Université d'Orléans National University Ho Chi Minh City
PUF Master 2015-2016

NATURAL FLOW MODELS

Numerical Schemes for Shallow Water Equations

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PUF Master **2015-2016**

Practical test
Numerical schemes for Shallow Water Equations

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We shall present the answer to question 4 first of all since question 4 looks like independent with the others.

Question 4[1]:

Consider the Shallow water equations:

$$\begin{cases} \partial_t h + \partial_x(hu) = 0 \\ \partial_t(hu) + \partial_x(hu^2 + gh^2/2) = -\tau hu - gh\partial_x z \end{cases}$$

and a topography of the form:

$$z = h_0\left(\frac{x^2}{a^2} - 1\right) \qquad \text{(B.18)}$$

(For the simplification, we shall omit the term $\frac{L}{2}$ in section 4.2.1 of [1])

The topography is independent of time, so we can write the conservation equation of the mass in the form:

$$\partial_t(h+z) + \partial_x(hu) = 0 \qquad \text{(B.19)}$$

and the equation of conservation of momentum gives us

$$\partial_t u + u\partial_x u + \tau u + g\partial_x(h+z) = 0 \qquad \text{(B.20)}$$

We search speed as a function of time:

$$u = u_0(t) \qquad \text{(B.21)}$$

Substitute (B.21) into (B.20) we get

$$\partial_t u_0(t) + \tau u_0 + g\partial_x(h+z) = 0$$

at any time t, the free surface is planar

[1] Based on Olivier Delestre's PhD thesis [2]

$h + z = F_0(t) + F_1(t, x) = F_0(t) - \frac{x}{g}(\partial_t u_0 + \tau u_0)$ (B.22)

using (B.18) and (B.22) in (B.19) we get:

$$\partial_t \left[F_0(t) - \frac{x}{g}(\partial_t u_0 + \tau u_0)\right] + \partial_x \left[\left(F_0(t) - \frac{x}{g}(\partial_t u_0 + \tau u_0) + h_0\left(1 - \frac{x^2}{a^2}\right)\right) u_0\right] = 0$$

Or

$$\partial_t F_0(t) - \frac{x}{g}(\partial_{t^2} u_0 + \tau \partial_t u_0) - \left(\frac{1}{g}(\partial_t u_0 + \tau u_0) + \frac{2h_0 x}{a^2}\right) u_0 = 0$$

by identifying the powers of x, we get two equations:

$\partial_t F_0(t) - \frac{u_0}{g}\partial_t u_0 - \frac{\tau}{g} u_0^2 = 0$ (B.23)

And

$\partial_{t^2} u_0(t) + \tau \partial_t u_0(t) + \frac{2h_0 x}{a^2} u_0(t) = 0$ (B.24)

The characteristic equation of (B.24) is

$\lambda^2 + \tau\lambda + \frac{2h_0 x}{a^2} = 0$ (B.25)

whose roots are

$$\lambda_1 = \frac{-\tau - \sqrt{\tau^2 - 4\omega^2}}{2}$$

$$\lambda_2 = \frac{-\tau + \sqrt{\tau^2 - 4\omega^2}}{2}$$

With

$$\omega = \sqrt{2gh_0/a^2}$$

Or

$\lambda_1 = \frac{-\tau - \sqrt{\tau^2 - p^2}}{2}$ and $\lambda_2 = \frac{-\tau + \sqrt{\tau^2 - p^2}}{2}$ (B.26)

Where

$$p = 2\omega$$

Three cases are possible:

$-0 \leq \tau < p$

$-\tau = p$

$-\tau > p$

For the first case, the two roots (B.26) of the characteristic equation (B.25) are complex conjugate, so the differential equation (B.24) has the following solution

$$u_0(t) = [A\cos(st) + B\sin(st)]e^{-\tau t/2}$$

Where A and B are constants and

$$s = \frac{\sqrt{p^2 - \tau^2}}{2} = \frac{\sqrt{4\omega^2 - \tau^2}}{2}$$

We assume zero initial velocity, so

$$u_0(t) = B\sin(st)\, e^{-\tau t/2} \tag{B.27}$$

Substitute (B.27) into (B.23) one gets

$$\partial_t F_0(t) = \frac{B^2}{g}\sin(st)\, e^{-\tau t/2}\left[s\cos(st) + \frac{\tau}{2}\sin(st)\right]$$

After integrating with respect to t and various calculations, we get

$$F_0(t) = -\frac{B^2}{g}\frac{e^{-\tau t}}{4s^2+\tau^2}\left(\sqrt{2}s\cos(st) + \frac{\tau}{\sqrt{2}}\sin(st)\right)^2 + C \tag{B.28}$$

Where C is an integration constant to be determined.

We have

$$F_1(t,x) = -\frac{x}{g}[\partial_t u_0 + \tau u_0]$$

Or

$$F_1(t,x) = -\frac{x}{g}Be^{-\tau t/2}\left(s\cos(st) - \frac{\tau}{2}\sin(st) + \tau\sin(st)\right) \tag{B.29}$$

Substitute (B.28) and (B.29) into (B.22), we get the following equation on the free surface:

$$h+z=-\frac{B^2}{g}\frac{e^{-\tau t}}{4s^2+\tau^2}\left(\sqrt{2}s\cos(st)+\frac{\tau}{\sqrt{2}}\sin(st)\right)^2-\frac{x}{g}Be^{-\frac{\tau t}{2}}\left(s\cos(st)+\frac{\tau}{2}\sin(st)\right)+C \tag{B.30}$$

$$\lim_{t\to+\infty} h+z=C$$

For $\tau > 0$, the free surface tends to a horizontal area with a rating C. We take $C = 0$. With (B.18) and (B.30), the equation of the water level is therefore written

$$h=-\frac{B^2}{g}\frac{e^{-\tau t}}{4s^2+\tau^2}\left(\sqrt{2}s\cos(st)+\frac{\tau}{\sqrt{2}}\sin(st)\right)^2-\frac{x}{g}Be^{-\frac{\tau t}{2}}\left(s\cos(st)+\frac{\tau}{2}\sin(st)\right)+h_0\left(1-\frac{x^2}{a^2}\right)$$

We have

$$4s^2+\tau^2=p^2=\frac{8gh_0}{a^2}$$

After computations we have

$$h=-\frac{h_0}{a^2}\left(x^2+\frac{Ba^2}{gh_0}e^{-\frac{\tau t}{2}}\left(s\cos(st)+\frac{\tau}{2}\sin(st)\right)x+\left(\frac{Ba^2}{2gh_0}\right)^2e^{-\tau t}\left(s\cos(st)+\frac{\tau}{2}\sin(st)\right)^2-a^2\right)$$

We seek the position of interfaces dry / wet over time, which comes to solve a function of x, the equation $h = 0$. Therefore

$$x^2+\frac{Ba^2}{gh_0}e^{-\frac{\tau t}{2}}\left(s\cos(st)+\frac{\tau}{2}\sin(st)\right)x+\left(\frac{Ba^2}{2gh_0}\right)^2e^{-\tau t}\left(s\cos(st)+\frac{\tau}{2}\sin(st)\right)^2-a^2=0$$

Or

$$\left(x+\frac{a^2e^{-\tau t/2}}{2gh_0}\left(Bs\cos(st)+\frac{\tau B}{2}\sin(st)\right)\right)^2-a^2=0$$

Both roots of this equation are

$$x_1=-\frac{a^2e^{-\tau t/2}}{2gh_0}\left(Bs\cos(st)+\frac{\tau B}{2}\sin(st)\right)-a$$

And

$$x_2=-\frac{a^2e^{-\tau t/2}}{2gh_0}\left(Bs\cos(st)+\frac{\tau B}{2}\sin(st)\right)+a$$

In summary, we thus have

$$h(t,x) = \begin{cases} -\dfrac{h_0}{a^2}\left(\left(x + \dfrac{a^2 e^{-\tau t/2}}{2gh_0}\left(Bs\cos(st) + \dfrac{\tau B}{2}\sin(st)\right)\right)^2 - a^2\right) \text{ if } x_1 < x < x_2 \\ 0 \text{ otherwise} \end{cases}$$

And

$$u(t,x) = \begin{cases} B\sin(st)\, e^{-\tau t/2} \text{ if } x_1 < x < x_2 \\ 0 \text{ otherwise} \end{cases}$$

In the particular case without friction ($\tau = 0$), the free surface is between the points

$$x_1 = -\frac{a^2}{2gh_0}(B\omega\cos(\omega t)) - a = -\frac{B}{\omega}\cos(\omega t) - a$$

And

$$x_2 = -\frac{a^2}{2gh_0}(B\omega\cos(\omega t)) + a = -\frac{B}{\omega}\cos(\omega t) + a$$

with the period

$$T = 2\pi/\omega$$

and pulsation

$$\omega = \sqrt{2gh_0/a^2}$$

We thus have

$$h(t,x) = \begin{cases} -\dfrac{h_0}{a^2}\left(\left(x + \dfrac{B}{\omega}\cos(\omega t)\right)^2 - a^2\right) \text{ if } x_1 < x < x_2 \\ 0 \text{ otherwise} \end{cases}$$

$$u(t,x) = \begin{cases} B\sin(\omega t) \text{ if } x_1 < x < x_2 \\ 0 \text{ otherwise} \end{cases}$$

Question 1:

Summary for n=2

(-)rarefaction

- $u(h) = -2\sqrt{gh} + u_L + 2\sqrt{gh_L}$
- Admissible part:

  $h_R < h_L, \quad u_R > u_L$
- $h(t,x) = \frac{1}{g}\left(-\frac{1}{3}\frac{x}{t} + \frac{1}{3}u_L + \frac{2}{3}\sqrt{gh_L}\right)^2$

  $u(t,x) = \frac{2}{3}\frac{x}{t} + \frac{1}{3}u_L + \frac{2}{3}\sqrt{gh_L}$

(+)rarefaction

- $u(h) = 2\sqrt{gh} + u_L - 2\sqrt{gh_L}$
- Admissible part:

  $h_R > h_L, \quad u_R > u_L$
- $h(t,x) = \frac{1}{g}\left(\frac{1}{3}\frac{x}{t} - \frac{1}{3}u_L + \frac{2}{3}\sqrt{gh_L}\right)^2$

  $u(t,x) = \frac{2}{3}\frac{x}{t} + \frac{1}{3}u_L - \frac{2}{3}\sqrt{gh_L}$

(-)shock

- $u(h) = u_L - (h_R - h_L)\sqrt{g\frac{h_R + h_L}{2h_R h_L}}$
- Admissible part:

  $h_R > h_L, \quad u_R < u_L$
- $\sigma = u_L - h_R\sqrt{g\frac{h_R + h_L}{2h_R h_L}}$

(+)shock

- $u(h) = u_L + (h_R - h_L)\sqrt{g\frac{h_R + h_L}{2h_R h_L}}$

- Admissible part:

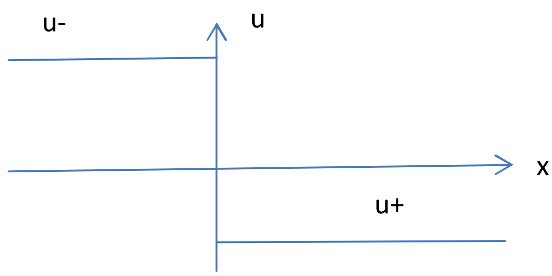
  $h_R < h_L, \quad u_R < u_L$

- $\sigma = u_L + h_R\sqrt{g\dfrac{h_R + h_L}{2h_R h_L}}$

<u>Case 1:</u>

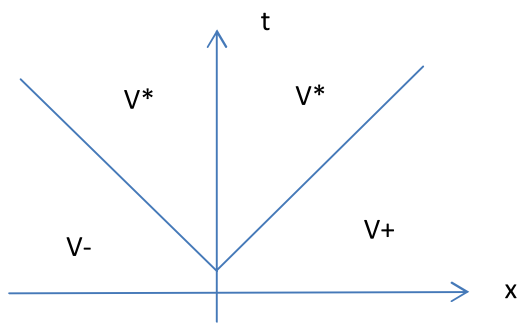
$u_+ = -u_- < 0, \quad h_+ = h_- = h_0$

When t=0



Solution for this Riemann problem

- $u_+ < u^* < u_- \to$ 2 outgoing shocks



Choose V^* such that

$$\begin{cases} V^* \in S_- \\ V_+ \in S_+ \end{cases}$$

$$\Leftrightarrow \begin{cases} u^* = u_- + (h_- - h_*)\sqrt{g\dfrac{h_- + h_*}{2h_- h_*}} \\ u_+ = u^* - (h_* - h_+)\sqrt{g\dfrac{h_+ + h_*}{2h_+ h_*}} \end{cases}$$

$$\Leftrightarrow \begin{cases} u^* = u_- + \left(h_0 - h_*\right)\sqrt{g\dfrac{h_0 + h_*}{2h_0h_*}} \\ -u_- = u^* - \left(h_* - h_0\right)\sqrt{g\dfrac{h_0 + h_*}{2h_0h_*}} \end{cases}$$

$$\Leftrightarrow \begin{cases} u^* = 0 \\ 2\left(h_* - h_0\right)\sqrt{g\dfrac{h_0 + h_*}{2h_0h_*}} = 2u_- \end{cases}$$

$$\Leftrightarrow \begin{cases} u^* = 0 \\ \left(h_* - h_0\right)^2 g\dfrac{h_0 + h_*}{2h_0h_*} = u_-{}^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} u^* = 0 \\ gh_*^3 - gh_0h_*^2 - \left(gh_0^2 + 2h_0u_-^2\right)h_* + gh_0^3 = 0 \quad (*) \end{cases}$$

Consider (*)

Let $f\left(x\right) = gx^3 - gh_0x^2 - \left(gh_0^2 + 2h_0u_-^2\right)x + gh_0^3, \quad x \geq 0$

$f'\left(x\right) = 3gx^2 - 2gh_0x - \left(gh_0^2 + 2h_0u_-^2\right)$

$$f'\left(x\right) = 0 \Leftrightarrow \left[\begin{array}{l} x = \dfrac{gh_0 - \sqrt{4g^2h_0^2 + 6gu_+^2}}{3g} := x_1 \\ x = \dfrac{gh_0 + \sqrt{4g^2h_0^2 + 6gu_+^2}}{3g} := x_2 \end{array}\right.$$

x	$-\infty$	x_1	0		x_2		$+\infty$
$f'(x)$				$-$	0	$+$	
$f(x)$				$\searrow$		$\nearrow$	

As x_2 is not solution of $f(x)=0$, this equation has 2 solutions on $[0,+\infty)$

However, $f(0)=gh_0^3>0,\quad f(h_0)=-2h_0u_-^2<0$ then there is $x_0\in(0,h_0)$ such that $f(x_0)=0$.

Therefore, there is unique $x_1\in(h_0,+\infty)$ such that $f(x_1)=0$.

Because $h_*>h_0$, we have unique value of h_*.

