Core 3 2015:

1). "Given that $\tan\theta = p$, $p \neq \pm 1$, use standard trigonometric identiti
to find in terms of p;"

a). <u>tan 2θ</u>: Identity: $\tan(2\theta) \equiv \dfrac{2\tan(\theta)}{1-\tan^2(\theta)}$

- Sob in for $p = \tan\theta$: $\dfrac{2p}{1-p^2}$

b). <u>cos θ</u>: Identity: $\sec^2\theta = 1 + \tan^2\theta$ ← * identity required involving tan, if used $\sin^2 + \cos^2$ or $\frac{\sin}{\cos}$ introducing a further trig function.

$$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta.$$

* cross-multiply:

$$\cos^2\theta = \frac{1}{1+\tan^2\theta}$$

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}}$$

$$\cos\theta = \frac{1}{\sqrt{1+p^2}}.$$

c). cot (θ−45) Identity: $\tan((\theta)-45) = \dfrac{\tan(\theta)-\tan(45)}{1+\tan(\theta)\tan(45)}$

<u>Note</u>: $\tan(45) = 1$

So, we have

$$\tan(\theta-45) \equiv \frac{\tan(\theta)-1}{1+\tan\theta}.$$

So $\cot(\theta-45)$

$$= \frac{1}{\tan(\theta-45)} = \frac{1}{\frac{\tan(\theta)-1}{1+\tan(\theta)}}$$

$$= \frac{1+\tan\theta}{1-\tan\theta} = \frac{p+1}{p-1}$$

2). "Given that $f(x) = 2e^x - 5$;
Sketch:"

i). $y = f(x)$

crosses x-axis when $y=0$ $\therefore 2e^x - 5 = 0$

$$2e^x = 5$$
$$e^x = 5/2$$
$$\ln e^x = \ln 5/2$$
$$x \ln e = \ln 5/2$$
$$\underline{\underline{x = \ln 5/2}}$$

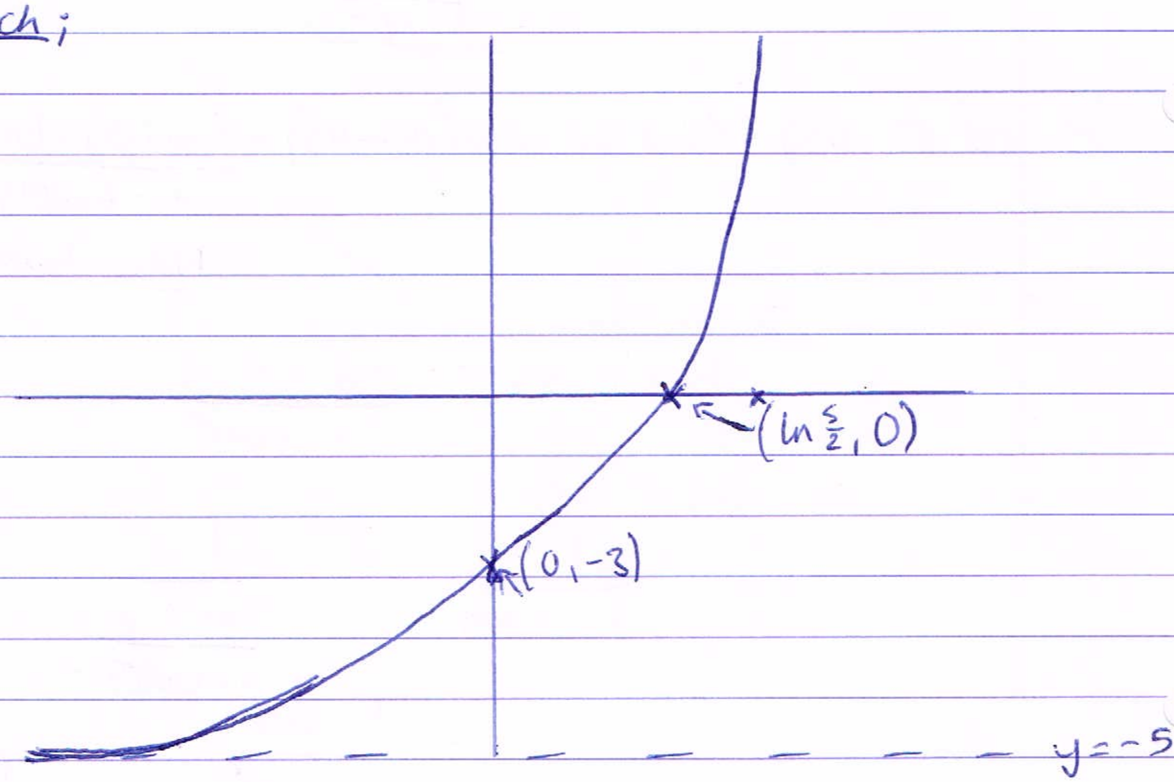
crosses y-axis when $x=0$ $\therefore y = 2e^0 - 5$

