Distance Integral Graphs Generated by Strong Sum and Strong Product

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Abstract　For two connected graphs G and H, the strong sum $G \oplus H$ is the graph with vertex set $V(G) \times V(H)$ and edge set $\{(u,v)(u',v') \mid uu' \in E(G), v = v'\} \cup \{(u,v)(u',v') \mid uu' \in E(G), vv' \in E(H)\}$, and the strong product $G \otimes H$ is the graph with vertex set $V(G) \times V(H)$ and edge set $\{(u,v)(u',v') \mid uu' \in E(G), v = v'\} \cup \{(u,v)(u',v') \mid uu' \in E(G), vv' \in E(H)\} \cup \{(u,v)(u',v') \mid u = u', vv' \in E(H)\}$. In this paper we completely obtain the distances in $G \oplus H$ and $G \otimes H$ when H has diameter less than 3. Furthermore, we get the distance spectra of $G \oplus H$ and $G \otimes H$ when G and H satisfy some conditions. As applications, some distance integral graphs generated by the strong sum and the strong product are obtained. Especially, we get a new infinite class of distance integral graphs generated by the strong product.

Key words　Distance spectrum　Distance integral graph　Strong sum　Strong product

图的强和与强积导出的距离整谱图

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摘　要　对于两个连通图 G 和 H, 它们的强和 $G \oplus H$ 是一个点集为 $V(G) \times V(H)$, 边集为 $\{(u,v)(u',v') \mid uu' \in E(G), v = v'\} \cup \{(u,v)(u',v') \mid uu' \in E(G), vv' \in E(H)\}$ 的图; 它们的强积 $G \otimes H$ 是一个点集为 $V(G) \times V(H)$, 边集为 $\{(u,v)(u',v') \mid uu' \in E(G), v = v'\} \cup \{(u,v)(u',v') \mid uu' \in E(G), vv' \in E(H)\} \cup \{(u,v)(u',v') \mid u = u', vv' \in E(H)\}$ 的图. 当 H 的直径小于 3 时, 本文完全确定了 $G \oplus H$ 和 $G \otimes H$ 中的距离. 进而, 当 G 和 H 满足某些条件时, 我们得到了 $G \oplus H$ 和 $G \otimes H$ 的距离谱. 作为应用，我们得到了一些由强和与强积导出的距离谱图. 特别地, 我们得到了一个新的由强积导出的距离整谱图无穷类.

关键词　距离谱　距离整谱图　强和　强积

1　Introduction

All graphs considered in this paper are finite, undirected and simple. Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. The distance between v_i and v_j, denoted by $d_G(v_i, v_j)$ (or $d(v_i, v_j)$ for short), is defined as the length of a shortest path between them. The diameter $d(G)$ of G is the largest distance in G, that is, $d(G) = \max_{v_i,v_j} d_G(v_i, v_j)$. For a vertex v_i, denote by $N_G^k(v_i)$ the set of vertices having distance k from v_i. It is clear that $V = \cup_{0 \geq k} N_G^k(v_i)$

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for any v_i. Note that the set $N_G^1(v_i)$ is the neighborhood of v_i, which is also denoted by $N_G(v_i)$. The distance matrix of G, denoted by $\mathcal{D}(G)$, is the $n \times n$ matrix whose (i,j)-entry is equal to $d_G(v_i, v_j)$ for $1 \leq i, j \leq n$. Since $\mathcal{D}(G)$ is a real symmetric matrix, all its eigenvalues are real and can be listed as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The multiset of such eigenvalues together with their multiplicities is called the distance spectrum of G, denoted by $\mathrm{Sp}_D(G) = \{[\lambda_1]^{m_1}, \ldots, [\lambda_s]^{m_s}\}$ where $\lambda_1, \ldots, \lambda_s$ are all the distinct eigenvalues and m_i is the multiplicity of λ_i. For more details about the distance matrix we refer the readers to [1].

As usual, we always write P_n, C_n and K_n for the path, the cycle and the complete graph of corresponding orders. For two graphs G and H, the strong sum $G \oplus H$ is the graph whose vertex set is $V(G) \times V(H)$, and two vertices (u_1, w_1) and (u_2, w_2) are adjacent if and only if $u_1u_2 \in E(G)$ and $w_1 = w_2$, or $u_1u_2 \in E(G)$ and $w_1w_2 \in E(H)$. Similarly, the strong product $G \otimes H$ is the graph whose vertex set is $V = V(G) \times V(H)$, vertices (u_1, w_1) and (u_2, w_2) being adjacent if and only if $u_1 = u_2$ and $w_1w_2 \in E(H)$, or $u_1u_2 \in E(G)$ and $w_1 = w_2$, or $u_1u_2 \in E(G)$ and $w_1w_2 \in E(H)$. A graph is called integral if all eigenvalues of its adjacency matrix are integers. The problem to characterize the integral graphs dates back to 1973 when Harary and Schwenk [2] posed the question "Which graphs have integral spectra ?". This problem initiated a significant investigation among algebraic graph theorists, trying to construct and classify integral graphs. Although this problem is easy to state, it turns out to be extremely hard. It has been attacked by many mathematicians during the past 40 years [8, 9, 10, 11, 12, 13], and it is still wide open. With respect to a distance matrix, a connected graph G is called distance integral if all eigenvalues of its distance matrix are integers. Although there is a huge amount of papers that study distance spectrum of graphs and their applications to distance energy of graphs, there are few researches on distance integral graphs. In 2010, Ilić [3] characterized the distance integral circulant graph. In 2011, Renteln [4] characterized the integral Cayley graphs over the Coxeter group. In 2015, Pokorný et al. [5] gave some conditions for the distance integrality on graphs similar to complete split graphs. Very recently, Huang [7] gave some necessary and sufficient conditions for the distance integrity of Cayley graphs over dihedral groups.

