

Lecture 5

Epipolar Geometry



1891

Professor Silvio Savarese

Computational Vision and Geometry Lab

Lecture 5

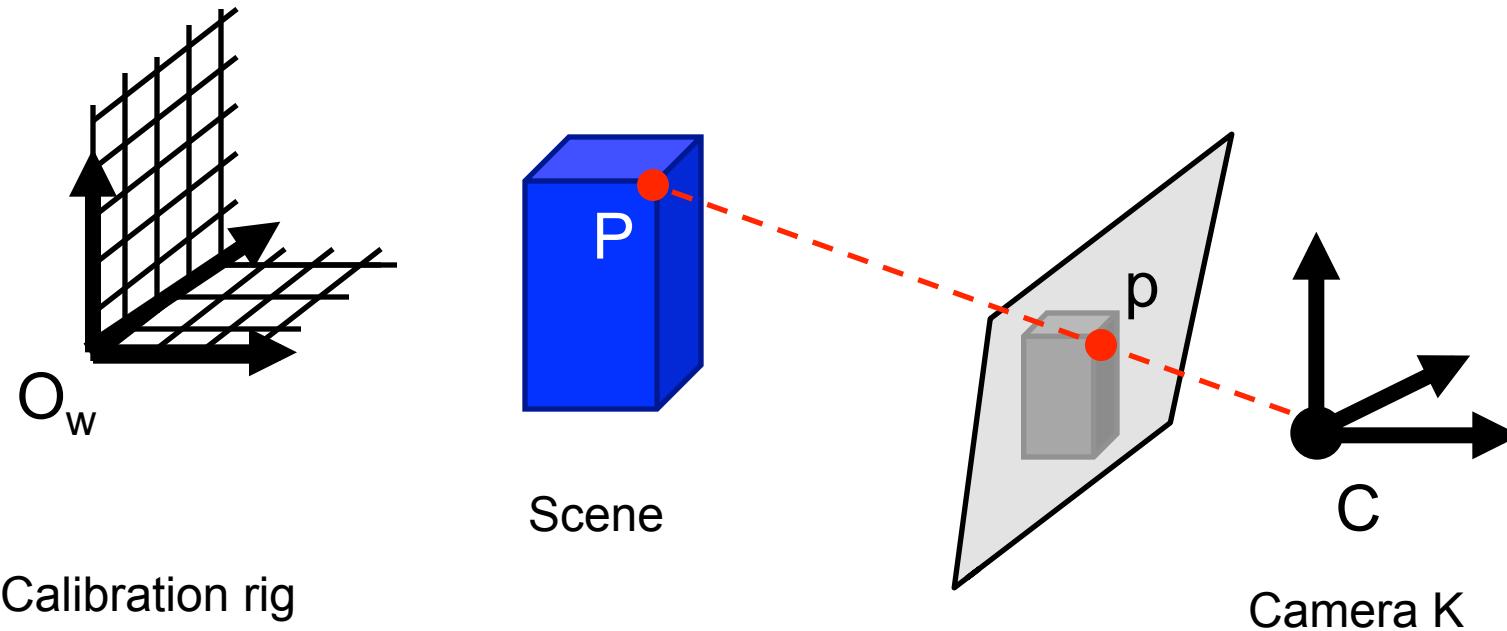
Epipolar Geometry



- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating F
- Examples

Reading: [AZ] Chapter: 4 “Estimation – 2D perspective transformations
Chapter: 9 “Epipolar Geometry and the Fundamental Matrix Transformation”
Chapter: 11 “Computation of the Fundamental Matrix F ”
[FP] Chapter: 7 “Stereopsis”
Chapter: 8 “Structure from Motion”

Recovering structure from a single view



From calibration rig

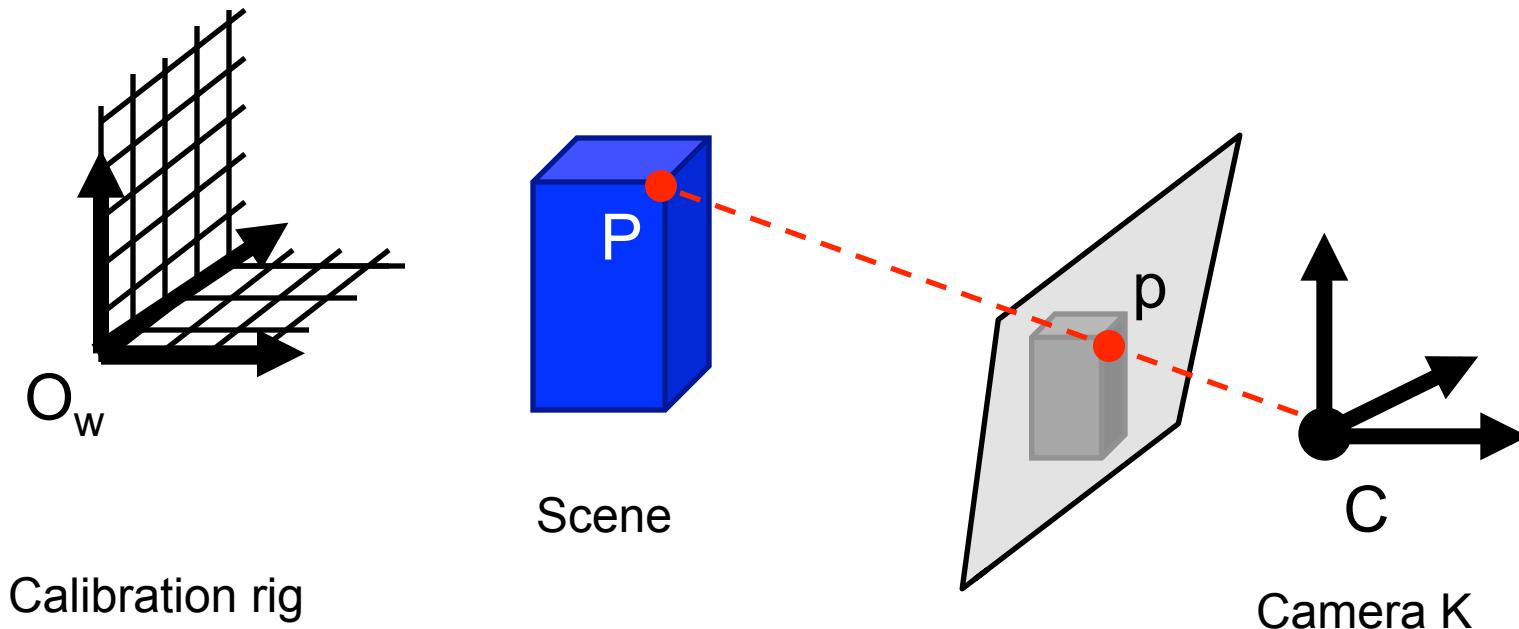
→ location/pose of the rig, K

From points and lines at infinity
+ orthogonal lines and planes

→ structure of the scene, K

Knowledge about scene (point correspondences, geometry of lines & planes, etc...)

Recovering structure from a single view



Why is it so difficult?

Intrinsic ambiguity of the mapping from 3D to image (2D)

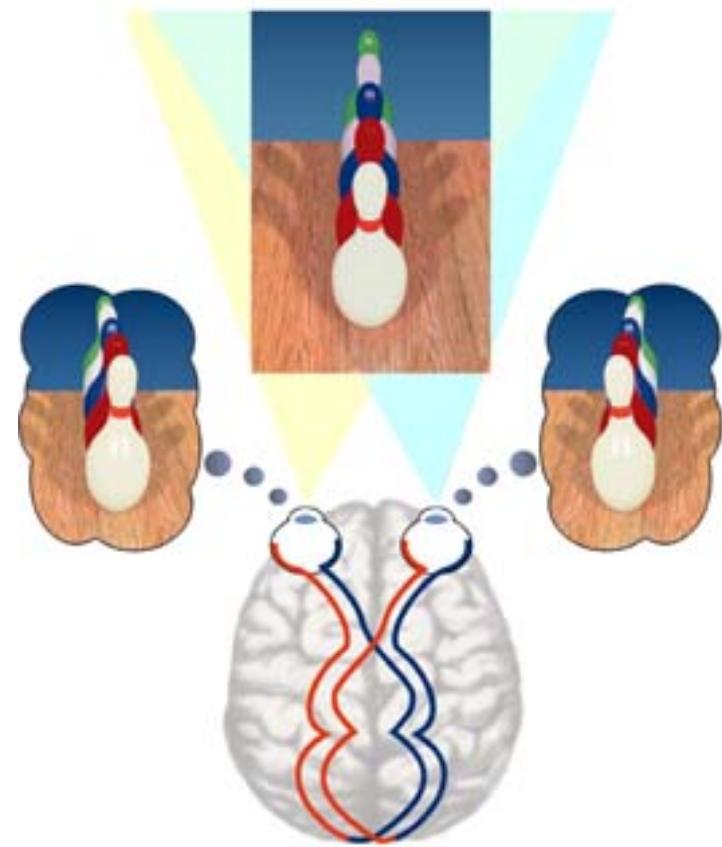
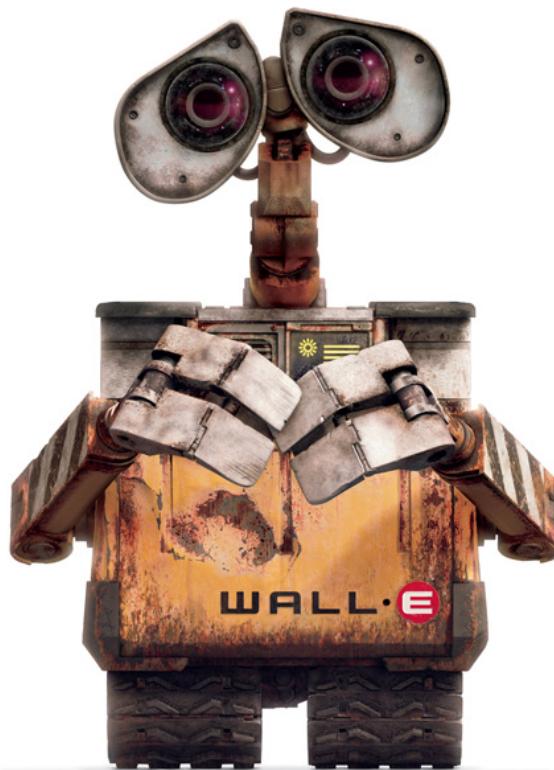
Recovering structure from a single view

Intrinsic ambiguity of the mapping from 3D to image (2D)



Courtesy slide S. Lazebnik

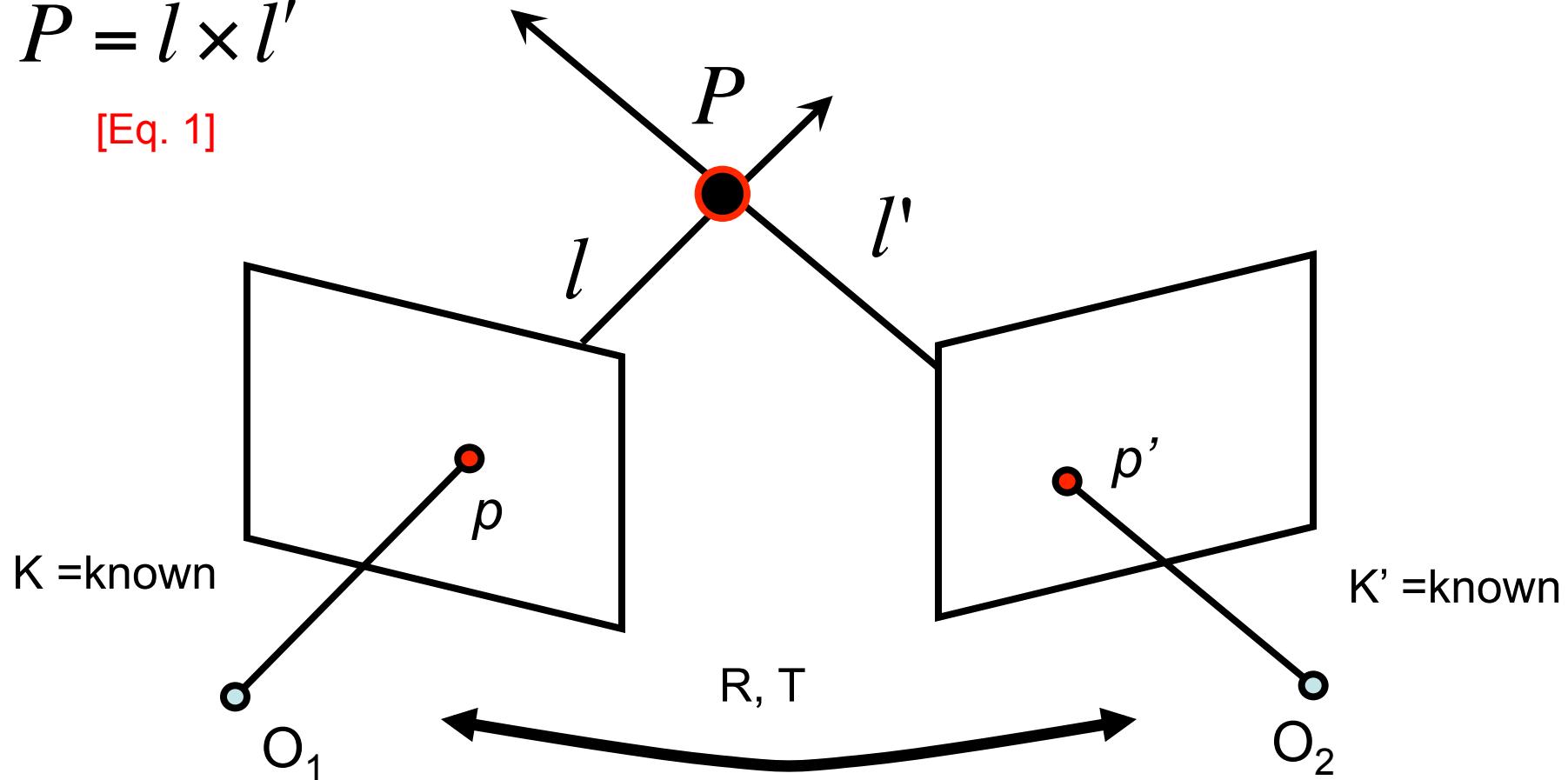
Two eyes help!



Two eyes help!

$$P = l \times l'$$

[Eq. 1]

