

Behaviour of the Serre Equations in the Presence of Steep Gradients Revisited

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Abstract

We use numerical methods to study the short term behaviour of the Serre equations in the presence of steep gradients because there are no known analytical solutions for these problems. In keeping with the literature we study a class of initial condition problems that are a smooth approximation to the initial conditions of the dam-break problem. This class of initial condition problems allow us to observe the behaviour of the Serre equations with varying steepness of the initial conditions. The numerical solutions of the Serre equations are justified by demonstrating that as the resolution increases they converge to a solution with little error in conservation of mass, momentum and energy independent of the numerical method. We observe and justify four different structures of the converged numerical solutions depending on the steepness of the initial conditions. Two of these structures were observed in the literature, with the other two not being commonly found in the literature. The numerical solutions are then used to assess how well the analytical solution of the shallow water wave equations captures the mean behaviour of the solution of the Serre equations for the dam-break problem. Lastly the numerical solutions are compared to asymptotic results in the literature to approximate the depth and location of the front of an undular bore.

Keywords: Serre equations, steep gradients, dam break

1. Introduction

The behaviour of flows containing steep gradients are important to a range of problems in shallow water such as the propagation of a bore, the dam-break problem and shoaling waves on a beach.

The Serre equations are used as a compromise between the non-dispersive shallow water wave equations and the incompressible inviscid Euler equations for modelling dispersive waves of the free surface in the presence of steep gradients, which are present for the Euler equations [1] but not for the shallow water wave equations. The

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9 Serre equations like the shallow water wave equations produce methods [2–4] that are
10 computationally easier and quicker to solve than the best methods for the Euler equa-
11 tions. The Serre equations are considered the most appropriate approximation to the
12 Euler equations for modelling dispersive waves up to the shore line [5, 6]. Therefore,
13 understanding the behaviour of the Serre equations in the presence of steep gradients
14 offers some insight into the behaviour of steep gradients for fluids more generally.

15 There are no known analytical solutions to problems containing steep gradients for
16 the Serre equations. To infer the structure of solutions to problems containing steep
17 gradients in the near term we have to resort to investigating numerical solutions of the
18 Serre equations for these problems. Whereas utilising modulation theory the long term
19 behaviour of steep gradient flows for dispersive equations can be understood; with El
20 and Hoefer [7] providing a review of these results for dispersive shock waves.

21 There are few examples in the literature which depict the short term behaviour of
22 numerical solutions to the Serre equations in the presence of steep gradients [1–4, 8,
23 9]. These papers all present problems with discontinuous initial conditions [2–4] or a
24 smooth approximation to them when the numerical method requires some smoothness
25 of the solutions [1, 8, 9]. Among these papers there are differences in the structures
26 of the numerical solutions, with some demonstrating undulations in depth and velocity
27 throughout the bore [3, 4, 8] and others showing a constant depth and velocity state in
28 the middle of the bore [1, 2, 9].

29 The mean behaviour of numerical solutions to the dam-break problem for the Serre
30 equations is consistent across the literature [1–4, 8, 9] as it is given by the analytical
31 solution to the dam-break problem by the shallow water wave equations [7].

32 Utilising modulation theory expressions for the long term leading wave amplitude
33 and speed of an undular bore for the Serre equations were derived and verified for a
34 range of undular bores by El et al. [8]. These expressions were also shown to be valid
35 for all the different structures found in the literature [8, 9].

36 The first aim of this paper is to investigate and explain why different short term be-
37 haviour has been published in the literature for numerical solutions of the Serre equa-
38 tions for problems containing steep gradients. We find that the undulations of a bore
39 can be damped to a constant depth and velocity state by the numerical diffusion in-
40 troduced by the method, as is the case for Le Métayer et al. [2]. Oscillation damping
41 can also occur due to the particular smoothing of the initial conditions, as is the case
42 for Mitsotakis et al. [1], El et al. [8] and Mitsotakis et al. [9]. Our results demonstrate
43 that in the short term our ‘growth’ structure is the structure that should be observed for
44 the solution of the dam-break problem for the Serre equations. While over long time
45 periods the Serre equations damps this ‘growth’ structure, which is consistent with
46 the expected long term behaviour. However, this natural decay is dominated by other
47 factors in the literature for numerical solutions over short time spans.

48 The second aim of this paper is to compare our numerical solutions with analytical
49 solution of the shallow water wave equations and the Whitham modulation results of El
50 et al. [8]. We find that for a range of dam-break problems the analytical solution of the
51 shallow water wave equations captures the mean depth and velocity of the numerical
52 solution of the Serre equations, as in the literature [2, 7, 9]. It was also found that the
53 results of El et al. [8] are a good approximation to our numerical solutions. However,
54 unlike the results of El et al. [8] and Mitsotakis et al. [9] we demonstrate that the

55 Whitham modulation results can underestimate the leading wave height and speed in
 56 the near term, even when the ratio of the two water depths is below the critical value.

57 The first aim of this paper is achieved by demonstrating that our numerical solutions
 58 are good approximations to the true solutions of the Serre equations. This is accom-
 59 plished by demonstrating that as the resolution of a particular method is increased, the
 60 numerical solutions converge to a numerical solution with little error in the conserva-
 61 tion of mass, momentum and energy. The numerical solution is also consistent across
 62 the five different numerical methods. Three of the methods are the first, second and
 63 third-order methods presented by Zoppou et al. [4]. The first-order method is equiva-
 64 lent to the method of Le Métayer et al. [2]. The fourth method is a recreation of the
 65 second-order method used by El et al. [8]. Lastly, the fifth method is a second-order
 66 finite difference approximation to the Serre equations.

67 The second aim is accomplished by comparing our verified numerical solutions to
 68 the analytical solutions of the shallow water wave equations and the Whitham modula-
 69 tion results presented by El et al. [8].

70 The paper is organised as follows, in Section 2 the Serre equations and the quan-
 71 tities they conserve are presented. In Section 3 the smoothed dam-break problem is
 72 defined and the notation for the measures of the relative difference between numerical
 73 solutions and the relative error in the conserved quantities is introduced. An analytical
 74 solution of the shallow water wave equations and the notation for the amplitude and
 75 speed of the leading wave of an undular bore are presented. In Section 4 the numerical
 76 methods and their important properties are presented. In Section 5 the four different
 77 structures in the solutions of smoothed dam-break problem for the Serre equations are
 78 determined using verified numerical solutions. The verified numerical solutions are
 79 also used to compare the analytical solution of the shallow water wave equations to the
 80 mean behaviour of the solution of the Serre equations for the dam-break problem. The
 81 Whitham modulations results are also compared to the verified numerical solutions.

82 2. Serre Equations

83 The Serre equations can be derived by integrating the full inviscid incompressible
 84 Euler equations over the water depth [10]. They can also be derived as an asymptotic
 85 expansion of the Euler equations [11]. Assuming a constant horizontal bed, the one-
 86 dimensional Serre equations are [12]

$$87 \quad \frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0 \quad (1a)$$

88 and

$$90 \quad \underbrace{\frac{\partial(uh)}{\partial t} + \frac{\partial}{\partial x} \left(u^2 h + \frac{gh^2}{2} \right)}_{\text{Shallow Water Wave Equations}} + \underbrace{\frac{\partial}{\partial x} \left(\frac{h^3}{3} \left[\frac{\partial u}{\partial x} \frac{\partial u}{\partial x} - u \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t} \right] \right)}_{\text{Dispersion Terms}} = 0. \quad (1b)$$

91 Serre Equations

92 Where $u(x, t)$ is the horizontal velocity over the depth of water $h(x, t)$, g is the acceler-
 93 ation due to gravity, x is the horizontal spatial variable and t is time.

94 The Serre equations are conservation laws for ‘mass’ (1a), ‘momentum’ (1b) and
 95 the Hamiltonian [13, 14]

$$96 \quad \mathcal{H}(x, t) = \frac{1}{2} \left(hu^2 + \frac{h^3}{3} \left(\frac{\partial u}{\partial x} \right)^2 + gh^2 \right) \quad (2)$$

97
 98 which is the total energy.

99 3. Smoothed Dam Break Problem

100 In this section we define a class of initial condition problems, called the smoothed
 101 dam-break problem that we use throughout our numerical investigation. This class of
 102 initial conditions are used in the literature [1, 9] to smoothly approximate the discontin-
 103 uous initial conditions of the dam-break problem, as some numerical methods require
 104 smoothness of the solutions.

105 The smoothed dam-break problem has the following initial conditions

$$106 \quad h(x, 0) = h_0 + \frac{h_1 - h_0}{2} \left(1 + \tanh \left(\frac{x_0 - x}{\alpha} \right) \right) m, \quad (3a)$$

107
 108 and

$$109 \quad u(x, 0) = 0.0 \text{ m/s}. \quad (3b)$$

110
 111 This represents a smooth transition centred around x_0 between a water depth of h_0
 112 on the right which is smaller than the water depth of h_1 on the left. Here α measures
 113 the distance over which approximately 46% of that smooth transition between the two
 114 heights occurs. These are the same h_0 and h_1 values as those of the smoothed dam-
 115 break problem of El et al. [8] and the dam-break problem of Le Métayer et al. [2].

116 There are no known analytical solutions of the Serre equations for the dam-break
 117 problem or an arbitrary smoothed dam-break problem. Therefore, to demonstrate that
 118 our numerical solutions converge we use the relative difference between numerical
 119 solutions L_1^h and L_1^u for the primitive variables h and u respectively. To demonstrate that
 120 our numerical solutions conserve the quantities h , uh and \mathcal{H} well we use the relative
 121 error of their conservation C_1^h , C_1^{uh} and $C_1^{\mathcal{H}}$ respectively.