In this paper, we derive the distance spectrums of $G \oplus H$ and $G \otimes H$ when G and H are regular with diameters not greater than 2. Particularly, we can remove the restrictions on the graph G when $H = K_n$. From this, we can get a series of distance integral graphs generated by the strong sum and the strong product.

2 Strong sum

In this section we obtain the distance spectrum of the strong sum of two graphs and some distance integral graphs generated by the strong sum. We first prove the following lemma on distance in the strong sum of graphs.

Lemma 2.1 Let G and H be two connected graphs with at least two vertices. If $d(H) \leq 2$, then for any $u = (u_1, w_1), v = (u_2, w_2) \in V(G) \times V(H)$, we have

$$d_{G\oplus H}(u,v)=\begin{cases} 0, & \text{if } u_1=u_2 \text{ and } w_1=w_2,\\ 2, & \text{if } u_1=u_2 \text{ and } w_1\neq w_2,\\ d_G(u_1,u_2), & \text{if } u_1\neq u_2 \text{ and } u_1\nsim u_2,\\ 1, & \text{if } u_1\neq u_2,\, u_1\sim u_2, \text{ and } w_1=w_2 \text{ or } w_1\sim w_2,\\ 2, & \text{if } u_1\neq u_2,\, u_1\sim u_2,\, w_1\neq w_2,\, w_1\nsim w_2 \text{ and } N_G(u_1)\cap N_G(u_2)\neq\emptyset,\\ 3, & \text{if } u_1\neq u_2,\, u_1\sim u_2,\, w_1\neq w_2,\, w_1\nsim w_2 \text{ and } N_G(u_1)\cap N_G(u_2)=\emptyset. \end{cases}$$

Proof It is clear that $d_{G\oplus H}(u,v)=0$ if $u_1=u_2$ and $w_1=w_2$.

Now suppose $u_1=u_2$ and $w_1\neq w_2$. Then by the definition of $G\oplus H$, we have $u\nsim v$, that is, $d_{G\oplus H}(u,v)\geq 2$. Let u_k be a vertex in G such that $u_k\sim u_1$. If $w_1\sim w_2$, then $\{u=(u_1,w_1),(u_k,w_2),(u_1,w_2)=v\}$ forms a uv-path of length 2. If $w_1\nsim w_2$, then there exist w_k such that $w_k\sim w_1$ and $w_k\sim w_2$ since $d(H)\leq 2$. It means that $\{u=(u_1,w_1),(u_k,w_k),(u_1,w_2)=v\}$ is a uv-path of length 2. Thus $d_{G\oplus H}(u,v)=2$.

In what follows, we consider the case that $u_1\neq u_2$. Assume that $d_G(u_1,u_2)=t\geq 1$ and $u_1=s_0s_1\cdots s_t=u_2$ is a shortest u_1u_2-path in G.

Suppose that $t\geq 2$. If $w_1=w_2$, then $u=(s_0,w_1)(s_1,w_1)\cdots(s_t,w_1)=v$ is a uv path of length t. If $w_1\sim w_2$, then $u=(s_0,w_1)(s_1,w_1)\cdots(s_{t-1},w_1)(s_t,w_2)=v$ is a uv-path of length t. If $w_1\nsim w_2$, then there exist a w_k such that w_k is adjacent to w_1 and w_2 due to $d(H)\leq 2$. It follows that $u=(s_0,w_1)(s_1,w_1)\cdots(s_{t-2},w_1)(s_{t-1},w_k)(s_t,w_2)=v$ is a uv-path of length t. Thus $d_{G\oplus H}(u,v)\leq d_G(u_1,u_2)$. Let $u=(x_0,y_0)(x_1,y_1)\cdots(x_p,y_p)=v$ be a shortest uv-path in $G\oplus H$. Therefore, $u_1=x_0x_1\cdots x_p=u_2$ is a u_1u_2-walk by the definition of $G\oplus H$. It implies that $d_G(u_1,u_2)\leq d_{G\oplus H}(u,v)$. Hence $d_{G\oplus H}(u,v)=d_G(u_1,u_2)$.

Suppose that $t=1$, i.e., $u_1\sim u_2$. If $w_1=w_2$ or $w_1\sim w_2$, then $u\sim v$, that is $d_{G\oplus H}(u,v)=1$. If $w_1\neq w_2$ and $w_1\nsim w_2$, then $d_{G\oplus H}(u,v)\geq 2$ and there exist a $w_k\in N_G(w_1)\cap N_G(w_2)$ due to $d(H)\leq 2$. If $N_G(u_1)\cap N_G(u_2)\neq\emptyset$, then there exist a $u_k\in N_G(u_1)\cap N_G(u_2)$. It follows that $u=(u_1,w_1)(u_k,w_k)(u_2,w_2)=v$ is a uv-path of length 2, and thus $d_{G\oplus H}(u,v)=2$. If $N_G(u_1)\cap N_G(u_2)=\emptyset$, then $u=(u_1,w_1)(u_2,w_k)(u_1,w_k)(u_2,w_2)=v$ is a uv-path of length 3, and thus $d_{G\oplus H}(u,v)\leq 3$. We claim $d_{G\oplus H}(u,v)=3$. Otherwise, there exist a uv-path $u=(u_1,w_1)(u',w_k)(u_2,w_2)=v$ of length 2, and thus $u'\in N_G(u_1)\cap N_G(u_2)$, a contradiction.