Finally, one gets

$$u(x,t)=\begin{cases} u_- & ,x\le\left(u_--h_*\sqrt{g\dfrac{h_*+h_-}{2h_*h_-}}\right)t \\ 0 & ,\left(u_--h_*\sqrt{g\dfrac{h_*+h_-}{2h_*h_-}}\right)t<x<\left(u_*+h_+\sqrt{g\dfrac{h_*+h_+}{2h_+h_*}}\right)t \\ u_+ & ,x\ge\left(u_*+h_+\sqrt{g\dfrac{h_*+h_+}{2h_+h_*}}\right)t \end{cases}$$

$$h(x,t)=\begin{cases} h_- & ,x\le\left(u_--h_*\sqrt{g\dfrac{h_*+h_-}{2h_*h_-}}\right)t \\ h_* & ,\left(u_--h_*\sqrt{g\dfrac{h_*+h_-}{2h_*h_-}}\right)t<x<\left(u_*+h_+\sqrt{g\dfrac{h_*+h_+}{2h_+h_*}}\right)t \\ h_+ & ,x\ge\left(u_*+h_+\sqrt{g\dfrac{h_*+h_+}{2h_+h_*}}\right)t \end{cases}$$

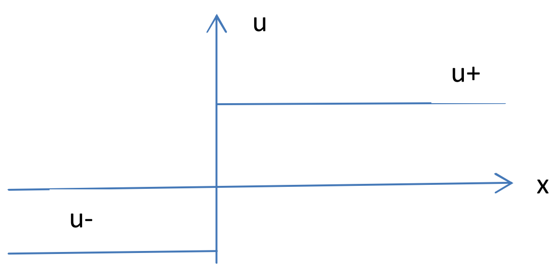
where h_* is solution of $gx^3-gh_0x^2-\left(gh_0^2+2h_0u_-^2\right)x+gh_0^3=0$.

- $u^* < u_+ < u_-$ -> This is contradiction with $h_+ = h_- = h_0$ as from V_- to V_* is (-)shock, from V_* to V_+ is (+)rarefaction. Then one gets $h_* > h_-$ and $h_* < h_+$.
- $u_+ < u_* < u_-$ -> This is contradiction with $h_+ = h_- = h_0$ as from V_- to V_* is (-)rarefaction, from V_* to V_+ is (+)shock. Then one gets $h_* < h_-$ and $h_* > h_+$.

<u>Case 2:</u>

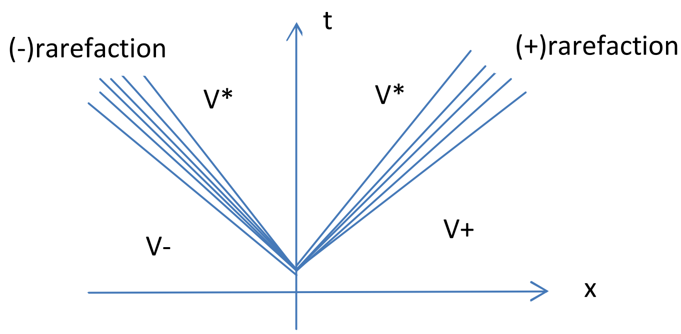
$u_+ = -u_- > 0;, \quad h_+ = h_- = h_0$

When t=0



Solution for this Riemann problem

- $u_- < u^* < u_+ \rightarrow$ 2 outgoing rarefactions



Choose V^* such that

$$\begin{cases} V^* \in R_- \\ V_+ \in R_+ \end{cases}$$

$$\Leftrightarrow \begin{cases} u^* = -2\sqrt{gh_*} + u_- + 2\sqrt{gh_-} \\ u_+ = 2\sqrt{gh_+} + u_* - 2\sqrt{gh_*} \end{cases}$$

$$\Leftrightarrow \begin{cases} u^* = -2\sqrt{gh_*} + u_- + 2\sqrt{gh_0} \\ -u_- = 2\sqrt{gh_0} + u_* - 2\sqrt{gh_*} \end{cases}$$

$$\Leftrightarrow \begin{cases} u^* = 0 \\ 4\sqrt{g}\left(\sqrt{h_*} - \sqrt{h_0}\right) = 2u_- \end{cases}$$

$$\Leftrightarrow \begin{cases} u^* = 0 \\ h_* = \left(\dfrac{u_- + 2\sqrt{gh_0}}{2\sqrt{g}}\right)^2 \end{cases}$$

Now for $\lambda_-(V_-) \le \dfrac{x}{t} \le \lambda_-(V_*)$, one gets

$$h(t,x) = \frac{1}{g}\left(-\frac{1}{3}\frac{x}{t} + \frac{1}{3}u_- + \frac{2}{3}\sqrt{gh_-}\right)^2$$

$$u(t,x) = \frac{2}{3}\frac{x}{t} + \frac{1}{3}u_- + \frac{2}{3}\sqrt{gh_-}$$

Now for $\lambda_+(V_*) \le \dfrac{x}{t} \le \lambda_+(V_+)$, one gets

$$h(t,x) = \frac{1}{g}\left(\frac{5}{3}\frac{x}{t} - \frac{1}{3}u_* + \frac{2}{3}\sqrt{gh_*}\right)^2 = \frac{1}{g}\left(\frac{5}{3}\frac{x}{t} + \frac{2}{3}\sqrt{gh_*}\right)^2$$

$$u(t,x) = -\frac{2}{3}\frac{x}{t} + \frac{1}{3}u_* - \frac{2}{3}\sqrt{gh_*} = -\frac{2}{3}\frac{x}{t} - \frac{2}{3}\sqrt{gh_*}$$

Finally, one gets

$$u(t,x)=\begin{cases} u_- & ,x\le \lambda_-(V_-)t \\ \frac{2}{3}\frac{x}{t}+\frac{1}{3}u_-+\frac{2}{3}\sqrt{gh_-} & ,\lambda_-(V_-)t\le x\le \lambda_-(V_*)t \\ u*, & \lambda_-(V_*)t\le x\le \lambda_+(V_*)t \\ \frac{2}{3}\frac{x}{t}+\frac{1}{3}u*-\frac{2}{3}\sqrt{gh_*} & ,\lambda_+(V_*)t\le x\le \lambda_+(V_+)t \\ u_+ & ,x\ge \lambda_+(V_+)t \end{cases}$$

$$h(t,x)=\begin{cases} h_- & ,x\le \lambda_-(V_-)t \\ \frac{1}{g}\left(-\frac{1}{3}\frac{x}{t}+\frac{1}{3}u_-+\frac{2}{3}\sqrt{gh_-}\right)^2 & ,\lambda_-(V_-)t\le x\le \lambda_-(V_*)t \\ u*, & \lambda_-(V_*)t\le x\le \lambda_+(V_*)t \\ \frac{1}{g}\left(\frac{1}{3}\frac{x}{t}-\frac{1}{3}u*+\frac{2}{3}\sqrt{gh_*}\right)^2 & ,\lambda_+(V_*)t\le x\le \lambda_+(V_+)t \\ h_+ & ,x\ge \lambda_+(V_+)t \end{cases}$$

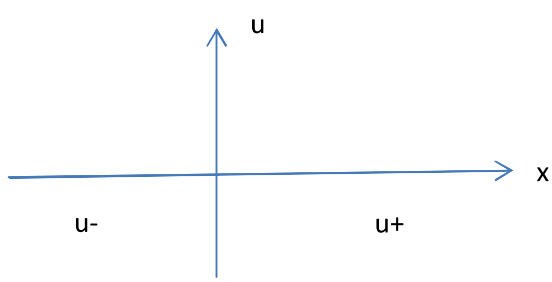
where $h_*=\left(\frac{u_-+2\sqrt{gh_0}}{2\sqrt{g}}\right)^2$.

- $u^*<u_-<u_+$ -> This is a contradiction with $h_+=h_-=h_0$ as from V_- to V_* is (-)shock, from V_* to V_+ is (+)rarefaction. Then one gets $h_*>h_-$ and $h_*<h_+$.
- $u_-<u_+<u^*$ -> This is a contradiction with $h_+=h_-=h_0$ as V_- from to V_* is (-)rarefaction, from V_* to V_+ is (+)shock. Then one gets $h_*<h_+$ and $h_*>h_-$.

Case 3:

$$u_+=u_-=0;\ h_+<h_-$$

When t=0



Solution for this Riemann problem

- $u^* > 0 \rightarrow$ From V_- to V_* is (-)rarefaction, from V_* to V_+ is (+)shock.
- $u^* < 0 \rightarrow$ This is contradiction with $h_+ < h_-$ as from V_- to V_* is (-)shock, from V_* to V_+ is (+)rarefaction. Then one gets $h_* > h_-$ and $h_* < h_+$.

$V_* \in R_-(\mathrm{V}_-)$:

$$\mathrm{u}_* = u_- + 2\sqrt{g}\left(\sqrt{h_-} - \sqrt{h_*}\right) \qquad (1)$$

$V_+ \in S_+(\mathrm{V}_*)$:

$$\mathrm{u}_+ = \mathrm{u}_* + \left(h_+ - h_*\right)\sqrt{\frac{g}{2}.\frac{h_+ + h_*}{2h_+h_*}} \qquad (2)$$

Add (1) and (2), we have:

$$\mathrm{u}_+ = u_- + 2\sqrt{g}\left(\sqrt{h_-} - \sqrt{h_*}\right) + \left(h_+ - h_*\right)\sqrt{\frac{g}{2}.\frac{h_+ + h_*}{h_+h_*}}$$

$$\overset{\mathrm{u}_+ = u_- = 0}{\Longrightarrow} 2\sqrt{g}\left(\sqrt{h_-} - \sqrt{h_*}\right) + \left(h_+ - h_*\right)\sqrt{\frac{g}{2}.\left(\frac{1}{h_+} + \frac{1}{h_*}\right)} = 0$$

$$\overset{g>0}{\Longrightarrow} 2\left(\sqrt{h_-} - \sqrt{h_*}\right) + \left(h_+ - h_*\right)\sqrt{\frac{1}{2}.\left(\frac{1}{h_+} + \frac{1}{h_*}\right)} = 0 \qquad (3)$$

Let

$t = \sqrt{h}$

Equation (3) is equivalent to

$$2\sqrt{2h_+}.\left(\sqrt{h_-}-t\right).t=-\left(h_+-t^2\right).\sqrt{t^2+h_+}$$

$$\Leftrightarrow 8h_+.\left(\sqrt{h_-}-t\right)^2.t^2=\left(h_+-t^2\right)^2\left(t^2+h_+\right)$$

$$\Leftrightarrow \left(8h_+\right).t^4+\left(-16h_+\sqrt{h_-}\right).t^3+\left(8h_-h_+\right)t^2=t^6+\left(-h_+\right)t^4+\left(-h_+^2\right)t^2+h_+^3$$

$$\Leftrightarrow t^6+\left(-9h_+\right)t^4+\left(16h_+\sqrt{h_-}\right)t^3+\left(-h_+^2-8h_+h_-\right)t^2+h_+^3=0$$

We need this equation when implement codes in question 2 and 3.

We define mapping f by:

$$f(h)=2\left(\sqrt{h_-}-\sqrt{h}\right)+\left(h_+-h\right)\sqrt{\frac{1}{2}.\left(\frac{1}{h_+}+\frac{1}{h}\right)}$$

$$f'(h)=-\frac{1}{\sqrt{h}}-\sqrt{\frac{1}{2}.\left(\frac{1}{h_+}+\frac{1}{h}\right)}-\frac{1}{2\sqrt{2}}\left(h_+-h\right)\frac{1}{h^2.\sqrt{\left(\frac{1}{h_+}+\frac{1}{h}\right)}}$$

$$=-\frac{1}{\sqrt{h}}+\left[2\left(\frac{1}{h_+}+\frac{1}{h}\right)-\frac{h_+-h}{h^2}\right].\frac{1}{2\sqrt{2}.\sqrt{\left(\frac{1}{h_+}+\frac{1}{h}\right)}}$$

$$=-\frac{1}{\sqrt{h}}-\frac{h^2+h_+^2}{h^2h_+}.\frac{1}{2\sqrt{2}.\sqrt{\left(\frac{1}{h_+}+\frac{1}{h}\right)}}$$

$$f'(h)<0\ \ \forall\ h_+<h_*<h_-$$

So (3) has a unique solution or system (1),(2) has a unique solution (h_*, u_*)

Therefore the solution of the given Riemann problem is $V=\begin{pmatrix} h \\ hu \end{pmatrix}$:

$$
\mathrm{h}(t,x)=\begin{cases} h_- & \text{if } \frac{x}{t}\le\lambda_-(V_-) \\ \frac{1}{g}\left(\frac{-1}{3}.\frac{x}{t}+\frac{1}{3}u_-+\frac{2}{3}\sqrt{gh_-}\right)^2 & \text{if } \lambda_-(V_-)\le\frac{x}{t}\le\lambda_-(V_*) \\ h_* & \text{if } \lambda_-(V_*)\le\frac{x}{t}\le\sigma_+ \\ h_+ & \text{if } \sigma_+\le\frac{x}{t} \end{cases}
$$

$$
\mathrm{u}(t,x)=\begin{cases} u_- & \text{if } \frac{x}{t}\le\lambda_-(V_-) \\ \frac{2}{3}.\frac{x}{t}+\frac{1}{3}u_-+\frac{2}{3}\sqrt{gh_-} & \text{if } \lambda_-(V_-)\le\frac{x}{t}\le\lambda_-(V_*) \\ u_* & \text{if } \lambda_-(V_*)\le\frac{x}{t}\le\sigma_+ \\ u_+ & \text{if } \sigma_+\le\frac{x}{t} \end{cases}
$$

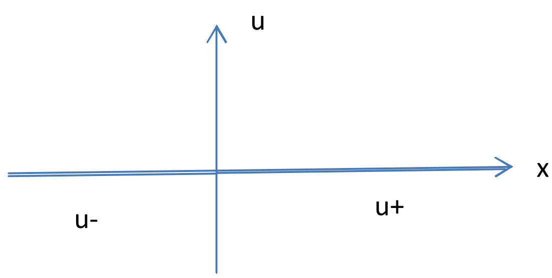
Where $\lambda_-(V_-)=u_--\sqrt{gh_-},\quad \lambda_-(V_*)=u_*-\sqrt{gh_*},\quad \sigma_+=u_*+h_+\sqrt{g.\frac{h_++h_*}{2h_+h_*}}$

And h_*,u_* satisfies equation (3)

Case 4:

$$
u_+=u_-=0;\ h_+>h_-
$$

When t=0



Solution to this Riemann problem

- $u^* < 0 \rightarrow$ From V_- to V_* is (-)shock, from V_* to V_+ is (+)rarefaction.
- $u^* > 0 \rightarrow$ This is contradiction with $h_+ > h_-$ as from V_- to V_* is (-)rarefaction, from V_* to V_+ is (+)shock. Then one gets $h_* < h_-$ and $h_* > h_+$.

