1 Elliptic Equation

The linearised elliptic equation is

$$G = uh3\frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

$$G = uh - h^2 h_x u_x - \frac{h^3}{3}u_{xx}$$

now we replace

$$u = U(t)e^{ikx}$$
$$h = H(t)e^{ikx}$$

$$G = UHe^{2ikx} - H^2 e^{2ikx} ikHe^{ikx} ikUe^{ikx} - \frac{1}{3}H^3 e^{3ikx}(-k^2)Ue^{ikx}$$

$$G = UHe^{2ikx} + k^2 H^3 Ue^{4ikx} + \frac{k^2}{3}UH^3 e^{4ikx}$$

$$G = \left(1 + \frac{4}{3}k^2 H^2 e^{2ikx}\right)UHe^{2ikx}$$

$$G_j = \left(1 + \frac{4}{3}k^2 H^2 e^{2ikx_j}\right)UHe^{2ikx_j}$$

2 Finite Difference

we have the derivatives

$$\left(\frac{\partial q}{\partial x}\right)_j = \frac{q_{j+1} - q_{j-1}}{2\Delta x} = \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}Q(t)e^{ikx_j} = \frac{i\sin(k\Delta x)}{\Delta x}Q(t)e^{ikx_j}$$

$$\left(\frac{\partial^2 q}{\partial x^2}\right)_j = \frac{q_{j+1} - 2q_j + q_{j-1}}{\Delta x^2} = \frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{\Delta x^2}Q(t)e^{ikx_j} = \frac{2\cos(k\Delta x) - 2}{\Delta x^2}Q(t)e^{ikx_j}$$

$$\left(\frac{\partial^2 q}{\partial x^2}\right)_j = -4\frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^2}Q(t)e^{ikx_j} = -\left(\frac{2\sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2 Q(t)e^{ikx_j}$$

$$G = uh - h^2 h_x u_x - \frac{h^3}{3}u_{xx}$$

$$G_j = UHe^{2ikx_j} - H^2 e^{2ikx_j}\frac{i\sin(k\Delta x)}{\Delta x}He^{ikx_j}\frac{i\sin(k\Delta x)}{\Delta x}Ue^{ikx_j} + \frac{1}{3}H^3 e^{3ikx_j}\left(\frac{2\sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2 Ue^{ikx_j}$$

$$G_j = UHe^{2ikx_j} + UH^3 e^{4ikx_j}\left(\frac{\sin(k\Delta x)}{\Delta x}\right)^2 + \frac{1}{3}UH^3 e^{4ikx_j}\left(\frac{2\sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2$$

$$G_j = \left(1 + H^2 e^{2ikx_j}\left(\frac{\sin(k\Delta x)}{\Delta x}\right)^2 + \frac{1}{3}H^2 e^{2ikx_j}\left(\frac{2\sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2\right)UHe^{2ikx_j}$$

$$G_j = \left(1 + \left[\left(\frac{\sin(k\Delta x)}{\Delta x}\right)^2 + \frac{1}{3}\left(\frac{2\sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2\right]H^2 e^{2ikx_j}\right)UHe^{2ikx_j}$$

So we want

$$\left[\left(\frac{\sin(k\Delta x)}{\Delta x}\right)^2 + \frac{1}{3}\left(\frac{2\sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2\right] \approx \frac{4}{3}k^2$$

Wolfram Alpha has

$$\left[\left(\frac{\sin(k\Delta x)}{\Delta x}\right)^2 + \frac{1}{3}\left(\frac{2\sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2\right] = \frac{4k^2}{3} - \frac{13k^4\Delta x^2}{36} + \frac{49k^6\Delta x^4}{1080} + O(\Delta x^6)$$

So we can see this scheme is second order.

3 Finite Element

$$G = uh - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = uhv - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)v$$

and then we integrate over the entire domain to get

$$\int_\Omega Gvdx = \int_\Omega uhvdx - \int_\Omega \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)vdx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_\Omega Gvdx = \int_\Omega uhvdx + \int_\Omega \frac{h^3}{3}u_x v_x dx$$

For u we are going to use $x_{j+1/2}$ as the nodes, which generate the basis functions $\phi_{j+1/2}$, which for us will be the space of continuous linear elements. While for G and h we will choose basis functions w that are linear from $[x_{j-1/2}, x_{j+1/2}]$ but discontinuous at the edges.

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gvdx = \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} uhvdx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^3}{3}u_x v_x dx$$

for all v

4 P1 FEM

We are going to cordinate tranform from x space the interval $[x_{j-1/2}, x_{j+1/2}, x_{j+3/2}]$ to the ξ space interval $[-1, 0, 1]$. To accomplish this we have the following relation

$$x = \xi \Delta x + x_{j+1/2}$$

Taking the derivatives we see
$dx = d\xi \Delta x$, $\frac{dx}{d\xi} = \Delta x$, $\frac{d\xi}{dx} = \frac{1}{\Delta x}$.

We can describe the basis functions in the ξ space

$$\phi_{j+1/2} = \begin{cases} 1+\xi & \xi < 0 \\ 1-\xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$\phi_{j-1/2} = \begin{cases} -\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$\phi_{j+3/2} = \begin{cases} \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

While the descriptions for w's is

$$w^{+}_{j+1/2} = \begin{cases} 1-\xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$w^{-}_{j+1/2} = \begin{cases} 1+\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$w^{+}_{j-1/2} = \begin{cases} -\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$w^{-}_{j+3/2} = \begin{cases} \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

We now replace our functions by our approximations to them

$$G \approx G' = \sum_j G_{j+1/2} w_{j+1/2}$$

$$u \approx u' = \sum_j u_{j+1/2} \phi_{j+1/2}$$

$$h \approx h' = \sum_j h_{j+1/2} w_{j+1/2}$$

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} G' \phi_{j+1/2} dx - \int_{x_{j-1/2}}^{x_{j+3/2}} u'h' \phi_{j+1/2} dx - \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{(h')^3}{3} u'_x (\phi_x)_{j+1/2} dx = 0$$

