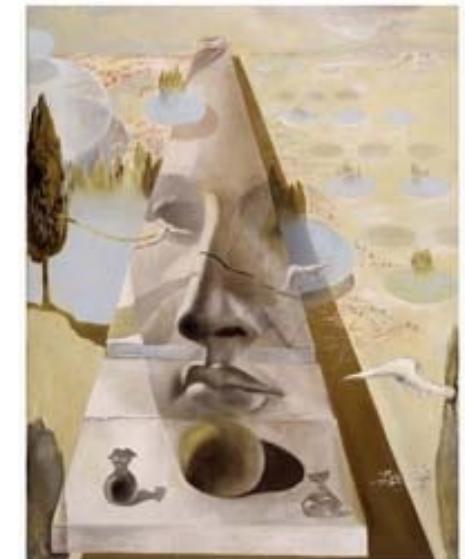


# Lecture 2

# Camera Models



1891

Professor Silvio Savarese

*Stanford Vision and Learning Lab*

# Lecture 2

## Camera Models

- Pinhole cameras
- Cameras & lenses
- The geometry of pinhole cameras



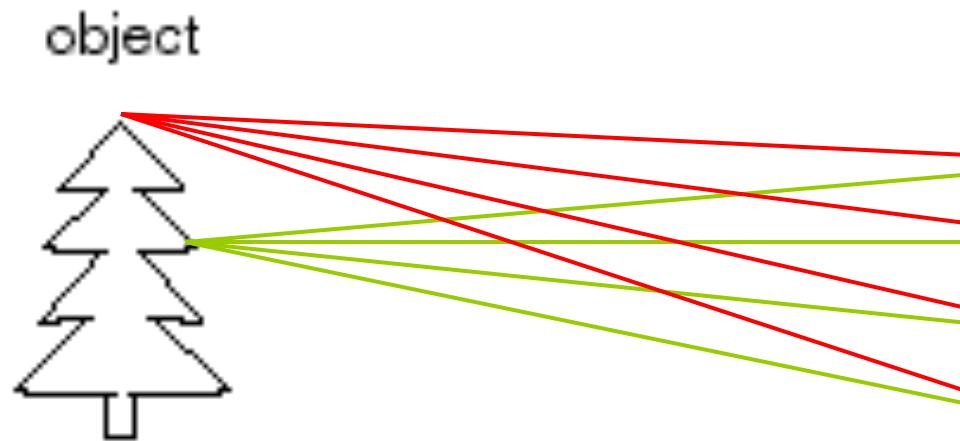
Reading:

[FP] Chapter 1, “Geometric Camera Models”

[HZ] Chapter 6 “Camera Models”

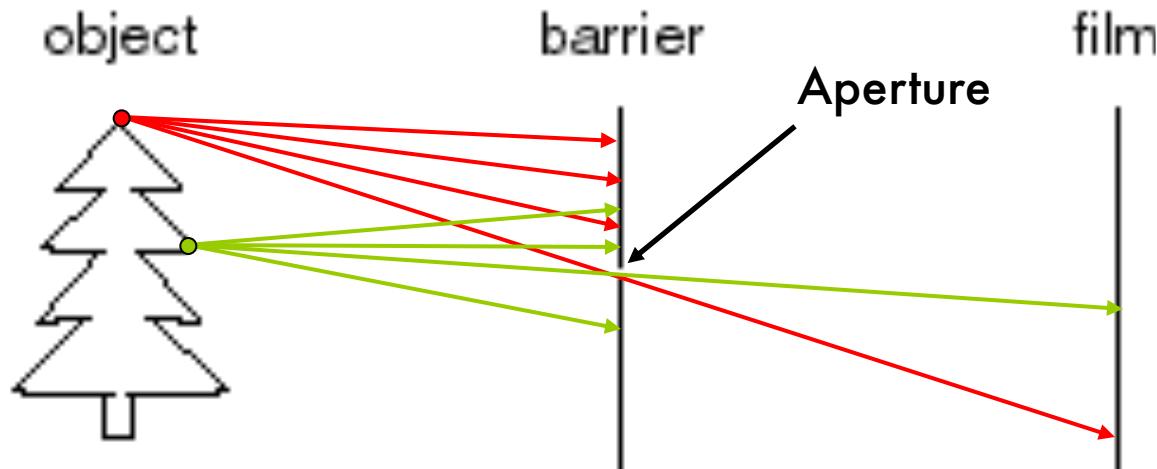
Some slides in this lecture are courtesy to Profs. J. Ponce, S. Seitz, F-F Li

# How do we see the world?



- Let's design a camera
  - Idea 1: put a piece of film in front of an object
  - Do we get a reasonable image?

# Pinhole camera



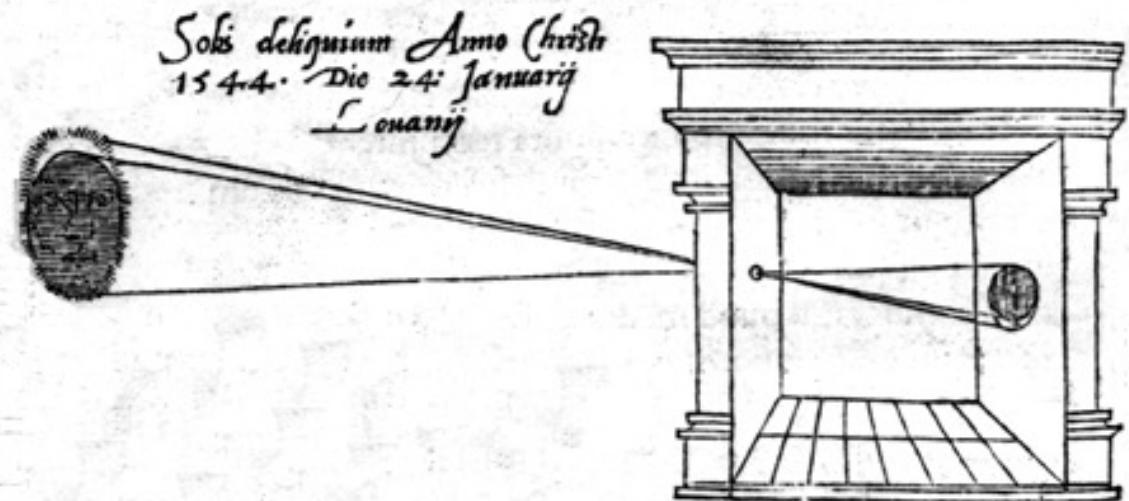
- Idea 2: Add a barrier to block off most of the rays
  - This reduces blurring
  - The opening known as the **aperture**

# Some history...

## Milestones:

- Leonardo da Vinci (1452-1519):  
first record of camera obscura (1502)

illum in tabula per radios Solis , quam in cœlo contin-  
git: hoc est, si in cœlo superior pars deliquiū patiatur, in  
radiis apparebit inferior deficere, ut ratio exigit optica.

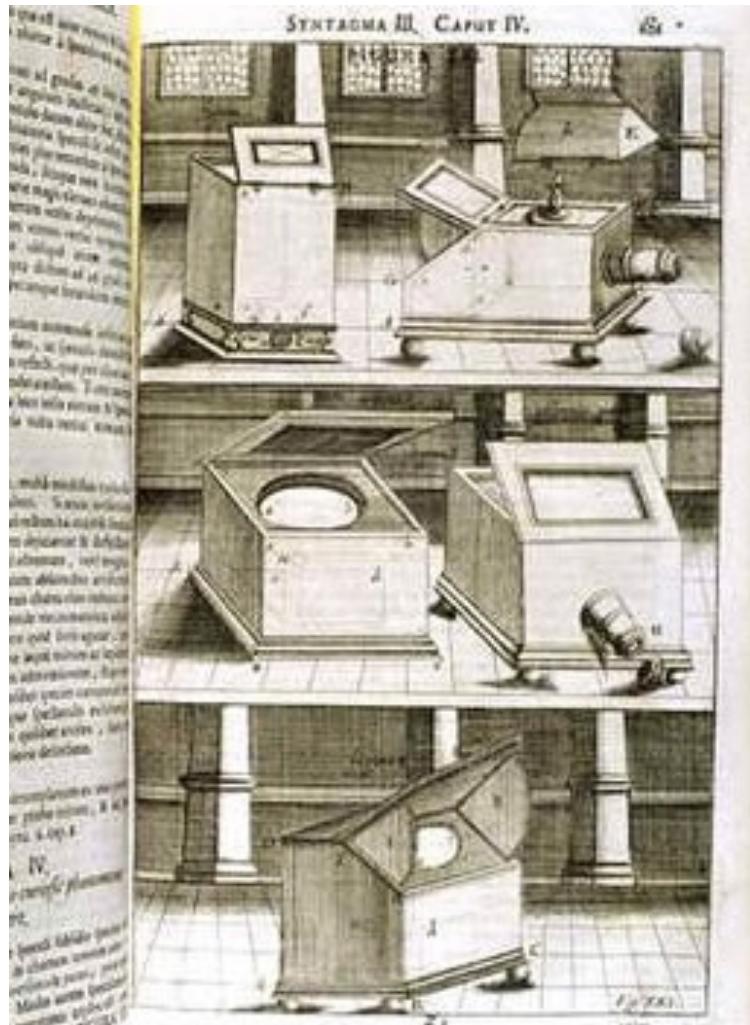


Sic nos exacte Anno .1544. Louanii eclipsim Solis  
obseruauimus, inuenimusq; deficere paulo plus q̄ dex-

# Some history...

## Milestones:

- Leonardo da Vinci (1452-1519): first record of camera obscura
- Johann Zahn (1685): first portable camera



# Some history...

## Milestones:

- Leonardo da Vinci (1452-1519): first record of camera obscura
- Johann Zahn (1685): first portable camera
- Joseph Nicéphore Niépce (1822): first photo - birth of photography

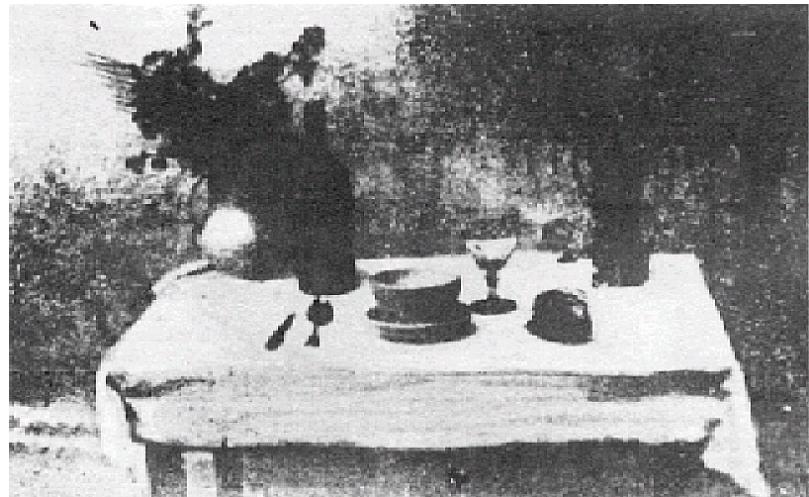


Photography (Niépce, "La Table Servie," 1822)

# Some history...

## Milestones:

- Leonardo da Vinci (1452-1519): first record of camera obscura
- Johann Zahn (1685): first portable camera
- Joseph Nicéphore Niépce (1822): first photo - birth of photography
- Daguerreotypes (1839)
- Photographic Film (Eastman, 1889)
- Cinema (Lumière Brothers, 1895)
- Color Photography (Lumière Brothers, 1908)



Photography (Niépce, "La Table Servie," 1822)