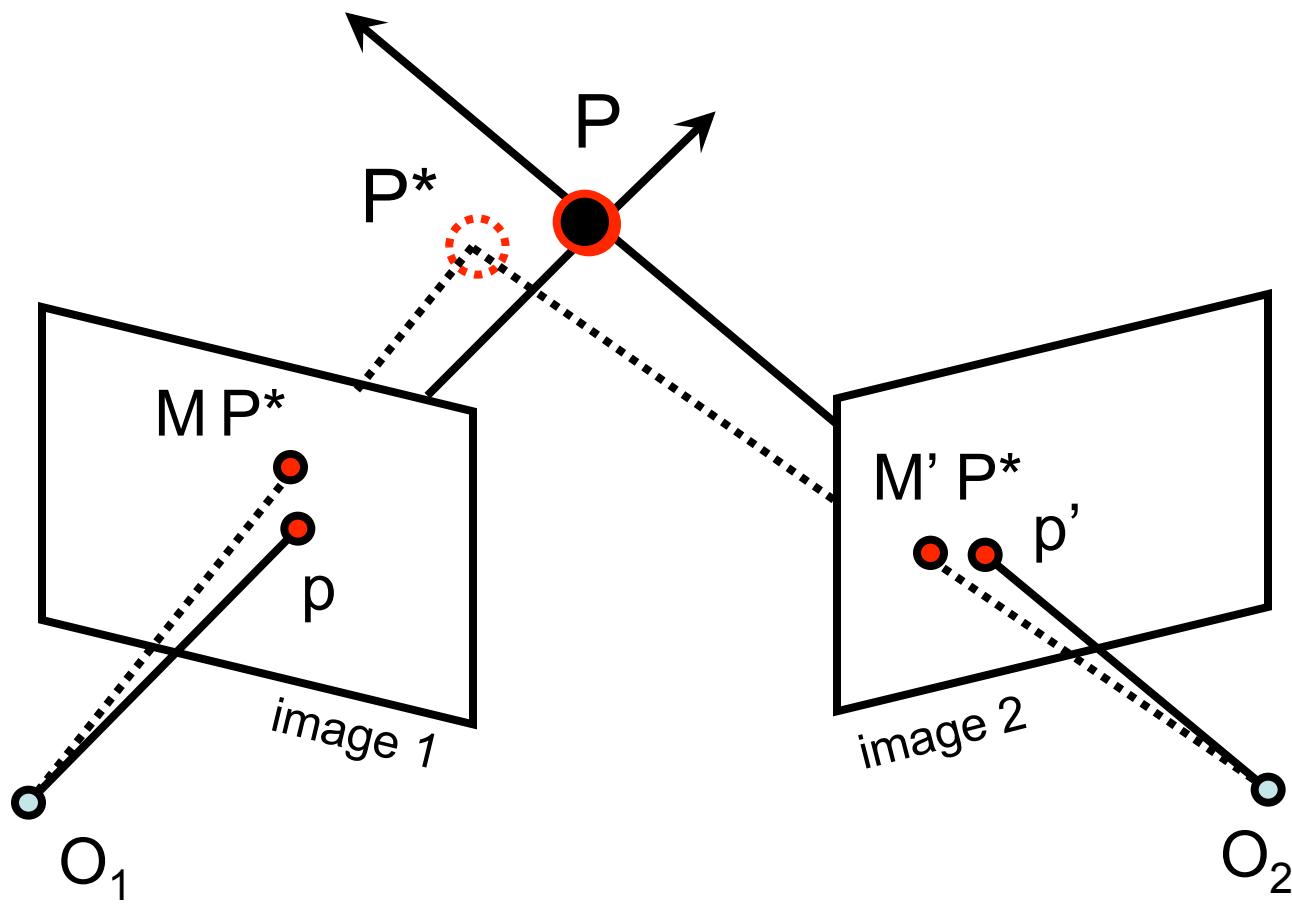


This is called **triangulation**

Triangulation

- Find P' that minimizes

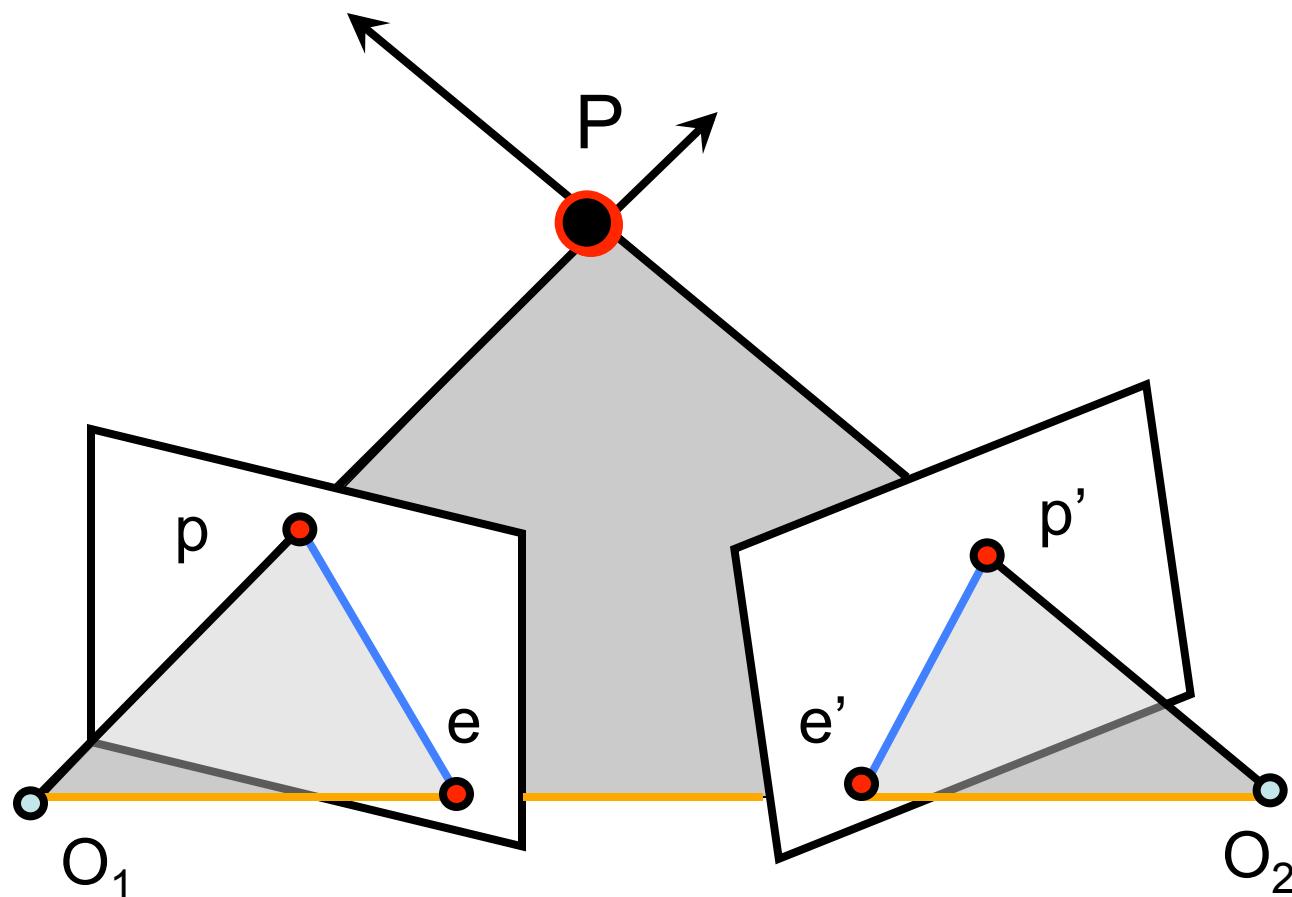
$$d(p, M P^*) + d(p', M' P^*) \quad [\text{Eq. 2}]$$



Stereo-view geometry

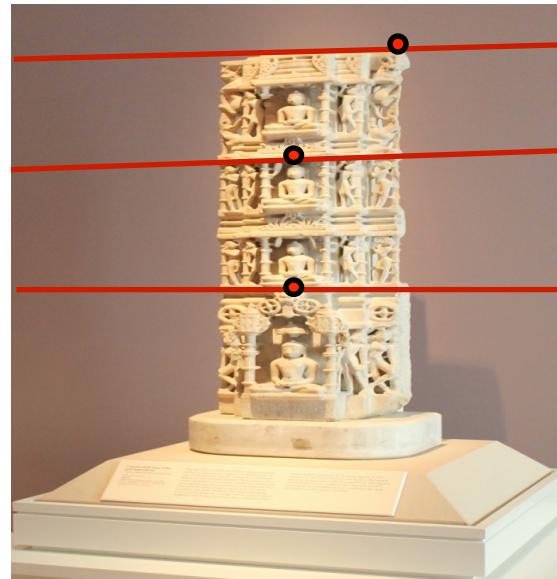
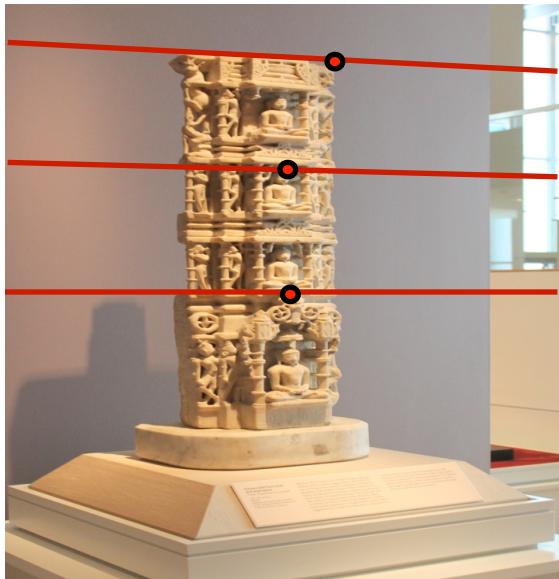
- **Correspondence:** Given a point p in one image, how can I find the corresponding point p' in another one?
- **Camera geometry:** Given corresponding points in two images, find camera matrices, position and pose.
- **Scene geometry:** Find coordinates of 3D point from its projection into 2 or multiple images.

Epipolar geometry

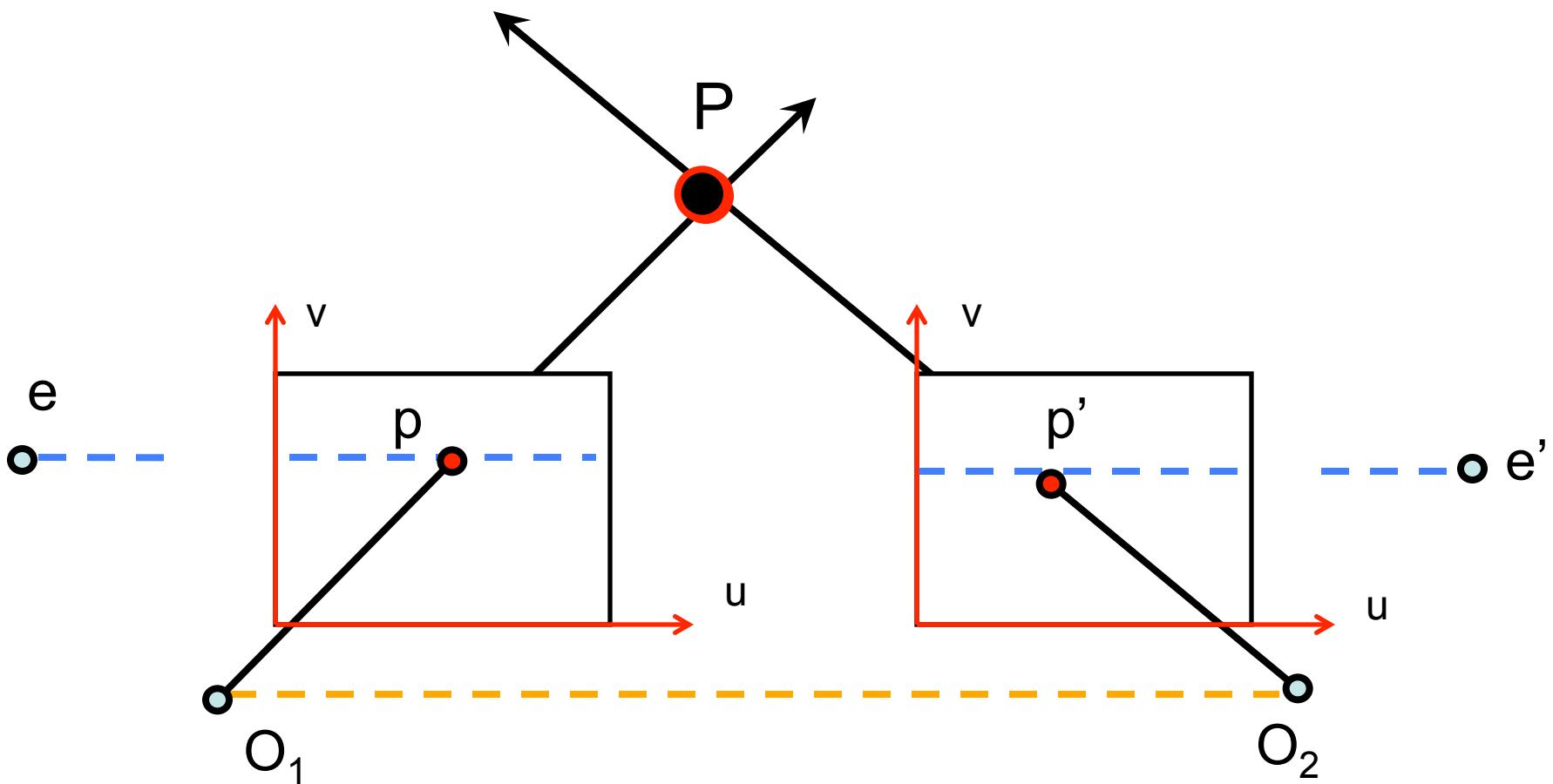


- Epipolar Plane
- Baseline
- Epipolar Lines
- Epipoles e, e'
 - = intersections of baseline with image planes
 - = projections of the other camera center

Example of epipolar lines



Example: Parallel image planes

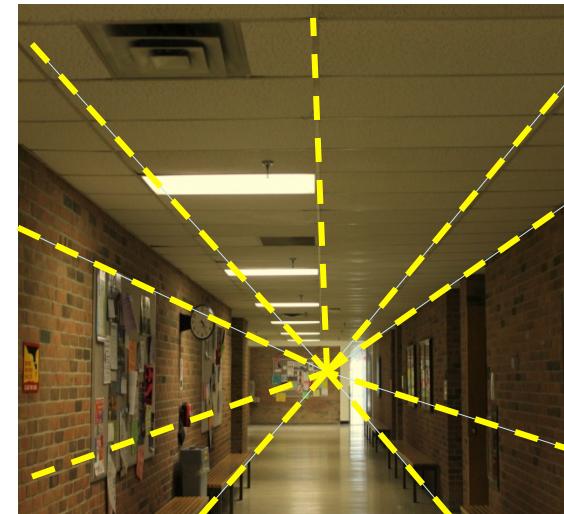
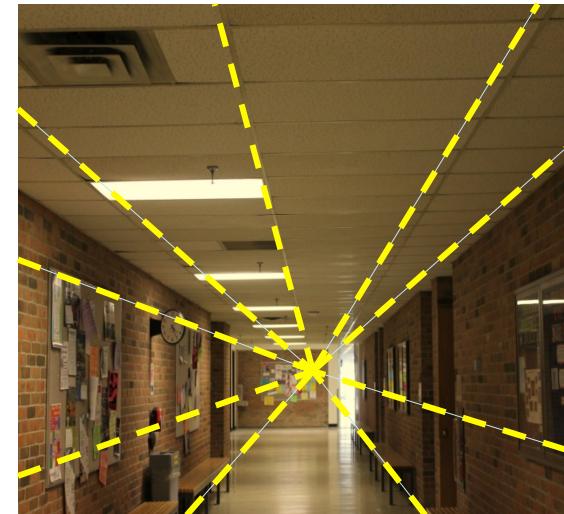
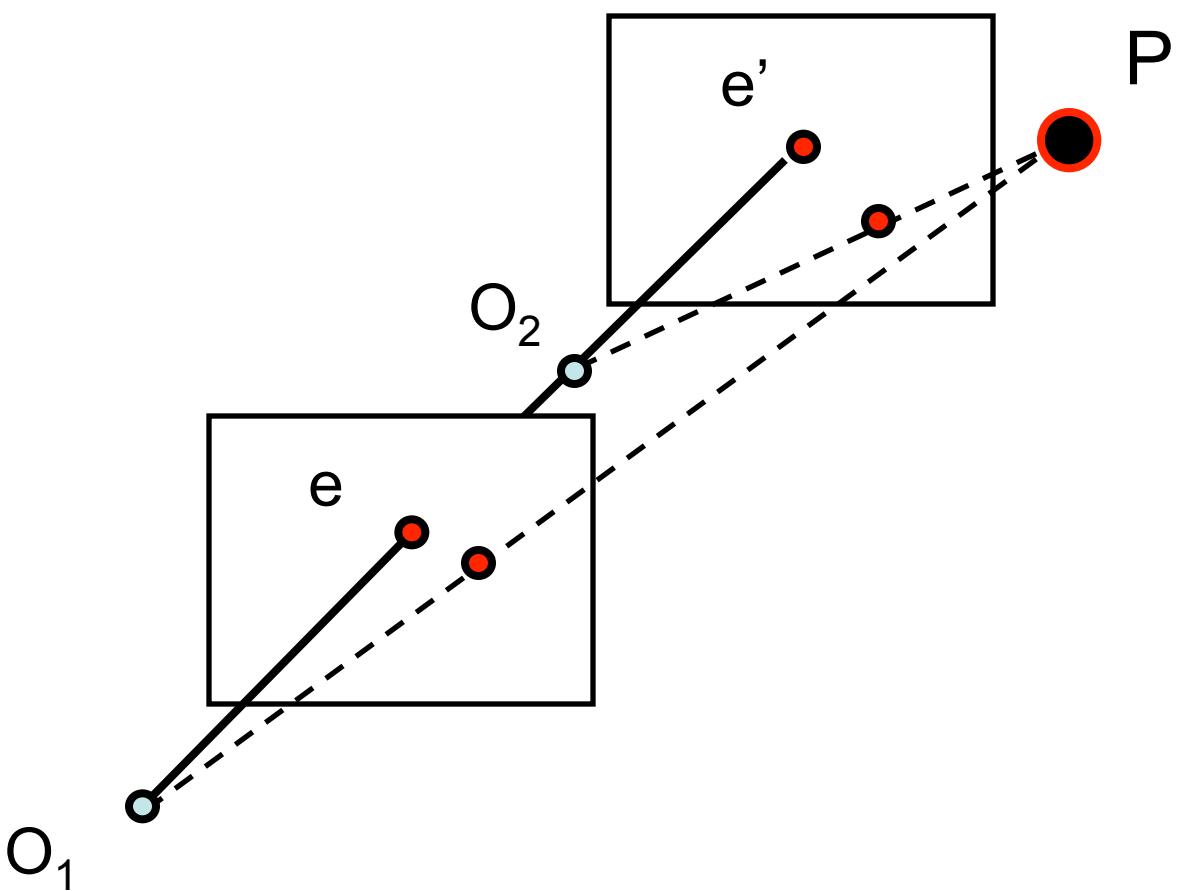


- Baseline intersects the image plane at infinity
- Epipoles are at infinity
- Epipolar lines are parallel to u axis