This completes the proof.

From Lemma 2.1, we obtain the following conclusion.

Theorem 2.1 Let G be an r_1-regular graph with adjacency spectrum $\{r_1,\lambda_2,\ldots,\lambda_m\}$ and H, an r_2-regular graph with adjacency spectrum $\{r_2,\gamma_2,\ldots,\gamma_n\}$. If G is triangle-free, $d(G)\leq 2$ and $d(H)\leq 2$, then the distance spectrum of $G\oplus H$ is

$$\{-2(1+\lambda_i+\lambda_i\gamma_j),\lambda_i n-2\lambda_i-2\lambda_i r_2-2,-2(1+r_1+r_1\gamma_j),$$
$$2mn+r_1n-2r_1-2r_1r_2-2 \mid 2\leq i\leq m,2\leq j\leq n\}.$$

Proof Let the adjacency matrices of G and H be A_1 and A_2 respectively. From Lemma 2.1, we have

$$d_{G\oplus H}(u,v)=\begin{cases} 0, & \text{if } u_1=u_2 \text{ and } w_1=w_2,\\ 2, & \text{if } u_1=u_2 \text{ and } w_1\neq w_2,\\ d_G(u_1,u_2), & \text{if } u_1\neq u_2,\, u_1\not\sim u_2,\\ 1, & \text{if } u_1\neq u_2,\, u_1\sim u_2, \text{ and } w_1=w_2 \text{ or } w_1\sim_2,\\ 3, & \text{if } u_1\neq u_2,\, u_1\sim u_2 \text{ and } w_1\not\sim w_2.\end{cases}$$

By a suitable ordering of vertices of $G\oplus H$, its distance matrix D can be written in the form

$$\begin{aligned} D &= 2I_m\otimes(A_2+\bar{A}_2)+(D_G-A_1)\otimes J_n+A_1\otimes(J_n+2\bar{A}_2)\\ &= 2I_m\otimes(A_2+\bar{A}_2)+D_G\otimes J_n+A_1\otimes 2\bar{A}_2,\end{aligned}$$

where D_G denotes the diatance matrix of G and $\bar{A}_2$ denotes the adjacency matrix of $\bar{H}$. Since $d(G)\leq 2$, we have $D_G=A_1+2\bar{A}_1$ where $\bar{A}_1$ denotes the adjacency matrix of $\bar{G}$. It follows that

$$D=2I_m\otimes(A_2+\bar{A}_2)+(A_1+2\bar{A}_1)\otimes J_n+A_1\otimes 2\bar{A}_2.$$

Assume that $j_m, x_2,\ldots,x_m$ are orthonormal eigenvectors of A_1 with $A_1x_i=\lambda_i x_i$ for $2\leq i\leq m$ and $j_n, y_2,\ldots,y_n$ are orthonormal eigenvectors of A_2 with $A_2y_j=\gamma_j y_j$ for $2\leq j\leq n$, where j_m and j_n are all-one vectors. It is clear that $\bar{A}_1j_m=(m-1-r_1)j_m$, $\bar{A}_1x_i=(-1-\lambda_i)x_i$ for $2\leq i\leq m$, $\bar{A}_2j_n=(n-1-r_2)j_n$ and $\bar{A}_2y_j=(-1-\gamma_j)y_j$ for $2\leq j\leq n$. Thus, we have

$$\begin{aligned} D\cdot(x_i\otimes y_j) &= (2I_m\otimes(A_2+\bar{A}_2)+(A_1+2\bar{A}_1)\otimes J_n+A_1\otimes 2\bar{A}_2)(x_i\otimes y_j)\\ &= 2I_mx_i\otimes(A_2+\bar{A}_2)y_j+(A_1+2\bar{A}_1)x_i\otimes J_ny_j+A_1x_i\otimes 2\bar{A}_2y_j\\ &= 2x_i\otimes -y_j+0+\lambda_ix_i\otimes -2(1+\gamma_j)y_j\\ &= -2x_i\otimes y_j-2\lambda_i(1+\gamma_j)x_i\otimes y_j\\ &= -2(1+\lambda_i+\lambda_i\gamma_j)x_i\otimes y_j,\end{aligned}$$

$$\begin{aligned} D\cdot(x_i\otimes j_n) &= (2I_m\otimes(A_2+\bar{A}_2)+(A_1+2\bar{A}_1)\otimes J_n+A_1\otimes 2\bar{A}_2)(x_i\otimes j_n)\\ &= 2I_mx_i\otimes(A_2+\bar{A}_2)j_n+(A_1+2\bar{A}_1)x_i\otimes J_nj_n+A_1x_i\otimes 2\bar{A}_2j_n\\ &= 2x_i\otimes(n-1)j_n-(\lambda_i+2)x_i\otimes nj_n+\lambda_ix_i\otimes 2(n-r_2-1)j_n\\ &= (2n-2)x_i\otimes j_n-(\lambda_in+2n)x_i\otimes j_n+(2\lambda_in-2\lambda_i-2\lambda_ir_2)x_i\otimes j_n\\ &= (\lambda_in-2\lambda_i-2\lambda_ir_2-2)x_i\otimes j_n,\end{aligned}$$

