Mark Scheme (Results)

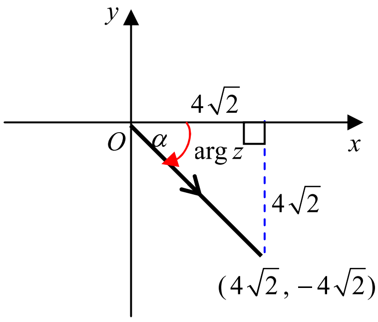
Summer 2009

GCE

GCE Mathematics (6668/01)

June 2009
6668 Further Pure Mathematics FP2 (new)
Mark Scheme

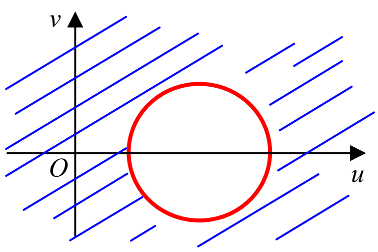
Question Number	Scheme		Marks
Q1 (a)	$\frac{1}{r(r+2)} = \underline{\frac{1}{2r} - \frac{1}{2(r+2)}}$	$\underline{\frac{1}{2r} - \frac{1}{2(r+2)}}$	B1 **aef**
			(1)
(b)	$\sum_{r=1}^{n} \frac{4}{r(r+2)} = \sum_{r=1}^{n} \left(\frac{2}{r} - \frac{2}{r+2}\right)$		
	$= \left(\underline{\frac{2}{1}} - \frac{2}{3}\right) + \left(\underline{\frac{2}{2}} - \frac{2}{4}\right) + \ \ldots\ldots$ $\ldots\ldots\ldots\ldots + \left(\frac{2}{n-1} - \underline{\frac{2}{n+1}}\right) + \left(\frac{2}{n} - \underline{\frac{2}{n+2}}\right)$	List the first two terms and the last two terms	M1
	$= \frac{2}{1} + \frac{2}{2}; - \frac{2}{n+1} - \frac{2}{n+2}$	Includes the first two underlined terms and includes the final two underlined terms.	M1
		$\frac{2}{1} + \frac{2}{2} - \frac{2}{n+1} - \frac{2}{n+2}$	A1
	$= 3 - \frac{2}{n+1} - \frac{2}{n+2}$		
	$= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{(n+1)(n+2)}$ $= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{(n+1)(n+2)}$	Attempt to combine to an at least 3 term fraction to a single fraction and an attempt to take out the brackets from their numerator.	M1
	$= \frac{3n^2 + 5n}{(n+1)(n+2)}$		
	$= \frac{n(3n+5)}{(n+1)(n+2)}$	Correct Result	A1 **cso AG**
			(5)
			[6]

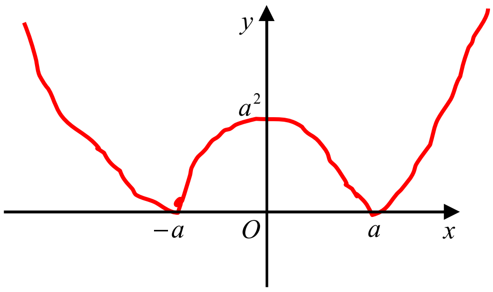
Question Number	Scheme		Marks
Q2 (a)	$z^3 = 4\sqrt{2} - 4\sqrt{2}\,\mathrm{i}\ ,\ -\pi < \theta \leq \pi$		
			