$V_* \in S_-(\mathrm{V}_-)$:

$$\mathrm{u}_* = u_- + \left(h_- - h_*\right)\sqrt{\frac{g}{2}.\frac{h_- + h_*}{2h_- h_*}} \qquad (1)$$

$V_+ \in R_+(\mathrm{V}_*)$:

$$\mathrm{u}_+ = \mathrm{u}_* + 2\sqrt{g}\left(\sqrt{h_+} - \sqrt{h_*}\right) \qquad (2)$$

Add (1) and (2), we have:

$$\mathrm{u}_+ = u_- + 2\sqrt{g}\left(\sqrt{h_+} - \sqrt{h_*}\right) + \left(h_- - h_*\right)\sqrt{\frac{g}{2}.\frac{h_- + h_*}{h_- h_*}}$$

$$\xRightarrow{\mathrm{u}_+ = u_- = 0} 2\sqrt{g}\left(\sqrt{h_+} - \sqrt{h_*}\right) + \left(h_- - h_*\right)\sqrt{\frac{g}{2}.\left(\frac{1}{h_-} + \frac{1}{h_*}\right)} = 0$$

$$\xRightarrow{g>0} 2\left(\sqrt{h_+} - \sqrt{h_*}\right) + \left(h_- - h_*\right)\sqrt{\frac{1}{2}.\left(\frac{1}{h_-} + \frac{1}{h_*}\right)} = 0 \qquad (3)$$

We define mapping f by:

$$f(h) = 2\left(\sqrt{h_+} - \sqrt{h}\right) + \left(h_- - h\right)\sqrt{\frac{1}{2}.\left(\frac{1}{h_-} + \frac{1}{h}\right)}$$

Let

$t = \sqrt{h}$

Equation (3) is equivalent to

$$2\sqrt{2h_-}.\left(\sqrt{h_+} - t\right).t = -\left(h_- - t^2\right).\sqrt{t^2 + h_-}$$
$$\Leftrightarrow 8h_-.\left(\sqrt{h_+} - t\right)^2.t^2 = \left(h_- - t^2\right)^2\left(t^2 + h_-\right)$$
$$\Leftrightarrow \left(8h_-\right).t^4 + \left(-16h_-\sqrt{h_+}\right).t^3 + \left(8h_+h_-\right)t^2 = t^6 + \left(-h_-\right)t^4 + \left(-h_-^2\right)t^2 + h_-^3$$
$$\Leftrightarrow t^6 + \left(-9h_-\right)t^4 + \left(16h_-\sqrt{h_+}\right)t^3 + \left(-h_-^2 - 8h_+h_-\right)t^2 + h_-^3 = 0$$

We need this equation when implement codes in question 2 and 3.

$$f'(h) = -\frac{1}{\sqrt{h}} - \sqrt{\frac{1}{2}.\left(\frac{1}{h_-} + \frac{1}{h}\right)} - \frac{1}{2\sqrt{2}}\left(h_- - h\right)\frac{1}{h^2.\sqrt{\left(\frac{1}{h_-} + \frac{1}{h}\right)}}$$

$$= -\frac{1}{\sqrt{h}} + \left[2\left(\frac{1}{h_-} + \frac{1}{h}\right) - \frac{h_- - h}{h^2}\right].\frac{1}{2\sqrt{2}.\sqrt{\left(\frac{1}{h_-} + \frac{1}{h}\right)}}$$

$$= -\frac{1}{\sqrt{h}} - \frac{h^2 + h_-^2}{h^2h_-}.\frac{1}{2\sqrt{2}.\sqrt{\left(\frac{1}{h_-} + \frac{1}{h}\right)}}$$

$$f'(h) < 0 \;\; \forall \;\; h_- < h_* < h_+$$

So (3) has a unique solution or system (1),(2) has a unique solution $(\mathrm{h}_*, \mathrm{u}_*)$

Therefore the solution of the given Riemann problem is $\mathrm{V} = \begin{pmatrix} h \\ hu \end{pmatrix}$:

$$
\text{h}(t,x)=\begin{cases}
h_- & if\ \ \dfrac{x}{t}\le\sigma_- \\
h_* & if\ \ \sigma_-\le\dfrac{x}{t}\le\lambda_+\left(V_*\right) \\
\dfrac{1}{g}\left(\dfrac{1}{3}.\dfrac{x}{t}-\dfrac{1}{3}u_*+\dfrac{2}{3}\sqrt{gh_*}\right)^2 & if\ \ \lambda_+\left(V_*\right)\le\dfrac{x}{t}\le\lambda_+\left(V_+\right) \\
h_+ & if\ \ \lambda_+\left(V_+\right)\le\dfrac{x}{t}
\end{cases}
$$

$$
\text{u}(t,x)=\begin{cases}
u_- & if\ \ \dfrac{x}{t}\le\sigma_- \\
u_* & if\ \ \sigma_-\le\dfrac{x}{t}\le\lambda_+\left(V_*\right) \\
\dfrac{2}{3}.\dfrac{x}{t}+\dfrac{1}{3}u_*-\dfrac{2}{3}\sqrt{gh_*} & if\ \ \lambda_+\left(V_*\right)\le\dfrac{x}{t}\le\lambda_+\left(V_+\right) \\
u_+ & if\ \ \lambda_+\left(V_+\right)\le\dfrac{x}{t}
\end{cases}
$$

Where $\lambda_+\left(V_*\right)=u_*+\sqrt{gh_*},\lambda_+\left(V_+\right)=u_++\sqrt{gh_+},\quad \sigma_-=u_--h_*\sqrt{g.\dfrac{h_-+h_*}{2h_-h_*}}$

And h_*,u_* satisfies equation (3)

Question 2

Running time:

We shall begin with the comparison between Rusanov and HLL schemes about their times using to run the simulations.

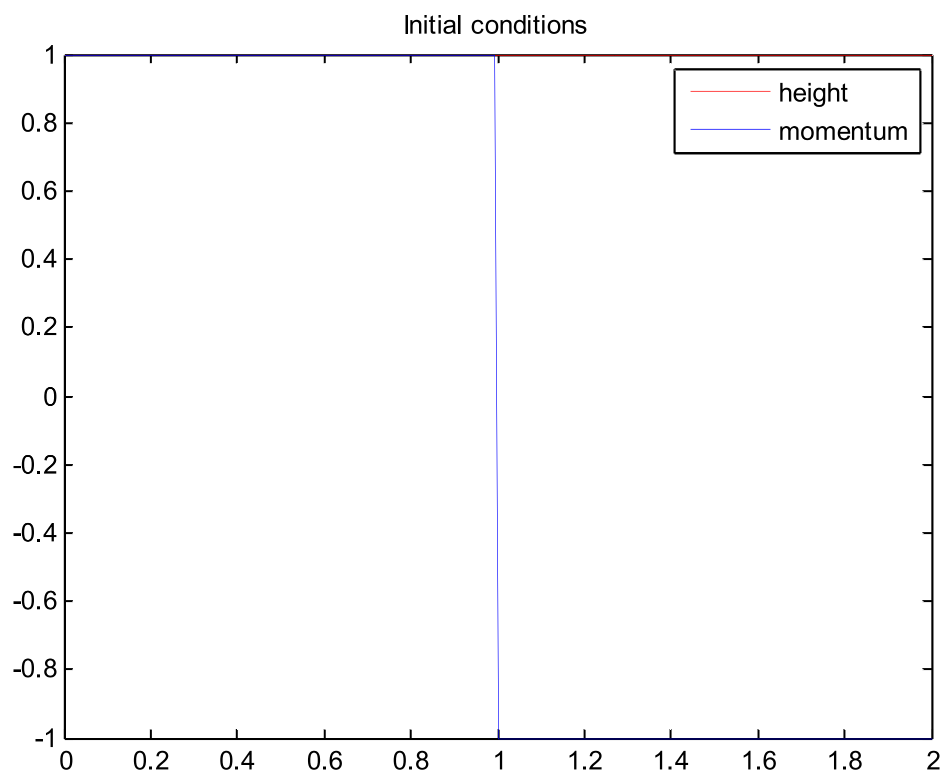
Case 1

Run the file Saint_Venant_two_shocks_Rusanov_flux.m

Elapsed time is 40.328784 seconds.

Our mesh is 400 space cells and this is the solution of the case two outgoing shocks, when t=0.1

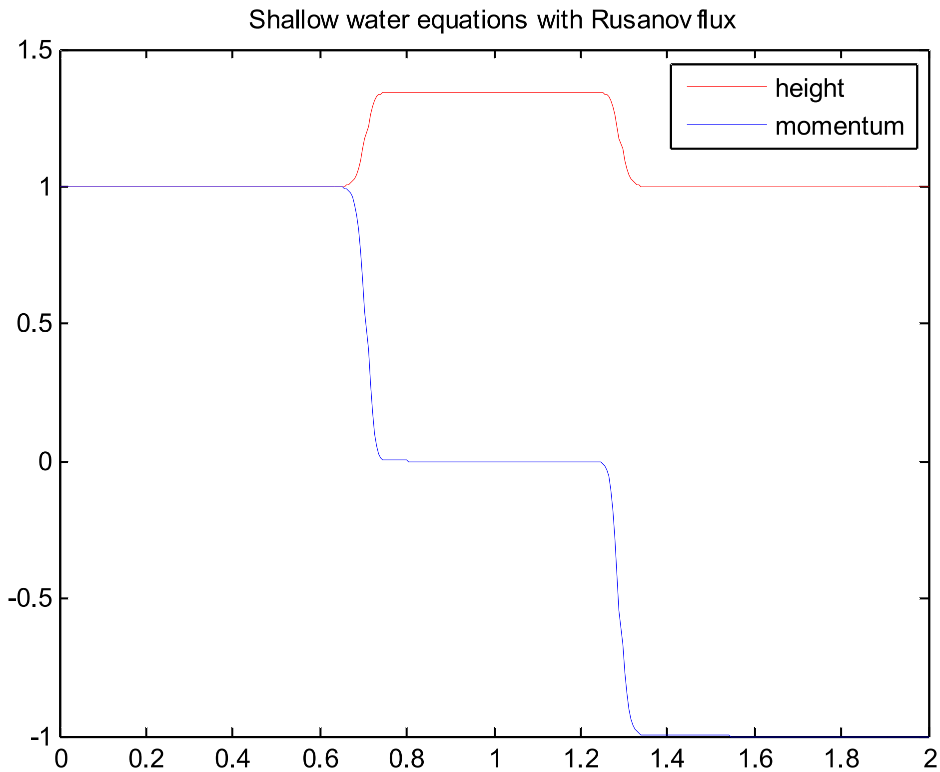
Our initial conditions:

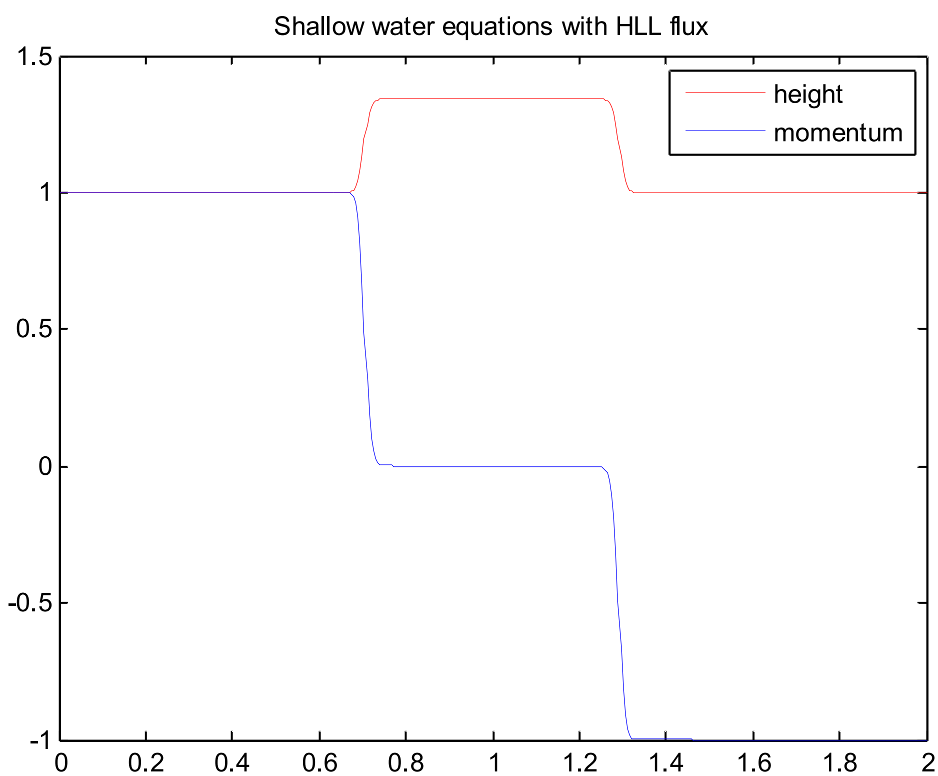


Now run the file Saint_Venant_two_shocks_HLL_flux.m, our mesh, initial conditions and the time when the solution is plotted are similar with Saint_Venant_two_shocks_Rusanov_flux.m

Elapsed time is 50.734321 seconds.

We see that HLL scheme is slower than Rusanov scheme.


Shallow water equations with Rusanov flux
height
momentum
1.5
1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

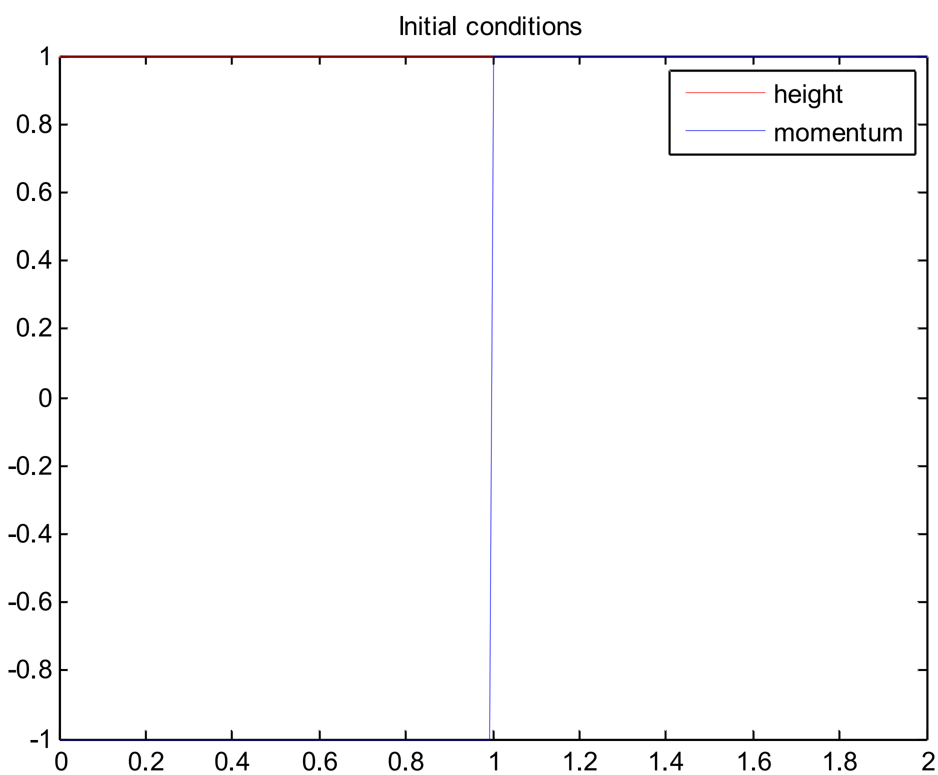

Shallow water equations with HLL flux
height
momentum
1.5
1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Case 2

Run Saint_Venant_two_rarefactions_Rusanov_flux.m

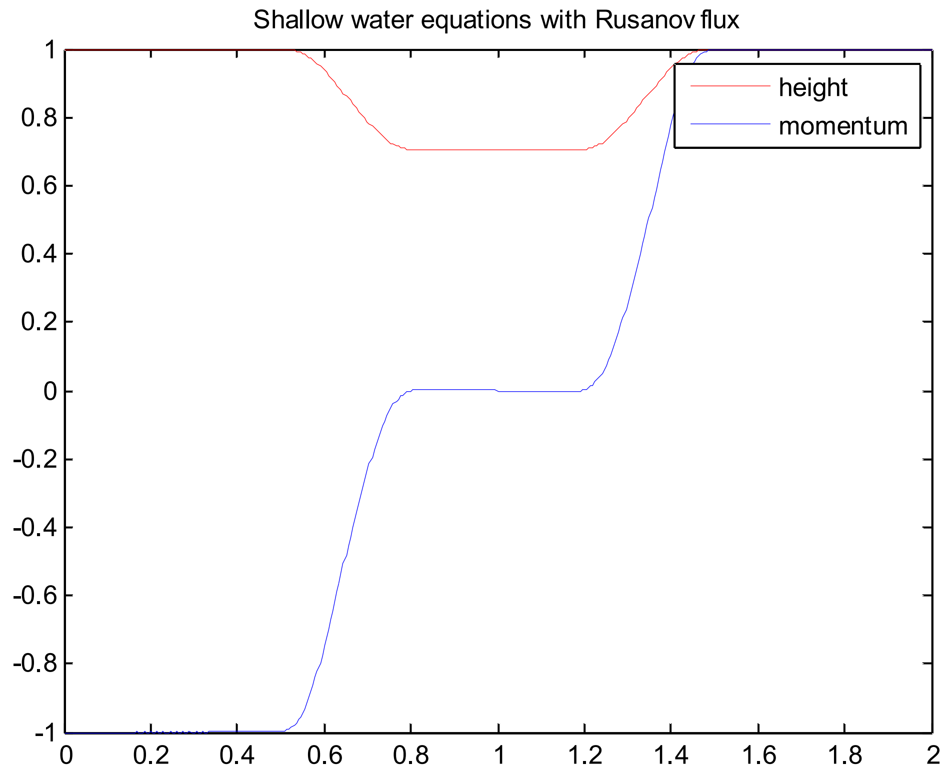
Elapsed time is 39.317180 seconds.