Example: Parallel Image Planes

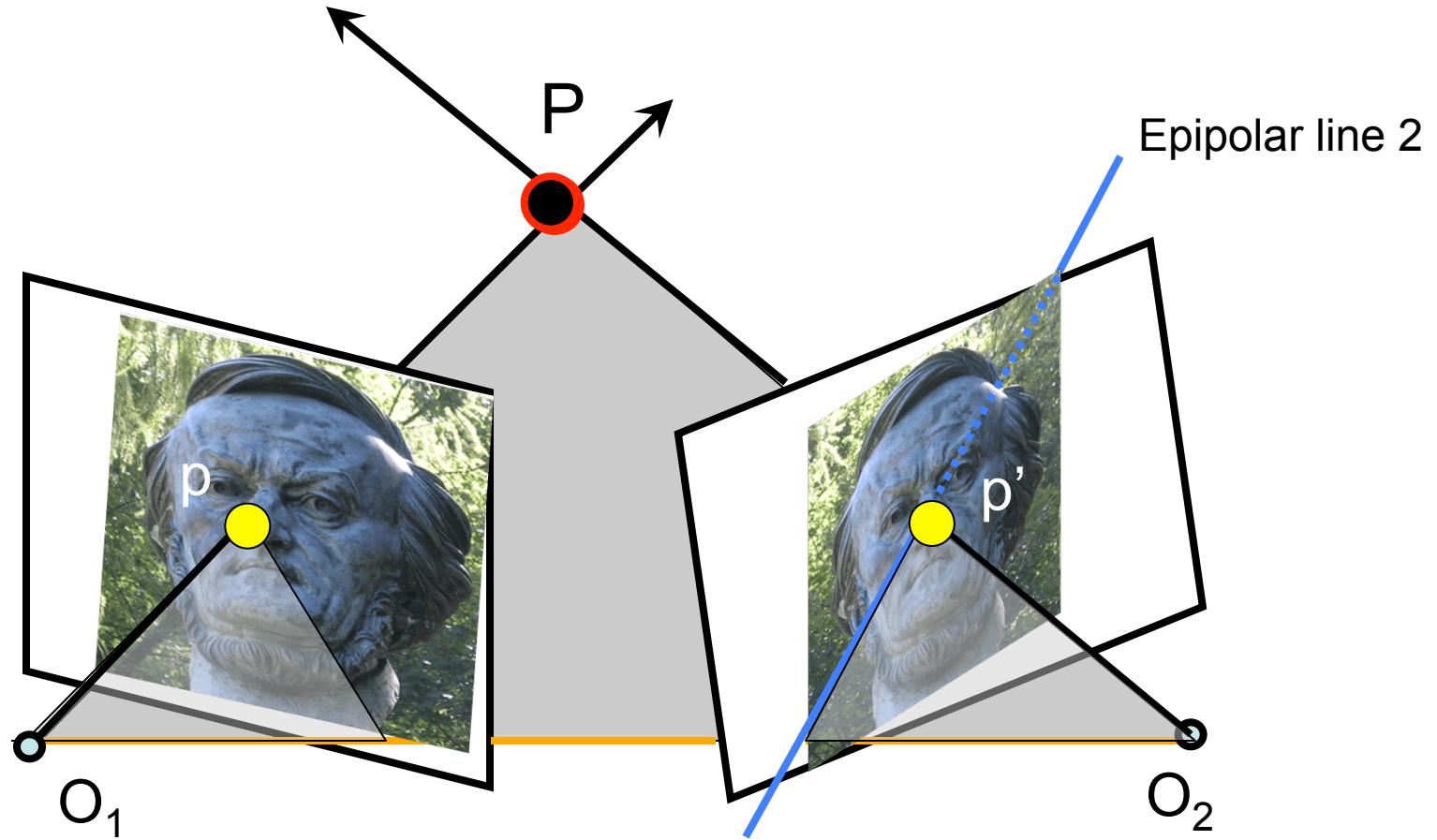


Example: Forward translation

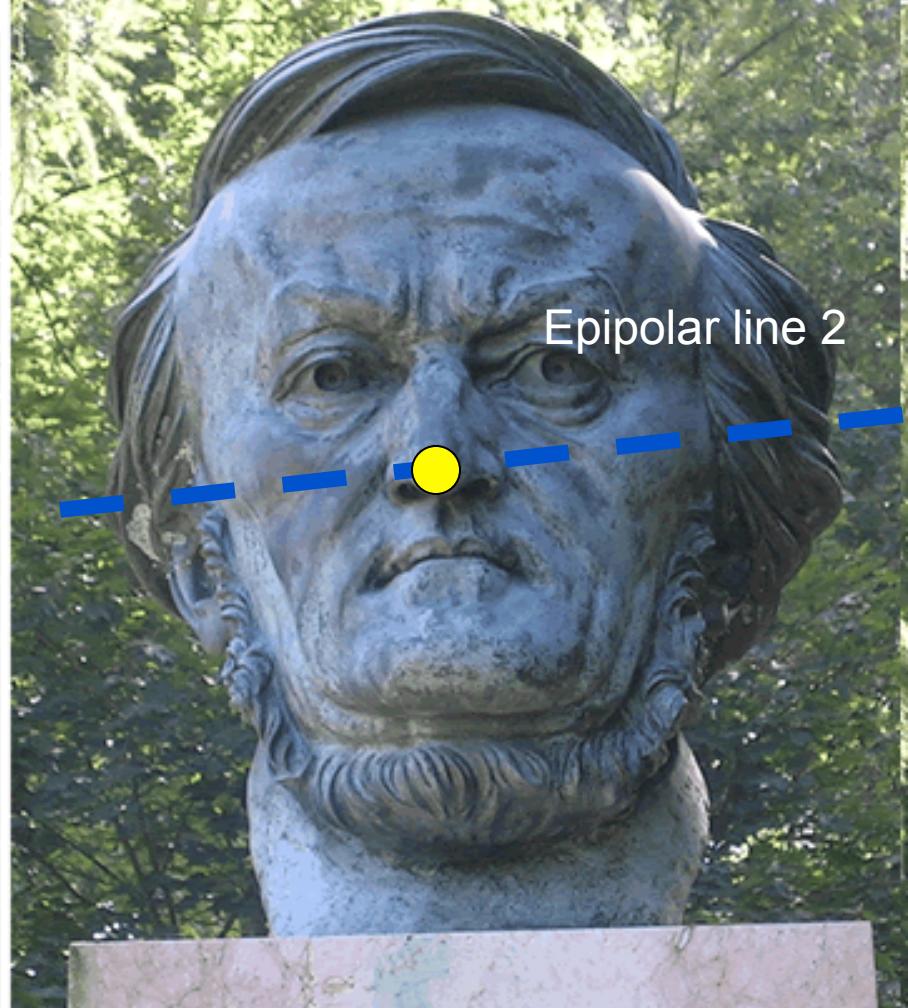


- The epipoles have same position in both images
- Epipole called FOE (focus of expansion)

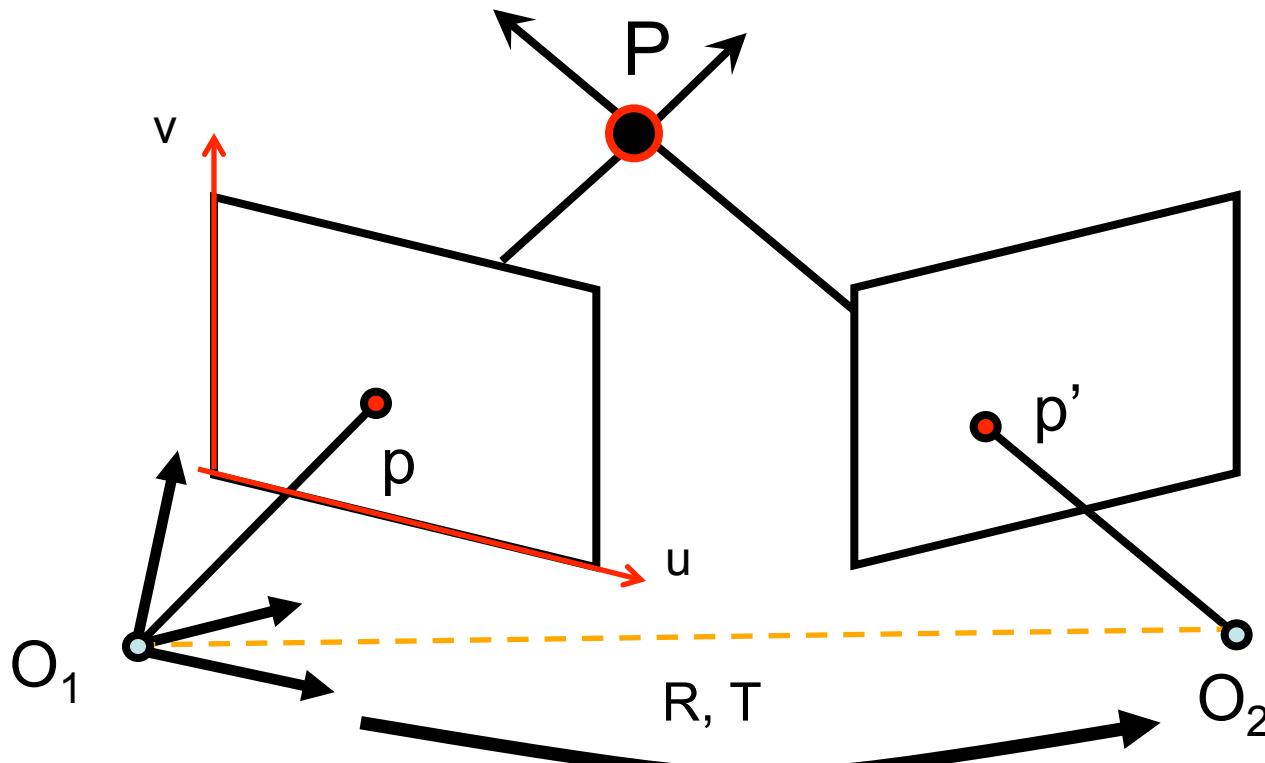
Epipolar geometry



Epipolar Constraint



Epipolar Constraint



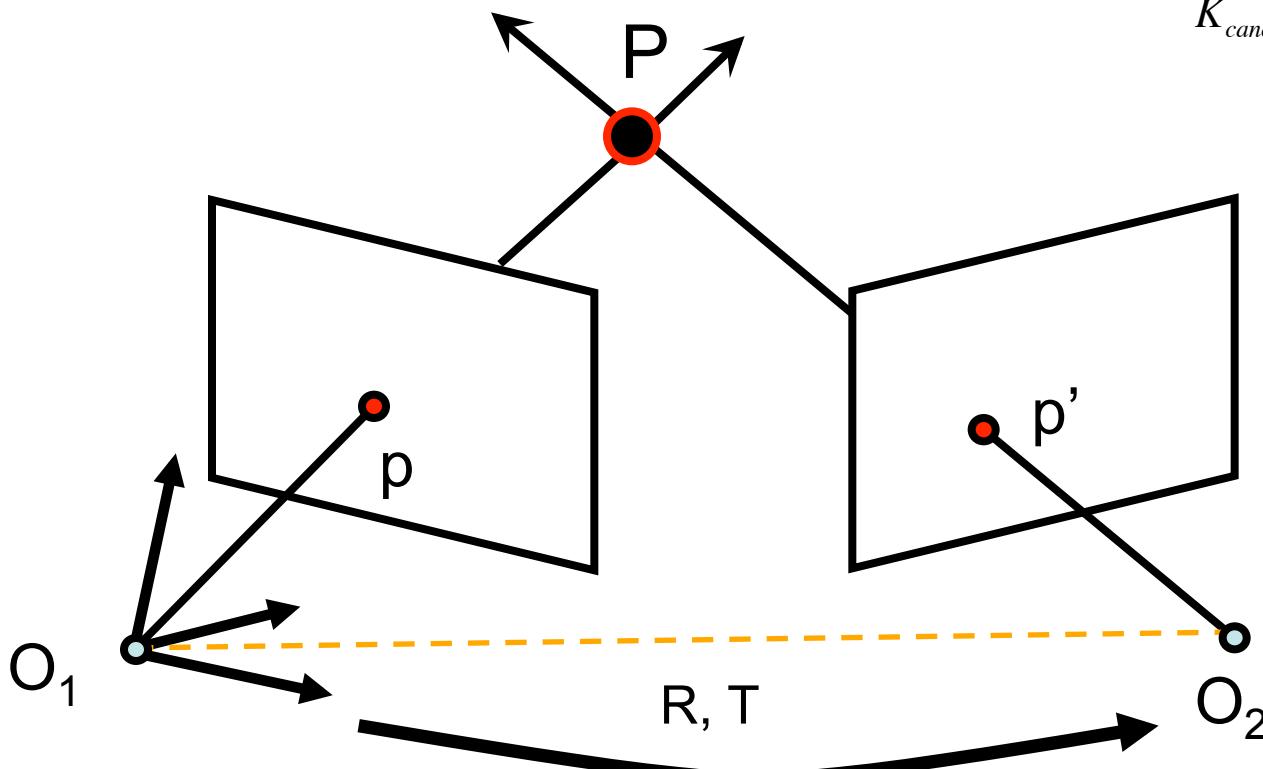
$$M = K \begin{bmatrix} I & 0 \end{bmatrix}$$

$$M P = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = p \quad [\text{Eq. 3}]$$

$$M' = K' \begin{bmatrix} R^T & -R^T T \end{bmatrix}$$

$$M' P = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = p' \quad [\text{Eq. 4}]$$

Epipolar Constraint



$$K_{\text{canonical}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = K \begin{bmatrix} I & 0 \end{bmatrix}$$

$$\downarrow$$

$$M = [I \quad 0] \quad [\text{Eq. 5}]$$

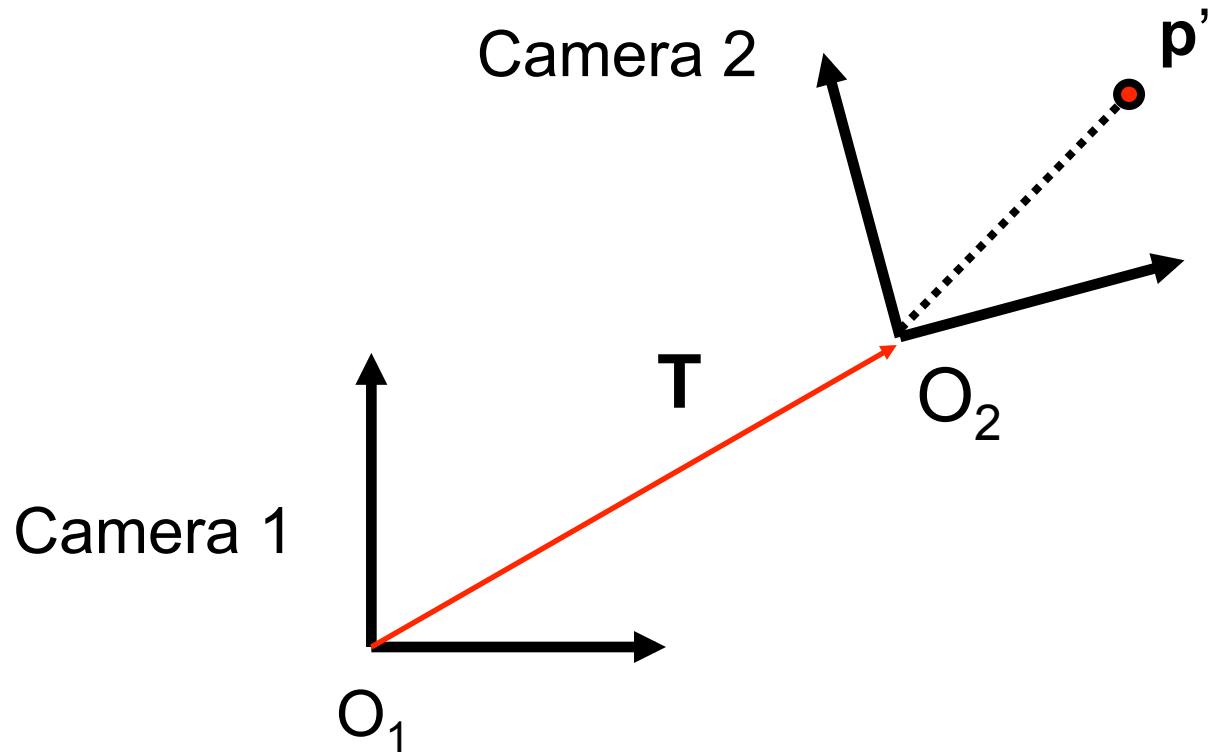
$K = K'$ are known
(canonical cameras)

$$M' = K' \begin{bmatrix} R^T & -R^T T \end{bmatrix}$$

$$\downarrow$$

$$M' = \begin{bmatrix} R^T & -R^T T \end{bmatrix} \quad [\text{Eq. 6}]$$

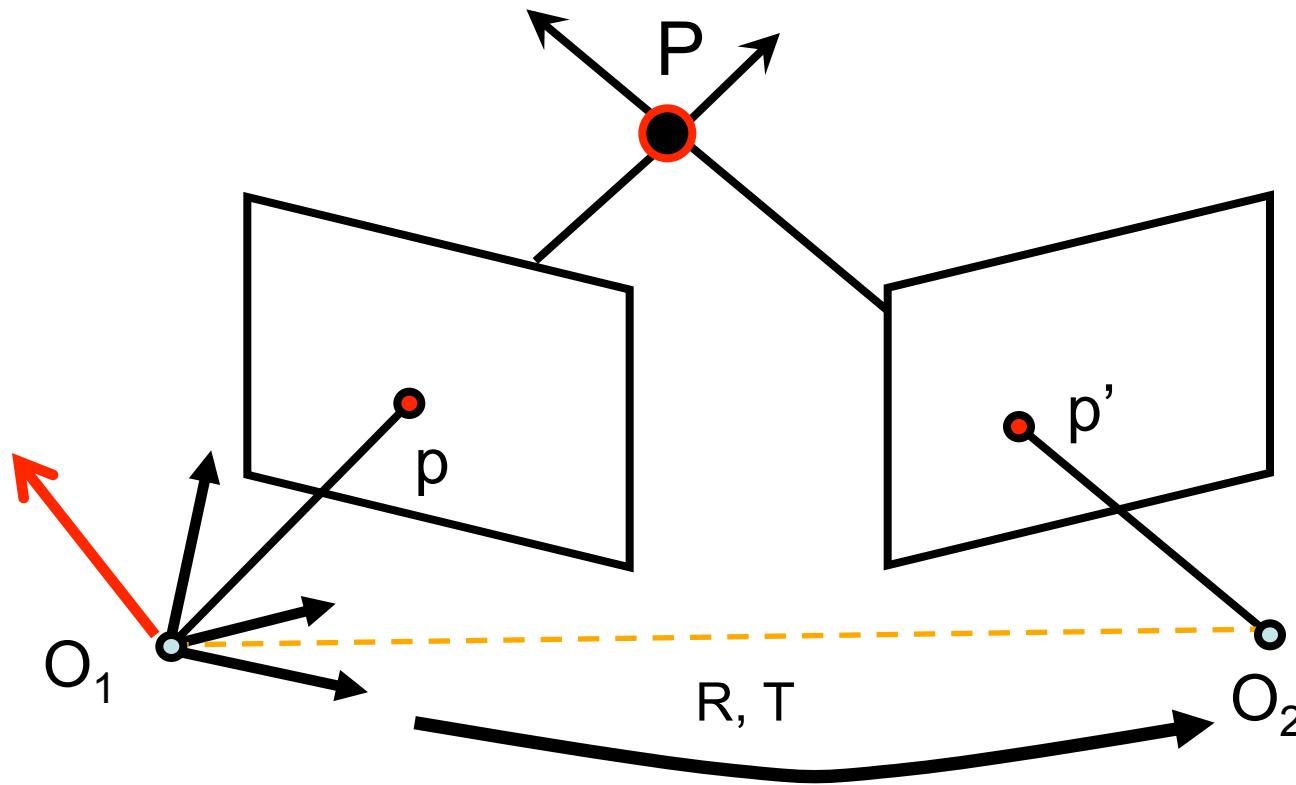
The cameras are related by R , T



$T = O_2$ in the camera 1 reference system

R is the rotation matrix such that a vector p' in the camera 2 is equal to $R p'$ in camera 1.