	$r = \sqrt{\left(4\sqrt{2}\right)^2 + \left(-4\sqrt{2}\right)^2} = \sqrt{32+32} = \sqrt{64} = 8$ $\theta = -\tan^{-1}\left(\frac{4\sqrt{2}}{4\sqrt{2}}\right) = -\frac{\pi}{4}$	A valid attempt to find the modulus and argument of $4\sqrt{2} - 4\sqrt{2}\,\mathrm{i}$.	M1
	$z^3 = 8\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$		
	So, $z = (8)^{\frac{1}{3}}\left(\cos\left(\frac{-\frac{\pi}{4}}{3}\right) + \mathrm{i}\sin\left(\frac{-\frac{\pi}{4}}{3}\right)\right)$	Taking the cube root of the modulus and dividing the argument by 3.	M1
	$\Rightarrow z = 2\left(\cos\left(-\frac{\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right)$	$2\left(\cos\left(-\frac{\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right)$	A1
	Also, $z^3 = 8\left(\cos\left(\frac{7\pi}{4}\right) + \mathrm{i}\sin\left(\frac{7\pi}{4}\right)\right)$ or $z^3 = 8\left(\cos\left(-\frac{9\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{9\pi}{4}\right)\right)$	Adding or subtracting 2π to the argument for z^3 in order to find other roots.	M1
	$\Rightarrow z = 2\left(\cos\frac{7\pi}{12} + \mathrm{i}\sin\frac{7\pi}{12}\right)$	Any one of the final two roots	A1
	and $z = 2\left(\cos\left(\frac{-3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{-3\pi}{4}\right)\right)$	Both of the final two roots.	A1
			[6]
	Special Case 1: Award SC: M1M1A1M1A0A0 for ALL three of $2\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)$, $2\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right)$ and $2\left(\cos\left(\frac{-7\pi}{12}\right) + \mathrm{i}\sin\left(\frac{-7\pi}{12}\right)\right)$.		
	Special Case 2: If r is incorrect (and not equal to 8) and candidate states the brackets ( ) correctly then give the first accuracy mark ONLY where this is applicable.		

<table>
<tr><th>Question Number</th><th>Scheme</th><th></th><th>Marks</th></tr>
<tr><td>Q3</td><td>$\sin x \frac{\mathrm{d}y}{\mathrm{d}x} - y\cos x = \sin 2x \sin x$</td><td></td><td></td></tr>
<tr><td></td><td>$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y\cos x}{\sin x} = \frac{\sin 2x \sin x}{\sin x}$</td><td>An attempt to divide every term in the differential equation by $\sin x$.
Can be implied.</td><td>M1</td></tr>
<tr><td></td><td>$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y\cos x}{\sin x} = \sin 2x$</td><td></td><td></td></tr>
<tr><td></td><td>Integrating factor $= \mathrm{e}^{\int -\frac{\cos x}{\sin x}\,\mathrm{d}x} = \mathrm{e}^{-\ln\sin x}$</td><td>$\mathrm{e}^{\int \pm\frac{\cos x}{\sin x}(\mathrm{d}x)}$ or $\mathrm{e}^{\int \pm \text{their } \mathrm{P}(x)\,(\mathrm{d}x)}$
$\mathrm{e}^{-\ln\sin x}$ or $\mathrm{e}^{\ln\operatorname{cosec} x}$</td><td>dM1
A1 aef</td></tr>
<tr><td></td><td>$= \frac{1}{\sin x}$</td><td>$\frac{1}{\sin x}$ or $(\sin x)^{-1}$ or $\operatorname{cosec} x$</td><td>A1 aef</td></tr>
<tr><td></td><td>$\left(\frac{1}{\sin x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y\cos x}{\sin^2 x} = \frac{\sin 2x}{\sin x}$</td><td></td><td></td></tr>
<tr><td></td><td>$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sin x}\right) = \sin 2x \times \frac{1}{\sin x}$</td><td>$\frac{\mathrm{d}}{\mathrm{d}x}(y \times \text{their I.F.}) = \sin 2x \times \text{their I.F}$</td><td>M1</td></tr>
<tr><td></td><td>$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sin x}\right) = 2\cos x$</td><td>$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sin x}\right) = 2\cos x$ or
$\frac{y}{\sin x} = \int 2\cos x\,(\mathrm{d}x)$</td><td>A1</td></tr>
<tr><td></td><td>$\frac{y}{\sin x} = \int 2\cos x\,\mathrm{d}x$</td><td></td><td></td></tr>
<tr><td></td><td>$\frac{y}{\sin x} = 2\sin x + K$</td><td>A credible attempt to integrate the RHS with/without $+K$</td><td>dddM1</td></tr>
<tr><td></td><td>$y = 2\sin^2 x + K\sin x$</td><td>$y = 2\sin^2 x + K\sin x$</td><td>A1 cao
[8]</td></tr>
</table>