122 3.1. Background for derived and observed comparisons

123 Because there are no known analytic solutions to the dam-break problem for the
 124 Serre equations the literature has relied on comparisons to simpler equations such as
 125 the shallow water wave equations [2, 9] or asymptotic solutions [8] to understand the
 126 behaviour of the Serre equations for the dam-break problem.

127 It was demonstrated by Le Métayer et al. [2] and Mitsotakis et al. [9] that the
 128 analytical solution of the shallow water wave equations for the dam-break problem
 129 captures the mean behaviour of the numerical solutions of the Serre equations to the
 130 dam-break problem [2] and the smoothed dam-break problem [9].

131 An example of the analytical solution of the shallow water wave equations for the
 132 dam-break problem is presented in Figure 1. Region I is the undisturbed water up-
 133 stream of the dam-break at constant height (h_1) and velocity (0m/s). Region II is the

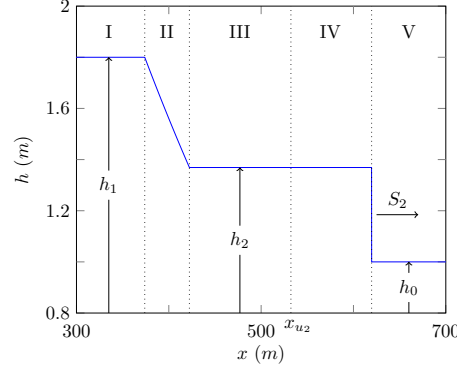


Figure 1: Analytical solution at $t = 30s$ of the dam-break problem for the shallow water wave equations with $h_0 = 1m$, $h_1 = 1.8m$ and $x_0 = 500m$.

rarefaction fan connecting regions I and III. Regions III and IV compose the constant height (h_2) and constant velocity (u_2) state, with the regions separated by $x_{u_2} = x_0 + u_2 t$. Region V is the undisturbed water downstream at constant height (h_0) and velocity ($0m/s$) separated from Region IV by a shock which travels at velocity S_2 . Expressions for the unknown quantities h_2 , u_2 and S_2 in terms of h_0 and h_1 are known, with the expressions of Wu et al. [15] requiring the solution of a single non-linear equation.

For our dam-break heights of interest; $h_0 = 1m$ and $h_1 = 1.8m$ we have $h_2 = 1.36898m$, $u_2 = 1.074975 m/s$, $S_2 = 3.98835 m/s$ and $x_{u_2} = 532.24925m$ which are shown in Figure 1 for $t = 30s$. The location of the front of the bore for the shallow water wave equations at time t is thus $x_{S_2} = x_0 + S_2 t$ so that $x_{S_2} = 619.6505m$ at $t = 30s$.

Utilizing Whitham modulation theory for a one-phase periodic travelling wave asymptotic expressions for the amplitude A^+ and speed S^+ of the leading wave were derived by El et al. [8]. These expressions were found to be most appropriate for dam-break aspect ratios $h_1/h_0 \leq 1.94$.

For our dam-break heights of interest $h_0 = 1m$ and $h_1 = 1.8m$ we obtain $A^+ = 1.73998m$ and $S^+ = 4.13148m/s$. The location of the leading wave of an undular bore at time t is then $x_{S^+} = x_0 + S^+ t$ so that $x_{S^+} = 623.9444m$ for $t = 30s$.

4. Numerical Methods

Five numerical schemes were used to investigate the behaviour of the Serre equations in the presence of steep gradients, the first (\mathcal{V}_1), second (\mathcal{V}_2) and third-order (\mathcal{V}_3) finite difference finite volume methods of Zoppou et al. [4], the second-order finite difference method of El et al. [8] (\mathcal{E}) and a second-order finite difference method (\mathcal{D}) that can be found in the Appendix.

The \mathcal{V}_i methods are stable under a Courant-Friedrichs-Lewy (CFL) condition presented by A. Harten [16]. The \mathcal{V}_i methods have demonstrated the appropriate order of convergence for smooth problems [4]. Furthermore, \mathcal{V}_2 and \mathcal{V}_3 have been validated against experimental data containing steep gradients [4]. The two methods \mathcal{D} and \mathcal{E} were found to be stable under the same CFL condition.

163 Generally, we found that \mathcal{V}_1 is the worst performing method due to its numeri-
 164 cal diffusion [4]. Of the high-order methods \mathcal{E} is the worst performing, introducing
 165 dispersive errors.

166 5. Numerical Results

167 We investigate the behaviour of the Serre equations in the presence of steep gradi-
 168 ents by numerically solving the smoothed dam-break problem while varying the steep-
 169 ness of the initial conditions. As $\Delta x \rightarrow 0$ our numerical solutions should represent
 170 a good approximation of the true solution of the Serre equations. If our numerical
 171 solutions to a smoothed dam-break problem converge to the same numerical solution
 172 with little error in conservation of mass, momentum and energy as $\Delta x \rightarrow 0$ for each
 173 method, then this numerical solution is considered an accurate approximate solution to
 174 that smoothed dam-break problem for the Serre equations.

175 This process validates our numerical solutions for the smoothed dam-break prob-
 176 lem, and thus validates our numerical methods to approximate the solution of the Serre
 177 equations in the presence of steep gradients, if it exists. With a validated model we can
 178 compare the numerical solution to the analytical solution of the shallow water wave
 179 equations for the dam-break problem and the results of El et al. [8].

180 Throughout most of this section we are interested in the numerical solution at $t =$
 181 $30s$ to the smoothed dam-break problem with $h_0 = 1m$, $h_1 = 1.8m$ and $x_0 = 500m$
 182 while allowing for different α values. All numerical methods used $\Delta t = 0.01\Delta x$ which
 183 is smaller than required by the CFL condition, ensuring stability of our schemes. The
 184 method \mathcal{V}_2 requires an input parameter to its slope limiter and this was chosen to be
 185 $\theta = 1.2$ [4]. The spatial domain was $[0m, 1000m]$ with the following Dirichlet boundary
 186 conditions, $u = 0m/s$ at both boundaries, $h = 1.8m$ on the left and $h = 1m$ on the right.

187 5.1. Observed Structures of the Numerical Solutions

188 We observe that there are four different structures for the converged to numerical
 189 solution depending on the chosen α . They are the ‘non-oscillatory’ structure \mathcal{S}_1 , the
 190 ‘flat’ structure \mathcal{S}_2 , the ‘node’ structure \mathcal{S}_3 and the ‘growth’ structure \mathcal{S}_4 . An example
 191 of each of these structures is shown in Figure 2 which were obtained using \mathcal{V}_3 with
 192 $\Delta x = 10/2^{11}m$.

193 The four structures are identified by the dominant features of the numerical solu-
 194 tions in regions III and IV. They also correspond to different structures in the numerical
 195 solutions that have been presented in the literature. From Figure 2 it can be seen that as
 196 α is decreased, steepening the initial conditions, the numerical solutions demonstrate
 197 an increase in the size and number of oscillations particularly around $x_{u_2} = x_0 + u_2t$. We
 198 observe that the difference between \mathcal{S}_2 , \mathcal{S}_3 and \mathcal{S}_4 is the amplitude of the oscillations
 199 in regions III and IV.

200 For the non-oscillatory and flat structures there is excellent agreement between all
 201 higher-order numerical methods at our highest resolution $\Delta x = 10/2^{11}m$. An illustra-
 202 tion of this agreement is given in Figure 3 for \mathcal{S}_2 which is the most difficult to resolve of
 203 the two structures. However, the first-order method \mathcal{V}_1 suppresses oscillations present
 204 in the numerical solutions of other methods due to its diffusive errors [4]. To resolve
 205 these oscillations with \mathcal{V}_1 lower values of Δx are required.

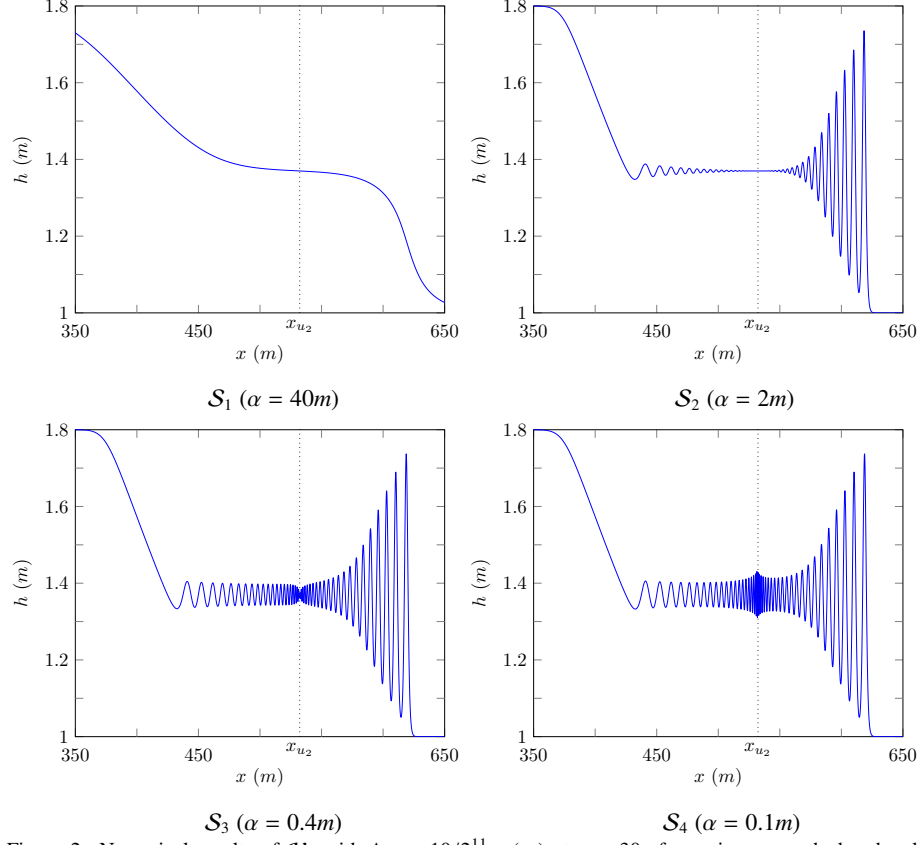


Figure 2: Numerical results of \mathcal{V}_3 with $\Delta x = 10/2^{11}m$ (—) at $t = 30s$ for various smooth dam-break problems demonstrating the different observed structures particularly around x_{u_2} (\cdots).

