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The access time of random walks on trees with given partition

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ABSTRACT

Denote by $\mathcal{T}(s,t)$ the set of trees, whose vertex set can be partitioned into two independent sets of sizes s and t respectively. Given a tree T with stationary distribution π and a vertex $v \in T$, the access time $H_T(\pi, v)$ is the expected length of optimal stopping rules from π to v. In this paper, we get a sharp upper bound for $\max_{v\in T} H_T(\pi, v)$ and a sharp lower bound for $\min_{v\in T} H_T(\pi, v)$ among $\mathcal{T}(s,t)$, respectively. The corresponding extremal graphs are also obtained. As a byproduct, it is proved that the path P_n maximizes $\max_{v\in T} H_T(\pi, v)$ and the star $K_{1,n-1}$ minimizes $\min_{v\in T} H_T(\pi, v)$ among all trees on n vertices.

1. Introduction

Assume that G is a simple graph with vertex set V and edge set E. The *distance* of two vertices $u, v \in V$ is the length of a shortest path from u to v, denoted by $d_G(u,v)$. The *diameter* $d(G)$ of G is the largest distance among the distances of all pairs of vertices in G, that is, $d(G) = \max\{d_G(u,v) \mid u, v \in V\}$. The *neighborhood* of u is the set of vertices adjacent to u, denote by $N_G(u)$. The degree $d_G(u)$ of u is the cardinality of $N_G(u)$, that is, $d_G(u) = |N_G(u)|$. If it will not arouse confusion, we delete the subscript G in the notations.

A *random walk* on a graph G is a sequence of vertices $(w_0, w_1, \ldots, w_t, \ldots)$ such that $\Pr[w_{t+1} = j \mid w_t = i]$ is $1/d(i)$ if i is adjacent to j, and 0 otherwise. We refer the reader to [6] and [7] for more details about random walks on graphs. For two vertices $u, v \in V$, the *hitting time* $H_G(u,v)$ is the expected number of steps before a walk started at u visits vertex v. If G is not bipartite, then the distribution of w_t tends to the *stationary distribution* $\pi^{(G)}$ when t tends to infinity, where $\pi_u^{(G)} = d(u)/2|E|$. For bipartite G, we have the same convergence if we consider lazy random walks in which we remain at the current state with probability $1/2$, at the cost of doubling the excepted length of any walk. For simplicity, we will consider non-lazy walks on trees. Throughout the paper, we always assume $n \geq 4$ since there is a unique tree structure for $n \leq 3$.

In this paper, we study some quantities under certain *exact stopping rules* in Markov chains. Given an initial distribution σ and a target distribution τ, a (σ,τ)-*stopping rule* halts a random walk whose initial state is drawn from σ so that the distribution of the final state is τ. The *access time* $H_G(\sigma,\tau)$ is the minimum expected length of all stopping rules that arrive

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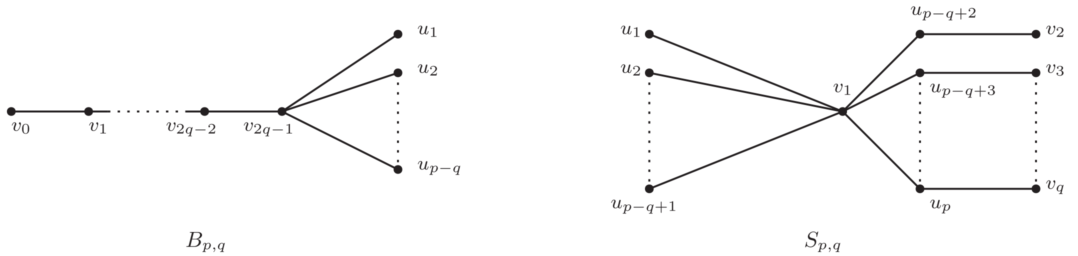

Fig. 1. The extremal graphs.

at the target distribution τ, when starts from the initial distribution σ. If $\sigma = u$ and $\tau = v$ are both single vertices, then $H_G(\sigma, \tau) = H_G(u, v)$ is just the hitting time. For more details, see [3,8,9].

The extremal graph structures with respect to random walks are well studied in the literature. Especially after the appearance of the two excellent books of Doyle and Snell [5] and Aldous [1], a large amount of attention has been paid to the random walks on graphs. Lawle [10] obtained an upper bound of hitting time in connected graph. Brightwell and Winkler [11] further determined the extremal graph with the maximum hitting time. Subsequently, many scholars study the hitting time on trees [13]. Beveridge and Meng [3] obtained some results about the access time starting from stable distribution to visit a single vertex. These results reveal that, most of results are focused on hitting time since the access time is more difficult to handle than hitting time. For other related topics, one may refer to [4].

Let T be a tree on n vertices. It is clear that T is a bipartite graph with partition $V(T) = V_1 \cup V_2$, where V_1 and V_2 are determined by T. If $|V_1| = s$ and $|V_2| = t$, then T is called an (s,t)-tree. Without loss of generality, we always assume that $s \geq t$. For two positive integers s and t with $s \geq t$, let $\mathcal{T}(s,t)$ be the set of all (s,t)-trees. In this paper, we focus on the access time from stationary distribution to a singleton of trees in $\mathcal{T}(s,t)$. In detail, we get a sharp upper bound for $\max_v H_T(\pi^T, v)$ and a sharp lower bound for $\min_v H_T(\pi^T, v)$ among $\mathcal{T}(s,t)$, respectively. The corresponding extremal graphs are also obtained. We list our main results below.

For two positive integers p and q with $p \geq q$, the *broom graph* is obtained by attaching $p-q$ pendent vertices to the end vertex v_{2q-1} of the path $P = v_0 v_1 \cdots v_{2q-1}$ (see Fig. 1). Clearly, $B_{p,q} \in \mathcal{T}(p,q)$ and its diameter $d(B_{p,q}) = 2q$. The vertex v_0 is called the *handle* of $B_{p,q}$ and the pendent vertices attached to v_{2q-1} are called the *tails* of $B_{p,q}$.

Theorem 1.1. *For any $T \in \mathcal{T}(p,q)$, it holds that*

$$\max_{v \in V(T)} H_T(\pi^T, v) \leq \frac{p-q}{2(p+q-1)} + \frac{(2p-1)(2q-1)(2p+2q-3)}{2(p+q-1)} + \frac{(2q-1)(2q-2)(2q-6p+3)}{6(p+q-1)}$$

with equality if and only if $T = B_{p,q}$ and the vertex achieving $\max\limits_{v} H_{B_{p,q}}(\pi^{B_{p,q}}, v)$ is the handle of $B_{p,q}$.

From Theorem 1.1, it is easy to obtain the graph maximizing $\max_{v \in V(T)} H_T(\pi^T, v)$ among all trees.

Corollary 1.2. *If T is a tree on n vertices, then it holds that*

$$\max_{v \in V(T)} H_T(\pi^T, v) \leq \frac{4n^3 - 12n^2 + 11n - 3}{6(n-1)},$$

with equality if and only if $T = P_n$.

Proof. For any tree T on n vertices, assume that $T \in \mathcal{T}(p,q)$ for some p and q with $p \geq q$ and $p + q = n$. Theorem 1.1 indicates that

$$\begin{aligned}
&\max\nolimits_{v \in V(T)} H_T(\pi^T, v) \\
\leq\ & \frac{p-q}{2(p+q-1)} + \frac{(2p-1)(2q-1)(2p+2q-3)}{2(p+q-1)} + \frac{(2q-1)(2q-2)(2q-6p+3)}{6(p+q-1)} \\
=\ & \frac{32q^3 - 48nq^2 + (24n^2 - 8)q - (12n^2 - 15n + 3)}{6(n-1)},
\end{aligned}$$

with equality if and only if $T = B_{p,q}$. Let $f(q) = 32q^3 - 48nq^2 + (24n^2 - 8)q - (12n^2 - 15n + 3)$. We have $f'(q) = 8(q - \frac{6n-1}{12})(q - \frac{6n+1}{12})$. Thus, by the image of the function $f(q)$, we have $f(q)$ increases along with q when $q \leq \frac{6n-1}{12}$. Note that $q \leq n/2$. Thus, the maximum value $f_{\max}(q) = f(n/2) = f((n-1)/2) = 4n^3 - 12n^2 + 11n - 3$. Note that $B(p,q) = P_n$ when $q = \lfloor n/2 \rfloor$. The result follows. □

For two positive integers $p \geq q$, let $S_{p,q}$ be the graph obtained from the star $K_{1,p}$ by attaching a new vertex to each pendent vertex in U, where U is the subset of pendent vertices of $K_{1,p}$ with $|U| = q-1$. The vertices of $S_{p,q}$ are labelled as in Fig. 1. Clearly, $S_{p,q} \in \mathcal{T}(p,q)$.

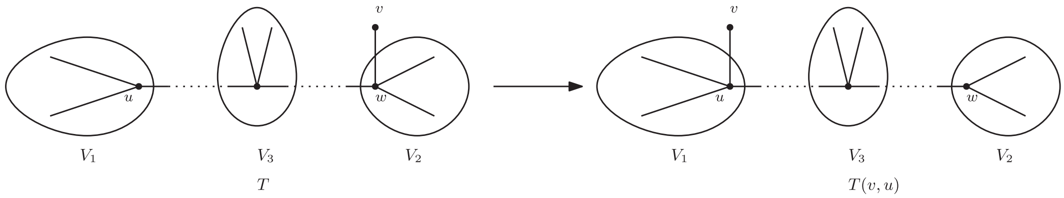

Fig. 2. The transformation from T to $T(v,u)$.

Theorem 1.3. *For any tree $T \in \mathcal{T}(p,q)$, it holds that*

$$\min_{v} H_T(\pi^T, v) \geq \frac{p+9q-9}{2(p+q-1)}$$

with equality if and only if $T = S_{p,q}$ and the vertex achieving $\min_{v\in V(T)} H_T(\pi^T, v)$ is the center vertex v_1.

From Theorem 1.3, it is easy to obtain the graph minimizing $\min_{v\in V(T)} H_T(\pi^T, v)$ among all trees.

Corollary 1.4. *If T is a tree on n vertices, then it holds that*

$$\min_{v} H_T(\pi^T, v) \geq \frac{n-1}{2n-1},$$

with equality if and only if $T = K_{1,n-1}$.

Proof. Assume that $T \in \mathcal{T}(p,q)$ with $p \geq q$ and $p+q=n$. Theorem 1.3 implies that

$$\min_{v} H_T(\pi^T, v) \geq \frac{n+8q-9}{2n-1},$$

with equality if and only if $T = S_{p,q}$. Note that $\frac{n+8q-9}{2n-1}$ achieving the minimum value $(n-1)/(2n-1)$ if and only if $q=1$, and $S(n-1,1)$ is just the star $K_{1,n-1}$ in this case. The result follows. □

2. Preliminary

In this part, we present some useful results which will be used frequently.

For a graph $G=(V,E)$, an initial distribution σ and a vertex $v\in V$, Lovász [9] obtained that

$$H_G(\sigma, v) = \sum_{u\in V} \sigma_u H_G(u,v).$$

Therefore, for the case of $G=T$ being a tree with order n and $\sigma = \pi^T$, we have

$$H_T(\pi^T, v) = \sum_{u\in V} \pi_u^T H_T(u,v) = \frac{1}{2(n-1)} \sum_{u\in V} d(u) H_T(u,v). \tag{1}$$

For a tree T and two vertices $u,v \in V(T)$, let $P_{u,v}^T$ be the unique path from u to v in T.