Question Number	Scheme		Marks
Q4	$A = \frac{1}{2}\int_0^{2\pi} (a + 3\cos\theta)^2 \, d\theta$	Applies $\frac{1}{2}\int_0^{2\pi} r^2 (\mathrm{d}\theta)$ with correct limits. Ignore $\mathrm{d}\theta$.	B1
	$(a + 3\cos\theta)^2 = a^2 + 6a\cos\theta + 9\cos^2\theta$		
	$= \underline{a^2 + 6a\cos\theta + 9\left(\frac{1+\cos 2\theta}{2}\right)}$	$\cos^2\theta = \frac{\pm 1 \pm \cos 2\theta}{2}$ Correct underlined expression.	M1 A1
	$A = \frac{1}{2}\int_0^{2\pi} \left(a^2 + 6a\cos\theta + \frac{9}{2} + \frac{9}{2}\cos 2\theta\right) d\theta$		
	$= \left(\frac{1}{2}\right)\left[a^2\theta + 6a\sin\theta + \frac{9}{2}\theta + \frac{9}{4}\sin 2\theta\right]_0^{2\pi}$	Integrated expression with at least 3 out of 4 terms of the form $\pm A\theta \pm B\sin\theta \pm C\theta \pm D\sin 2\theta$. Ignore the $\frac{1}{2}$. Ignore limits. $a^2\theta + 6a\sin\theta +$ correct ft integration. Ignore the $\frac{1}{2}$. Ignore limits.	M1* A1 ft
	$= \frac{1}{2}\left[\left(2\pi a^2 + 0 + 9\pi + 0\right) - (0)\right]$		
	$= \pi a^2 + \frac{9\pi}{2}$	$\pi a^2 + \frac{9\pi}{2}$	A1
	Hence, $\pi a^2 + \frac{9\pi}{2} = \frac{107}{2}\pi$	Integrated expression equal to $\frac{107}{2}\pi$.	dM1*
	$a^2 + \frac{9}{2} = \frac{107}{2}$		
	$a^2 = 49$		
	As $a > 0$, $a = 7$	$a = 7$	A1 cso **[8]**
	Some candidates may achieve $a = 7$ from incorrect working. Such candidates will not get full marks		

