{(1)}}, \ldots, y^{(i)}_{v_r^{(m)}})^T = 0$, we have $D(G)y^{(i)} = -(r+1)y^{(i)}$. Moreover, $y^{(2)}, \ldots, y^{(m)}$ are linearly independent. Thus, the result follows. □

If $S = \{v_1, \ldots, v_p\}$ $(p \geq 2)$ is an independent set of the connected graph G with $N(v_i) = N(v_j)$ for $1 \leq i, j \leq p$, we say that S is a *star independent set* of order p. The following two corollaries are special cases of Lemma 3.5 for $r = 1$ and $r = 2$, respectively.

Corollary 3.6 *Let G be a connected graph. If G contains a star independent set of order p, then -2 is an eigenvalue of $D(G)$ with multiplicity at least $p - 1$.*

Corollary 3.7 *Let G be a connected graph. If $S = mK_2$ $(m \geq 2)$ is an induced subgraph of G with $N(u)\backslash V(S) = N(v)\backslash V(S)$ for any $u, v \in V(S)$, then -3 is an eigenvalue of $D(G)$ with multiplicity at least $m - 1$.*

Let G be a graph with vertex set $V(G)$. For any $X \subseteq V(G)$, we say that X is *G-connected* if the subgraph $G[X]$ of G induced by X is connected.

Lemma 3.8 [8] *Let G be a graph. The following statements are equivalent:*

(1) G has no induced subgraph isomorphic to P_4.
(2) Every subset of $V(G)$ with more than one element is not G-connected or not $\bar{G}$-connected.

Let G be a connected graph containing no induced P_4. Then $V(G)$ is a subset of itself and so is G-connected, by Lemma 3.8, we know that $\bar{G}$ is disconnected. Therefore, we get the following result.

Lemma 3.9 *If G is a connected graph containing no induced P_4, then G must be a join of two graphs, i.e., $G \cong G_1 \vee G_2$, where G_1 and G_2 are non-null.*

From Lemma 3.9, we know that the diameter of a non-complete connected graph containing no induced P_4 is two. However, a graph with diameter two may contain induced P_4 such as the cycle C_5.

Denote by $\mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$ the set of non-complete connected graphs of order n $(n \geq 4)$ satisfying $-3 \leq \partial_n(G)$ and $\partial_3(G) \leq -1$. In the following, we try to characterize the graphs in $\mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$. We start with a list of forbidden induced subgraphs shown in Fig. 1.

Lemma 3.10 *No graph in $\mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$ has one of the graphs P_4, C_5, H_i $(i = 0, 1, \ldots, 6)$ and I_j $(j = 1, 2, 3, 4)$ (shown in Fig. 1) as an induced subgraph.*

Proof Let $G \in \mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$. Since the diameters of C_5, H_i $(i = 0, 1, \ldots, 6)$ and I_j $(j = 1, 2, 3, 4)$ are all less than 3, and each of these graphs has its third largest distance eigenvalue ∂_3 strictly greater than -1 or its smallest distance eigenvalue ∂_n strictly less than -3, by Lemma 3.2, none of C_5, H_i $(i = 0, \ldots, 6)$ and I_j $(j = 1, 2, 3, 4)$ can be an induced subgraph of G. In the following, it suffices to show that P_4 cannot be an induced subgraph of G.

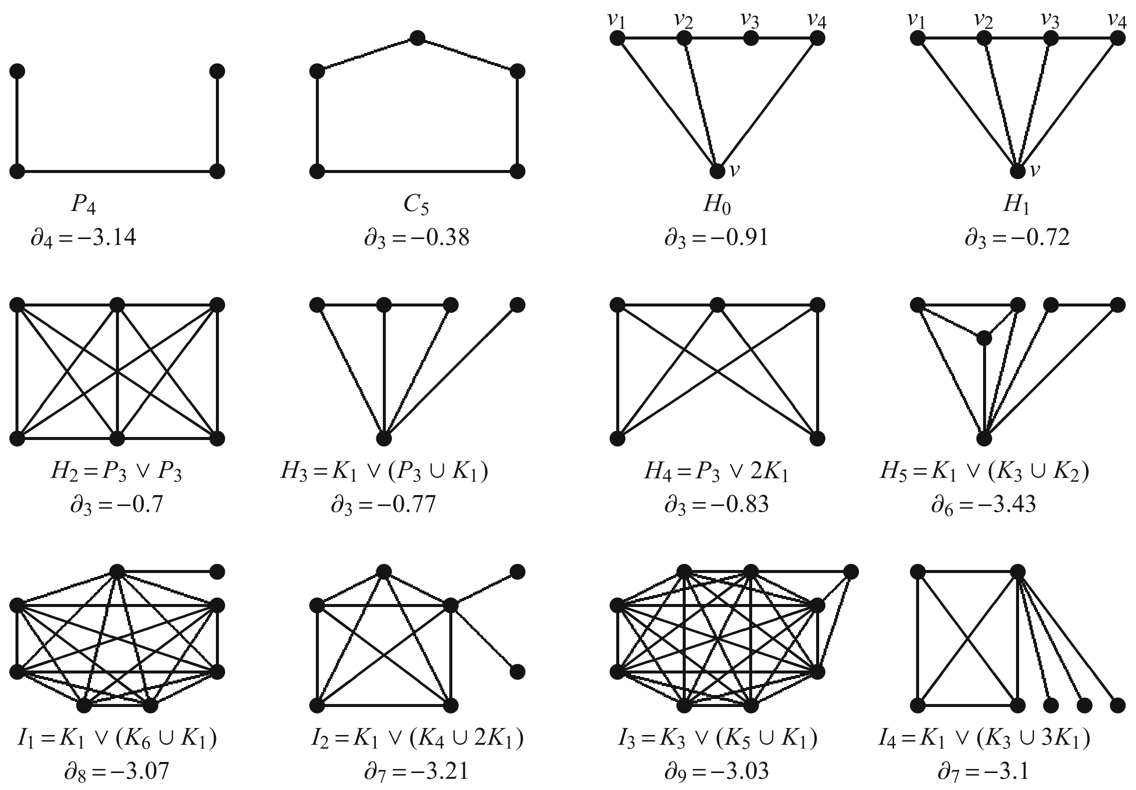

Fig. 1 Forbidden induced subgraphs

Assume by contradiction that G contains an induced $P_4 = v_1v_2v_3v_4$. If $N(v_1) \cap N(v_4) = \emptyset$, then $D(P_4)$ is a principal submatrix of $D(G)$. By Lemma 3.1, $-3 \leq \partial_n(G) \leq \lambda_4(D(P_4)) = -3.14$, a contradiction. Next, we assume that there exists $v \in N(v_1) \cap N(v_4)$. If $v \nsim v_2$ and $v \nsim v_3$, then C_5 will be an induced subgraph of G, a contradiction. If $v \sim v_2$ and $v \nsim v_3$ (see H_0 in Fig. 1), or $v \nsim v_2$ and $v \sim v_3$, then H_0 will be an induced subgraph of G, a contradiction. If $v \sim v_2$ and $v \sim v_3$ (see H_1 in Fig. 1), then H_1 will be an induced subgraph of G, a contradiction. □