Lemma 2.1 ([12])**.** *Let $T=(V,E)$ be a tree and $u,v\in V$. If $k \in V(P_{u,v}^T)$, then $H_T(u,v) = H_T(u,k) + H_T(k,v)$.*

Lemma 2.2 ([2])**.** *Let $T=(V,E)$ be a tree and $u,v\in V$. For any $w\in V(P_{u,v}^T)$, denote by $T_w^{u,v}$ the component of $T - E(P_{u,v}^T)$ containing w. If $|E(T_w^{u,v})| = m_w^T$, then $H_T(u,v) = d(u,v)^2 + 2\sum_{w\in V(P_{u,v}^T)} m_w^T d(w,v)$.*

From Lemma 2.2, one can easily get the following result.

Corollary 2.1. *Let T be a tree on order n. If v is a pendent vertex with neighbor u, then $H_T(u,v) = 2n-3$ and $H_T(v,u) = 1$.*

Proof. It is clear that $V(P_{u,v}^T) = \{u,v\}$. Therefore, Lemma 2.2 indicates that

$$\begin{cases} H_T(u,v) = d^2(u,v) + 2(m_u^T d(u,v) + m_v^T d(v,v)) = 1 + 2m_u^T, \\ H_T(v,u) = d^2(v,u) + 2(m_u^T d(u,u) + m_v^T d(v,u)) = 1 + 2m_v^T. \end{cases}$$

Note that $m_u^T = 2n-2$ and $m_v^T = 0$. It yields that $H_T(u,v) = 2n-3$ and $H_T(v,u) = 1$. □

In what follows, we introduce a transformation of a tree. Let $T=(V,E)$ be a tree, v a pendent vertex with neighbor w and u a non-pendent vertex distinct from w. The tree $T(v,u)$ is obtained from T by deleting the edge wv and adding the edge uv (see Fig. 2). For convenience, we write $T' = T(v,u)$. Let $V_1 = T_u^{u,w}, V_2 = T_w^{u,w} \setminus \{v\}$ and $V_3 = V \setminus (T_u^{u,w} \cup T_w^{u,w})$. Let $U_1 = \{x \in V_2 \cup V_3 \mid V(P_{x,v_2}^{T'}) \cap V(P_{u,w}^{T'}) \neq \emptyset\}$ and $U_2 = (V_2 \cup V_3) \setminus U_1$. In the following lemma, we calculate $H_{T'}(x,y)$ for $x,y \in V(T')$.

Lemma 2.3. *As the symbols above, for any $x, y \in V(T')$, we have*

$$H_{T'}(x,y) = \begin{cases} H_{T'}(x,u) + (2n-3), & \text{if } x \neq v \text{ and } y = v, \\ H_T(x,y), & \text{if } x, y \in V_1; \\ H_T(x,u) - 2d_T(w,u), & \text{if } x \in V_2 \text{ and } y = u, \\ H_T(u,y) + 2d_T(w,u), & \text{if } x = u \text{ and } y \in V_2, \\ H_T(x,y) + 2d_T(f_x, w), & \text{if } x \in U_1 \text{ and } y \in V_2, \\ H_T(x,y), & \text{if } x \in U_2 \text{ and } y \in V_2, \\ H_T(x,u) - 2d_T(g_x,u), & \text{if } x \in V_2 \cup V_3 \text{ and } y = u, \\ H_T(u,y) + 2d_T(u,h_u), & \text{if } x = u \text{ and } y \in V_3, \\ H_T(x,y) + 2d_T(l_x^+, l_x^-), & \text{if } x \in V_2 \cup V_3 \text{ and } y \in V_3, \end{cases}$$

where f_x is the first point that $P^{T'}_{x,y}$ meets $P^{T'}_{uw}, g_x$ is the first point that $P^{T'}_{x,u}$ meets $P^{T'}(w,u), h_u$ is the last point that $P^{T'}_{u,y}$ meets $P^{T'}(u,w)$, and $l_x^+, l_x^- \in V(P^{T'}_{x,y} \cap V(P^{T'}_{u,w}))$ are respectively the points nearest to w and u.

Proof. Since each path u lies on the path $P^{T'}_{x,v}$ for any $x \neq v$, by Lemma 2.1, we have $H_{T'}(x,v) = H_{T'}(x,u) + H_{T'}(u,v)$. Corollary 2.1 indicates that $H_{T'}(u,v) = 2n-3$. Therefore, $H_{T'}(x,v) = H_{T'}(x,u) + (2n-3)$. If $x, y \in V_1$, then $d_{T'}(x,y) = d_T(x,y), P^{T'}_{x,y} = P^T_{x,y}, d_{T'}(z,y) = d_T(z,y)$ and $m_z^{T'} = m_z^T$ for any $z \in V(P^{T'}_{x,y})$. By Lemma 2.2, we have

$$\begin{aligned} H_{T'}(x,y) &= d^2_{T'}(x,y) + 2\textstyle\sum_{z\in V(P^{T'}_{x,y})} m_z^{T'} d_{T'}(z,y) \\ &= d^2_T(x,y) + 2\textstyle\sum_{z\in V(P^{T}_{x,y})} m_z^{T} d_{T}(z,y) \\ &= H_T(x,y). \end{aligned}$$

Similarly, all other cases could be obtained by Lemma 2.2 and immediate calculations. □

Lemma 2.4 ([3]). *For any two vertices i, j in a tree T,*

$$H_T(i,j) = \sum_{k\in V} l_T(i,k;j) d(k), \tag{2}$$

where $l_T(i,k;j) = 1/2(d(i,j) + d(k,j) - d(i,k))$ measures the length of the intersection of the $P^T_{i,j}$ and the $P^T_{k,j}$.

Lemma 2.5 ([3]). *For any tree T, $\max_{i\in V} H_T(\pi^T, i)$ must be achieved at a leaf.*

Given a vertex i, i *-pessimal vertex* [3] denoted by i' is the vertex achieving $H(i', i) = \max_{j\in V} H(j,i)$.

Lemma 2.6 ([3]). *For any tree T, for any vertex u in T, the u-pessimal vertex u' must be a leaf.*

Lemma 2.7. *For a tree T and a vertex $v_0 \in V(T)$, assume that v_0' is a v_0-pessimal vertex. If $N(v_0') = \{z\}$ then all components of $T - z$ but the one that contains v_0 of $T - z$ are isolated vertices.*

Proof. Suppose to the contrary that there is another component W which is not an isolate vertex. Therefore, there exist $x, y \in W$ such that $x \sim z$ and $y \sim x$. By simple calculations, we have

$$H_T(y, v_0) = H_T(y,z) + H_T(z, v_0) > 1 + H_T(z, v_0) = H_T(v_0', v_0),$$

which contradicts the definition of v_0'. □

Let $T \in \mathcal{T}(s,t)$ be a tree with vertex $u \in V(T)$ satisfying $H_T(\pi^T, u) = \min_x H_T(\pi^T, x)$ and $T \neq P_n$. Let u' be a u-pessimal vertex. Let z be nearest vertex to u' on T with degree at least 3. Let x be a leaf in $V \setminus \{u, u'\}$ such that the $P^T_{x,u}$ contains z.

Lemma 2.8 ([3]). *For the above symbols, we have*
$$\max_{i\in V} H_T(\pi^T, i) = H_T(\pi^T, u) < H_{T(x,u')}(\pi^{T(x,u')}, u) \le \max_{j\in V(T(x,u'))} H_{T(x,u')}(\pi^{T(x,u')}, j).$$

Let $T \in \mathcal{T}(p,q)$ be a tree with vertex $u_0 \in V(T)$ satisfying $H_T(\pi^T, u_0) = \min_x H_T(\pi^T, x)$. Assume that $u_1, u_2, \cdots, u_m$ are neighbors of u_0 and they are not leaves. The forest $T - u_0$ has $d(u_0)$ components. Let $T_1, T_2, \cdots, T_m$ be the vertex sets of the components containing $u_1, u_2, \cdots, u_m$ respectively. If u_0' is on the $P^{T'}_{z,u_0}$, for some $1 \le i \le m$, let $V_1 = T^{u_i,z}_{u_i}, V_2 = T^{u_0',z}_{u_0'} \setminus V_1, V_4 = T_z^{u_i,z}$ and $V_3 = V \setminus (V_1 \cup V_2 \cup V_4)$.

Lemma 2.9. *For the symbols as above, if x is a pendent vertex of T_i not adjacent to u_i, denote by $T' = T(x, u_i), \pi' = \pi^{T'}$. Let z be the neighbor of x, then the following statements hold.*

For any $v_1 \in V_1 \setminus u_i$, we have

(i) $\pi_{v_1}^{T'} H_{T'}(v_1, u_0) - \pi_{v_1}^{T} H_T(v_1, u_0) = 0, \quad \pi_{v_1}^{T'} H_{T'}(v_1, u_0') - \pi_{v_1}^{T} H_T(v_1, u_0') > 0.$

For any $v_2 \in V_2$, we have

(ii) $\pi_{v_2}' H_{T'}(v_2, u_0') - \pi_{v_2}^{T} H_T(v_2, u_0') > 0.$

For any $v \in V_3 \cup V_4 \setminus x$, we have

(iii) $\pi_{v}' H_{T'}(v, u_0) - \pi_{v}^{T} H_T(v, u_0) = \pi_{v}' H_{T'}(v, u_0') - \pi_{v}^{T} H_T(v, u_0').$

For x and x', we have

(iv) $H_{T'}(x', u_0) - H_T(x, u_0) \le H_{T'}(x', u_0') - H_T(x, u_0').$

For z and u_i, we have

(v) $\pi_{z}' H_{T'}(z, u_0) - \pi_{z}^{T} H_T(z, u_0) + \pi_{u_i}' H_{T'}(u_i, u_0) - \pi_{u_i}^{T} H_T(u_i, u_0) < \pi_{z}' H_{T'}(z, u_0') - \pi_{z}^{T} H_T(z, u_0') + \pi_{u_i}' H_{T'}(u_i, u_0') - \pi_{u_i}^{T} H_T(u_i, u_0').$

Proof. (i) is obvious.

For any $v_2 \in V_2$, it holds that

$$
\begin{aligned}
&\pi_{v_2}' H_{T'}(v_2, u_0) = \pi_{v_2}^{T}(d^2(v_2, u_0) + 2\sum_{i\in V(P)} m_i' d(i, u_0)) \\
&= \pi_{v_2}^{T}[d^2(v_2, u_0) + 2\sum_{i\in V(P)\setminus\{u_i, v_2\}} m_i^T d(i, u_0) + 2(m_{u_i}^T + 2)d(u_i, u_0) + 2(m_{v_2}^T - 2)d(v_2, u_0) \\
&= \pi_{v_2}^{T} H_T(v_2, u_0) + d(u_i, u_0) - d(v_2, u_0) > \pi_{v_2}^{T} H_T(v_2, u_0).
\end{aligned}
$$

Thus (ii) holds.

For any $v \in V_3 \cup V_4 \setminus x$, it holds that

$$
\begin{aligned}
&\pi_{v}' H_{T'}(v, u_0) - \pi_{v}^{T} H_T(v, u_0) \\
&= \pi_{v}^{T}(H_{T'}(v, u_0') + H_{T'}(u_0', u_0) - H_T(v, u_0') - H_T(u_0', u_0)) \\
&< \pi_{v}^{T}(H_{T'}(v, u_0') - H_T(v, u_0')) = \pi_{v}' H_{T'}(v, u_0') - \pi_{v}^{T} H_T(v, u_0').
\end{aligned}
$$

Thus (iii) holds.