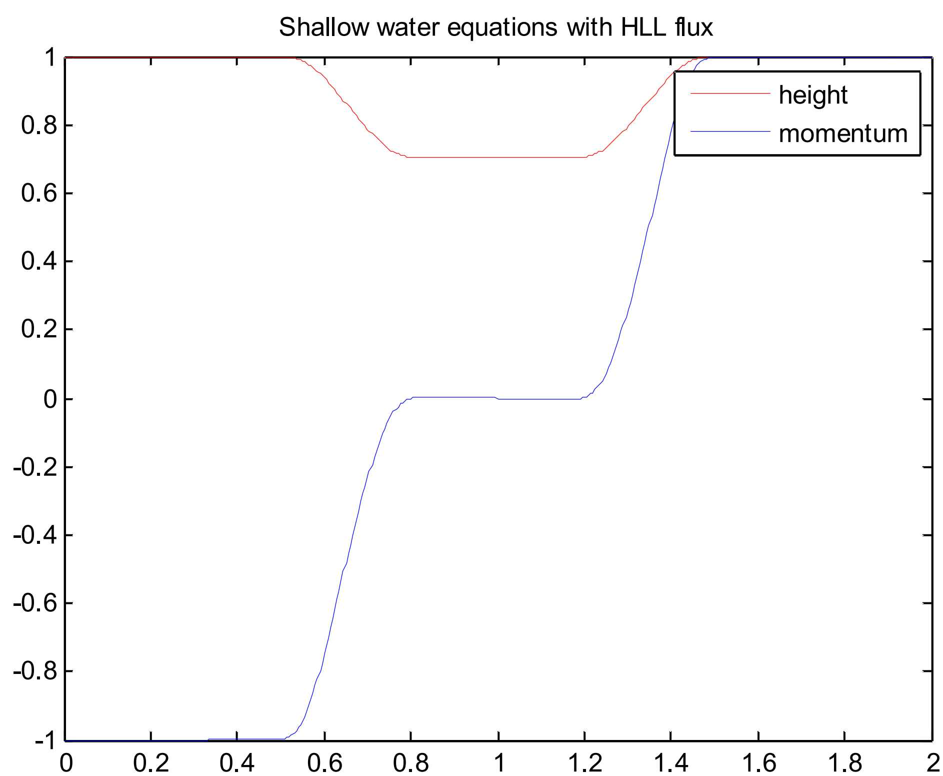
Initial conditions



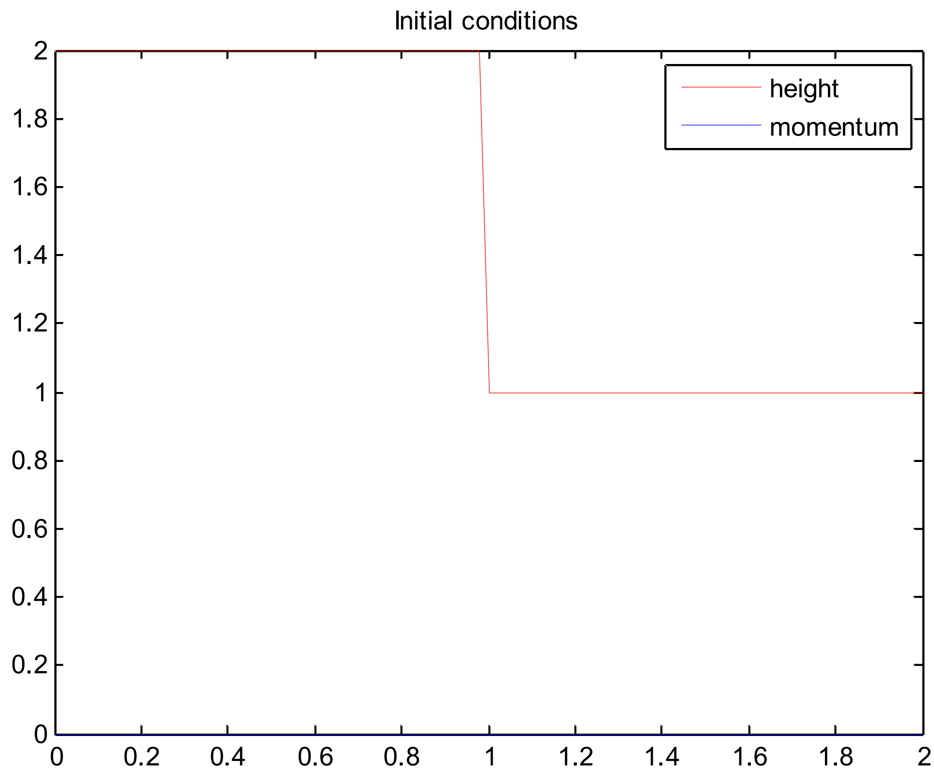
Saint_Venant_two_rarefactions_HLL_flux.m

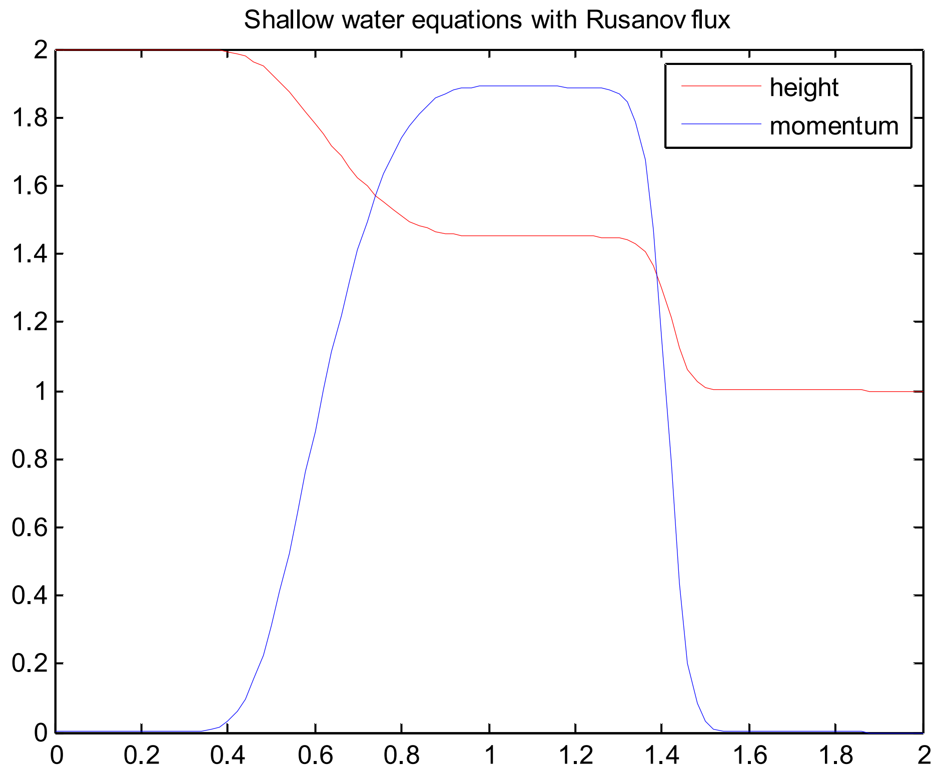
Elapsed time is 43.003601 seconds. Slower than Rusanov.

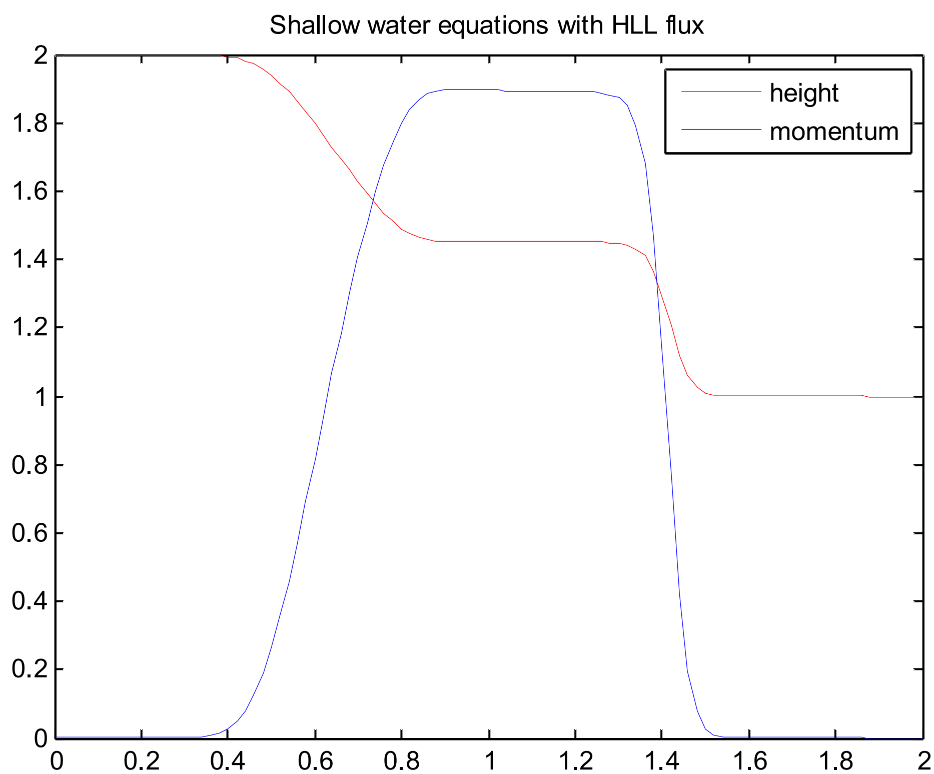

Shallow water equations with Rusanov flux
height
momentum
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


Shallow water equations with HLL flux
height
momentum
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Case 3: Dam break

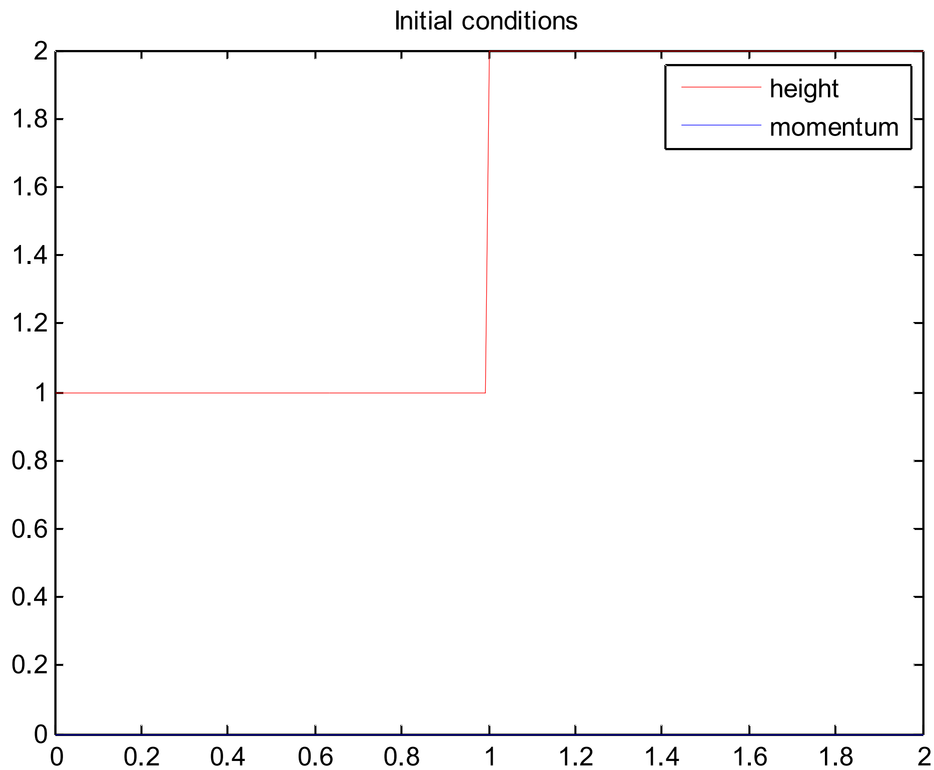



Shallow water equations with Rusanov flux
height
momentum
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

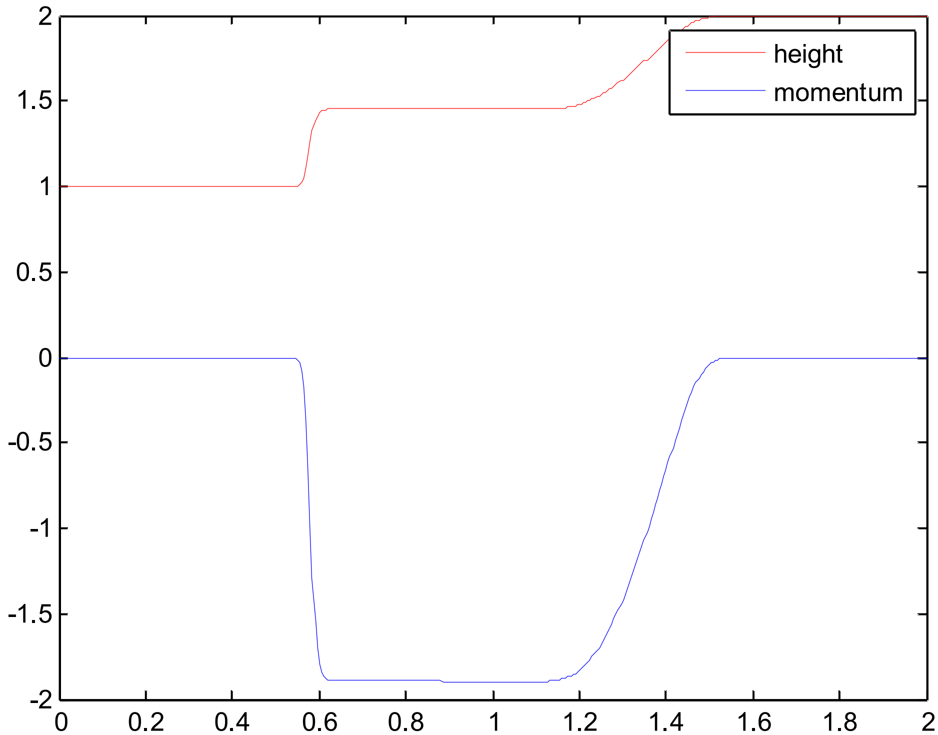

Shallow water equations with HLL flux
height
momentum
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Case 4: Reversed dam break