For all $\phi_{j+1/2}$. For this analysis we choose a particular basis function $\phi_{j+1/2}$ and we look at all the integrals. Begining from the right

$$\int_{x_{j-1/2}}^{x_{j+3/2}} G'(x)\phi_{j+1/2}dx = \int_{-1}^{1} G'(\xi)\phi_{j+1/2}(\xi)\frac{dx}{d\xi}d\xi$$

$$= \Delta x \int_{-1}^{1} \left(G^+_{j-1/2}w^+_{j-1/2} + G^-_{j+1/2}w^-_{j+1/2} + G^+_{j+1/2}w^+_{j+1/2} + G^-_{j-3/2}w^-_{j-3/2}\right)\phi_{j+1/2}d\xi$$

$$\begin{aligned} = \Delta x[G^+_{j-1/2}\int_{-1}^{1} w^+_{j-1/2}\phi_{j+1/2}d\xi + G^-_{j+1/2}\int_{-1}^{1} w^-_{j+1/2}\phi_{j+1/2}d\xi \\ + G^+_{j+1/2}\int_{-1}^{1} w^+_{j+1/2}\phi_{j+1/2}d\xi + G^-_{j+3/2}\int_{-1}^{1} w^-_{j+3/2}\phi_{j+1/2}d\xi] \end{aligned} \quad (8)$$

We have that

$$\int_{-1}^{1} w^+_{j-1/2}\phi_{j+1/2}d\xi = \int_{-1}^{1} w^-_{j+3/2}\phi_{j+1/2}d\xi = \frac{1}{6}$$

and

$$\int_{-1}^{1} w^-_{j+1/2}\phi_{j+1/2}d\xi = \int_{-1}^{1} w^+_{j+1/2}\phi_{j+1/2}d\xi = \frac{1}{3}$$

So

$$\begin{aligned} &= \Delta x\left[\frac{1}{6}G^+_{j-1/2} + \frac{1}{3}G^-_{j+1/2} + \frac{1}{3}G^+_{j+1/2} + \frac{1}{6}G^-_{j+3/2}\right] \\ &= \frac{\Delta x}{6}\left[G^+_{j-1/2} + 2G^-_{j+1/2} + 2G^+_{j+1/2} + G^-_{j+3/2}\right] \end{aligned}$$

Next we have

$$\int_{x_{j-1/2}}^{x_{j+3/2}} h'u'\phi_{j+1/2}dx = \Delta x\int_{-1}^{1} h'(\xi)u'(\xi)\phi_{j+1/2}(\xi)d\xi$$

$$\begin{aligned} = \Delta x\int_{-1}^{1} \left(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2} + h^+_{j+1/2}w^+_{j+1/2} + h^-_{j-3/2}w^-_{j-3/2}\right) \\ \left(u_{j-1/2}\phi_{j-1/2} + u_{j+1/2}\phi_{j+1/2} + u_{j+3/2}\phi_{j+3/2}\right)\phi_{j+1/2}d\xi \end{aligned} \quad (9)$$

$$= \Delta x \int_{-1}^{1} \left(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2} + h^+_{j+1/2}w^+_{j+1/2} + h^-_{j-3/2}w^-_{j-3/2}\right) u_{j-1/2}\phi_{j-1/2}\phi_{j+1/2}$$
$$+\left(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2} + h^+_{j+1/2}w^+_{j+1/2} + h^-_{j-3/2}w^-_{j-3/2}\right) u_{j+1/2}\phi_{j+1/2}\phi_{j+1/2}$$
$$+\left(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2} + h^+_{j+1/2}w^+_{j+1/2} + h^-_{j-3/2}w^-_{j-3/2}\right) u_{j+3/2}\phi_{j+3/2}\phi_{j+1/2}d\xi \tag{10}$$

If one of the terms w_k, ϕ_l , ϕ_m is 0 then $w_k\phi_l\phi_m = 0$

$$= \Delta x \int_{-1}^{1} \left(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2}\right) u_{j-1/2}\phi_{j-1/2}\phi_{j+1/2}$$
$$+\left(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2} + h^+_{j+1/2}w^+_{j+1/2} + h^-_{j-3/2}w^-_{j-3/2}\right) u_{j+1/2}\phi_{j+1/2}\phi_{j+1/2}$$
$$+ \left(h^+_{j+1/2}w^+_{j+1/2} + h^-_{j-3/2}w^-_{j-3/2}\right) u_{j+3/2}\phi_{j+3/2}\phi_{j+1/2}d\xi \tag{11}$$

$$= \Delta x \int_{-1}^{1} u_{j-1/2}h^+_{j-1/2}w^+_{j-1/2}\phi_{j-1/2}\phi_{j+1/2} + u_{j-1/2}h^-_{j+1/2}w^-_{j+1/2}\phi_{j-1/2}\phi_{j+1/2}$$
$$+ u_{j+1/2}h^+_{j-1/2}w^+_{j-1/2}\phi_{j+1/2}\phi_{j+1/2} + u_{j+1/2}h^-_{j+1/2}w^-_{j+1/2}\phi_{j+1/2}\phi_{j+1/2}$$
$$+ u_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}\phi_{j+1/2}\phi_{j+1/2} + u_{j+1/2}h^-_{j-3/2}w^-_{j-3/2}\phi_{j+1/2}\phi_{j+1/2}$$
$$+ u_{j+3/2}h^+_{j+1/2}w^+_{j+1/2}\phi_{j+3/2}\phi_{j+1/2} + u_{j+3/2}h^-_{j-3/2}w^-_{j-3/2}\phi_{j+3/2}\phi_{j+1/2}d\xi \tag{12}$$