For x and x', it is clear that

$$
\begin{aligned}
&H_T(x, u_0) - H_T(x, u_0') = H_T(u_0', u_0) \\
&H_{T'}(x', u_0) - H_{T'}(x', u_0') \le 1 + H_{T'}(u_i, u_0) - 1 = H_T(u_i, u_0) \le H_T(u_0', u_0).
\end{aligned}
$$

Therefore, $H_{T'}(x', u_0) - H_T(x, u_0) \le H_{T'}(x', u_0') - H_T(x, u_0')$. Thus (iv) holds.

At last, we consider the vertices z and u_i. It holds that

$$
\begin{aligned}
&\pi_{z}' H_{T'}(z, u_0) - \pi_{z}^{T} H_T(z, u_0) \\
&= \pi_{z}' H_{T'}(z, u_0') - \pi_{z}^{T} H_T(z, u_0') + (\pi_z^T - \frac{1}{2|E|}) H_{T'}(u_0', u_0) - \pi_{z}^{T} H_T(u_0', u_0) \\
&< \pi_{z}' H_{T'}(z, u_0') - \pi_{z}^{T} H_T(z, u_0') - \frac{1}{2|E|}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\pi_{u_i}' H_{T'}(u_i, u_0) - \pi_{u_i}^{T} H_T(u_i, u_0) \\
&= \pi^{T}(H_{T'}(u_i, u_0) - H_T(u_i, u_0)) + \frac{1}{2|E|} H_{T'}(u_i, u_0) \\
&= \frac{1}{2|E|} H_{T'}(u_i, u_0) < \frac{1}{2|E|} H_{T'}(u_i, u_0) + \pi_{u_i}' H_{T'}(u_i, u_0') - \pi_{u_i}^{T} H_T(u_i, u_0').
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\pi_{z}' H_{T'}(z, u_0) - \pi_{z}^{T} H_T(z, u_0) + \pi_{u_i}' H_{T'}(u_i, u_0) - \pi_{u_i}^{T} H(u_i, u_0) \\
&< \pi_{z}' H_{T'}(z, u_0') - \pi_{z}^{T} H_T(z, u_0') + \pi_{u_i}' H_{T'}(u_i, u_0') - \pi_{u_i}^{T} H_T(u_i, u_0').
\end{aligned}
$$

Thus (v) holds. □

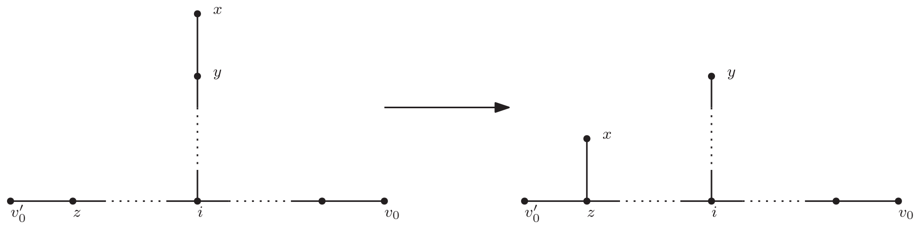

Fig. 3. The operation in Lemma 3.1.

3. Proof of Theorem 1.1

Before proving Theorem 1.1, we need the following results. Let $T \in \mathcal{T}(p,q)$ be a tree different from $B_{p,q}$. Assume that $H_T(\pi^T, v_0) = \max_v H_T(\pi^T, v)$ and v_0' is a v_0-pessimal vertex. By Lemmas 2.5 and 2.6, v_0 and v_0' are pendent verties. Moreover, Lemma 2.7 implies that there exists a vertex different from z in the (z, v_0)-path with degree at least 3 since $T \neq B_{p,q}$, and assume that i is the such vertex nearest to z. Let x be a pendent vertex in T_i^{z,v_0} and y the neighbor of x. Recall that the tree $T(x,z)$ is obtained from T by deleting the edge xy and adding the edge xz (see Fig. 3).

Lemma 3.1. *As the above symbols, it holds that*

$$\max_v H_T(\pi^T, v) = H_T(\pi^T, v_0) < H_{T(x,z)}(\pi^{T(x,z)}, v_0) \le \max_v H_{T(x,z)}(\pi^{T(x,z)}, v).$$

Proof. For convenience, denote by $T' = T(x,z), V = V(T) = V(T')$ and $n = |V| = p+q$. Let $T_i(z)$ and $T_i(x)$ be the components of $T - i$ containing z and x, respectively. Let $X = \{x\}, V_z = V(T_i(z)), V_i = \{i\} \cup (V(T_i(x)) \setminus \{x\})$ and $W = V \setminus (X \cup V_z \cup V_i)$. Clearly, $V = X \cup V_z \cup V_x \cup W$ is a partition of V. By simple observations, we have $d_{T'}(y) = d_T(y) - 1, d_{T'}(z) = d_T(z) + 1$ and $d_{T'}(v) = d_T(v)$ for any $v \in V \setminus \{y, z\}$.

By Lemma 2.4 and simple calculations, we have

$$H_{T'}(v,i) - H_T(v,i) = \begin{cases} 0, & \text{for any } v \in W, \\ -l_T(v,y;i) - l_T(v,x;i) = -2l_T(v,y;i), & \text{for any } v \in V_i, \\ l_T(v,z;i) + l_{T'}(v,x;i) = 2l_T(v,z;i), & \text{for any } v \in V_z. \end{cases} \tag{3}$$

Moreover, by Lemma 2.2, we also have

$$H_{T'}(i, v_0) = H_T(i, v_0). \tag{4}$$

In what follows, we show that $H_{T'}(\pi^{T'}, v_0) > H_T(\pi^T, v_0)$ by calculating $(2n-1)(H_{T'}(\pi^{T'}, v_0) - H_T(\pi^T, v_0))$. By Eq. (1), we have

$$(2n-1)(H_{T'}(\pi^{T'}, v_0) - H_T(\pi^T, v_0)) = \sum_{v \in V} (d_{T'}(v) H_{T'}(v, v_0) - d_T(v) H_T(v, v_0)). \tag{5}$$

For any $v \in W$, it is clear that $d_{T'}(v) = d_T(v)$, and one can easily verify that $H_{T'}(v, v_0) = H_T(v, v_0)$ due to Lemma 2.2. Thus, $\sum_{v \in W} (d_{T'}(v) H_{T(x,z)}(v, v_0) - d_T(v) H_T(v, v_0)) = 0$.

Now we consider the vertices in V_i. Since $d_{T'}(y) = d_T(y) - 1$ and $d_{T'}(v) = d_T(v)$ for any $v \in V_i \setminus \{y\}$, we have

$$\begin{aligned}
&\textstyle\sum_{v \in V_i} (d_{T'}(v) H_{T'}(v, v_0) - d_T(v) H_T(v, v_0)) \\
&= \textstyle\sum_{v \in V_i} d_{T'}(v)(H_{T'}(v, v_0) - H_T(v, v_0)) - H_T(y, v_0) \\
&= \textstyle\sum_{v \in V_i} d_{T'}(v)(H_{T'}(v, i) + H_{T'}(i, v_0) - H_T(v, i) - H_T(i, v_0)) - H_T(y, v_0) \\
&= \textstyle\sum_{v \in V_i} d_{T'}(v)(H_{T'}(v, i) - H_T(v, i)) - H_T(y, v_0) \\
&= -2 \textstyle\sum_{v \in V_i} d_{T'}(v)(l_T(v, y; i)) - H_T(y, v_0) \\
&= -2 \textstyle\sum_{v \in V} d_{T'}(v)(l_T(y, v; i)) - H_T(y, v_0) \\
&= -2 H_{T'}(y, i) - H_T(y, v_0),
\end{aligned}$$

where the second equality is due to Lemma 2.1, the third equality is due to Eq. (4), the fourth equality is due to Eq. (3), and the last two equalities are due to the fact that $H_{T'}(y,i) = \sum_{v \in V} l_{T'}(y, v; i) d_{T'}(v) = \sum_{v \in V_i} l_{T'}(y, v; i) d_{T'}(v) = \sum_{v \in V_i} l_{T'}(v, y; i) d_{T'}(v)$. Similarly, by considering the vertices in V_z, we have

$$\sum_{v \in V_z} (d_{T'}(v) H_{T'}(v, v_0) - d_T(v) H_T(v, v_0)) = H_{T'}(z, v_0) + 2H_T(z, i).$$

Moreover, we also have

$$
\begin{aligned}
& d_{T'}(x)H_{T'}(x,v_0) - d_T(x)H_T(x,v_0) \\
&= (H_{T'}(x,z) + H_{T'}(z,i) + H_{T'}(i,v_0)) - (H_T(x,y) + H_T(y,i) + H_T(i,v_0)) \\
&= H_{T'}(z,i) - H_T(y,i).
\end{aligned}
$$

Thus, by arguments above, Eq. (5) can be written as

$$
\begin{aligned}
& (2n-1)(H_{T'}(\pi^{T'},v_0) - H_T(\pi^T,v_0)) = \sum_{v\in V}(d_{T'}(v)H_{T'}(v,v_0) - d_T(v)H_T(v,v_0)) \\
&= \sum_{v\in W}(d_{T'}(v)H_{T'}(v,v_0) - d_T(v)H_T(v,v_0)) + \sum_{v\in V_i}(d_{T'}(v)H_{T'}(v,v_0) - d_T(v)H_T(v,v_0)) \\
&\quad + \sum_{v\in V_z}(d_{T'}(v)H_{T'}(v,v_0) - d_T(v)H_T(v,v_0)) + (d_{T'}(x)H_{T'}(x,v_0) - d_T(x)H_T(x,v_0)) \\
&= 0 + (-2H_{T'}(y,i) - H_T(y,v_0)) + (H_{T'}(z,v_0) + 2H_T(z,i)) + (H_{T'}(z,i) - H_T(y,i)) \qquad (6) \\
&= -2H_{T'}(y,i) - H_T(y,i) - H_T(i,v_0) + H_{T'}(z,i) + H_{T'}(i,v_0) + 2H_T(z,i) + H_{T'}(z,i) - H_T(y_i) \\
&= 2(H_{T'}(z,i) - H_{T'}(y,i)) + 2(H_T(z,i) - H_T(y,i)) \\
&= 2(H_{T'}(z,v_0) - H_{T'}(y,v_0)) + 2(H_T(z,v_0) - H_T(y,v_0)).
\end{aligned}
$$

Note that

$$
H_T(z,v_0) - H_T(y,v_0) = (H_T(v_0',v_0) - 1) - (H_T(x,v_0) - 1) = H_T(v_0',v_0) - H_T(x,v_0) \geq 0. \qquad (7)
$$

Moreover, by Lemma 2.2, we have

$$
\begin{aligned}
& H_{T'}(y,v_0) - H_T(y,v_0) \\
&= d^2_{T'}(y,v_0) + 2\sum_{v\in V(P^{T'}_{y,v_0})} m^{T'}_v d_{T'}(v,v_0) - d^2_T(y,v_0) - 2\sum_{v\in V(P^T_{y,v_0})} m^T_v d_T(v,v_0) \\
&= 2d_T(i,v_0) - 2d_T(y,v_0) = -2d_T(y,i) \leq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
& H_{T'}(v_0',v_0) - H_T(v_0',v_0) \\
&= d^2_{T'}(v_0',v_0) + 2\sum_{v\in V(P^{T'}_{v_0',v_0})} m^{T'}_v d_{T'}(v,v_0) \\
&\quad - d^2_T(v_0',v_0) - 2\sum_{v\in V(P^T_{v_0',v_0})} m^{T(x,z)}_v d_T(v,v_0) \\
&= 2(\sum_{v\in V(P^{T'}_{v_0',v_0})} m^{T'}_v d_{T'}(v,v_0) - \sum_{v\in V(P^T_{v_0',v_0})} m^{T(x,z)}_v d_T(v,v_0)) \\
&= 2(m^{T'}_z d_{T'}(z,v_0) + m^{T'}_i d_{T'}(i,v_0) - m^T_z d_T(z,v_0) + m^T_i d_T(i,v_0)) \\
&= 2(d_T(z,v_0) - d_T(i,v_0)) = 2d_T(z,i) > 0.
\end{aligned}
$$