Lemma 3.11 *If $G \in \mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$, then there exist two proper subgraphs G_1 and G_2 at most one of them containing an induced P_3 such that $G = G_1 \vee G_2$. Furthermore, if G_1 contains an induced P_3, then G_1 is connected and G_2 is complete.*

Proof By Lemma 3.10, G contains no induced P_4 and so $G = G_1 \vee G_2$ by Lemma 3.9. By Lemma 3.10 again, G contains no induced H_2 ($\cong P_3 \vee P_3$) and so at most one of G_1 and G_2 contains an induced P_3. Furthermore, if G_1 contains an induced P_3 but G_2 does not, then G_1 is connected since otherwise G will contain an induced H_3 ($\cong (P_3 \cup K_1) \vee K_1$), and G_2 is a union of complete graphs since it contains no induced P_3. In fact, $G_2 \cong K_s$ for some $s \geq 1$, since otherwise G will contain an induced H_4 ($\cong P_3 \vee 2K_1$), which contradicts Lemma 3.10. □

Lemma 3.11 gives a sketch for the graph G in $\mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$, that is, $G = G_1 \vee G_2$. Now we give a precise characterization of G_1 and G_2 in the following theorem.

Theorem 3.12 *Let* $G = G_1 \vee G_2 \in \mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$.

(i) If both G_1 *and* G_2 *contain no induced* P_3*, then* G *is one of the following:*
  (a) $G \cong K_r \vee (K_s \cup tK_1)$ *where* $r, t \geq 1$ *and* $s \geq 3$*;*
  (b) $G \cong K_r \vee (mK_2 \cup tK_1)$ *where* $r \geq 1$ *and* $m + t \geq 2$*;*
  (c) $G \cong (K_{s_1} \cup t_1K_1) \vee (K_{s_2} \cup t_2K_1)$ *where* $s_i \geq 3$ *and* $t_i \geq 1$ *for* $i = 1, 2$*;*
  (d) $G \cong (K_s \cup t_1K_1) \vee (mK_2 \cup t_2K_1)$ *where* $s \geq 3$, $t_1 \geq 1$ *and* $m + t_2 \geq 2$*;*
  (e) $G \cong (m_1K_2 \cup t_1K_1) \vee (m_2K_2 \cup t_2K_1)$ *where* $m_i + t_i \geq 2$ *for* $i = 1, 2$.

(ii) If one of G_1 *and* G_2 *contains an induced* P_3*, then there exists* $r \geq 1$ *such that* $G = K_r \vee F_1 \vee F_2$*, where* F_1 *and* F_2 *are non-null containing no induced* P_3.

Proof By Lemma 3.11, there exists two proper subgraphs G_1 and G_2 at most one of them containing induced P_3 such that $G = G_1 \vee G_2$. Now, we divide our proof into two cases.

Case 1 Both G_1 and G_2 contain no induced P_3.

In this case, both G_1 and G_2 are unions of some complete graphs. Then at most one of G_1 and G_2 is connected since otherwise G will be complete. We consider two subcases below.

Subcase 1.1 One of G_1 and G_2 is connected.

Without loss of generality, we assume that G_1 is connected but G_2 is disconnected. Then $G_1 \cong K_r$ for some $r \geq 1$.

If G_2 contains K_3, then $G_2 \cong K_s \cup tK_1$, where $s \geq 3, t \geq 1$. Since otherwise, G_2 will contain induced $K_3 \cup K_2$, and then $G = K_r \vee G_2$ will contain induced H_5 ($\cong K_1 \vee (K_3 \cup K_2)$), which contradicts Lemma 3.10. Thus (a) follows.

If G_2 contains no K_3, then $G_2 \cong mK_2 \cup tK_1$, where $m + t \geq 2$. It follows (b).

Subcase 1.2 Both of G_1 and G_2 are disconnected.

If both of G_1 and G_2 contain K_3, then $G_1 \cong K_{s_1} \cup t_1K_1$ and $G_2 \cong K_{s_2} \cup t_2K_1$, where $s_i \geq 3$ and $t_i \geq 1$ for $i = 1, 2$. Since otherwise, G_1 or G_2 will contain induced $K_3 \cup K_2$, and then G will contain induced H_5, which contradicts Lemma 3.10. It follows (c).

If just one of G_1 and G_2 contains K_3, say G_1, then $G_2 \cong mK_2 \cup t_2K_1$, where $m + t_2 \geq 2$. We claim that $G_1 \cong K_s \cup t_1K_1$ for $s \geq 3, t \geq 1$. Since otherwise, G_1 will contain induced $K_3 \cup K_2$, and thus G will contain induced H_5, which contradicts Lemma 3.10. It follows (d).

If both G_1 and G_2 contain no K_3, then $G_1 \cong m_1K_2 \cup t_1K_1$ and $G_2 \cong m_2K_2 \cup t_2K_1$, where $m_i + t_i \geq 2$ for $i = 1, 2$. It follows (e).

Case 2 Exactly one of G_1 and G_2 contains an induced P_3.

Without loss of generality, suppose G_1 contains an induced P_3 but G_2 does not. By Lemma 3.11, G_1 is connected and $G_2 \cong K_s$ for some $s \geq 1$. By Lemma 3.10, G contains no induced P_4, so G_1 contains no induced P_4. By Lemma 3.9, G_1 is the join of two non-null graphs, and so the diameter of G_1 is less than 3. Now, we obtain that $G = K_s \vee G_1$, where $d(G_1) < 3$. Thus, the eigenvalues of $D(G_1)$ interlace those of $D(G)$ by Corollary 3.3, and so $G_1 \in \mathcal{G}[-3 \leq \partial_{|G_1|}, \partial_3 \leq -1]$. Again by Lemma 3.11, we have $G_1 = G_1' \vee G_2'$, in which at most one of G_1' and G_2' contains induced P_3. Thus, $G = K_s \vee G_1' \vee G_2'$, where $s \geq 1$.