Evaluating the integral

$$\int_{-1}^{1} u_{j-1/2}h^+_{j-1/2}w^+_{j-1/2}\phi_{j-1/2}\phi_{j+1/2} + u_{j-1/2}h^-_{j+1/2}w^-_{j+1/2}\phi_{j-1/2}\phi_{j+1/2}$$
$$+ u_{j+1/2}h^+_{j-1/2}w^+_{j-1/2}\phi_{j+1/2}\phi_{j+1/2} + u_{j+1/2}h^-_{j+1/2}w^-_{j+1/2}\phi_{j+1/2}\phi_{j+1/2}$$
$$+ u_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}\phi_{j+1/2}\phi_{j+1/2} + u_{j+1/2}h^-_{j-3/2}w^-_{j-3/2}\phi_{j+1/2}\phi_{j+1/2}$$
$$+ u_{j+3/2}h^+_{j+1/2}w^+_{j+1/2}\phi_{j+3/2}\phi_{j+1/2} + u_{j+3/2}h^-_{j-3/2}w^-_{j-3/2}\phi_{j+3/2}\phi_{j+1/2}d\xi \tag{13}$$

$$
\begin{aligned}
&= u_{j-1/2}h^+_{j-1/2}\int_{-1}^{1} w^+_{j-1/2}\phi_{j-1/2}\phi_{j+1/2}d\xi + u_{j-1/2}h^-_{j+1/2}\int_{-1}^{1} w^-_{j+1/2}\phi_{j-1/2}\phi_{j+1/2}d\xi \\
&+u_{j+1/2}h^+_{j-1/2}\int_{-1}^{1} w^+_{j-1/2}\phi_{j+1/2}\phi_{j+1/2}d\xi + u_{j+1/2}h^-_{j+1/2}\int_{-1}^{1} w^-_{j+1/2}\phi_{j+1/2}\phi_{j+1/2}d\xi \\
&+u_{j+1/2}h^+_{j+1/2}\int_{-1}^{1} w^+_{j+1/2}\phi_{j+1/2}\phi_{j+1/2}d\xi + u_{j+1/2}h^-_{j-3/2}\int_{-1}^{1} w^-_{j-3/2}\phi_{j+1/2}\phi_{j+1/2}d\xi \\
&+u_{j+3/2}h^+_{j+1/2}\int_{-1}^{1} w^+_{j+1/2}\phi_{j+3/2}\phi_{j+1/2}d\xi + u_{j+3/2}h^-_{j-3/2}\int_{-1}^{1} w^-_{j-3/2}\phi_{j+3/2}\phi_{j+1/2}d\xi
\end{aligned}
\tag{14}
$$

Now we evaluate the integrals

$$\int_{-1}^{1} w^+_{j-1/2}\phi_{j-1/2}\phi_{j+1/2}d\xi = \int_{-1}^{0} (-\xi)(-\xi)(1+\xi) = \frac{1}{12}$$

$$\int_{-1}^{1} w^-_{j+1/2}\phi_{j-1/2}\phi_{j+1/2}d\xi = \int_{-1}^{0} (1+\xi)(-\xi)(1+\xi)\,d\xi = \frac{1}{12}$$

$$\int_{-1}^{1} w^+_{j-1/2}\phi_{j+1/2}\phi_{j+1/2}d\xi = \int_{-1}^{0} (-\xi)(1+\xi)(1+\xi)\,d\xi = \frac{1}{12}$$

$$\int_{-1}^{1} w^-_{j+1/2}\phi_{j+1/2}\phi_{j+1/2}d\xi = \int_{-1}^{0} (1+\xi)(1+\xi)(1+\xi)\,d\xi = \frac{1}{4}$$

$$\int_{-1}^{1} w^+_{j+1/2}\phi_{j+1/2}\phi_{j+1/2}d\xi = \int_{0}^{1} (1-\xi)(1-\xi)(1-\xi)\,d\xi = \frac{1}{4}$$

$$\int_{-1}^{1} w^-_{j-3/2}\phi_{j+1/2}\phi_{j+1/2}d\xi = \int_{0}^{1} (\xi)(1-\xi)(1-\xi)\,d\xi = \frac{1}{12}$$

$$\int_{-1}^{1} w^+_{j+1/2}\phi_{j+3/2}\phi_{j+1/2}d\xi = \int_{0}^{1} (1-\xi)(\xi)(1-\xi)\,d\xi = \frac{1}{12}$$

$$\int_{-1}^{1} w^-_{j-3/2}\phi_{j+3/2}\phi_{j+1/2}d\xi = \int_{0}^{1} (\xi)(\xi)(1-\xi)\,d\xi = \frac{1}{12}$$

Note that these sum to the same fractions as in the linear case if the h is constant.

$$= u_{j-1/2}h^+_{j-1/2}\frac{1}{12} + u_{j-1/2}h^-_{j+1/2}\frac{1}{12} \\ + u_{j+1/2}h^+_{j-1/2}\frac{1}{12} + u_{j+1/2}h^-_{j+1/2}\frac{1}{4} \\ + u_{j+1/2}h^+_{j+1/2}\frac{1}{4} + u_{j+1/2}h^-_{j-3/2}\frac{1}{12} \\ + u_{j+3/2}h^+_{j+1/2}\frac{1}{12} + u_{j+3/2}h^-_{j-3/2}\frac{1}{12} \quad (15)$$

$$= \frac{1}{12}\Big[u_{j-1/2}h^+_{j-1/2} + u_{j-1/2}h^-_{j+1/2} + u_{j+1/2}h^+_{j-1/2} + 3u_{j+1/2}h^-_{j+1/2} \\ + 3u_{j+1/2}h^+_{j+1/2} + u_{j+1/2}h^-_{j-3/2} + u_{j+3/2}h^+_{j+1/2} + u_{j+3/2}h^-_{j-3/2}\Big] \quad (16)$$

Therefore

$$\int_{x_{j-1/2}}^{x_{j+3/2}} h'u'\phi_{j+1/2}dx = \frac{\Delta x}{12}\Big[u_{j-1/2}h^+_{j-1/2} + u_{j-1/2}h^-_{j+1/2} + u_{j+1/2}h^+_{j-1/2} + 3u_{j+1/2}h^-_{j+1/2} \\ + 3u_{j+1/2}h^+_{j+1/2} + u_{j+1/2}h^-_{j-3/2} + u_{j+3/2}h^+_{j+1/2} + u_{j+3/2}h^-_{j-3/2}\Big] \quad (17)$$