Question Number	Scheme		Marks
Q5	$y = \sec^2 x = (\sec x)^2$		
(a)	$\dfrac{dy}{dx} = 2(\sec x)^1(\sec x\tan x) = 2\sec^2 x\tan x$	Either $2(\sec x)^1(\sec x\tan x)$ or $2\sec^2 x\tan x$	B1 aef
	Apply product rule: $\left\{\begin{matrix} u = 2\sec^2 x & v = \tan x \\ \dfrac{du}{dx} = 4\sec^2 x\tan x & \dfrac{dv}{dx} = \sec^2 x \end{matrix}\right\}$		
	$\dfrac{d^2y}{dx^2} = 4\sec^2 x\tan^2 x + 2\sec^4 x$	Two terms added with one of either $A\sec^2 x\tan^2 x$ or $B\sec^4 x$ in the correct form.	M1
		Correct differentiation	A1
	$= 4\sec^2 x(\sec^2 x - 1) + 2\sec^4 x$		
	Hence, $\dfrac{d^2y}{dx^2} = 6\sec^4 x - 4\sec^2 x$	Applies $\tan^2 x = \sec^2 x - 1$ leading to the correct result.	A1 **AG**
			(4)
(b)	$y_{\frac{\pi}{4}} = \left(\sqrt{2}\right)^2 = \underline{2}$, $\left(\dfrac{dy}{dx}\right)_{\frac{\pi}{4}} = 2\left(\sqrt{2}\right)^2(1) = \underline{4}$	Both $y_{\frac{\pi}{4}} = \underline{2}$ and $\left(\dfrac{dy}{dx}\right)_{\frac{\pi}{4}} = \underline{4}$	B1
	$\left(\dfrac{d^2y}{dx^2}\right)_{\frac{\pi}{4}} = 6\left(\sqrt{2}\right)^4 - 4\left(\sqrt{2}\right)^2 = 24 - 8 = 16$	Attempts to substitute $x = \frac{\pi}{4}$ into both terms in the expression for $\dfrac{d^2y}{dx^2}$.	M1
	$\dfrac{d^3y}{dx^3} = 24\sec^3 x(\sec x\tan x) - 8\sec x(\sec x\tan x)$	Two terms differentiated with either $24\sec^4 x\tan x$ or $-8\sec^2 x\tan x$ being correct	M1
	$= 24\sec^4 x\tan x - 8\sec^2 x\tan x$		
	$\left(\dfrac{d^2y}{dx^2}\right)_{\frac{\pi}{4}} = 24\left(\sqrt{2}\right)^4(1) - 8\left(\sqrt{2}\right)^2(1) = 96 - 16 = 80$	$\left(\dfrac{d^3y}{dx^3}\right)_{\frac{\pi}{4}} = \underline{80}$	B1
	$\sec x \approx 2 + 4\left(x - \frac{\pi}{4}\right) + \frac{16}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{80}{6}\left(x - \frac{\pi}{4}\right)^3 + \ldots$	Applies a Taylor expansion with at least 3 out of 4 terms ft correctly.	M1
		Correct Taylor series expansion.	A1
			(6)
	$\left\{\sec x \approx 2 + 4\left(x - \frac{\pi}{4}\right) + 8\left(x - \frac{\pi}{4}\right)^2 + \frac{40}{3}\left(x - \frac{\pi}{4}\right)^3 + \ldots\right\}$		
			[10]

Question Number	Scheme		Marks
Q6	$w = \dfrac{z}{z+\mathrm{i}}$, $z = -\mathrm{i}$		
(a)	$w(z+\mathrm{i}) = z \Rightarrow wz + \mathrm{i}w = z \Rightarrow \mathrm{i}w = z - wz$ $\Rightarrow \mathrm{i}w = z(1-w) \Rightarrow z = \dfrac{\mathrm{i}w}{(1-w)}$	Complete method of rearranging to make z the subject. $z = \dfrac{\mathrm{i}w}{(1-w)}$	M1 A1 **aef**
	$\lvert z\rvert = 3 \Rightarrow \left\lvert \dfrac{\mathrm{i}w}{1-w}\right\rvert = 3$	Putting $\lvert z$ in terms of their $w\rvert = 3$	dM1
	$\left\{ \begin{array}{l} \lvert \mathrm{i}w\rvert = 3\lvert 1-w\rvert \Rightarrow \lvert w\rvert = 3\lvert w-1\rvert \Rightarrow \lvert w\rvert^2 = 9\lvert w-1\rvert^2 \\ \Rightarrow \lvert u+\mathrm{i}v\rvert^2 = 9\lvert u+\mathrm{i}v-1\rvert^2 \end{array} \right\}$		
	$\Rightarrow u^2 + v^2 = 9\left[(u-1)^2 + v^2\right]$	Applies $w = u + \mathrm{i}v$, and uses Pythagoras correctly to get an equation in terms of u and v without any i's. Correct equation.	ddM1 A1
	$\left\{ \begin{array}{l} \Rightarrow u^2 + v^2 = 9u^2 - 18u + 9 + 9v^2 \\ \Rightarrow 0 = 8u^2 - 18u + 8v^2 + 9 \end{array} \right\}$		
	$\Rightarrow 0 = u^2 - \frac{9}{4}u + v^2 + \frac{9}{8}$	Simplifies down to $u^2 + v^2 \pm \alpha u \pm \beta v \pm \delta = 0$.	dddM1
	$\Rightarrow \left(u - \frac{9}{8}\right)^2 - \frac{81}{64} + v^2 + \frac{9}{8} = 0$		
	$\Rightarrow \left(u - \frac{9}{8}\right)^2 + v^2 = \frac{9}{64}$		
	{Circle} centre $\left(\frac{9}{8}, 0\right)$, radius $\frac{3}{8}$	One of centre or radius correct. Both centre and radius correct.	A1 A1 (8)
(b)		Circle indicated on the Argand diagram in the correct position in follow through quadrants. Ignore plotted coordinates. Region outside a circle indicated only.	B1ft B1 (2)
			[10]