Epipolar Constraint



p' in first camera reference system is $= R p' + T$

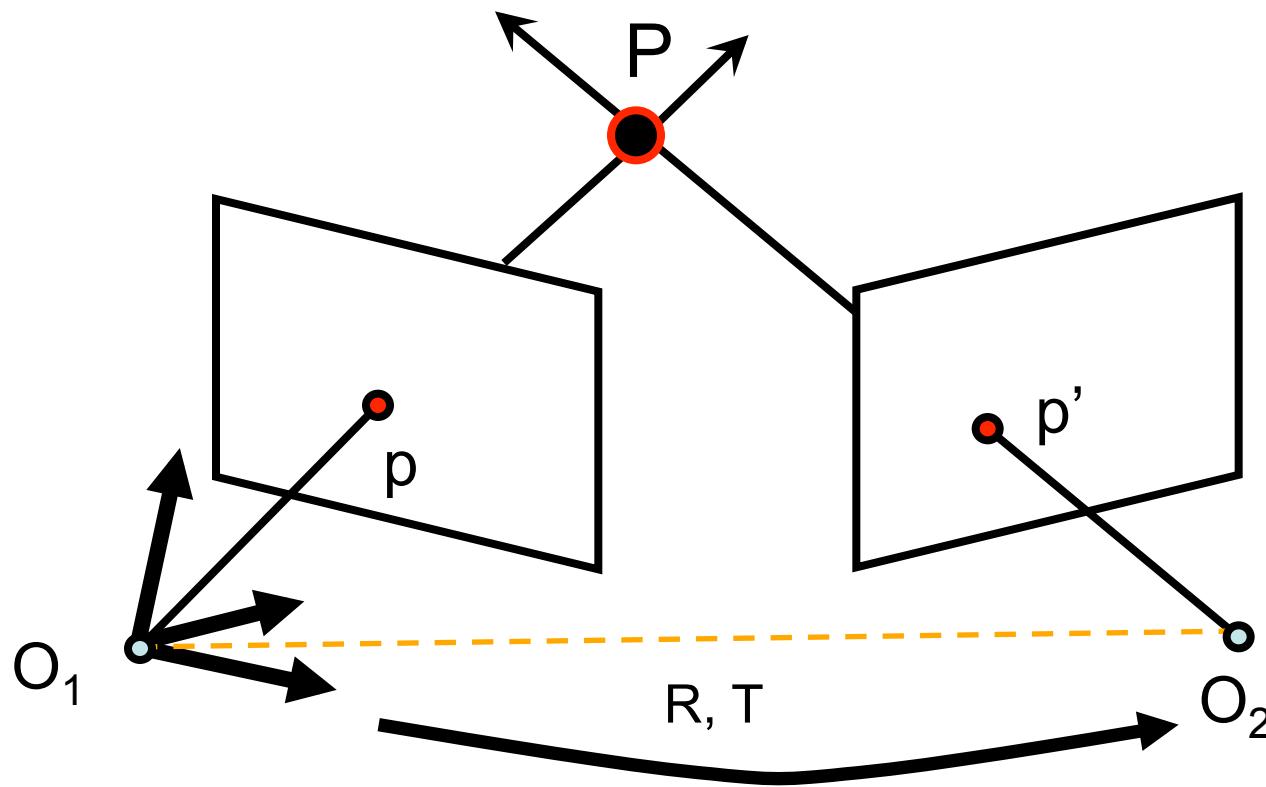
$T \times ((R p') + T) = T \times (R p')$ is perpendicular to epipolar plane

$$\rightarrow p^T \cdot [T \times (R p')] = 0 \quad [\text{Eq. 7}]$$

Cross product as matrix multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times] \mathbf{b}$$

Epipolar Constraint



$$p^T \cdot [T \times (R p')] = 0 \rightarrow p^T \cdot [T_x] \cdot R p' = 0 \quad [Eq. 9]$$

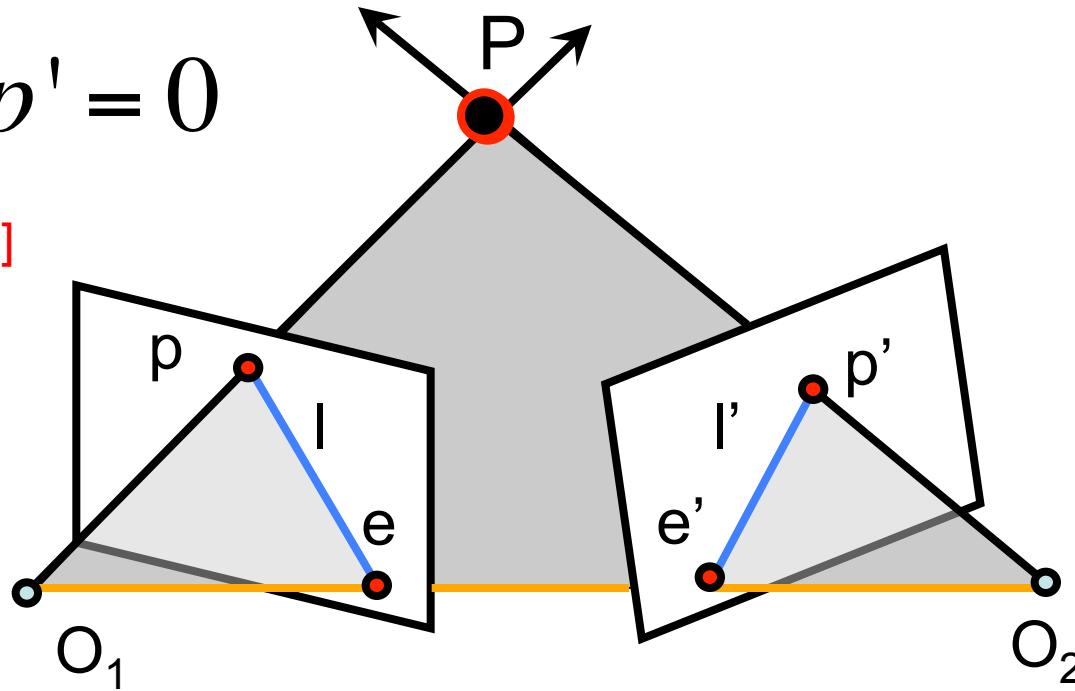
$E = \text{Essential matrix}$
(Longuet-Higgins, 1981)

[Eq. 8]

Epipolar Constraint

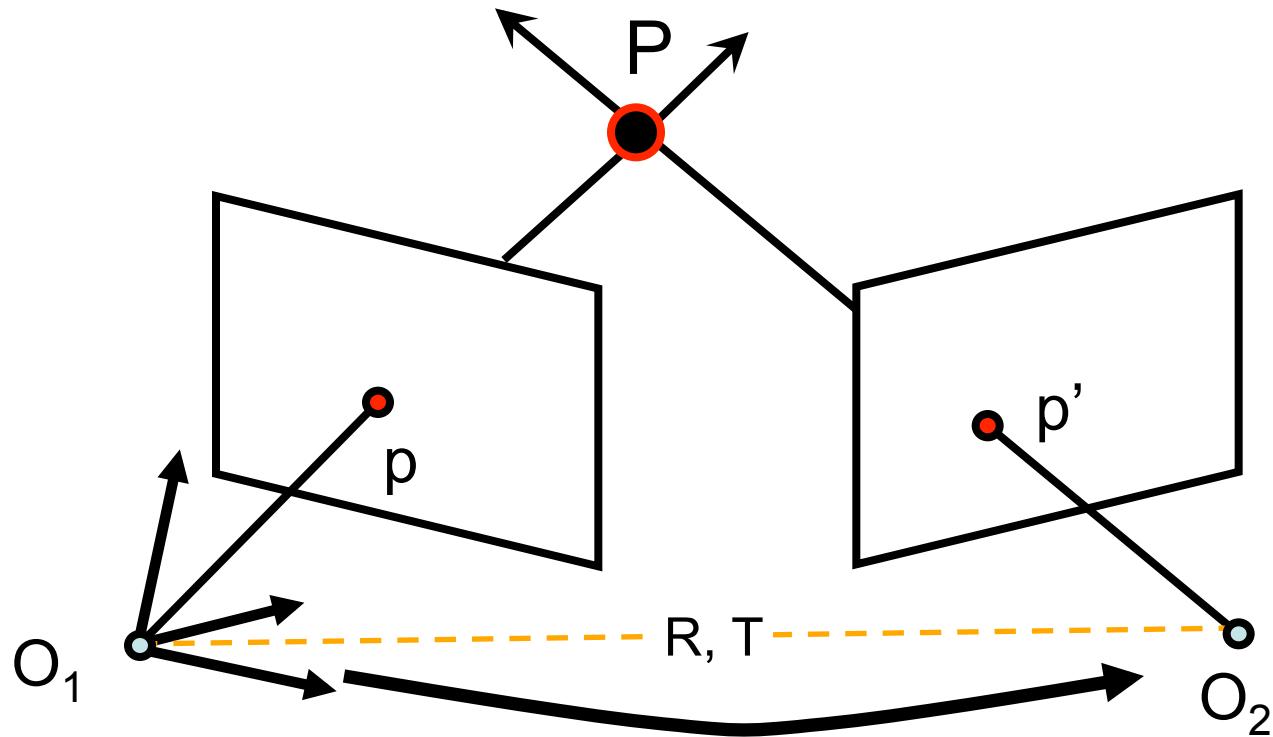
$$p^T \cdot E p' = 0$$

[Eq. 10]



- $l = E p'$ is the epipolar line associated with p'
- $l' = E^T p$ is the epipolar line associated with p
- $E e' = 0$ and $E^T e = 0$
- E is 3×3 matrix; 5 DOF
- E is singular (rank two)

Epipolar Constraint



$$M = K \begin{bmatrix} I & 0 \end{bmatrix}$$

$$p_c = K^{-1} p \quad [\text{Eq. 11}]$$

$$M' = K' \begin{bmatrix} R^T & -R^T T \end{bmatrix}$$

$$p'_c = K'^{-1} p' \quad [\text{Eq. 12}]$$

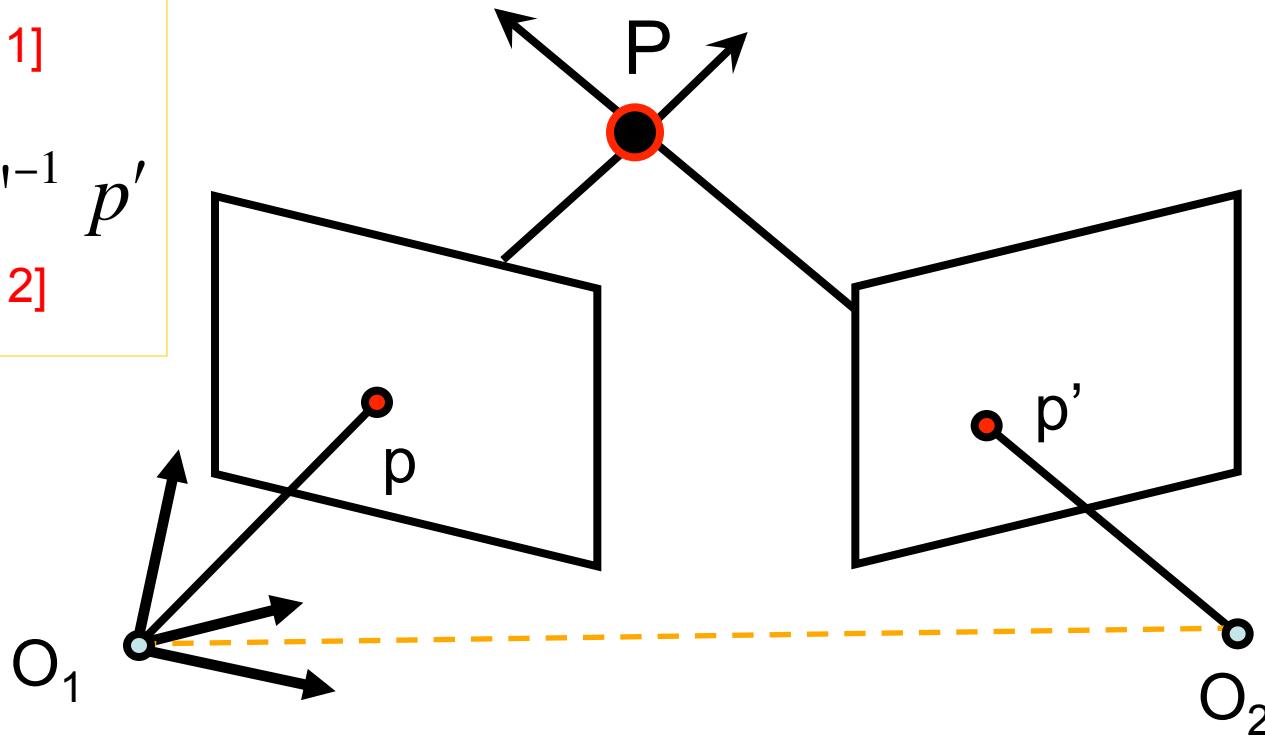
Epipolar Constraint

$$p_c = K^{-1} p$$

[Eq. 11]

$$p'_c = K'^{-1} p'$$

[Eq. 12]



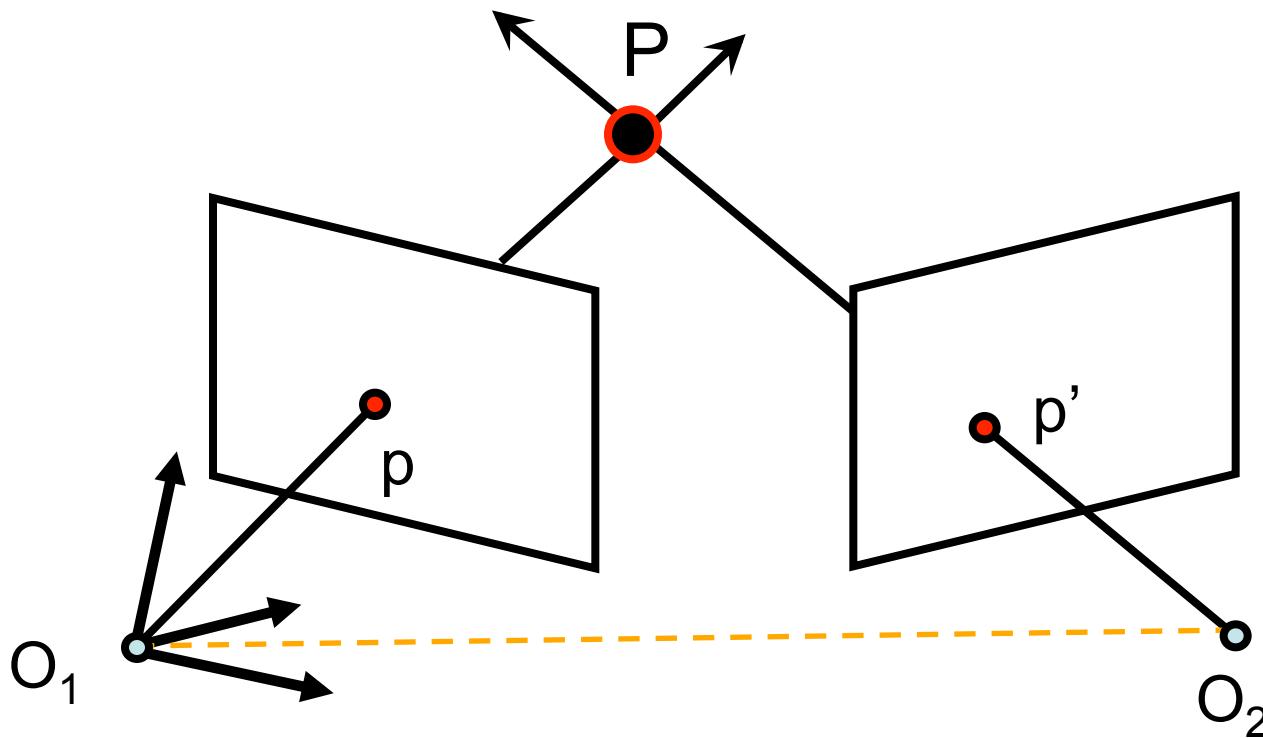
[Eq. 9]

$$p_c^T \cdot [T_x] \cdot R p'_c = 0 \rightarrow (K^{-1} p)^T \cdot [T_x] \cdot R K'^{-1} p' = 0$$

$$p^T [K^{-T} \cdot [T_x] \cdot R K'^{-1}] p' = 0 \rightarrow p^T [F] p' = 0$$

[Eq. 13]

Epipolar Constraint



[Eq. 13]

$$p^T F p' = 0$$

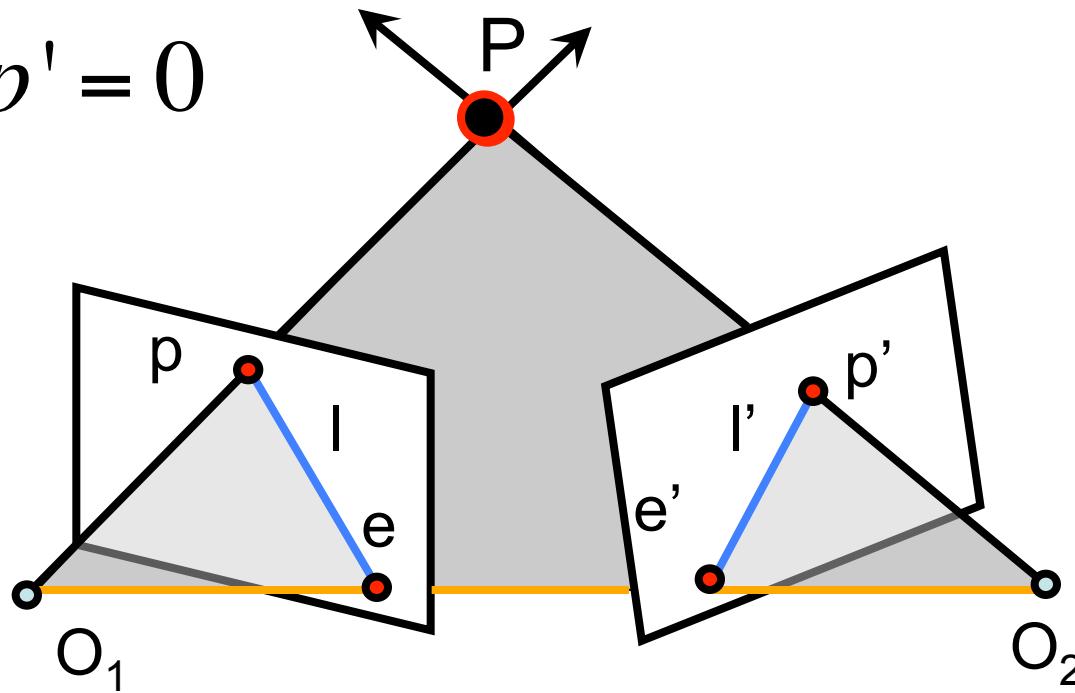
$$F = K^{-T} \cdot [T_x] \cdot R \cdot K'^{-1}$$

F = Fundamental Matrix
(Faugeras and Luong, 1992)

[Eq. 14]