It leads to

$$
H_{T'}(z,v_0) - H_{T'}(y,v_0) \geq (H_{T'}(v_0',v_0) - 1) - (H_T(x,v_0) - 1) \geq H_{T'}(v_0',v_0) - H_T(v_0',v_0) > 0. \qquad (8)
$$

Combining Eqs. (6), (7) and (8), we have

$$
(2n-1)(H_{T'}(\pi^{T'},v_0) - H_T(\pi^T,v_0)) > 0.
$$

The proof is completed. □

Lemma 3.2. *Let $T = B_{p,q}$ be a broom graph with $p \geq q$. If v_0 is the handle of $B_{p,q}$ then*

$$
H_T(\pi^T,v_0) = \max_v H_T(\pi^T,v) = \frac{p-q}{2(p+q-1)} + \frac{(2p-1)(2q-1)(2p+2q-3)}{2(p+q-1)} + \frac{(2q-1)(2q-2)(2q-6p+3)}{6(p+q-1)}.
$$

Proof. By Lemma 2.5, we know that for any tree T, $\max_{i\in V} H_T(\pi^T, i)$ must be achieved at a leaf. So, $\max_{i\in V(B)} H_B(\pi^B, i)$ must be achieved at v_0 or v_{2q}. Firstly, we calculate $H_B(\pi^B, v_0)$,

$$
\begin{aligned}
& H_B(\pi^B, v_0) \\
&= (p-q)\frac{1}{2|E|}H(v_{2q-1}, v_0) + \frac{p-q+1}{2|E|}H(v_{2q-2}, v_0) + \frac{2}{2|E|}\sum_{i=1}^{2q-2} H(v_{2q-2-i}, v_0) \\
&= (p-q)\frac{1}{2|E|}(1 + H(v_{2q-2}, v_0)) + \frac{p-q+1}{2|E|}H(v_{2q-2}, v_0) + \frac{2}{2|E|}\sum_{i=1}^{2q-2} H(v_{2q-2-i}, v_0) \\
&= \frac{p-q}{2|E|} + \frac{2p-2q+1}{2|E|}H(v_{2q-2}, v_0) + \frac{2}{2|E|}\sum_{i=1}^{2q-2}((2q-1-i)^2 + 2(p-q+i)(2q-1-i)) \\
&= \frac{p-q}{2|E|} + \frac{2p-2q+1}{2|E|}((2q-1)^2 + 2(p-q)(2q-1)) + \frac{2}{2|E|}\sum_{i=1}^{2q-2}(2q-1-i)(2p+i-1) \\
&= \frac{p-q}{2|E|} + \frac{2p-2q+1}{2|E|}(2q-1)(2p-1) + \frac{2}{2|E|}\sum_{i=1}^{2q-2}((2p-1)(2q-1) + (2q-2p)i - i^2) \\
&= \frac{p-q}{2|E|} + \frac{(2p-2q+1)(2q-1)(2p-1)}{2|E|} + \frac{(2p-1)(2q-1)(2q-2)}{|E|} \\
&\quad + \frac{2(q-p)(2q-1)(q-1)}{|E|} - \frac{(q-1)(2q-1)(4q-3)}{3|E|} \\
&= \frac{p-q}{2(p+q-1)} + \frac{(2p-1)(2q-1)(2p+2q-3)}{2(p+q-1)} + \frac{(2q-1)(2q-2)(2q-6p+3)}{6(p+q-1)}.
\end{aligned}
$$

Then, we calculate $H_B(\pi^B, v_{2q})$,

$$
\begin{aligned}
& H_B(\pi^B, v_{2q}) \\
&= \frac{p-q+1}{2|E|}H(v_{2q-1}, v_{2q}) + \frac{p-q-1}{2|E|}(1 + H(v_{2q-1}, v_{2q})) + \sum_{j=0}^{2q-2}\frac{d(v_j)}{2|E|}(H(v_j, v_{2q-1}) + H(v_{2q-1}, v_2q)) \\
&= \frac{p-q+1}{2|E|}(1 + 2(p+q-2)) + \frac{p-q-1}{2|E|}(1 + 1 + 2(p+q-2)) \\
&\quad + \frac{2}{2|E|}\sum_{j=0}^{2q-2}((2q-1-j)^2 + 2j(2q-1-j) + 1 + 2(p+q-2)) - \frac{1}{2|E|}H(v_0, v_{2q}) \\
&= \frac{p-q-1}{2|E|} + (2p+2q-3)\frac{2p-2q}{2|E|} \\
&\quad + \frac{1}{|E|}\sum_{j=0}^{2q-2}((2q-1-j)(2q-1+j) + 2p + 2q - 3) - \frac{1}{2|E|}(4q^2 + 2(p-q-1)) \\
&= \frac{p-q-1 + 2(p-q)(2p+2q-3) - 4q^2 - 2(p-q-1)}{2|E|} \\
&\quad + \frac{1}{|E|}\sum_{j=0}^{2q-2}((2q-1)^2 - j^2) + \frac{(2q-1)(2p+2q-3)}{|E|} \\
&= \frac{q-p+1 + 2(p-q)(2p+2q-3) - 4q^2}{2|E|} + \frac{(2q-1)^3}{|E|} - \frac{(2q-2)(2q-1)(4q-3)}{6|E|} \\
&= \frac{q-p+1 + 2(p-q)(2p+2q-3) - 4q^2}{2|E|} + \frac{(2q-1)^2}{|E|} \\
&\quad + \frac{(2q-1)^2(2q-2)}{|E|} - \frac{(2q-2)(2q-1)(4q-3)}{6|E|} \\
&= \frac{q-p+1 + 4p^2 - 6p - 2q + 2}{2|E|} + \frac{(2q-1)(2q-2)(12q-6-4q+3)}{6|E|} \\
&= \frac{4p^2 - 7p - q + 3}{2|E|} + \frac{(2q-1)(2q-2)(8q-3)}{6|E|}.
\end{aligned}
$$

Finally,

$$H_B(\pi^B, v_0) - H_B(\pi^B, v_{2q})$$

$$
\begin{aligned}
&= \frac{-4p^2+8p-3}{2|E|} + \frac{(2p-1)(2q-1)(2p+2q-3)}{2|E|} + \frac{(2q-1)(2q-2)(-6p-6q+6)}{6|E|} \\
&= \frac{-4p^2+8p-3}{2|E|} + \frac{(2q-1)(4p^2-4q^2-p+6q-1}{2|E|} \\
&= \frac{8p^2q-8q^3-8p^2+16q^2-2pq+9p-8q-2}{2|E|} \\
&= \frac{8p^2q-16p^2-(8q^3-16q^2)+8p^2-2pq+9p-8q-2}{2|E|} \\
&= \frac{(q-2)(8p^2-8q^2)+(p-q)(2p+8)+6p^2+p-2}{2|E|} > 0.
\end{aligned}
$$

Therefore, when $p > q \geq 2, H_B(\pi^B, v_0) = \max_{i \in V(B)} H_B(\pi^B, i)$. $\square$

From Lemma 3.2, we get the following result by immediate calculations.

Corollary 3.3. *If p, q, s, t are positive integers satisfying $p \geq q, s \geq t, q \geq t$ and $s = p-1, t = q+1$, then $\max_v H_{B_{p,q}}(\pi^{B(p,q)}, v) \geq \max_v H_{B_{s,t}}(\pi^{B_{s,t}}, v)$.*

Proof. Let $T' = B_{p,q}(u_0, u_{q-1})$, then $T' = B_{s,t}$. From Lemma 3.2, we have

$$
\begin{aligned}
&\max_v H_{T'}(\pi^{T'}, v) = H_{T'}(\pi^{T'}, u_1) \\
&= \frac{p-q+1}{2|E|}(1 + H_{T'}(u_{2q-1}, u_1)) + \frac{p-q+2}{2|E|}H_{T'}(u_{2q-1}, u_1) + \frac{2}{2|E|}\sum_{i=1}^{2q-3} H_{T'}(u_{2q-1-i}, u_1) \\
&= \frac{p-q+1}{2|E|} + \frac{2p-2q+3}{2|E|}H_{T'}(u_{2q-1}, u_1) + \frac{2}{2|E|}\sum_{i=1}^{2q-3}((2q-2-i)^2 + 2(p-q+1+i)(2q-2-i)) \\
&= \frac{p-q+1}{2|E|} + \frac{2p-2q+3}{2|E|}((2q-2)^2 + 2(p-q+1)(2q-2)) \\
&+ \frac{2}{2|E|}\sum_{i=1}^{2q-3}((2q-2-i)^2 + 2(p-q+1+i)(2q-2-i)).
\end{aligned}
$$

It leads to that

$$
\begin{aligned}
&\max_v H_B(\pi^B, v) = H_B(\pi^B, u_0) \\
&= \frac{p-q}{2|E|}(1 + H_B(u_{2q-1}, u_0)) + \frac{p-q+1}{2|E|}H_B(u_{2q-1}, u_0) + \frac{2}{2|E|}\sum_{i=1}^{2q-2} H_{T'}(u_{2q-1-i}, u_0) \\
&= \frac{p-q}{2|E|} + \frac{2p-2q+1}{2|E|}H_B(u_{2q-1}, u_0) + \frac{2}{2|E|}\sum_{i=1}^{2q-2}((2q-1-i)^2 + 2(p-q+i)(2q-1-i)) \\
&= \frac{p-q}{2|E|} + \frac{2p-2q+1}{2|E|}((2q-1)^2 + 2(p-q)(2q-1)) \\
&+ \frac{2}{2|E|}\sum_{i=1}^{2q-2}((2q-1-i)^2 + 2(p-q+1+i)(2q-1-i)).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&H_B(\pi^B, u_0) - H_{T'}(\pi^{T'}, u_1) \\
&= -\frac{1}{2|E|} + \frac{2p-2q+1}{2|E|}(4q-3+2(p-q+2)) \\
&- \frac{2}{2|E|}((2q-2)^2 + 2(p-q+1)(2q-2)) + \frac{2}{2|E|}((2q-1)^2 + 2(p-q)(2q-1)) \\
&+ \frac{2}{2|E|}\sum_{i=1}^{2q-2}((2q-1-i)^2 + 2(p-q+1+i)(2q-1-i)) \\
&- \frac{2}{2|E|}\sum_{i=1}^{2q-3}((2q-2-i)^2 + 2(p-q+1+i)(2q-2-i))
\end{aligned}
$$

$$= -\frac{1}{2|E|} + \frac{(2p-2q+1)(2p+3q+1)}{2|E|} > 0.$$

Therefore, $\max_v H_{B_{p,q}}(\pi^{B(p,q)}, v) \geq \max_v H_{B_{s,t}}(\pi^{B_{s,t}}, v)$. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1.. According to Lemma 3.2, it suffices to show that $\max_v H_T(\pi^T, v) < \max_v H_{B_{p,q}}(\pi^{B_{p,q}}, v)$ for any $T \in \mathcal{T}(p,q)$ different from $B(p,q)$. Since $T \in \mathcal{T}(p,q)$, its diameter is $d \leq 2q$. Note that Lemma 3.1 implies that there always exists a pendent vertex x and a non-pendent vertex z such that $\max_v H_T(\pi^T, v) < \max_v H_{T(x,z)}(\pi^{T(x,z)}, v)$ whenever T is not a broom graph. Therefore, by repeatedly applying Lemma 3.1, we will ultimately get a broom graph $B_{s,t}$ with $s \geq t$ such that $\max_v H_T(\pi^T, v) < \max_v H_{B_{s,t}}(\pi^{B_{s,t}}, v)$. By noticing that the operation in Lemma 3.1 do not increase the diameter of a tree, we have $2t \leq d \leq 2q$. Thus, Corollary 3.3 indicates that

$$\max_v H_T(\pi^T, v) < \max_v H_{B_{s,t}}(\pi^{B_{s,t}}, v) \leq \max_v H_{B_{p,q}}(\pi^{B_{p,q}}, v).$$

It completes the proof. □

4. Proof of Theorem 1.3

In order to prove Theorem 1.3, we need the following technical lemmas.