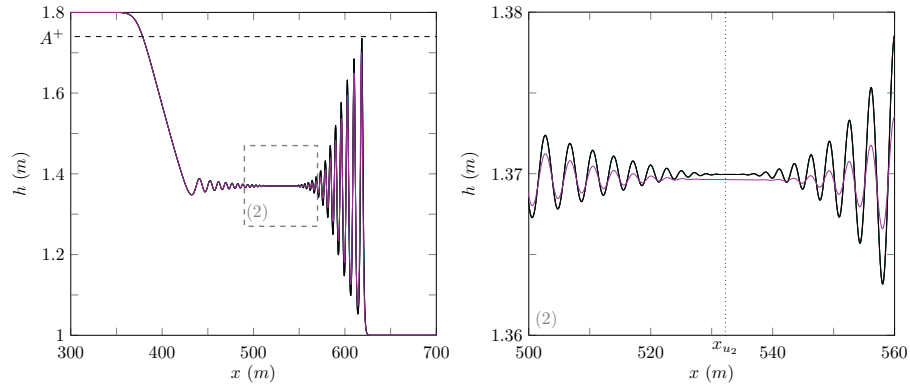


Figure 3: Numerical solutions of \mathcal{D} (—), \mathcal{E} (—), \mathcal{V}_3 (—), \mathcal{V}_2 (—) and \mathcal{V}_1 (—) with $\Delta x = 10/2^{11}m$ at $t = 30s$ for the smooth dam-break problem with $\alpha = 2m$. The Whitham modulation result for the leading wave height A^+ (—) and x_{u_2} (\cdots) are presented for comparison.

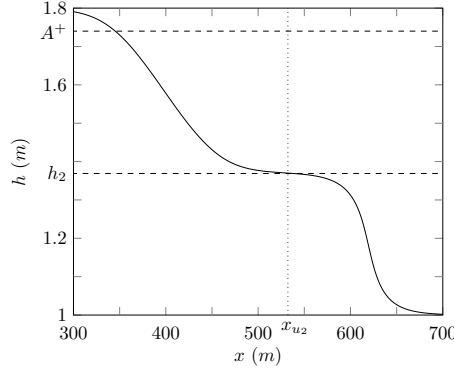


Figure 4: Numerical solutions of \mathcal{V}_3 at $t = 30s$ for smooth dam-break problem with $\alpha = 40m$ for $\Delta x = 10/2^{10}m$ (—), $10/2^8m$ (—), $10/2^6m$ (—) and $10/2^4m$ (—). The important quantities A^+ (—), h_2 (—) and x_{u2} (···) are also presented.

5.1.1. Non-oscillatory Structure

The \mathcal{S}_1 “non-oscillatory” structure is the result of a large α , which causes the front of this flow to not be steep enough to generate undulations over short time periods. As the system evolves the front will steepen due to non-linearity and undulations will develop.

The structure \mathcal{S}_1 is not present in the literature as no authors chose large enough α because, such a large α poorly approximates the dam-break problem. An example of this structure can be seen in Figure 4 for $\alpha = 40m$ using \mathcal{V}_3 with various Δx values. Because this is not a very steep problem all numerical results are visually identical for all $\Delta x < 10/2^4m$.

From Table 1 it can be seen that not only have these solutions converged visually but the L_1 measures demonstrate that we have reached convergence to round-off error by $\Delta x = 10/2^8m$ after which the relative difference between numerical solutions plateau.

Table 1 also demonstrates that the error in conservation of the numerical solutions are at round-off error for h and \mathcal{H} . The conservation of uh is poor because the smoothed dam-break has such a large α that $h(0m) \neq 1.8m$ and $h(1000m) \neq 1m$, causing unequal fluxes in momentum at the boundaries.

As stated above when $\Delta x = 10/2^{11}m$ the numerical solutions from all methods are identical for this smoothed dam-break problem.

The convergence of the numerical solutions as $\Delta x \rightarrow 0$ to a numerical solution with small error in conservation, independent of the method demonstrates that we have accurately solved the smoothed dam-break problem with $\alpha = 40m$. Therefore, the \mathcal{S}_1 structure should be observed in the solutions of the Serre equations for the smoothed dam-break problem for sufficiently large α .

5.1.2. Flat Structure

The most common structure observed in the literature [1, 2, 9] is the “flat structure” \mathcal{S}_2 . It is observed when the initial conditions are steep enough such that the bore that develops has undulations. This structure consists of oscillations in regions III and

| α | Δx | C_1^h | C_1^{uh} | $C_1^{\mathcal{H}}$ | L_1^h | L_1^u |
|----------|-------------|-----------------------|----------------------|-----------------------|------------------------------|------------------------------|
| 40 | $10/2^4$ | $2.00 \cdot 10^{-11}$ | $1.77 \cdot 10^{-6}$ | $1.23 \cdot 10^{-8}$ | $1.74 \cdot 10^{-7}$ | $2.90 \cdot 10^{-6}$ |
| 40 | $10/2^6$ | $1.07 \cdot 10^{-11}$ | $1.50 \cdot 10^{-6}$ | $1.49 \cdot 10^{-10}$ | $2.57 \cdot 10^{-9}$ | $4.19 \cdot 10^{-8}$ |
| 40 | $10/2^8$ | $8.77 \cdot 10^{-13}$ | $5.49 \cdot 10^{-7}$ | $3.77 \cdot 10^{-13}$ | $6.08 \cdot 10^{-11}$ | $5.28 \cdot 10^{-10}$ |
| 40 | $10/2^{10}$ | $1.77 \cdot 10^{-11}$ | $2.21 \cdot 10^{-8}$ | $3.56 \cdot 10^{-11}$ | $2.54 \cdot 10^{-11}$ | $6.49 \cdot 10^{-11}$ |
| 2 | $10/2^4$ | $4.90 \cdot 10^{-14}$ | $5.10 \cdot 10^{-3}$ | $8.69 \cdot 10^{-4}$ | $5.02 \cdot 10^{-3}$ | $6.77 \cdot 10^{-2}$ |
| 2 | $10/2^6$ | $2.51 \cdot 10^{-13}$ | $2.18 \cdot 10^{-4}$ | $6.58 \cdot 10^{-5}$ | $4.14 \cdot 10^{-4}$ | $5.20 \cdot 10^{-3}$ |
| 2 | $10/2^8$ | $9.81 \cdot 10^{-13}$ | $7.72 \cdot 10^{-7}$ | $5.01 \cdot 10^{-7}$ | $6.00 \cdot 10^{-6}$ | $7.59 \cdot 10^{-5}$ |
| 2 | $10/2^{10}$ | $3.95 \cdot 10^{-12}$ | $5.56 \cdot 10^{-9}$ | $6.13 \cdot 10^{-9}$ | $1.76 \cdot 10^{-7}$ | $2.33 \cdot 10^{-6}$ |
| 0.4 | $10/2^4$ | $9.00 \cdot 10^{-14}$ | $4.82 \cdot 10^{-3}$ | $1.02 \cdot 10^{-3}$ | $6.79 \cdot 10^{-3} \dagger$ | $9.93 \cdot 10^{-2} \dagger$ |
| 0.4 | $10/2^6$ | $2.40 \cdot 10^{-13}$ | $2.41 \cdot 10^{-4}$ | $1.11 \cdot 10^{-4}$ | $8.89 \cdot 10^{-4} \dagger$ | $1.13 \cdot 10^{-2} \dagger$ |
| 0.4 | $10/2^8$ | $9.68 \cdot 10^{-13}$ | $7.57 \cdot 10^{-7}$ | $2.25 \cdot 10^{-6}$ | $1.53 \cdot 10^{-5} \dagger$ | $1.91 \cdot 10^{-4} \dagger$ |
| 0.4 | $10/2^{10}$ | $3.91 \cdot 10^{-12}$ | $4.95 \cdot 10^{-9}$ | $2.01 \cdot 10^{-8}$ | $3.61 \cdot 10^{-7} \dagger$ | $5.00 \cdot 10^{-6} \dagger$ |
| 0.1 | $10/2^4$ | $7.60 \cdot 10^{-14}$ | $4.82 \cdot 10^{-3}$ | $1.06 \cdot 10^{-3}$ | $7.04 \cdot 10^{-3} \dagger$ | $1.02 \cdot 10^{-1} \dagger$ |
| 0.1 | $10/2^6$ | $2.40 \cdot 10^{-13}$ | $2.39 \cdot 10^{-4}$ | $1.44 \cdot 10^{-4}$ | $1.02 \cdot 10^{-3} \dagger$ | $1.28 \cdot 10^{-2} \dagger$ |
| 0.1 | $10/2^8$ | $9.79 \cdot 10^{-13}$ | $2.21 \cdot 10^{-7}$ | $1.20 \cdot 10^{-5}$ | $2.86 \cdot 10^{-5} \dagger$ | $3.46 \cdot 10^{-4} \dagger$ |
| 0.1 | $10/2^{10}$ | $3.92 \cdot 10^{-12}$ | $4.46 \cdot 10^{-8}$ | $7.61 \cdot 10^{-7}$ | $4.99 \cdot 10^{-7} \dagger$ | $6.40 \cdot 10^{-6} \dagger$ |

Table 1: All errors in conservation C_1^q for the conserved quantities and relative differences L_1^q of the primitive variables for numerical solutions of \mathcal{V}_3 . L_1^q uses the numerical solution with $\Delta x = 10/2^{11}m$ as the high resolution basis of comparison and \dagger indicates the omission of the interval $[520m, 540m]$ from the comparison.