The next integral is

$$\int_{x_{j-1/2}}^{x_{j+3/2}} \frac{(h')^3}{3}u'_x(\phi_x)_{j+1/2}dx = \frac{1}{\Delta x}\int_{-1}^{1}\frac{(h'(\xi))^3}{3}u'_\xi(\phi_\xi)_{j+1/2}dx \\ = \frac{1}{3\Delta x}\int_{-1}^{1}(h'(\xi))^3u'_\xi(\phi_\xi)_{j+1/2}d\xi$$

were now going to expand andd use the superscript $'$ to denote derivatives

$$= \frac{1}{3\Delta x}\int_{-1}^{1}\left(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2} + h^+_{j+1/2}w^+_{j+1/2} + h^-_{j-3/2}w^-_{j-3/2}\right)^3 \\ \left(u_{j-1/2}\phi'_{j-1/2} + u_{j+1/2}\phi'_{j+1/2} + u_{j+3/2}\phi'_{j+3/2}\right)\phi'_{j+1/2}d\xi \quad (18)$$

$$\left(h^+_{j-1/2}w^+_{j-1/2}\right)$$
$$\left(h^-_{j+1/2}w^-_{j+1/2}\right)$$
$$\left(h^+_{j+1/2}w^+_{j+1/2}\right)$$
$$\left(h^-_{j-3/2}w^-_{j-3/2}\right)$$

$$
\begin{aligned}
&= \frac{1}{3\Delta x}\int_{-1}^{1}\Big[\left(h^+_{j-1/2}w^+_{j-1/2}\right)^3 + 3\left(h^+_{j-1/2}w^+_{j-1/2}\right)^2\left(h^-_{j+1/2}w^-_{j+1/2}\right) \\
&+ 3\left(h^+_{j-1/2}w^+_{j-1/2}\right)^2\left(h^+_{j+1/2}w^+_{j+1/2}\right) + 3\left(h^+_{j-1/2}w^+_{j-1/2}\right)^2\left(h^-_{j-3/2}w^-_{j-3/2}\right) \\
&+3\left(h^+_{j-1/2}w^+_{j-1/2}\right)\left(h^-_{j+1/2}w^-_{j+1/2}\right)^2+6\left(h^+_{j-1/2}w^+_{j-1/2}\right)\left(h^-_{j+1/2}w^-_{j+1/2}\right)\left(h^+_{j+1/2}w^+_{j+1/2}\right) \\
&+6\left(h^+_{j-1/2}w^+_{j-1/2}\right)\left(h^-_{j+1/2}w^-_{j+1/2}\right)\left(h^-_{j-3/2}w^-_{j-3/2}\right)+3\left(h^+_{j-1/2}w^+_{j-1/2}\right)\left(h^+_{j+1/2}w^+_{j+1/2}\right)^2 \\
&+6\left(h^+_{j-1/2}w^+_{j-1/2}\right)\left(h^+_{j+1/2}w^+_{j+1/2}\right)\left(h^-_{j-3/2}w^-_{j-3/2}\right)+3\left(h^+_{j-1/2}w^+_{j-1/2}\right)\left(h^-_{j-3/2}w^-_{j-3/2}\right)^2 \\
&+\left(h^-_{j+1/2}w^-_{j+1/2}\right)^3+3\left(h^-_{j+1/2}w^-_{j+1/2}\right)^2\left(h^+_{j+1/2}w^+_{j+1/2}\right)+3\left(h^-_{j+1/2}w^-_{j+1/2}\right)^2\left(h^-_{j-3/2}w^-_{j-3/2}\right) \\
&+3\left(h^-_{j+1/2}w^-_{j+1/2}\right)\left(h^+_{j+1/2}w^+_{j+1/2}\right)^2+6\left(h^-_{j+1/2}w^-_{j+1/2}\right)\left(h^+_{j+1/2}w^+_{j+1/2}\right)\left(h^-_{j-3/2}w^-_{j-3/2}\right) \\
&+3\left(h^-_{j+1/2}w^-_{j+1/2}\right)\left(h^-_{j-3/2}w^-_{j-3/2}\right)^2+\left(h^+_{j+1/2}w^+_{j+1/2}\right)^3+3\left(h^+_{j+1/2}w^+_{j+1/2}\right)^2\left(h^-_{j-3/2}w^-_{j-3/2}\right) \\
&+ 3\left(h^+_{j+1/2}w^+_{j+1/2}\right)\left(h^-_{j-3/2}w^-_{j-3/2}\right)^2 + \left(h^-_{j-3/2}w^-_{j-3/2}\right)^3\Big] \\
&\left(u_{j-1/2}\phi'_{j-1/2} + u_{j+1/2}\phi'_{j+1/2} + u_{j+3/2}\phi'_{j+3/2}\right)\phi'_{j+1/2}d\xi \quad (19)
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{3\Delta x} \int_{-1}^{1} \Bigg[& \left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^{3} + 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^{2} \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right) \\
& + 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right) \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^{2} \\
& + \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^{3} \\
& + \left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right)^{3} + 3\left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right)^{2} \left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right) \\
& + 3\left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right) \left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right)^{2} + \left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right)^{3} \Bigg] \\
& \left(u_{j-1/2} \phi'_{j-1/2} + u_{j+1/2} \phi'_{j+1/2} + u_{j+3/2} \phi'_{j+3/2}\right) \phi'_{j+1/2} d\xi \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{3\Delta x} \int_{-1}^{1} \Bigg[& \left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^{3} + 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^{2} \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right) \\
& + 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right) \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^{2} + \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^{3} \\
& + \left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right)^{3} + 3\left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right)^{2} \left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right) \\
& + 3\left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right) \left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right)^{2} + \left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right)^{3} \Bigg] \\
& \left(u_{j-1/2} \phi'_{j-1/2} + u_{j+1/2} \phi'_{j+1/2} + u_{j+3/2} \phi'_{j+3/2}\right) \phi'_{j+1/2} d\xi \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{3\Delta x} \int_{-1}^{1} \Bigg[& \left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^{3} + 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^{2} \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right) \\
& + 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)\left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^{2} + \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^{3} \Bigg] \\
& \left(u_{j-1/2} \phi'_{j-1/2}\right) \phi'_{j+1/2} \\
& + \Bigg[\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^{3} + 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^{2} \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right) \\
& + 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)\left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^{2} + \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^{3} \\
& + \left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right)^{3} + 3\left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right)^{2} \left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right) \\
& + 3\left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right)\left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right)^{2} + \left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right)^{3} \Bigg] \\
& \left(u_{j+1/2} \phi'_{j+1/2}\right) \phi'_{j+1/2} \\
& + \Bigg[\left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right)^{3} + 3\left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right)^{2} \left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right) \\
& + 3\left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right)\left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right)^{2} + \left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right)^{3} \Bigg] \\
& \left(u_{j+3/2} \phi'_{j+3/2}\right) \phi'_{j+1/2} d\xi \qquad (22)
\end{aligned}
$$

