1 Elliptic Equation

The linearised elliptic equation is

$$G = uh - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

$$G = uh - h^2 h_x u_x - \frac{h^3}{3}u_{xx}$$

now we replace

$$u = U(t)e^{ikx}$$
$$h = H(t)e^{ikx}$$

$$G = UHe^{2ikx} - H^2 e^{2ikx} ikHe^{ikx} ikUe^{ikx} - \frac{1}{3}H^3 e^{3ikx}(-k^2)Ue^{ikx}$$

$$G = UHe^{2ikx} + k^2 H^3 Ue^{4ikx} + \frac{k^2}{3}UH^3 e^{4ikx}$$

$$G = \left(1 + \frac{4}{3}k^2 H^2 e^{2ikx}\right)UHe^{2ikx}$$

$$G_j = \left(1 + \frac{4}{3}k^2 H^2 e^{2ikx_j}\right)UHe^{2ikx_j}$$

2 Finite Difference

we have the derivatives

$$\left(\frac{\partial q}{\partial x}\right)_j = \frac{q_{j+1} - q_{j-1}}{2\Delta x} = \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}Q(t)e^{ikx_j} = \frac{i\sin(k\Delta x)}{\Delta x}Q(t)e^{ikx_j}$$

$$\left(\frac{\partial^2 q}{\partial x^2}\right)_j = \frac{q_{j+1} - 2q_j + q_{j-1}}{\Delta x^2} = \frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{\Delta x^2}Q(t)e^{ikx_j} = \frac{2\cos(k\Delta x) - 2}{\Delta x^2}Q(t)e^{ikx_j}$$

$$\left(\frac{\partial^2 q}{\partial x^2}\right)_j = -4\frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^2}Q(t)e^{ikx_j} = -\left(\frac{2\sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2 Q(t)e^{ikx_j}$$

$$G = uh - h^2 h_x u_x - \frac{h^3}{3}u_{xx}$$

$$G_j = UHe^{2ikx_j} - H^2 e^{2ikx_j}\frac{i\sin(k\Delta x)}{\Delta x}He^{ikx_j}\frac{i\sin(k\Delta x)}{\Delta x}Ue^{ikx_j} + \frac{1}{3}H^3 e^{3ikx_j}\left(\frac{2\sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2 Ue^{ikx_j}$$

$$G_j = UHe^{2ikx_j} + UH^3 e^{4ikx_j}\left(\frac{\sin(k\Delta x)}{\Delta x}\right)^2 + \frac{1}{3}UH^3 e^{4ikx_j}\left(\frac{2\sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2$$

$$G_j = \left(1 + H^2 e^{2ikx_j}\left(\frac{\sin(k\Delta x)}{\Delta x}\right)^2 + \frac{1}{3}H^2 e^{2ikx_j}\left(\frac{2\sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2\right)UHe^{2ikx_j}$$

$$G_j = \left(1 + \left[\left(\frac{\sin(k\Delta x)}{\Delta x}\right)^2 + \frac{1}{3}\left(\frac{2\sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2\right]H^2 e^{2ikx_j}\right)UHe^{2ikx_j}$$

So we want

$$\left[\left(\frac{\sin(k\Delta x)}{\Delta x}\right)^2 + \frac{1}{3}\left(\frac{2\sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2\right] \approx \frac{4}{3}k^2$$

Wolfram Alpha has

$$\left[\left(\frac{\sin(k\Delta x)}{\Delta x}\right)^2 + \frac{1}{3}\left(\frac{2\sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2\right] = \frac{4k^2}{3} - \frac{13k^4\Delta x^2}{36} + \frac{49k^6\Delta x^4}{1080} + O(\Delta x^6)$$

So we can see this scheme is second order.

3 Finite Element

$$G = uh - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = uhv - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)v$$

and then we integrate over the entire domain to get

$$\int_\Omega Gvdx = \int_\Omega uhvdx - \int_\Omega \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)vdx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_\Omega Gvdx = \int_\Omega uhvdx + \int_\Omega \frac{h^3}{3}u_x v_x dx$$

For u we are going to use $x_{j+1/2}$ as the nodes, which generate the basis functions $\phi_{j+1/2}$, which for us will be the space of continuous linear elements. While for G and h we will choose basis functions w that are linear from $[x_{j-1/2}, x_{j+1/2}]$ but discontinuous at the edges.