$$\begin{aligned} D\cdot(j_m\otimes y_j) &= (2I_m\otimes(A_2+\bar{A}_2)+(A_1+2\bar{A}_1)\otimes J_n+A_1\otimes 2\bar{A}_2)(j_m\otimes y_j)\\ &= 2I_mj_m\otimes(A_2+\bar{A}_2)y_j+(A_1+2\bar{A}_1)j_m\otimes J_ny_j+A_1j_m\otimes 2\bar{A}_2y_j\\ &= 2j_m\otimes -j_n+0+r_1j_m\otimes -2(1+\gamma_j)y_j\\ &= -2j_m\otimes y_j-2r_1(1+\gamma_j)j_m\otimes y_j\\ &= -2(1+r_1+r_1\gamma_j)j_m\otimes y_j,\end{aligned}$$

and

$$\begin{aligned} D\cdot(j_m\otimes j_n) &= (2I_m\otimes(A_2+\bar{A}_2)+(A_1+2\bar{A}_1)\otimes J_n+A_1\otimes\bar{A}_2)(j_m\otimes j_n)\\ &= 2I_mj_m\otimes(A_2+\bar{A}_2)j_n+(A_1+2\bar{A}_1)j_m\otimes J_nj_n+A_1j_m\otimes\bar{A}_2j_n\\ &= 2j_m\otimes(n-1)j_n+(2m-r_1-2)j_m\otimes nj_n+r_1j_m\otimes(n-r_2-1)j_n\\ &= (2n-2)j_m\otimes j_n+(2mn-r_1n-2n)j_m\otimes j_n+2r_1(n-r_2-1)j_m\otimes j_n\\ &= (2mn+r_1n-2r_1-2r_1r_2-2)j_m\otimes j_n. \end{aligned}$$

Therefore, $-2(1+\lambda_i+\lambda_i\gamma_j)$, $\lambda_in-2\lambda_i-2\lambda_ir_2-2$, $-2(1+r_1+r_1\gamma_j)$ and $2mn+r_1n-2r_1-2r_1r_2-2$ are eigenvalues of D with eigenvectors $x_i\otimes y_j$, $x_i\otimes j_n$, $j_m\otimes y_j$ and $j_m\otimes j_n$ respectively. As such eigenvectors are linearly independent, the result follows.

From Theorem 2.1, we get the following result immediately.

Corollary 2.1 Let G and H be two integral regular graphs. If G is triangle-free, $d(G)\le 2$ and $d(H)\le 2$, then $G\oplus H$ is distance integral.

Similar to Theorem 2.1, we can also obtain the following result from Lemma 2.1.

Theorem 2.2 Let G be an r_1-regular graph with adjacency spectrum $\{r_1,\lambda_2,\dots,\lambda_m\}$ and H, an r_2-regular graph with adjacency spectrum $\{r_2,\gamma_2,\dots,\gamma_n\}$. If $d(H)\le 2$ and $N_G(u_1)\cap N_G(u_2)\neq\emptyset$ for any $u_1,u_2\in V(G)$, then the distance spectrum of $G\oplus H$ is

$$\{-2-\lambda_i-\lambda_i\gamma_j,-\lambda_i-\lambda_ir_2-2,-2-r_1-r_1\gamma_j,2mn-r_1-r_1r_2-2\mid 2\le i\le m,2\le j\le n\}.$$

Proof Let the adjacency matrices of G and H be A_1 and A_2 respectively. From Lemma 2.1, we have

$$d_{G\oplus H}(u,v)=\begin{cases}0, & \text{if } u_1=u_2 \text{ and } w_1=w_2,\\ 2, & \text{if } u_1=u_2 \text{ and } w_1\neq w_2,\\ d_G(u_1,u_2), & \text{if } u_1\neq u_2,\ u_1\not\sim u_2,\\ 1, & \text{if } u_1\neq u_2,\ u_1\sim u_2, \text{ and } w_1=w_2 \text{ or } w_1\sim w_2,\\ 2, & \text{if } u_1\neq u_2,\ u_1\sim u_2 \text{ and } w_1\not\sim w_2.\end{cases}$$

By a suitable ordering of vertices of $G\oplus H$, its distance matrix D can be written in the form

$$\begin{aligned} D &= 2I_m\otimes(A_2+\bar{A}_2)+(D_G-A_1)\otimes J_n+A_1\otimes(J_n+\bar{A}_2)\\ &= 2I_m\otimes(A_2+\bar{A}_2)+D_G\otimes J_n+A_1\otimes\bar{A}_2, \end{aligned}$$

where D_G denotes the diatance matrix of G and $\bar{A}_2$ denotes the adjacency matrix of $\bar{H}$.

Since $N_G(u_1)\cap N_G(u_2)\neq\emptyset$ for any $u_1,u_2\in V(G)$, we have $d(G)\le 2$ and thus $D_G=A_1+2\bar{A}_1$ where $\bar{A}_1$ denotes the adjacency matrix of $\bar{G}$. It follows that

$$D=2I_m\otimes(A_2+\bar{A}_2)+(A_1+2\bar{A}_1)\otimes J_n+A_1\otimes\bar{A}_2.$$