Lemma 4.1. *Let $T \in \mathcal{T}(p,q)$ be a tree with vertex $u \in V(T)$ satisfying $H_T(\pi^T, u) = \min_x H_T(\pi^T, x)$. If v is a pendent vertex of T not adjacent to u, then*

$$\min_{x \in V(T(x,u))} H_{T(v,u)}(\pi^{T(v,u)}, x) = H_{T(v,u)}(\pi^{T(v,u)}, u) < H_T(\pi^T, u) = \min_x H_T(\pi^T, x).$$

Proof. For convenience, denote by $T' = T(v,u), \pi' = \pi^{T'}, V = V(T) = V(T(v,u))$ and $n = s+t$. Assume that w is the neighbor of $v, V_1 = T_u^{u,w}$, $V_2 = T_w^{u,w} \setminus \{v\}$ and $V_3 = V(T) \setminus (T_u^{u,w} \cup T_w^{u,w})$. We first prove that $\min_{x \in V(T')} H_{T'}(\pi', x) = H_{T'}(\pi', u)$. It suffices to show that $H_{T'}(\pi', x) > H_{T'}(\pi', u)$ for any $x \neq u$. It is divided into the following cases to discuss.

Case 1. $x = v$.

In this case, by Eq. (1), Lemma 2.1 and Corollary 2.1, we have

$$\begin{aligned}
&2(n-1)H_{T'}(\pi', v) = \textstyle\sum_{z \in V} d_{T'}(z) H_{T'}(z, v) = \sum_{z \in V \setminus \{v\}} d_{T'}(z) H_{T'}(z, v) \\
&= \textstyle\sum_{z \in V \setminus \{v\}} d_{T'}(z) H_{T'}(z, u) + (2n-3) \sum_{z \in V \setminus \{v\}} d_{T'}(z) \\
&= \textstyle\sum_{z \in V} d_{T'}(z) H_{T'}(z, u) - H_{T'}(v, u) + (2n-3)^2 \\
&= 2(n-1)H_{T'}(\pi', u) + (2n-3)^2 - 1 > 0.
\end{aligned}$$

It follows that $H_{T'}(\pi', v) > H_{T'}(\pi', u)$.

Case 2. $x = v_1 \in V_1 \setminus \{u\}$.

In this case, by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&2(n-1)H_{T'}(\pi', v_1) = \textstyle\sum_{z \in V} d_{T'}(z) H_{T'}(z, v_1) \\
&= \textstyle\sum_{z \in V \setminus V_1} d_{T'}(z) H_{T'}(z, v_1) + \sum_{z \in V_1} d_{T'}(z) H_{T'}(z, v_1) \\
&= \textstyle\sum_{z \in V \setminus V_1} d_{T'}(z)(H_{T'}(z, u) + H_{T'}(u, v_1)) + \sum_{z \in V_1} d_{T'}(z) H_{T'}(z, v_1) \\
&= \textstyle\sum_{z \in V \setminus V_1} d_{T'}(z) H_{T'}(z, u) + \sum_{z \in V \setminus V_1} d_{T'}(z) H_{T'}(u, v_1) + \sum_{z \in V_1} d_{T'}(z) H_{T'}(z, v_1) \\
&= \textstyle\sum_{z \in V} d_{T'}(z) H_{T'}(z, u) + \sum_{z \in V \setminus V_1} d_{T'}(z) H_{T'}(u, v_1) + \sum_{z \in V_1} d_{T'}(z) H_{T'}(z, v_1) - \sum_{z \in V_1} d_{T'}(z) H_{T'}(z, u)
\end{aligned}$$

From Lemma 2.3(ii), we have $H_{T'}(u, v_1) = H_T(u, v_1), H_{T'}(z, v_1) = H_T(z, v_1)$ and $H_{T'}(z, u) = H_T(z, u)$ for any $z \in V_1$. Therefore, we have

$$\begin{aligned}
&2(n-1)H_{T'}(\pi', v_1) \\
&= \textstyle\sum_{z \in V} d_{T'}(z) H_{T'}(z, u) + \sum_{z \in V \setminus V_1} d_T(z) H_T(u, v_1) + \sum_{z \in V_1} d_T(z) H_T(z, v_1) - \sum_{z \in V_1} d_T(z) H_T(z, u) \\
&= 2(n-1)H_{T'}(\pi', u) + \textstyle\sum_{z \in V \setminus V_1} d_T(z)(H_T(z, v_1) - H_T(z, u)) \\
&\quad + \textstyle\sum_{z \in V_1} d_T(z) H_T(z, v_1) - \sum_{z \in V_1} d_T(z) H_T(z, u) \\
&= 2(n-1)H_{T'}(\pi', u) + \textstyle\sum_{z \in V} d_T(z) H_T(z, v_1) - \sum_{z \in V} d_T(z) H_T(z, u) \\
&= 2(n-1)H_{T'}(\pi', u) + 2(n-1)H_T(\pi, v_1) - 2(n-1)H_T(\pi, u).
\end{aligned}$$

It means that $H_{T'}(\pi', v_1) > 2(n-1)H_{T'}(\pi', u)$ since $H_T(\pi, v_1) > H_T(\pi, u)$.

Case 3. $x = v_2 \in V_2$.

In this case, by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
2(n-1)H_{T'}(\pi', v_2) &= \textstyle\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(z,v_2) + \sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_2)\\
&= \textstyle\sum_{z\in V_1\cup\{v\}} d_{T'}(z)(H_{T'}(z,u) + H(u,v_2)) + \sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_2)\\
&= \textstyle\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(z,u) + \sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(u,v_2)\\
&\quad \textstyle+ \sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_2)\\
&= \textstyle\sum_{z\in V} d_{T'}(z)H_{T'}(z,u) - \sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,u)\\
&\quad \textstyle+ \sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(u,v_2) + \sum_{z\in V_2\cup V_3} d_{T}(z)H_{T'}(z,v_2)\\
&= \textstyle 2(n-1)H_{T'}(\pi',u) - \sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,u)\\
&\quad \textstyle+ \sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(u,v_2) + \sum_{z\in V_2\cup V_3} d_{T}(z)H_{T'}(z,v_2).
\end{aligned}$$

It suffices to show that

$$-\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,u) + \sum_{z\in V_1\cup\{v\}} d_{T'}(z)H(u,v_2) + \sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_2) > 0. \tag{9}$$

Firstly, we consider the term $\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,u)$. Note that $d_{T'}(z) = d_T(z)$ for any $z \in (V_2\cup V_3)\setminus\{w\}$, and $d_{T'}(w) = d_T(w) - 1$. We have

$$\begin{aligned}
\textstyle\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,u) &= \textstyle\sum_{z\in (V_2\cup V_3)\setminus\{w\}} d_{T'}(z)H_{T'}(z,u) + d_{T'}(w)H_{T'}(w,u)\\
&= \textstyle\sum_{z\in (V_2\cup V_3)\setminus\{w\}} d_{T}(z)H_{T'}(z,u) + (d_T(w)-1)H_{T'}(w,u)\\
&= \textstyle\sum_{z\in V_2\cup V_3} d_{T}(z)H_{T'}(z,u) - H_{T'}(w,u).
\end{aligned}$$

Therefore, from Lemma 2.3(vi), we have

$$\begin{aligned}
&\textstyle\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,u)\\
&= \textstyle\sum_{z\in V_2\cup V_3} d_{T}(z)(H_T(z,u) - 2d_T(g_z,u)) - H_T(w,u) + 2d_T(w,u)\\
&\le \textstyle\sum_{z\in V_2\cup V_3} d_{T}(z)H_T(z,u).
\end{aligned} \tag{10}$$

Next, we consider the term $\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_2)$. Denote by $U_1 = \{z \in V_2\cup V_3 \mid V(P^{T'}_{z,v_2}) \cap V(P^{T'}_{u,w}) \neq \emptyset\}$ and $U_2 = (V_2\cup V_3)\setminus U_1$. It is clear that $V_3 \subseteq U_1$. For each $z \in U_1$, let f_z be the first point such that the path $P^{T'}_{z,v_2}$ meets the path $P^{T'}_{u,w}$. It is clear that $f_u = u$ and $f_w = w$. Therefore, we have

$$\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_2) = \sum_{z\in U_1} d_{T'}(z)H_{T'}(z,v_2) + \sum_{z\in U_2} d_{T'}(z)H_{T'}(z,v_2).$$

Therefore, from Lemma 2.3(v), we have

$$\begin{aligned}
&\textstyle\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_2)\\
&= \textstyle\sum_{z\in U_1\setminus\{w\}} d_{T'}(z)H_{T'}(z,v_2) + d_{T'}(w)H_{T'}(w,v_2) + \sum_{z\in U_2} d_{T'}(z)H_{T'}(z,v_2)\\
&= \textstyle\sum_{z\in U_1} d_{T}(z)H_{T'}(z,v_2) - H_{T'}(w,v_2) + \sum_{z\in U_2} d_{T}(z)H_{T'}(z,v_2)\\
&= \textstyle\sum_{z\in U_1} d_{T}(z)(H_T(z,v_2) + 2d_T(f_z,v_2)) - (H_T(w,v_2) + 2d_T(w,v_2)) + \sum_{z\in U_2} d_{T}(z)H_{T}(z,v_2)\\
&\ge \textstyle\sum_{z\in V_2\cup V_3} d_{T}(z)H_{T}(z,v_2) - H_T(w,v_2).
\end{aligned} \tag{11}$$

At last, we consider the term $\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(u,v_2)$. Since $d_{T'}(u) = d_T(u) + 1$ and $d_{T'}(z) = d_T(z)$ for any $z \in (V_1\cup\{v\})\setminus\{u\}$, we have

$$\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(u,v_2) = \sum_{z\in V_1\cup\{v\}} d_{T}(z)H_{T'}(u,v_2) + H_{T'}(u,v_2).$$

Therefore, from Lemmas 2.1 and 2.3(iv), we have

$$\begin{aligned}
&\textstyle\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(u,v_2)\\
&= \textstyle\sum_{z\in V_1\cup\{v\}} d_T(z)(H_T(u,v_2)+2d_T(u,w))+H_{T'}(u,v_2)\\
&= \textstyle\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(u,v_2)+2\sum_{z\in V_1\cup\{v\}} d_T(z)d_T(u,w)+H_{T'}(u,v_2)\\
&= \textstyle\sum_{z\in V_1} d_T(z)H_T(u,v_2)+H_T(u,v_2)+H_{T'}(u,v_2)+2\sum_{z\in V_1\cup\{v\}} d_T(z)d_T(u,w)\\
&= \textstyle\sum_{z\in V_1} d_T(z)(H_T(z,v_2)-H_T(z,u))+H_T(v,v_2)-H_T(v,u)+H_T(v,u)-H_T(v,v_2)\\
&\quad \textstyle +H_T(u,v_2)+H_{T'}(u,v_2)+2\sum_{z\in V_1\cup\{v\}} d_T(z)d_T(u,w)\\
&= \textstyle\sum_{V_1\cup\{v\}} d_T(z)H_T(z,v_2)-\sum_{V_1\cup\{v\}} d_T(z)H_T(z,u)+(H_T(w,u)-H_T(w,v_2))\\
&\quad \textstyle +(H_T(u,w)+H_T(w,v_2))+H_{T'}(u,v_2)+2\sum_{z\in V_1\cup\{v\}} d_T(z)d_T(u,w)\\
&= \textstyle\sum_{V_1\cup\{v\}} d_T(z)H_T(z,v_2)-\sum_{V_1\cup\{v\}} d_T(z)H_T(z,u)+H_T(w,u)+H_T(u,w)\\
&\quad \textstyle +H_T(u,v_2)+2d_T(u,w)+2\sum_{z\in V_1\cup\{v\}} d_T(z)d_T\\
&> \textstyle\sum_{V_1\cup\{v\}} d_T(z)H_T(z,v_2)-\sum_{V_1\cup\{v_1\}} d_T(z)H_T(z,u)+H_T(w,v_2)
\end{aligned} \tag{12}$$

Combining Eqs. (10), (11) and (12), we have

$$\begin{aligned}
&\textstyle -\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,u)+\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H(u,v_2)+\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_2)\\
&> \textstyle\sum_{z\in V} d_T(z)H_T(z,v_2)-\sum_{z\in V} d_T(z)H_T(z,u)\\
&= H_T(\pi^T,v_2)-H_T(\pi^T,u)\geq 0,
\end{aligned}$$

and thus (9) holds.

