

# Behaviour of the Dam-Break Problem for the Serre Equations

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## ABSTRACT

**Keywords:** dispersive waves, conservation laws, Serre equation, finite volume method, finite difference method

## 1 INTRODUCTION

## 2 SERRE EQUATIONS

The Serre equations can be derived as an approximation to the full Euler equations by depth integration similar to (Su and Gardner 1969). They can also be seen as an asymptotic expansion to the Euler equations as well (Lannes and Bonneton 2009). The former is more consistent with the perspective from which numerical methods will be developed while the latter indicates the appropriate regions in which to use these equations as a model for fluid flow. The set up of the scenario under which the Serre approximation is made consists of a two dimensional  $\mathbf{x} = (x, z)$  fluid over a bottom topography as in Figure 1 acting under gravity. Consider a fluid particle at depth  $\xi(\mathbf{x}, t) = z - h(x, t) - z_b(x)$  below the water surface, see Figure 1. Where the water depth is  $h(x, t)$  and  $z_b(x)$  is the bed elevation. The fluid particle is subject to the pressure,  $p(\mathbf{x}, t)$  and gravitational acceleration,  $\mathbf{g} = (0, g)^T$  and has a velocity  $\mathbf{u} = (u(\mathbf{x}, t), w(\mathbf{x}, t))$ , where  $u(\mathbf{x}, t)$  is the velocity in the  $x$ -coordinate and  $w(\mathbf{x}, t)$  is the velocity in the  $z$ -coordinate and  $t$  is time. Assuming that  $z_b(x)$  is constant the Serre equations read (Li et al. 2014)

$$\frac{\partial h}{\partial t} + \frac{\partial(\bar{u}h)}{\partial x} = 0 \quad (1a)$$

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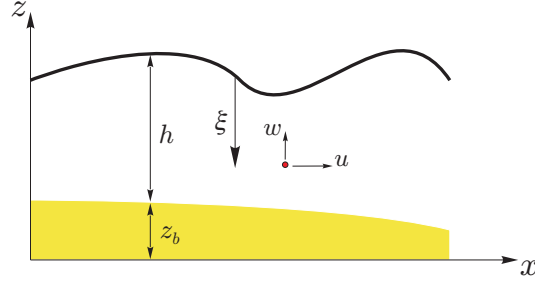


FIG. 1: The notation used for one-dimensional flow governed by the Serre equation.

$$\underbrace{\frac{\partial(\bar{u}h)}{\partial t} + \frac{\partial}{\partial x} \left( \bar{u}^2 h + \frac{gh^2}{2} \right)}_{\text{Shallow Water Wave Equations}} + \underbrace{\frac{\partial}{\partial x} \left( \frac{h^3}{3} \left[ \frac{\partial \bar{u}}{\partial x} \frac{\partial \bar{u}}{\partial x} - \bar{u} \frac{\partial^2 \bar{u}}{\partial x^2} - \frac{\partial^2 \bar{u}}{\partial x \partial t} \right] \right)}_{\text{Dispersion Terms}} = 0. \quad (1b)$$

Serre Equations

Where  $\bar{u}$  means the average of  $u$  over the depth of water.

## FINITE DIFFERENCE AND LAX WENDROFF

This method was used in El et al. (2006) for the Serre equations. It consists of a lax-wendroff update for  $h$  and a spatio-temporal second order approximation to  $[\ ]$  which results in a fully second-order method. To make this method precise it will be presented here in sufficient replicable detail.

Note that  $[\ ]$  is in conservative law form for  $h$  where the Jacobian is  $u$ , where the bar has been dropped to simplify the notation. Thus assuming a fixed resolution discretisation for space and time which will be represented as follows  $q_i^n = q(x_i, t^n)$  for some quantity  $q$  the lax-wendroff update for  $h$  obtained is

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{2\Delta x} ((uh)_{i+1}^n - (uh)_{i-1}^n) + \frac{\Delta t^2}{2\Delta x^2} \left( \frac{u_{i+1}^n - u_i^n}{2} ((uh)_{i+1}^n - (uh)_i^n) - \frac{u_i^n - u_{i-1}^n}{2} ((uh)_i^n - (uh)_{i-1}^n) \right) \quad (2)$$