Assume that $j_m,x_2,\dots,x_m$ are orthonormal eigenvectors of A_1 with $A_1x_i=\lambda_ix_i$ for $2\le i\le m$ and $j_n,y_2,\dots,y_n$ are orthonormal eigenvectors of A_2 with $A_2y_j=\gamma_jy_j$ for $2\le j\le n$. As similar to the proof of Theorem 2.1, we have

$$\begin{aligned} D\cdot(x_i\otimes y_j) &= (2I_m\otimes(A_2+\bar{A}_2)+(A_1+2\bar{A}_1)\otimes J_n+A_1\otimes\bar{A}_2)(x_i\otimes y_j)\\ &= 2I_mx_i\otimes(A_2+\bar{A}_2)y_j+(A_1+2\bar{A}_1)x_i\otimes J_ny_j+A_1x_i\otimes\bar{A}_2y_j\\ &= 2x_i\otimes -y_j+0+\lambda_ix_i\otimes-(1+\gamma_j)y_j\\ &= -2x_i\otimes y_j-\lambda_i(1+\gamma_j)x_i\otimes y_j\\ &= (-2-\lambda_i-\lambda_i\gamma_j)x_i\otimes y_j, \end{aligned}$$

$$\begin{aligned} D\cdot(x_i\otimes j_n) &= (2I_m\otimes(A_2+\bar{A}_2)+(A_1+2\bar{A}_1)\otimes J_n+A_1\otimes\bar{A}_2)(x_i\otimes j_n)\\ &= 2I_mx_i\otimes(A_2+\bar{A}_2)j_n+(A_1+2\bar{A}_1)x_i\otimes J_nj_n+A_1x_i\otimes\bar{A}_2j_n\\ &= 2x_i\otimes(n-1)j_n-(\lambda_i+2)x_i\otimes nj_n+\lambda_ix_i\otimes(n-r_2-1)j_n\\ &= (2n-2)x_i\otimes j_n-(\lambda_in+2n)x_i\otimes j_n+(\lambda_in-\lambda_i-\lambda_ir_2)x_i\otimes j_n\\ &= (-\lambda_i-\lambda_ir_2-2)x_i\otimes j_n, \end{aligned}$$

$$\begin{aligned} D\cdot(j_m\otimes y_j) &= (2I_m\otimes(A_2+\bar{A}_2)+(A_1+2\bar{A}_1)\otimes J_n+A_1\otimes\bar{A}_2)(j_m\otimes y_j)\\ &= 2I_mj_m\otimes(A_2+\bar{A}_2)y_j+(A_1+2\bar{A}_1)j_m\otimes J_ny_j+A_1j_m\otimes\bar{A}_2y_j\\ &= 2j_m\otimes -j_n+0+r_1j_m\otimes-(1+\gamma_j)y_j\\ &= -2j_m\otimes y_j-r_1(1+\gamma_j)j_m\otimes y_j\\ &= (-2-r_1-r_1\gamma_j)j_m\otimes y_j, \end{aligned}$$

and

$$\begin{aligned} D\cdot(j_m\otimes j_n) &= (2I_m\otimes(A_2+\bar{A}_2)+(A_1+2\bar{A}_1)\otimes J_n+A_1\otimes\bar{A}_2)(j_m\otimes j_n)\\ &= 2I_mj_m\otimes(A_2+\bar{A}_2)j_n+(A_1+2\bar{A}_1)j_m\otimes J_nj_n+A_1j_m\otimes\bar{A}_2j_n\\ &= 2j_m\otimes(n-1)j_n+(2m-r_1-2)j_m\otimes nj_n+r_1j_m\otimes(n-r_2-1)j_n\\ &= (2n-2)j_m\otimes j_n+(2mn-r_1n-2n)j_m\otimes j_n+r_1(n-r_2-1)j_m\otimes j_n\\ &= (2mn-r_1-r_1r_2-2)j_m\otimes j_n. \end{aligned}$$

Therefore $-2-\lambda_i-\lambda_i\gamma_j$, $-\lambda_i-\lambda_ir_2-2$, $-2-r_1-r_1\gamma_j$ and $2mn-r_1-r_1r_2-2$ are eigenvalues of D with eigenvectors $x_i\otimes y_j$, $x_i\otimes j_n$, $j_m\otimes y_j$ and $j_m\otimes j_n$ respectively. The result follows.

Corollary 2.2 Let G and H be two integral regular graphs. If $d(H)\le 2$ and $N_G(u_1)\cap N_G(u_2)\neq\emptyset$ for any $u_1,u_2\in V(G)$, then $G\oplus H$ is distance integral.

By taking $G=K_m$ in Theorem 2.2, the following results are obtained immediately.

Corollary 2.3 Let H be an r-regular graph with adjacency spectrum $\{r,\gamma_2,\ldots,\gamma_n\}$. If $d(H)\le 2$ then the distance spectrum of $K_n\oplus H$ is

$$\{[-1+\gamma_j]^{m-1},[r-1]^{m-1},-m-1-(m-1)\gamma_j,2mn-m-(m-1)r_2-1\mid 2\le j\le n\}.$$

Corollary 2.4 Let H be an integral regular graph. If $d(H)\le 2$ then $K_m\oplus H$ is distance integral.

Note that the case will be very different if we take $H=K_n$ in the former results.

Theorem 2.3 Let G be a graph with distance spectrum $\{\mu_1, \mu_2, \ldots, \mu_m\}$. Then

$$\mathrm{Sp}_D(G \oplus K_n) = \{n\mu_i + 2(n-1), [-2]^{m(n-1)} \mid 1 \le i \le m\}.$$

Proof From Lemma 2.1, we have

$$d_{G\oplus K_n}(u,v) = \begin{cases} 0, & \text{if } u_1 = u_2 \text{ and } w_1 = w_2, \\ 2, & \text{if } u_1 = u_2 \text{ and } w_1 \neq w_2, \\ d_G(u_1,u_2), & \text{if } u_1 \neq u_2. \end{cases}$$

By a suitable ordering of vertices of $G \oplus K_n$, its distance matrix D can be written in the form

$$D = 2I_m \otimes A + D_G \otimes J_n,$$

where A denotes the adjacency matrix of K_n and D_G denotes the distance matrix of G.