234 IV which are separated by a constant height state around x_{u_2} . An example of the S_2
 235 structure can be seen in the numerical solutions presented in Figure 5 where $\alpha = 2m$.

236 As Δx decreases the numerical solutions converge so that by $\Delta x = 10/2^8 m$ the
 237 solutions for higher Δx are visually identical. Table 1 demonstrates that although we
 238 have convergence visually, the L_1 measures are still decreasing and are larger than
 239 round-off error. Likewise the C_1 measures are still decreasing and have only reached
 240 round-off error for h . This indicates that to attain full convergence of the numerical
 241 solutions of this smoothed dam-break problem down to round-off error using \mathcal{V}_3 would
 242 require an even smaller Δx . The relative difference between numerical solutions is
 243 small and the numerical solutions exhibit good conservation. Therefore, our highest
 244 resolution numerical solution is a good approximation to any numerical solutions with
 245 lower Δx values. Figure 3 demonstrates that at $\Delta x = 10/2^{11} m$ the numerical solutions
 246 of all higher order methods are the same.

247 These results demonstrate that our highest resolution numerical solution is an accu-
 248 rate approximate solution of the Serre equations for the smoothed dam-break problem
 249 with $\alpha = 2m$. This implies that the S_2 structure should be observed in solutions of the
 250 Serre equations for smooth dam-break problems with similar α values.

251 These numerical solutions compare well with those of Mitsotakis et al. [9] who
 252 use the same α but different h_0 and h_1 values and observe the S_2 structure. We found
 253 that we observed this structure for all numerical method's numerical solutions to the
 254 smoothed dam-break problem with α values as low as $1m$ and $\Delta x = 10/2^{11} m$. The
 255 numerical solutions of Mitsotakis et al. [1] use $\alpha = 1m$ but different heights and observe
 256 the structure S_2 . Therefore Mitsotakis et al. [1] and Mitsotakis et al. [9] observe the S_2
 257 structure in their numerical results due to their choice of α for the smoothed dam-break
 258 problem.

259 The first-order method \mathcal{V}_1 is diffusive [4] and damps oscillations that are present
 260 in the numerical solutions of higher-order methods as in Figure 3. We find that for any
 261 smoothed dam-break problem with $\alpha \leq 4m$ and the dam-break problem only the S_2
 262 structure is observed for the numerical solutions of \mathcal{V}_1 at $t = 30s$ with $\Delta x = 10/2^{11} m$.
 263 This is evident in Figure 6 with the numerical solutions of \mathcal{V}_1 using our finest grid
 264 where $\Delta x = 10/2^{11} m$ on our steepest initial conditions where $\alpha = 0.001m$. There-
 265 fore, Le Métayer et al. [2] using the diffusive \mathcal{V}_1 with their chosen Δx and Δt , which
 266 are larger than our Δx and Δt could only observe the S_2 structure for the dam-break
 267 problem.

268 5.1.3. Node Structure

269 The “node” structure, S_3 was observed by El et al. [8]. The S_3 structure has os-
 270 cillations throughout regions III and IV that decay to a node at x_{u_2} as can be seen in
 271 Figure 7 where $\alpha = 0.4m$.

272 Figure 7 demonstrates that our numerical solutions have not converged, however
 273 this is only in the area around x_{u_2} . Due to the large difference in numerical solutions
 274 around x_{u_2} the L_1 measure over the area around x_{u_2} would not be insightful. However,
 275 by omitting this region we can gain some knowledge about how well our solutions
 276 agree away from x_{u_2} . This was performed for the relevant L_1 measures in Table 1
 277 by omitting the interval $[520m, 540m]$. These modified L_1 measures demonstrate that

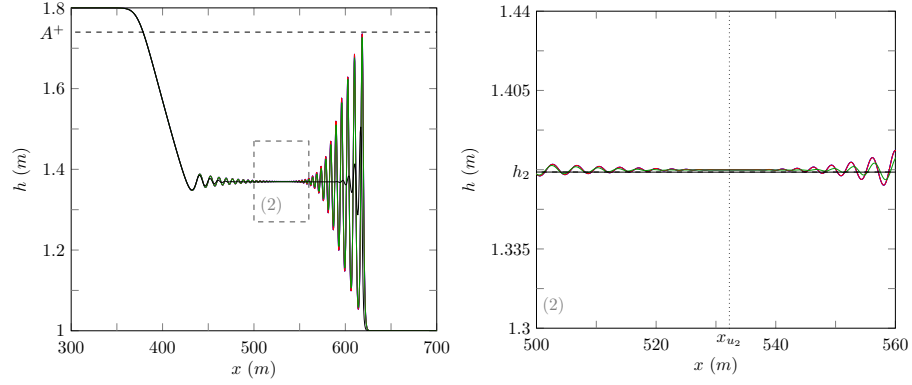


Figure 5: Numerical solutions of \mathcal{V}_3 at $t = 30s$ for the smooth dam-break problem with $\alpha = 2m$ for $\Delta x = 10/2^{10}m$ (—), $10/2^8m$ (—), $10/2^6m$ (—) and $10/2^4m$ (—). The important quantities A^+ (—), h_2 (—) and x_{u_2} (···) are also presented.

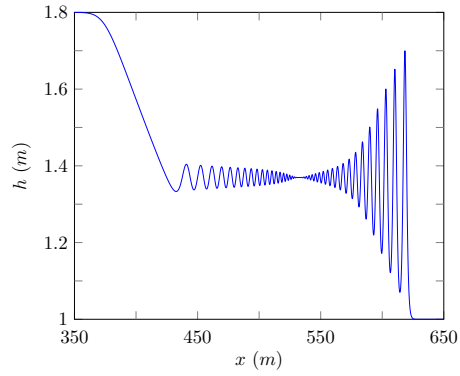


Figure 6: Numerical solution of \mathcal{V}_1 at $t = 30s$ for the smooth dam-break problem with $\alpha = 0.001m$ for $\Delta x = 10/2^{11}m$ (—).

278 while our numerical results have visually converged outside this interval, they have not
 279 converged down to round-off error.

280 Table 1 demonstrates that the C_1 measures are still decreasing and have only at-
 281 tained round-off error for h . Therefore, to resolve the desired convergence of the nu-
 282 merical solutions to one with small error in conservation using \mathcal{V}_3 would require even
 283 smaller Δx values.

284 There is good agreement across different numerical methods for $\Delta x = 10/2^{11}m$ as
 285 can be seen in Figure 8. In particular all the higher-order methods exhibit the same
 286 structure and only disagree in a very small region around x_{u_2} . We observe that the
 287 numerical solution of the worst higher-order method \mathcal{E} has not converged well to the
 288 numerical solutions of the other higher-order methods.

289 We have only obtained a good approximation to the desired numerical solution as
 290 $\Delta x \rightarrow 0$ away from x_{u_2} . However, our highest resolution numerical solutions from
 291 various higher-order methods are very similar. This suggests that again although we
 292 do not have full convergence, our highest resolution numerical solution is a good ap-
 293 proximation to the desired numerical solution over the whole domain. Therefore, our
 294 highest resolution numerical solutions are an accurate representation of the solutions of
 295 the Serre equations for this smoothed dam-break problem. Therefore, the \mathcal{S}_3 structure
 296 should be observed in the solutions of the Serre equations for the smoothed dam-break
 297 problem with $\alpha = 0.4m$.

298 These numerical solutions support the findings of El et al. [8] who also use some
 299 smoothing [7] but do not report what smoothing was performed. Using their method \mathcal{E}
 300 and similar Δx to El et al. [8] we observe the \mathcal{S}_4 “growth” structure in the numerical
 301 solution for α values smaller than $0.1m$, indicating that the smoothing performed by El
 302 et al. [8] limited their observed behaviour to just the \mathcal{S}_3 structure.

303 5.1.4. Growth Structure

304 The \mathcal{S}_4 “growth” structure, which has hitherto not been commonly published in the
 305 literature features a growth in the oscillation amplitude around x_{u_2} . This growth is a
 306 result of the interaction of the left and right wave trains, which were predicted by the
 307 linear theory of Dougalis et al. [17] to be separate.

308 An example of the growth structure can be seen for \mathcal{V}_3 ’s numerical solutions in
 309 Figure 9 to the smoothed dam-break problem with $\alpha = 0.1m$. This structure was
 310 observed in the numerical solutions of \mathcal{V}_3 for $\Delta x = 10/2^{11}m$ at $t = 30s$ for α values
 311 as low as $0.001m$ and even for the dam-break problem. The growth structure has also
 312 been observed using the finite element method of Mitsotakis et al. [18].