Now, we may assume that K_r $(r \geq 1)$ is the maximum clique such that $G = K_r \vee F_1 \vee F_2$, where F_1 and F_2 are non-null and at most one of them contains induced P_3. Finally, we show that F_1 and F_2 contain no induced P_3. Suppose, by contradiction, that F_1 contains induced P_3 but F_2 does not. Let $F = F_1 \vee F_2$. Since the diameter of F is less than 3, we have $\partial_3(F) \leq \partial_3(G) \leq -1$ and $\partial_{|F|}(F) \geq \partial_n(G) \geq -3$ by Corollary 3.3. By Lemma 3.11, F_1 is connected and F_2 is complete, say $F_2 = K_{r'}$ $(r' \geq 1)$. By Lemma 3.10, G contains no induced P_4, nor does F_1. Thus, there exists non-null graphs F_1', F_2' such that $F_1 = F_1' \vee F_2'$ by Lemma 3.9, and so $G \cong K_{r+r'} \vee (F_1' \vee F_2')$, which is a contradiction since $r + r' > r$. □

Theorem 3.12 tells us that those graphs belonging to $\mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$ are included in the set of graphs described in Theorem 3.12. Conversely, the graphs described in Theorem 3.12 may not be in $\mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$. In fact, the graph $G = K_1 \vee (K_6 \cup K_1)$ has the form $K_r \vee (K_s \cup K_1)$ characterized in Theorem 3.12(i)(a), however $G \notin \mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$ since $\mathrm{Spec}_D(G) = [8.78, -0.70, (-1)^5, -3.07]$. Naturally, we try to give a complete characterization of the graphs in $\mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$. We first classify those graphs described in Theorem 3.12 into two types.

By simple observation of Theorem 3.12, all graphs characterized in Theorem 3.12(i) can be written as $K_0 \vee F_1 \vee F_2$, where F_1 and F_2 are non-null and contain no induced P_3. Therefore, we get that $G \cong K_r \vee F_1 \vee F_2$ if $G \in \mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$, where $r \geq 0$, F_1 and F_2 are non-null and contain no induced P_3. A graph F containing no induced P_3 will be a complete graph if it is connected and will be a union of some complete graphs otherwise. Therefore, we get the following result.

Corollary 3.13 *If $G \in \mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$, then one of the following cases holds:*

(I) $G \cong K_r \vee (K_{s_1} \cup \cdots \cup K_{s_i})$, where $r \geq 1$ and $i \geq 2$;
(II) $G \cong (K_{s_1} \cup \cdots \cup K_{s_i}) \vee (K_{t_1} \cup \cdots \cup K_{t_j})$, where $i \geq 2$ and $j \geq 2$.

Proof First, we know that $G = K_r \vee F_1 \vee F_2$, where F_i $(i = 1, 2)$ is the union of some complete graphs and $r \geq 0$. Since G is not complete, at most one of F_1 and F_2 is connected. If exactly one of F_1 and F_2 is connected, then $G \cong K_r \vee (K_{s_1} \cup \cdots \cup K_{s_i})$, where $r \geq 1$ and $i \geq 2$. Thus (I) holds. If both of F_1 and F_2 are disconnected, then $G \cong K_r \vee (K_{s_1} \cup \cdots \cup K_{s_i}) \vee (K_{t_1} \cup \cdots \cup K_{t_j})$, where $r \geq 0$, $i \geq 2$ and $j \geq 2$. By Lemma 3.10, H_4 $(= K_1 \vee 2K_1 \vee 2K_1)$ cannot be an induced subgraph of G, we claim that $r = 0$ and so (II) holds. □

We say that $G \in \mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$ is of type-I and type-II if G satisfies (I) and (II) in Corollary 3.13, respectively.

Next, we give a complete characterization of the graphs in $\mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$ by using the forbidden subgraphs $H_4, H_5, I_1, I_2, I_3, I_4$. Denote by $S(m, n) = (mK_2 \cup nK_1) \vee (K_5 \cup K_1)$ $(m + n \geq 1)$, $T_1 = K_4 \cup K_1$, $T_2 = K_3 \cup 2K_1$, $T_3 = K_3 \cup K_1$ and $T_4(m, n) = mK_2 \cup nK_1$ $(m + n \geq 2)$. Moreover, denote by $\mathcal{S} = \{S(m, n) \mid m + n \geq 1\}$, $\mathcal{T}_1 = \{K_r \vee T_i \mid r \geq 1, 1 \leq i \leq 3\} \cup \{K_r \vee T_4(m, n)\}$ and $\mathcal{T}_2 = \{T_i \vee T_j \mid 1 \leq i, j \leq 3\} \cup \{T_i \vee T_4(m, n), T_4(m_1, n_1) \vee T_4(m_2, n_2) \mid 1 \leq i \leq 3\}$. Now, we state one of our main results (Theorem 3.14), which follows from Lemma 3.15 and Lemma 3.16 below.

Table 1 Subclasses of $\mathcal{S} \cup \mathcal{T}_1 \cup \mathcal{T}_2$

	Type-I	Type-II
$\mathcal{S}$	$S(1,0)$, $S(0,1)$	$S(m,0)$ $(m \geq 2)$, $S(0,n)$ $(n \geq 2)$, $S(m,n)$ $(m,n \geq 1)$
$\mathcal{T}_1$	$K_r \vee T_1$, $K_r \vee T_2$, $K_r \vee T_3$, $K_r \vee T_4(m,0)$, $K_r \vee T_4(0,n)$, $K_r \vee T_4(m,n)$	
$\mathcal{T}_2$		$T_1 \vee T_1$,$T_1 \vee T_2$, $T_1 \vee T_3$,$T_1 \vee T_4(m,n)$,$T_1 \vee T_4(m,0)$,$T_1 \vee T_4(0,n)$, $T_2 \vee T_2$,$T_2 \vee T_3$,$T_2 \vee T_4(m,n)$,$T_2 \vee T_4(m,0)$,$T_2 \vee T_4(0,n)$,$T_3 \vee T_3$,$T_3 \vee T_4(m,n)$,$T_3 \vee T_4(m,0)$,$T_3 \vee T_4(0,n)$,$T_4(m_1,n_1) \vee T_4(m_2,n_2)$,$T_4(m_1,n_1) \vee T_4(m_2,0)$,$T_4(m_1,n_1) \vee T_4(0,n_2)$,$T_4(m_1,0) \vee T_4(m_2,0)$,$T_4(m_1,0) \vee T_4(0,n_2)$,$T_4(0,n_1) \vee T_4(0,n_2)$

Theorem 3.14 $\mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1] = \mathcal{S} \cup \mathcal{T}_1 \cup \mathcal{T}_2$.