# Let's also not forget...



**Mozi**  
**(468-376 BC)**

Oldest existent  
book on  
geometry in  
China

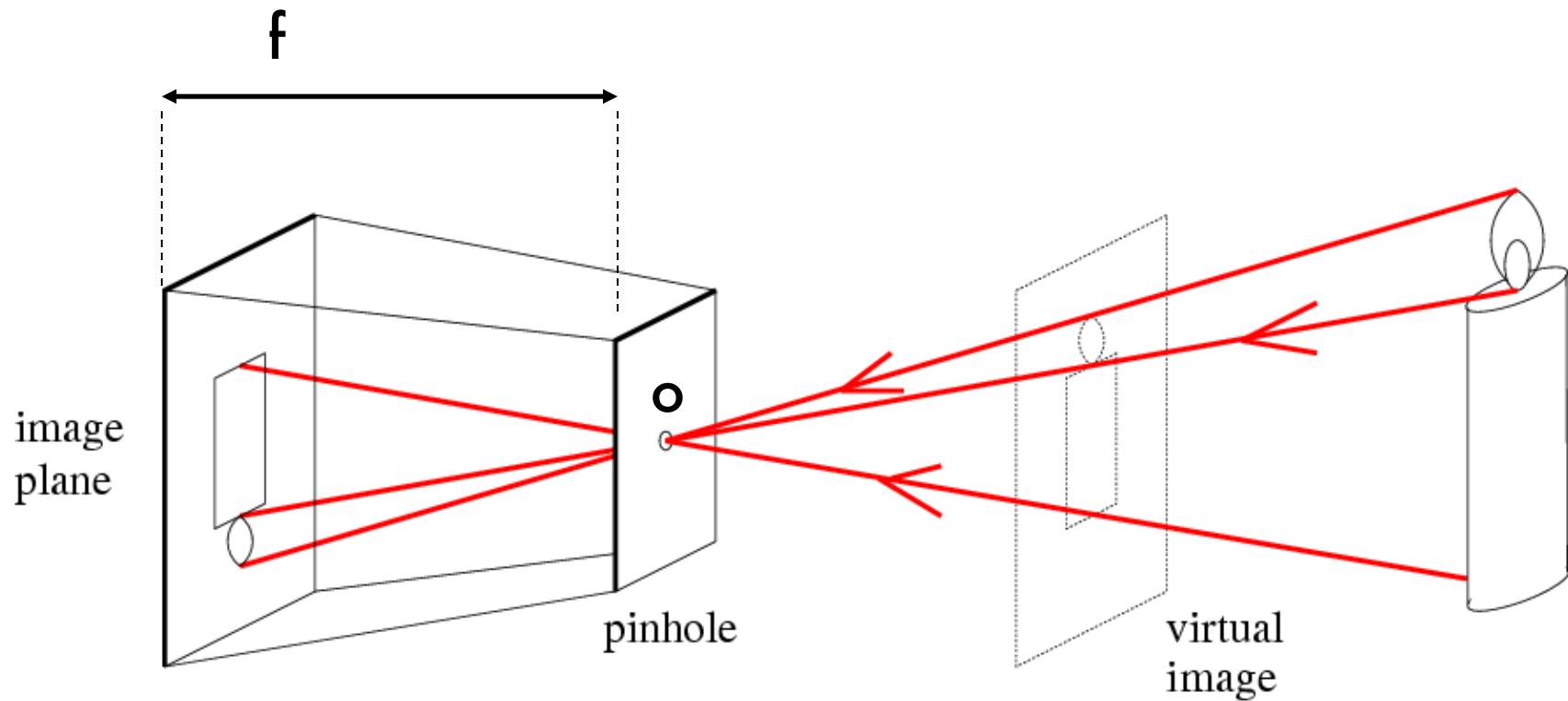


**Aristotle**  
**(384-322 BC)**  
Also: Plato, Euclid



**Al-Kindi (c. 801-873)**  
**Ibn al-Haitham**  
**(965-1040)**

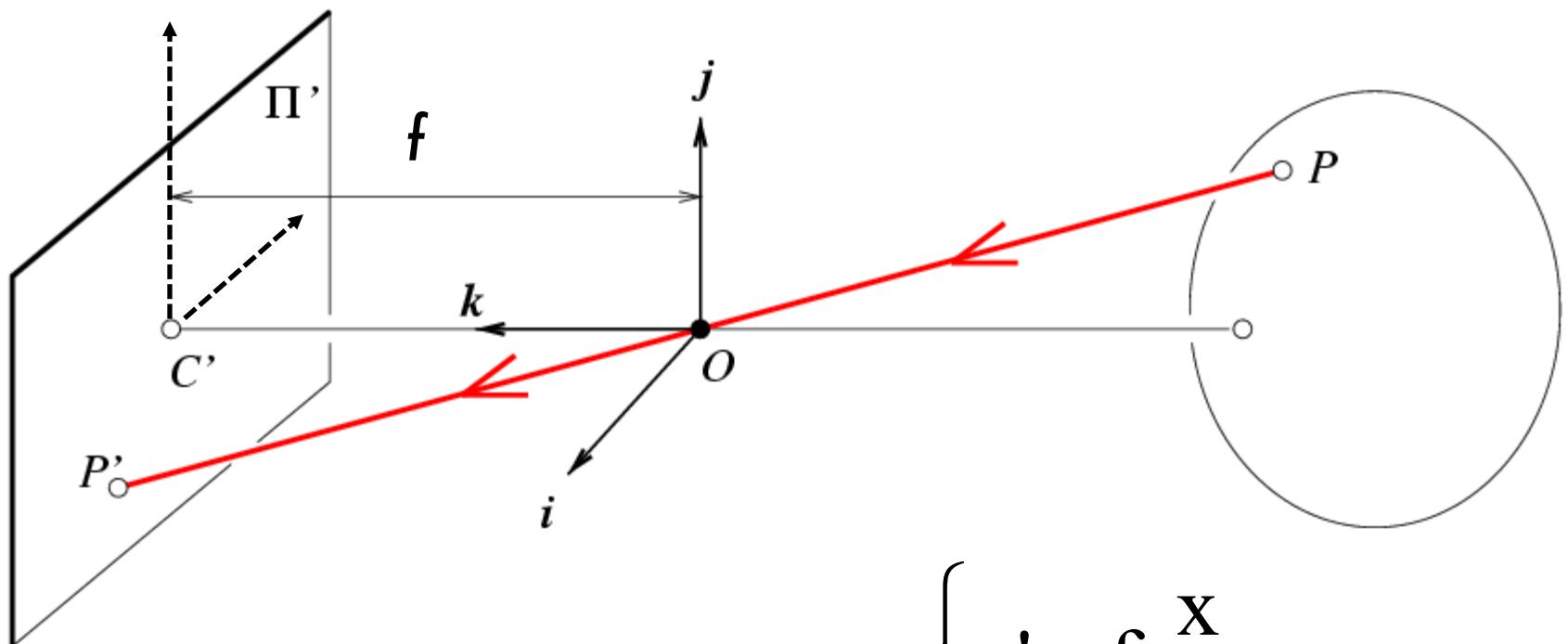
# Pinhole camera



$f$  = focal length

$o$  = aperture = pinhole = center of the camera

# Pinhole camera



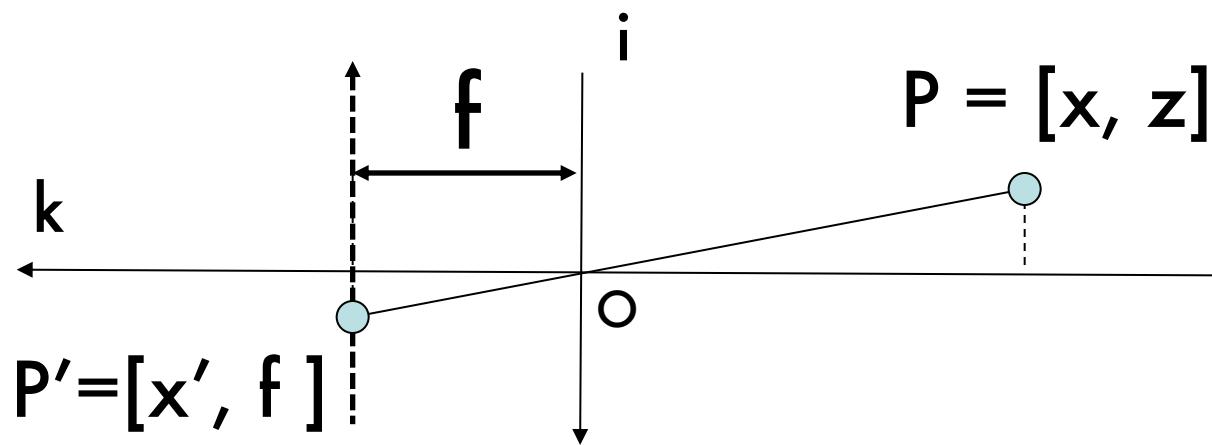
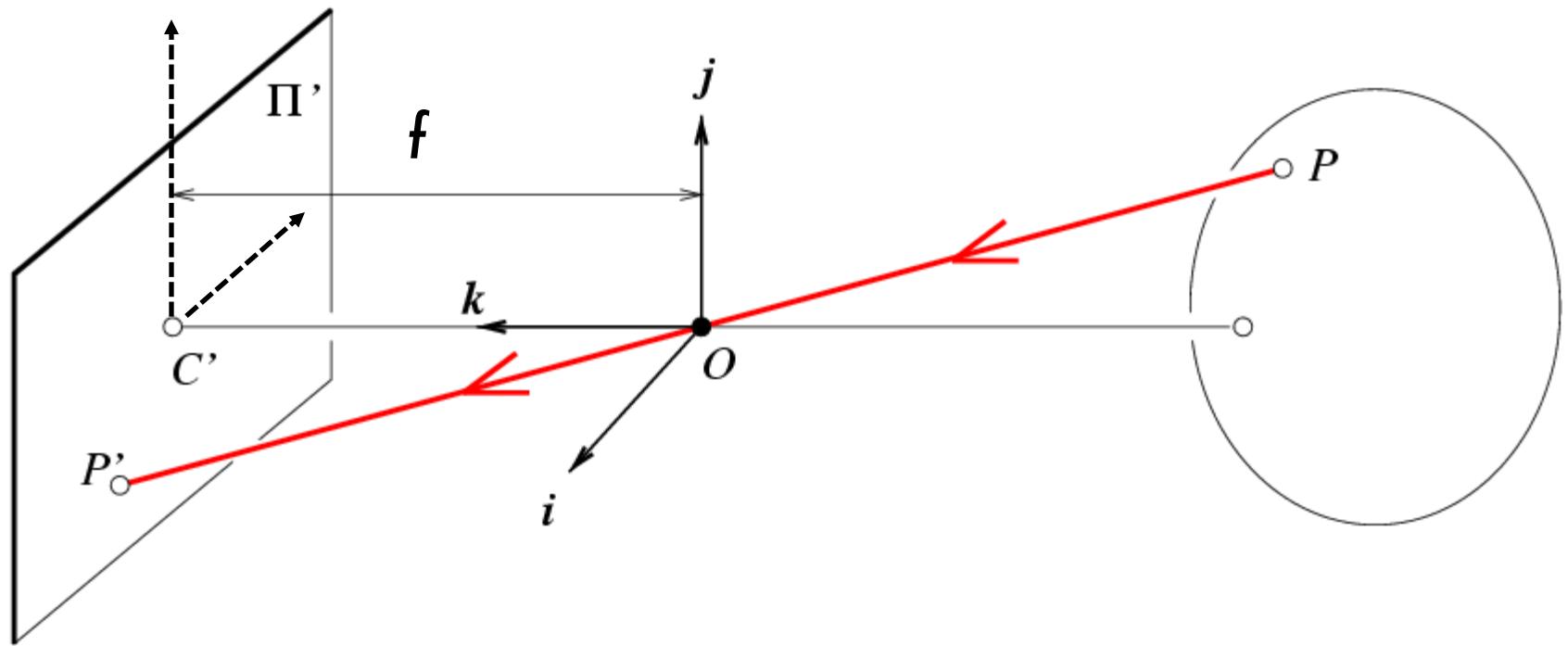
$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow P' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\left\{ \begin{array}{l} x' = f \frac{x}{z} \\ y' = f \frac{y}{z} \end{array} \right.$$

[Eq. 1]

Derived using similar triangles

# Pinhole camera

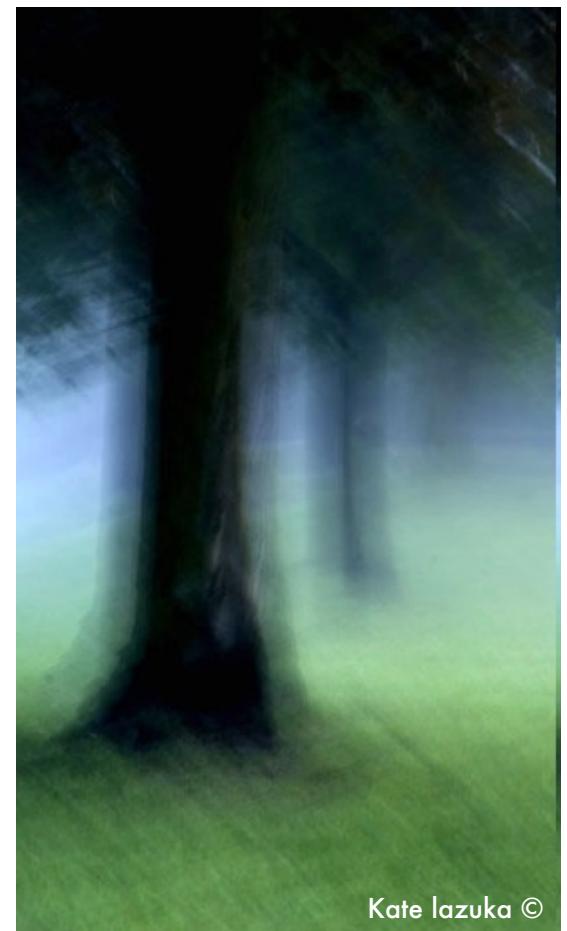
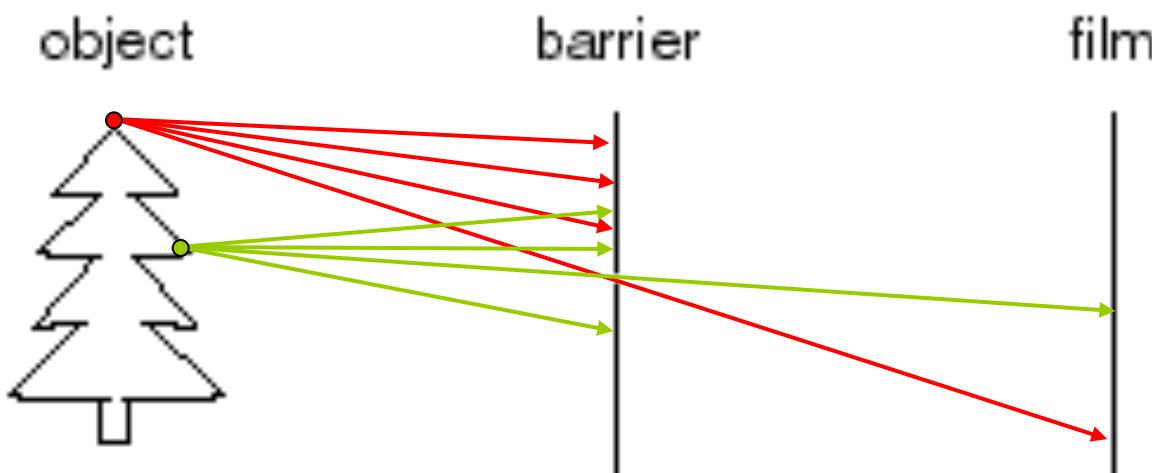


[Eq. 2]

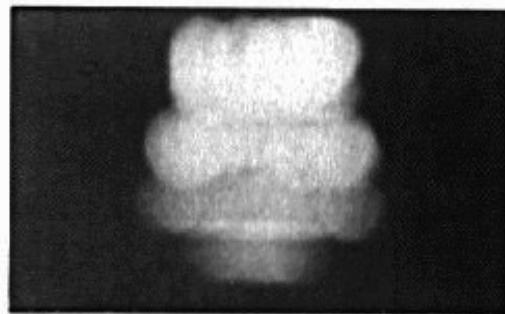
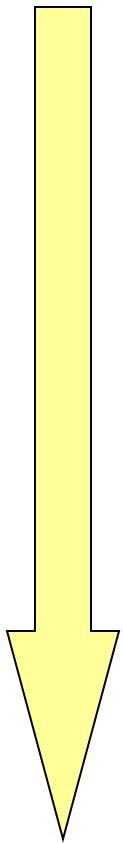
$$\frac{x'}{f} = \frac{x}{z}$$

# Pinhole camera

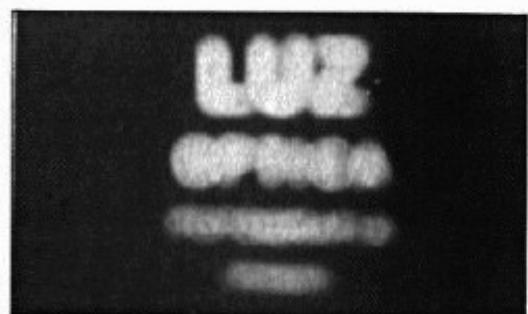
Is the size of the aperture important?



Shrinking  
aperture  
size



2 mm



1 mm



0.6mm



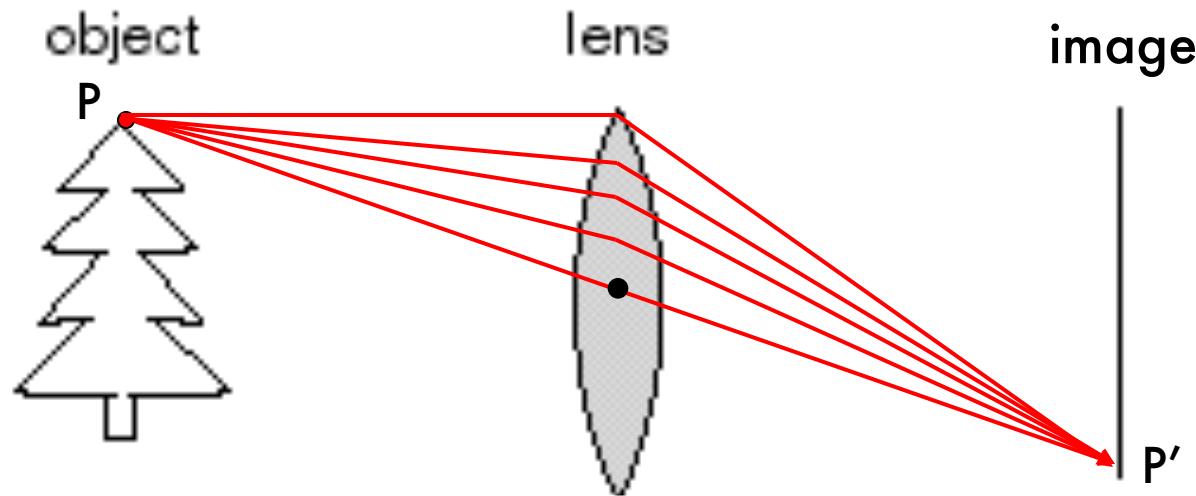
0.35 mm

-What happens if the aperture is too small?