Case 4. $x=v_3\in V_3$.

In this case, by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
2(n-1)H_{T'}(\pi',v_3) &= \textstyle\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(z,v_3)+\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_3)\\
&= \textstyle\sum_{z\in V_1\cup\{v\}} d_{T'}(z)(H_{T'}(z,u)+H_{T'}(u,v_3))+\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_3)\\
&= \textstyle\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(z,u)+\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(u,v_3)\\
&\quad \textstyle +\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_3)\\
&= \textstyle\sum_{z\in V} d_{T'}(z)H_{T'}(z,u)-\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,u)\\
&\quad \textstyle +\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(u,v_3)+\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_3)\\
&= \textstyle 2(n-1)H_{T'}(\pi',z)-\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,u)\\
&\quad \textstyle +\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(u,v_3)+\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_3).
\end{aligned}$$

It suffices to show that

$$-\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,u)+\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(u,v_3)+\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_3)>0. \tag{13}$$

From Eq. (10), we have

$$\begin{aligned}
&\textstyle\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,u)\\
&\leq \textstyle\sum_{z\in V_2\cup V_3} d_T(z)H_T(z,u)-H(w,u)\\
&= \textstyle\sum_{z\in V_2\cup V_3} d_T(z)H_T(z,u)-(H_T(w,h_u)+H_T(h_u,u))\\
&\leq \textstyle\sum_{z\in V_2\cup V_3} d_T(z)H_T(z,u)-H(w,f_u).
\end{aligned} \tag{14}$$

Next, we consider the term $\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(u,v_3)$. Since $d_{T'}(u)=d_T(u)+1$ and $d_{T'}(z)=d_T(z)$ for any $z\in(V_1\cup\{v\})\setminus\{u\}$, we have

$$\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(u,v_3)=\sum_{z\in V_1\cup\{v\}} d_T(z)H_{T'}(u,v_3)+H_{T'}(u,v_3).$$

Therefore, from Lemma 2.3(vii), we have

$$\begin{aligned}
&\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(u,v_3)\\
=&\sum_{z\in V_1\cup\{v\}} d_T(z)(H_T(u,v_3)+2d_T(u,h_u))+H_{T'}(u,v_3)\\
=&\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(u,v_3)+2d_T(u,h_u)\sum_{z\in V_1\cup\{v\}} d_T(z)+H_{T'}(u,v_3)\\
=&\sum_{z\in V_1} d_T(z)(H_T(z,v_3)-H_T(z,u))+H_T(v,v_3)-H_T(v,u)+H_T(v,u)\\
&-H_T(v,v_3)+H_T(u,v_3)+2d_T(u,h_u)\sum_{z\in V_1\cup\{v\}} d_T(z)+H_{T'}(u,v_3)\\
=&\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,v_3)-\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,u)+H_T(h_u,u)\\
&-H_T(h_u,v_3)+H_T(u,v_3)+2d_T(u,h_u)\sum_{z\in V_1\cup\{v\}} d_T(z)+H_{T'}(u,v_3)\\
=&\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,v_3)-\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,u)+H_T(h_u,u)\\
&-H_T(h_u,v_3)+H_T(u,h_u)+H_T(h_u,v_3)+2d_T(u,h_u)\sum_{z\in V_1\cup\{v\}} d_T(z)+H_{T'}(u,v_3)\\
>&\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,v_3)-\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,u)+H_{T'}(u,v_3)\\
=&\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,v_3)-\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,u)+H_T(u,v_3)+2d_T(u,h_u)\\
\geq&\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,v_3)-\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,u)+H_T(u,v_3)\\
=&\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,v_3)-\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,u)+H_T(u,h_u)+H_T(h_u,v_3)\\
\geq&\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,v_3)-\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,u)+H_T(h_u,v_3).
\end{aligned}\tag{15}$$

At last, we consider the term $\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_3)$. Since $d_{T'}(w)=d_T(w)-1$ and $d_{T'}(z)=d_T(z)$ for any $z\in(V_2\cup V_3)\setminus\{w\}$, we have

$$\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_3)=\sum_{z\in V_2\cup V_3} d_T(z)H_{T'}(z,v_3)-H_{T'}(w,v_3).$$

Therefore, from Lemma 2.3(viii), we have

$$\begin{aligned}
&\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_3)\\
=&\sum_{z\in V_2\cup V_3} d_T(z)(H_T(z,v_3)+2d_T(l_z^+,l_z^-))-(H_T(w,v_3)+2d_T(l_w^+,l_w^-))\\
\geq&\sum_{z\in V_2\cup V_3} d_T(z)H_T(z,v_3)-H_T(w,v_3).
\end{aligned}\tag{16}$$

By noticing that $H_T(w,v_3)=H_T(w,h_u)+H_T(h_u,v_3)$, combining Eqs. (14), (15) and (16), we have

$$\begin{aligned}
&-\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,u)+\sum_{z\in V_1\cup\{v\}} d_{T'}(z)H_{T'}(u,v_3)+\sum_{z\in V_2\cup V_3} d_{T'}(z)H_{T'}(z,v_3)\\
>&-\sum_{z\in V_2\cup V_3} d_T(z)H_T(z,u)+H(w,h_u)+\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,v_3)\\
&-\sum_{z\in V_1\cup\{v\}} d_T(z)H_T(z,u)+H_T(h_u,v_3)+\sum_{z\in V_2\cup V_3} d_T(z)H_T(z,v_3)-H_T(w,v_3)\\
=&H_T(\pi,v_3)-H_T(\pi,u)\geq 0,
\end{aligned}$$

and thus (13) holds.

The proof is completed. □

Let $T\in\mathcal{T}(p,q)$ be a tree with vertex $u_0\in V(T)$ satisfying $H_T(\pi^T,u_0)=\min_x H_T(\pi^T,x)$. Assume that $u_1,u_2,\cdots,u_m$ are neighbors of u_0 and they're not leaves. The forest $T-u_0$ has $d(u_0)$ components. Let $T_1,T_2,\cdots,T_m$ be the vertex sets of the components containing $u_1,u_2,\cdots,u_m$ respectively.

Lemma 4.2. *For the symbols as above, if x is a pendent vertex of T_i not adjacent to u_i, then* $\min\limits_{v\in V(T)} H_T(\pi^T,v)=H_T(\pi^T,u_0)>H_{T(x,u_i)}(\pi^{T(x,u_i)},u_0)=\min\limits_{v\in V(T(x,u_i))} H_{T(x,u_i)}(\pi^{T(x,u_i)},u_0)$.

Proof. We first prove that $H_T(\pi^T,u_0)>H_{T'}(\pi',u_0)$, where $T'=T(x,u_i)$ and $\pi'=\pi^{T'}$. Let z be the neighbor of x, $V_1=T_{u_i}^{u_i,z}$, $V_2=T_z^{u_i,z}\setminus\{v\}$ and $V_3=V\setminus(T_{u_i}^{u_i,z}\cup T_z^{u_i,z})$. By Lemma 2.3 (*ii*), we have that for any $v_1\in V_1$, $H_{T'}(v_1,u_0)=H_T(v_1,u_0)$. By Lemma 2.3 (*iii*), we have that for any $v_2\in V_2$, $H_{T'}(v_2,u_i)=H_T(v_2,u_i)-2d_T(z,u_i)$. By Lemma 2.3 (*vi*), we have that for any $v_3\in V_3$, $H_{T'}(v_3,u_i)=H_T(v_3,u_i)-2d_T(g_{v_3},u_i)$ where g_{v_3} is the first point that $P_{v_3,u_i}^{T'}$ meets $P_{z,u_i}^{T'}$. For a vertex $v\in V_2\cup V_3$, we have $H_{T'}(v,u_0)=H_{T'}(v,u_i)+H_{T'}(u_i,u_0)$ and $H_T(v,u_0)=H_T(v,u_i)+H_T(u_i,u_0)$. Therefore, we have $H_{T'}(v_2,u_0)=H_T(v_2,u_0)-2d_T(z,u_i)$ for any $v_2\in V_2$, and $H_{T'}(v_3,u_0)=H_T(v_3,u_0)-2d_T(g_{v_3},u_i)$ for any $v_3\in V_3$, where g_{v_3} is the first point that $P_{v_3,u_i}^{T'}$ meets $P_{z,u_i}^{T'}$. Thus, we have $H_T(\pi^T,u_0)>H_{T'}(\pi',u_0)$.

Next, we show that $H_{T'}(\pi',u_0)=\min_{v\in V(T')} H_{T'}(\pi',v)$. As shown in Fig. 4, suppose that there exists $u_0'\in V(T')(u_0'\neq u_0)$ such that $\min_{v\in V(T')} H_{T'}(\pi',v)$ is achieved at the vertex u_0'. In what follows, we divide three cases to discuss.

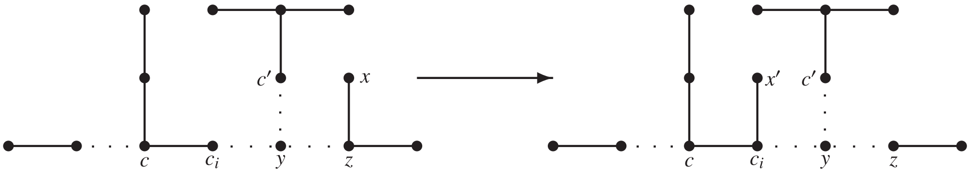

Fig. 4. The transformation in Lemma 4.2.

Case 1. The vertex u_0 is on the path $P^{T'}_{x,u_0'}$.

Let $V_1 = T^{u_i,z}_{u_i}$ and $V_2 = V \setminus T^{u_i,z}_{u_i}$. For any $v_1 \in V_1$, we have $H_{T'}(v_1, u_0') = H_T(v_1, u_0')$ and $H_{T'}(v_1, u_0) = H_T(v_1, u_0)$. For any $v_2 \in V_2$, we have $H_{T'}(v_2, u_0') = H_{T'}(v_2, u_0) + H_{T'}(c, u_0') = H_{T'}(v_2, u_0) + H_T(c, u_0')$. Therefore, we have $H_{T'}(\pi', u_0) - H_T(\pi^T, u_0) = H_{T'}(\pi', u_0') - H(\pi, u_0')$, and thus $H_{T'}(\pi', u_0) \le H_{T'}(\pi', u_0')$. It indicates $H_{T'}(\pi', u_0) = \min_{v\in V(T')} H_{T'}(\pi', v)$.