For the convenience, we first partition the graphs in $\mathcal{S} \cup \mathcal{T}_1 \cup \mathcal{T}_2$ into 32 subclasses in terms of parameters m and n in Table 1.

We calculate all the distance spectra of the graphs in Table 1, which are listed in Appendix. In fact, we concretely calculate some distance eigenvalues of them in details in Lemma 3.16 and the others will be obtained by the same method.

Lemma 3.15 $\mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1] \subseteq \mathcal{S} \cup \mathcal{T}_1 \cup \mathcal{T}_2$.

Proof Let $G \in \mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$. Then G is of type-I or type-II by Corollary 3.13.

Suppose that G is of type-I, i.e., $G \cong K_r \vee (K_{s_1} \cup \cdots \cup K_{s_i})$, where $r \geq 1$ and $i \geq 2$. Without loss of generality, we assume that $s_1 \geq \cdots \geq s_i$. Since I_1 $(= K_1 \vee (K_6 \cup K_1))$ cannot be an induced subgraph of G, we have $s_1 \leq 5$. If $s_1 = 5$, then $s_k \leq 1$ for $k = 2, \ldots, i$ because G contains no induced H_5 $(= K_1 \vee (K_3 \cup K_2))$. Moreover, since I_2 $(= K_1 \vee (K_4 \cup 2K_1))$ cannot be an induced subgraph of G, we have $s_3 = 0$, which implies that $G = K_r \vee (K_5 \cup K_1)$. Note that I_3 $(= K_3 \vee (K_5 \cup K_1))$ cannot be an induced subgraph of G, we obtain that $r = 1$ or $r = 2$, that is, $G = K_1 \vee (K_5 \cup K_1) = S(0,1)$ or $G = K_2 \vee (K_5 \cup K_1) = S(1,0)$. If $s_1 = 4$, then $s_k \leq 1$ for $k = 2, \ldots, i$ because G contains no induced H_5. Moreover, since I_2 cannot be an induced subgraph of G, we have $s_3 = 0$, which implies that $G = K_r \vee (K_4 \cup K_1) = K_r \vee T_1$. Similarly, if $s_1 = 3$, then $G = K_r \vee (K_3 \cup 2K_1) = K_r \vee T_2$ or $G = K_r \vee (K_3 \cup K_1) = K_r \vee T_3$ because H_5 and I_4 $(= K_1 \vee (K_3 \cup 3K_1))$ cannot be induced subgraphs of G. If $s_1 \leq 2$ then $G = K_r \vee (mK_2 \cup nK_1) = K_r \vee T_4(m,n)$.

Suppose that G is of type-II, i.e., $G \cong (K_{s_1} \cup \cdots \cup K_{s_i}) \vee (K_{t_1} \cup \cdots \cup K_{t_j})$, where $i \geq 2$ and $j \geq 2$. Without loss of generality, we assume that $s_1 \geq \cdots \geq s_i$, $t_1 \geq \cdots \geq t_j$ and $s_1 \geq t_1$. Since I_1 $(= K_1 \vee (K_6 \cup K_1))$ cannot be an induced subgraph of G, we have $s_1 \leq 5$. If $s_1 = 5$, then $s_2 = 1$ and $s_3 = 0$ because H_5 and I_2 cannot be induced subgraphs of G. Moreover, we have $t_1 \leq 2$ because I_3 cannot be an induced subgraph of G. Thus $G = (K_5 \cup K_1) \vee (mK_2 \cup nK_1) = S(m,n)$ $(m+n \geq 2)$. Similarly, if $s_1 = 4$, then $G = T_1 \vee T_i$ for $1 \leq i \leq 3$ or $G = T_1 \vee T_4(m,n)$ because H_5, I_2 and I_4

cannot be induced subgraphs of G; if $s_1 = 3$, then $G = T_i \vee T_j$ or $G = T_i \vee T_4(m, n)$ for for $2 \leq i, j \leq 3$ because H_5 and I_4 cannot be induced subgraphs of G; if $s_1 \leq 2$, then $G = T_4(m_1, n_1) \vee T_4(m_2, n_2)$. □

Lemma 3.16 $\mathcal{S} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \subseteq \mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$.

Proof By calculating the distance spectra, we obtain that $\{S(0,1), S(1,0)\} \cup \{T_i \vee T_j \mid 1 \leq i, j \leq 3\} \subseteq \mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$ (see Appendix). It suffices to consider the remaining situations.

First, we consider the graph $K_r \vee T_4(m, n) = K_r \vee (mK_2 \cup nK_1) \in \mathcal{T}_1$, where $m+n \geq 2$. If $m, n \neq 0$, it is easy to see that Π: $V(K_r \vee T_4(m, n)) = V(K_r) \cup V(mK_2) \cup V(nK_1)$ is a distance equitable partition of $K_r \vee T_4(m, n)$ with the distance divisor matrix

$$B_1 = \begin{pmatrix} r-1 & 2m & n \\ r & 4m-3 & 2n \\ r & 4m & 2n-2 \end{pmatrix}.$$

By Theorem 2.3, $\det(xI - B_1) \mid \det(xI - D(K_r \vee T_4(m, n)))$, which implies that the roots of the polynomial

$$\begin{aligned} f_1(x) = \det(xI - B_1) &= x^3 + (6 - 2n - 4m - r)x^2 \\ &\quad + (2mr - 8n - 5r - 12m + nr + 11)x \\ &\quad - (8m + 6n + 6r - 4mr - 3nr - 6) \end{aligned}$$

are distance eigenvalues of $K_r \vee T_4(m, n)$. Note that $f_1(-1) = 2r(m + n - 1) > 0 > f_1(-3) = -8m - 2mr$. By Corollary 2.4, the largest root of $f_1(x)$ is just the distance spectral radius of $K_r \vee T_4$, which is simple and greater than 0. Therefore, by the function image of $f_1(x)$, the roots of $f_1(x)$ satisfy $-3 < \partial_3 < -1 < \partial_2 < \partial_1$. Moreover, by Lemma 3.4, Corollaries 3.6 and 3.7, we obtain that -1, -2 and -3 are distance eigenvalues of G with multiplicities at least $r-1$, $n-1$ and $m-1$, respectively. Thus $\mathrm{Spec}_D(K_r \vee T_4(m, n)) = [\partial_1, \partial_2, (-1)^{m+r-1}, (-2)^{n-1}, \partial_3, (-3)^{m-1}]$ and so $K_r \vee T_4(m, n) \in \mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$. If $m = 0$ or $n = 0$, by using the same method, we get the distance spectrum of $K_r \vee T_4(m, 0)$ and $K_r \vee T_4(0, n)$ (see Appendix), and thus we have $K_r \vee T_4(m, 0)$, $K_r \vee T_4(0, n) \in \mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$ as well. Similarly, we get the distance spectra of $K_r \vee T_i$ for $i = 1, 2, 3$ (see Appendix). Clearly, all these graphs belong to $\mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$.

