1 Method for $\mathcal{A}$

Fourth order centered difference approximation

$$G_i = u_i h_i - h_i^2 \left(\frac{-h_{i+2} + 8h_{i+1} - 8h_{i-1} + h_{i-2}}{12\Delta x}\right)\left(\frac{-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}}{12\Delta x}\right) - \frac{h_i^3}{3}\left(\frac{-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}}{12\Delta x^2}\right). \tag{1}$$

Can be rearranged into a matrix equation with the following form

$$\begin{bmatrix} G_0 \\ \vdots \\ G_m \end{bmatrix} = A \begin{bmatrix} u_0 \\ \vdots \\ u_m \end{bmatrix}.$$

Where

$$A = \begin{bmatrix} c_0 & d_0 & e_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ b_1 & c_1 & d_1 & e_1 & 0 & \cdots & \cdots & \cdots & 0 \\ a_2 & b_2 & c_2 & d_2 & e_2 & 0 & \cdots & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & d_3 & e_3 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & a_{m-3} & b_{m-3} & c_{m-3} & d_{m-3} & e_{m-3} & 0 \\ 0 & \cdots & \cdots & 0 & a_{m-2} & b_{m-2} & c_{m-2} & d_{m-2} & e_{m-2} \\ 0 & \cdots & \cdots & \cdots & 0 & a_{m-1} & b_{m-1} & c_{m-1} & d_{m-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & a_m & b_m & c_m \end{bmatrix}$$

For a fourth-order scheme these are given by

$$a_i = -\left(\frac{-h_{i+2} + 8h_{i+1} - 8h_{i-1} + h_{i-2}}{144\Delta x^2}\right)h_i^2 + \frac{h_i^3}{36\Delta x^2}, \tag{2a}$$

$$b_i = 8\left(\frac{-h_{i+2} + 8h_{i+1} - 8h_{i-1} + h_{i-2}}{144\Delta x^2}\right)h_i^2 - 16\frac{h_i^3}{36\Delta x^2}, \tag{2b}$$

$$c_i = h_i + \frac{30h_i^3}{36\Delta x^2}, \tag{2c}$$

$$d_i = -8\left(\frac{-h_{i+2} + 8h_{i+1} - 8h_{i-1} + h_{i-2}}{144\Delta x^2}\right)h_i^2 - 16\frac{h_i^3}{36\Delta x^2}, \tag{2d}$$

$$e_i = \left(\frac{-h_{i+2} + 8h_{i+1} - 8h_{i-1} + h_{i-2}}{144\Delta x^2}\right)h_i^2 + \frac{h_i^3}{36\Delta x^2}. \tag{2e}$$