-Less light passes through

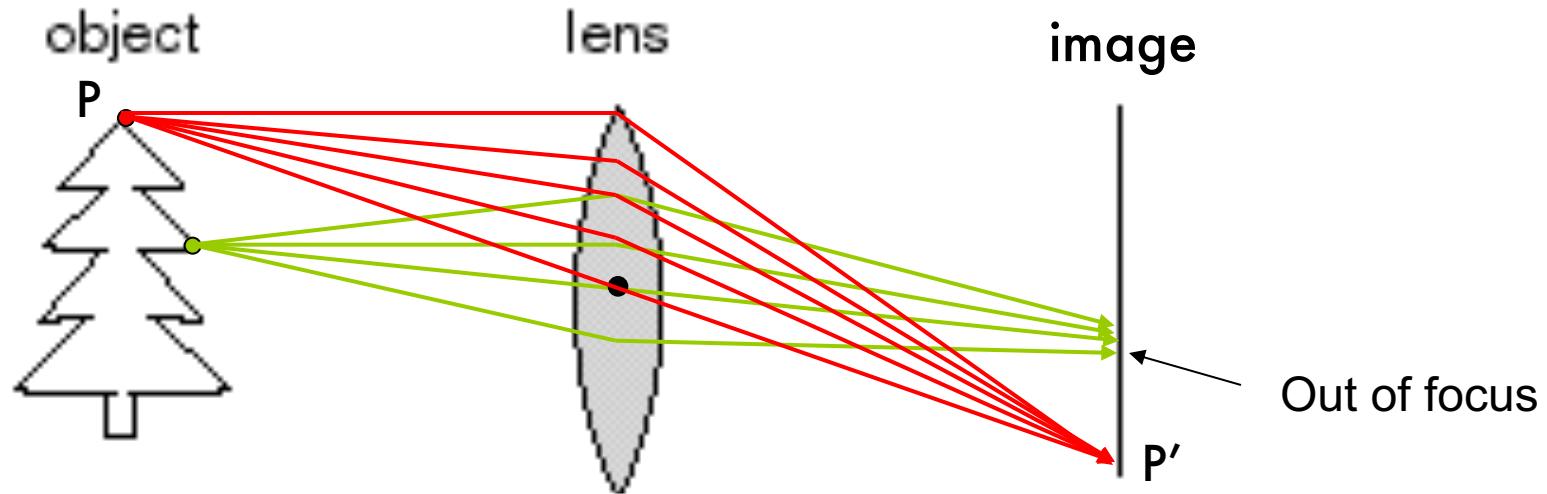
Adding lenses!

# Cameras & Lenses



- A lens focuses light onto the film

# Cameras & Lenses



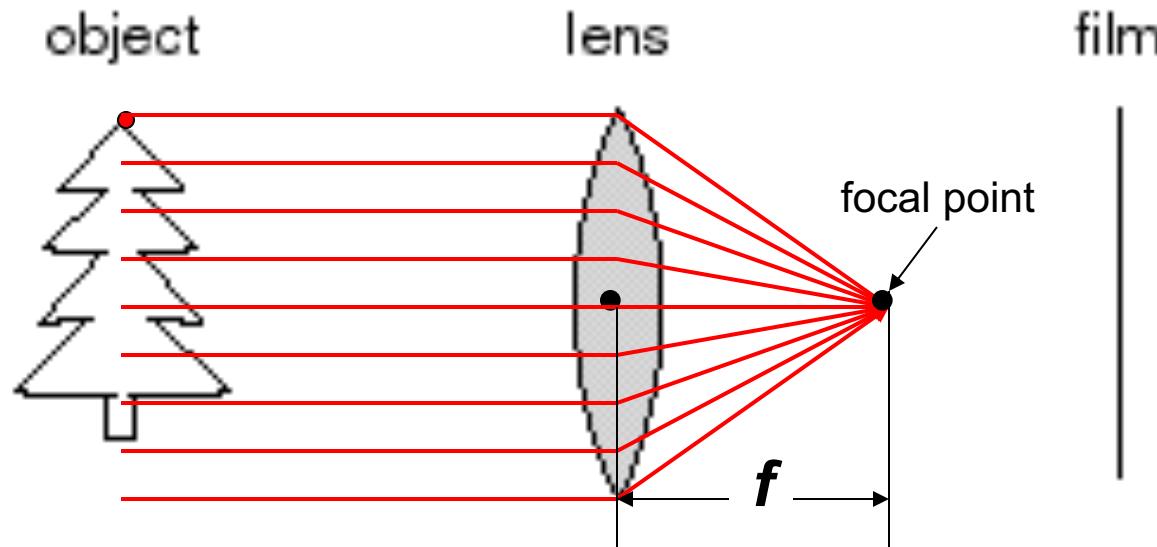
- A lens focuses light onto the film
  - There is a specific distance at which objects are “in focus”
  - Related to the concept of depth of field

# Cameras & Lenses



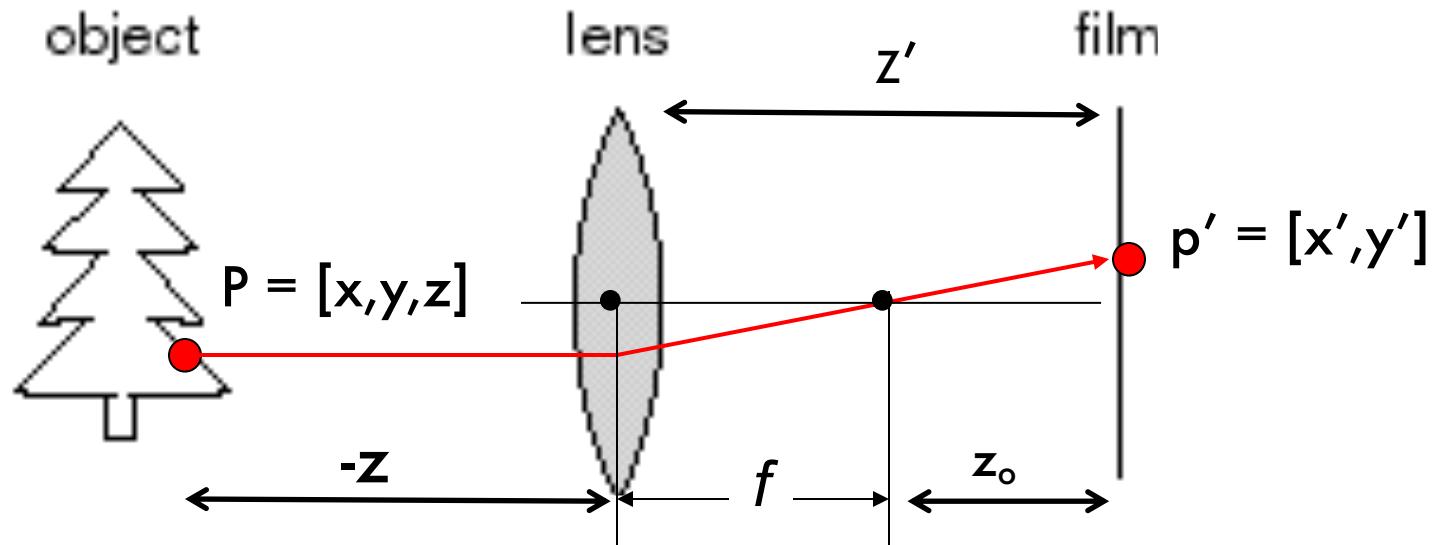
- A lens focuses light onto the film
  - There is a specific distance at which objects are “in focus”
  - Related to the concept of depth of field

# Cameras & Lenses



- A lens focuses light onto the film
- All rays parallel to the optical (or principal) axis converge to one point (the *focal point*) on a plane located at the *focal length*  $f$  from the center of the lens.
- Rays passing through the center are not deviated

# Paraxial refraction model



From Snell's law:

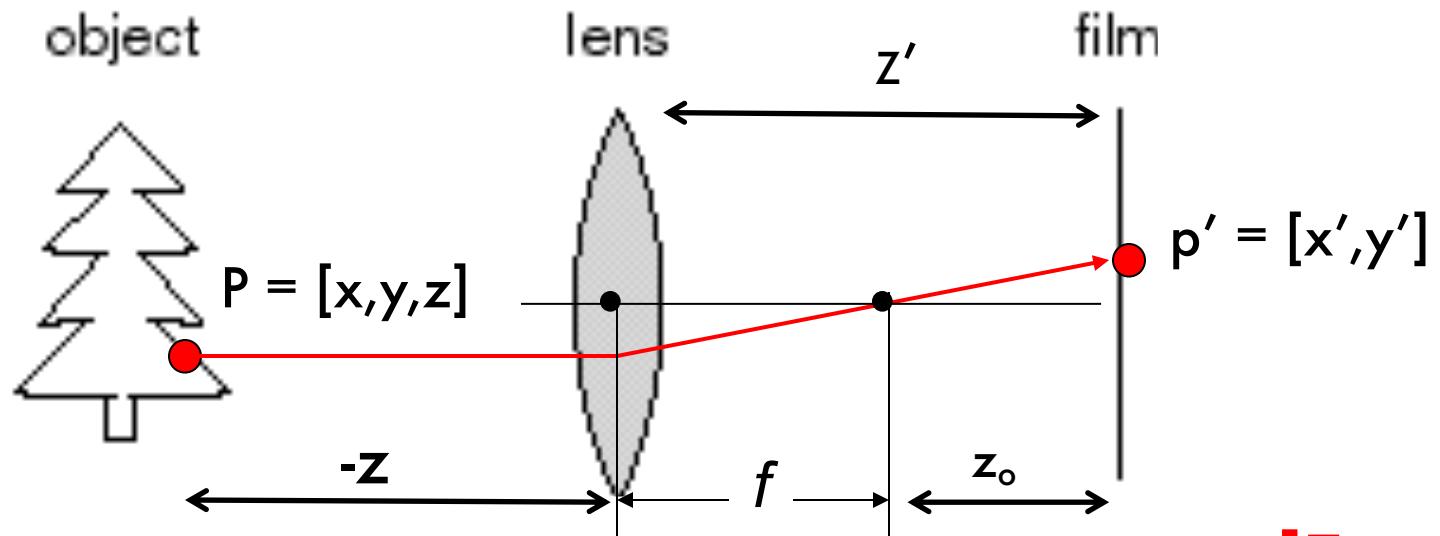
[Eq. 3]

$$\begin{cases} x' = z' \frac{x}{z} \\ y' = z' \frac{y}{z} \end{cases}$$

$$\begin{cases} x' = f \frac{x}{z} \\ y' = f \frac{y}{z} \end{cases}$$

[Eq. 1]

# Paraxial refraction model



[Eq. 4]

From Snell's law:

[Eq. 3]

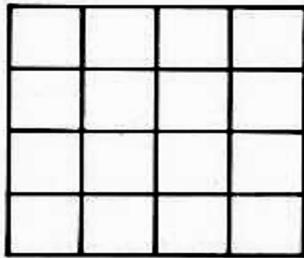
$$\left\{ \begin{array}{l} x' = z' \frac{x}{z} \\ y' = z' \frac{y}{z} \end{array} \right.$$

$$z' = f + z_o$$

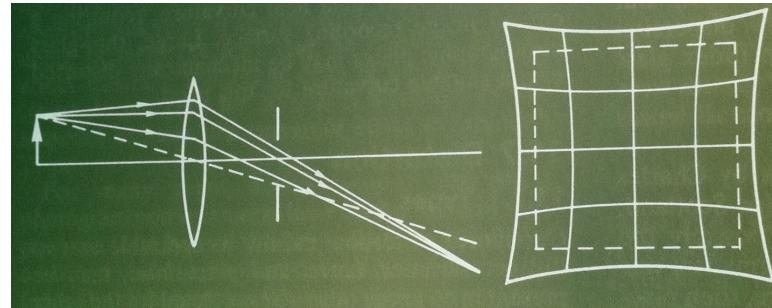
$$f = \frac{R}{2(n-1)}$$

# Issues with lenses: Radial Distortion

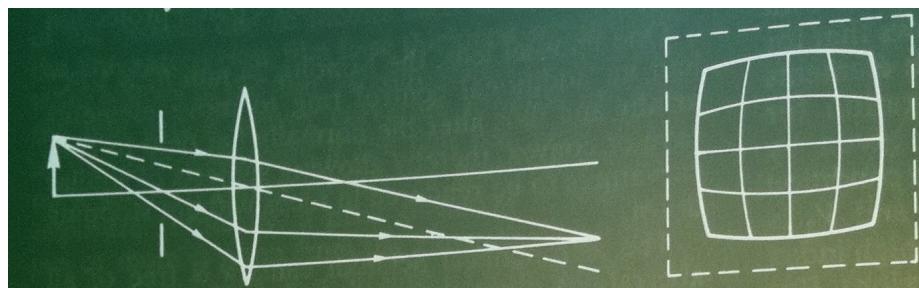
- Deviations are most noticeable for rays that pass through the edge of the lens



No distortion



Pin cushion



Barrel (fisheye lens)

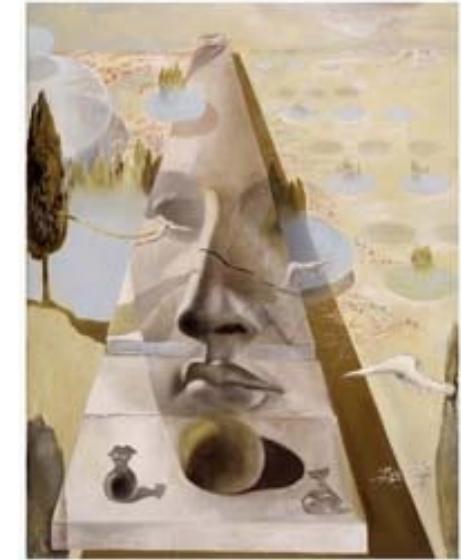


Image magnification decreases with distance from the optical axis

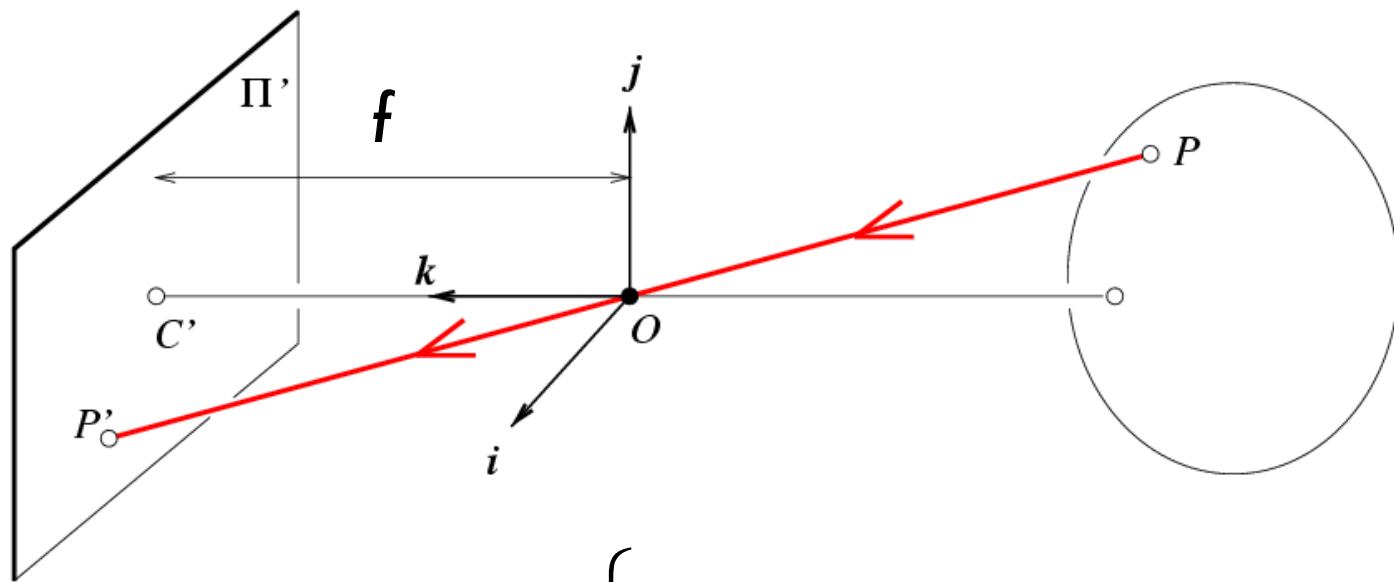
# Lecture 2

# Camera Models

- Pinhole cameras
- Cameras & lenses
- The geometry of pinhole cameras
  - Intrinsic
  - Extrinsic