Next, we consider the graph $S(m, n) = (K_5 \cup K_1) \vee (mK_2 \cup nK_1) \in \mathcal{S}$, where $m + n \geq 2$. If $m, n \neq 0$, it is easy to see that Π: $V(S(m, n)) = V(K_5) \cup V(K_1) \cup V(mK_2) \cup V(nK_1)$ is a distance equitable partition of $S(m, n)$ with the distance divisor matrix

$$B_2 = \begin{pmatrix} 4 & 2 & 2m & n \\ 10 & 0 & 2m & n \\ 5 & 1 & 4m-3 & 2n \\ 5 & 1 & 4m & 2n-2 \end{pmatrix}.$$

By Theorem 2.3, $\det(xI - B_2) | \det(xI - D(S(m, n)))$. It follows that the roots of the polynomial

$$f_2(x) = (x + 3)(x^3 - (2n + 4m + 2)x^2 + (2n + 8m - 28)x + 32m + 24n - 40)$$

are distance eigenvalues of $S(m, n)$. One root of $f_2(x)$ is -3, and the others are the roots of $g(x) = x^3 - (2n + 4m + 2)x^2 + (2n + 8m - 28)x + 32m + 24n - 40$. Note that $g(-1) = 20(m+n) - 15 > 0 > g(-3) = -28n - 1$ and the largest root of $g(x)$ is greater than 0. Thus, by the function image of $g(x)$, the roots of $g(x)$ satisfy $-3 < \partial_3 < -1 < \partial_2 < \partial_1$. Moreover, by Lemma 3.4, Corollaries 3.6 and 3.7, we obtain that -1, -2 and -3 are distance eigenvalues of G with multiplicity at least $m+4$, $n-1$ and $m-1$, respectively. Thus $\mathrm{Spec}_D(S(m, n)) = [\partial_1, \partial_2, (-1)^{m+4}, (-2)^{n-1}, \partial_3, (-3)^m]$, and so $S(m, n) \in \mathcal{G}[-3 \le \partial_n, \partial_3 \le -1]$. If $m = 0$ or $n = 0$, by using the same method, we also get the distance spectrum of $S(m, n)$ (see Appendix), and so $S(m, n) \in \mathcal{G}[-3 \le \partial_n, \partial_3 \le -1]$.

At last, we consider the graph $T_1 \vee T_4(m, n) = (K_4 \cup K_1) \vee (mK_2 \cup nK_1) \in \mathcal{T}_2$, where $m + n \ge 2$. If $m, n \ne 0$, it is easy to see that Π: $V(T_1 \vee T_4(m, n)) = V(K_4) \cup V(K_1) \cup V(mK_2) \cup V(nK_1)$ is a distance equitable partition of $T_1 \vee T_4(m, n)$ with the distance divisor matrix

$$B_3 = \begin{pmatrix} 3 & 2 & 2m & n \\ 8 & 0 & 2m & n \\ 4 & 1 & 4m - 3 & 2n \\ 4 & 1 & 4m & 2n - 2 \end{pmatrix}.$$

By Theorem 2.3, $\det(xI - B_3) | \det(xI - D(T_1 \vee T_4(m, n)))$. Thus the roots of the polynomial

$$\begin{aligned} f_3(x) = x^4 + (2 - 2n - 4m)x^3 - (6m + 5n + 25)x^2 \\ + (42m + 22n - 98)x + 76m + 57n - 96 \end{aligned}$$

are distance eigenvalues of $T_1 \vee T_4$. Note that the derivative of $f_3(x)$ is

$$f_3'(x) = 4x^3 - (12m + 6n - 6)x^2 - 2(6m + 5n + 25)x + 42m + 22n - 98.$$

By simple computation, we have $f_3(-1) = 32m + 32n - 24 > 0$, $f_3(-3) = 4m > 0$, $f_3'(-3) = -30m - 2n - 2 < 0$, $f_3'(-1) = 42m + 26n - 46 > 0$ and $f_3'(3) = -102m - 62n - 86 < 0$. By the function image of $f_3'(x)$, the roots of $f_3'(x)$, denoted by μ_1, μ_2, μ_3, satisfy $-3 < \mu_3 < -1 < \mu_2 < 3 < \mu_1$. Therefore $f_3(x)$ monotonically decreases when $x < -3$. Since $f_3(-3) > 0$, we have $f_3(x) > 0$ for $x \le -3$. Moreover, since $-3 < \mu_3 < -1$ and $f_3(-1), f_3(-3) > 0$, by the function image of $f_3(x)$, we obtain that two roots of $f_3(x)$ lie in the interval $(-3, -1)$, and the other two roots lie in $(-1, +\infty)$. Combining Lemma 3.4, Corollaries 3.6 and 3.7, we obtain the distance spectrum of $T_1 \vee T_4(m, n)$ (see Appendix), and so $T_1 \vee T_4(m, n) \in \mathcal{G}[-3 \le \partial_n, \partial_3 \le -1]$. If $m = 0$ or $n = 0$, by using the same method, we also get the distance spectrum of $T_1 \vee T_4(m, 0)$ and $T_1 \vee T_4(0, n)$ (see Appendix), and

$T_1 \vee T_4(m, 0)$, $T_1 \vee T_4(0, n) \in \mathcal{G}[-3 \le \partial_n, \partial_3 \le -1]$ as well. Similarly, we get the distance spectra of $T_i \vee T_4(m, n)$ for $i = 2, 3, 4$ (see Appendix), and all these graphs belong to $\mathcal{G}[-3 \le \partial_n, \partial_3 \le -1]$. □

Remark 1 In the proof of Lemma 3.16, we compute the distance spectra of three subclasses in Table 1 in detail. In general, for a graph G in Table 1, by Theorem 2.3, we first get some distance eigenvalues by analyzing the roots of the polynomial of the corresponding distance divisor matrix. Then, we get the other distance eigenvalues, which are -1, -2 or -3, by using Lemma 3.4, Corollaries 3.6 and 3.7. At last, by noticing that the number of distance eigenvalues equals to the order of G, we get the distance spectrum of G. Repeating these process, we obtain the spectra of all graphs in Table 1 and we list them in Appendix.