Since K_n is $(n-1)$-regular, the all-one column vector j_n of order $n \times 1$ is an eigenvector of A to the eigenvalue $n-1$. Let $\{y_j \mid 2 \le j \le n\}$ be the family of $n-1$ linearly independent eigenvectors associated with the eigenvalue -1 of A, then ${j_n}^T y_j = 0$ for $2 \le j \le n$. Let z_i be an eigenvector corresponding to the eigenvalue μ_i of D_G, then $D_G z_i = \mu_i z_i$. Thus we have

$$\begin{aligned} D\cdot(z_i \otimes y_j) &= (2I_m \otimes A + D_G \otimes J_n)(z_i \otimes y_j) \\ &= 2I_m z_i \otimes A y_j + D_G z_i \otimes J_n y_j \\ &= 2z_i \otimes -y_j + 0 \\ &= -2z_i \otimes y_j, \end{aligned}$$

and

$$\begin{aligned} D\cdot(z_i \otimes j_n) &= (2I_m \otimes A + D_G \otimes J_n \\ &= 2I_m z_i \otimes A j_n + D_G z_i \otimes J_n j_n \\ &= 2z_i \otimes (n-1) j_n + \mu_i z_i \otimes n j_n \\ &= 2(n-1)x_i \otimes j_n + \mu_i n z_i \otimes j_n \\ &= (n\mu_i + 2(n-1)) z_i \otimes j_n. \end{aligned}$$

Therefore -2 and $n\mu_i + 2(n-1)$ are eigenvalues of D with eigenvectors $z_i \otimes y_j$ and $z_i \otimes j_n$ respectively. As the eigenvectors belonging to different eigenvalues are linearly independent, the result follows.

From Theorem 2.3, we get the following corollary immediately.

Corollary 2.5 If G is distance integral, then $G \oplus K_n$ is also distance integral.

Remark 2.1 This result has been proved by M. Pokorný et al. [5] by another method.

3 Strong product

In this section we obtain the distance spectrum of the strong product of two graphs and a new class of distance integral graphs generated by the strong product is obtained. We first prove the following lemma on distance in strong product of graphs.

Lemma 3.1 Let G and H be two connected graphs. If $d(H) \leq 2$ then for any $u = (u_1, w_1), v = (u_2, w_2) \in V(G) \times V(H)$, we have

$$d_{G\oplus H}(u,v) = \begin{cases} 0, & \text{if } u_1 = u_2 \text{ and } w_1 = w_2, \\ 1, & \text{if } u_1 = u_2 \text{ and } w_1 \sim w_2, \\ 2, & \text{if } u_1 = u_2 \text{ and } w_1 \not\sim w_2, \\ d_G(u_1,u_2), & \text{if } u_1 \neq u_2 \text{ and } u_1 \not\sim u_2, \\ 1, & \text{if } u_1 \neq u_2, u_1 \sim u_2, \text{ and } w_1 = w_2 \text{ or } w_1 \sim w_2, \\ 2, & \text{if } u_1 \neq u_2, u_1 \sim u_2 \text{ and } w_1 \not\sim w_2. \end{cases}$$

Proof It is clear that $d_{G\oplus H}(u,v) = 0$ if $u_1 = u_2$ and $w_1 = w_2$. Now suppose $u_1 = u_2$ and $w_1 \neq w_2$. If $w_1 \sim w_2$, then we have that $u \sim v$ by the definition of $G \otimes H$, that is, $d_{G\otimes H}(u,v) = 1$. If $w_1 \not\sim w_2$, then we have that $u \not\sim v$ by the definition of $G \otimes H$, that is, $d_{G\otimes H}(u,v) \geq 2$. Since $d(H) \leq 2$, there exists a w_k such that $w_k \sim w_1$ and $w_k \sim w_2$. Then $u = (u_1,w_1)(u_1,w_k)(u_1,w_2) = v$ is a uv-path of length 2. Thus $d_{G\otimes H}(u,v) = 2$.

In what follows, we consider the case of $u_1 \neq u_2$. Let $d_G(u_1,u_2) = t \geq 1$ and $u_1 = s_0 s_1 \cdots s_t = u_2$ be a shortest u_1u_2-path in G. Suppose that $t \geq 2$. If $w_1 = w_2$, then $u = (s_0,w_1)(s_1,w_1)\cdots(s_t,w_1) = v$ is a uv-path of length t. If $w_1 \sim w_2$, then we have $u = (s_0,w_1)(s_1,w_1)\cdots(s_{t-1},w_1)(s_t,w_2) = v$ is a uv-path of length t. If $w_1 \not\sim w_2$, then there exists a w_k such that $w_k \sim w_1, w_2$ since $d(H) \leq 2$. It follows that $u = (s_0,w_1)(s_1,w_1)\cdots(s_{t-2},w_1)(s_{t-1},w_k)(s_t,w_2) = v$ is a uv-path of length t. Thus $d_{G\otimes H}(u,v) \leq d_G(u_1,u_2)$. Let $u = (x_0,y_0),(x_1,y_1)\cdots(x_p,y_p) = v$ be a shortest uv-path in $G\otimes H$, then either $x_i = x_{i+1}$ or $x_i \sim x_{i+1}$ for $0 \leq i \leq p-1$. It follows that $u_2 = x_p$ is reachable from $u_1 = x_0$ by no more than p steps. Thus, $d_G(u_1,u_2) \leq d_{G\otimes H}(u,v)$. Hence $d_{G\otimes H}(u,v) = d_G(u_1,u_2)$. Suppose $t = 1$, i.e., $u_1 \sim u_2$. If $w_1 = w_2$ or $w_1 \sim w_2$, then $u \sim v$. If $w_1 \not\sim w_2$, then $u \not\sim v$, that is $d_{G\otimes H}(u,v) \geq 2$. Since $d(H) \leq 2$, there exists a $w_k \in N_G(w_1) \cap N_G(w_2)$. It follows that $u = (u_1,w_1)(u_2,w_k)(u_2,w_2) = v$ is a uv-path of length 2. Thus, $d_{G\otimes H}(u,v) = 2$.