# Pinhole camera



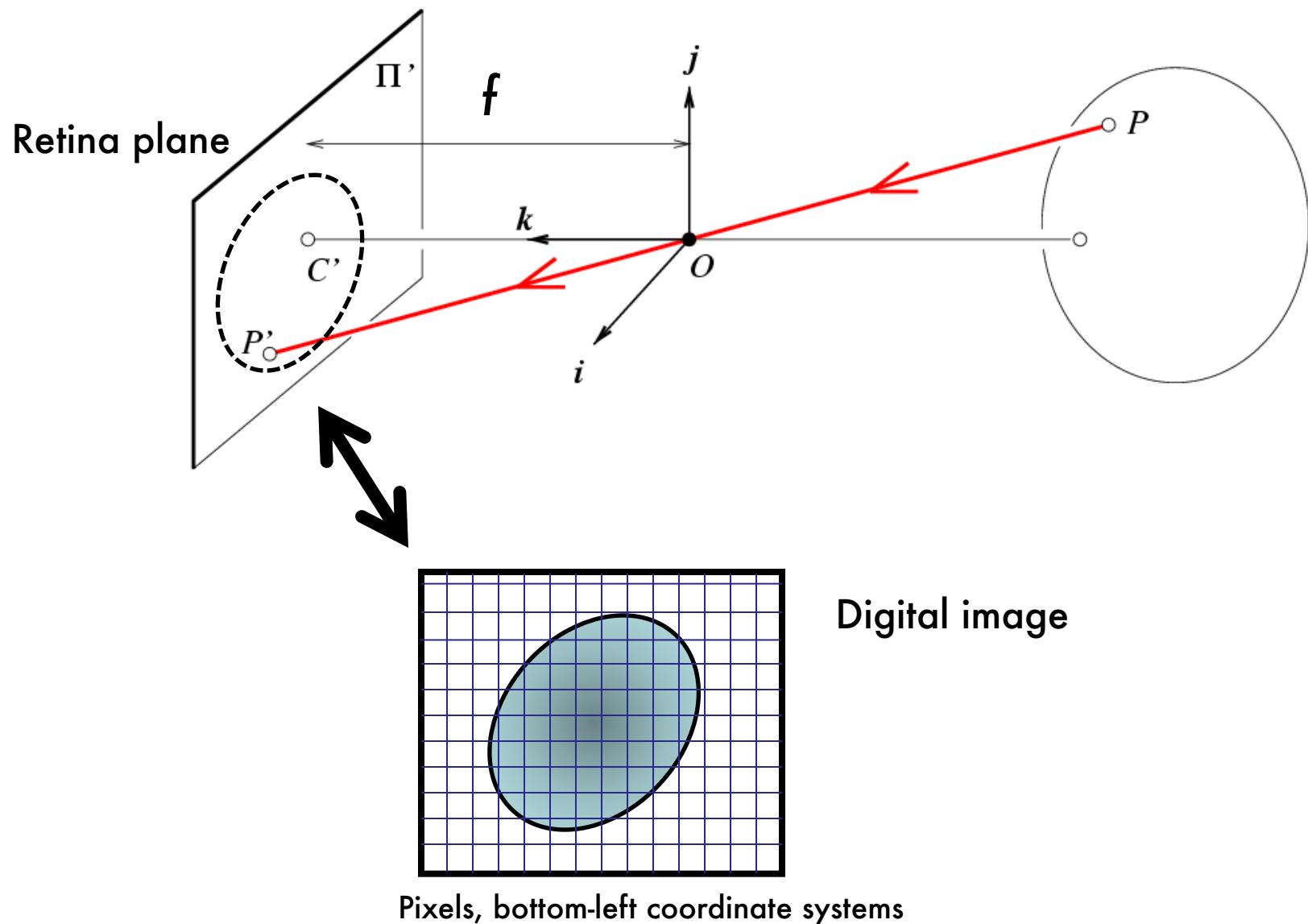
$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \left\{ \begin{array}{l} x' = f \frac{x}{z} \\ y' = f \frac{y}{z} \end{array} \right. \quad \mathfrak{R}^3 \xrightarrow{E} \mathfrak{R}^2$$

[Eq. 1]

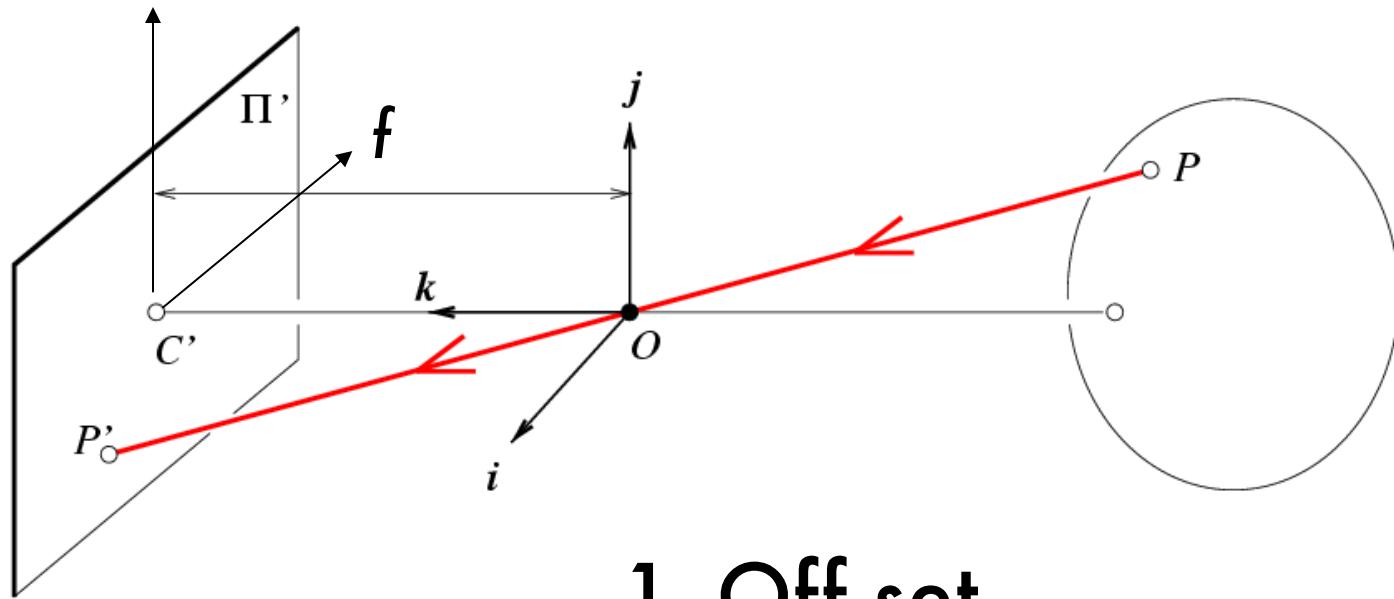
$f$  = focal length

$O$  = center of the camera

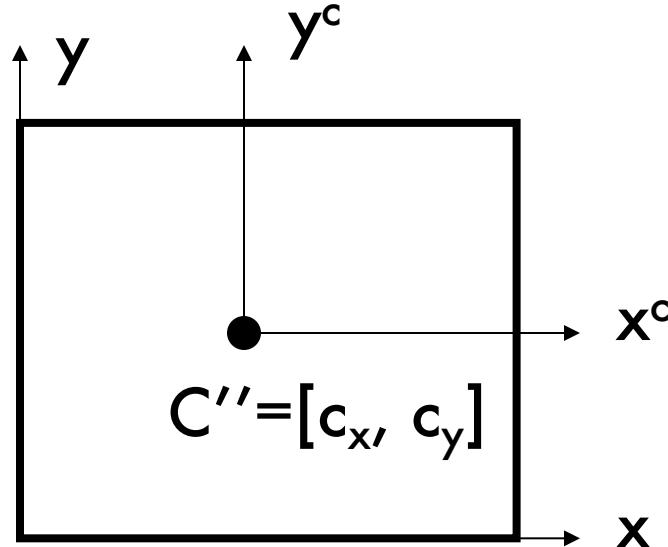
# From retina plane to images



# Coordinate systems



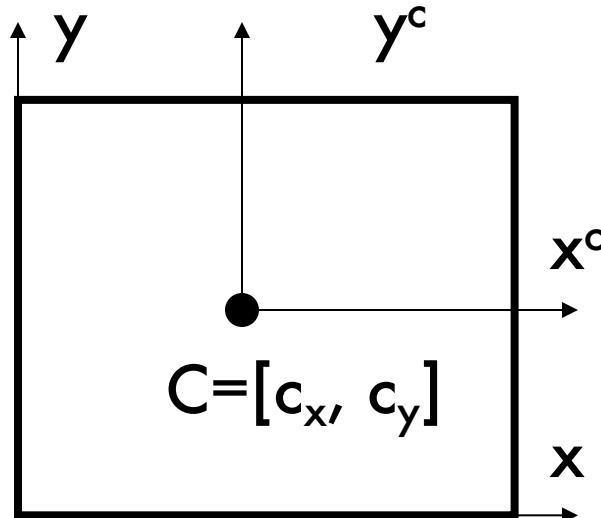
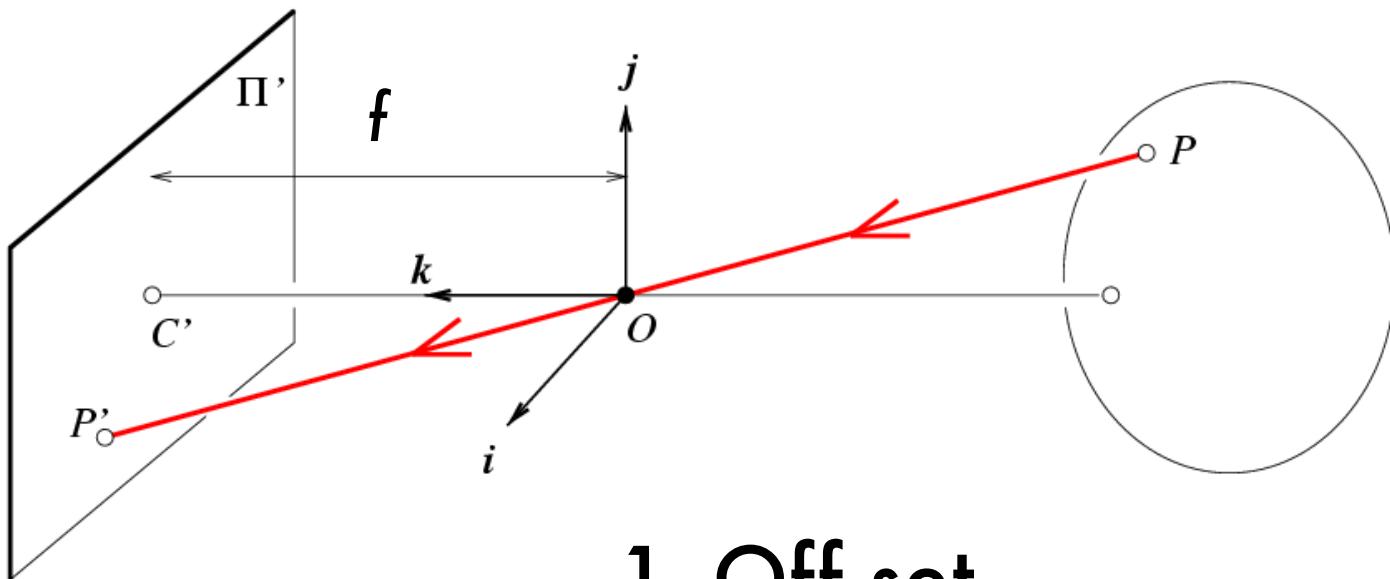
## 1. Off set



$$(x, y, z) \rightarrow \left( f \frac{x}{z} + c_x, f \frac{y}{z} + c_y \right)$$

[Eq. 5]

# Converting to pixels



1. Off set
2. From metric to pixels

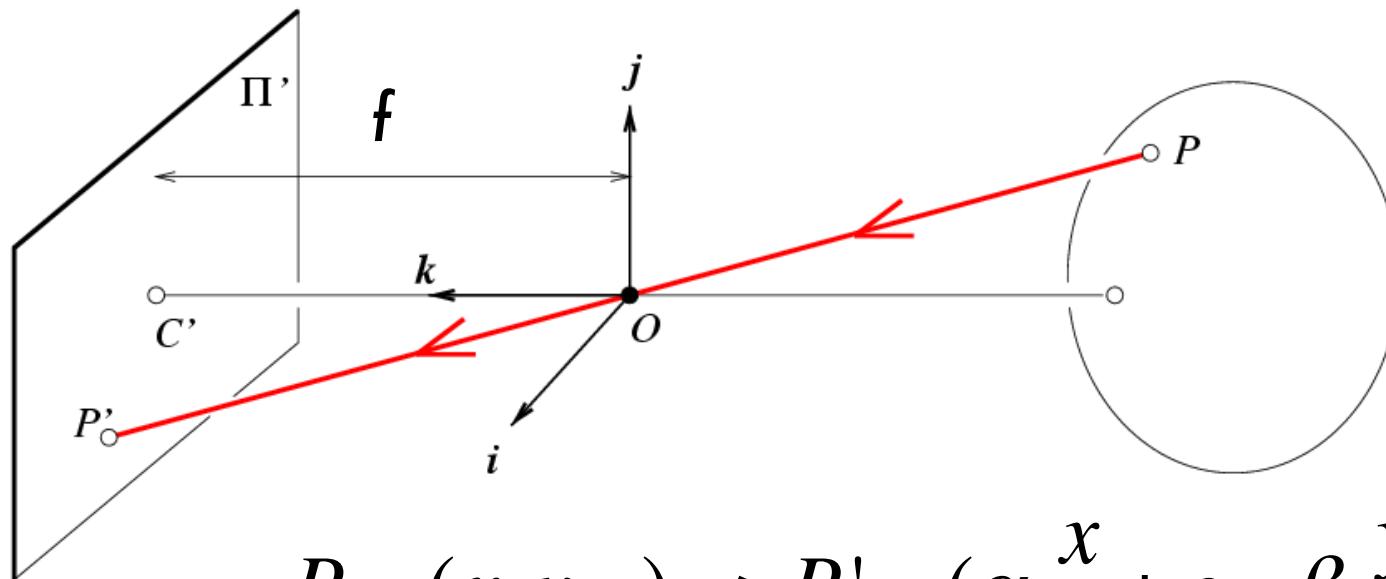
$$(x, y, z) \rightarrow \left( f \frac{k}{z} + c_x, f \frac{l}{z} + c_y \right)$$

[Eq. 6]

Units:  $k, l$  : pixel/m  
 $f$  : m

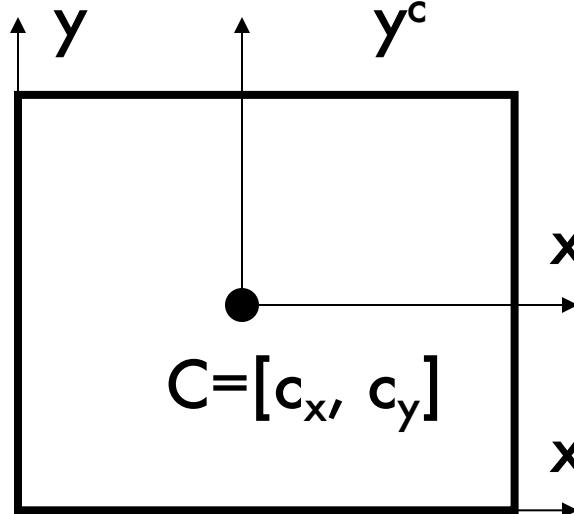
Non-square pixels  
 $\alpha, \beta$  : pixel

# Is this projective transformation linear?



$$P = (x, y, z) \rightarrow P' = (\alpha \frac{x}{z} + c_x, \beta \frac{y}{z} + c_y)$$

[Eq. 7]



$$C = [c_x, c_y]$$

- Is this a linear transformation?  
No – division by  $z$  is nonlinear
- Can we express it in a matrix form?