4 The graphs with exactly two distance eigenvalues different from $-1, -3$

Let $\mathcal{F}$ denote the set of connected graphs with all but two adjacency eigenvalues equal to ± 1. Then $\mathcal{F}$ is a special family of graphs with exactly four distinct adjacency eigenvalues. Cioabă et al. [2] completely characterized the graphs in $\mathcal{F}$. Motivated by their work, we try to characterize a special family of graphs with exactly four distinct distance eigenvalues, that is, the graphs having exactly two distance eigenvalues (counting multiplicity) different from -1 and -3. Let $\mathcal{H}$ denote the set of such graphs. In this section, we will give a complete characterization of the graphs in $\mathcal{H}$.

Lemma 4.1 *The set $\mathcal{H}$ is a subset of $\mathcal{G}[-3 \le \partial_n, \partial_3 \le -1]$.*

Proof Let $G \in \mathcal{H}$, and let α, β ($\alpha > \beta$) be the two distance eigenvalues of G different from $-1, -3$. We claim that $\beta > -1$. Otherwise, we have $\partial_2(G) = -1 < -0.73 = \partial_2(P_3)$ ($\mathrm{Spec}_D(P_3) = [2.73, -0.73, -2]$). This implies that G contains no induced P_3 and so is a complete graph, which is impossible because complete graphs have only two distinct distance eigenvalues. Hence, $G \in \mathcal{G}[-3 \le \partial_n, \partial_3 \le -1]$ and our results follows. □

According to Lemma 4.1, we determine all graphs belonging to $\mathcal{H}$ in the following theorem.

Theorem 4.2 *A connected graph G has exactly two distance eigenvalues (counting multiplicity) different from -1 and -3 (i.e., $G \in \mathcal{H}$) if and only if*

(i) $G \cong (K_5 \cup K_1) \vee mK_2$ ($m \ge 1$), *or*
(ii) $G \cong K_r \vee mK_2$ ($r \ge 1, m \ge 2$), *or*
(iii) $G \cong m_1K_2 \vee m_2K_2$ ($m_1, m_2 \ge 2$).

Proof By Appendix, we see that $(K_5 \cup K_1) \vee mK_2$, $K_r \vee mK_2$ and $m_1K_2 \vee m_2K_2$ have exactly two distance eigenvalues different from -1 and -3. Thus the sufficiency follows. Conversely, let G be a graph with exactly two distance eigenvalues different from -1 and -3, i.e., $G \in \mathcal{H}$. By Lemma 4.1, we know that $G \in \mathcal{G}[-3 \le \partial_n, \partial_3 \le -1]$. Thus, the necessity follows because $(K_5 \cup K_1) \vee mK_2$, $K_r \vee mK_2$ and $m_1K_2 \vee m_2K_2$

are the only graphs in $\mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$ with exactly two distance eigenvalues different from -1 and -3 by Appendix. □

By Theorem 4.2, we obtain the following result.

Theorem 4.3 *All graphs with exactly two distance eigenvalues (counting multiplicity) different from -1 and -3 are determined by their distance spectra.*

Proof By Theorem 4.2, every graph in $\mathcal{H}$ has the form $(K_5 \cup K_1) \vee mK_2$ $(m \geq 1)$, or $K_r \vee mK_2$ $(r \geq 1, m \geq 2)$, or $m_1K_2 \vee m_2K_2$ $(m_1, m_2 \geq 2)$. By Appendix, we get the distance spectra of these graphs:

$$\begin{cases} \mathrm{Spec}_D((K_5 \cup K_1) \vee mK_2) = [2m + 2 \pm 2\sqrt{m^2 - 2m + 6}, (-1)^{m+4}, (-3)^m] \\ \mathrm{Spec}_D(K_r \vee mK_2) = [2m + \frac{r}{2} - 2 \pm \frac{\sqrt{(4m-2)^2+(r+2)^2-4}}{2}, (-1)^{m+r-1}, (-3)^{m-1}] \\ \mathrm{Spec}_D(m_1K_2 \vee m_2K_2) = [2m_1 + 2m_2 - 3 \pm 2\sqrt{m_1^2 - m_1m_2 + m_2^2}, \\ \quad (-1)^{m_1+m_2}, (-3)^{m_1+m_2-2}] \end{cases} \tag{3}$$

It is easy to verify that any two graphs of the same form with different parameters cannot share the same distance spectrum. Thus, we only need to consider the distance spectra of these graphs of distinct forms.

First, suppose that $\mathrm{Spec}_D((K_5 \cup K_1) \vee mK_2) = \mathrm{Spec}_D(K_r \vee m'K_2)$. By counting the multiplicities of -3 and -1, we have $m' - 1 = m$ and $m' + r - 1 = m + 4$, which leads to $m' = m + 1$ and $r = 4$. Furthermore, by comparing the distance spectral radius we get

$$2m + 2 + 2\sqrt{m^2 - 2m + 16} = 2m' + \frac{r}{2} - 2 + \frac{\sqrt{(4m' - 2)^2 + (r + 2)^2 - 4}}{2}.$$

Putting $m' = m + 1$ and $r = 4$ into the above equation, we obtain $m = \frac{5}{4}$, which is impossible.

Next, suppose that $\mathrm{Spec}_D((K_5 \cup K_1) \vee mK_2) = \mathrm{Spec}_D(m_1K_2 \vee m_2K_2)$. By counting the multiplicities of -3 and -1, we have $m_1 + m_2 - 2 = m$ and $m_1 + m_2 = m + 4$. Therefore, $m + 2 = m + 4$, a contradiction.