$$y = 2 - 5$$
$$\underline{\underline{y = -3}}$$

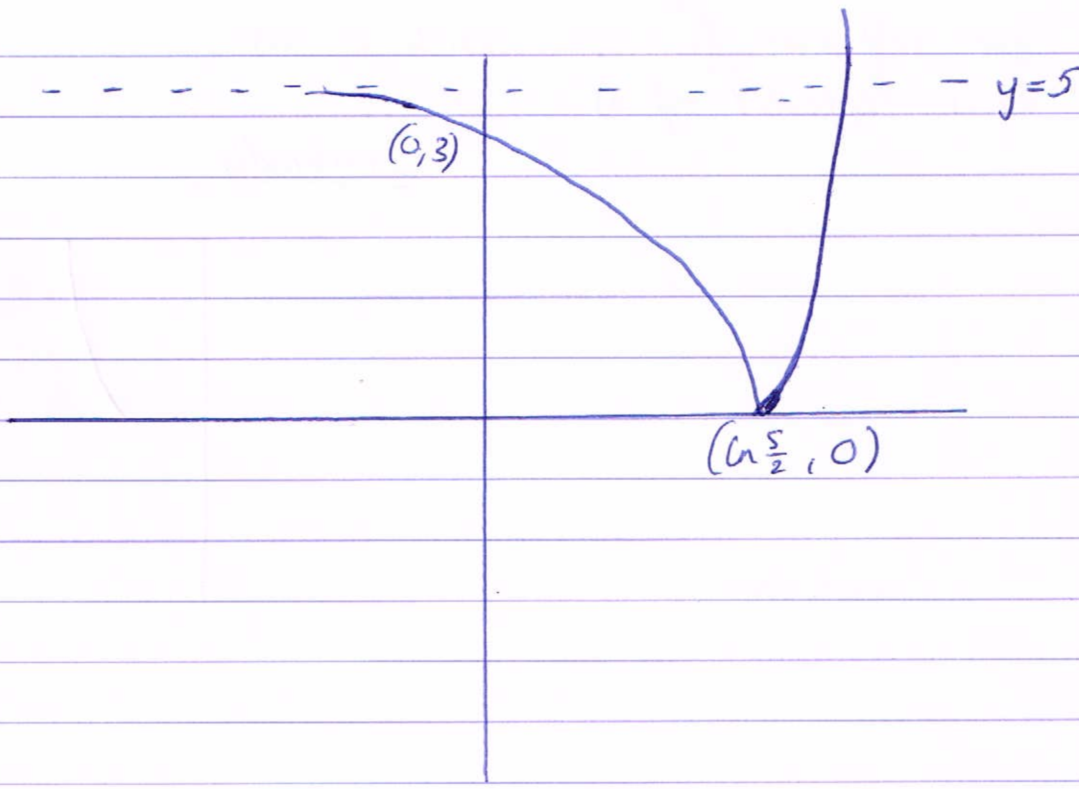
$f(x)$ represents a transformation of $y = e^x$, since as x varies, this output in y always stays +ive, there is an asymptope at $y=0$

but, the transformation lowers the graph by 5 units, hence new asymptope of $y = -5$.

Sketch;



ii). $y = |f(x)|$: mod function gives us the +ive value / absolute value of our output. So for all $f(x) < 0$, we want $-f(x)$ and so we reflect these points in the x-axis, along with our asymptope.