# Homogeneous coordinates

E → H

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous image  
coordinates

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

homogeneous scene  
coordinates

- Converting back from homogeneous coordinates

H → E

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

# Projective transformation in the homogenous coordinate system

$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$P_h$

# Projective transformation in the homogenous coordinate system

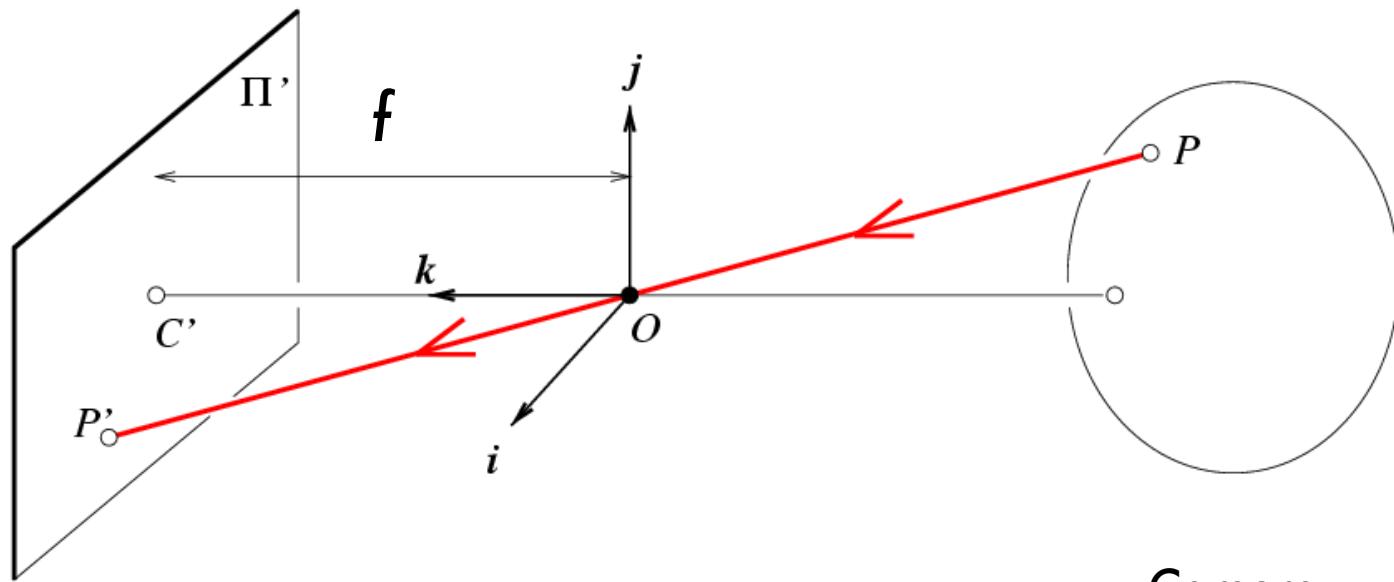
$$P_h' = \begin{bmatrix} \alpha x + c_x z \\ \beta y + c_y z \\ z \end{bmatrix} = \begin{bmatrix} \alpha & 0 & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad P_h \quad [\text{Eq.8}]$$

Homogenous                                  Euclidian

$$\overbrace{P_h'}^{} \rightarrow P' = (\alpha \frac{x}{z} + c_x, \beta \frac{y}{z} + c_y)$$

$$M = \begin{bmatrix} \alpha & 0 & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

# The Camera Matrix



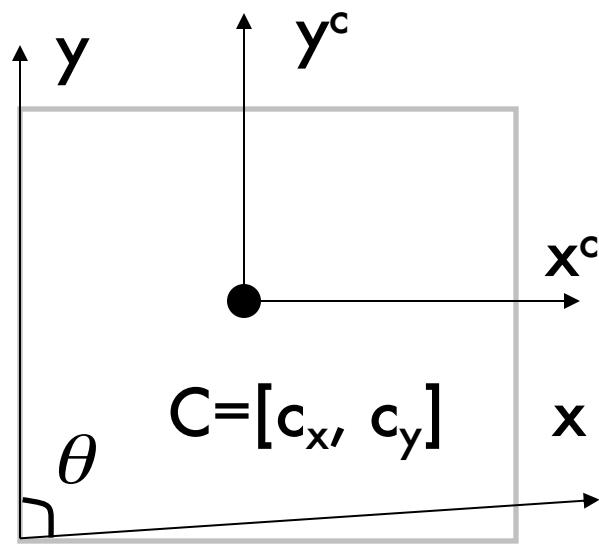
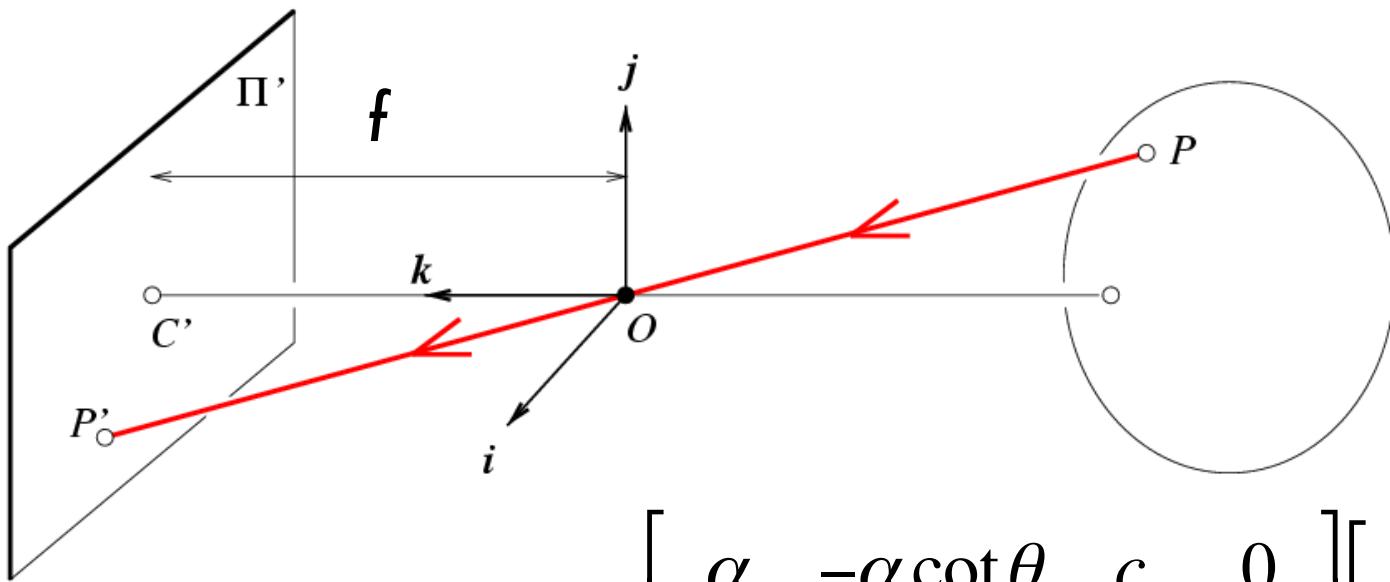
[Eq.9]

$$\begin{aligned}P' &= M P \\&= K \begin{bmatrix} I & 0 \end{bmatrix} P\end{aligned}$$

Camera matrix  $K$

$$P' = \begin{bmatrix} \alpha & 0 & c_x \\ 0 & \beta & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

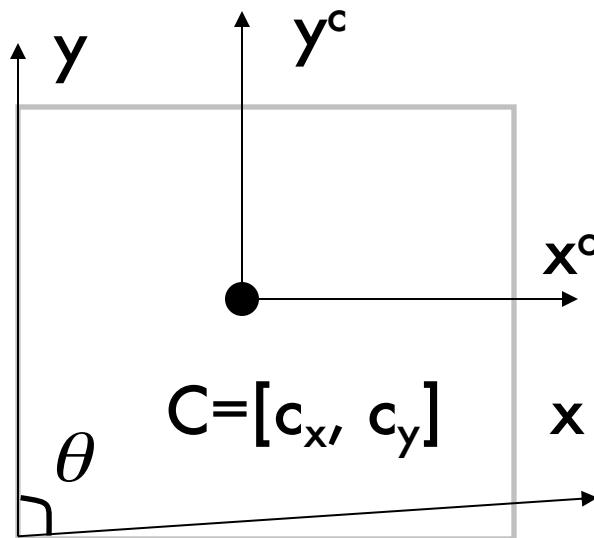
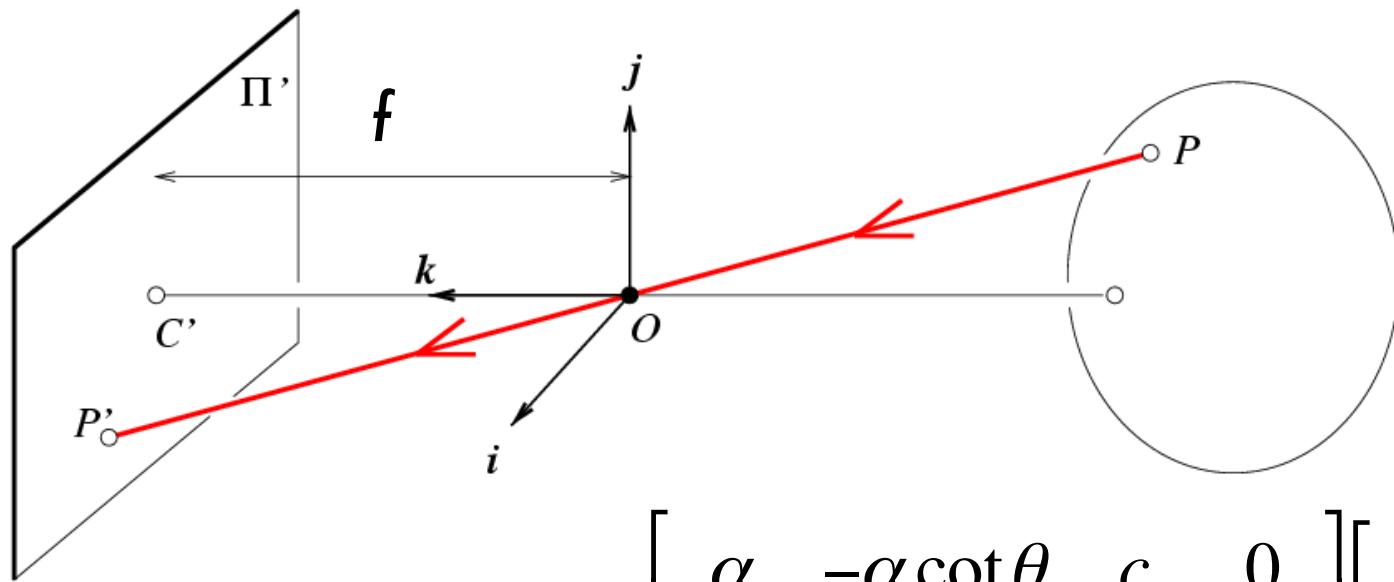
# Camera Skewness



$$P' = \begin{bmatrix} \alpha & -\alpha \cot \theta & c_x & 0 \\ 0 & \frac{\beta}{\sin \theta} & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

sec. 1.2.2 [FP] or sec. 6.2.4 [HZ]

# Degrees of freedom of K



$$P' = \begin{bmatrix} \alpha & -\alpha \cot \theta & c_x & 0 \\ 0 & \frac{\beta}{\sin \theta} & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

How many degrees of freedom does K have?  
5 degrees of freedom!

# Canonical Projective Transformation

$$P' = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_M \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad [Eq.10]$$
$$P' = M P$$
$$\mathbb{R}^4 \xrightarrow{H} \mathbb{R}^3$$

$$P_i' = \begin{bmatrix} \frac{x}{z} \\ \frac{y}{z} \end{bmatrix}$$

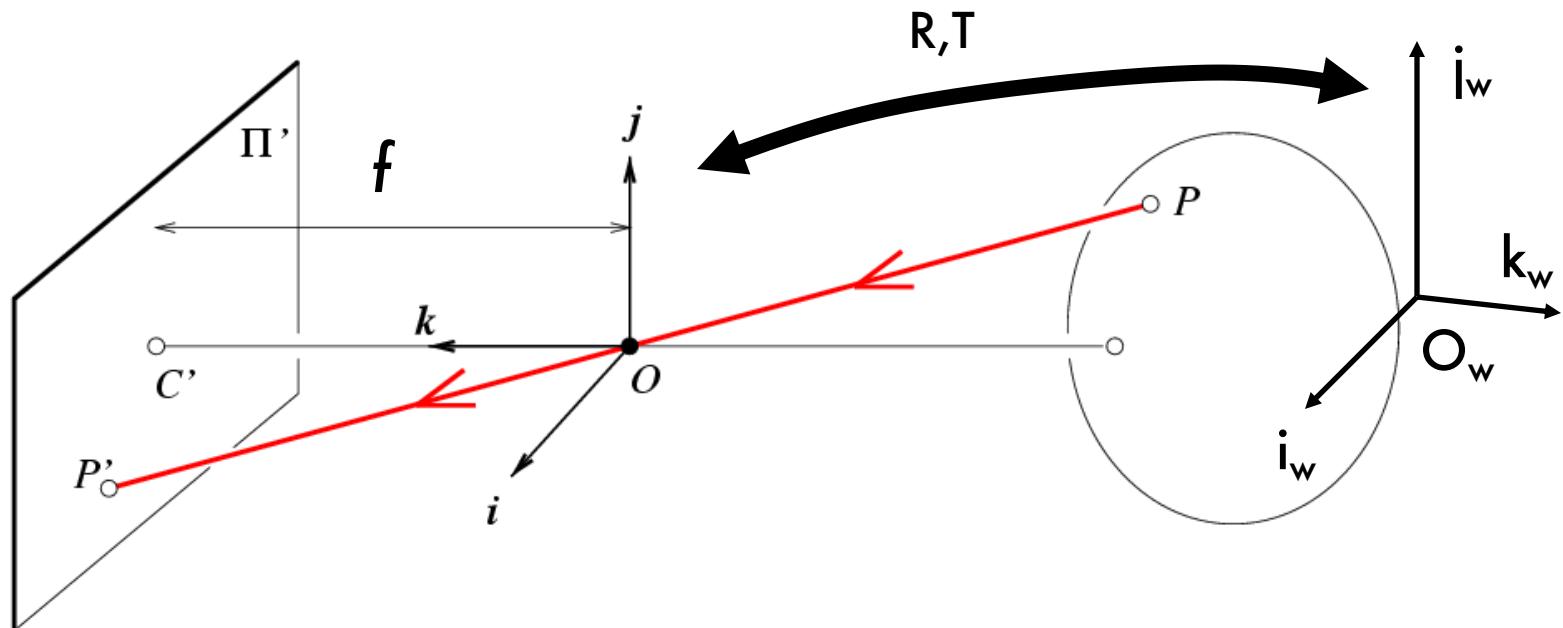
# Lecture 2

# Camera Models

- Pinhole cameras
- Cameras & lenses
- The geometry of pinhole cameras
  - Intrinsic
  - Extrinsic
- Other camera models