At last, we suppose that $\mathrm{Spec}_D(K_r \vee m'K_2) = \mathrm{Spec}_D(m_1K_2 \vee m_2K_2)$. By counting the multiplicities of -3 and -1, we have $m_1 + m_2 - 2 = m' - 1$ and $m_1 + m_2 = m' + r - 1$, which implies that $m_1 + m_2 = m' + 1$ and $r = 2$. Furthermore, by comparing the distance spectral radius, we have

$$\begin{aligned} 2m' + \frac{r}{2} - 2 + \frac{\sqrt{(4m' - 2)^2 + (r + 2)^2 - 4}}{2} \\ = 2m_1 + 2m_2 - 3 + 2\sqrt{m_1^2 - m_1m_2 + m_2^2}. \end{aligned}$$

Putting $m_1 + m_2 = m' + 1$ and $r = 2$ into the above equation, we obtain that $m_1m_2 = m' = m_1 + m_2 - 1$, which is impossible because $m_1m_2 \geq m_1 + m_2$ due to $m_1, m_2 \geq 2$. □

Remark 2 By Theorem 4.2, we know that $\mathcal{H}$ contains exactly three classes of graphs: $(K_5 \cup K_1) \vee mK_2 \in \mathcal{S}$, $K_r \vee mK_2 \in \mathcal{T}_1$ and $m_1K_2 \vee m_2K_2 \in \mathcal{T}_2$. By Theorem 4.3, all these graphs are determined by their distance spectra. In fact, we have completely characterized the graphs in $\mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$ in Theorem 3.14 and listed their spectra in Appendix. At last, we conjecture that every graph in $\mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$ is determined by its distance spectrum.

Notice that the friendship graph $F_k = K_1 \vee kK_2$ is included in $K_r \vee mK_2$. The following result follows from Theorem 4.3 immediately.

Corollary 4.4 *The friendship graph* $F_k = K_1 \vee kK_2$ *is determined by its distance spectrum.*

Corollary 4.4 provides a witness that distance spectrum is stronger than adjacency spectrum since the friendship graph F_{16} is not DAS.

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5 Appendix: Spectra of graphs in $\mathcal{G}[-3 \leq \partial_n, \partial_3 \leq -1]$

Graphs	Distance Spectra	$f(x)$
$S(0,1) = K_1 \vee (K_5 \cup K_1)$	$[7.66, -0.71, (-1)^4, -2.96]$	–
$S(1,0) = K_2 \vee (K_5 \cup K_1)$	$[8.47, -0.47, (-1)^5, -3]$	–
$S(m,n) = (K_5 \cup K_1) \vee (mK_2 \cup nK_1)\ (m,n \neq 0)$	$[\partial_1, \partial_2, (-1)^{m+4}, (-2)^{n-1}, \partial_3, (-3)^m]$	$[x^3-(2n+4m+2)x^2+(2n+8m-28)x+(32m+24n-40)]$
$S(m,0) = (K_5 \cup K_1) \vee mK_2\ (m \geq 2)$	$[\partial_1, \partial_2, (-1)^{m+4}, (-3)^m]$	$[-x^2+(4m+4)x+20-16m]$
$S(0,n) = (K_5 \cup K_1) \vee nK_1\ (n \geq 2)$	$[\partial_1, \partial_2, (-1)^4, \partial_3, (-2)^{n-1}]$	$[x^3-(2n+2)x^2+(2n-28)x+(24n-40)]$
$K_r \vee T_1 = K_r \vee (K_4 \cup K_1)$	$[\partial_1, \partial_2, (-1)^{r+2}, \partial_3](\partial_3 > -3)$	$[x^3-(r+2)x^2-(2r+19)x+(3r-16)]$
$K_r \vee T_2 = K_r \vee (K_3 \cup 2K_1)$	$[\partial_1, \partial_2, (-1)^{r+1}, -2, \partial_3](\partial_3 > -3)$	$[x^3-(r+3)x^2-(r+24)x+(6r-20)]$
$K_r \vee T_3 = K_r \vee (K_3 \cup K_1)$	$[\partial_1, \partial_2, (-1)^{r+1}, \partial_3](\partial_3 > -3)$	$[x^3-(r+1)x^2-(2r+14)x+(2r-12)]$
$K_r \vee T_4(m,n) = K_r \vee (mK_2 \cup nK_1)\ (n,m \neq 0)$	$[\partial_1, \partial_2, (-1)^{m+r-1}, (-2)^{n-1}, \partial_3, (-3)^{m-1}]$	$[x^3+(6-2n-4m-r)x^2+(2mr-8n-5r-12m+nr+11)x-(8m+6n+6r-4mr-3nr-6)]$
$K_r \vee T_4(m,0) = K_r \vee mK_2\ (m \geq 2)$	$[\partial_1, \partial_2, (-1)^{m+r-1}, (-3)^{m-1}]$	$[x^2+(4-r-4m)x+(2mr-4m-3r+3)]$
$K_r \vee T_4(0,n) = K_r \vee nK_1\ (n \geq 2)$	$[\partial_1, \partial_2, (-1)^{r-1}, (-2)^{n-1}]$	$[x^2+(3-r-2n)x+(nr-2n-2r+2)]$
$T_1 \vee T_1 = (K_4 \cup K_1) \vee (K_4 \cup K_1)$	$[10.71, 1, (-1)^6, -2.71, -3]$	–
$T_1 \vee T_2 = (K_4 \cup K_1) \vee (K_3 \cup 2K_1)$	$[11.32, 1.46, (-1)^5, -2, -2.78, -3]$	–
$T_1 \vee T_3 = (K_4 \cup K_1) \vee (K_3 \cup K_1)$	$[9.65, 0.85, (-1)^5, -2.6, -2.9]$	–
$T_1 \vee T_4(m,n) = (K_4 \cup K_1) \vee (mK_2 \cup nK_1)\ (m,n \neq 0)$	$[\partial_1, \partial_2, (-1)^{m+3}, (-2)^{n-1}, \partial_3, \partial_4, (-3)^{m-1}]$	$[x^4+(2-2n-4m)x^3-(6m+5n+25)x^2+(42m+22n-98)x+76m+57n-96]$
$T_1 \vee T_4(m,0) = (K_4 \cup K_1) \vee mK_2\ (m \geq 2)$	$[\partial_1, \partial_2, (-1)^{m+3}, \partial_3, (-3)^{m-1}]$	$[x^3-4mx^2+(2m-25)x+38m-48]$
$T_1 \vee T_4(0,n) = (K_4 \cup K_1) \vee nK_1\ (n \geq 2)$	$[\partial_1, \partial_2, (-1)^3, (-2)^{n-1}, \partial_3](\partial_3 > -3)$	$[x^3-(2n+1)x^2+(n-22)x+19n-32]$
$T_2 \vee T_2 = (K_3 \cup 2K_1) \vee (K_3 \cup 2K_1)$	$[11.87, 2, (-1)^4, (-2)^2, -2.87, -3]$	–
$T_2 \vee T_3 = (K_3 \cup 2K_1) \vee (K_3 \cup K_1)$	$[10.34, 1.25, (-1)^4, -2, -2.63, -2.95]$	–
$T_2 \vee T_4(m,n) = (K_3 \cup 2K_1) \vee (mK_2 \cup nK_1)\ (m,n \neq 0)$	$[\partial_1, \partial_2, (-1)^{m+2}, (-2)^n, \partial_3, \partial_4, (-3)^{m-1}]$	$[x^4+(1-2n-4m)x^3-(2m+3n+34)x^2+(64m+35n-124)x+104m+78n-120]$