Case 2. The vertex u_0' is on the path $P^{T'}_{x,u_0}$.

Let $V_1 = T^{u_i,z}_{u_i}$, $V_2 = T^{c',z}_{c'} \setminus V_1, V_4 = T^{u_i,z}_z$ and $V_3 = V \setminus (V_1 \cup V_2 \cup V_4)$. By Lemma 2.9, for any $v_1 \in V_1 \setminus u_i$, we have

$$\pi^{T'}_{v_1} H_{T'}(v_1, u_0) - \pi^T_{v_1} H_T(v_1, u_0) = 0, \text{ and } \pi^{T'}_{v_1} H_{T'}(v_1, u_0') - \pi^T_{v_1} H_T(v_1, u_0') > 0;$$

for any $v_2 \in V_2$, we have

$$\pi'_{v_2} H_{T'}(v_2, u_0') - \pi^T_{v_2} H_T(v_2, u_0') > 0;$$

for any $v \in V_3 \cup V_4 \setminus x$, we have

$$\pi'_v H_{T'}(v, u_0) - \pi^T_v H_T(v, u_0) = \pi'_v H_{T'}(v, u_0') - \pi^T_v H_T(v, u_0');$$

for the vertices x and x', we have

$$H_{T'}(x', u_0) - H_T(x, u_0) \le H_{T'}(x', u_0') - H_T(x, u_0');$$

for z and u_i, we have

$$\begin{aligned} &\pi'_z H_{T'}(z, u_0) - \pi^T_z H(z, u_0) + \pi'_{u_i} H_{T'}(u_i, u_0) - \pi^T_{u_i} H(u_i, u_0) \\ &< \pi'_z H_{T'}(z, u_0') - \pi^T_z H(z, u_0') + \pi'_{u_i} H_{T'}(u_i, u_0') - \pi^T_{u_i} H_T(u_i, u_0'). \end{aligned}$$

Combining these equations, we have $H_{T'}(\pi', u_0) - H_T(\pi^T, u_0) \le H_{T'}(\pi', u_0') - H(\pi, u_0')$, and thus $H_{T'}(\pi', u_0) \le H_{T'}(\pi', u_0')$. It implies $H_{T'}(\pi', u_0) = \min\limits_{v\in V(T')} H_{T'}(\pi', v)$.

Case 3. There is a vertex $y(\neq c, u_0')$, which is the first point that P'_{x,u_0} meets P'_{c',u_0}.

Let z be the unique neighbor of x. The forest $T_{p,q} - y$ has $d(y)$ components. Let V_1 be the vertex set of $T_{p,q} - T_i, V_2$ and V_3 the vertex sets of the components containing z and u_0' respectively, and let V_4 be the vertex set of $T_i - V_2 - V_3$. Thus, we have a partition Φ: $V = V_1 \cup V_2 \cup V_3 \cup V_4$. It is clear that $d_{T'}(v) = d_T(v)$ for $v \in V \setminus \{u_i, z\}, d_{T'}(z) = d_T(z) - 1, d_{T'}(u_i) = d_T(u_i) + 1$ and $d_{T'}(x') = d_T(x) = 1$.

For $v \in V_1$, we have

$$\begin{cases} \sum\limits_{v\in V_1} \pi'_v H_{T'}(v, u_0) - \sum\limits_{v\in V_1} \pi^T_v H_T(v, u_0) = 0, \\ \sum\limits_{v\in V_1} \pi'_v H_{T'}(v, u_0') - \sum\limits_{v\in V_1} \pi^T_v H_T(v, u_0') = \sum\limits_{v\in V_1} 2\pi^T_v \left(d(u_i, u_0') - d(y, u_0')\right) \ge 0, \end{cases}$$

and thus $\sum\limits_{v\in V_1} \pi'_v H_{T'}(v, u_0) - \sum\limits_{v\in V_1} \pi'_v H_{T'}(v, u_0') \le \sum\limits_{v\in V_1} \pi^T_v H_T(v, u_0) - \sum\limits_{v\in V_1} \pi^T_v H_T(v, u_0')$.

Note that,

$$\begin{aligned} &H_{T'}(y, u_0) - H_T(y, u_0) \\ &= \pi'_y (d^2_{T'}(y, u_0) + 2\sum_{i\in V(P)} m'_i d_{T'}(i, u_0)) - \pi^T_y (d^2_T(y, u_0) - 2\sum_{i\in V(P)} m^T_i d_T(i, u_0)) \\ &= 2(d(u_i - c) - d(y, u_0)) \le 0, \end{aligned}$$

and $H_{T'}(y, u_0') = H_T(y, u_0')$. Therefore, for $v \in V_2$, we have

$$\begin{aligned} &\textstyle\sum_{v\in V_2\setminus\{x,z\}} \pi'_v H_{T'}(v, u_0) - \sum_{v\in V_2\setminus\{x,z\}} \pi'_v H_{T'}(v, u_0') \\ &= \textstyle\sum_{v\in V_2\setminus\{x,z\}} \pi^T_v \left(H_{T'}(v, y) + H_{T'}(y, u_0) - H_{T'}(v, y) + H_{T'}(y, u_0')\right) \\ &= \textstyle\sum_{v\in V_2\setminus\{x,z\}} \pi^T_v \left(H_{T'}(y, u_0) - H_{T'}(y, u_0')\right) \\ &\le \textstyle\sum_{v\in V_2\setminus\{x,z\}} \pi^T_v H_T(y, u_0) - \sum_{v\in V_2\setminus\{x,z\}} \pi^T_v H_T(y, u_0'). \end{aligned}$$

For $v \in V_3$, we have

$$\begin{aligned}
&\sum_{v\in V_3} \pi'_v H_{T'}(v,u_0) - \sum_{v\in V_3} \pi^T_v H_T(v,u_0)\\
&= \sum_{v\in V_3} (\pi'_v (d^2_{T'}(v,u_0) + 2\sum_{i\in V(P)} m'_i d_{T'}(i,u_0))) - \sum_{v\in V_3} (\pi^T_v (d^2_T(v,u_0) + 2\sum_{i\in V(P)} m^T_i d_T(i,u_0)))\\
&= \sum_{v\in V_3} (\pi^T_v (d^2_T(v,u_0) + 2\sum_{i\in V(P)} m'_i d_T(i,u_0))) - \sum_{v\in V_3} (\pi^T_v (d^2_T(v,u_0) + 2\sum_{i\in V(P)} m^T_i d_T(i,u_0)))\\
&= \sum_{v\in V_3} 2\pi^T_v (d(u_i - c) - d(y,u_0)) \le 0,
\end{aligned}$$

and $\sum_{v\in V_3} \pi'_v H_{T'}(v,u'_0) - \sum_{v\in V_3} \pi^T_v H_T(v,u'_0) = 0$. Therefore, we have $\sum_{v\in V_3} \pi'_v H_{T'}(v,u_0) - \sum_{v\in V_3} \pi'_v H_{T'}(v,u'_0) \le \sum_{v\in V_3} \pi^T_v H_T(v,u_0) - \sum_{v\in V_3} \pi^T_v H_T(v,u'_0)$.

For $v \in V_4 \setminus \{u_i\}$, we have

$$\begin{aligned}
&\sum_{v\in V_4\setminus\{u_i\}} \pi'_v H_{T'}(v,u_0) - \sum_{v\in V_4\setminus\{u_i\}} \pi^T_v H_T(v,u_0)\\
&= \sum_{v\in V_4\setminus\{u_i\}} (\pi'_v (d^2_{T'}(v,u_0) + 2\sum_{i\in V(P)} m'_i d_{T'}(i,u_0))) - \sum_{v\in V_4\setminus\{u_i\}} (\pi^T_v (d^2_T(v,u_0) + 2\sum_{i\in V(P)} m^T_i d_T(i,u_0)))\\
&= \sum_{v\in V_4\setminus\{u_i\}} (\pi^T_v (d^2_T(v,u_0) + 2\sum_{i\in V(P)} m'_i d_T(i,u_0))) - \sum_{v\in V_4\setminus\{u_i\}} (\pi^T_v (d^2_T(v,u_0) + 2\sum_{i\in V(P)} m^T_i d_T(i,u_0)))\\
&= \sum_{v\in V_4\setminus\{u_i\}} 2\pi^T_v (d(u_i - c) - d(v,u_0)) \le 0,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{v\in V_4\setminus\{u_i\}} \pi'_v H_{T'}(v,u'_0) - \sum_{v\in V_4\setminus\{u_i\}} \pi^T_v H_T(v,u'_0)\\
&= \sum_{v\in V_4\setminus\{u_i\}} (\pi'_v (d^2_{T'}(v,u'_0) + 2\sum_{i\in V(P)} m'_i d_{T'}(i,u'_0))) - \sum_{v\in V_4\setminus\{u_i\}} (\pi^T_v (d^2_T(v,u'_0) + 2\sum_{i\in V(P)} m^T_i d_T(i,u'_0)))\\
&= \sum_{v\in V_4\setminus\{u_i\}} (\pi^T_v (d^2_T(v,u'_0) + 2\sum_{i\in V(P)} m'_i d_T(i,u'_0))) - \sum_{v\in V_4\setminus\{u_i\}} (\pi^T_v (d^2_T(v,u'_0) + 2\sum_{i\in V(P)} m^T_i d_T(i,u'_0)))\\
&= \sum_{v\in V_4\setminus\{u_i\}} 2\pi^T_v \left(d(v - c') - d(y,u'_0)\right) \ge 0.
\end{aligned}$$

Therefore, we have $\sum\limits_{v\in V_4\setminus\{u_i\}} \pi'_v H_{T'}(v,u_0) - \sum\limits_{v\in V_4\setminus\{u_i\}} \pi'_v H_{T'}(v,u'_0) \le \sum\limits_{v\in V_4\setminus\{u_i\}} \pi^T_v H_T(v,u_0) - \sum\limits_{v\in V_4\setminus\{u_i\}} \pi^T_v H_T(v,u'_0)$.

For the vertices z and u_i, we have

$$\begin{aligned}
&\pi'_z H_{T'}(z,u_0) + \pi'_{u_i} H_{T'}(u_i,u_0) - \pi'_z H_{T'}(z,u'_0) - \pi'_{u_i} H_{T'}(u_i,u'_0)\\
&\le \pi_z\left(H_T(z,u_0) - H_T(z,u'_0)\right) + \pi_{u_i}\left(H_T(u_i,u_0) - H_T(u_i,u'_0)\right)\\
&\quad + \frac{1}{2|E|}\left(H_T(u_i,u_0) - H_{T'}(y,u_0) + H_T(y,u'_0) - H_{T'} u_i, y) - H_{T'}(y,u'_0)\right)\\
&< \pi_z\left(H_T(z,u_0) - H_T(z,u'_0)\right) + \pi_{u_i}\left(H_T(u_i,u_0) - H_T(u_i,u'_0)\right).
\end{aligned}$$

For the vertex x and the vertex x', we have $H_T(x,u_0) - H_T(x,u'_0) = H_T(y,u_0) - H_T(y,u'_0)$ and $H_{T'}(x',u_0) - H_{T'}(x',u'_0) = H_{T'}(x',u_0) - H_{T'}(x',y) - H_{T'}(y,u'_0)$.