To get a second-order approximation to  $[\ ]$  is built by first expanding all the derivatives out and making use of the continuity equation  $[\ ]$ , this results in:

$$h \frac{\partial u}{\partial t} + X - h^2 \frac{\partial^2 u}{\partial x \partial t} - \frac{h^3}{3} \frac{\partial^3 u}{\partial x^2 \partial t} = 0 \quad (3a)$$

37 where  $X$  contains only spatial derivatives and is

$$38 \quad X = uh \frac{\partial u}{\partial x} + gh \frac{\partial h}{\partial x} + h^2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \frac{h^3}{3} \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} - h^2 u \frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3} u \frac{\partial^3 u}{\partial x^3}. \quad (3b)$$

40 Then taking second-order approximations to the time derivatives for [] gives

$$41 \quad h^n \frac{u^{n+1} - u^{n-1}}{2\Delta t} + X^n - (h^n)^2 \frac{\left(\frac{\partial u}{\partial x}\right)^{n+1} - \left(\frac{\partial u}{\partial x}\right)^{n-1}}{2\Delta t} - \frac{(h^n)^3}{3} \frac{\left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1} - \left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1}}{2\Delta t} = 0 \quad (4)$$

$$44 \quad h^n (u^{n+1} - u^{n-1}) + 2\Delta t X^n - (h^n)^2 \left( \left(\frac{\partial u}{\partial x}\right)^{n+1} - \left(\frac{\partial u}{\partial x}\right)^{n-1} \right) - \frac{(h^n)^3}{3} \left( \left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1} - \left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1} \right) = 0 \quad (5)$$

$$46 \quad h^n u^{n+1} - (h^n)^2 \left(\frac{\partial u}{\partial x}\right)^{n+1} - \frac{(h^n)^3}{3} \left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1} + 2\Delta t X^n - h^n u^{n-1} + (h^n)^2 \left(\frac{\partial u}{\partial x}\right)^{n-1} + \frac{(h^n)^3}{3} \left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1} = 0 \quad (6)$$

48 Let

$$49 \quad Y^n = 2\Delta t X^n - h^n u^{n-1} + (h^n)^2 \left(\frac{\partial u}{\partial x}\right)^{n-1} + \frac{(h^n)^3}{3} \left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1} \quad (7)$$

51 Taking second-order approximations to the spatial derivatives gives

$$52 \quad h_i^n u_i^{n+1} - (h_i^n)^2 \left( \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} \right) - \frac{(h_i^n)^3}{3} \left( \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} \right) = -Y_i^n \quad (8)$$

54 This can be rearranged into a tri-diagonal matrix that updates  $u$  given its current and previous values. So that

$$56 \quad \begin{bmatrix} u_0^{n+1} \\ \vdots \\ u_m^{n+1} \end{bmatrix} = A^{-1} \begin{bmatrix} -Y_0^n \\ \vdots \\ -Y_m^n \end{bmatrix} =: \mathcal{G}_u(\mathbf{u}^n, \mathbf{h}^n, \mathbf{u}^{n-1}, \Delta t).$$

58 Where

$$59 \quad A = \begin{bmatrix} b_0 & c_0 & & & & \\ a_0 & b_1 & c_1 & & & \\ & a_1 & b_2 & c_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{m-3} & b_{m-2} & c_{m-2} \\ & & & & a_{m-2} & b_{m-1} & c_{m-1} \\ & & & & & a_{m-1} & b_m \end{bmatrix}$$

with

$$a_{i-1} = \frac{(h_i^n)^2}{2\Delta x} \frac{h_{i+1}^n - h_{i-1}^n}{2\Delta x} - \frac{(h_i^n)^3}{3\Delta x^2}, \quad (9a)$$

$$b_i = h_i^n + \frac{2h_i^n}{3\Delta x^2} \quad (9b)$$

and

$$c_i = -\frac{(h_i^n)^2}{2\Delta x} \frac{h_{i+1}^n - h_{i-1}^n}{2\Delta x} - \frac{(h_i^n)^3}{3\Delta x^2}. \quad (9c)$$