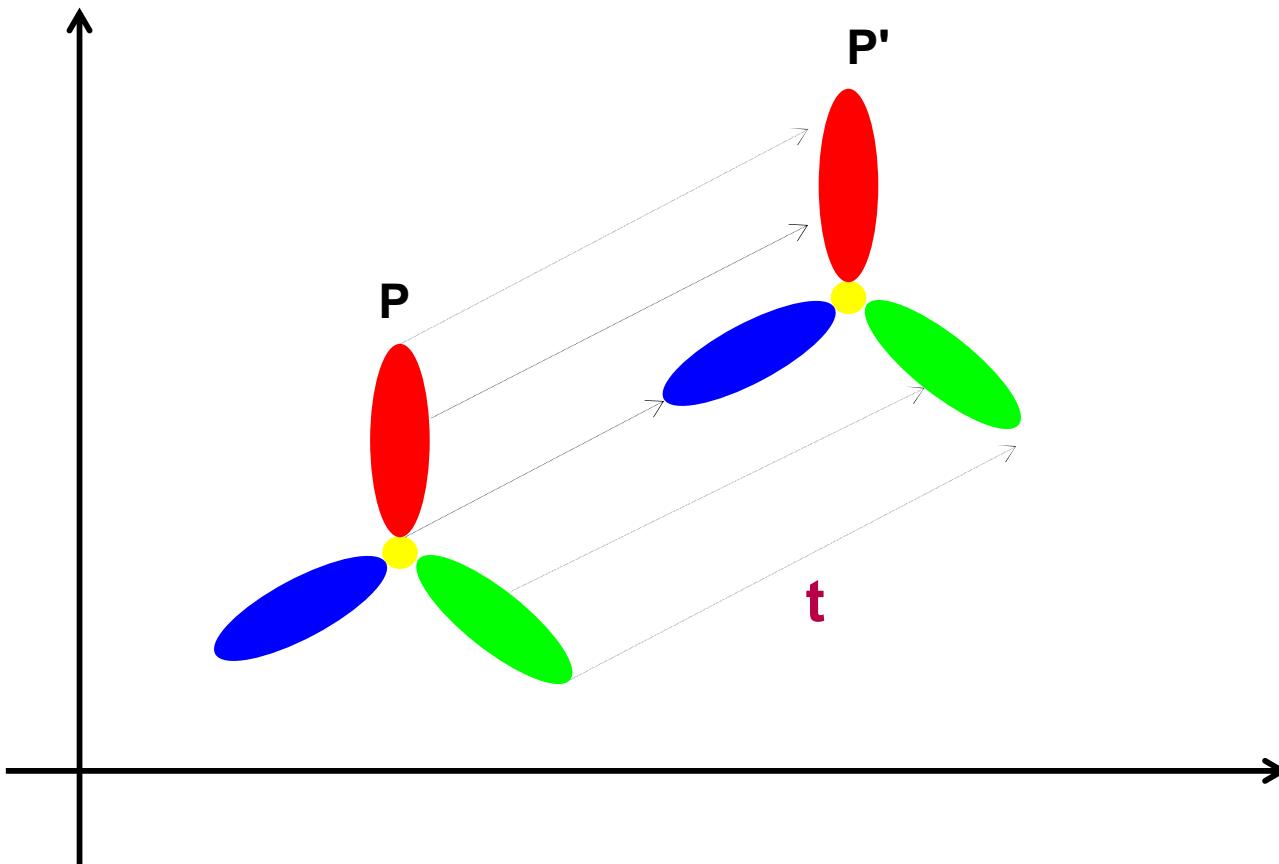
# World reference system



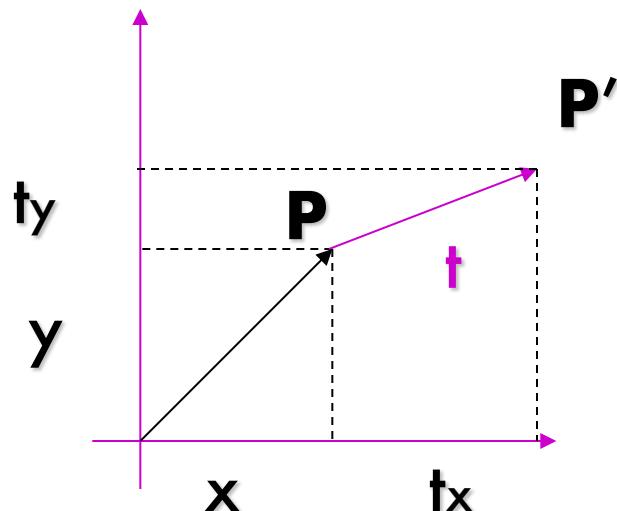
- The mapping so far is defined within the camera reference system
- What if an object is represented in the world reference system?
- Need to introduce an additional mapping from world ref system to camera ref system

Please refer to CA session  
on transformations for  
more details

# 2D Translation



# 2D Translation Equation

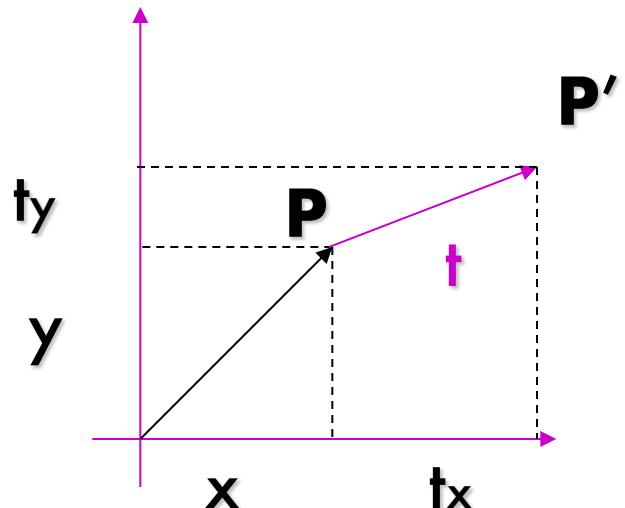


$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' = \mathbf{P} + \mathbf{t} = (x + t_x, y + t_y)$$

# 2D Translation using Homogeneous Coordinates

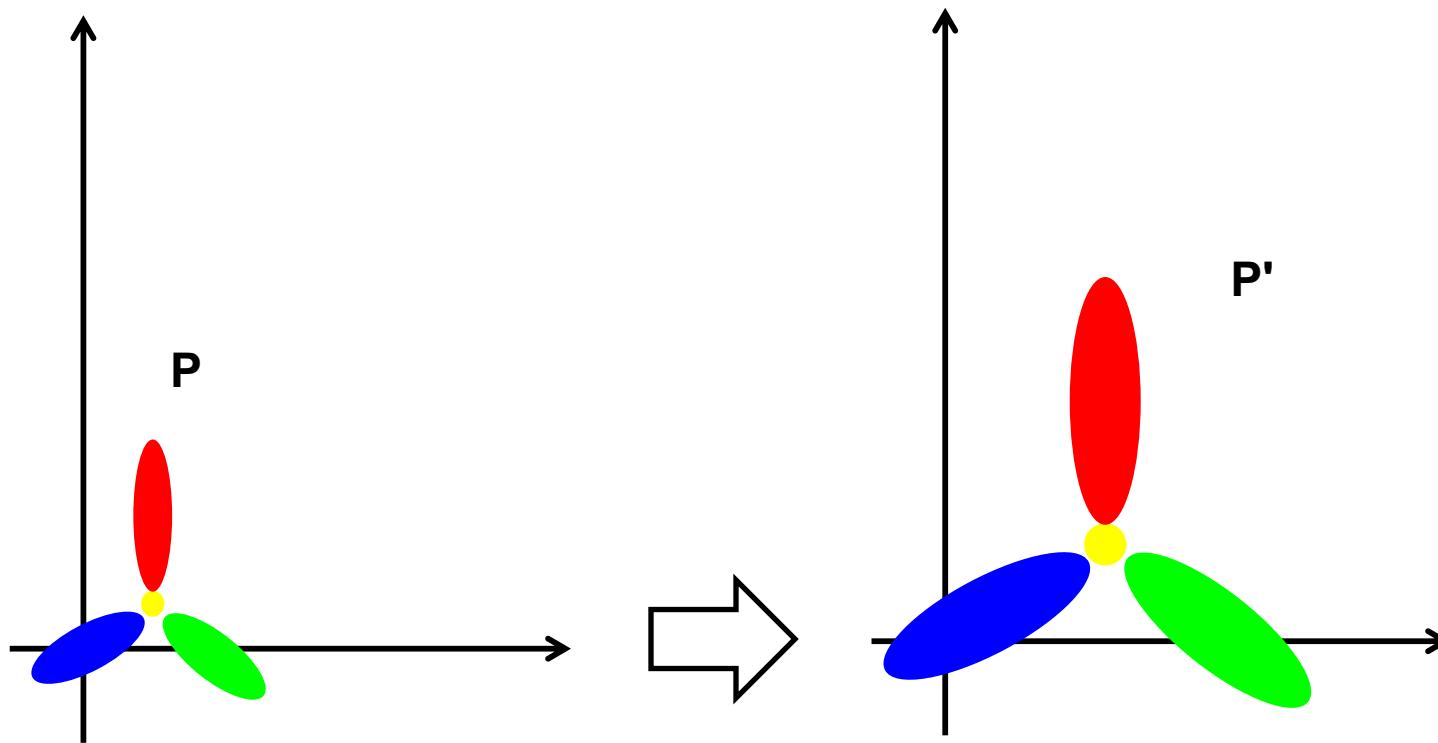


$$P = (x, y) \rightarrow (x, y, 1)$$

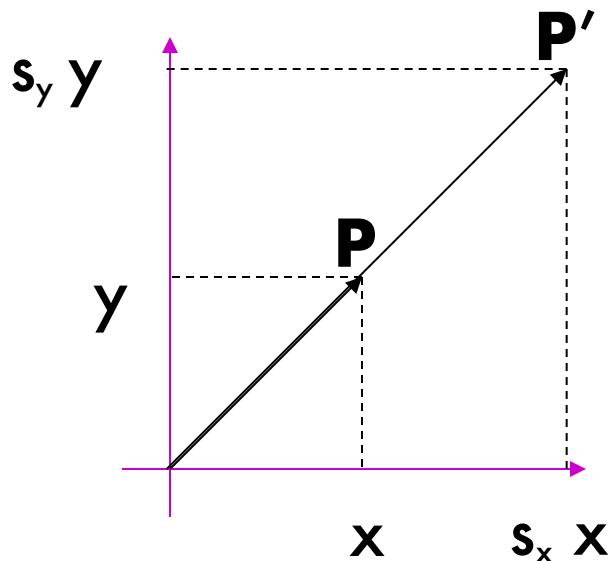
$$P' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} ? \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} I & t \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = T \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# Scaling



# Scaling Equation

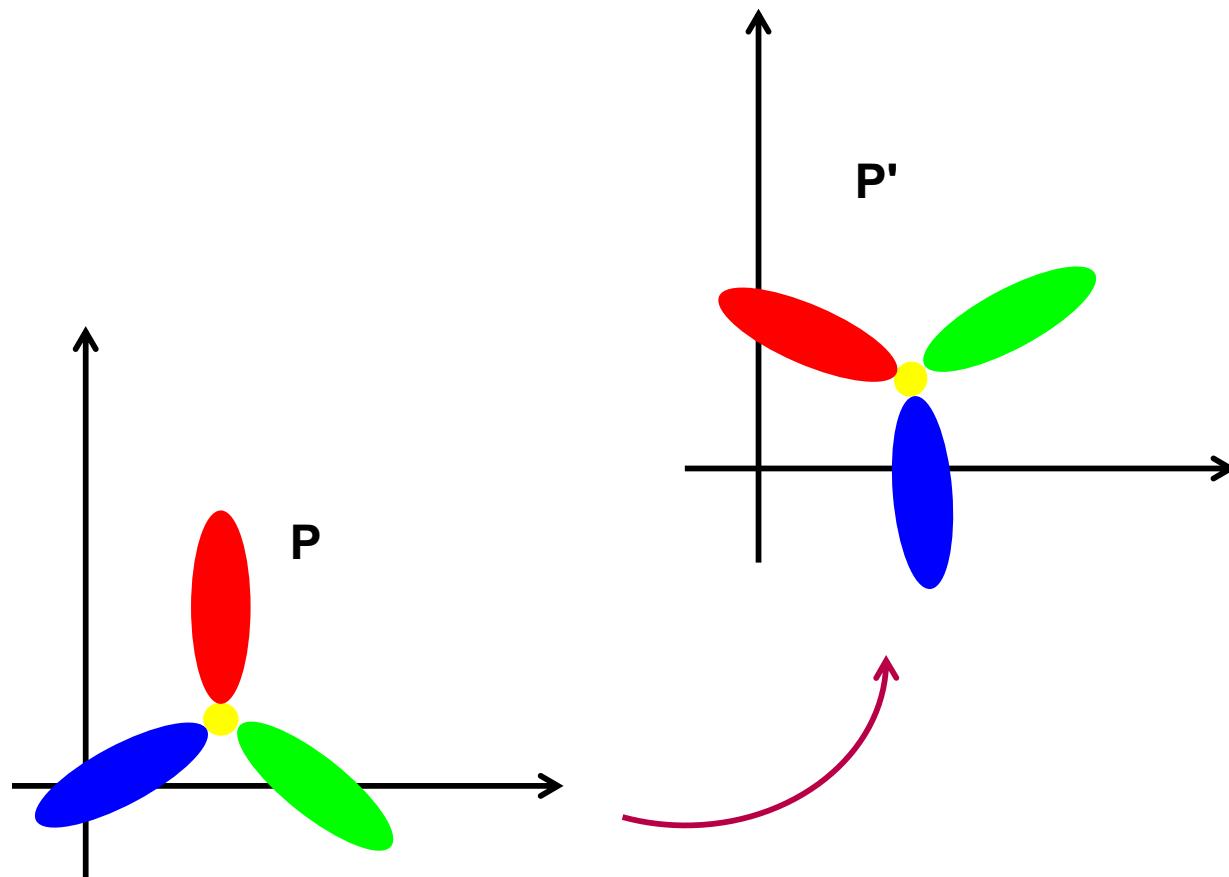


$$P = (x, y) \rightarrow P' = (s_x x, s_y y)$$

$$P = (x, y) \rightarrow (x, y, 1)$$

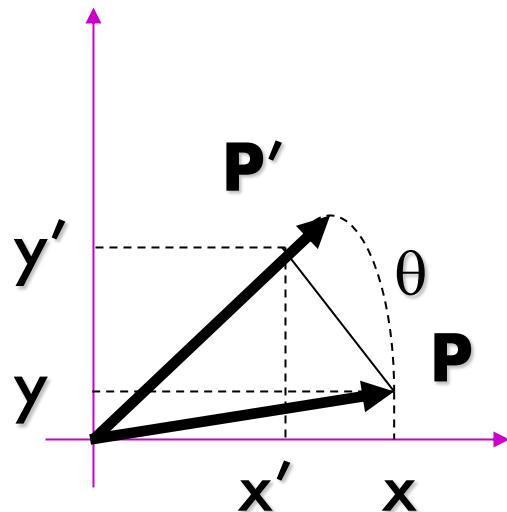
$$P' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s' & 0 \\ \mathbf{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{S} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# Rotation