313 Figure 9 shows that this structure can only be observed for $\Delta x = 10/2^{10}m$, with poor
 314 convergence of the numerical results around x_{u_2} . Again our L_1 measures in Table 1 omit
 315 the interval $[520m, 540m]$ in the numerical solutions. This demonstrates that although
 316 we have visual convergence away from x_{u_2} our numerical solutions have not converged
 317 to round-off error as $\Delta x \rightarrow 0$. The C_1 measures in Table 1 are still decreasing and have
 318 only attained round-off error for h , although for uh and \mathcal{H} the errors in conservation
 319 are small. These measures continue the trend in Table 1 where smaller α ’s and thus
 320 steeper initial conditions lead to larger L_1 and C_1 measures because steeper problems
 321 are more difficult to solve accurately.

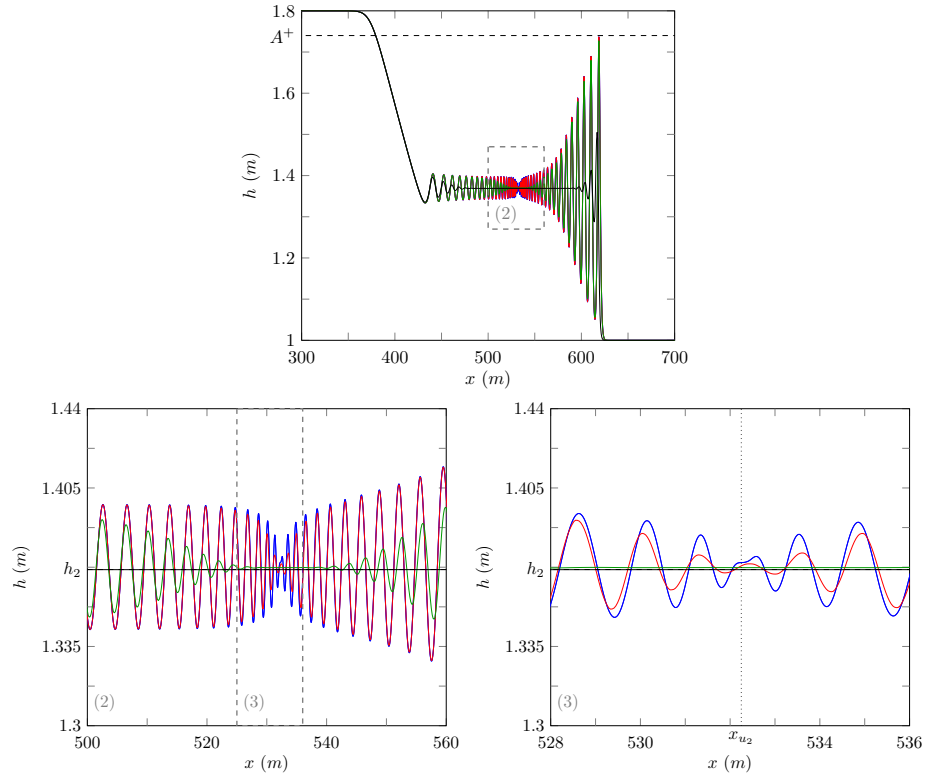


Figure 7: Numerical solutions of \mathcal{V}_3 at $t = 30s$ for the smooth dam-break problem with $\alpha = 0.4m$ for $\Delta x = 10/2^{10}m$ (—), $10/2^8m$ (—), $10/2^6m$ (—) and $10/2^4m$ (—). The important quantities A^+ (—), h_2 (—) and x_{u_2} (···) are also presented.

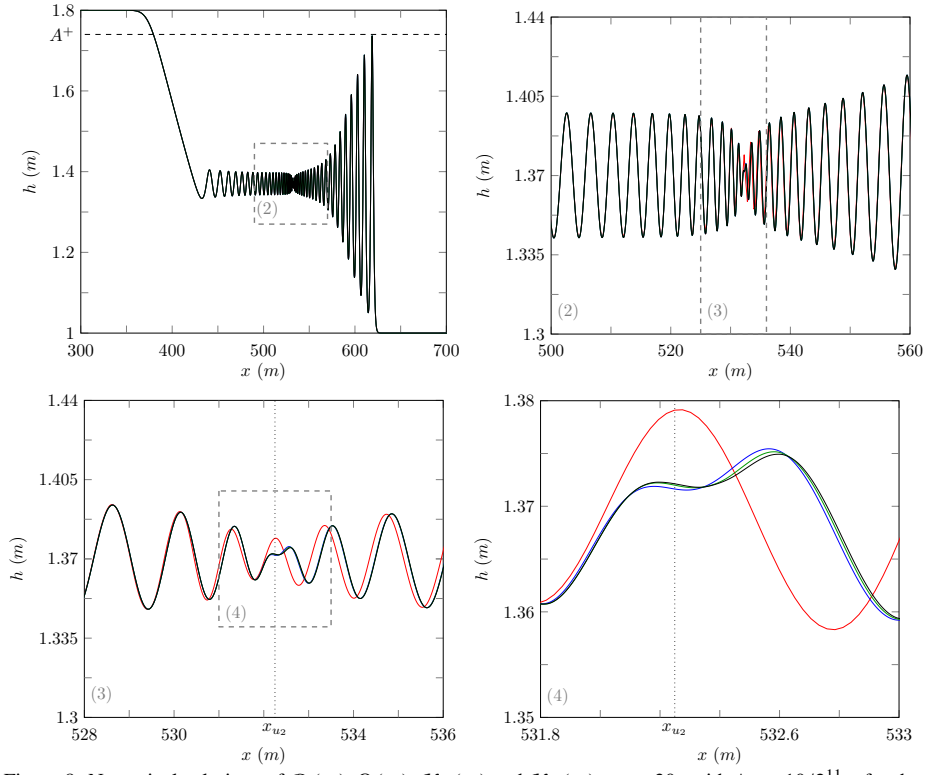


Figure 8: Numerical solutions of \mathcal{D} (—), \mathcal{E} (—), \mathcal{V}_3 (—) and \mathcal{V}_2 (—) at $t = 30s$ with $\Delta x = 10/2^{11}m$ for the smoothed dam-break problem with $\alpha = 0.4m$. The important quantities A^+ (—) and x_{u_2} (···) are also presented.

Figure 10 demonstrates that our numerical solutions for $\Delta x = 10/2^{11}m$ with the best methods \mathcal{D} , \mathcal{V}_3 and \mathcal{V}_2 disagree for only a few oscillations around x_{u_2} . Since both \mathcal{D} and \mathcal{E} are second-order finite difference methods their errors are dispersive. These dispersive errors cause the numerical solutions to overestimate the oscillation amplitude of the true solution, particularly around x_{u_2} . Because the dispersive errors of \mathcal{E} are larger than \mathcal{D} more oscillations are observed for the numerical solutions produced by \mathcal{E} . The \mathcal{V}_3 method was shown to be diffusive by Zoppou et al. [4] and therefore its numerical solutions underestimate the oscillation amplitude in the true solution. Therefore, the true solution of the Serre equations should be between the dispersive method \mathcal{D} and the diffusive method \mathcal{V}_3 , and thus will possess the \mathcal{S}_4 structure.

The numerical solutions of \mathcal{D} and \mathcal{V}_3 acting as upper and lower bounds respectively for the oscillation amplitude as Δx is reduced is demonstrated in Figure 11 using the maximum of h in the interval $[520m, 540m]$. From this figure it is clear that the amplitudes of the numerical solutions of \mathcal{D} converge down to the limit as the resolution is increased while the numerical solution amplitudes of \mathcal{V}_3 converge up to it. This shows that we have effectively bounded the true solution of the Serre equations. Unfortunately, \mathcal{V}_3 could not be run in reasonable computational times with lower Δx , but the numerical solutions of \mathcal{D} show that doing so is unnecessary.

These results indicate that the solutions of the Serre equations to the smoothed dam-break problem with sufficiently small α values should exhibit a growth structure at $t = 30s$, even though we have not precisely resolved all the oscillations in our numerical solutions.

It was found that decreasing α did increase the amplitude of the oscillations around x_{u_2} . For \mathcal{V}_3 with $\Delta x = 10/2^{11}m$ and $\alpha = 0.001m$ the oscillations in h were bounded by the interval $[1.28m, 1.46m]$. Of particular note is that the number of oscillations are the same in Figures 8 and 10 for the best methods even though they have different structures.

By changing the interval and desired time for the numerical solution, Δx could be lowered further so that by $t = 3s$ our numerical solutions have fully converged for α values as low as $0.001m$. This allows us to show that the height of the oscillations around x_{u_2} for the solution of the Serre equation to the smoothed dam-break problem are bounded at $t = 3s$ as $\alpha \rightarrow 0$. Figure 12 demonstrates this for the numerical solutions of \mathcal{V}_3 with $\Delta x = 10/2^{13}m$.

5.2. Shallow water wave equation comparison

The dispersive shock fitting method outlined in the review article of El and Hoefler [7], demonstrates that the means of h and u inside the undular bore of the Serre equations are equal to the height and velocity inside the bore of the shallow water wave equations. This is the basis for the comparison in the literature [2, 9] between the numerical solution of the Serre equations and the analytic solutions of the shallow water wave equations for the dam-break problem.

