1 Linearised Equations

We begin with the linearised equations from Chris's thesis/papers.

continuity:

$$\frac{\partial h_1}{\partial t} + h_0 \frac{\partial u_1}{\partial x} + u_0 \frac{\partial h_1}{\partial x} = 0$$

velocity:

$$\frac{\partial u_1}{\partial t} + g \frac{\partial h_1}{\partial x} + u_0 \frac{\partial u_1}{\partial x} - \frac{h_0^2}{3}\left(u_0 \frac{\partial^3 u_1}{\partial x^3} + \frac{\partial^3 u_1}{\partial x^3 \partial t}\right) = 0$$

Also G

$$G = u_0 h_0 + u_0 h_1 + h_0 u_1 - \frac{h_0^3}{3}\frac{\partial^2 u_1}{\partial x^2}$$

Now for simplicity, and because its all we need, we assume the water is still (except for the pertubations) so that $u_0 = 0$ thus we get:

$$\frac{\partial h_1}{\partial t} + h_0 \frac{\partial u_1}{\partial x} = 0$$

$$\frac{\partial u_1}{\partial t} + g \frac{\partial h_1}{\partial x} - \frac{h_0^2}{3}\frac{\partial^3 u_1}{\partial x^3 \partial t} = 0$$

$$G = h_0 u_1 - \frac{h_0^3}{3}\frac{\partial^2 u_1}{\partial x^2}$$

Importantly by multiplying the velocity by h_0 to get the momentum equation we have

$$\frac{\partial u_1}{\partial t} h_0 + g \frac{\partial h_1}{\partial x} h_0 - \frac{h_0^3}{3}\frac{\partial^3 u_1}{\partial x^3 \partial t} = 0$$

and thus

$$\frac{\partial G}{\partial t} + g \frac{\partial h_1}{\partial x} h_0 = 0$$

So we finally have

$$\frac{\partial h_1}{\partial t} + h_0 \frac{\partial u_1}{\partial x} = 0$$

$$\frac{\partial G}{\partial t} + g\frac{\partial h_1}{\partial x}h_0 = 0$$

$$G = h_0 u_1 - \frac{h_0^3}{3}\frac{\partial^2 u_1}{\partial x^2}$$

For convenience I will make the following notational changes $H = h_0$, $h = h_1$ and $u = u_1$. So that

$$\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial G}{\partial t} + gH\frac{\partial h}{\partial x} = 0$$

$$G = Hu - \frac{H^3}{3}\frac{\partial^2 u}{\partial x^2}$$

We know that the dispersion relation is given by

$$\omega = \pm k\sqrt{gH}\sqrt{\frac{3}{k^2H^2+3}}$$

2 Source Term in Mass Equation

We want to solve this for $u(x_0, t)$ for a given $h(x_0, t)$

$$h_t + Hu_x = 0$$

$$u_t H + gHh_x - \frac{H^3}{3}u_{xxt} = 0$$

$$G = Hu - \frac{H^3}{3}\frac{\partial^2 u}{\partial x^2}$$

The $h(x_0, t)$ is

$$\begin{aligned} h(x,t) = a_0 \cos\left(2\pi\left(\frac{x}{\lambda} - \frac{t}{T_0}\right)\right) + \frac{\pi a_0^2}{\lambda}\cos\left(4\pi\left(\frac{x}{\lambda} - \frac{t}{T_0}\right)\right) \\ - \frac{a_0^3\pi^2}{2\lambda^2}\left[\cos\left(2\pi\left(\frac{x}{\lambda} - \frac{t}{T_0}\right)\right) - \cos\left(6\pi\left(\frac{x}{\lambda} - \frac{t}{T_0}\right)\right)\right] \end{aligned} \quad (1)$$

So

$$h_t(x,t) = \frac{2\pi a_0}{T_0}\sin\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right) + \frac{4\pi^2 a_0^2}{\lambda T_0}\sin\left(4\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right)$$
$$-\frac{2a_0^3\pi^3}{2\lambda^2 T_0}\left[\sin\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right) - 3\sin\left(6\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right)\right] \quad (2)$$

So we have

$$u_x = -\frac{1}{H}\left[\frac{2\pi a_0}{T_0}\sin\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right) + \frac{4\pi^2 a_0^2}{\lambda T_0}\sin\left(4\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right)\right]$$
$$+\frac{1}{H}\left[\frac{2a_0^3\pi^3}{2\lambda^2 T_0}\left[\sin\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right) - 3\sin\left(6\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right)\right]\right] \quad (3)$$

$$u_x = -\frac{1}{H}\left[\frac{2\pi a_0}{T_0}\sin\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right) + 2\frac{4\pi^2 a_0^2}{\lambda T_0}\sin\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right)\cos\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right)\right]$$
$$+\frac{1}{H}\left[\frac{2a_0^3\pi^3}{2\lambda^2 T_0}\left[\sin\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right) - 3\left(-4\sin^3\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right) + 3\sin\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right)\right)\right]\right] \quad (4$$

$$u_x = -\frac{\sin\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right)}{H}\left[\frac{2\pi a_0}{T_0} + 2\frac{4\pi^2 a_0^2}{\lambda T_0}\cos\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right)\right]$$
$$+\frac{\sin\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right)}{H}\left[\left[1 - 3\left(-4\sin^2\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right) + 3\right)\right]\right] \quad (5)$$

$$u_x = -\frac{\sin\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right)}{H}\left[\frac{2\pi a_0}{T_0} + 2\frac{4\pi^2 a_0^2}{\lambda T_0}\cos\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right)\right]$$
$$+\frac{\sin\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right)}{H}\left[\left[12\sin^2\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right) - 8\right]\right] \quad (6)$$

$$u_x = -\frac{2\sin\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right)}{H}\left[\frac{\pi a_0}{T_0} - 4 + \frac{4\pi^2 a_0^2}{\lambda T_0}\cos\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right) + 6\sin^2\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T_0}\right)\right)\right]$$