Note that $H_{T'}(x',u_0) - H_{T'}(x',y) \le 1 + H_{T'}(u_i,u_0) - 1 = H_T(u_i,u_0) \le H_T(y,u_0)$. We have $H_{T'}(x',u_0) - H_{T'}(x',u'_0) \le H_T(x,u_0) - H_T(x,u'_0)$.

Thus, by the inequalities above, we have

$$\begin{aligned}
&H_{T'}(\pi',u_0) - H_{T'}(\pi',u'_0) = \sum_{v\in V(T')} \left(\pi'_v H_{T'}(v,u_0) - \pi'_v H_{T'}(v,u'_0)\right)\\
&= \sum_{v\in V(T)-\{x\}+\{x'\}} \left(\pi'_v H_{T'}(v,u_0) - \pi'_v H_{T'}(v,u'_0)\right)\\
&< \sum_{v\in V(T)} \left(\pi^T_v H_T(v,u_0) - \pi^T_v H_T(v,u'_0)\right) - \frac{1}{2|E|}\left(H_T(x,u_0) - H_T(x,u'_0) - H_{T'}(x',u_0) + H_{T'}(x',u'_0)\right)\\
&\le \sum_{v\in V(T)} \left(\pi^T_v H_T(v,u_0) - \pi^T_v H_T(v,u'_0)\right) = H_T(\pi^T,u_0) - H_T(\pi,u'_0) \le 0,
\end{aligned}$$

and thus $H_{T'}(\pi',u'_0) > H_{T'}(\pi',u_0)$, which is impossible because we assume that $\min\limits_{v\in V(T')} H_{T'}(\pi',v)$ is achieved at the vertex u'_0.

The proof is completed. □

Let $\mathcal{R}(p,q) \subseteq \mathcal{T}(p,q)$ be the set of trees with a vertex $c \in V(R)$ satisfying $H(\pi,c) = \min_{x\in V} H_R(\pi,x)$ and the other vertices being either neighbors of c or pendent vertices adjacent to some neighbors of c (see Fig. 5).

For a graph $R(p,q) \in \mathcal{R}(p,q)$, assume that $c_1, c_2, \cdots, c_m$ are neighbors of c. The forest $R_{p,q} - c$ has $d(c)$ components. Let $W_1, W_2, \cdots, W_m$ be the vertex sets of the components containing $c_1, c_2, \cdots, c_m$ respectively.

Lemma 4.3. *For the symbols as above, if $|W_j| - |W_i| \ge 2$ for some $1 \le i, j \le m$, then there is at least one neighbor x of c_j in W_j. Let $T(x,c_i)$ be the tree obtained from T by deleting x and adding a new vertex x' adjacent to c_i. Then we have*

$$\min_{v\in V(T)} H_T(\pi^T,v) = H_T(\pi^T,c) > H_{T(x,c_i)}(\pi^{T(x,c_i)},c) = \min_{v\in V(T(x,c_i))} H_{T(x,c_i)}(\pi^{T(x,c_i)},c).$$

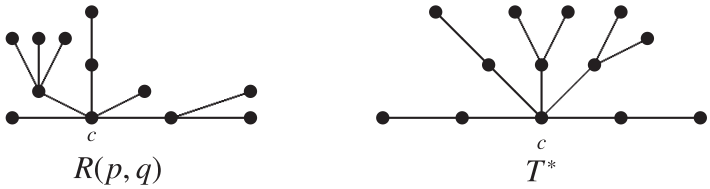

Fig. 5. The graphs $R_{p,q}$ and T^*.

Proof. It's obvious that $H_{T(x,c_i)}(\pi^{T(x,c_i)},c) = \min_{v\in V(T(x,c_i))} H_{T(x,c_i)}(\pi^{T(x,c_i)},c)$. Therefore, it suffices to show that $H_T(\pi^T,c) > H_{T(x,c_i)}(\pi^{T(x,c_i)},c)$.

For $v\in W_j$, we have

$$\begin{aligned}
&\textstyle\sum_{v\in W_j\setminus\{x\}} \pi_v^{T(x,c_i)} H_{T(x,c_i)}(v,c) - \sum_{v\in W_j\setminus\{x\}} \pi_v^T H_T(v,c)\\
&= -\frac{1}{2|E|}H_{T(x,c_i)}(c_j,c) + \textstyle\sum_{v\in W_j\setminus\{x\}} \pi_v^T \left(H_{T(x,c_i)}(v,c_j) + H_{T(x,c_i)}(c_j,c) - H_T(v,c_j) - H_T(c_j,c)\right)\\
&= -\frac{1}{2|E|}H_{T(x,c_i)}(c_j,c)\\
&\quad + \textstyle\sum_{v\in W_j\setminus\{x\}} \pi_v^T \left(d^2_{T(x,c_i)}(v,c_j) + 2\sum_{i\in V(P)} m_i^{T(x,c_i)} d_{T(x,c_i)}(i,c_j) - d_T^2(v,c_j) - 2\sum_{i\in V(P)} m_i^T d_T(i,c_j)\right)\\
&\quad + \textstyle\sum_{v\in W_j\setminus\{x\}} \pi_v^T \left(d^2_{T(x,c_i)}(c_j,c) + 2\sum_{i\in V(P)} m_i^{T(x,c_i)} d_{T(x,c_i)}(i,c) - d_T^2(c_j,c) - 2\sum_{i\in V(P)} m_i^T d_T(i,c)\right)\\
&= -\frac{1}{2|E|}\left(1+2(|W_j|-3)\right) - 2\textstyle\sum_{v\in W_j\setminus\{x\}} \pi_v^T\\
&= -\tfrac{1+2(|W_j|-3)}{2|E|} - \tfrac{2(|W_j|-1)-1}{|E|}.
\end{aligned}$$

For $v\in W_i$, we have

$$\begin{aligned}
&\textstyle\sum_{v\in W_i} \pi_v^{T(x,c_i)} H_{T(x,c_i)}(v,c) - \sum_{v\in W_i} \pi_v^T H_T(v,c)\\
&= \frac{1}{2|E|}H_{T(x,c_i)}(c_i,c) + \textstyle\sum_{v\in W_i} \pi_v^T \left(H_{T(x,c_i)}(v,c_i) + H_{T(x,c_i)}(c_i,c) - H_T(v,c_i) - H_T(c_i,c)\right)\\
&= \frac{1}{2|E|}H_{T(x,c_i)}(c_i,c)\\
&\quad + \textstyle\sum_{v\in W_i} \pi_v^T \left(d^2_{T(x,c_i)}(v,c_i) + 2\sum_{i\in V(P)} m_i^{T(x,c_i)} d_{T(x,c_i)}(i,c_i) - d_T^2(v,c_i) - 2\sum_{i\in V(P)} m_i^T d_T(i,c_i)\right)\\
&\quad + \textstyle\sum_{v\in W_i} \pi_v^T \left(d^2_{T(x,c_i)}(c_i,c) + 2\sum_{i\in V(P)} m_i^{T(x,c_i)} d_{T(x,c_i)}(i,c) - d_T^2(c_i,c) - 2\sum_{i\in V(P)} m_i^T d_T(i,c)\right)\\
&= \frac{1}{2|E|}(1+2(|W_i|-1)) - 2\textstyle\sum_{v\in W_i} \pi_v^T\\
&= \tfrac{1+2(|W_i|-1)}{2|E|} + \tfrac{2(|W_i|-1)}{|E|}.
\end{aligned}$$

For the vertex x, we have

$$\begin{aligned}
&\pi_x^{T(x,c_i)} H_{T(x,c_i)}(x,c) - \pi_x^T H_T(x,c)\\
&= \frac{1}{2|E|}\left(2^2 + 2(|W_i|-2) - 2^2 - 2(|W_j|-3)\right)\\
&= \frac{|W_i|-|W_j|+1}{|E|} < 0.
\end{aligned}$$

By Lemma 2.2, it is easy to obtain

$$\sum_{v\in V(T)\setminus(W_i\cup W_j)} \pi_v^{T(x,c_i)} H_{T(x,c_i)}(v,c) = \sum_{v\in V(T)\setminus(W_i\cup W_j)} \pi_v^T H_T(v,c).$$

Therefore, we have

$$\begin{aligned}
&H_{T(x,c_i)}(\pi^{T(x,c_i)},c) - H_T(\pi^T,c)\\
&= \textstyle\sum_{v\in V(T(x,c_i))} \pi_v^{T(x,c_i)} H_{T(x,c_i)}(v,c) - \sum_{v\in V(T)} \pi_v^T H_T(v,c)\\
&= \textstyle\sum_{v\in W_i\cup W_j} \pi_v^{T(x,c_i)} H_{T(x,c_i)}(v,c) - \sum_{v\in W_i\cup W_j} \pi_v^T H_T(v,c) + \pi_x^{T(x,c_i)} H_{T(x,c_i)}(x',c) - \pi_x^{T(x,c_i)} H_T(x,c) < 0,
\end{aligned}$$

and thus $H_{T(x,c_i)}(\pi^{T(x,c_i)}, c) < H_T(\pi^T, c)$.

The proof is completed. □

Proof of Theorem 1.3.. Let $T^* \in \mathcal{R}(p,q)$ be the graph with a vertex c such that

$$\min_{v\in V(T^*)} H^*(\pi^*, v) = H^*(\pi^*, c),$$

and the neighbors $\{c_1, c_2, \cdots, c_q\}$ of c satisfying $|deg(c_i) - deg(c_j)| \le 1$ for any $1 \le i, j \le q$ (see Fig. 5).

Let $T \in \mathcal{T}(p,q)$ be a tree such that $\min_{v\in V} H_T(\pi^T, v) = H_T(\pi^T, c)$ for some vertex c, and let $c_1, c_2, \ldots, c_k$ be the neighbors of c and T_i the component of $T_{p,q} - c$ that contains c_i. For any pendent vertex x in T_i different from c_i, if x and c are in same partition, then we construct the new tree T' from T by deleting x and adding a new pendent vertex x' adjacent to c_i; if x and c are in different partition, then we construct the new tree T' from T by deleting x and adding a new pendent vertex x' adjacent to c. Therefore, by Lemmas 4.1 and 4.2, we have $\min_{v\in V} H_{T'}(\pi^{T'}, v) = H_{T'}(\pi^{T'}, c)$ and $H_{T'}(\pi^{T'}, c) < H_T(\pi^T, c)$. Moreover, it is clear that $T' \in \mathcal{T}(p,q)$. By repeating this process, we ulitimately obtain a graph $R_{p,q} \in \mathcal{R}(p,q)$.

For any graph $R_{p,q} \in \mathcal{R}(p,q)$, by applying Lemma 4.3 repeatedly, we will ultimately get either the graph $S_{p,q} \in \mathcal{T}(p,q)$ or the graph $T^* \in \mathcal{T}(p,q)$ such that $\min_{v\in V} H(\pi, v)$ decreases.

By immediate calculations, we have $\min_{v\in V(S_{p,q})} H_{S(p,q)}(\pi^{S_{p,q}}, v) \le \min_{v\in V(T^*)} H_{T^*}(\pi^*, v)$, and

$$\begin{aligned} &\min\nolimits_{v\in V} H_{S_{p,q}}(\pi, v) = H_{S_{p,q}}(\pi, c) = \sum_{v\in V} \pi_v H_{S_{p,q}}(v, c) \\ &= \frac{p-q+1}{2(p+q-1)} \times 1 + \frac{q-1}{2(p+q-1)} \times 4 + \frac{2(q-1)}{2(p+q-1)} \times 3 \\ &= \frac{p+9q-9}{2(p+q-1)}. \end{aligned}$$

The proof is completed. □

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