From Lemma 3.1, we obtain the following result.

Theorem 3.1 Let G be an r_1-regular graph with adjacency spectrum $\{r_1, \lambda_2, \ldots, \lambda_m\}$ and H, an r_2-regular graph with adjacency spectrum $\{r_2, \gamma_2, \ldots, \gamma_n\}$. If $d(G), d(H) \leq 2$, then the distance spectrum of $G \otimes H$ is

$$\{-2-\gamma_j-\lambda_i-\lambda_i\gamma_j, -\lambda_i-\lambda_i r_2-r_2-2, -2-\gamma_j-r_1-r_1\gamma_j, 2mn-r_1-r_2-r_1r_2-2 \mid 2 \leq i \leq m, 2 \leq j \leq n\}.$$

Proof Let the adjacency matrices of G and H be A_1 and A_2 reapectively. From Lemma 3.1, we have

$$d_{G\oplus H}(u,v) = \begin{cases} 0, & \text{if } u_1 = u_2 \text{ and } w_1 = w_2, \\ 1, & \text{if } u_1 = u_2 \text{ and } w_1 \sim w_2, \\ 2, & \text{if } u_1 = u_2 \text{ and } w_1 \not\sim w_2, \\ d_G(u_1,u_2), & \text{if } u_1 \neq u_2 \text{ and } u_1 \not\sim u_2, \\ 1, & \text{if } u_1 \neq u_2, u_1 \sim u_2, \text{ and } w_1 = w_2 \text{ or } w_1 \sim w_2, \\ 2, & \text{if } u_1 \neq u_2, u_1 \sim u_2 \text{ and } w_1 \not\sim w_2. \end{cases}$$

By a suitable ordering of vertices of $G \otimes H$, its distance matrix D can be written in the form

$$\begin{aligned} D &= I_m \otimes (A_2 + 2\bar{A}_2) + (D_G - A_1) \otimes J_n + A_1 \otimes (J_n + \bar{A}_2) \\ &= I_m \otimes (A_2 + 2\bar{A}_2) + D_G \otimes J_n + A_1 \otimes \bar{A}_2, \end{aligned}$$

where D_G denotes the diatance matrix of G and $\bar{A}_2$ denotes the adjacency matrix of $\bar{H}$. Since $d(G) \leq 2$, we have $D_G = A_1 + 2\bar{A}_1$ where $\bar{A}_1$ denotes the adjacency matrix of $\bar{G}$. It follows that

$$D = I_m \otimes (A_2 + 2\bar{A}_2) + (A_1 + 2\bar{A}_1) \otimes J_n + A_1 \otimes \bar{A}_2.$$

Assume that $j_m, x_2, \ldots, x_m$ are orthonormal eigenvectors of A_1 with $A_1 x_i = \lambda_i x_i$ for $2 \leq i \leq m$ and $j_n, y_2, \ldots, y_n$ are orthonormal eigenvectors of A_2 with $A_2 y_j = \gamma_j y_j$ for $2 \leq j \leq n$. We have

$$\begin{aligned} D \cdot (x_i \otimes y_j) &= (I_m \otimes (A_2 + 2\bar{A}_2) + (A_1 + 2\bar{A}_1) \otimes J_n + A_1 \otimes \bar{A}_2)(x_i \otimes y_j) \\ &= I_m x_i \otimes (A_2 + 2\bar{A}_2) y_j + (A_1 + 2\bar{A}_1) x_i \otimes J_n y_j + A_1 x_i \otimes \bar{A}_2 y_j \\ &= x_i \otimes -(2 + \gamma_j) y_j + 0 + \lambda_i x_i \otimes -(1 + \gamma_j) y_j \\ &= -(2 + \gamma_j) x_i \otimes y_j - \lambda_i (1 + \gamma_j) x_i \otimes y_j \\ &= (-2 - \gamma_j - \lambda_i - \lambda_i \gamma_j) x_i \otimes y_j, \end{aligned}$$

$$\begin{aligned} D \cdot (x_i \otimes j_n) &= (I_m \otimes (A_2 + 2\bar{A}_2) + (A_1 + 2\bar{A}_1) \otimes J_n + A_1 \otimes \bar{A}_2)(x_i \otimes j_n) \\ &= I_m x_i \otimes (A_2 + 2\bar{A}_2) j_n + (A_1 + 2\bar{A}_1) x_i \otimes J_n j_n + A_1 x_i \otimes \bar{A}_2 j_n \\ &= x_i \otimes (2n - r_2 - 2) j_n - (\lambda_i + 2) x_i \otimes n j_n + \lambda_i x_i \otimes (n - r_2 - 1) j_n \\ &= (2n - r_2 - 2) x_i \otimes j_n - (\lambda_i n + 2n) x_i \otimes j_n + (\lambda_i n - \lambda_i - \lambda_i r_2) x_i \otimes j_n \\ &= (-\lambda_i - \lambda_i r_2 - r_2 - 2) x_i \otimes j_n, \end{aligned}$$