This relationship was investigated for our numerical solutions by using the mean bore depth and mean fluid velocity in the interval $[x_{u_2} - 50m, x_{u_2} + 50m]$ from our numerical solutions to the smoothed dam-break problem with various height ratios. These means were compared to the analytical solution of the dam-break problem for

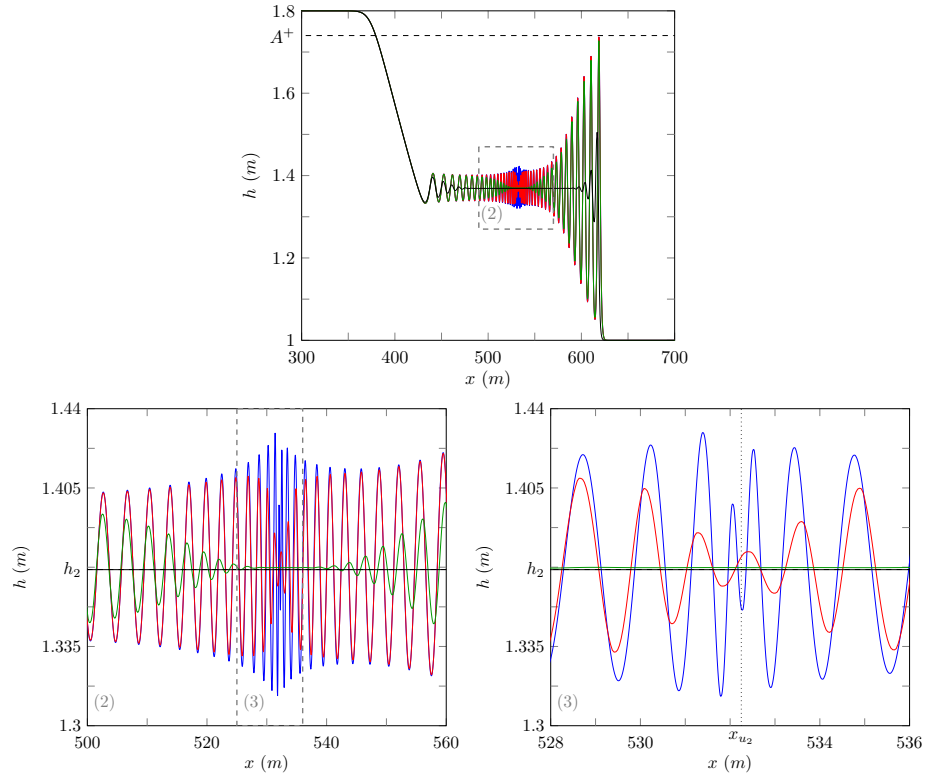


Figure 9: Numerical solutions of \mathcal{V}_3 at $t = 30s$ for the smooth dam-break problem with $\alpha = 0.1m$ for $\Delta x = 10/2^{10}m$ (—), $10/2^8m$ (—), $10/2^6m$ (—) and $10/2^4m$ (—). The important quantities A^+ (—), h_2 (—) and x_{u_2} (···) are also presented.

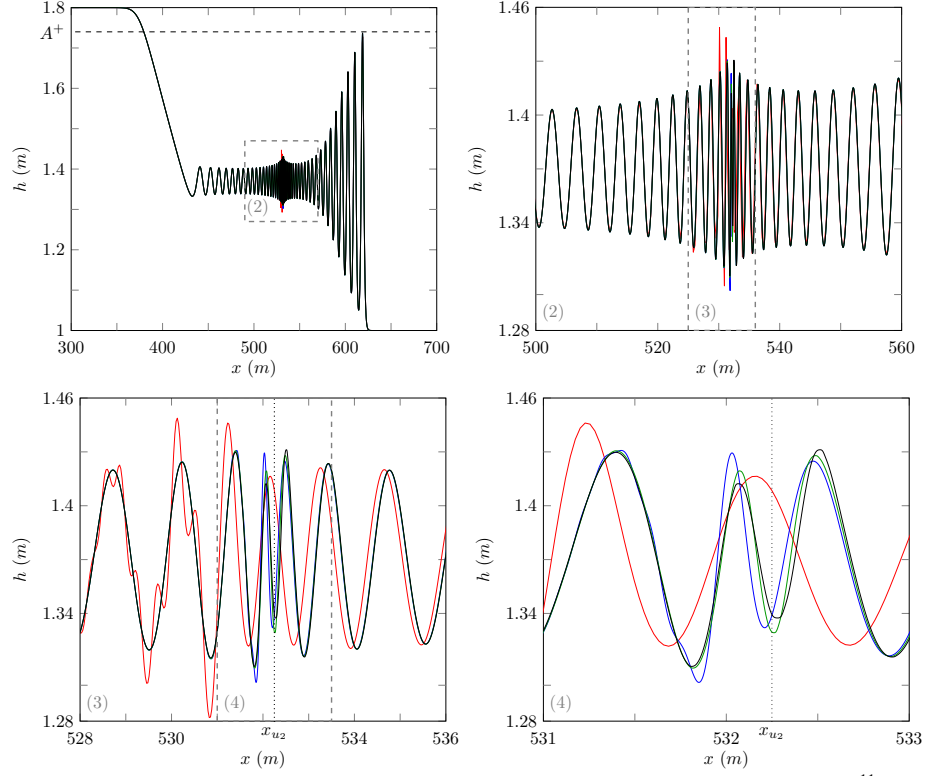


Figure 10: Numerical solutions of \mathcal{D} (—), \mathcal{E} (—), \mathcal{V}_3 (—) and \mathcal{V}_2 (—) at $t = 30s$ with $\Delta x = 10/2^{11}m$ for the smoothed dam-break problem with $\alpha = 0.1m$. The important quantities A^+ (—) and x_{u_2} (···) are also presented.

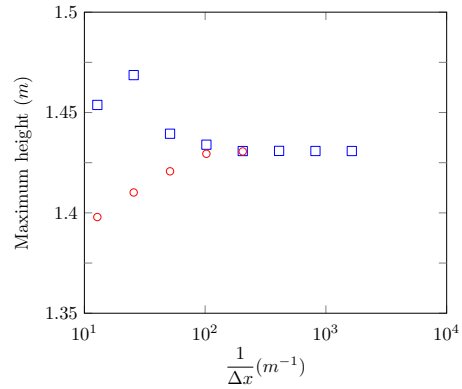


Figure 11: Maximum height of numerical solution of the smoothed dam-break problem with $\alpha = 0.4m$ at $t = 30s$ inside the interval $[520m, 540m]$ using \mathcal{D} (□) and \mathcal{V}_3 (○).

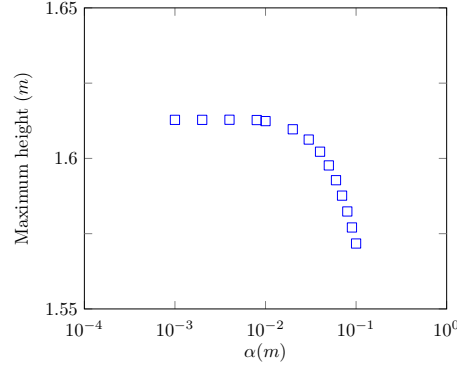


Figure 12: Maximum height of numerical solution around x_{u_2} at $t = 3s$ of various smoothed dam-break problem as α decreases, using \mathcal{V}_3 (\square) with $\Delta x = 10/2^{13}m$.

the shallow water wave equations, which gives the height h_2 and velocity u_2 inside the bore. The results of this can be seen in Figure 13 for numerical solutions of \mathcal{V}_3 with $\Delta x = 10/2^9m$ to the smoothed dam-break problem at $t = 100s$ with $\alpha = 0.1m$ where h_0 is fixed and h_1 is varied.

We use a final time of $t = 100s$ as it allows the internal structure of the bore to develop more fully giving a more reliable mean estimate, as a consequence we resort to a coarser grid to keep the run-times reasonable. We find that decreasing Δx does not significantly alter the mean of h and u . We also find that increasing α also does not significantly alter the mean of h and u . Therefore, the mean behaviour of the true solution of the Serre equations to the dam-break problem is captured by these numerical solutions, if it exists.

It can be seen that h_2 and u_2 are very close to the mean behaviour of the fluid inside the bore for a range of different aspect ratios. Although, as h_1/h_0 increases this approximation becomes worse, so that h_2 becomes an underestimate and consequently u_2 is an overestimate. Most likely this is due to the increasing influence of non-linearity on the undulations.

We find that for $h_1/h_0 = 1.8$ the mean values of h and u inside the bore for the Serre equations are not equal to h_2 and u_2 . This can be seen in Figure 14 for the numerical solutions of \mathcal{V}_3 with $\Delta x = 10/2^9m$ to the smoothed dam-break problem with $\alpha = 0.1m$ at $t = 300s$. It can be seen that h_2 is an underestimate of h and u_2 is an overestimate of u although the difference between these values and the mean behaviour of the Serre equations is small and only noticeable over long time periods. This agreement is quite good given that, we are numerically solving the smoothed dam-break problem and the equality of these values is derived from asymptotic results.

The location of the leading wave of the Serre equations slowly diverges from the location of the front of a bore in the shallow water wave equations over long periods of time. This is because the Rankine Hugoniot conditions that determine the front of the bore for the shallow water wave equations are not applicable to dispersive shock waves [7]. This divergence causes the small difference evident in \mathcal{V}_3 's numerical solution to the smoothed dam-break problem with $\alpha = 0.1m$ at $t = 300s$ using $\Delta x = 10/2^9m$, which is shown in Figure 15.