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gvdx = \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} uhvdx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^3}{3}u_x v_x dx$$

for all v

4 P1 FEM

We are going to cordinate tranform from x space the interval $[x_{j-1/2}, x_{j+1/2}, x_{j+3/2}]$ to the ξ space interval $[-1, 0, 1]$. To accomplish this we have the following relation

$$x = \xi \Delta x + x_{j+1/2}$$

Taking the derivatives we see
$dx = d\xi \Delta x$, $\frac{dx}{d\xi} = \Delta x$, $\frac{d\xi}{dx} = \frac{1}{\Delta x}$.

We can describe the basis functions in the ξ space

$$\phi_{j+1/2} = \begin{cases} 1+\xi & \xi < 0 \\ 1-\xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$\phi_{j-1/2} = \begin{cases} -\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$\phi_{j+3/2} = \begin{cases} \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

While the descriptions for w's is

$$w^{+}_{j+1/2} = \begin{cases} 1-\xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$w^{-}_{j+1/2} = \begin{cases} 1+\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$w^{+}_{j-1/2} = \begin{cases} -\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$w^{-}_{j+3/2} = \begin{cases} \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

We now replace our functions by our approximations to them

$$G \approx G' = \sum_j G_{j+1/2} w_{j+1/2}$$

$$u \approx u' = \sum_j u_{j+1/2} \phi_{j+1/2}$$

$$h \approx h' = \sum_j h_{j+1/2} w_{j+1/2}$$

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} G' \phi_{j+1/2} dx - \int_{x_{j-1/2}}^{x_{j+3/2}} u' h' \phi_{j+1/2} dx - \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{(h')^3}{3} u'_x (\phi_x)_{j+1/2} dx = 0$$

For all $\phi_{j+1/2}$. For this analysis we choose a particular basis function $\phi_{j+1/2}$ and we look at all the integrals. Begining from the right

$$\int_{x_{j-1/2}}^{x_{j+3/2}} G'(x)\phi_{j+1/2}dx = \int_{-1}^{1} G'(\xi)\phi_{j+1/2}(\xi)\frac{dx}{d\xi}d\xi$$

$$= \Delta x \int_{-1}^{1} \left(G^{+}_{j-1/2}w^{+}_{j-1/2} + G^{-}_{j+1/2}w^{-}_{j+1/2} + G^{+}_{j+1/2}w^{+}_{j+1/2} + G^{-}_{j-3/2}w^{-}_{j-3/2}\right)\phi_{j+1/2}d\xi$$

$$\begin{aligned} = \Delta x[G^{+}_{j-1/2}\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j+1/2}d\xi + G^{-}_{j+1/2}\int_{-1}^{1} w^{-}_{j+1/2}\phi_{j+1/2}d\xi \\ + G^{+}_{j+1/2}\int_{-1}^{1} w^{+}_{j+1/2}\phi_{j+1/2}d\xi + G^{-}_{j+3/2}\int_{-1}^{1} w^{-}_{j+3/2}\phi_{j+1/2}d\xi] \end{aligned} \quad (8)$$

We have that

$$\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j+1/2}d\xi = \int_{-1}^{1} w^{-}_{j+3/2}\phi_{j+1/2}d\xi = \frac{1}{6}$$

and

$$\int_{-1}^{1} w^{-}_{j+1/2}\phi_{j+1/2}d\xi = \int_{-1}^{1} w^{+}_{j+1/2}\phi_{j+1/2}d\xi = \frac{1}{3}$$

So

$$\begin{aligned} &= \Delta x\left[\frac{1}{6}G^{+}_{j-1/2} + \frac{1}{3}G^{-}_{j+1/2} + \frac{1}{3}G^{+}_{j+1/2} + \frac{1}{6}G^{-}_{j+3/2}\right] \\ &= \frac{\Delta x}{6}\left[G^{+}_{j-1/2} + 2G^{-}_{j+1/2} + 2G^{+}_{j+1/2} + G^{-}_{j+3/2}\right] \end{aligned}$$