# Rotation Equations

- Counter-clockwise rotation by an angle  $\theta$



$$x' = \cos \theta x - \sin \theta y$$

$$y' = \cos \theta y + \sin \theta x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{R} \ \mathbf{P}$$

How many degrees of freedom? 1

$$\mathbf{P}' \rightarrow \left[ \begin{array}{c|c} & \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ \hline \mathbf{R} & \end{array} \right]$$

# Scale + Rotation + Translation

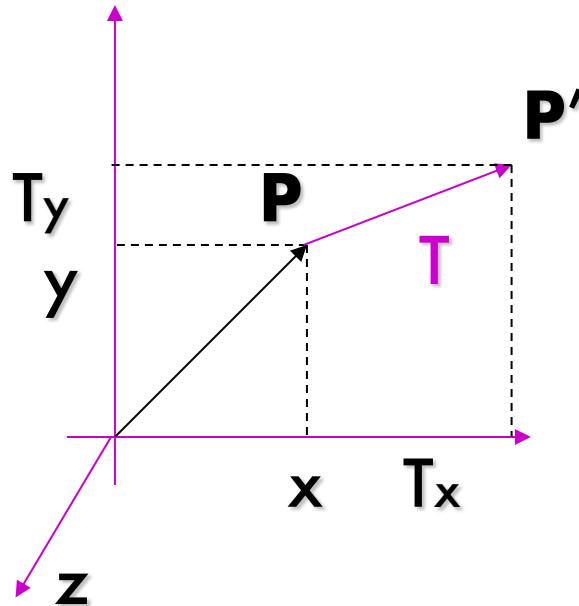
$$\mathbf{P}' \rightarrow \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} \mathbf{R} & \mathbf{S} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

If  $s_x = s_y$ , this is a similarity transformation

# 3D Translation of Points



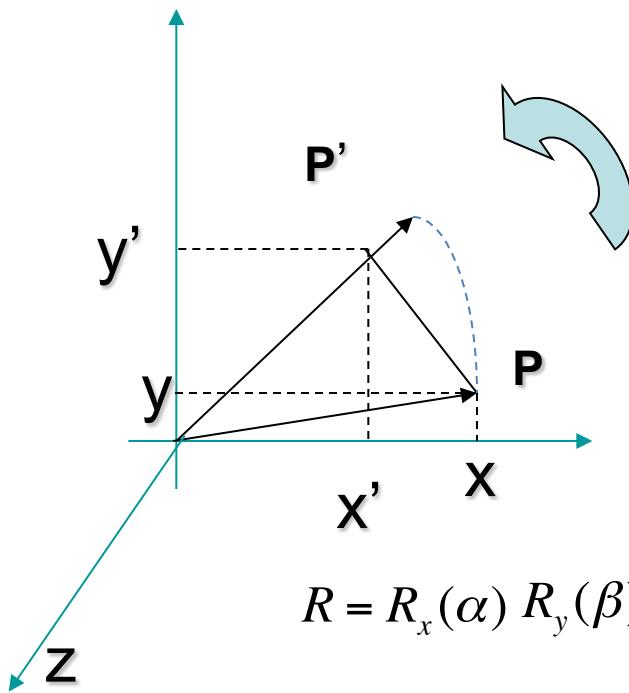
$$T = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

$$P' \rightarrow \begin{bmatrix} I & T \\ 0 & 1 \end{bmatrix}_{4 \times 4} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

A translation vector in 3D has 3 degrees of freedom

# 3D Rotation of Points

Rotation around the coordinate axes, counter-clockwise:



$$R = R_x(\alpha) R_y(\beta) R_z(\gamma)$$

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' \rightarrow \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}_{4 \times 4} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

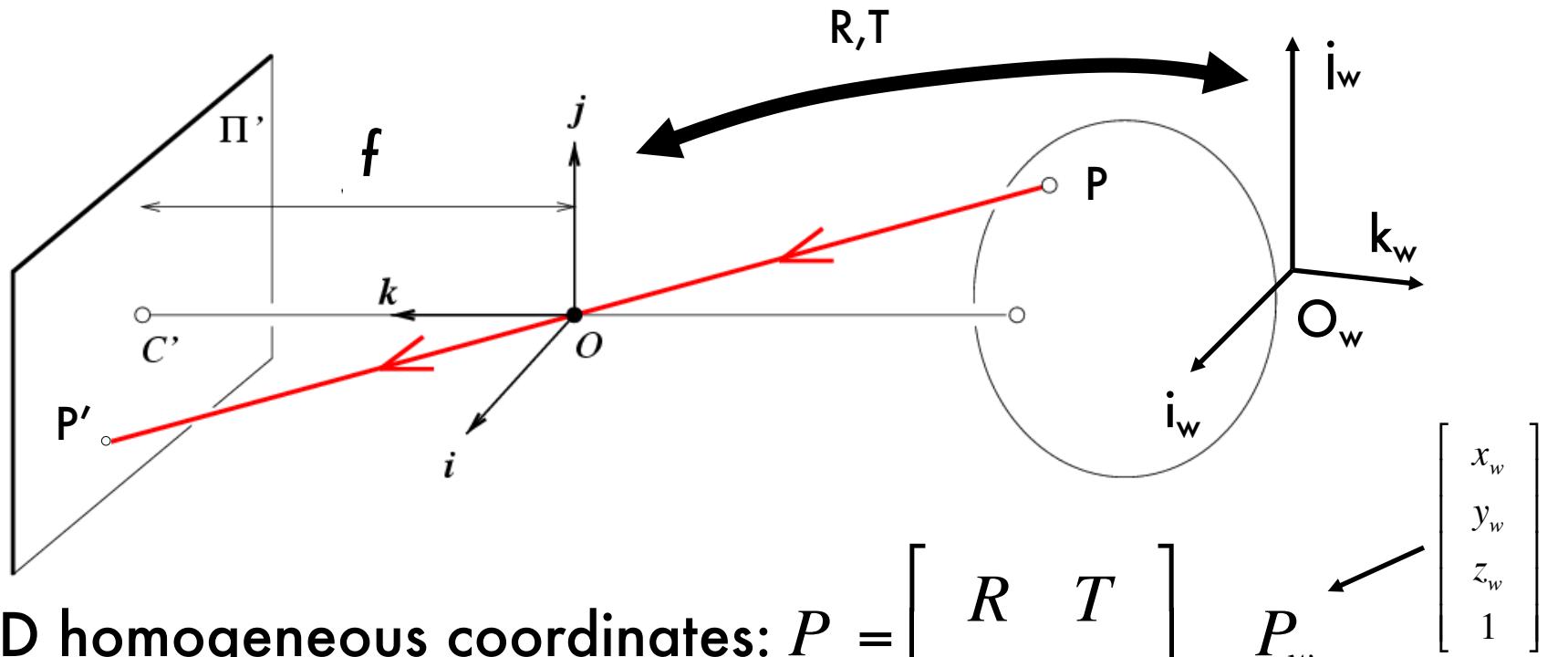
A rotation matrix in 3D has 3 degrees of freedom

# 3D Translation and Rotation

$$R = R_x(\alpha) \ R_y(\beta) \ R_z(\gamma) \quad T = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

$$P' \rightarrow \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4 \times 4} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

# World reference system



In 4D homogeneous coordinates:  $P = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4 \times 4} P_w$

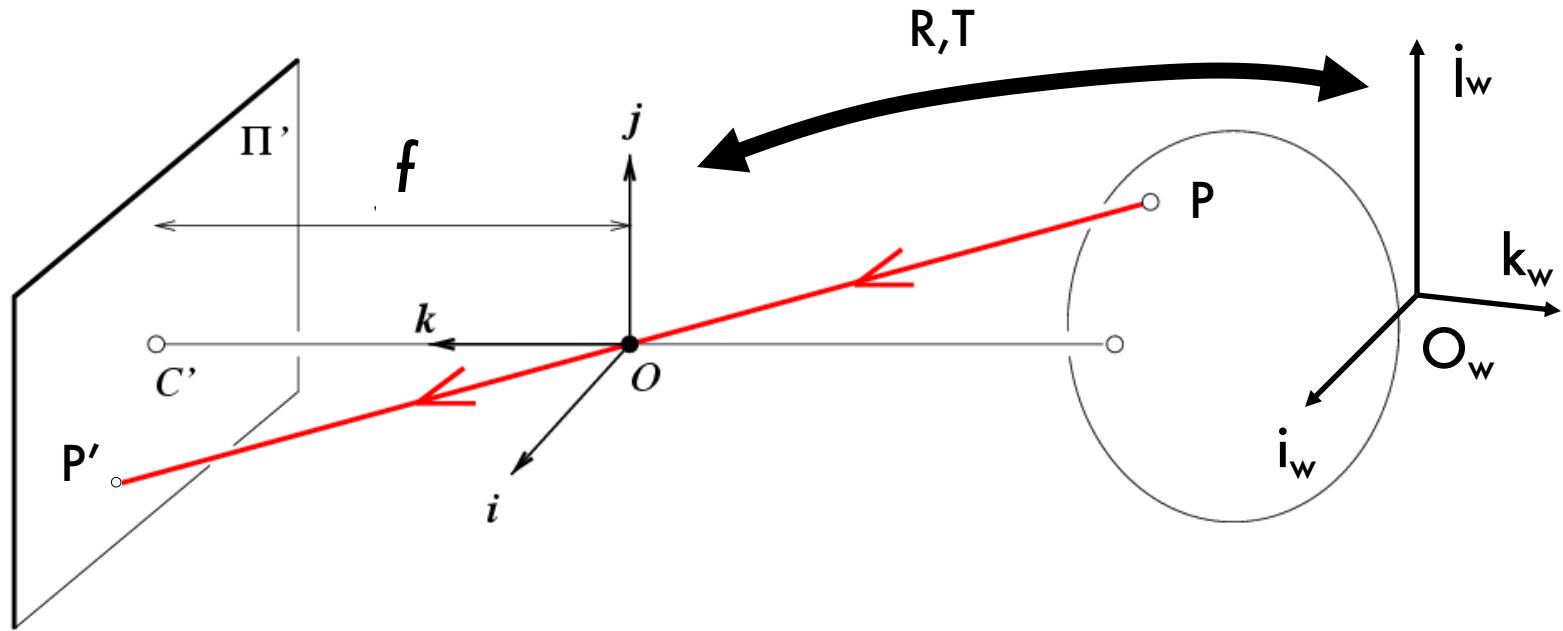
[Eq.9]

$P' = K \begin{bmatrix} I & 0 \end{bmatrix} P = K \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4 \times 4} P_w = K \begin{bmatrix} R & T \end{bmatrix} P_w$

Internal parameters      External parameters

[Eq.11]

# The projective transformation

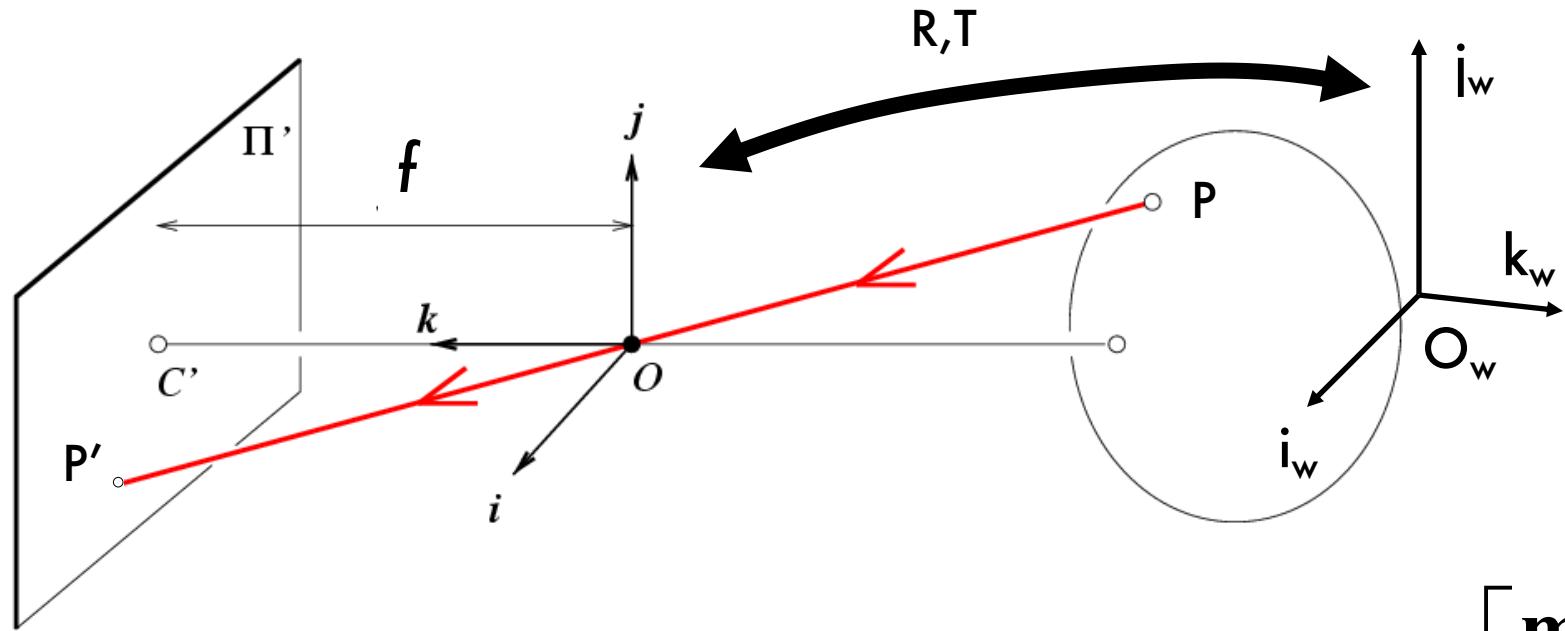


$$P'_{3 \times 1} = M_{3 \times 4} P_w = K_{3 \times 3} \begin{bmatrix} R & T \end{bmatrix}_{3 \times 4} P_w^{4 \times 1} \quad [\text{Eq.11}]$$

How many degrees of freedom does  $M$  have?