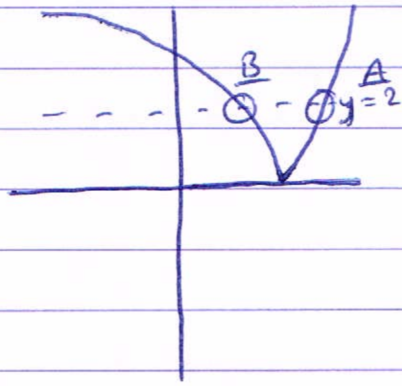
b). "Deduce a set of values of x for which $f(x) = |f(x)|$

From our sketches: it becomes apparent that the +ive output of $f(x)$ remains unchanged in $|f(x)|$.

$\therefore$ this is true for all +ive values of $f(x)$ that is, $x \geqslant \ln\frac{5}{2}$.

c). "Find exact solutions of $|f(x)| = 2$.

Graphically: $y = |f(x)|$



Note - solution A is also a solution of $y = f(x) = 2$.
- Solution B is a solution of $y = -f(x) = 2$
Since reflected in x-axis.

Algebraically:

A: $2e^x - 5 = 2$

$2e^x = 7$

$e^x = 7/2$ $\quad$ $\ln e^x = \ln 7/2$

$x \ln e = \ln 7/2$

$\underline{x = \ln 7/2}$

B: $-(2e^x - 5) = 2$

$-2e^x + 5 = 2$

$-2e^x = -3$

$e^x = 3/2$ ... $\underline{x = \ln 3/2}$

Alt: since is a reflection you can solve it for $f(x) = -2$.

ie. $2e^x - 5 = -2$

$2e^x = 3$

$e^x = 3/2$ $\quad$ $\underline{x = \ln 3/2}$

3). $g(\theta) = 4\cos(2\theta) + 2\sin(2\theta)$

"Given that $g(\theta) = R\cos(2\theta - \alpha)$ $R>0,\ 0<\alpha<90$;

a). "Find value of R and α to 2 d.p."

we have, $4\cos(2\theta) + 2\sin(2\theta) = R\cos(2\theta - \alpha)$ ← by $\cos(A-B)$ identity

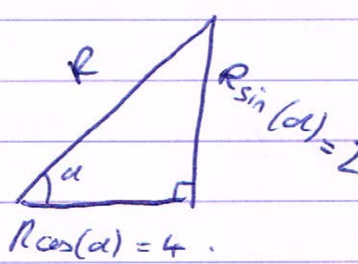
$$4\cos(2\theta) + 2\sin(2\theta) = R\cos(2\theta)\cos(\alpha) + R\sin(2\theta)\sin(\alpha)$$

then comparing LHS with RHS: for α and R

we get;

$$4\cos(2\theta) = R\cos(2\theta)\cos(\alpha)$$
$$\Rightarrow 4 = R\cos(\alpha) \quad ①$$
$$2\sin(2\theta) = R\sin(2\theta)\sin(\alpha)$$
$$2 = R\sin(\alpha) \quad ②$$

representing on a triangle:



then $R^2 = 4^2 + 2^2$

$R = \sqrt{20}$ $\quad R = 2\sqrt{5}$

and ② ÷ ① gives: $\frac{1}{2} = \tan(\alpha)$

$\alpha = 26.57°$

$\therefore = 2\sqrt{5}\cos(2\theta - 26.57)$.

b). "Hence solve for $-90° < \theta < 90°$,
$4\cos 2\theta + 2\sin 2\theta = 1$."

We can now say from a). ;

intervals :

$$2\sqrt{5}\cos(2\theta - 26.57) = 1$$
$$\cos(2\theta - 26.57) = \frac{1}{2\sqrt{5}}$$

$[-90, 90]$ = required interval.

then;

$(2\theta - 26.57) = 77.08, -77.08$ $[-206.57, 153.43]$

$(2\theta) = 103.65, -50.51$ $[-180, 180]$

$\theta = 51.8, -25.3$ $[-90, 90]$

c). "Given k is a constant and $g(\theta) = k$ has no solutions - state possible range of values of k":

By inspection: $\cos(2\theta - 26.57) = \frac{k}{2\sqrt{5}}$

Since for b). we solved for $k = 1$.

but $\cos(x)$ takes values in range
$-1 \leq \cos(x) \leq 1$.