Next we have

$$\int_{x_{j-1/2}}^{x_{j+3/2}} h'u'\phi_{j+1/2}dx = \Delta x\int_{-1}^{1} h'(\xi)u'(\xi)\phi_{j+1/2}(\xi)d\xi$$

$$\begin{aligned} = \Delta x\int_{-1}^{1} \left(h^{+}_{j-1/2}w^{+}_{j-1/2} + h^{-}_{j+1/2}w^{-}_{j+1/2} + h^{+}_{j+1/2}w^{+}_{j+1/2} + h^{-}_{j-3/2}w^{-}_{j-3/2}\right) \\ \left(u_{j-1/2}\phi_{j-1/2} + u_{j+1/2}\phi_{j+1/2} + u_{j+3/2}\phi_{j+3/2}\right)\phi_{j+1/2}d\xi \end{aligned} \quad (9)$$

$$= \Delta x \int_{-1}^{1} \Big(h^{+}_{j-1/2} w^{+}_{j-1/2} + h^{-}_{j+1/2} w^{-}_{j+1/2} + h^{+}_{j+1/2} w^{+}_{j+1/2} + h^{-}_{j-3/2} w^{-}_{j-3/2} \Big) u_{j-1/2} \phi_{j-1/2} \phi_{j+1/2}$$
$$+ \Big(h^{+}_{j-1/2} w^{+}_{j-1/2} + h^{-}_{j+1/2} w^{-}_{j+1/2} + h^{+}_{j+1/2} w^{+}_{j+1/2} + h^{-}_{j-3/2} w^{-}_{j-3/2} \Big) u_{j+1/2} \phi_{j+1/2} \phi_{j+1/2}$$
$$+ \Big(h^{+}_{j-1/2} w^{+}_{j-1/2} + h^{-}_{j+1/2} w^{-}_{j+1/2} + h^{+}_{j+1/2} w^{+}_{j+1/2} + h^{-}_{j-3/2} w^{-}_{j-3/2} \Big) u_{j+3/2} \phi_{j+3/2} \phi_{j+1/2} d\xi \tag{10}$$

If one of the terms w_k, ϕ_l , ϕ_m is 0 then $w_k \phi_l \phi_m = 0$

$$= \Delta x \int_{-1}^{1} \Big(h^{+}_{j-1/2} w^{+}_{j-1/2} + h^{-}_{j+1/2} w^{-}_{j+1/2} \Big) u_{j-1/2} \phi_{j-1/2} \phi_{j+1/2}$$
$$+ \Big(h^{+}_{j-1/2} w^{+}_{j-1/2} + h^{-}_{j+1/2} w^{-}_{j+1/2} + h^{+}_{j+1/2} w^{+}_{j+1/2} + h^{-}_{j-3/2} w^{-}_{j-3/2} \Big) u_{j+1/2} \phi_{j+1/2} \phi_{j+1/2}$$
$$+ \Big(h^{+}_{j+1/2} w^{+}_{j+1/2} + h^{-}_{j-3/2} w^{-}_{j-3/2} \Big) u_{j+3/2} \phi_{j+3/2} \phi_{j+1/2} d\xi \tag{11}$$

$$= \Delta x \int_{-1}^{1} u_{j-1/2} h^{+}_{j-1/2} w^{+}_{j-1/2} \phi_{j-1/2} \phi_{j+1/2} + u_{j-1/2} h^{-}_{j+1/2} w^{-}_{j+1/2} \phi_{j-1/2} \phi_{j+1/2}$$
$$+ u_{j+1/2} h^{+}_{j-1/2} w^{+}_{j-1/2} \phi_{j+1/2} \phi_{j+1/2} + u_{j+1/2} h^{-}_{j+1/2} w^{-}_{j+1/2} \phi_{j+1/2} \phi_{j+1/2}$$
$$+ u_{j+1/2} h^{+}_{j+1/2} w^{+}_{j+1/2} \phi_{j+1/2} \phi_{j+1/2} + u_{j+1/2} h^{-}_{j-3/2} w^{-}_{j-3/2} \phi_{j+1/2} \phi_{j+1/2}$$
$$+ u_{j+3/2} h^{+}_{j+1/2} w^{+}_{j+1/2} \phi_{j+3/2} \phi_{j+1/2} + u_{j+3/2} h^{-}_{j-3/2} w^{-}_{j-3/2} \phi_{j+3/2} \phi_{j+1/2} d\xi \tag{12}$$