Graphs	Distance spectra	$f(x)$
$T_2 \vee T_4(m,0) = (K_3 \cup 2K_1) \vee mK_2$ $(m \geq 2)$	$[\partial_1, \partial_2, (-1)^{m+2}, -2, \partial_3, (-3)^{m-1}]$	$[x^3 - (4m+1)x^2 + (6m-32)x + 52m - 60]$
$T_2 \vee T_4(0,n) = (K_3 \cup 2K_1) \vee nK_1$ $(n \geq 2)$	$[\partial_1, \partial_2, (-1)^2, (-2)^n, \partial_3](\partial_3 > -3)$	$[x^3 - (2n+2)x^2 + (3n-28)x + 26n - 40]$
$T_3 \vee T_3 = (K_3 \cup K_1) \vee (K_3 \cup K_1)$	$[8.57, 0.73, (-1)^4, -2.57, -2.73]$	–
$T_3 \vee T_4(m,n) = (K_3 \cup K_1) \vee (mK_2 \cup nK_1)$ $(m, n \neq 0)$	$[\partial_1, \partial_2, (-1)^{m+2}, (-2)^{n-1}, \partial_3, \partial_4, (-3)^{m-1}]$	$[x^4 + (3 - 2n - 4m)x^3 - (8m + 6n + 16)x^2 + (28m + 14n - 72)x + 56m + 42n - 72]$
$T_3 \vee T_4(m,0) = (K_3 \cup K_1) \vee mK_2$ $(m \geq 2)$	$[\partial_1, \partial_2, (-1)^{m+2}, \partial_3, (-3)^{m-1}]$	$[x^3 - (4m-1)x^2 - 18x + 28m - 36]$
$T_3 \vee T_4(0,n) = (K_3 \cup K_1) \vee nK_1$ $(n \geq 2)$	$[\partial_1, \partial_2, (-1)^2, (-2)^{n-1}, \partial_3](\partial_3 > -3)$	$[x^3 - 2nx^2 - 16x + 14n - 24]$
$T_4(m_1,n_1) \vee T_4(m_2,n_2) = (m_1K_2 \cup n_1K_1) \vee (m_2K_2 \cup n_2K_1)$ $(m_1, m_2, n_1, n_2 \neq 0)$	$[\partial_1, \partial_2, (-1)^{m_1+m_2}, (-2)^{n_1+n_2-2}, \partial_3, \partial_4, (-3)^{m_1+m_2-2}]$	$[x^4 + (10 - 4(m_1+m_2) - 2(n_1+n_2))x^3 + (12m_1m_2 - 28(m_1+m_2) - 16(n_1+n_2) + 6(m_1n_2 + m_2n_1) + 37)x^2 + (48m_1m_2 - 64(m_1+m_2) - 42(n_1+n_2) + 30(m_1n_2 + m_2n_1) + 18n_1n_2 + 60)x - (48(m_1+m_2-m_1m_2) + 36(n_1+n_2) - 36(m_1n_2 + m_2n_1) - 27n_1n_2 + 36)]$
$T_4(m_1,n_1) \vee T_4(m_2,0) = (m_1K_2 \cup n_1K_1) \vee m_2K_2 (m_1, n_1 \neq 0, m_2 \geq 2)$	$[\partial_1, \partial_2, (-1)^{m_1+m_2}, (-2)^{n_1-1}, \partial_3, (-3)^{m_1+m_2-2}]$	$[x^3 + (8 - 4(m_1+m_2) - 2n_1)x^2 + (12m_1m_2 - 20(m_1+m_2) - 12n_1 + 6m_2n_1 + 21)x - 24(m_1+m_2) + 18m_2n_1 + 24m_1m_2 - 18n_1 + 18]$
$T_4(m_1,n_1) \vee T_4(0,n_2) = (m_1K_2 \cup n_1K_1) \vee n_2K_1 (m_1, n_1 \neq 0, n_2 \geq 2)$	$[\partial_1, \partial_2, (-1)^{m_1}, (-2)^{n_1+n_2-2}, \partial_3, (-3)^{m_1-1}]$	$[x^3 + (7 - 2(n_1+n_2) - 4m_1)x^2 + (6m_1n_2 - 10(n_1+n_2) - 16m_1 + 3n_1n_2 + 16)x - 12(n_1+n_2) + 12m_1n_2 + 9n_1n_2 - 16m_1 + 12]$
$T_4(m_1,0) \vee T_4(m_2,0) = m_1K_2 \vee m_2K_2$ $(m_1, m_2 \geq 2)$	$[\partial_1, \partial_2, (-1)^{m_1+m_2}, (-3)^{m_1+m_2-2}]$	$[x^2 + (6 - 4(m_1+m_2))x - 12(m_1+m_2) + 12m_1m_2 + 9]$
$T_4(m_1,0) \vee T_4(0,n_2) = m_1K_2 \vee n_2K_1$ $(m_1, n_2 \geq 2)$	$[\partial_1, \partial_2, (-1)^{m_1}, (-2)^{n_2-1}, (-3)^{m_1-1}]$	$[x^2 + (6 - 4(m_1+m_2))x - 12(m_1+m_2) + 12m_1m_2 + 9]$
$T_4(0,n_1) \vee T_4(0,n_2) = n_1K_2 \vee n_2K_1$ $(n_1, n_2 \geq 2)$	$[\partial_1, \partial_2, (-2)^{n_1+n_2-2}]$	$[x^2 + (4 - 2(n_1+n_2))x - 4(n_1+n_2) + 3n_1n_2 + 4]$

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