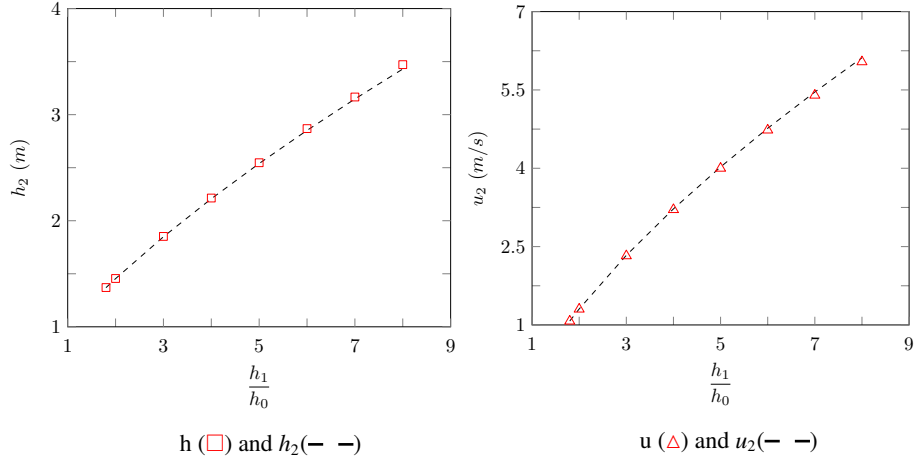


Figure 13: Comparison between mean behaviour inside the bore of the Serre equations and the analytical solution of the shallow water wave equations for a range of different aspect ratios.

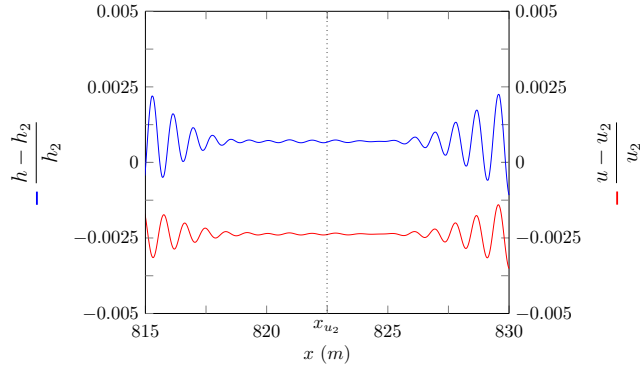


Figure 14: The relative difference between h and u and their comparisons h_2 and u_2 plotted around x_{u_2} (\cdots) for \mathcal{V}_3 's solutions with $\Delta x = 10/2^9 m$ for the smoothed dam-break problem with $\alpha = 0.1m$ at $t = 300s$.

397 We note that the \mathcal{S}_4 structure present in the numerical solutions using this method
 398 and parameters at $t = 30s$ in Figure 9 has decayed away by $t = 300s$ in Figure 14. This
 399 is a trend throughout our numerical solutions where oscillation amplitude decreases
 400 over time around x_{u_2} , changing the structure of the solution. This can be seen by ob-
 401 taining full convergence of the numerical solutions to the smoothed dam-break problem
 402 at $t = 3s$. The converged to numerical solutions for \mathcal{V}_3 are shown in Figure 16. From
 403 this figure it can be seen that the oscillation amplitudes for the numerical solutions for
 404 the smoothed dam-break problems with $\alpha = 0.4m$ and $\alpha = 0.1m$ are much larger at
 405 $t = 3s$ than they are at $t = 30s$ in Figure 2. Since we have demonstrated that our nu-
 406 merical solutions are good approximations to the true solution of the Serre equations at
 407 $t = 30s$ and $t = 3s$, decreasing oscillation amplitude around x_{u_2} over time is probably a
 408 property of the Serre equations. This implies that bounding the oscillation amplitudes
 409 at time $t = 3s$ as was done above, bounds the oscillation amplitudes at all later times.

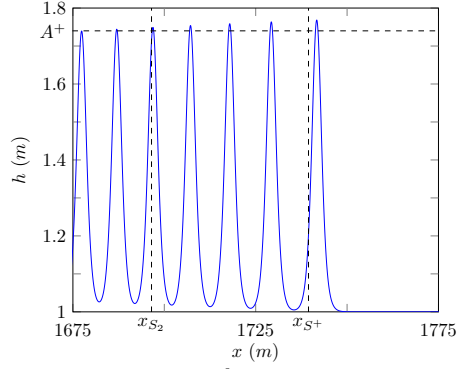


Figure 15: Numerical solution of \mathcal{V}_3 with $\Delta x = 10/2^9 m$ for the smoothed dam-break problem with $\alpha = 0.1m$ at $t = 300s$ around the front of the undular bore. The important quantities A^+ (— —), x_{S_2} (— —) and x_{S+} (— —) are also presented.

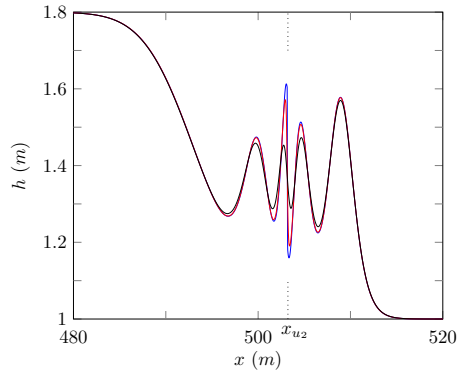


Figure 16: Numerical solution of \mathcal{V}_3 with $\Delta x = 10/2^{13}m$ for the smoothed dam-break problem with $\alpha = 0.001m$ (—), $0.1m$ (—) and 0.4 (—) at $t = 3s$. For comparison x_{u_2} (···) is also plotted.

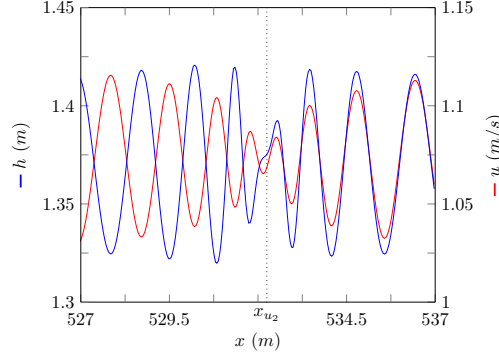


Figure 17: Numerical solution of \mathcal{V}_3 's with $\Delta x = 10/2^9 m$ for the smoothed dam-break problem with $\alpha = 0.1m$ at $t = 30s$ around the contact discontinuity close to x_{u_2} (\cdots).

5.2.1. Contact discontinuity

El et al. [8] noted the presence of a ‘degenerate contact discontinuity’ which is where the two wave trains interact in the \mathcal{S}_3 and \mathcal{S}_4 structures; it travels at the mean fluid velocity in the bore.

We observe that as our numerical solutions evolve over time, oscillations appear to be released from the contact discontinuity and travel away from it in both directions, leading to decay of amplitudes around the contact discontinuity. Therefore, the contact discontinuity is an important feature and its behaviour determines the structure of the oscillations in regions III and IV.

The different speeds of the oscillations are determined by the phase velocity, which for the Serre equations linearised around the mean height \bar{h} and mean velocity \bar{u} in regions III and IV is

$$v_p = \bar{u} \pm \sqrt{g\bar{h}} \sqrt{\frac{3}{\bar{h}^2 k^2 + 3}} \quad (4)$$

with wave number k . It can be seen that as $k \rightarrow \infty$ then $v_p \rightarrow \bar{u}$ and as $k \rightarrow 0$ then $v_p \rightarrow \bar{u} \pm \sqrt{g\bar{h}}$. The left wave train corresponds to the $-$ branch of the phase velocity while the right wave train corresponds to the $+$ branch of the phase velocity. The contact discontinuity corresponds to the high wave-number limit of these two wave trains. Because the two wave trains correspond to the $-$ and $+$ branches of the phase velocity for the left wave train h and u are in-phase, while for the right wave train h and u are out-of-phase. This can be seen in Figure 17 for the numerical solutions of \mathcal{V}_3 with $\Delta x = 10/2^9 m$ for the smoothed dam-break problem with $\alpha = 0.1m$ at $t = 30s$.

The $+$ branch of the phase velocity is strictly increasing in k , while the $-$ branch is strictly decreasing. It can also be shown that $+$ branch and $-$ branch of the group velocity behave in the same way with the same limits as their phase velocity counterparts. Therefore as in Dougalis et al. [17] the linear theory predicts separate dispersive wave trains. However, we have demonstrated that in the short term these wave trains are not be separate and so the interaction of these wave trains in the \mathcal{S}_3 and \mathcal{S}_4 structures is caused by non-linear effects.

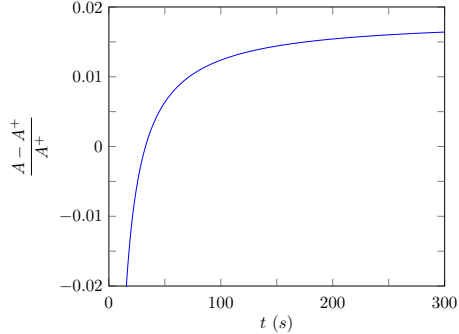


Figure 18: Relative difference between Whitham modulation result A^+ and the leading wave amplitude A from our numerical solutions of \mathcal{V}_3 with $\Delta x = 10/2^9 m$ for the smoothed dam-break problem with $\alpha = 0.1m$ over time.