Question Number	Scheme		Marks
Q7	$y = \lvert x^2 - a^2 \rvert,\ a > 1$		
(a)		Correct Shape. Ignore cusps. Correct coordinates.	B1 B1
			(2)
(b)	$\lvert x^2 - a^2 \rvert = a^2 - x\ ,\ a > 1$		
	$\{\lvert x \rvert > a\},\quad x^2 - a^2 = a^2 - x$	$x^2 - a^2 = a^2 - x$	M1 aef
	$\Rightarrow x^2 + x - 2a^2 = 0$		
	$\Rightarrow x = \dfrac{-1 \pm \sqrt{1 - 4(1)(-2a^2)}}{2}$	Applies the quadratic formula or completes the square in order to find the roots.	M1
	$\Rightarrow x = \dfrac{-1 \pm \sqrt{1 + 8a^2}}{2}$	Both correct "simplified down" solutions.	A1
	$\{\lvert x \rvert < a\},\quad -x^2 + a^2 = a^2 - x$	$-x^2 + a^2 = a^2 - x$ or $x^2 - a^2 = x - a^2$	M1 aef
	$\{\Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0\}$		
	$\Rightarrow x = 0\ ,1$	$x = 0$ $x = 1$	B1 A1
			(6)
(c)	$\lvert x^2 - a^2 \rvert > a^2 - x\ ,\ a > 1$		
	$x < \dfrac{-1 - \sqrt{1 + 8a^2}}{2}\quad \{\text{or}\}\quad x > \dfrac{-1 + \sqrt{1 + 8a^2}}{2}$	x is less than their least value x is greater than their maximum value	B1 ft B1 ft
	$\{\text{or}\}\quad 0 < x < 1$	For $\{\lvert x \rvert < a\}$, Lowest $< x <$ Highest $0 < x < 1$	M1 A1
			(4)
			[12]