So
$$-1 \leq \frac{k}{2\sqrt{5}} \leq 1$$

means $2\sqrt{5} \leq k \leq 2\sqrt{5}$

4). "Water heated in kettle. temp = θ°C, of water, t seconds is modelled by : $\theta = 120 - 100e^{-\lambda t}$."

a). State value of θ when t=0 :

$$\theta = 120 - 100e^{0}$$
$$\theta = 120 - 100(1)$$
$$\underline{\theta = 20}$$

b). "Given temp water = 70°C when t = 40, find lamba in form $\frac{\ln a}{b}$". $(a, b \in \mathbb{Z}^{+})$.

We get: $70 = 120 - 100e^{-\lambda(40)}$

$$-50 = -100e^{-40\lambda}$$
$$-1 = -2e^{-40\lambda}$$
$$1 = 2e^{-40\lambda}$$

* dont simplify to ½ as you will take ln(½) and need a and b to be integers.

$$\ln(1) = \ln(2e^{-40\lambda})$$
$$\ln(1) = \ln(2) + \ln(e^{-40\lambda})$$
$$\ln(1) = \ln(2) + -40\lambda \ln e$$
$$\ln(1) = \ln(2) - 40\lambda$$
$$40\lambda = \ln(2) - \ln(1)$$
$$\lambda = \frac{\ln(2) - \ln(1)}{40}$$

$$\lambda = \frac{\ln\left(\frac{2}{1}\right)}{40}$$
$$\lambda = \frac{\ln(2)}{40}.$$

c). "When $t=T$, $\theta=100$ find t to nearest integer."

$$\therefore\ 100 = 120 - 100e^{-\frac{\ln 2}{40}t}$$

$$-20 = -100e^{-\frac{\ln 2}{40}t}$$

$$1 = 5e^{-\frac{\ln 2}{40}t}$$

$$\frac{1}{5} = e^{-\frac{\ln 2}{40}t}$$

$$\ln\left(\tfrac{1}{5}\right) = \ln e^{-\frac{\ln 2}{40}t}$$

$$\ln\left(\tfrac{1}{5}\right) = -\frac{\ln 2}{40}t \ln e$$

$$\ln\left(\tfrac{1}{5}\right) = -\frac{\ln 2}{40}t \qquad t = -\frac{\ln\left(\frac{1}{5}\right)}{\frac{\ln 2}{40}}$$

$$\underline{t = 93 \text{ secs}}$$

5). "$P(p, \frac{\pi}{2})$ lies on $x = (4y - \sin 2y)^2$."

a). find p:

$$\therefore\ f(y) = (4y - \sin 2y)^2$$

$$f\left(\tfrac{\pi}{2}\right) = (2\pi - \sin\pi)^2$$

$$= (2\pi)^2 = \underline{4\pi^2}$$

b). "Tangent to curve at P meets y-axis at A. find A:"

$$\frac{dx}{dy} \text{ of } (4y - \sin 2y)^2$$

* $f(y)$ is a function of a function of x i.e. $x = (u)^2$

$\therefore$ Chain rule $\qquad u = 4y - \sin 2y$.

$$\frac{dx}{dy} = 2u \cdot \frac{du}{dx}$$

$\frac{d}{dx} \sin = \cos$ + chain rule as $\sin(u)$ when u function of y.

$$\frac{dx}{dy} = 2(4y - \sin 2y) \cdot (4 - 2\cos 2y)$$

$$\left.\frac{dx}{dy}\right|_{y=\frac{\pi}{2}} = \left(8\left(\tfrac{\pi}{2}\right) - 2\sin(\pi)\right) \cdot (4 - 2\cos\pi)$$

$$= (4\pi - 0) \cdot (4 - (-2)) = 4\pi \cdot 6$$

$$\left.\frac{dx}{dy}\right|_{y=\frac{\pi}{2}} = \underline{24\pi} \quad \ldots$$

but gradient of tangent is $\frac{dy}{dx}$.