Evaluating the integral

$$\int_{-1}^{1} u_{j-1/2} h^{+}_{j-1/2} w^{+}_{j-1/2} \phi_{j-1/2} \phi_{j+1/2} + u_{j-1/2} h^{-}_{j+1/2} w^{-}_{j+1/2} \phi_{j-1/2} \phi_{j+1/2}$$
$$+ u_{j+1/2} h^{+}_{j-1/2} w^{+}_{j-1/2} \phi_{j+1/2} \phi_{j+1/2} + u_{j+1/2} h^{-}_{j+1/2} w^{-}_{j+1/2} \phi_{j+1/2} \phi_{j+1/2}$$
$$+ u_{j+1/2} h^{+}_{j+1/2} w^{+}_{j+1/2} \phi_{j+1/2} \phi_{j+1/2} + u_{j+1/2} h^{-}_{j-3/2} w^{-}_{j-3/2} \phi_{j+1/2} \phi_{j+1/2}$$
$$+ u_{j+3/2} h^{+}_{j+1/2} w^{+}_{j+1/2} \phi_{j+3/2} \phi_{j+1/2} + u_{j+3/2} h^{-}_{j-3/2} w^{-}_{j-3/2} \phi_{j+3/2} \phi_{j+1/2} d\xi \tag{13}$$

$$= u_{j-1/2} h^+_{j-1/2} \int_{-1}^{1} w^+_{j-1/2} \phi_{j-1/2} \phi_{j+1/2} d\xi + u_{j-1/2} h^-_{j+1/2} \int_{-1}^{1} w^-_{j+1/2} \phi_{j-1/2} \phi_{j+1/2} d\xi$$
$$+ u_{j+1/2} h^+_{j-1/2} \int_{-1}^{1} w^+_{j-1/2} \phi_{j+1/2} \phi_{j+1/2} d\xi + u_{j+1/2} h^-_{j+1/2} \int_{-1}^{1} w^-_{j+1/2} \phi_{j+1/2} \phi_{j+1/2} d\xi$$
$$+ u_{j+1/2} h^+_{j+1/2} \int_{-1}^{1} w^+_{j+1/2} \phi_{j+1/2} \phi_{j+1/2} d\xi + u_{j+1/2} h^-_{j-3/2} \int_{-1}^{1} w^-_{j-3/2} \phi_{j+1/2} \phi_{j+1/2} d\xi$$
$$+ u_{j+3/2} h^+_{j+1/2} \int_{-1}^{1} w^+_{j+1/2} \phi_{j+3/2} \phi_{j+1/2} d\xi + u_{j+3/2} h^-_{j-3/2} \int_{-1}^{1} w^-_{j-3/2} \phi_{j+3/2} \phi_{j+1/2} d\xi \tag{14}$$

Now we evaluate the integrals

$$\int_{-1}^{1} w^+_{j-1/2} \phi_{j-1/2} \phi_{j+1/2} d\xi = \int_{-1}^{0} (-\xi)(-\xi)(1+\xi) = \frac{1}{12}$$
$$\int_{-1}^{1} w^-_{j+1/2} \phi_{j-1/2} \phi_{j+1/2} d\xi = \int_{-1}^{0} (1+\xi)(-\xi)(1+\xi)\, d\xi = \frac{1}{12}$$
$$\int_{-1}^{1} w^+_{j-1/2} \phi_{j+1/2} \phi_{j+1/2} d\xi = \int_{-1}^{0} (-\xi)(1+\xi)(1+\xi)\, d\xi = \frac{1}{12}$$