61 Lastly the final expression for  $Y_i^n$  is given by:

$$Y_i^n = 2\Delta t X_i^n - h_i^n u_i^{n-1} + (h_i^n)^2 \frac{u_{i+1}^{n-1} - u_{i-1}^{n-1}}{2\Delta x} + \frac{(h_i^n)^3}{3} \frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{\Delta x^2} \quad (10)$$

$$\begin{aligned} Y_i^n = 2\Delta t & \left[ u_i^n h_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + g h_i^n \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{2\Delta x} + (h_i^n)^2 \left( \frac{u_{i+1}^{n-1} - u_{i-1}^{n-1}}{2\Delta x} \right)^2 \right. \\ & + \frac{(h_i^n)^3}{3} \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} - (h_i^n)^2 u_i^n \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \\ & \left. - \frac{(h_i^n)^3}{3} u_i^n \frac{u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n}{2\Delta x^3} \right] \\ & - h_i^n u_i^{n-1} + (h_i^n)^2 \frac{u_{i+1}^{n-1} - u_{i-1}^{n-1}}{2\Delta x} + \frac{(h_i^n)^3}{3} \frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{\Delta x^2} \end{aligned} \quad (11)$$

## 66 SECOND ORDER FINITE DIFFERENCE METHOD

67 Above a second order finite difference method for updating  $u$  was given, thus replacing  
68 the numerical method for  $h$  by replacing derivatives with second order finite differences  
69 will give another full finite difference method. From (1a) we expand derivatives and then  
70 approximate them by second order finite differences to give

$$\frac{h_i^{n+1} - h_i^{n-1}}{2\Delta t} + u_i^n \frac{h_{i+1}^n - h_{i-1}^n}{2\Delta x} + h_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0 \quad (12)$$

73 After rearranging this to give an update formula one obtains

$$74 \quad h_i^{n+1} = h_i^{n-1} - \Delta t \left( u_i^n \frac{h_{i+1}^n - h_{i-1}^n}{\Delta x} + h_i^n \frac{u_{i+1}^n - u_{i-1}^n}{\Delta x} \right) \quad (13)$$

75

76 Combining this with the update formula for  $u$  [] gives a full finite difference method  
77 for the Serre equations.

## 78 **A HYBRID FINITE DIFFERENCE-VOLUME METHOD FOR SERRE** 79 **EQUATIONS IN CONSERVATIVE FORM**

80 [] also offer another family of numerical methods which can be constructed by first  
81 rearranging the equations into conservative form and then using both a finite difference  
82 and a finite volume method to solve these equations. This paper will make use of the  
83 first-, second- and third-order versions of this method as set out in []. These have been  
84 validated for both smooth and discontinuous problems and their orders of accuracy have  
85 been verified for smooth solutions so they are of particular interest for the comparisons  
86 that will be investigated in this paper.

## 87 **NUMERICAL SIMULATIONS**

88 In this section the methods introduced in this paper will be validated by using them  
89 to approximate an analytic solution of the Serre equations, this will also be used to verify  
90 their order of accuracy. Then an in depth comparison of using these methods for a smooth  
91 approximation to the discontinuous dam break problem will be provided to investigate the  
92 behaviour of these equations in the presence of discontinuities. This is a problem that so  
93 far has only received a proper treatment in (El et al. 2006), with other research giving only  
94 a cursory look into the topic.

## 95 **SMOOTHED DAM-BREAK**

96 The discontinuous dam-break problem can be approximated by a smooth function us-  
97 ing the hyperbolic tan function []. Such an approximation will be called a smoothed dam-  
98 break problem and will be defined as such

$$99 \quad h(x, 0) = h_0 + \frac{h_1 - h_0}{2} (1 + \tanh(a(x_0 - x))),$$

100

$$102 \quad u(x, 0) = 0.0m/s.$$

103

104 Where  $a$  is given and controls the width of the transition between the two dam-break  
105 heights of  $h_0$  and  $h_1$ . For large  $a$  the width is small and vice versa. For a fixed  $\Delta x$  there