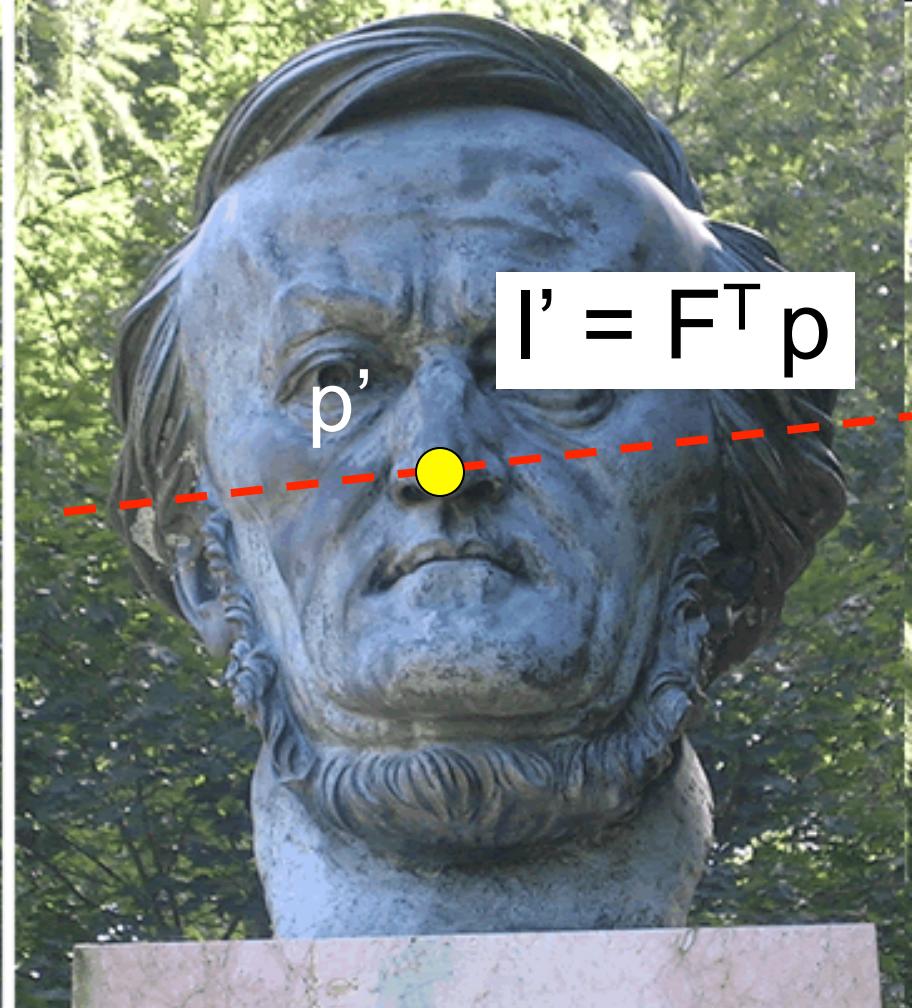
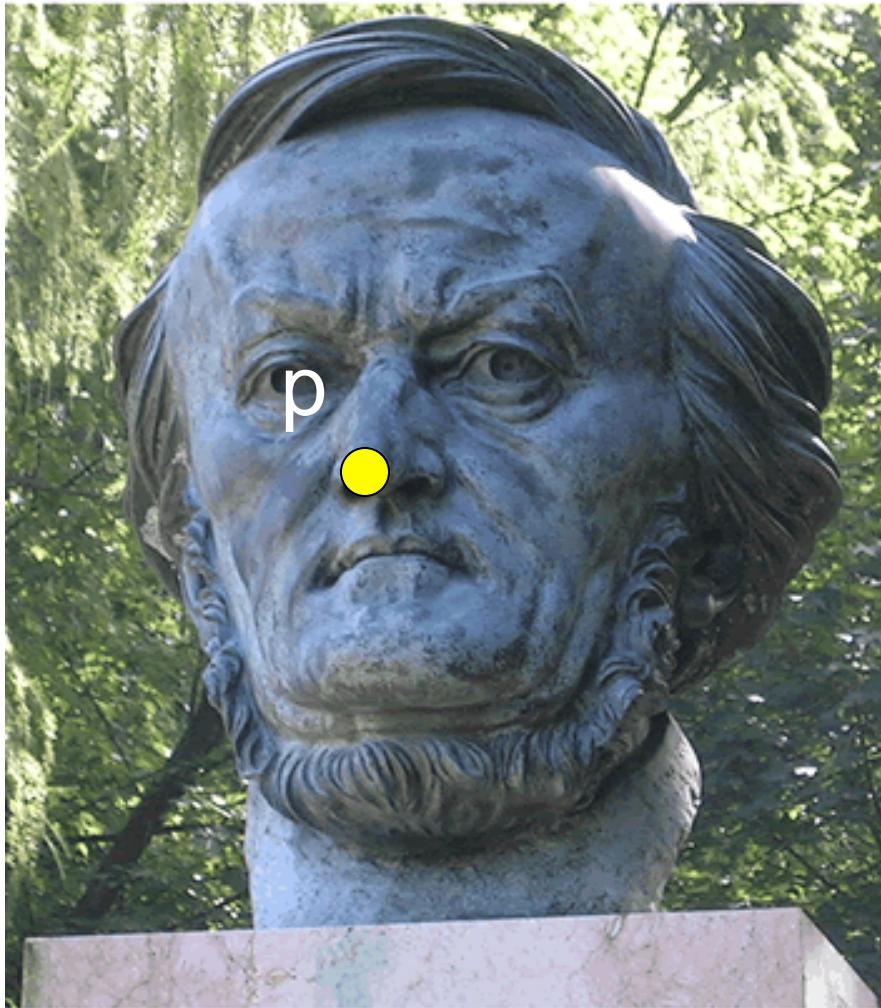
Epipolar Constraint

$$p^T \cdot F p' = 0$$



- $l = F p'$ is the epipolar line associated with p'
- $l' = F^T p$ is the epipolar line associated with p
- $F e' = 0$ and $F^T e = 0$
- F is 3×3 matrix; 7 DOF
- F is singular (rank two)

Why F is useful?



- Suppose F is known
- No additional information about the scene and camera is given
- Given a point on left image, we can compute the corresponding epipolar line in the second image

Why F is useful?

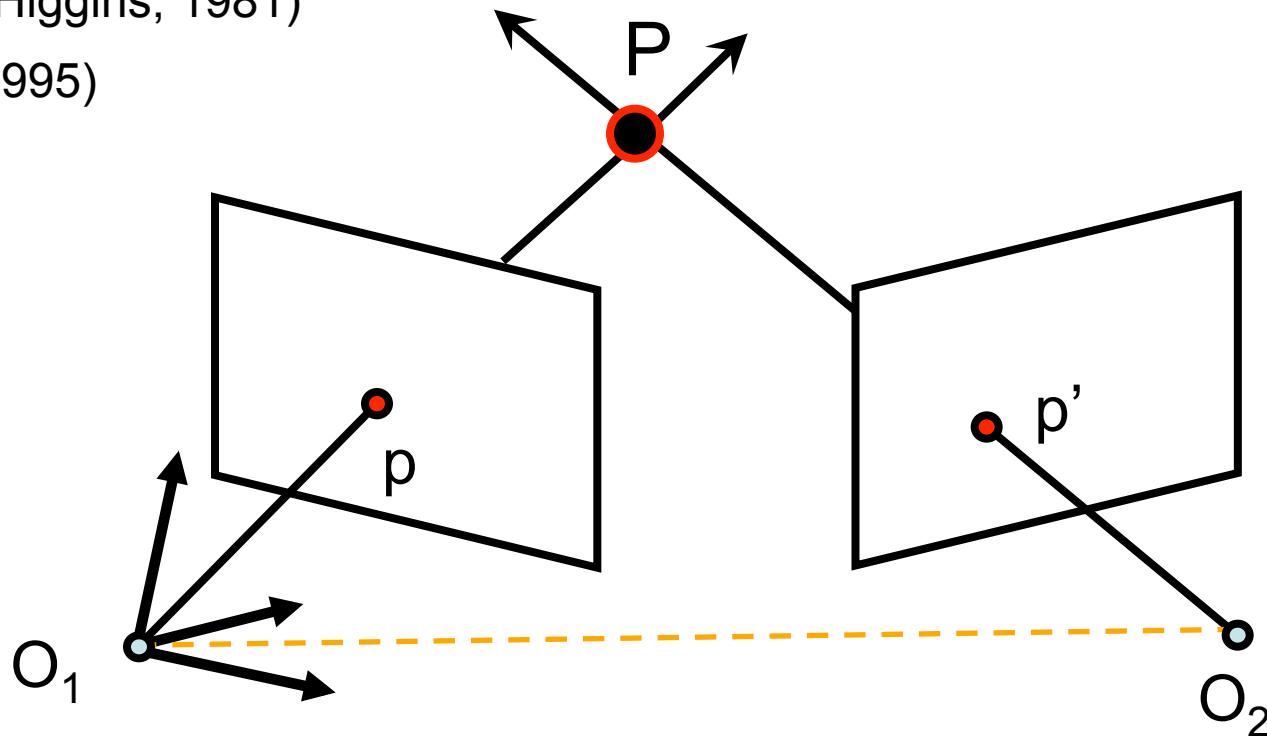
- F captures information about the epipolar geometry of 2 views + camera parameters
- **MORE IMPORTANTLY:** F gives constraints on how the scene changes under view point transformation (without reconstructing the scene!)
- Powerful tool in:
 - 3D reconstruction
 - Multi-view object/scene matching

Estimating F

The Eight-Point Algorithm

(Longuet-Higgins, 1981)

(Hartley, 1995)



$$p^T F p' = 0$$

Estimating F

$$[\text{Eq. 13}] \quad p^T F p' = 0 \quad \rightarrow$$

$$p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

$$(u, v, 1) \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

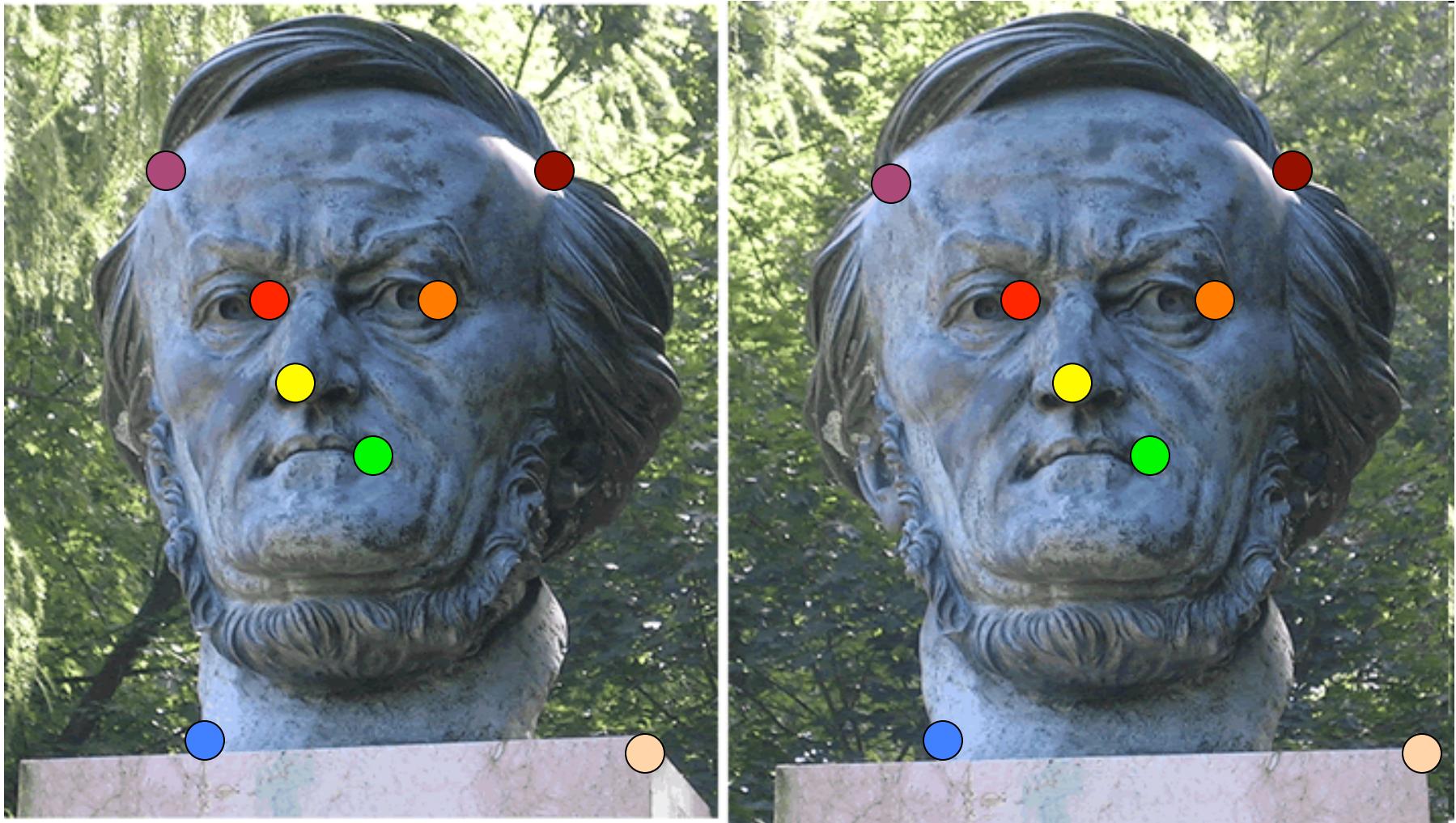


$$(uu', uv', u, vu', vv', v, u', v', 1) \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

[Eq. 14]

Let's take 8 corresponding points

Estimating F



Estimating F

$$\begin{pmatrix} u_i u'_i, u_i v'_i, u_i, v_i u'_i, v_i v'_i, v_i, u'_i, v'_i, 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0 \quad [\text{Eq. 14}]$$

Estimating F

$$\mathbf{W} \begin{pmatrix} u_1 u'_1 & u_1 v'_1 & u_1 & v_1 u'_1 & v_1 v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2 u'_2 & u_2 v'_2 & u_2 & v_2 u'_2 & v_2 v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3 u'_3 & u_3 v'_3 & u_3 & v_3 u'_3 & v_3 v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4 u'_4 & u_4 v'_4 & u_4 & v_4 u'_4 & v_4 v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5 u'_5 & u_5 v'_5 & u_5 & v_5 u'_5 & v_5 v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6 u'_6 & u_6 v'_6 & u_6 & v_6 u'_6 & v_6 v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7 u'_7 & u_7 v'_7 & u_7 & v_7 u'_7 & v_7 v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8 u'_8 & u_8 v'_8 & u_8 & v_8 u'_8 & v_8 v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0 \quad [\text{Eqs. 15}]$$

- Homogeneous system $\mathbf{W} \mathbf{f} = 0$
- Rank 8 \rightarrow A non-zero solution exists (unique)
- If $N > 8$ \rightarrow Lsq. solution by SVD! $\rightarrow \hat{\mathbf{F}}$
 $\|\mathbf{f}\| = 1$

\hat{F} satisfies: $p^T \hat{F} p' = 0$

and estimated \hat{F} may have full rank ($\det(\hat{F}) \neq 0$)

But remember: fundamental matrix is Rank2

Find F that minimizes $\|F - \hat{F}\| = 0$

Frobenius norm (*)

Subject to $\det(F) = 0$

SVD (again!) can be used to solve this problem

(*) Sq. root of the sum of squares of all entries

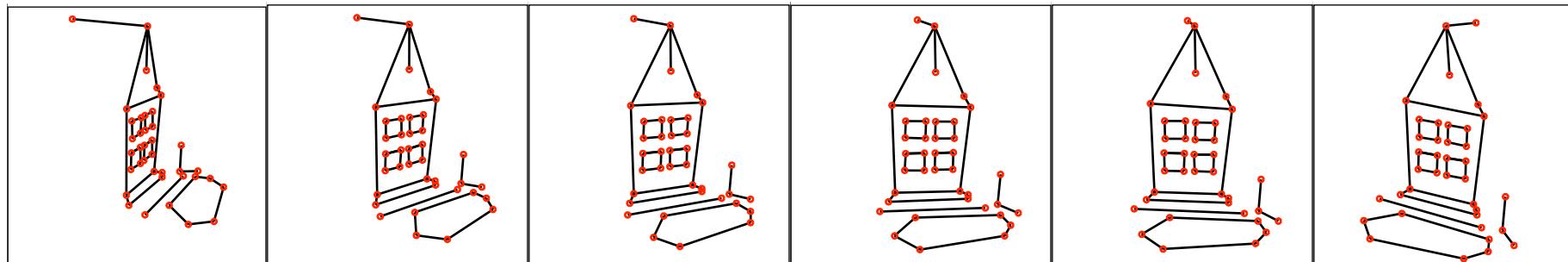
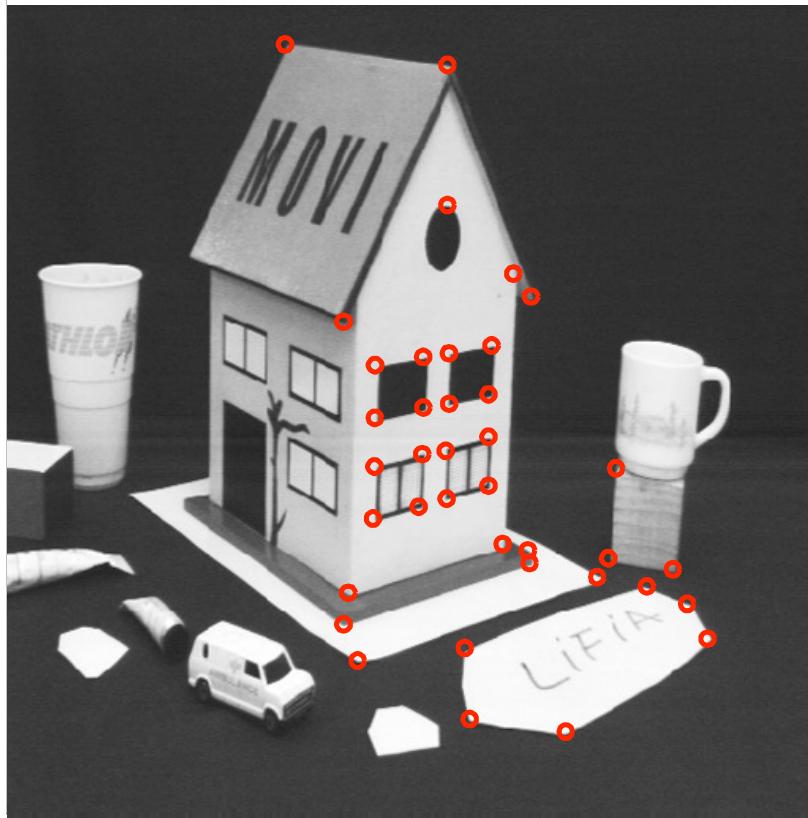
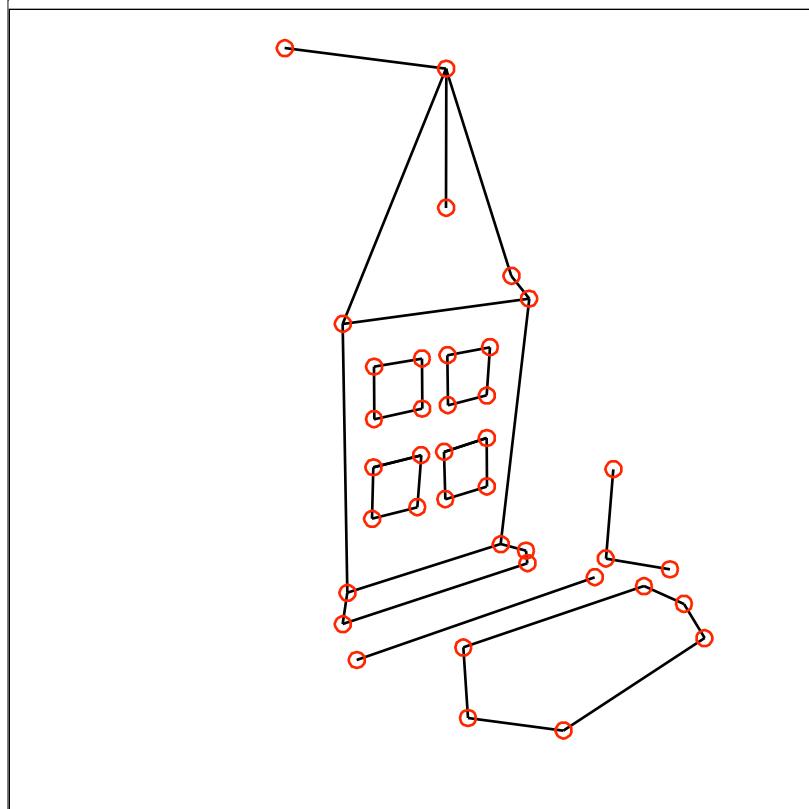
Find F that minimizes $\|F - \hat{F}\| = 0$
Frobenius norm (*)