Now to we break it up into the separate ϕ terms so that

4.0.1 p1

$$
\begin{aligned}
\int_{-1}^{1} \Bigg[& \left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^{3} + 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^{2} \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right) \\
& + 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)\left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^{2} + \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^{3} \Bigg] \\
& \left(u_{j-1/2} \phi'_{j-1/2}\right) \phi'_{j+1/2} \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
= \int_{-1}^{1} \left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^3 & u_{j-1/2}\phi'_{j-1/2}\phi'_{j+1/2} \\
&+ 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^2 \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right) u_{j-1/2}\phi'_{j-1/2}\phi'_{j+1/2} \\
&+ 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right) \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^2 u_{j-1/2}\phi'_{j-1/2}\phi'_{j+1/2} \\
&\qquad + \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^3 u_{j-1/2}\phi'_{j-1/2}\phi'_{j+1/2} d\xi
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
= \left(h^{+}_{j-1/2}\right)^3 u_{j-1/2} & \int_{-1}^{1} \left(w^{+}_{j-1/2}\right)^3 \phi'_{j-1/2}\phi'_{j+1/2} d\xi \\
&+ 3\left(h^{+}_{j-1/2}\right)^2 h^{-}_{j+1/2} u_{j-1/2} \int_{-1}^{1} \left(w^{+}_{j-1/2}\right)^2 w^{-}_{j+1/2}\phi'_{j-1/2}\phi'_{j+1/2} d\xi \\
&+ 3h^{+}_{j-1/2}\left(h^{-}_{j+1/2}\right)^2 u_{j-1/2} \int_{-1}^{1} w^{+}_{j-1/2}\left(w^{-}_{j+1/2}\right)^2 \phi'_{j-1/2}\phi'_{j+1/2} d\xi \\
&\qquad + \left(h^{-}_{j+1/2}\right)^3 u_{j-1/2} \int_{-1}^{1} \left(w^{-}_{j+1/2}\right)^3 \phi'_{j-1/2}\phi'_{j+1/2} d\xi
\end{aligned} \tag{25}
$$

$$
\int_{-1}^{1} \left(w^{+}_{j-1/2}\right)^3 \phi'_{j-1/2}\phi'_{j+1/2} d\xi = \int_{-1}^{0} (-\xi)^3 (-1)(1) d\xi = -\frac{1}{4}
$$

$$
\int_{-1}^{1} \left(w^{+}_{j-1/2}\right)^2 w^{-}_{j+1/2}\phi'_{j-1/2}\phi'_{j+1/2} d\xi = \int_{-1}^{0} (-\xi)^2(1+\xi)(-1)(1) d\xi = -\frac{1}{12}
$$

$$
\int_{-1}^{1} w^{+}_{j-1/2}\left(w^{-}_{j+1/2}\right)^2 \phi'_{j-1/2}\phi'_{j+1/2} d\xi = \int_{-1}^{1} (-\xi)(1+\xi)^2 (-1)(1) d\xi = -\frac{1}{12}
$$

$$
\int_{-1}^{1} \left(w^{-}_{j+1/2}\right)^3 \phi'_{j-1/2}\phi'_{j+1/2} d\xi = \int_{-1}^{1} (1+\xi)^3 (-1)(1) d\xi = -\frac{1}{4}
$$

So then

$$
\begin{aligned}
= \left(h^{+}_{j-1/2}\right)^{3} u_{j-1/2}(-\frac{1}{4}) \\
+ 3\left(h^{+}_{j-1/2}\right)^{2} h^{-}_{j+1/2} u_{j-1/2}(-\frac{1}{12}) \\
+ 3 h^{+}_{j-1/2}\left(h^{-}_{j+1/2}\right)^{2} u_{j-1/2}(-\frac{1}{12}) \\
+ \left(h^{-}_{j+1/2}\right)^{3} u_{j-1/2}(-\frac{1}{4})
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
= -\frac{1}{4}\Bigg(\left(h^{+}_{j-1/2}\right)^{3} u_{j-1/2} + \left(h^{+}_{j-1/2}\right)^{2} h^{-}_{j+1/2} u_{j-1/2} \\
+ h^{+}_{j-1/2}\left(h^{-}_{j+1/2}\right)^{2} u_{j-1/2} + \left(h^{-}_{j+1/2}\right)^{3} u_{j-1/2} \Bigg)
\end{aligned} \tag{27}
$$