$$\begin{aligned} D \cdot (j_m \otimes y_j) &= (I_m \otimes (A_2 + 2\bar{A}_2) + (A_1 + 2\bar{A}_1) \otimes J_n + A_1 \otimes \bar{A}_2)(j_m \otimes y_j) \\ &= I_m j_m \otimes (A_2 + 2\bar{A}_2) y_j + (A_1 + 2\bar{A}_1) j_m \otimes J_n y_j + A_1 j_m \otimes \bar{A}_2 y_j \\ &= j_m \otimes -(2 + \gamma_j) j_n + 0 + r_1 j_m \otimes -(1 + \gamma_j) y_j \\ &= -(2 + \gamma_j) j_m \otimes y_j - r_1 (1 + \gamma_j) j_m \otimes y_j \\ &= (-2 - \gamma_j - r_1 - r_1 \gamma_j) j_m \otimes y_j, \end{aligned}$$

and

$$\begin{aligned} D \cdot (j_m \otimes j_n) &= (I_m \otimes (A_2 + 2\bar{A}_2) + (A_1 + 2\bar{A}_1) \otimes J_n + A_1 \otimes \bar{A}_2)(j_m \otimes j_n) \\ &= I_m j_m \otimes (A_2 + 2\bar{A}_2) j_n + (A_1 + 2\bar{A}_1) j_m \otimes J_n j_n + A_1 j_m \otimes \bar{A}_2 j_n \\ &= j_m \otimes (2n - r_2 - 2) j_n + (2m - r_1 - 2) j_m \otimes n j_n + r_1 j_m \otimes (n - r_2 - 1) j_n \\ &= (2n - r_2 - 2) j_m \otimes j_n + (2mn - r_1 n - 2n) j_m \otimes j_n + r_1 (n - r_2 - 1) j_m \otimes j_n \\ &= (2mn - r_1 - r_2 - r_1 r_2 - 2) j_m \otimes j_n. \end{aligned}$$

Therefore $-2 - \gamma_j - \lambda_i - \lambda_i \gamma_j$, $-\lambda_i - \lambda_i r_2 - r_2 - 2$, $-2 - \gamma_j - r_1 - r_1 \gamma_j$ and $2mn - r_1 - r_2 - r_1 r_2 - 2$ are eigenvalues of D with eigenvectors $x_i \otimes y_j$, $x_i \otimes j_n$, $j_m \otimes y_j$ and $j_m \otimes j_n$ respectively. As such eigenvectors are linearly independent, the result follows.

Corollary 3.1 Let G and H be two integral regular graphs. If $d(G), d(H) \leq 2$ then $G \otimes H$ is distance integral.

Note that $G \otimes H = H \otimes G$. If one of G and H is complete, we get the following result.

Theorem 3.2 Let G be a graph with distance spectrum $\{\mu_1, \mu_2, \ldots, \mu_m\}$. Then the distance spectrum of $G \otimes K_n$ is

$$\{n\mu_i + n - 1, [-1]^{m(n-1)} \mid 1 \leq i \leq m\}.$$

Proof Since $d(K_n) = 1 \leq 2$, we have

$$d_{G\otimes K_n}(u,v) = \begin{cases} 0, & \text{if } u_1 = u_2 \ and\ w_1 = w_2, \\ 1, & \text{if } u_1 = u_2 \ and\ w_1 \neq w_2, \\ d_G(u_1,u_2), & \text{if } u_1 \neq u_2, \end{cases}$$

from Lemma 3.1. By a suitable ordering of vertices of $G \otimes K_n$, its distance matrix D can be written in the form

$$D = I_m \otimes A + D_G \otimes J_n,$$

where A denotes the adjacency matrix of K_n and D_G denotes the distance matrix of G.

Since K_n is $(n-1)$-regular, the all-one column vector j_n of order $n \times 1$ is an eigenvector of A to the eigenvalue $n-1$. Let $\{y_j \mid 2 \leq j \leq n\}$ be the family of $n-1$ linearly independent eigenvectors associated with the eigenvalue -1 of A, then we have ${j_n}^T y_j = 0$ for $2 \leq j \leq n$. Let z_i be an eigenvector corresponding to the eigenvalue μ_i of D_G, then we have $D_G z_i = \mu_i z_i$.

Now we have

$$\begin{aligned} D\cdot(z_i \otimes y_j) &= (I_m \otimes A + D_G \otimes J_n)(z_i \otimes y_j) \\ &= I_m z_i \otimes A y_j + D_G z_i \otimes J_n y_j \\ &= z_i \otimes -y_j + 0 \\ &= -1 z_i \otimes y_j, \end{aligned}$$

and

$$\begin{aligned} D\cdot(z_i \otimes j_n) &= (I_m \otimes A + D_G \otimes J_n \\ &= I_m z_i \otimes A j_n + D_G z_i \otimes J_n j_n \\ &= z_i \otimes (n-1) j_n + \mu_i z_i \otimes n j_n \\ &= (n-1) x_i \otimes j_n + \mu_i n z_i \otimes j_n \\ &= (n\mu_i + n - 1) z_i \otimes j_n. \end{aligned}$$

Therefore -1 and $n\mu_i + n - 1$ are eigenvalues of D with eigenvectors $z_i \otimes y_j$ and $z_i \otimes j_n$ respectively. As such eigenvectors are linearly independent, the theorem follows.

Corollary 3.2 If G is distance integral, then $G \otimes K_n$ is also distance integral.

Remark 3.1 From Corollary 3.2, we could construct a series of distance integral graphs. For example, the cycle C_6 is distance integral, then $C_6 \otimes K_n$ is distance integral for any positive integer n.

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