Subject to $\det(F) = 0$

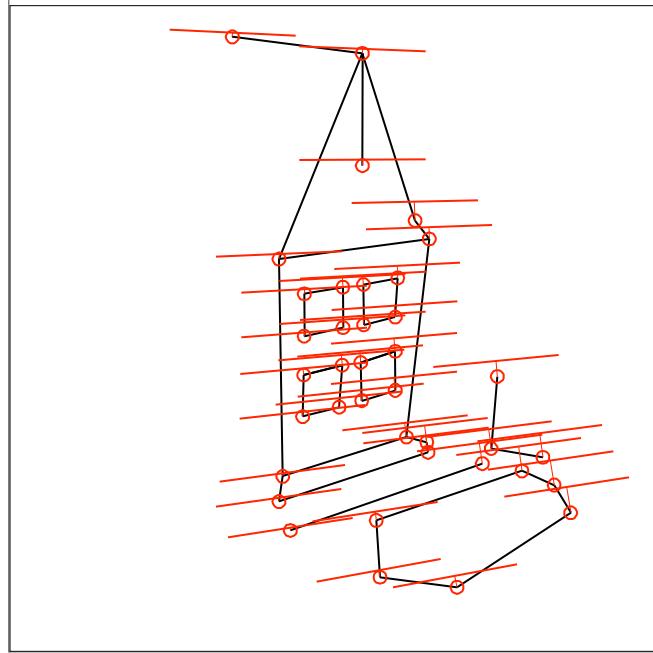
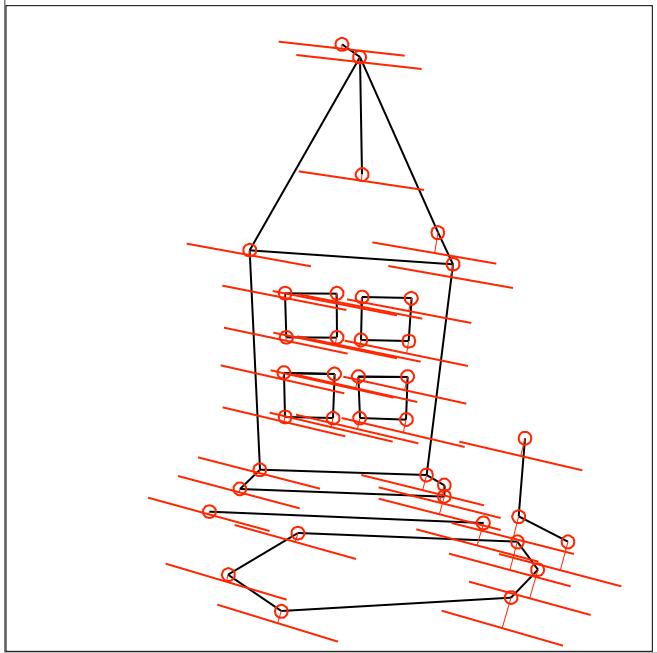
$$F = U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

Where:

$$U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} V^T = SVD(\hat{F})$$

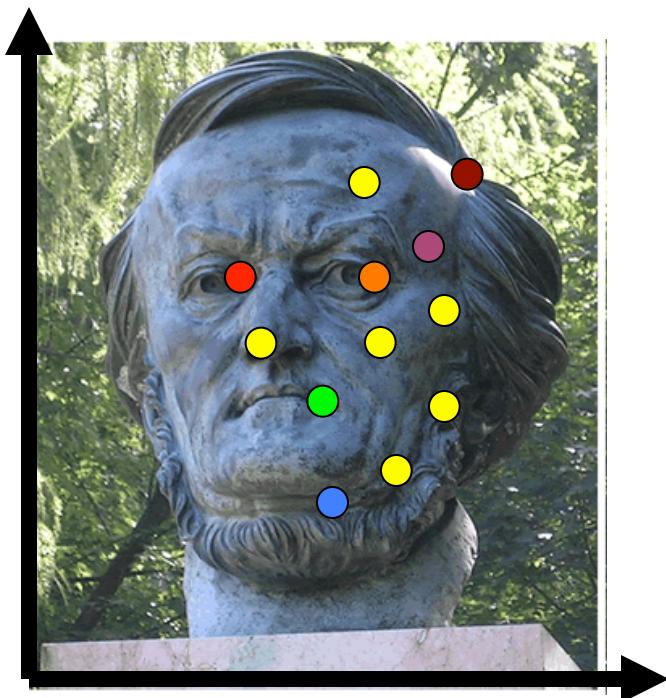


Data courtesy of R. Mohr and B. Boufama.



Mean errors:
10.0pixel
9.1pixel

Problems with the 8-Point Algorithm



$$W f = 0,$$

$$\|f\| = 1$$

Lsq solution
by SVD

$$F$$

- Recall the structure of W :
 - do we see any potential (numerical) issue?

Problems with the 8-Point Algorithm

$$\mathbf{Wf} = 0$$

$$\begin{pmatrix} u_1 u'_1 & u_1 v'_1 & u_1 & v_1 u'_1 & v_1 v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2 u'_2 & u_2 v'_2 & u_2 & v_2 u'_2 & v_2 v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3 u'_3 & u_3 v'_3 & u_3 & v_3 u'_3 & v_3 v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4 u'_4 & u_4 v'_4 & u_4 & v_4 u'_4 & v_4 v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5 u'_5 & u_5 v'_5 & u_5 & v_5 u'_5 & v_5 v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6 u'_6 & u_6 v'_6 & u_6 & v_6 u'_6 & v_6 v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7 u'_7 & u_7 v'_7 & u_7 & v_7 u'_7 & v_7 v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8 u'_8 & u_8 v'_8 & u_8 & v_8 u'_8 & v_8 v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

- Highly un-balanced (not well conditioned)
- Values of W must have similar magnitude
- This creates problems during the SVD decomposition

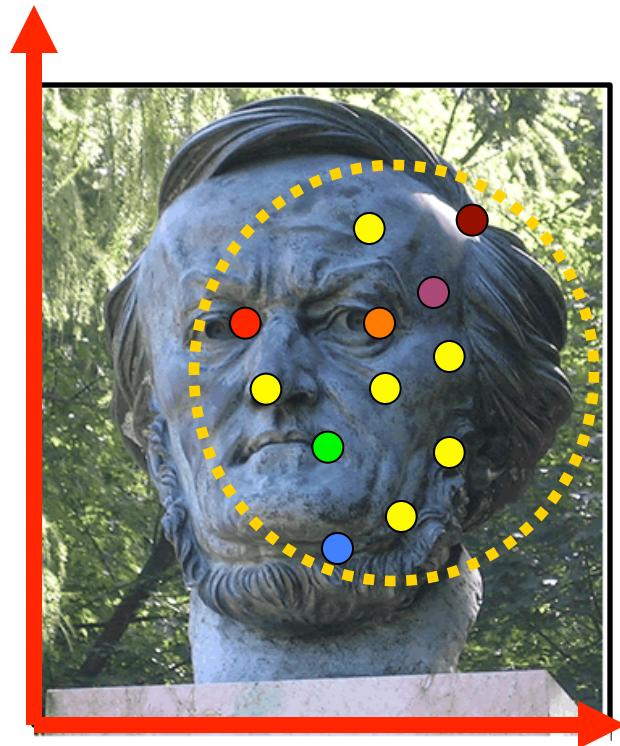
Normalization

IDEA: Transform image coordinates such that the matrix \mathbf{W} becomes better conditioned (**pre-conditioning**)

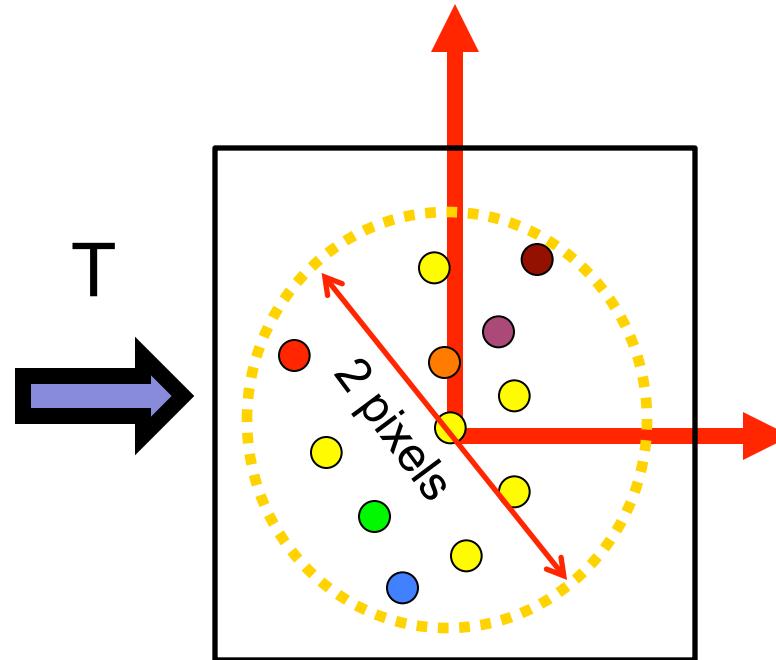
For each image, apply a following transformation T (translation and scaling) acting on image coordinates such that:

- Origin = centroid of image points
- Mean square distance of the image points from origin is ~ 2 pixels

Example of normalization



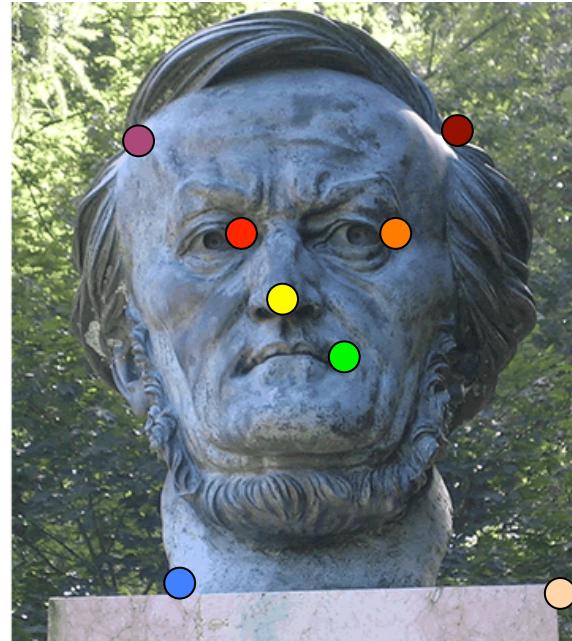
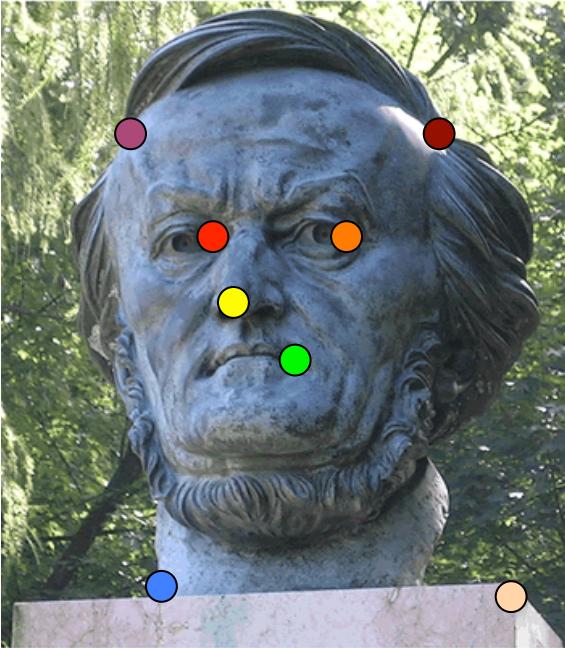
Coordinate system of the image before applying T



Coordinate system of the image after applying T

- Origin = centroid of image points
- Mean square distance of the image points from origin is ~ 2 pixels

Normalization



$$q_i = T \ p_i$$

$$q'_i = T' \ p'_i$$

The Normalized Eight-Point Algorithm

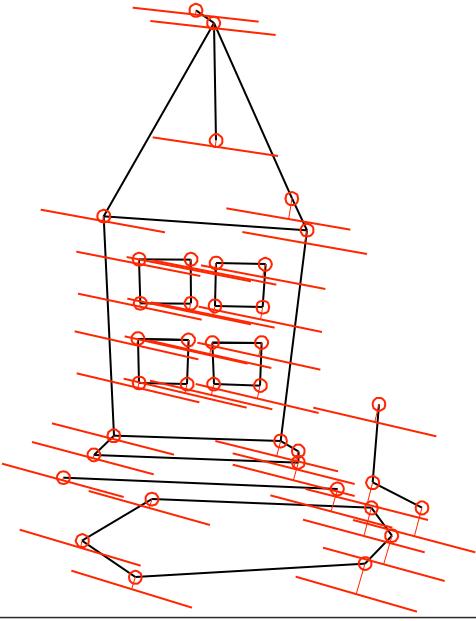
0. Compute T and T' for image 1 and 2, respectively
1. Normalize coordinates in images 1 and 2:

$$q_i = T p_i \quad q'_i = T' p'_i$$

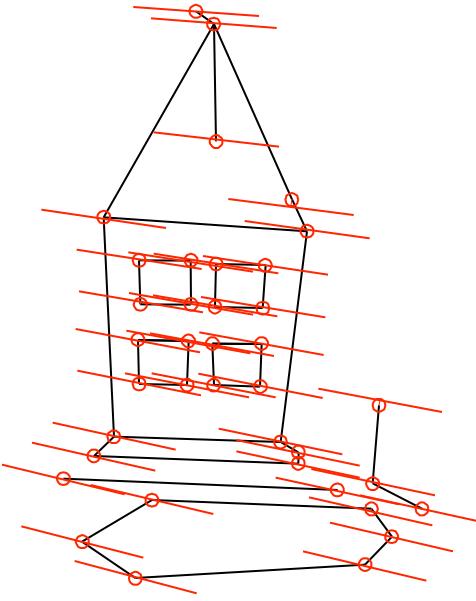
2. Use the eight-point algorithm to compute \hat{F}_q from the corresponding points q and q' .

1. Enforce the rank-2 constraint. $\rightarrow F_q$ such that:
$$\begin{cases} q^T F_q q' = 0 \\ \det(F_q) = 0 \end{cases}$$
2. De-normalize F_q : $F = T^T F_q T'$

Without normalization



With normalization



Mean errors:
10.0pixel
9.1pixel

Mean errors:
1.0pixel
0.9pixel

Lecture 5

Epipolar Geometry

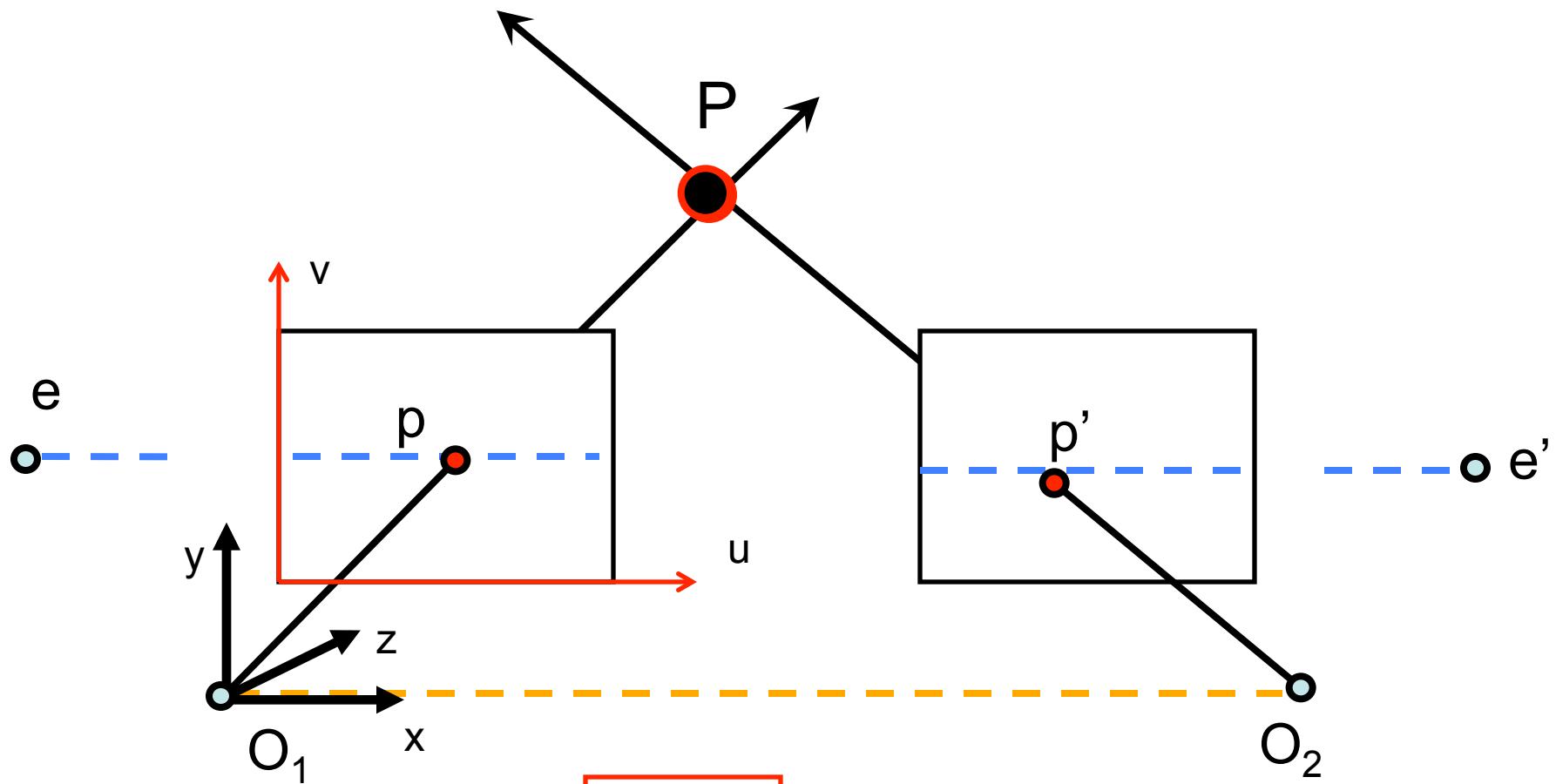


- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating F
- Examples

Reading: [AZ] Chapter: 4 “Estimation – 2D perspective transformations
Chapter: 9 “Epipolar Geometry and the Fundamental Matrix Transformation”
Chapter: 11 “Computation of the Fundamental Matrix F ”