5.3. Whitham Modulation Comparison

El et al. [8] demonstrated that their Whitham modulation results approximated the numerical solutions of the smoothed dam-break problem well for a range of aspect ratios. We observed that the Whitham modulation results are an underestimate compared to our numerical solutions.

This can be seen in Figure 18 as the relative difference between A^+ from El et al. [8] and the leading wave amplitude of our numerical solution A does not converge to 0 over time. Since we find that the numerical solutions for the smoothed dam-break problem with $\alpha = 0.1m$ have converged for the front of the undular bore by $\Delta x = 10/2^8 m$ as in Figure 9, our numerical solutions for A are considered reliable. We also note that unlike the oscillations around x_{u_2} the leading wave amplitude increases over time.

The Whitham modulation results for the location of the leading wave x_{S^+} is a better approximation than that given by the shallow water wave equations x_{S_2} , as can be seen in Figure 15. This is due to the inapplicability of the Rankine Hugoniot conditions from which S_2 is derived for dispersive shock waves [7].

The agreement between the Whitham modulation results and our numerical solutions is actually quite good given that the Whitham modulations results are an asymptotic solution to the dam-break problem. We include them here as a counter point to the results of El et al. [8] and Mitsotakis et al. [9], as from their results one would expect that numerical solutions should only ever underestimate A^+ and S^+ when the ratio h_1/h_0 is less than the critical value 1.94.

6. Conclusions

Utilising two finite difference methods of second-order and three finite difference finite volume methods of various orders to solve the nonlinear weakly dispersive Serre equations an investigation into the smoothed dam-break problem with varying steepness was performed. Four different structures of the numerical solutions were observed and demonstrated to be valid, the general trend of these structures is that an increase in steepness increases the size and number of oscillations in the solution. This study explains the different structures exhibited by the numerical results in the literature for

the smoothed dam-break problem for the Serre equations and uncovers a new result. These results demonstrate that other methods in the literature could replicate our results if their simulations are extended. Furthermore, these results suggest that this new result and its associated structure is to be expected for the solution of the Serre equation to the dam-break problem at least for short enough time spans, if it exists.

We confirm that the analytical solution of the shallow water wave equations for the dam-break problem provides a reasonable approximation to the mean height and velocity inside the bore formed by the smoothed dam-break problem for the Serre equations across a range of aspect ratios. Finally, we observe that the Whitham modulations results for the leading wave of an undular bore can underestimate the leading wave amplitude and speed when the aspect ratio is below the critical value.

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525 Appendix A.

526 The methods \mathcal{E} and \mathcal{D} use the centred second-order finite difference approximation
 527 to the momentum equation (1b), denoted as \mathcal{D}_u . For the mass equation (1a) \mathcal{E} uses the
 528 two step Lax-Wendroff method, denoted as \mathcal{E}_h while \mathcal{D} uses a centred second-order
 529 finite difference approximation, denoted as \mathcal{D}_h .

530 Appendix A.1. \mathcal{D}_u for the Momentum Equation

531 First (1b) is expanded to get

$$532 \quad h \frac{\partial u}{\partial t} - h^2 \frac{\partial^2 u}{\partial x \partial t} - \frac{h^3}{3} \frac{\partial^3 u}{\partial x^2 \partial t} = -X$$

534 where X contains only spatial derivatives and is

$$535 \quad X = uh \frac{\partial u}{\partial x} + gh \frac{\partial h}{\partial x} + h^2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \frac{h^3}{3} \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} - h^2 u \frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3} u \frac{\partial^3 u}{\partial x^3}.$$

537 All derivatives are approximated by second-order centred finite difference approxi-
 538 mations on a uniform grid in space and time, which after rearranging into an update
 539 formula becomes

$$540 \quad h_i^n u_i^{n+1} - (h_i^n)^2 \left(\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} \right) - \frac{(h_i^n)^3}{3} \left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} \right) = -Y_i^n \quad (\text{A.1})$$

542 where

$$543 \quad Y_i^n = 2\Delta t X_i^n - h_i^n u_i^{n-1} + (h_i^n)^2 \left(\frac{u_{i+1}^{n-1} - u_{i-1}^{n-1}}{2\Delta x} \right) + \frac{(h_i^n)^3}{3} \left(\frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{\Delta x^2} \right)$$

545 and

$$546 \quad X_i^n = u_i^n h_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + gh_i^n \frac{h_{i+1}^n - h_{i-1}^n}{2\Delta x} + (h_i^n)^2 \left(\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \right)^2$$

$$547 \quad + \frac{(h_i^n)^3}{3} \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} - (h_i^n)^2 u_i^n \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

$$548 \quad - \frac{(h_i^n)^3}{3} u_i^n \frac{u_{i+2}^n - 2u_{i+1}^n + 2u_{i-1}^n - u_{i-2}^n}{2\Delta x^3}.$$

551 Equation (A.1) can be rearranged into an explicit update scheme \mathcal{D}_u for u given its
 552 current and previous values, so that

$$553 \quad \begin{bmatrix} u_0^{n+1} \\ \vdots \\ u_m^{n+1} \end{bmatrix} = A^{-1} \begin{bmatrix} -Y_0^n \\ \vdots \\ -Y_m^n \end{bmatrix} =: \mathcal{D}_u(\mathbf{u}^n, \mathbf{h}^n, \mathbf{u}^{n-1}, \Delta x, \Delta t) \quad (\text{A.2})$$

555 where A is a tri-diagonal matrix.

556 *Appendix A.2. Numerical Methods for the Mass Equation*

557 The two step Lax-Wendroff update \mathcal{E}_h for h is

$$558 \quad h_{i+1/2}^{n+1/2} = \frac{1}{2} (h_{i+1}^n + h_i^n) - \frac{\Delta t}{2\Delta x} (u_{i+1}^n h_{i+1}^n - h_i^n u_i^n),$$

559

$$561 \quad h_{i-1/2}^{n+1/2} = \frac{1}{2} (h_i^n + h_{i-1}^n) - \frac{\Delta t}{2\Delta x} (u_i^n h_i^n - h_{i-1}^n u_{i-1}^n)$$

562

563 and

$$564 \quad h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x} (u_{i+1/2}^{n+1/2} h_{i+1/2}^{n+1/2} - u_{i-1/2}^{n+1/2} h_{i-1/2}^{n+1/2}).$$

565

566 The quantities $u_{i\pm 1/2}^{n+1/2}$ are calculated using u^{n+1} obtained by applying \mathcal{D}_u (A.2) to u^n
567 then linearly interpolating in space and time to give

$$568 \quad u_{i+1/2}^{n+1/2} = \frac{u_{i+1}^{n+1} + u_{i+1}^n + u_i^{n+1} + u_i^n}{4}$$

569

570 and

$$571 \quad u_{i-1/2}^{n+1/2} = \frac{u_i^{n+1} + u_i^n + u_{i-1}^{n+1} + u_{i-1}^n}{4}.$$

572

573 Thus we have the following update scheme \mathcal{E}_h for (1a)

$$574 \quad \mathbf{h}^{n+1} = \mathcal{E}_h(\mathbf{u}^n, \mathbf{h}^n, \mathbf{u}^{n+1}, \Delta x, \Delta t). \quad (\text{A.3})$$

575

576 The second order centered finite difference approximation to the conservation of
577 mass equation (1a) is

$$578 \quad h_i^{n+1} = h_i^{n-1} - \Delta t \left(u_i^n \frac{h_{i+1}^n - h_{i-1}^n}{\Delta x} + h_i^n \frac{u_{i+1}^n - u_{i-1}^n}{\Delta x} \right).$$

579

580 Thus we have an update scheme \mathcal{D}_h for all i

$$581 \quad \mathbf{h}^{n+1} = \mathcal{D}_h(\mathbf{u}^n, \mathbf{h}^n, \mathbf{h}^{n-1}, \Delta x, \Delta t). \quad (\text{A.4})$$

582

583 *Appendix A.3. Complete Method*

584 The method \mathcal{E} is the combination of (A.3) for (1a) and (A.2) for (1b) in the follow-
585 ing way

$$586 \quad \left. \begin{aligned} \mathbf{u}^{n+1} &= \mathcal{D}_u(\mathbf{u}^n, \mathbf{h}^n, \mathbf{u}^{n-1}, \Delta x, \Delta t) \\ \mathbf{h}^{n+1} &= \mathcal{E}_h(\mathbf{u}^n, \mathbf{h}^n, \mathbf{u}^{n+1}, \Delta x, \Delta t) \end{aligned} \right\} \mathcal{E}(\mathbf{u}^n, \mathbf{h}^n, \mathbf{u}^{n-1}, \mathbf{h}^{n-1}, \Delta x, \Delta t). \quad (\text{A.5})$$

587

588 The method \mathcal{D} is the combination of (A.4) for (1a) and (A.2) for (1b) in the follow-
589 ing way

$$590 \quad \left. \begin{aligned} \mathbf{h}^{n+1} &= \mathcal{D}_h(\mathbf{u}^n, \mathbf{h}^n, \mathbf{h}^{n-1}, \Delta x, \Delta t) \\ \mathbf{u}^{n+1} &= \mathcal{D}_u(\mathbf{u}^n, \mathbf{h}^n, \mathbf{u}^{n-1}, \Delta x, \Delta t) \end{aligned} \right\} \mathcal{D}(\mathbf{u}^n, \mathbf{h}^n, \mathbf{u}^{n-1}, \mathbf{h}^{n-1}, \Delta x, \Delta t). \quad (\text{A.6})$$

591