$$\int_{-1}^{1} w^-_{j+1/2} \phi_{j+1/2} \phi_{j+1/2} d\xi = \int_{-1}^{0} (1+\xi)(1+\xi)(1+\xi)\, d\xi = \frac{1}{4}$$
$$\int_{-1}^{1} w^+_{j+1/2} \phi_{j+1/2} \phi_{j+1/2} d\xi = \int_{0}^{1} (1-\xi)(1-\xi)(1-\xi)\, d\xi = \frac{1}{4}$$
$$\int_{-1}^{1} w^-_{j-3/2} \phi_{j+1/2} \phi_{j+1/2} d\xi = \int_{0}^{1} (\xi)(1-\xi)(1-\xi)\, d\xi = \frac{1}{12}$$
$$\int_{-1}^{1} w^+_{j+1/2} \phi_{j+3/2} \phi_{j+1/2} d\xi = \int_{0}^{1} (1-\xi)(\xi)(1-\xi)\, d\xi = \frac{1}{12}$$
$$\int_{-1}^{1} w^-_{j-3/2} \phi_{j+3/2} \phi_{j+1/2} d\xi = \int_{0}^{1} (\xi)(\xi)(1-\xi)\, d\xi = \frac{1}{12}$$

Note that these sum to the same fractions as in the linear case if the h is constant.

$$= u_{j-1/2}h^+_{j-1/2}\frac{1}{12} + u_{j-1/2}h^-_{j+1/2}\frac{1}{12} + u_{j+1/2}h^+_{j-1/2}\frac{1}{12} + u_{j+1/2}h^-_{j+1/2}\frac{1}{4} + u_{j+1/2}h^+_{j+1/2}\frac{1}{4} + u_{j+1/2}h^-_{j-3/2}\frac{1}{12} + u_{j+3/2}h^+_{j+1/2}\frac{1}{12} + u_{j+3/2}h^-_{j-3/2}\frac{1}{12} \quad (15)$$

$$= \frac{1}{12}\Big[u_{j-1/2}h^+_{j-1/2} + u_{j-1/2}h^-_{j+1/2} + u_{j+1/2}h^+_{j-1/2} + 3u_{j+1/2}h^-_{j+1/2} + 3u_{j+1/2}h^+_{j+1/2} + u_{j+1/2}h^-_{j-3/2} + u_{j+3/2}h^+_{j+1/2} + u_{j+3/2}h^-_{j-3/2}\Big] \quad (16)$$

Therefore

$$\int_{x_{j-1/2}}^{x_{j+3/2}} h'u'\phi_{j+1/2}dx = \frac{\Delta x}{12}\Big[u_{j-1/2}h^+_{j-1/2}+u_{j-1/2}h^-_{j+1/2}+u_{j+1/2}h^+_{j-1/2}+3u_{j+1/2}h^-_{j+1/2} + 3u_{j+1/2}h^+_{j+1/2} + u_{j+1/2}h^-_{j-3/2} + u_{j+3/2}h^+_{j+1/2} + u_{j+3/2}h^-_{j-3/2}\Big] \quad (17)$$

The next integral is

$$\int_{x_{j-1/2}}^{x_{j+3/2}} \frac{(h')^3}{3}u'_x(\phi_x)_{j+1/2}dx = \Delta x\int_{-1}^{1}\frac{(h'(\xi))^3}{3}u'_\xi(\phi_\xi)_{j+1/2}dx$$
$$= \frac{\Delta x}{3}\int_{-1}^{1}(h'(\xi))^3u'_\xi(\phi_\xi)_{j+1/2}d\xi$$

were now going to expand andd use the superscript $'$ to denote derivatives

$$= \frac{\Delta x}{3}\int_{-1}^{1}\left(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2} + h^+_{j+1/2}w^+_{j+1/2} + h^-_{j-3/2}w^-_{j-3/2}\right)^3 \left(u_{j-1/2}\phi'_{j-1/2} + u_{j+1/2}\phi'_{j+1/2} + u_{j+3/2}\phi'_{j+3/2}\right)\phi'_{j+1/2}d\xi \quad (18)$$