[FP] Chapter: 7 “Stereopsis”
Chapter: 8 “Structure from Motion”

Example: Parallel image planes



$K_1 = K_2 = \text{known}$

x parallel to O_1O_2

$$E=?$$

Hint :

$$R = I \quad T = (T, 0, 0)$$

Essential matrix for parallel images

$$\mathbf{E} = [\mathbf{T}_\times] \cdot \mathbf{R}$$

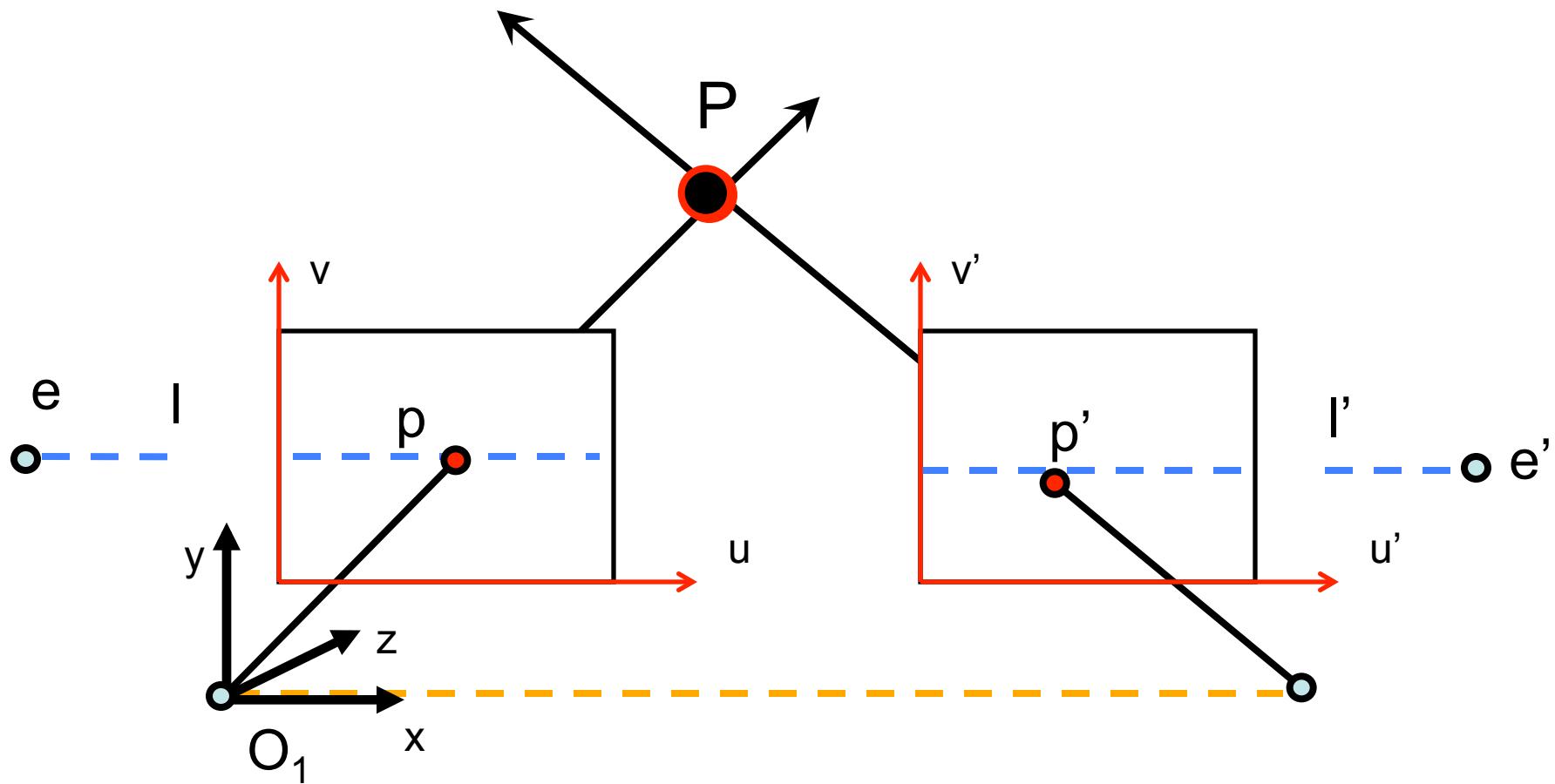
$$\mathbf{E} = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix}$$

[Eq. 20]

$$\mathbf{T} = [\mathbf{T} \ 0 \ 0]$$

$$\mathbf{R} = \mathbf{I}$$

Example: Parallel image planes

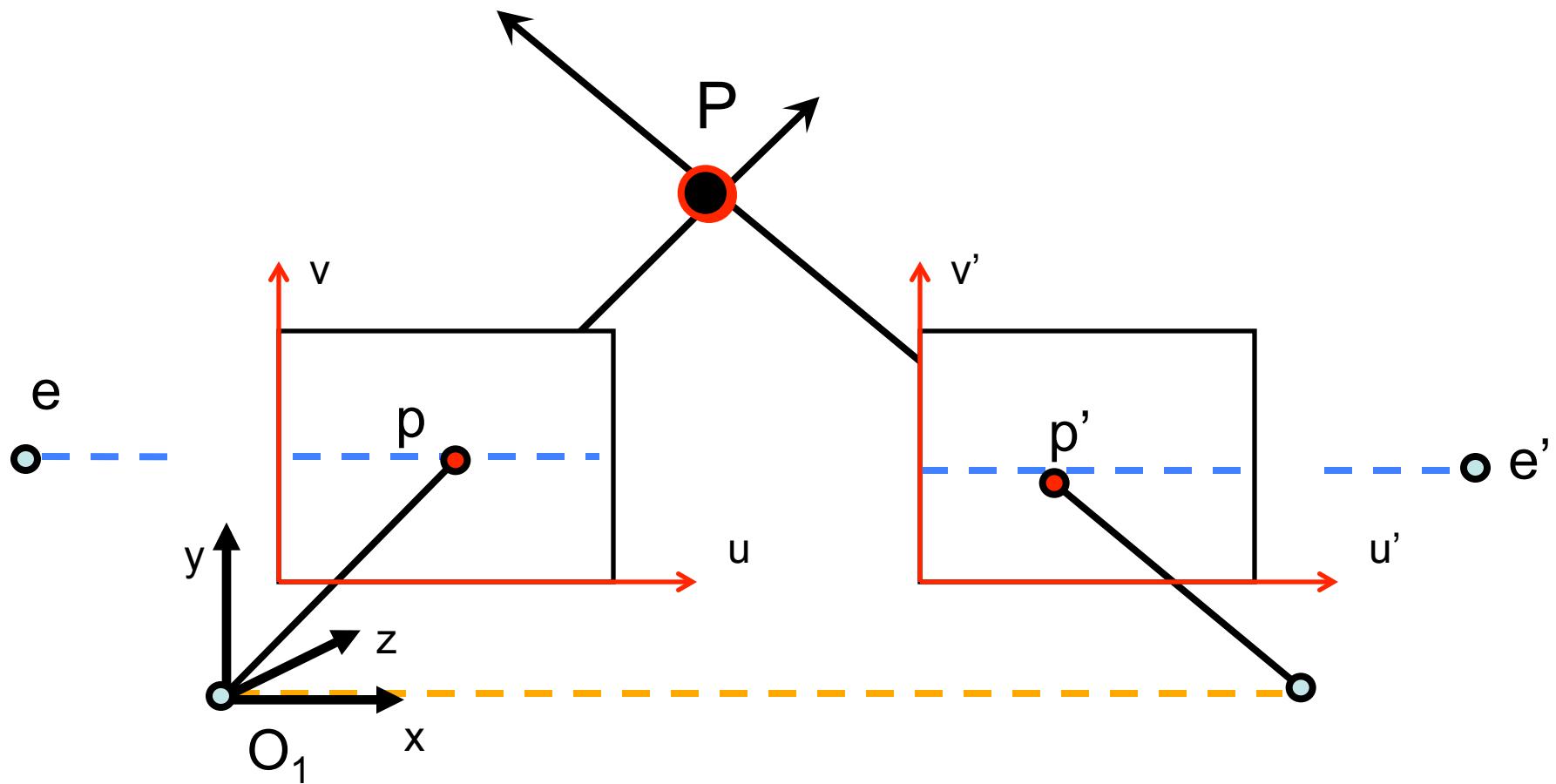


What are the directions of epipolar lines?

$$l = E p' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -T \\ T v' \end{bmatrix}$$

horizontal!

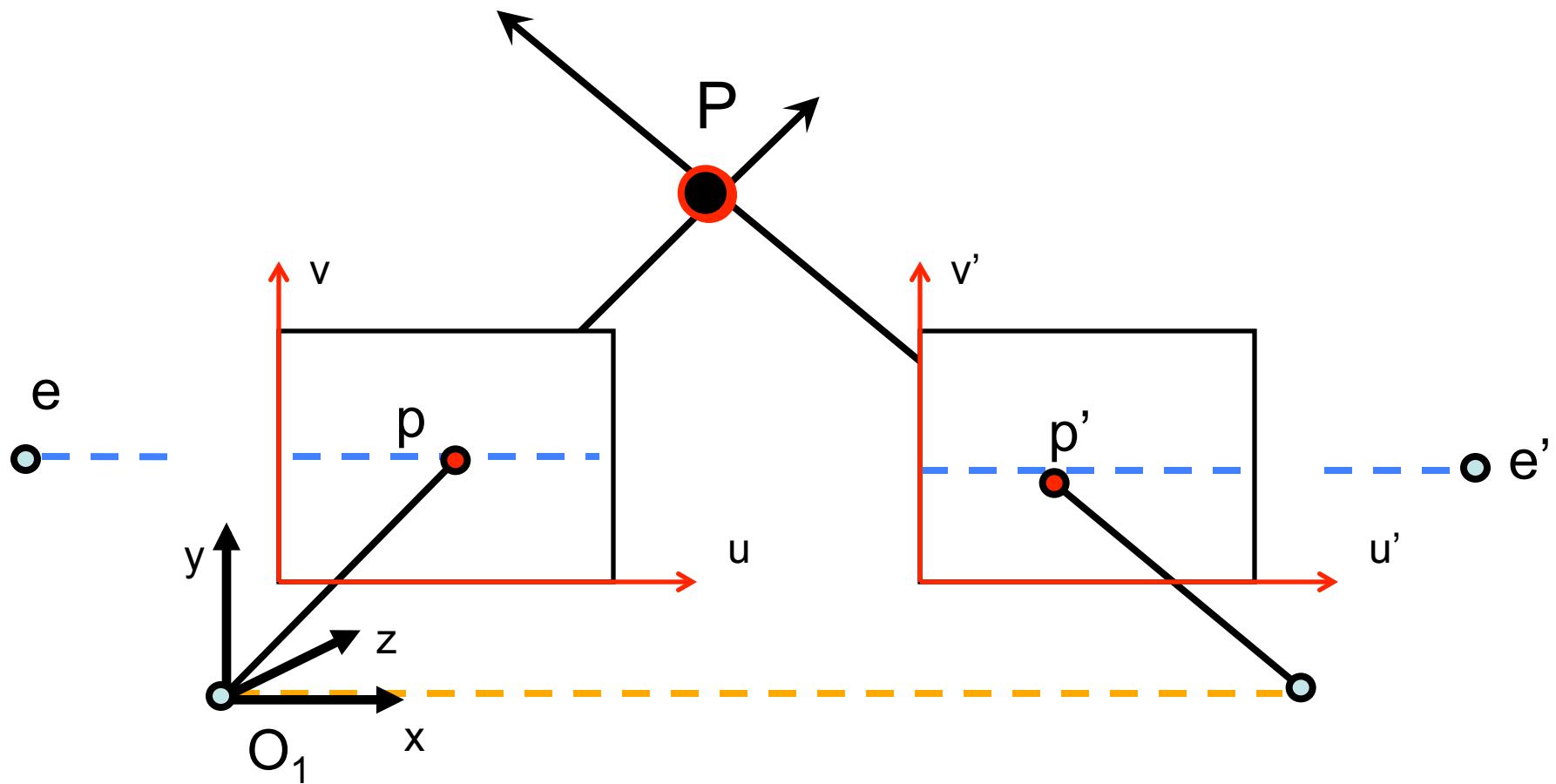
Example: Parallel image planes



How are p
and p'
related?

$$p^T \cdot E \ p' = 0$$

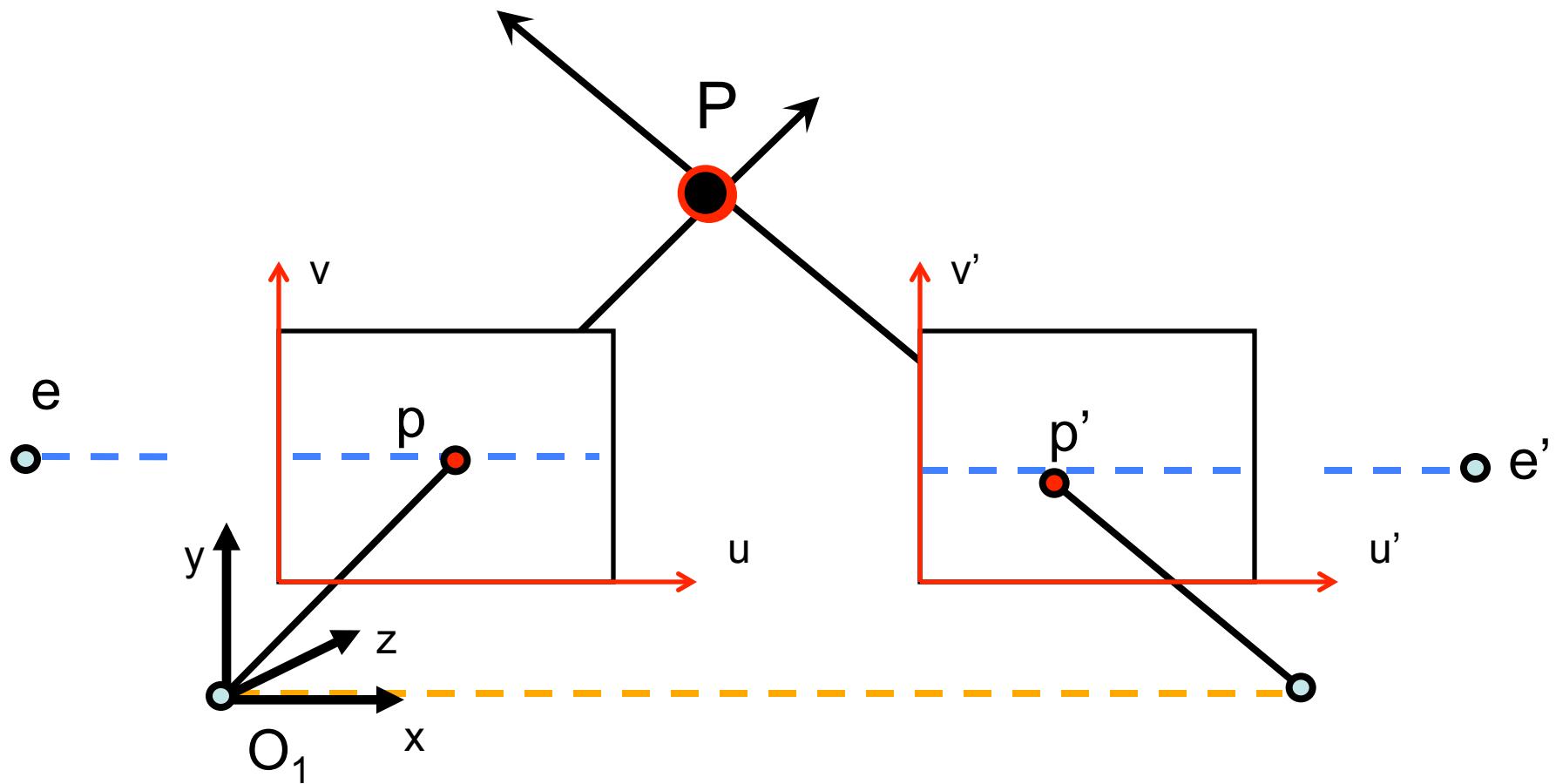
Example: Parallel image planes



How are p
and p'
related?

$$\Rightarrow \begin{pmatrix} u & v & 1 \end{pmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} u & v & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -T \\ Tv' \end{pmatrix} = 0 \Rightarrow Tv = Tv' \Rightarrow v = v'$$

Example: Parallel image planes



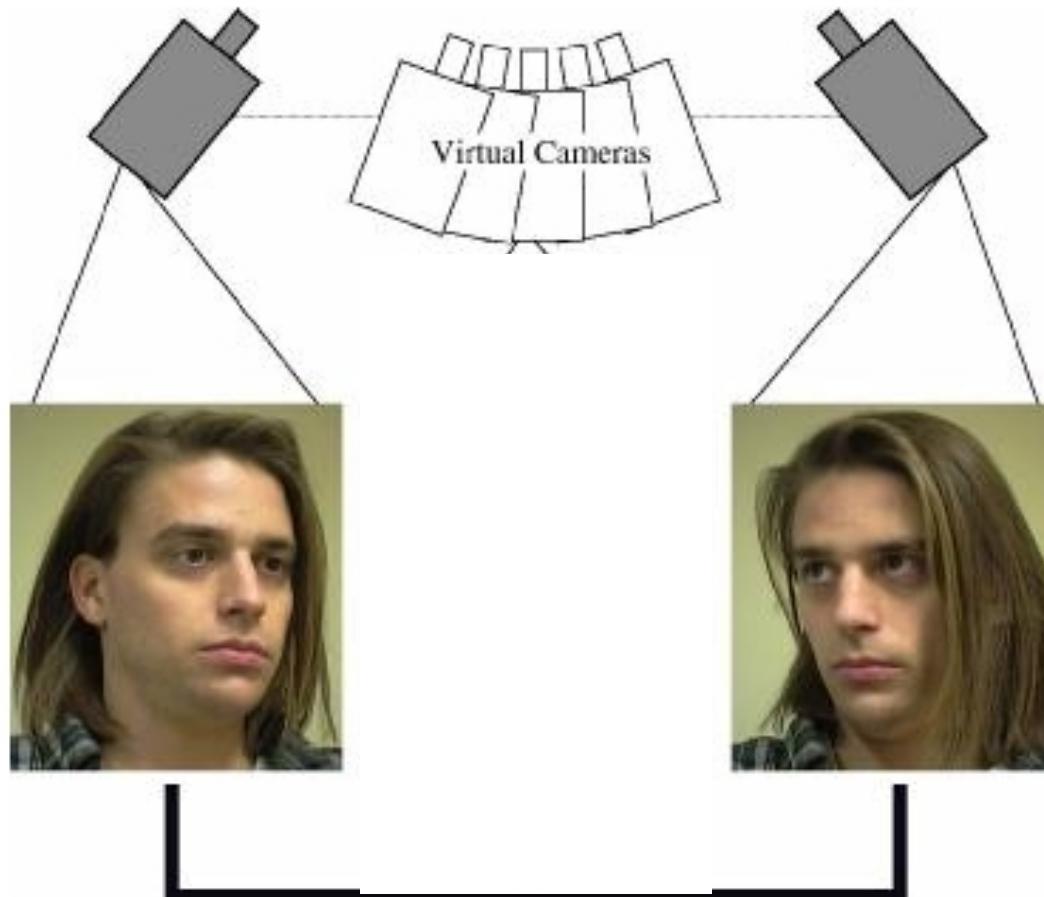
Rectification: making two images “parallel”

Why it is useful?