4.0.2 p2

$$
\begin{aligned}
\int_{-1}^{1} \Bigg[\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^{3} + 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^{2}\left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right) \\
+ 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)\left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^{2} + \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^{3} \\
+ \left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right)^{3} + 3\left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right)^{2}\left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right) \\
+ 3\left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right)\left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right)^{2} + \left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right)^{3} \Bigg] \\
u_{j+1/2}\left(\phi'_{j+1/2}\right)^{2}
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
= \int_{-1}^{1} & \left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^{3} u_{j+1/2} \left(\phi'_{j+1/2}\right)^{2} \\
& + 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right)^{2} \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right) u_{j+1/2} \left(\phi'_{j+1/2}\right)^{2} \\
& + 3\left(h^{+}_{j-1/2} w^{+}_{j-1/2}\right) \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^{2} u_{j+1/2} \left(\phi'_{j+1/2}\right)^{2} \\
& + \left(h^{-}_{j+1/2} w^{-}_{j+1/2}\right)^{3} u_{j+1/2} \left(\phi'_{j+1/2}\right)^{2} \\
& + \left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right)^{3} u_{j+1/2} \left(\phi'_{j+1/2}\right)^{2} \\
& + 3\left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right)^{2} \left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right) u_{j+1/2} \left(\phi'_{j+1/2}\right)^{2} \\
& + 3\left(h^{+}_{j+1/2} w^{+}_{j+1/2}\right) \left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right)^{2} u_{j+1/2} \left(\phi'_{j+1/2}\right)^{2} \\
& + \left(h^{-}_{j-3/2} w^{-}_{j-3/2}\right)^{3} u_{j+1/2} \left(\phi'_{j+1/2}\right)^{2} d\xi
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
= & \left(h^{+}_{j-1/2}\right)^{3} u_{j+1/2} \int_{-1}^{1} \left(w^{+}_{j-1/2}\right)^{3} \left(\phi'_{j+1/2}\right)^{2} d\xi \\
& + 3\left(h^{+}_{j-1/2}\right)^{2} h^{-}_{j+1/2} u_{j+1/2} \int_{-1}^{1} \left(w^{+}_{j-1/2}\right)^{2} \left(w^{-}_{j+1/2}\right) \left(\phi'_{j+1/2}\right)^{2} d\xi \\
& + 3h^{+}_{j-1/2} \left(h^{-}_{j+1/2}\right)^{2} u_{j+1/2} \int_{-1}^{1} w^{+}_{j-1/2} \left(w^{-}_{j+1/2}\right)^{2} \left(\phi'_{j+1/2}\right)^{2} d\xi \\
& + \left(h^{-}_{j+1/2}\right)^{3} u_{j+1/2} \int_{-1}^{1} \left(w^{-}_{j+1/2}\right)^{3} \left(\phi'_{j+1/2}\right)^{2} d\xi \\
& + \left(h^{+}_{j+1/2}\right)^{3} u_{j+1/2} \int_{-1}^{1} \left(w^{+}_{j+1/2}\right)^{3} \left(\phi'_{j+1/2}\right)^{2} d\xi \\
& + 3\left(h^{+}_{j+1/2}\right)^{2} \left(h^{-}_{j-3/2}\right) u_{j+1/2} \int_{-1}^{1} \left(w^{+}_{j+1/2}\right)^{2} \left(w^{-}_{j-3/2}\right) \left(\phi'_{j+1/2}\right)^{2} d\xi \\
& + 3\left(h^{+}_{j+1/2}\right) \left(h^{-}_{j-3/2}\right)^{2} u_{j+1/2} \int_{-1}^{1} \left(w^{+}_{j+1/2}\right) \left(w^{-}_{j-3/2}\right)^{2} \left(\phi'_{j+1/2}\right)^{2} d\xi \\
& + \left(h^{-}_{j-3/2}\right)^{3} u_{j+1/2} \int_{-1}^{1} \left(w^{-}_{j-3/2}\right)^{3} \left(\phi'_{j+1/2}\right)^{2} d\xi
\end{aligned} \tag{30}
$$

$$\int_{-1}^{1}\left(w_{j-1/2}^{+}\right)^{3}\left(\phi_{j+1/2}'\right)^{2}d\xi=\int_{-1}^{0}(-\xi)^{3}(1)^{2}d\xi=\frac{1}{4}$$

$$\int_{-1}^{1}\left(w_{j-1/2}^{+}\right)^{2}\left(w_{j+1/2}^{-}\right)\left(\phi_{j+1/2}'\right)^{2}d\xi=\int_{-1}^{0}(-\xi)^{2}(1+\xi)(1)^{2}d\xi=\frac{1}{12}$$

$$\int_{-1}^{1}w_{j-1/2}^{+}\left(w_{j+1/2}^{-}\right)^{2}\left(\phi_{j+1/2}'\right)^{2}d\xi=\int_{-1}^{0}(-\xi)(1+\xi)^{2}(1)^{2}d\xi=\frac{1}{12}$$

$$\int_{-1}^{1}\left(w_{j+1/2}^{-}\right)^{3}\left(\phi_{j+1/2}'\right)^{2}d\xi=\int_{-1}^{0}(1+\xi)^{3}(1)^{2}d\xi=\frac{1}{4}$$

$$\int_{-1}^{1}\left(w_{j+1/2}^{+}\right)^{3}\left(\phi_{j+1/2}'\right)^{2}d\xi=\int_{0}^{1}(1-\xi)^{3}(-1)^{2}d\xi=\frac{1}{4}$$

$$\int_{-1}^{1}\left(w_{j+1/2}^{+}\right)^{2}\left(w_{j-3/2}^{-}\right)\left(\phi_{j+1/2}'\right)^{2}d\xi=\int_{0}^{1}(1-\xi)^{2}(\xi)(-1)^{2}d\xi=\frac{1}{12}$$

$$\int_{-1}^{1}\left(w_{j+1/2}^{+}\right)\left(w_{j-3/2}^{-}\right)^{2}\left(\phi_{j+1/2}'\right)^{2}d\xi=\int_{0}^{1}(1-\xi)(\xi)^{2}(-1)^{2}d\xi=\frac{1}{12}$$

$$\int_{-1}^{1}\left(w_{j-3/2}^{-}\right)^{3}\left(\phi_{j+1/2}'\right)^{2}d\xi=\int_{0}^{1}(\xi)^{3}(-1)^{2}d\xi=\frac{1}{4}$$