$$5 + 3 + 3 = 11!$$

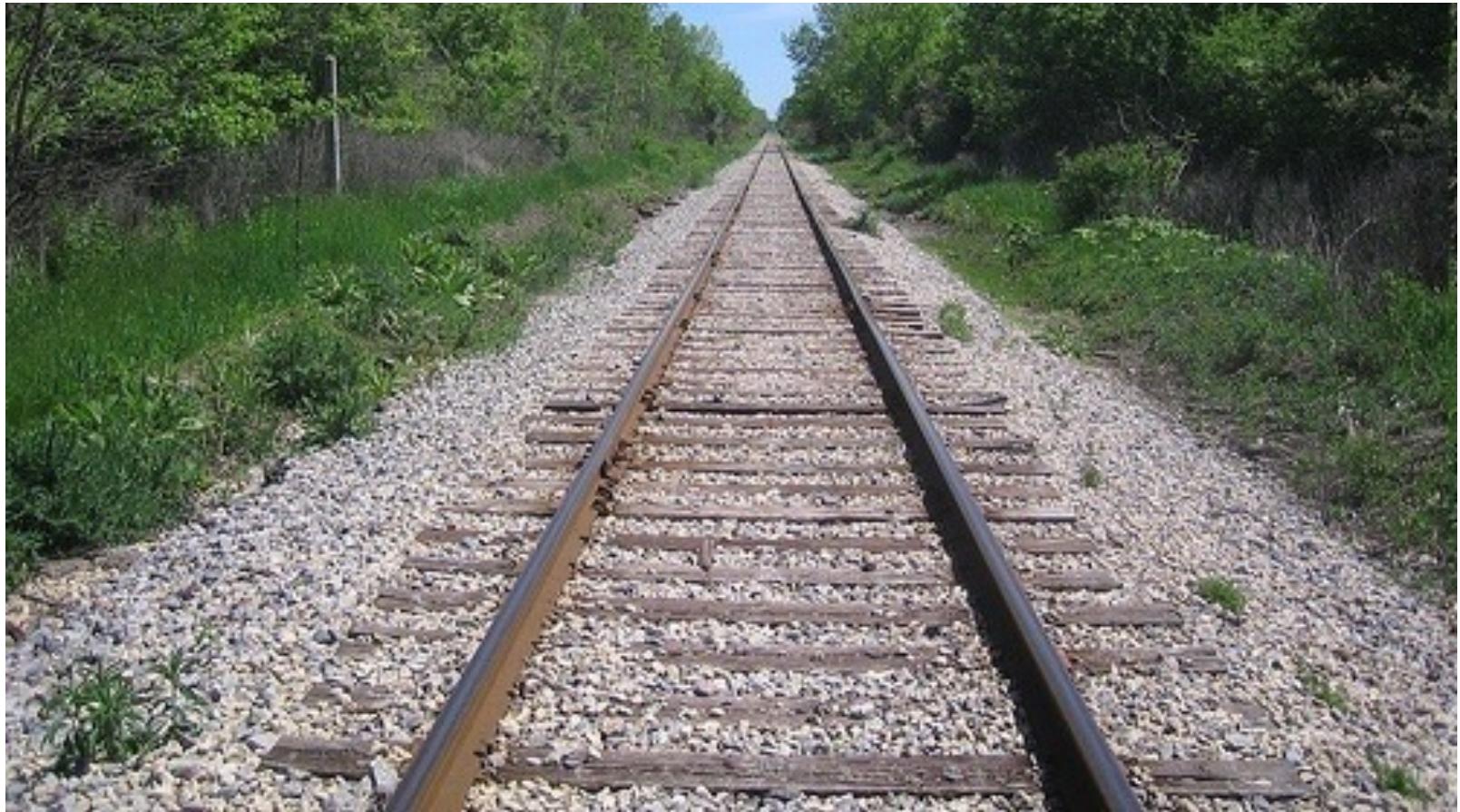
# The projective transformation



$$\begin{aligned}
 P'_{3 \times 1} &= M P_w = K_{3 \times 3} \begin{bmatrix} R & T \end{bmatrix}_{3 \times 4} P_w_{4 \times 1} & M = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} \\
 &= \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} P_w = \begin{bmatrix} \mathbf{m}_1 P_w \\ \mathbf{m}_2 P_w \\ \mathbf{m}_3 P_w \end{bmatrix} & \xrightarrow{E} \left( \frac{\mathbf{m}_1 P_w}{\mathbf{m}_3 P_w}, \frac{\mathbf{m}_2 P_w}{\mathbf{m}_3 P_w} \right) \quad [\text{Eq.12}]
 \end{aligned}$$

# Properties of projective transformations

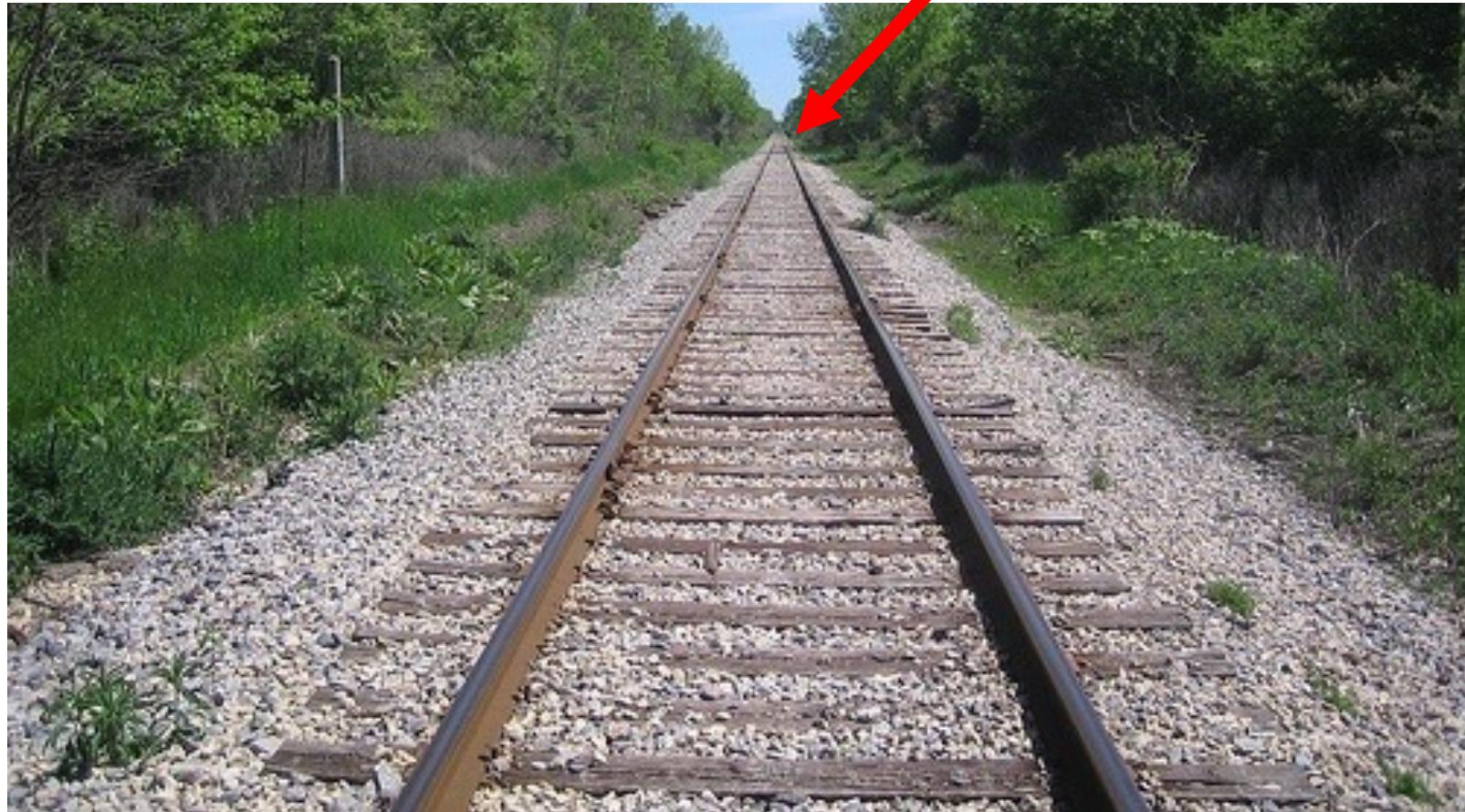
- Points project to points
- Lines project to lines
- Distant objects look smaller



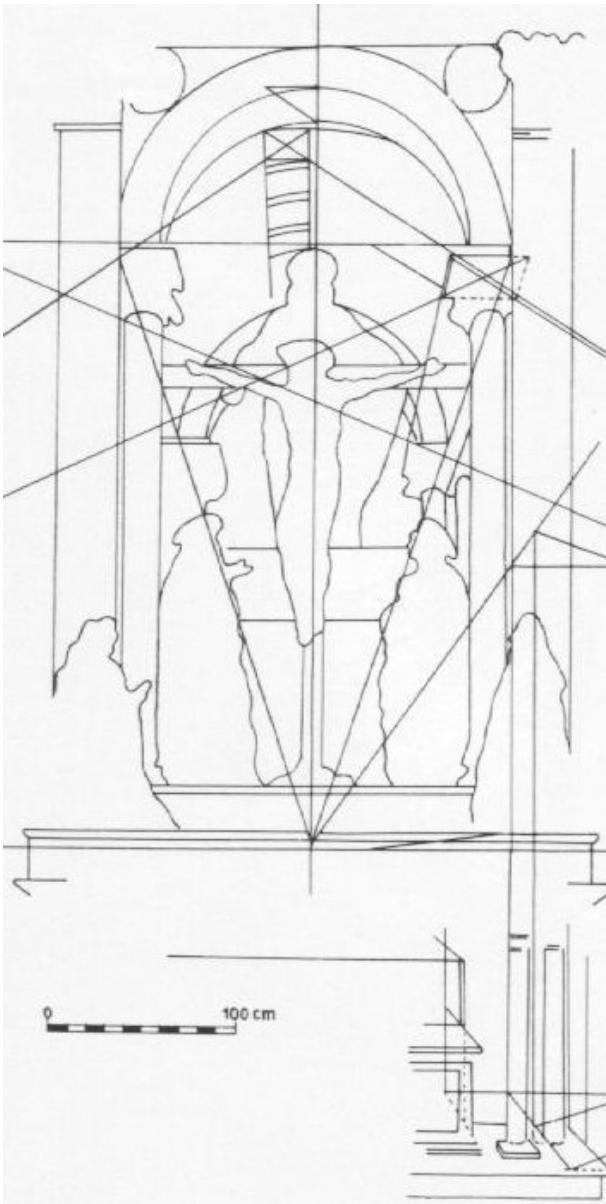
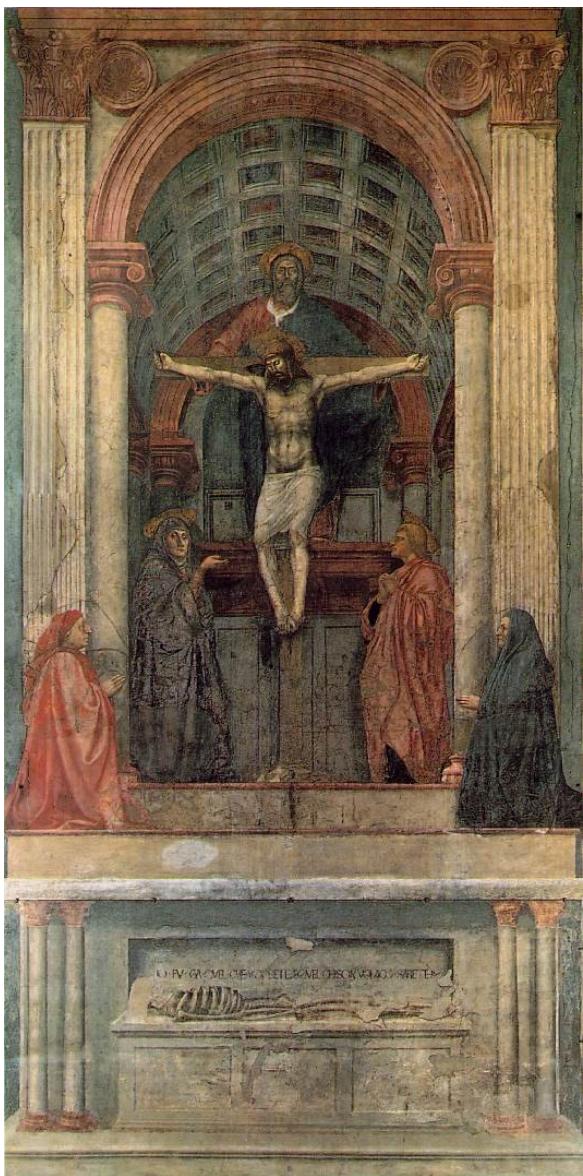
# Properties of Projection

- Angles are not preserved
- Parallel lines meet!

Parallel lines in the world intersect in the image at a “vanishing point”



# One-point perspective



- **Masaccio, Trinity, Santa Maria Novella, Florence, 1425-28**

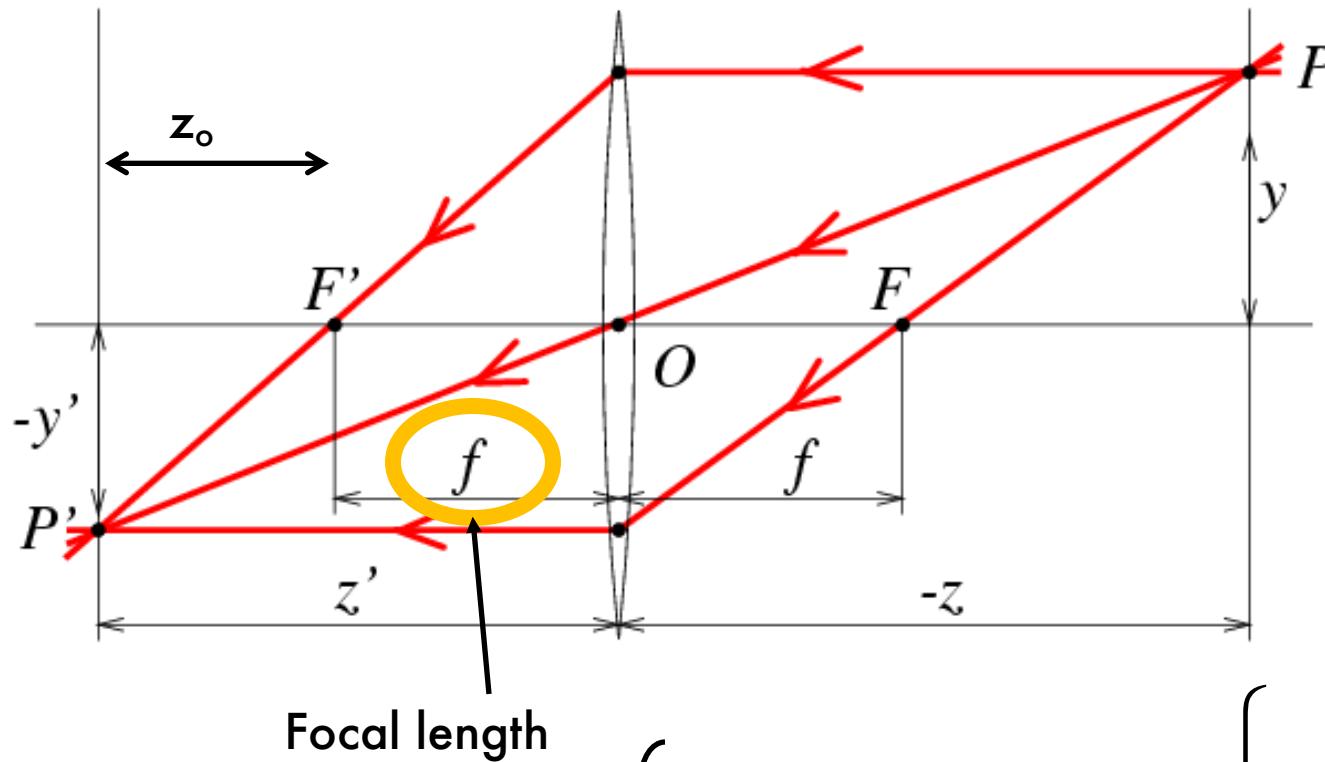
# Next lecture

- How to calibrate a camera?

# **Supplemental material**

# Thin Lenses

[FP] sec 1.1, page 8.



$$z' = f + z_0$$

$$f = \frac{R}{2(n-1)}$$

**Snell's law:**

$$n_1 \sin \alpha_1 = n_2 \sin \alpha_2$$



$$\left\{ \begin{array}{l} \text{Small angles:} \\ n_1 \alpha_1 \approx n_2 \alpha_2 \\ n_1 = n \text{ (lens)} \\ n_1 = 1 \text{ (air)} \end{array} \right.$$



$$\left\{ \begin{array}{l} x' = z' \frac{x}{z} \\ y' = z' \frac{y}{z} \end{array} \right.$$