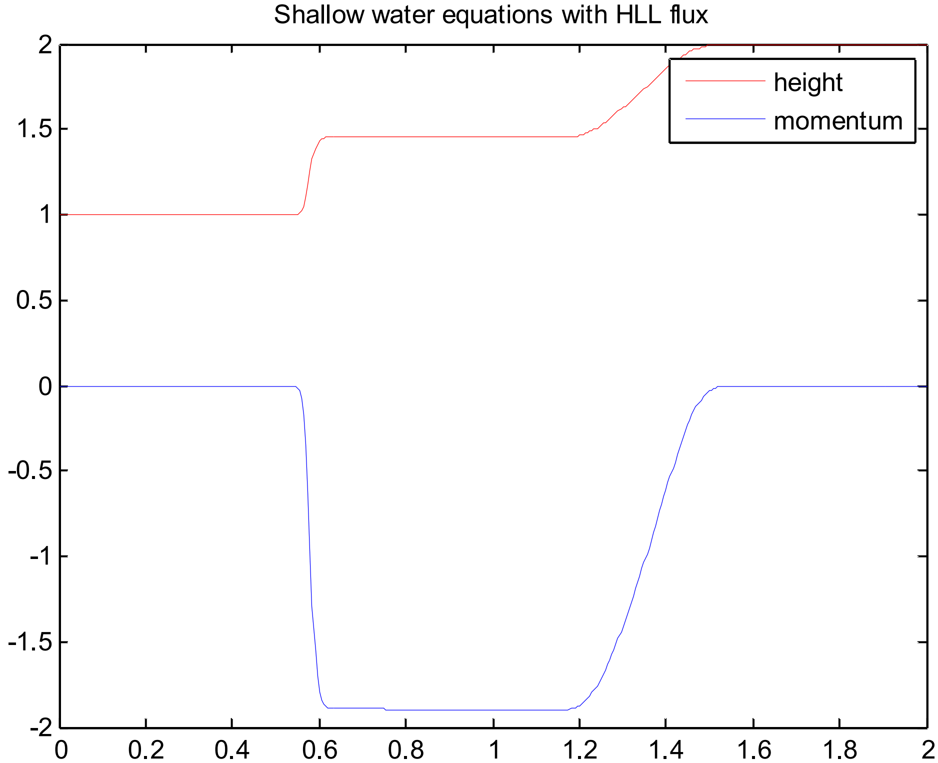
Initial conditions



Shallow water equations with Rusanov flux



Shallow water equations with HLL flux



We have the following summary in running time

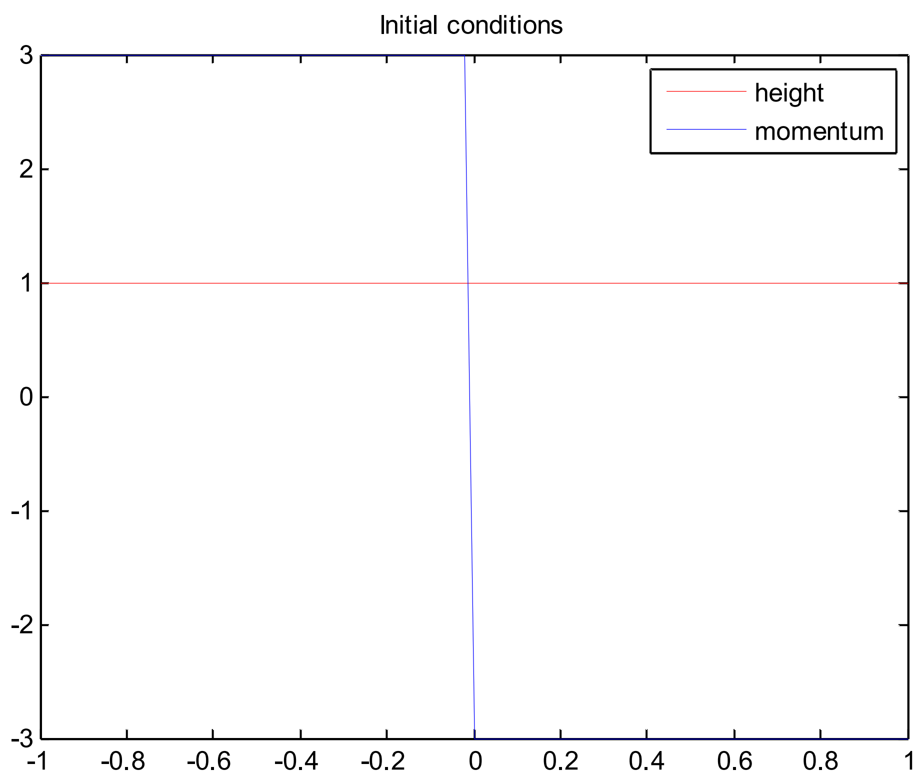
Running time	Rusanov	HLL
Case 1	40.328784 seconds	50.734321 seconds.
Case 2	39.317180 seconds	43.003601 seconds
Case 3	9.388192 seconds	9.609466 seconds
Case 4	47.788977 seconds	52.797057 seconds

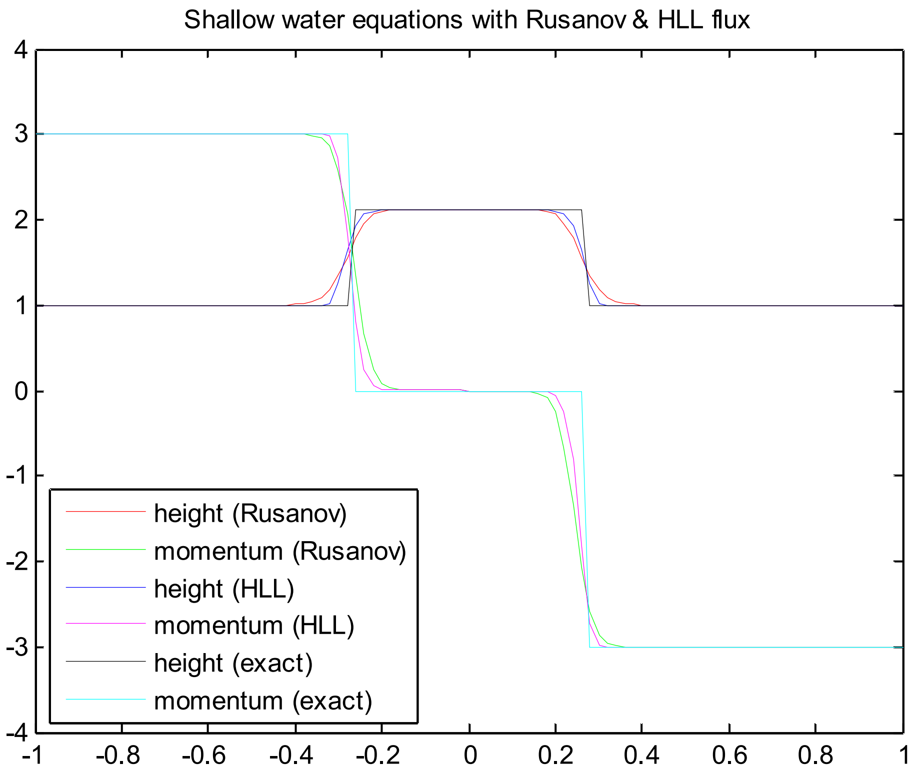
Approximate exact solution

In approximating exact solution (we 've got it by question 1), we see that HLL scheme is better than Rusanov scheme.

Our mesh is 100 space cells.

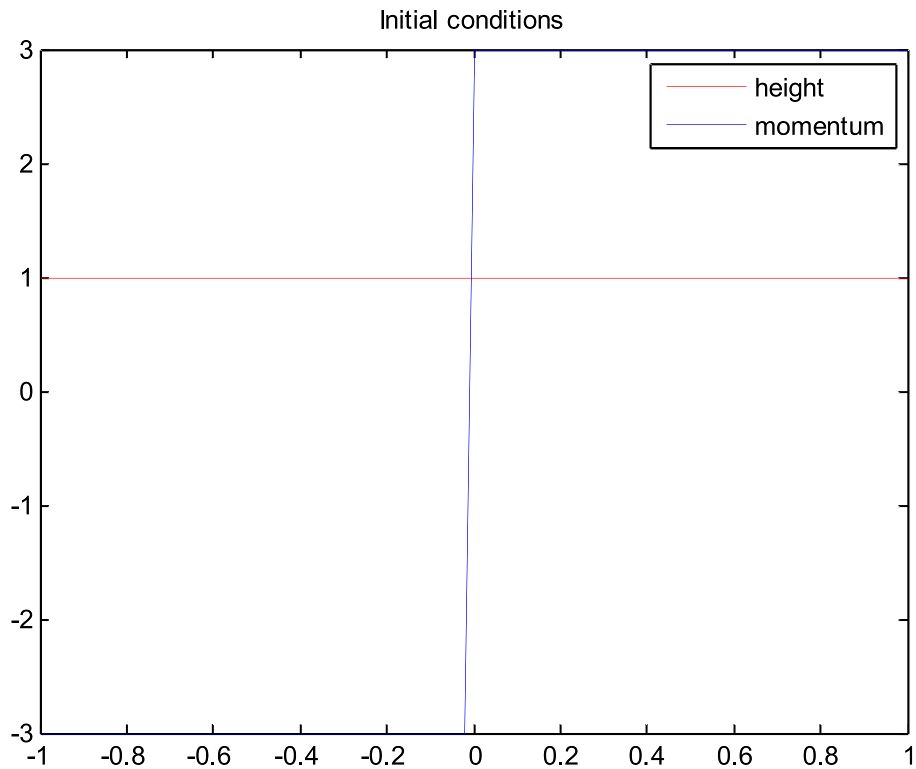
<u>Case 1</u>

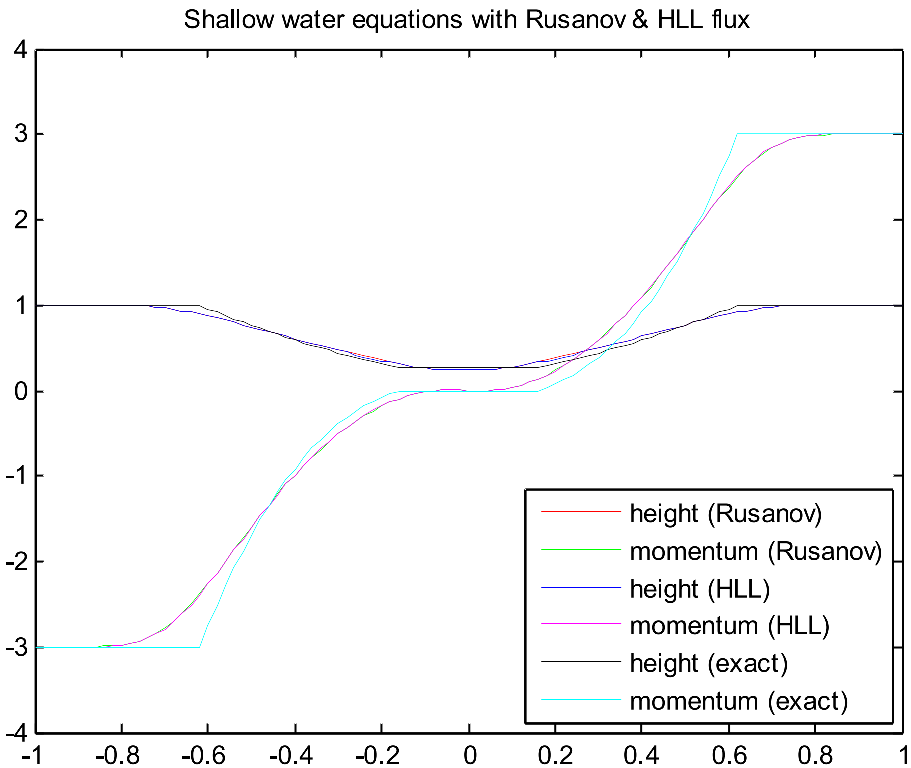




The solution obtained by HLL scheme is closer to the exact solution than Rusanov scheme.

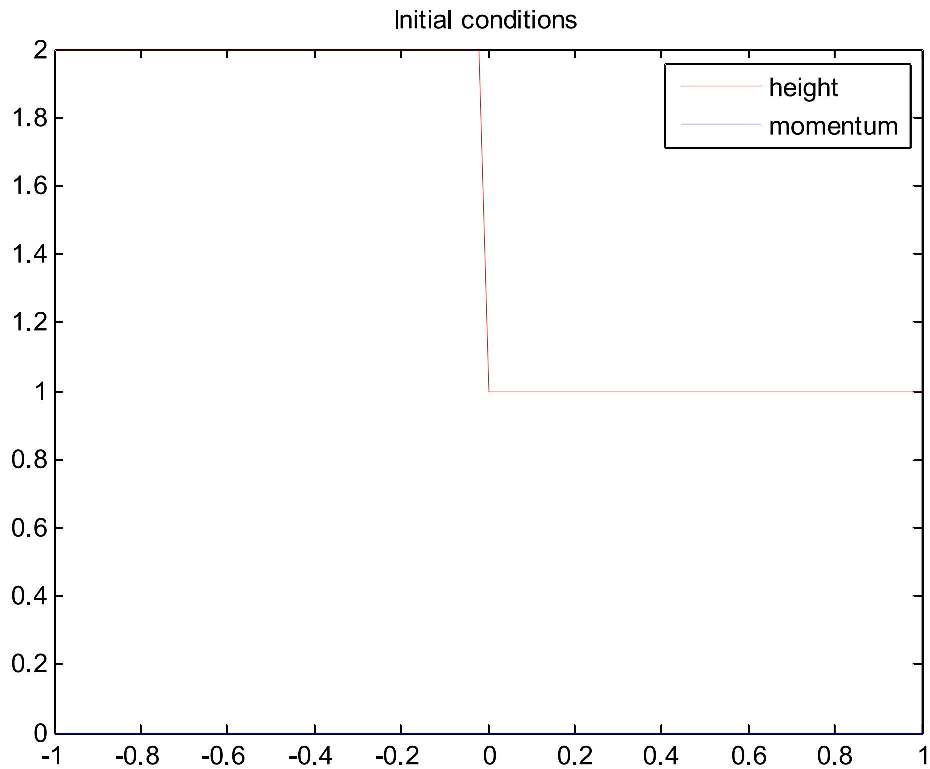
Case 2


Initial conditions
height
momentum
3
2
1
0
-1
-2
-3
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

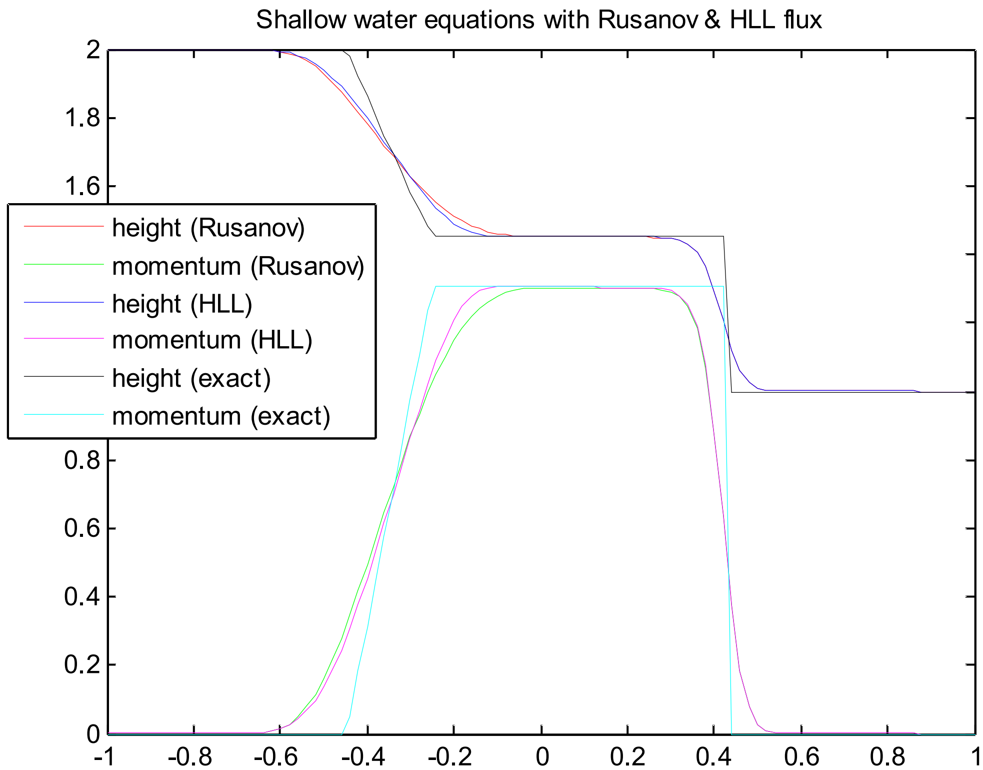


In this case they (Rusanov and HLL solution) look like the same.