So

$$\begin{aligned}=\frac{1}{4}\Bigg[&\left(h_{j-1/2}^{+}\right)^{3}u_{j+1/2}+\left(h_{j-1/2}^{+}\right)^{2}h_{j+1/2}^{-}u_{j+1/2}\\&+h_{j-1/2}^{+}\left(h_{j+1/2}^{-}\right)^{2}u_{j+1/2}+\left(h_{j+1/2}^{-}\right)^{3}u_{j+1/2}\\&+\left(h_{j+1/2}^{+}\right)^{3}u_{j+1/2}+3\left(h_{j+1/2}^{+}\right)^{2}\left(h_{j-3/2}^{-}\right)u_{j+1/2}\\&+3\left(h_{j+1/2}^{+}\right)\left(h_{j-3/2}^{-}\right)^{2}u_{j+1/2}+\left(h_{j-3/2}^{-}\right)^{3}u_{j+1/2}\Bigg]\end{aligned}\tag{31}$$

4.0.3 p3

Similarly the integral seperates into

$$\left(h^+_{j+1/2}\right)^3 u_{j+3/2} \int_{-1}^{1} \left(w^+_{j+1/2}\right)^3 \phi'_{j+1/2}\phi'_{j+3/2} d\xi \\ + 3\left(h^+_{j+1/2}\right)^2 \left(h^-_{j-3/2}\right) u_{j+3/2} \int_{-1}^{1} \left(w^+_{j+1/2}\right)^2 \left(w^-_{j-3/2}\right) \phi'_{j+1/2}\phi'_{j+3/2} d\xi \\ + 3\left(h^+_{j+1/2}\right) \left(h^-_{j-3/2}\right)^2 u_{j+3/2} \int_{-1}^{1} \left(w^+_{j+1/2}\right) \left(w^-_{j-3/2}\right)^2 \phi'_{j+1/2}\phi'_{j+3/2} d\xi \\ + \left(h^-_{j-3/2}\right)^3 u_{j+3/2} \int_{-1}^{1} \left(w^-_{j-3/2}\right)^3 \phi'_{j+1/2}\phi'_{j+3/2} d\xi \quad (32)$$

We have that

$$\int_{-1}^{1} \left(w^+_{j+1/2}\right)^3 \phi'_{j+1/2}\phi'_{j+3/2} d\xi = \int_{0}^{1} (1-\xi)^3 (-1)(1) d\xi = -\frac{1}{4}$$

$$\int_{-1}^{1} \left(w^+_{j+1/2}\right)^2 \left(w^-_{j-3/2}\right) \phi'_{j+1/2}\phi'_{j+3/2} d\xi = \int_{0}^{1} (1-\xi)^2 (\xi)(-1)(1) d\xi = -\frac{1}{12}$$

$$\int_{-1}^{1} \left(w^+_{j+1/2}\right) \left(w^-_{j-3/2}\right)^2 \phi'_{j+1/2}\phi'_{j+3/2} d\xi = \int_{0}^{1} (1-\xi)(\xi)^2 (-1)(1) d\xi = -\frac{1}{12}$$

$$\int_{-1}^{1} \left(w^-_{j-3/2}\right)^3 \phi'_{j+1/2}\phi'_{j+3/2} d\xi = \int_{0}^{1} (\xi)^3 (-1)(1) d\xi = -\frac{1}{4}$$

So we have

$$= -\frac{1}{4}\Big[\left(h^+_{j+1/2}\right)^3 u_{j+3/2} + 3\left(h^+_{j+1/2}\right)^2 \left(h^-_{j-3/2}\right) u_{j+3/2} \\ + 3\left(h^+_{j+1/2}\right) \left(h^-_{j-3/2}\right)^2 u_{j+3/2} + \left(h^-_{j-3/2}\right)^3 u_{j+3/2}\Big] \quad (33)$$