Question Number	Scheme		Marks
Q8	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + 6x = 2\mathrm{e}^{-t}$, $x = 0, \frac{\mathrm{d}x}{\mathrm{d}t} = 2$ at $t = 0$.		
(a)	AE, $m^2 + 5m + 6 = 0 \Rightarrow (m+3)(m+2) = 0$ $\Rightarrow m = -3, -2.$		
	So, $x_{\mathrm{CF}} = A\mathrm{e}^{-3t} + B\mathrm{e}^{-2t}$	$A\mathrm{e}^{m_1 t} + B\mathrm{e}^{m_2 t}$, where $m_1 \neq m_2$. $A\mathrm{e}^{-3t} + B\mathrm{e}^{-2t}$	M1 A1
	$\left\{ x = k\mathrm{e}^{-t} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = -k\mathrm{e}^{-t} \Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = k\mathrm{e}^{-t} \right\}$		
	$\Rightarrow k\mathrm{e}^{-t} + 5(-k\mathrm{e}^{-t}) + 6k\mathrm{e}^{-t} = 2\mathrm{e}^{-t} \Rightarrow 2k\mathrm{e}^{-t} = 2\mathrm{e}^{-t}$ $\Rightarrow k = 1$	Substitutes $k\mathrm{e}^{-t}$ into the differential equation given in the question. Finds $k = 1$.	M1 A1
	$\left\{ \text{So, } x_{\mathrm{PI}} = \mathrm{e}^{-t} \right\}$		
	So, $x = A\mathrm{e}^{-3t} + B\mathrm{e}^{-2t} + \mathrm{e}^{-t}$	their x_{CF} + their x_{PI}	M1*
	$\frac{\mathrm{d}x}{\mathrm{d}t} = = -3A\mathrm{e}^{-3t} - 2B\mathrm{e}^{-2t} - \mathrm{e}^{-t}$	Finds $\frac{\mathrm{d}x}{\mathrm{d}t}$ by differentiating their x_{CF} and their x_{PI}	dM1*
	$t = 0, x = 0 \Rightarrow \quad 0 = A + B + 1$ $t = 0, \frac{\mathrm{d}x}{\mathrm{d}t} = 2 \Rightarrow \quad 2 = -3A - 2B - 1$	Applies $t = 0$, $x = 0$ to x and $t = 0, \frac{\mathrm{d}x}{\mathrm{d}t} = 2$ to $\frac{\mathrm{d}x}{\mathrm{d}t}$ to form simultaneous equations.	ddM1*
	$\left\{ \begin{array}{l} 2A + 2B = -2 \\ -3A - 2B = 3 \end{array} \right\}$		
	$\Rightarrow A = -1, B = 0$		
	So, $x = -\mathrm{e}^{-3t} + \mathrm{e}^{-t}$	$x = -\mathrm{e}^{-3t} + \mathrm{e}^{-t}$	A1 **cao** (8)

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<tr><td></td><td>$x = -e^{-3t} + e^{-t}$</td><td></td><td></td></tr>
<tr><td>(b)</td><td>$\frac{dx}{dt} = 3e^{-3t} - e^{-t} = 0$</td><td>Differentiates their x to give $\frac{dx}{dt}$ and puts $\frac{dx}{dt}$ equal to 0.</td><td>M1</td></tr>
<tr><td></td><td>$3 - e^{2t} = 0$
$\Rightarrow t = \frac{1}{2}\ln 3$</td><td>A credible attempt to solve.
$t = \frac{1}{2}\ln 3$ or $t = \ln\sqrt{3}$ or awrt 0.55</td><td>dM1*
A1</td></tr>
<tr><td></td><td>So, $x = -e^{-\frac{3}{2}\ln 3} + e^{-\frac{1}{2}\ln 3} = -e^{\ln 3^{-\frac{3}{2}}} + e^{\ln 3^{-\frac{1}{2}}}$
$x = -3^{-\frac{3}{2}} + 3^{-\frac{1}{2}}$</td><td>Substitutes their t back into x and an attempt to eliminate out the ln's.</td><td>ddM1</td></tr>
<tr><td></td><td>$= -\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}} = \underline{\frac{2\sqrt{3}}{9}}$</td><td>uses exact values to give $\frac{2\sqrt{3}}{9}$</td><td>A1 **AG**</td></tr>
<tr><td></td><td>$\frac{d^2x}{dt^2} = -9e^{-3t} + e^{-t}$
At $t = \frac{1}{2}\ln 3$, $\frac{d^2x}{dt^2} = -9e^{-\frac{3}{2}\ln 3} + e^{-\frac{1}{2}\ln 3}$</td><td>Finds $\frac{d^2x}{dt^2}$ and substitutes their t into $\frac{d^2x}{dt^2}$</td><td>dM1*</td></tr>
<tr><td></td><td>$= -9(3)^{-\frac{3}{2}} + 3^{-\frac{1}{2}} = -\frac{9}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = -\frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}}$</td><td></td><td></td></tr>
<tr><td></td><td>As $\frac{d^2x}{dt^2} = -\frac{9}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = \left\{-\frac{2}{\sqrt{3}}\right\} < 0$
then x is maximum.</td><td>$-\frac{9}{3\sqrt{3}} + \frac{1}{\sqrt{3}} < 0$ and maximum conclusion.</td><td>A1
(7)
[15]</td></tr>
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