$\therefore$ use result that $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

$$\therefore \left.\frac{dy}{dx}\right|_{y=\frac{\pi}{2}} = \frac{1}{24\pi}$$

Using $y - y_1 = m(x - x_1)$;

$$y - \frac{\pi}{2} = \frac{1}{24\pi}(x - 4\pi^2)$$

<u>at A</u> : $(x=0)$.

$$\therefore y - \frac{\pi}{2} = \frac{1}{24\pi}(-4\pi^2)$$

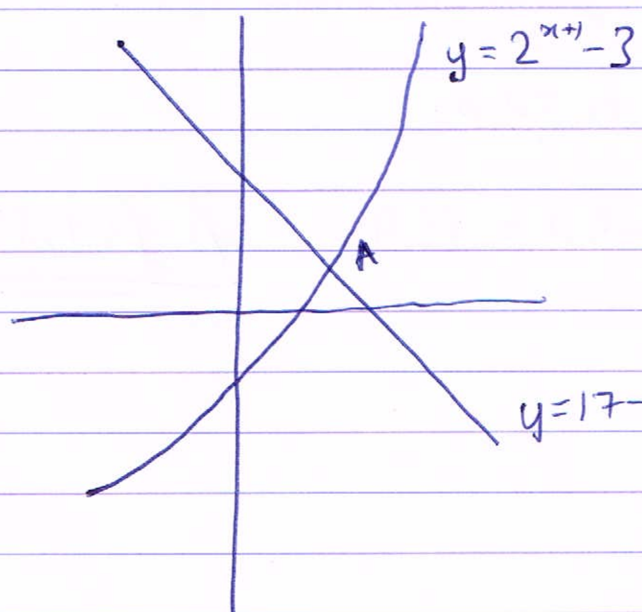
$$24\pi y - 12\pi^2 = -4\pi^2$$

$$24\pi y = 8\pi^2$$

$$3y = \pi$$

$$y = \pi/3 \quad \therefore \underline{\underline{A\left(0, \frac{\pi}{3}\right)}}$$

6).



a). Show that x coordinate of A satisfies $x = \frac{\ln(20-x)}{\ln 2} - 1$.

...

By intersection: $17 - x = 2^{x+1} - 3$.

$2^{x+1} = 20 - x$.

$\ln(2^{x+1}) = \ln(20-x)$.

$(x+1)\ln 2 = \ln(20-x)$

$(x+1) = \dfrac{\ln(20-x)}{\ln(2)}$

$x = \dfrac{\ln(20-x)}{\ln(2)} - 1$, as required.

b). "Use $x_{n+1} = \dfrac{\ln(20-x_n)}{\ln(2)} - 1$ to generate x_1, x_2, x_3 to 3 d.p". with $x_0 = 3$.

$x_1 = \dfrac{\ln(17)}{\ln(2)} - 1 = 3.087$.

$x_2 = \ldots\ldots = 3.080$.

$x_3 = \ldots\ldots = 3.081$.

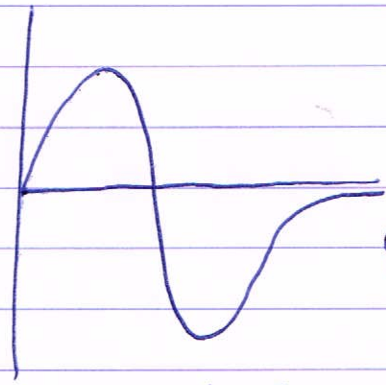
c). "Use part b). to deduce coordinates of A".

then x is 3.1 to 2d.p.

for $f(x) = 17 - x$

$f(3.1) = 17 - 3.1 = 13.9$ A: $(3.1, 13.9)$

7).



$g(x) = x^2(1-x)e^{-2x} \quad , x \geq 0 .$

a). Show $g'(x) = f(x)e^{-2x}$ where $f(x)$ is a cubic to be found."

$$g(x) = (x^2 - x^3) \cdot e^{-2x}$$

this is a product

$\therefore g'(x) = e^{-2x}(2x - 3x^2) + (x^2 - x^3)(-2e^{-2x})$ by product rule.

$$g'(x) = e^{-2x}(2x - 3x^2) + e^{-2x}(-2x^2 + 2x^3)$$

$$g'(x) = (2x^3 - 5x^2 + 2x)\, e^{-2x}$$

where $f(x) = 2x^3 - 5x^2 + 2x$

b). "Hence find range of g".