Case 3

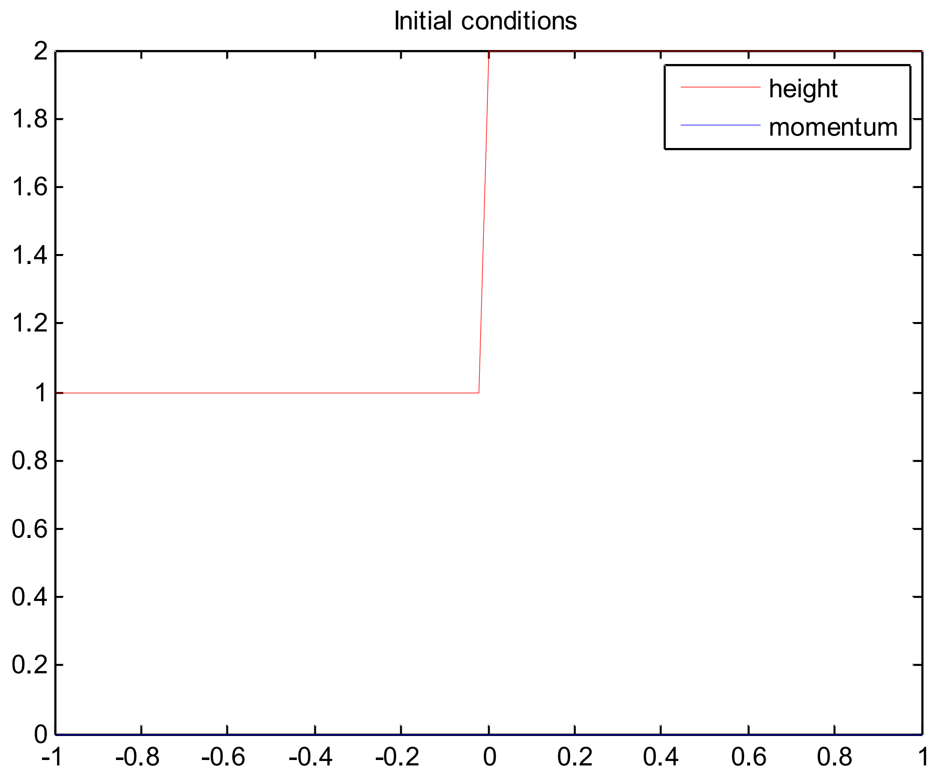


Initial conditions
height
momentum
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

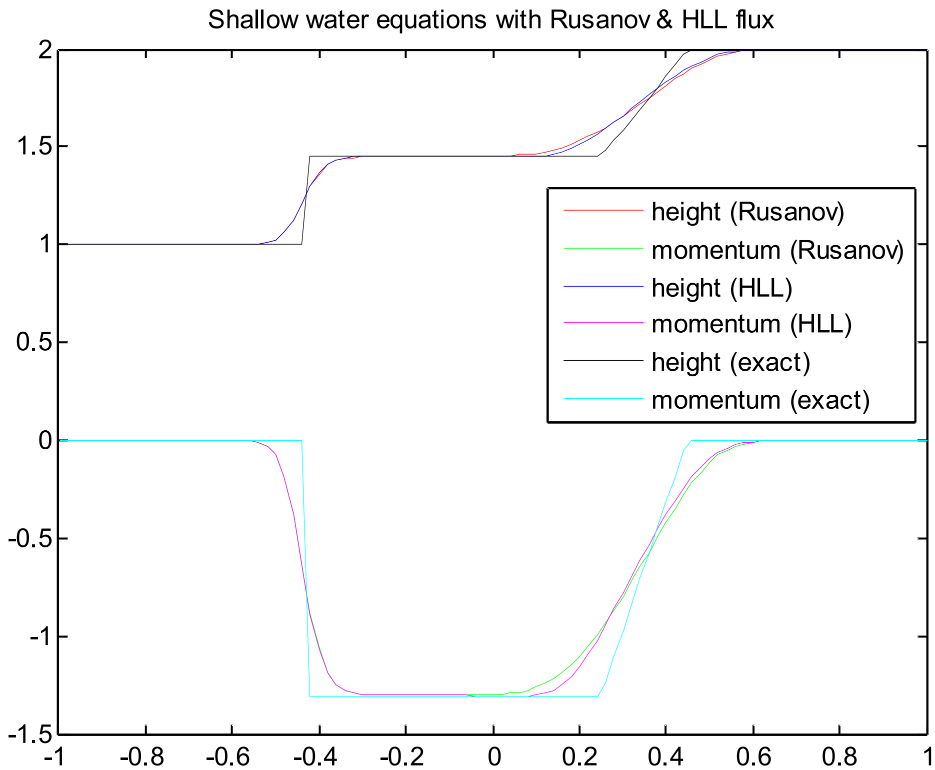


Once again HLL is better.

Case 4



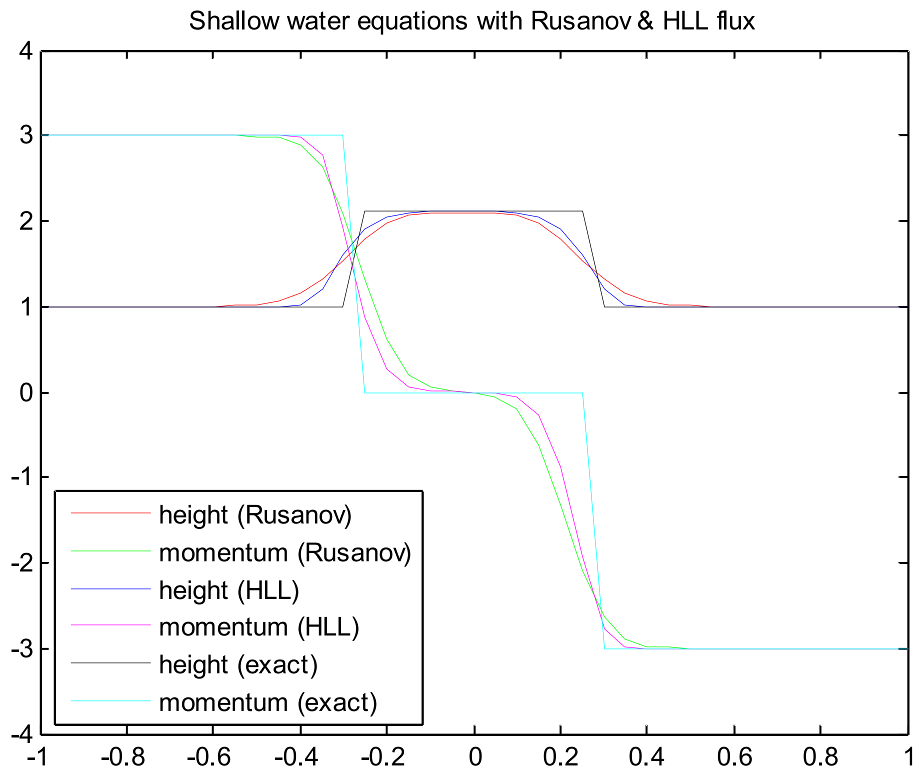
Initial conditions
height
momentum
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1



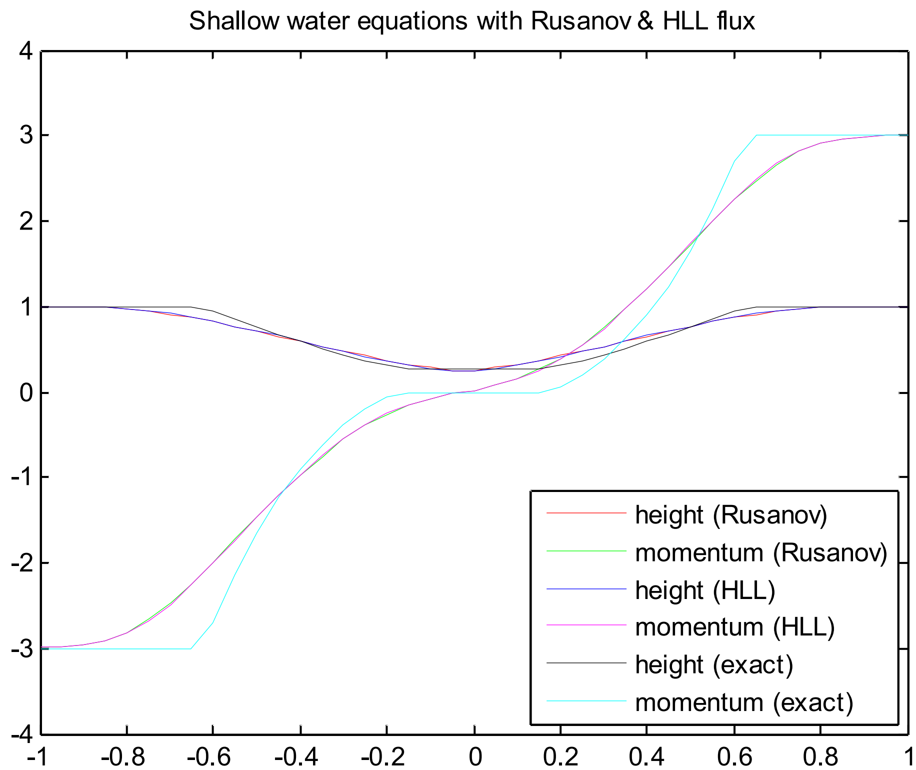
Once again HLL is better.

Now our mesh is 40 space cells.

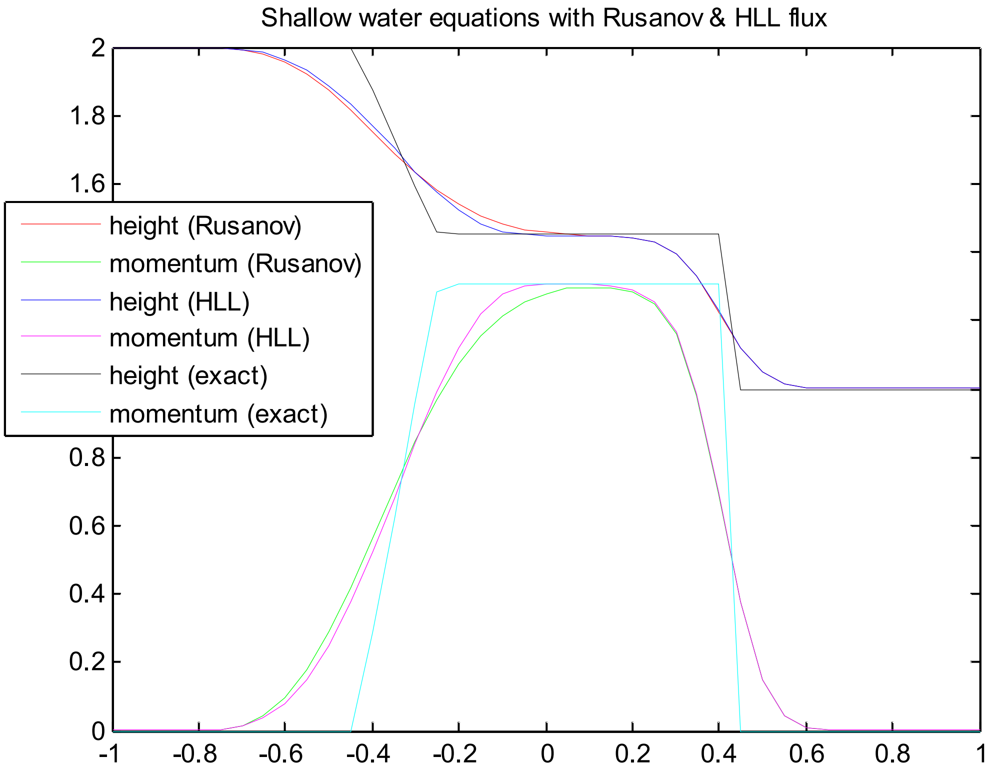
Two outgoing shocks: we see clearly that HLL is better.