I just noticed that $j-3/2$ is $j+3/2$
So our equation becomes

$$
\begin{aligned}
&\frac{\Delta x}{6}\left[G^{+}_{j-1/2} + 2G^{-}_{j+1/2} + 2G^{+}_{j+1/2} + G^{-}_{j+3/2}\right] \\
&\quad = \frac{\Delta x}{12}\Bigg[u_{j-1/2}h^{+}_{j-1/2} + u_{j-1/2}h^{-}_{j+1/2} + u_{j+1/2}h^{+}_{j-1/2} + 3u_{j+1/2}h^{-}_{j+1/2} \\
&\qquad + 3u_{j+1/2}h^{+}_{j+1/2} + u_{j+1/2}h^{-}_{j-3/2} + u_{j+3/2}h^{+}_{j+1/2} + u_{j+3/2}h^{-}_{j-3/2}\Bigg] \\
&\qquad\quad + \frac{1}{12\Delta x}\Bigg[-\Bigg[\left(h^{+}_{j-1/2}\right)^{3} u_{j-1/2} + \left(h^{+}_{j-1/2}\right)^{2} h^{-}_{j+1/2}u_{j-1/2} \\
&\qquad\qquad + h^{+}_{j-1/2}\left(h^{-}_{j+1/2}\right)^{2} u_{j-1/2} + \left(h^{-}_{j+1/2}\right)^{3} u_{j-1/2}\Bigg] \\
&\qquad\qquad + \Bigg[\left(h^{+}_{j-1/2}\right)^{3} u_{j+1/2} + \left(h^{+}_{j-1/2}\right)^{2} h^{-}_{j+1/2}u_{j+1/2} \\
&\qquad\qquad + h^{+}_{j-1/2}\left(h^{-}_{j+1/2}\right)^{2} u_{j+1/2} + \left(h^{-}_{j+1/2}\right)^{3} u_{j+1/2} \\
&\qquad\qquad + \left(h^{+}_{j+1/2}\right)^{3} u_{j+1/2} + 3\left(h^{+}_{j+1/2}\right)^{2}\left(h^{-}_{j-3/2}\right)u_{j+1/2} \\
&\qquad\qquad + 3\left(h^{+}_{j+1/2}\right)\left(h^{-}_{j-3/2}\right)^{2} u_{j+1/2} + \left(h^{-}_{j-3/2}\right)^{3} u_{j+1/2}\Bigg] \\
&\qquad\qquad - \Bigg[\left(h^{+}_{j+1/2}\right)^{3} u_{j+3/2} + 3\left(h^{+}_{j+1/2}\right)^{2}\left(h^{-}_{j-3/2}\right)u_{j+3/2} \\
&\qquad\qquad\quad + 3\left(h^{+}_{j+1/2}\right)\left(h^{-}_{j-3/2}\right)^{2} u_{j+3/2} + \left(h^{-}_{j-3/2}\right)^{3} u_{j+3/2}\Bigg]\Bigg]
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
&2\left[G^{+}_{j-1/2}+2G^{-}_{j+1/2}+2G^{+}_{j+1/2}+G^{-}_{j+3/2}\right] \\
&\quad=\Big[u_{j-1/2}h^{+}_{j-1/2}+u_{j-1/2}h^{-}_{j+1/2}+u_{j+1/2}h^{+}_{j-1/2}+3u_{j+1/2}h^{-}_{j+1/2} \\
&\quad+3u_{j+1/2}h^{+}_{j+1/2}+u_{j+1/2}h^{-}_{j+3/2}+u_{j+3/2}h^{+}_{j+1/2}+u_{j+3/2}h^{-}_{j+3/2}\Big] \\
&\qquad+\frac{1}{\Delta x^2}\Bigg[-\Bigg[\left(h^{+}_{j-1/2}\right)^3u_{j-1/2}+\left(h^{+}_{j-1/2}\right)^2h^{-}_{j+1/2}u_{j-1/2} \\
&\qquad\quad+h^{+}_{j-1/2}\left(h^{-}_{j+1/2}\right)^2u_{j-1/2}+\left(h^{-}_{j+1/2}\right)^3u_{j-1/2}\Bigg] \\
&\qquad\quad+\Bigg[\left(h^{+}_{j-1/2}\right)^3u_{j+1/2}+\left(h^{+}_{j-1/2}\right)^2h^{-}_{j+1/2}u_{j+1/2} \\
&\qquad\quad+h^{+}_{j-1/2}\left(h^{-}_{j+1/2}\right)^2u_{j+1/2}+\left(h^{-}_{j+1/2}\right)^3u_{j+1/2} \\
&\qquad\quad+\left(h^{+}_{j+1/2}\right)^3u_{j+1/2}+\left(h^{+}_{j+1/2}\right)^2\left(h^{-}_{j+3/2}\right)u_{j+1/2} \\
&\qquad\quad+\left(h^{+}_{j+1/2}\right)\left(h^{-}_{j+3/2}\right)^2u_{j+1/2}+\left(h^{-}_{j+3/2}\right)^3u_{j+1/2}\Bigg] \\
&\qquad\quad-\Bigg[\left(h^{+}_{j+1/2}\right)^3u_{j+3/2}+\left(h^{+}_{j+1/2}\right)^2\left(h^{-}_{j+3/2}\right)u_{j+3/2} \\
&\qquad\qquad+\left(h^{+}_{j+1/2}\right)\left(h^{-}_{j+3/2}\right)^2u_{j+3/2}+\left(h^{-}_{j+3/2}\right)^3u_{j+3/2}\Bigg]\Bigg]
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
&2\left[G^{+}_{j-1/2} + 2G^{-}_{j+1/2} + 2G^{+}_{j+1/2} + G^{-}_{j+3/2}\right] \\
&\qquad = \Big[u_{j-1/2}h^{+}_{j-1/2} + u_{j-1/2}h^{-}_{j+1/2} + u_{j+1/2}h^{+}_{j-1/2} + 3u_{j+1/2}h^{-}_{j+1/2} \\
&\qquad + 3u_{j+1/2}h^{+}_{j+1/2} + u_{j+1/2}h^{-}_{j+3/2} + u_{j+3/2}h^{+}_{j+1/2} + u_{j+3/2}h^{-}_{j+3/2}\Big] \\
&+\frac{1}{\Delta x^2}\Bigg[-\left[\left(h^{+}_{j-1/2}\right)^3 + \left(h^{+}_{j-1/2}\right)^2 h^{-}_{j+1/2} + h^{+}_{j-1/2}\left(h^{-}_{j+1/2}\right)^2 + \left(h^{-}_{j+1/2}\right)^3\right]u_{j-1/2} \\
&\qquad + \Big[\left(h^{+}_{j-1/2}\right)^3 + \left(h^{+}_{j-1/2}\right)^2 h^{-}_{j+1/2} + h^{+}_{j-1/2}\left(h^{-}_{j+1/2}\right)^2 + \left(h^{-}_{j+1/2}\right)^3 \\
&+\left(h^{+}_{j+1/2}\right)^3 + \left(h^{+}_{j+1/2}\right)^2\left(h^{-}_{j+3/2}\right) + \left(h^{+}_{j+1/2}\right)\left(h^{-}_{j+3/2}\right)^2 + \left(h^{-}_{j+3/2}\right)^3\Big]u_{j+1/2} \\
&-\left[\left(h^{+}_{j+1/2}\right)^3 + \left(h^{+}_{j+1/2}\right)^2\left(h^{-}_{j+3/2}\right) + \left(h^{+}_{j+1/2}\right)\left(h^{-}_{j+3/2}\right)^2 + \left(h^{-}_{j+3/2}\right)^3\right]u_{j+3/2}\Bigg]
\end{aligned}
\tag{36}
$$

if h is constant then the coefficients are the same as before.