"We see that curve has 4 Turning points as $g'(x)$ is cubic $\Rightarrow g(x)$ is quartic,

but $g(x)$ takes values in range of max/min shown on graph, we see max before min so, no need to find $\frac{d^2y}{dx^2}$.

$g'(x) = 0$ when $(2x^3 - 5x^2 + 2x)e^{-2x} = 0$

then $e^{-2x} = 0$ no solution

$$2x^3 - 5x^2 + 2x = 0$$

$$x(2x^2 - 5x + 2) = 0$$

$$x(2x-1)(x-2) = 0$$

$\therefore x = 1/2, 2$ (and $x = 0$).

$g\left(\frac{1}{2}\right) = \frac{1}{8e}$ $\therefore$ $-\frac{4}{e^4} \le g(x) \le \frac{1}{8e}$.

$g(2) = -\frac{4}{e^4}$

c). "State why function $g^{-1}(x)$ can't exist."

- $g(x)$ is many to one
  $\therefore$ $g^{-1}(x)$ is one to many, as inverse functions describe the relation from the output to the input of the original function.
  $\therefore$ $g^{-1}(x)$ isn't a function as many to one.

8

a). "Prove: $\sec 2A + \tan 2A \equiv \frac{\cos A + \sin A}{\cos A - \sin A}$.

we have: $\frac{1}{\cos 2A} + \tan 2A$

$= \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A}$

$= \frac{1 + \sin 2A}{\cos 2A}$ * double angle identities.

$= \frac{1 + 2\sin A\cos A}{\cos^2 A - \sin^2 A}$ * $\sin^2 A + \cos^2 A = 1$.

$= \frac{(\sin^2 A + \cos^2 A) + 2\sin A\cos A}{\cos^2 A - \sin^2 A}$

$= \frac{(\cos A + \sin A)(\cos A + \sin A)}{(\cos A + \sin A)(\cos A - \sin A)}$ * diff. of 2 squares.

$$= \frac{(\cos A + \sin A)}{(\cos A - \sin A)}$$ as required.

b). Hence solve for $0 \leq \theta < 2\pi$;

$\sec 2\theta + \tan 2\theta = \frac{1}{2}$ to 3 d.p.

$$\therefore \frac{\cos A + \sin A}{\cos A - \sin A} = \frac{1}{2}$$

$$\therefore 2(\cos A + \sin A) = \cos A - \sin A$$

$$2\cos A + 2\sin A = \cos A - \sin A$$

$$(\div \cos A) \quad = 2 + 2\tan A = 1 - \tan A$$

$$3\tan A = -1$$

$$\tan A = -\frac{1}{3}$$

$$\Rightarrow A = 2.820, \ (\pi + 2.820)$$

$$\underline{\underline{A = 2.820, 5.961}}$$

9 "Given k negative .... $f(x) = 2 - \frac{(x-5k)(x-k)}{x^2-3kx+2k^2}$.

a) Show that $f(x) = \frac{x+k}{x-2k}$

$$f(x) = 2 - \frac{(x-5k)(x-k)}{(x-k)(x-2k)}$$

$$f(x) = \frac{2(x-2k)}{(x-2k)} - \frac{(x-5k)}{(x-2k)} = \frac{2x-4k-x+5k}{(x-2k)}$$

$$= \frac{x+k}{x-2k}$$

b). "find $f'(x)$:"

by quotient rule:

$$f'(x) = \frac{(x-2k)(1)-(x+k)(1)}{(x-2k)^2}$$

$$= \frac{(x-2k)-(x+k)}{(x-2k)^2} = \frac{-3k}{(x-2k)^2}$$

c). State whether $f(x)$ increasing / decreasing.

$f'(x) = \frac{-3k}{(x-2k)^2}$ $\quad (x-2k)^2$ always +ive.

Since k -ive

then $-3k$ also +ive.

$\therefore f'(x)$ +ive for all x.

if gradient always +ive, then increasing.

$\Rightarrow$ Every subsequent increase in input (x) leads to an increase in output $(f(x))$ $\therefore$ the function is strictly increasing.