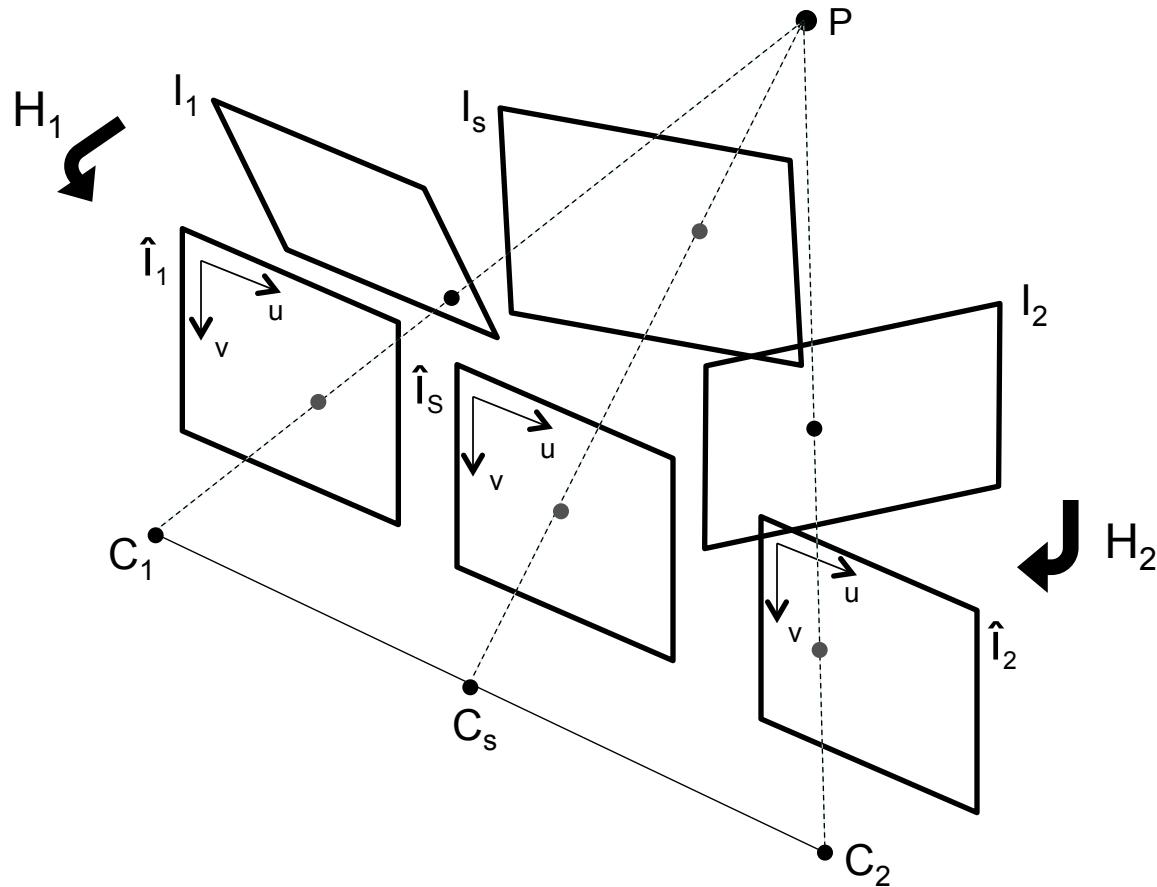
- Epipolar constraint $\rightarrow v = v'$
- New views can be synthesized by linear interpolation

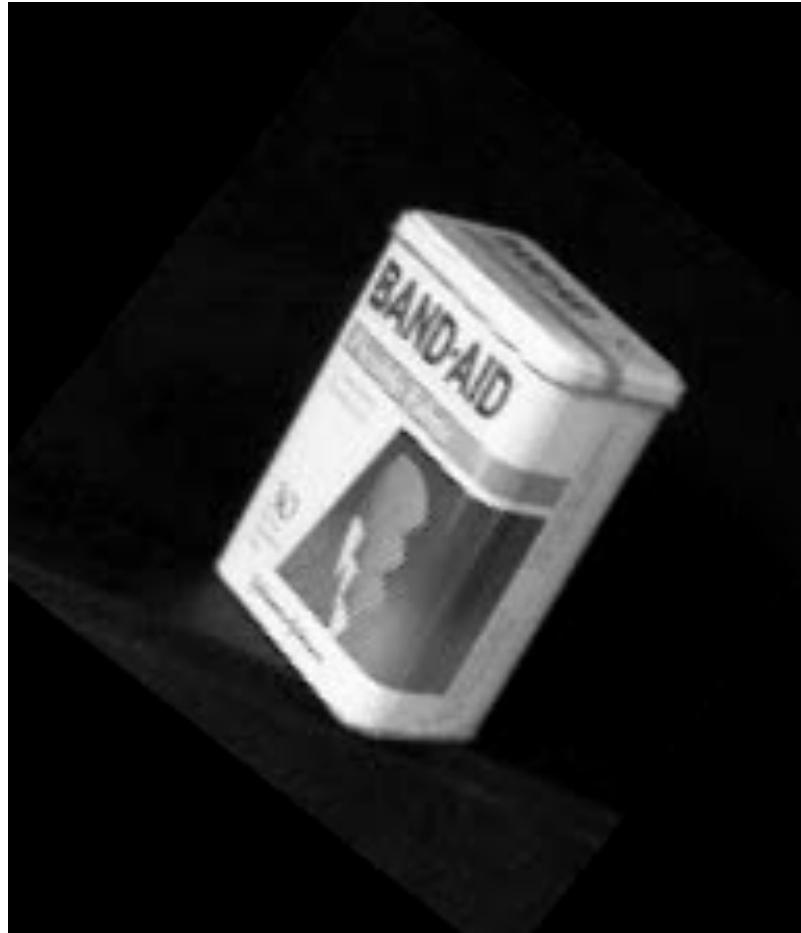
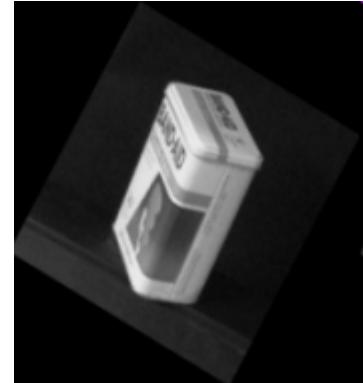
Application: view morphing

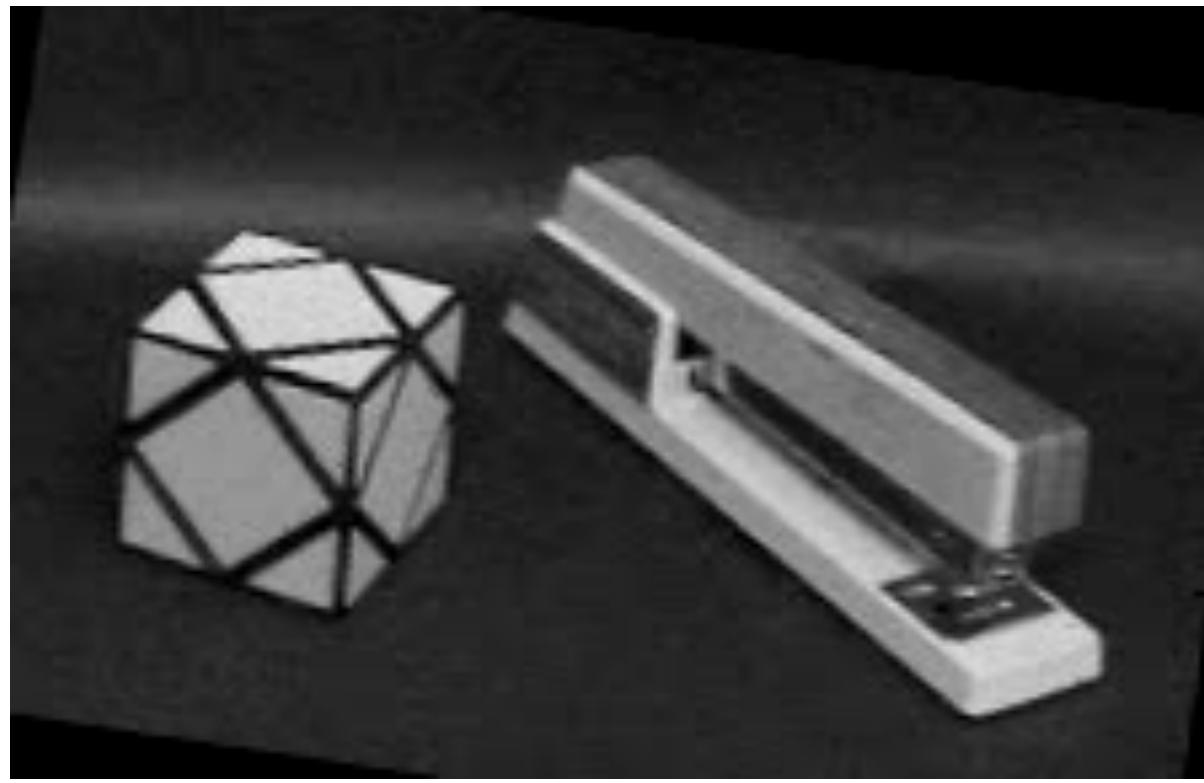
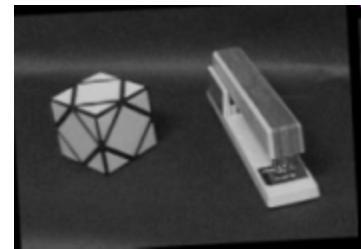
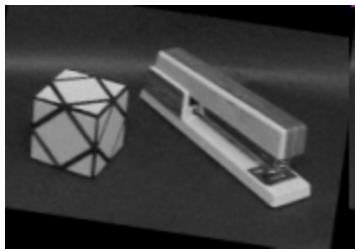
S. M. Seitz and C. R. Dyer, *Proc. SIGGRAPH 96*, 1996, 21-30



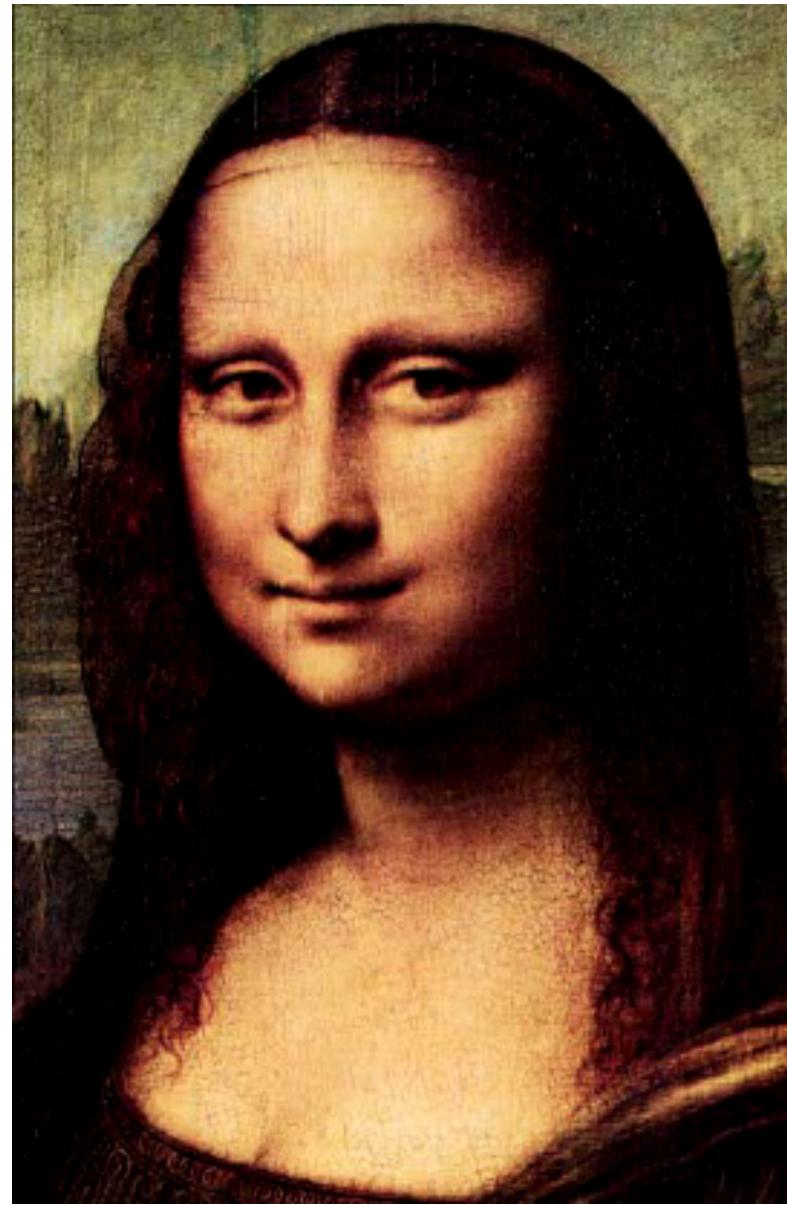
Rectification



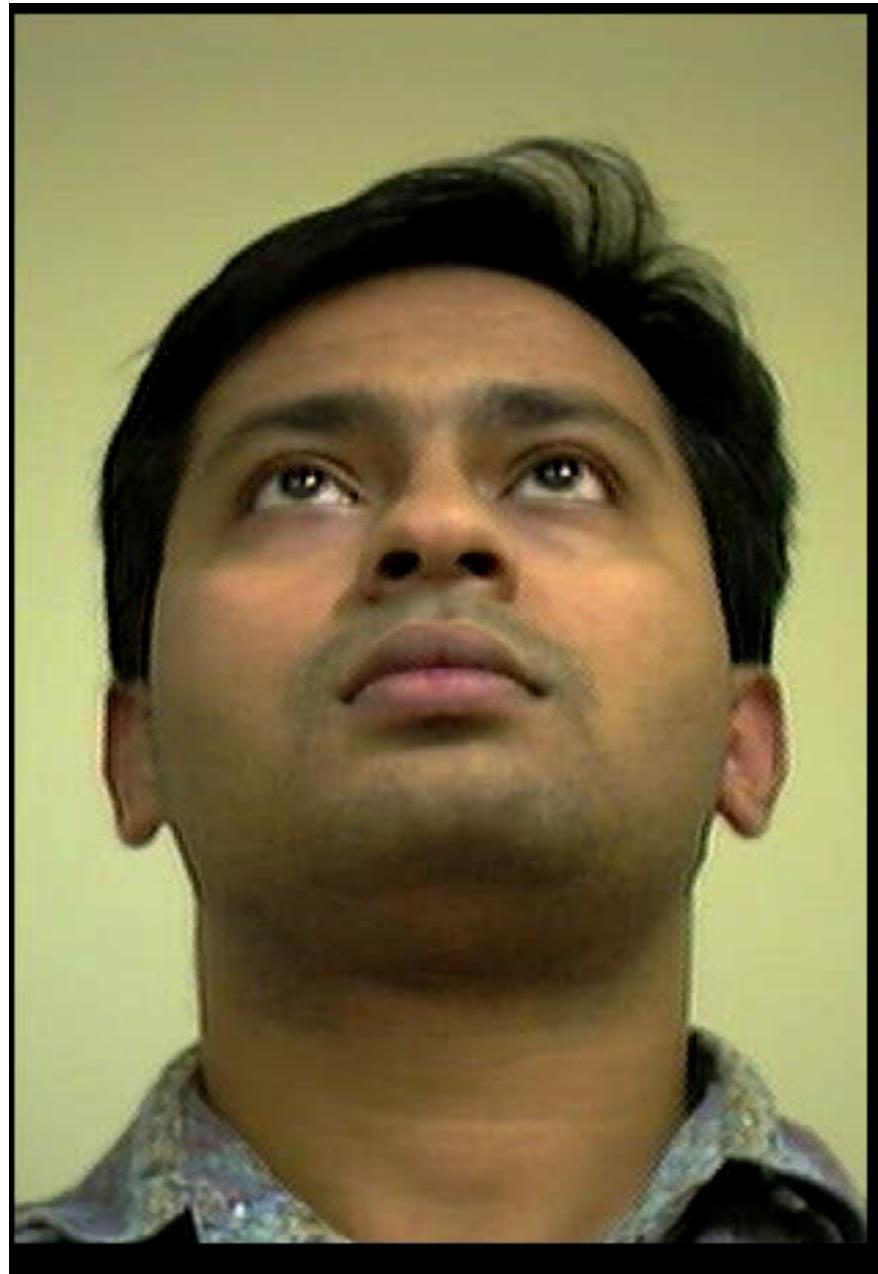








From its reflection!



The Fundamental Matrix Song