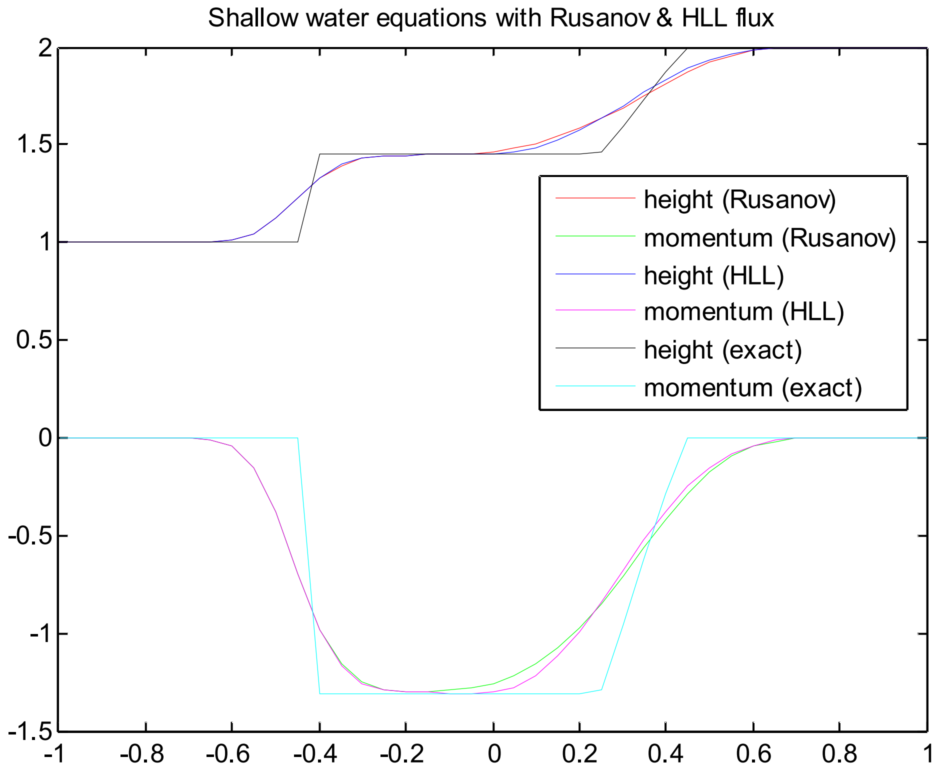
Two rarefactions: look like the same



Dam break: HLL is better



Reversed dam break: HLL is better

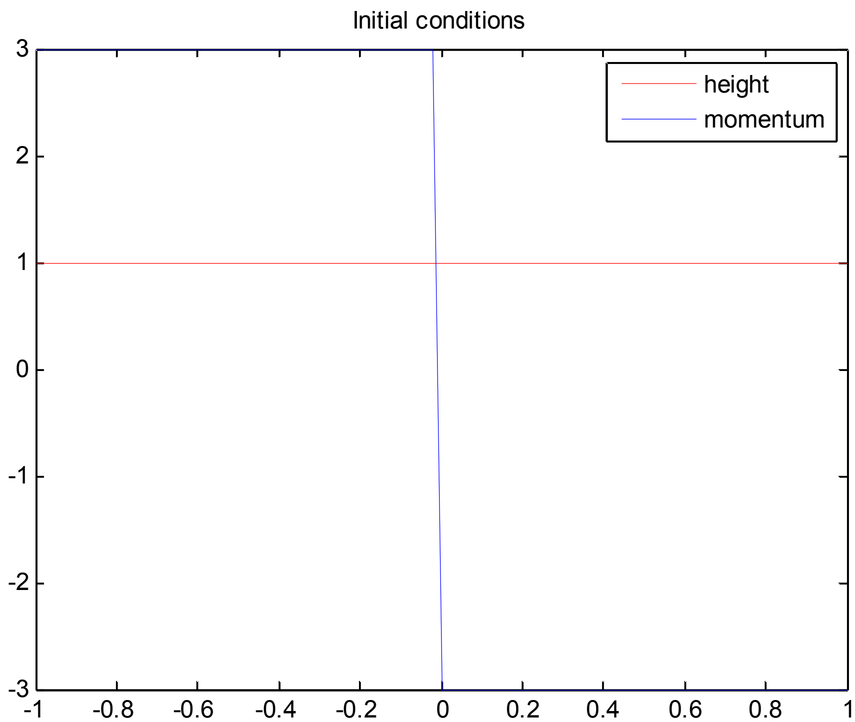


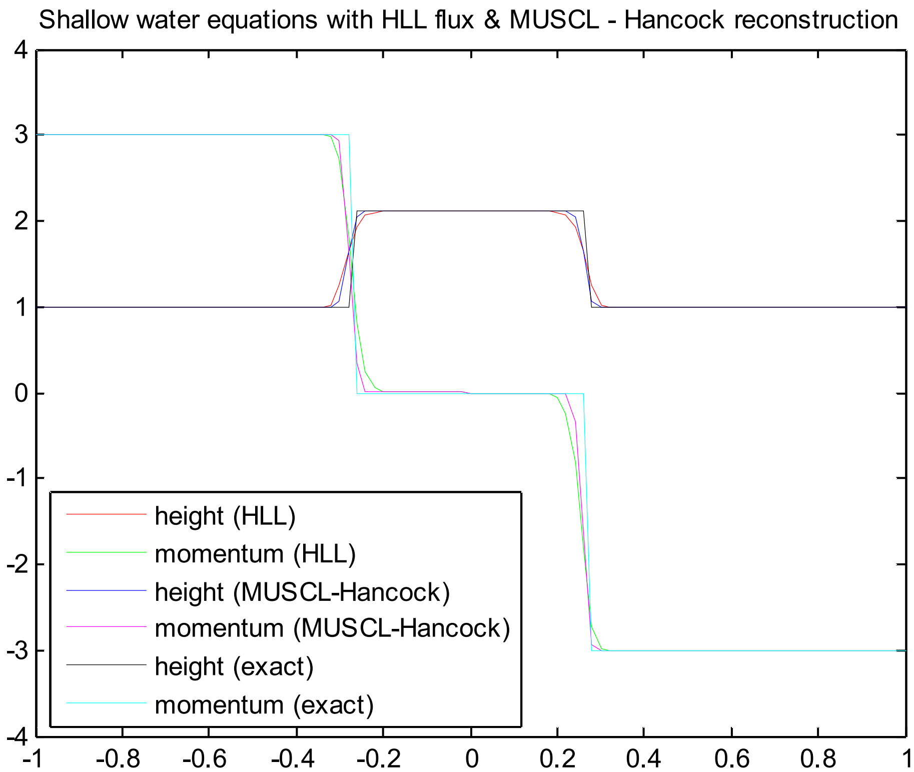
Question 3

We will see that MUSCL-Hancock is better than HLL.

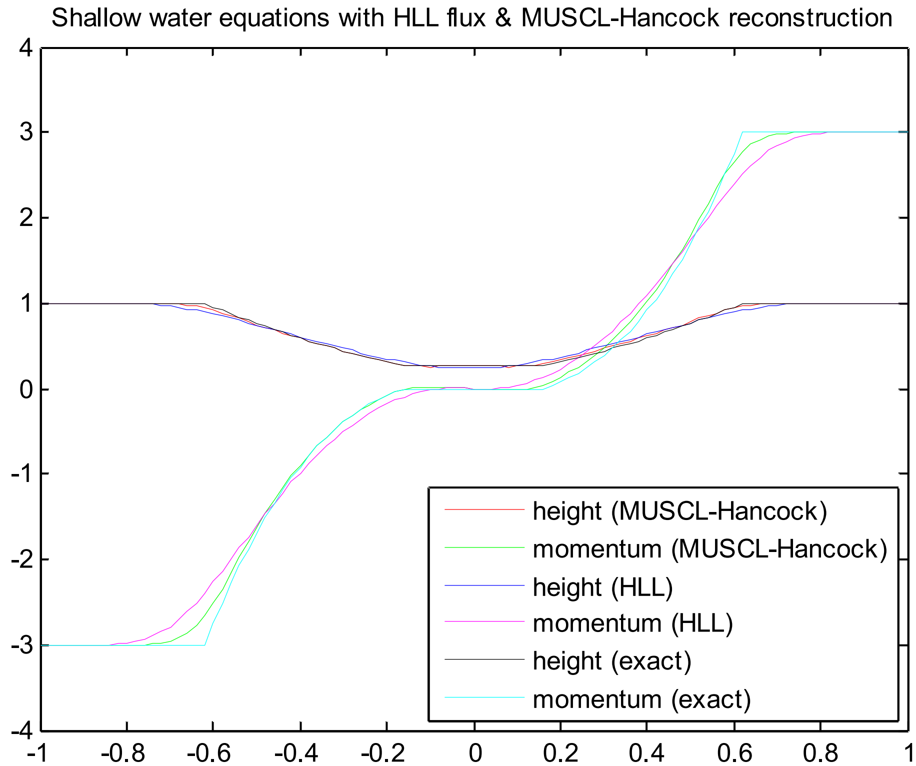
Firstly our mesh is 100 space cells.

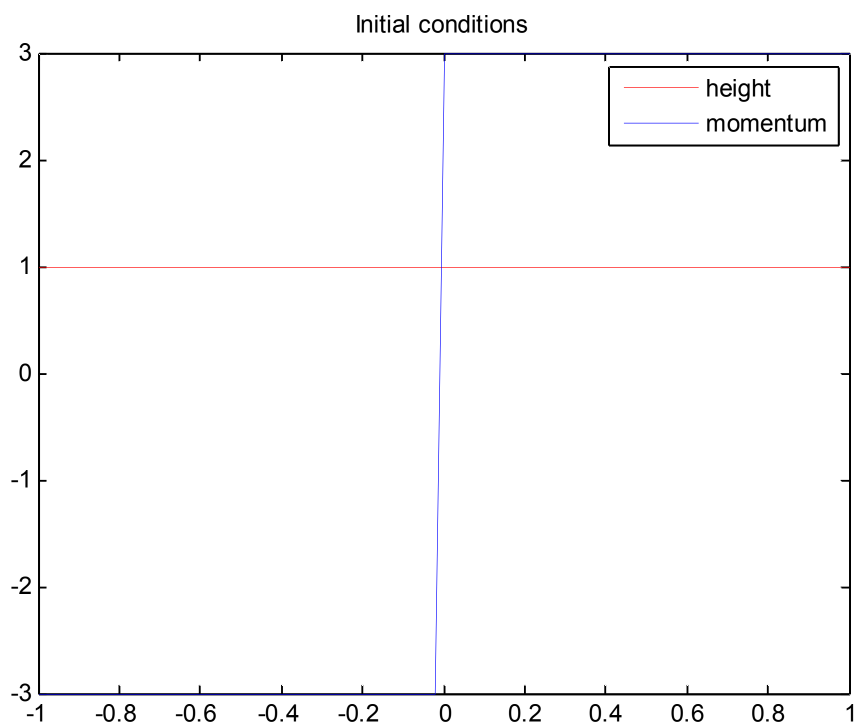
Two outgoing shocks: MUSCL-Hancock is better



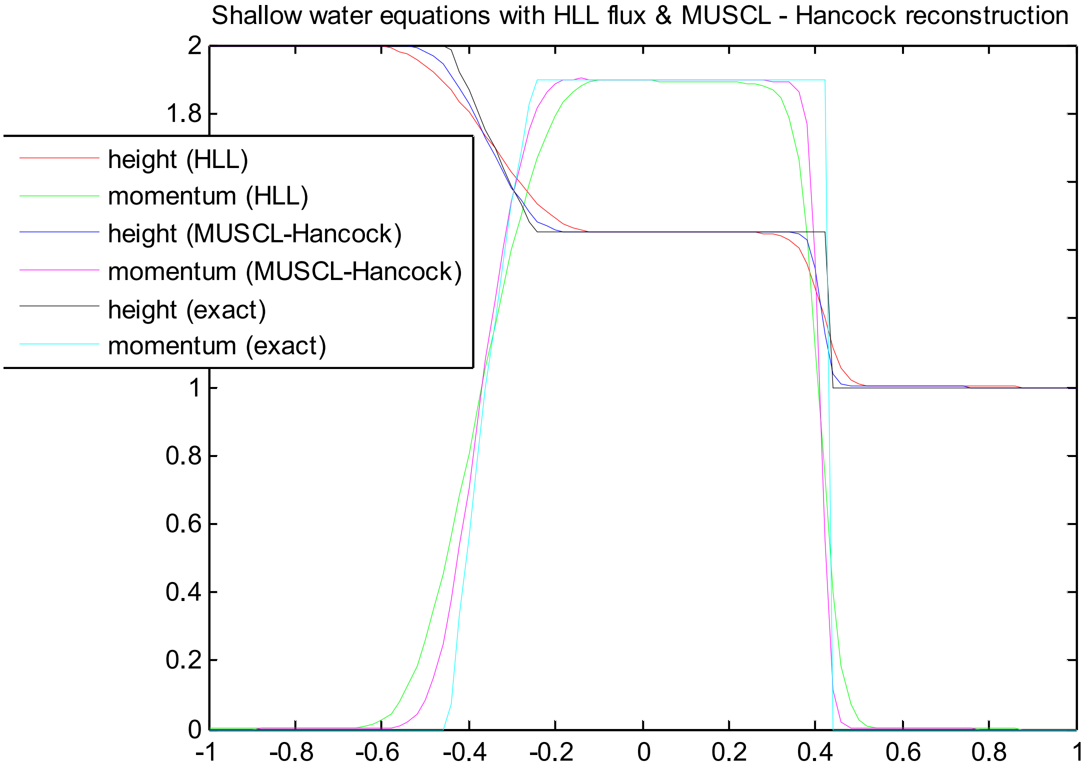

Shallow water equations with HLL flux & MUSCL - Hancock reconstruction
4
3
2
1
0
-1
-2
-3
-4
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
height (HLL)
momentum (HLL)
height (MUSCL-Hancock)
momentum (MUSCL-Hancock)
height (exact)
momentum (exact)

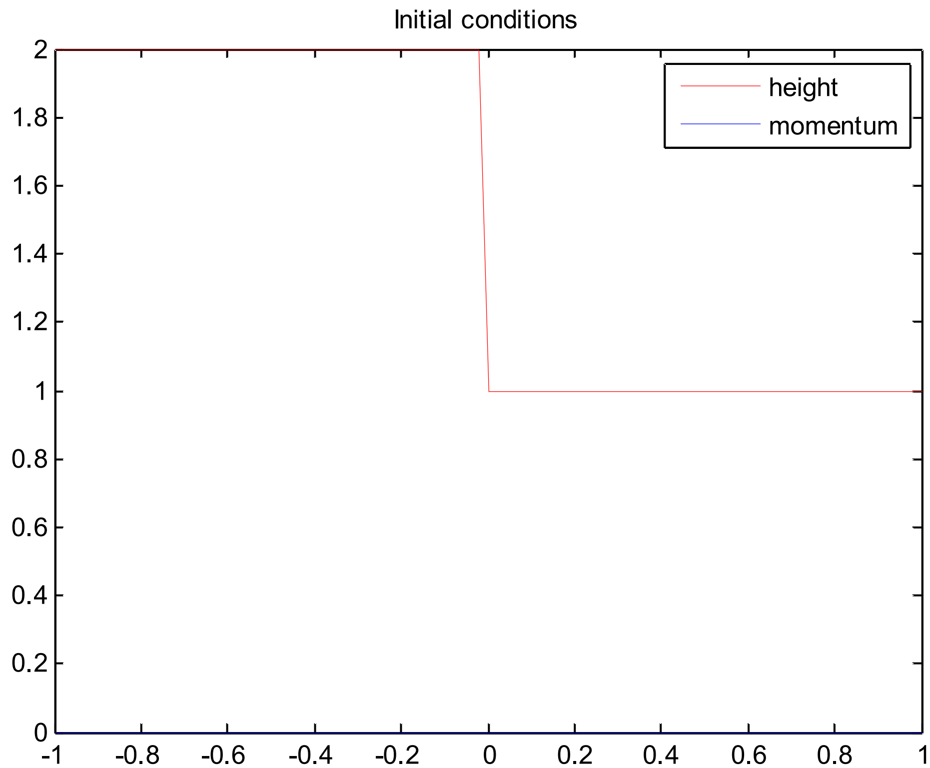
Two outgoing rarefactions: MUSCL-Hancock is better



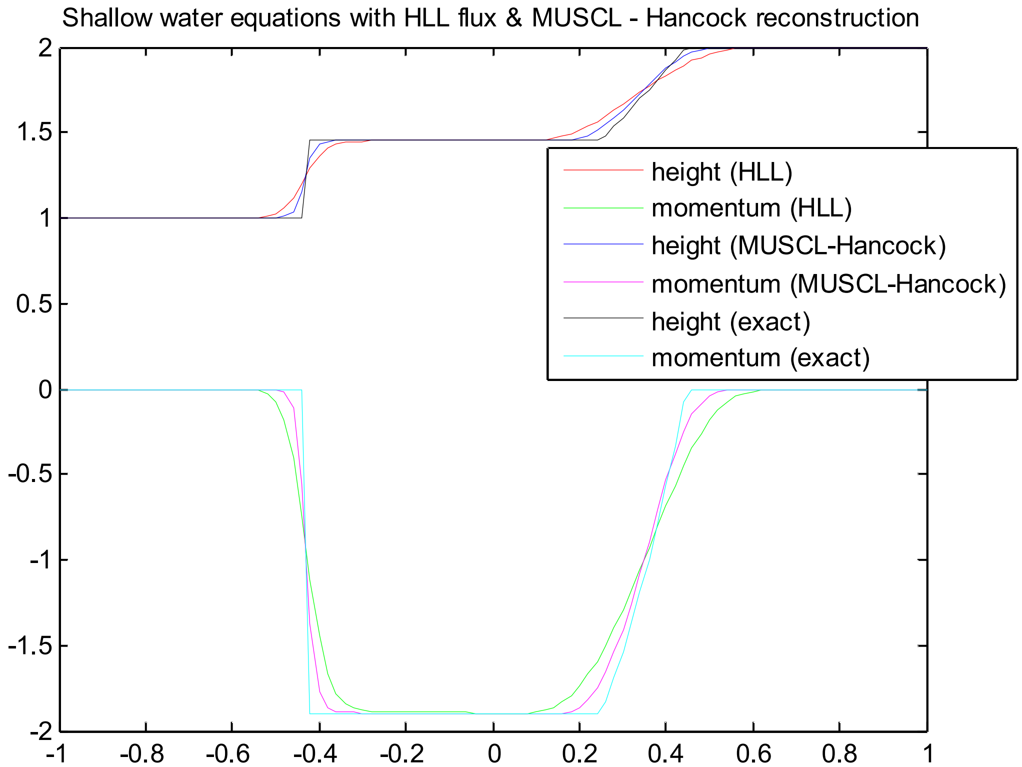


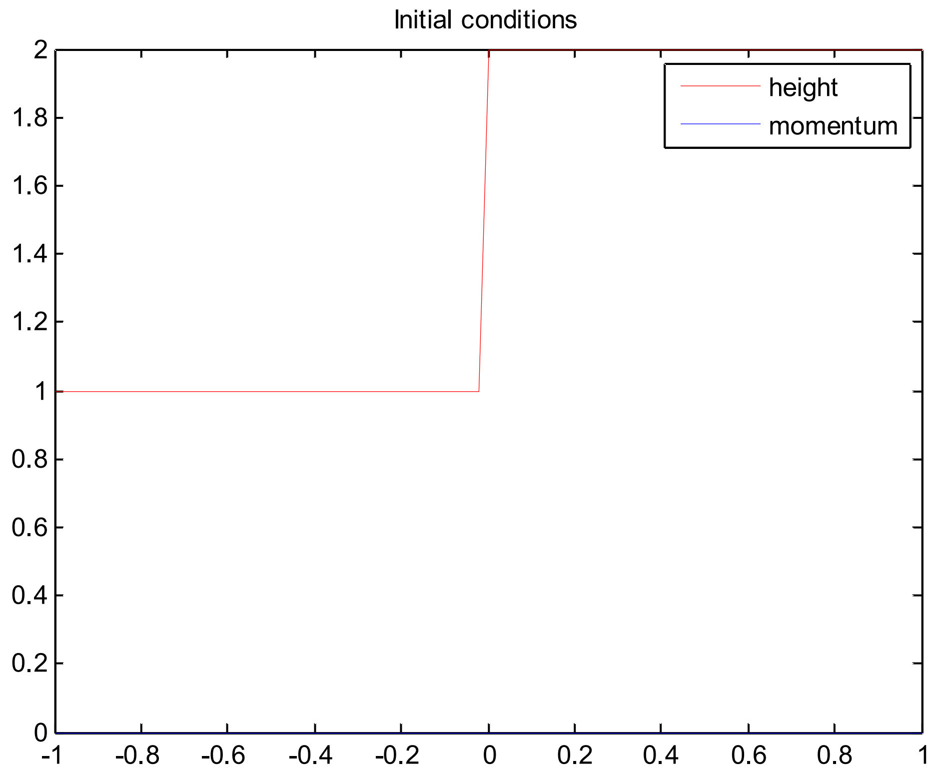
Dam break: MUSCL-Hancock is better




Initial conditions
height
momentum
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

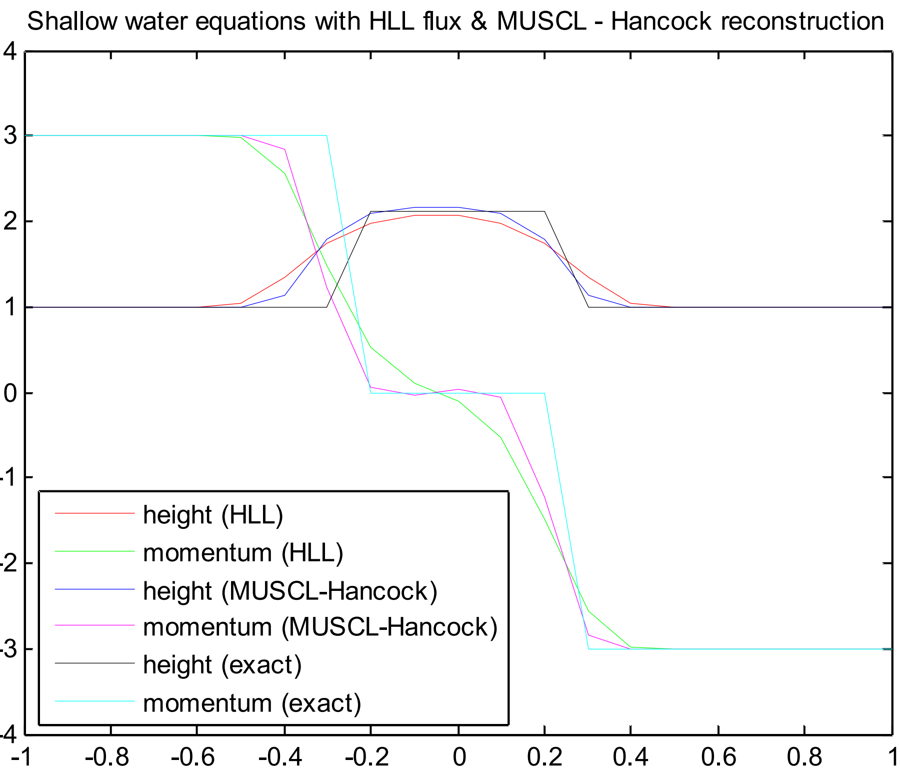
Reversed dam break: MUSCL-Hancock is better



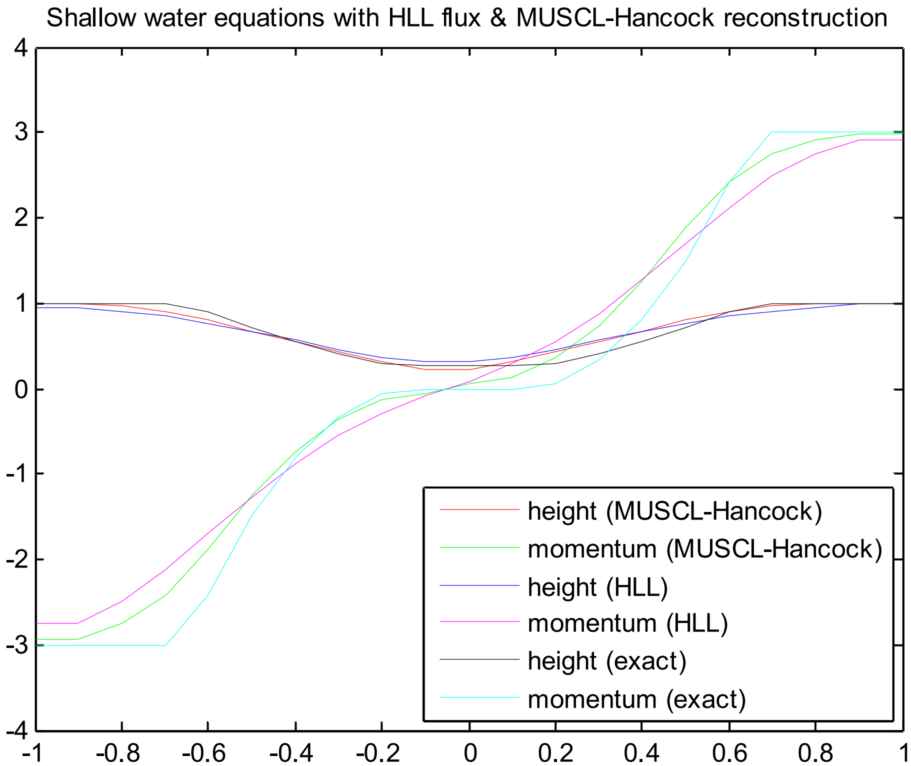

Initial conditions
height
momentum
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Now our mesh is 20 space cells

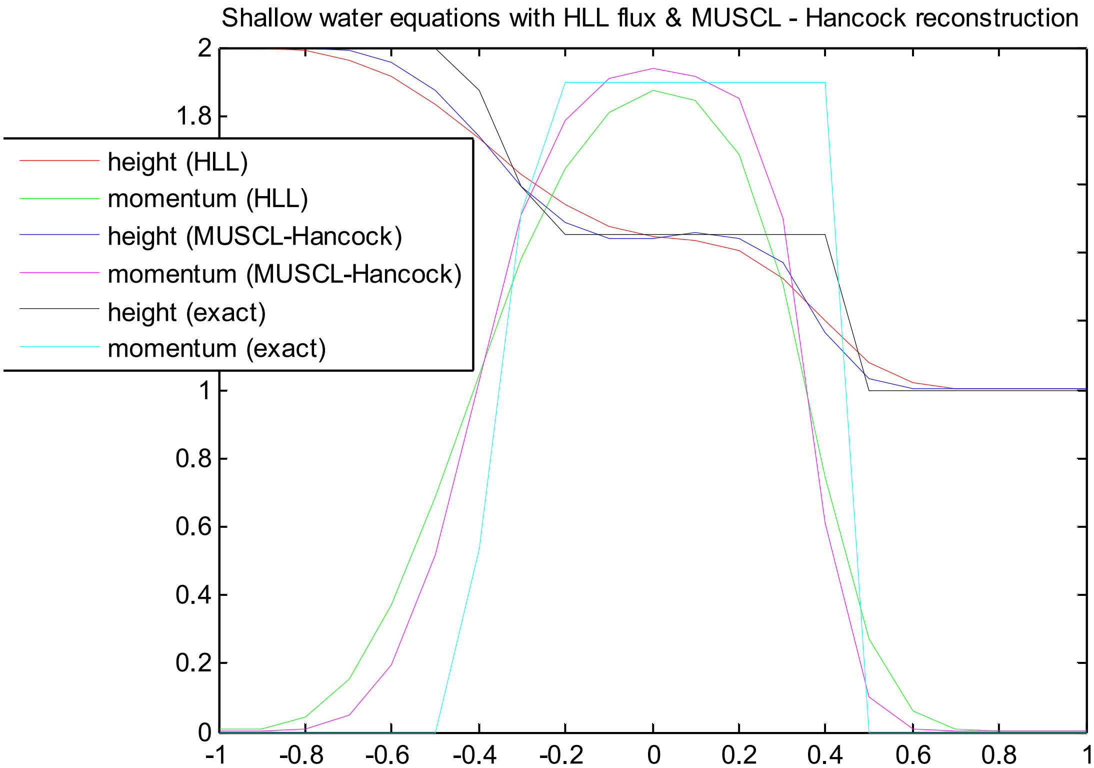
Two outgoing shocks: clearly MUSCL-Hancock is better



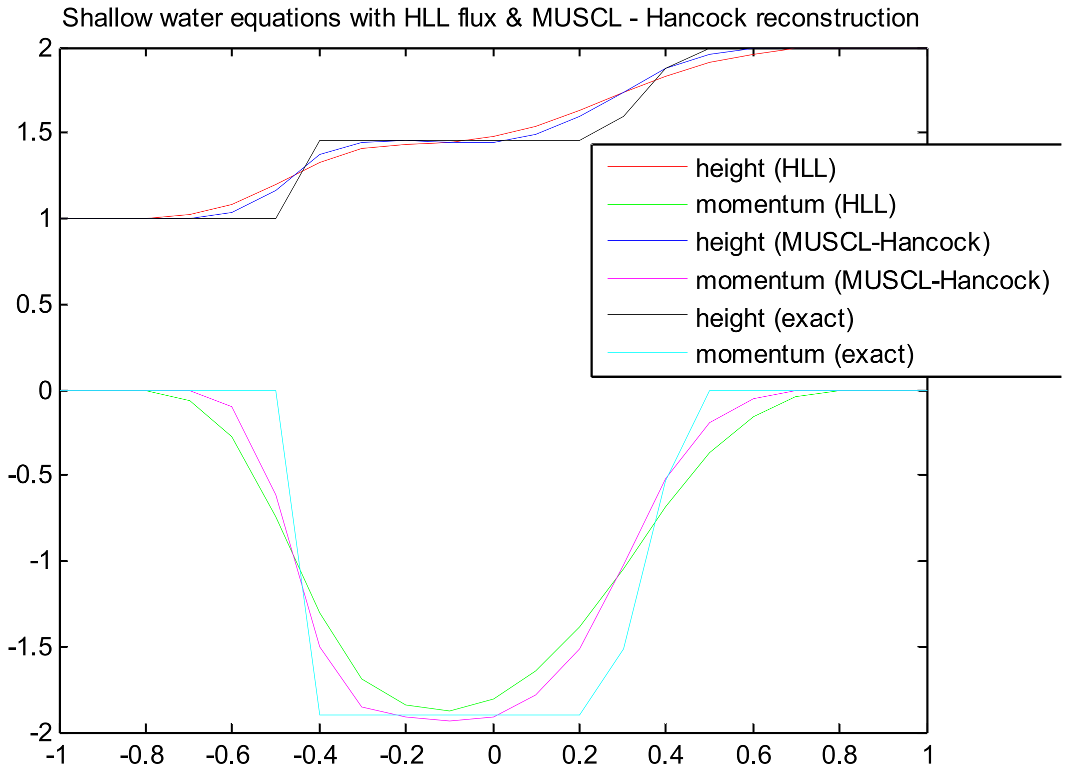
Two outgoing rarefactions: MUSCL-Hancock is better



Dam break: MUSCL-Hancock is better because the solution obtained by MUSCL-Hancock is closer than HLL



Reversed dam break: MUSCL-Hancock is better because the solution obtained by MUSCL-Hancock is closer than HLL



<u>Note:</u> the file Saint_Venant.m is originally a sample Scilab code for Saint Venant – Shallow water equations for an exercise in class of PUF Master 2015-2016. From that I ‘ve developed many codes to solve 4 Riemann problems for this Practical Test.

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[1] O. Delestre, C. Lucas, P.-A. Ksinant, F. Darboux, Ch. Laguerre, T. Vo Thi Ngoc, F. James, S. Cordier, *SWASHES: a library of Shallow Water Analytic Solutions for Hydraulic and Environmental Studies*, Int. J. Numer. Meth. Fluids, 72 (2013), 269-300, https://sourcesup.cru.fr/projects/swashes/

[2] Olivier Delestre. Simulation du ruissellement d'eau de pluie sur des surfaces agricoles. PhD thesis, Université d'Orléans, Orléans, France, July 2010.

[3] O. Delestre, F. Darboux, F. James, C. Lucas, Ch. Laguerre, S. Cordier, *FullSWOF: A free software package for the simulation of shallow water flows*, http://hal.archives-ouvertes.fr/hal-00932234,

https://sourcesup.cru.fr/projects/fullswof-2d/

[4] S. Kokh, Lecture notes Master HCMC http://samuel.kokh.free.fr/10bWMoUf/TPHCM-lecture.html

[5] http://www.univ-orleans.fr/mapmo/membres/james/Postscripts/Cours15-16.pdf