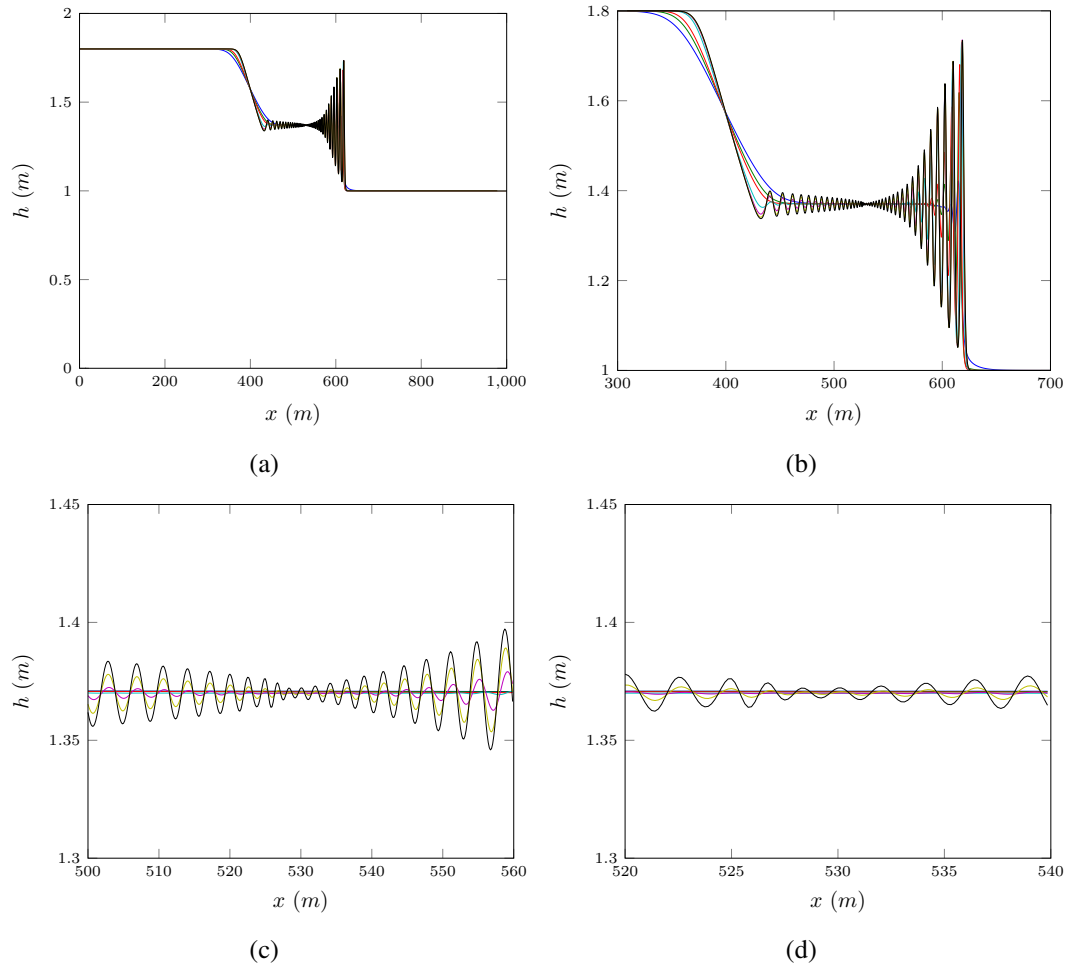


FIG. 2: Smooth dam break problem for FDcent [] with  $dx = 10/2^6 m$  for  $a = 0.05$  (– blue),  $a = 0.075$  (– green),  $a = 0.1$  (– red),  $a = 0.25$  (– cyan),  $a = 0.5$  (– magenta),  $a = 0.75$  (– yellow),  $a = 1.00$  (– black)

are large enough  $a$  values such that the transition width is zero. This experiment was run for both of the methods described in this paper and the 3 different order finite difference-volume methods described in []. In this particular simulation  $h_0 = 1.0m$ ,  $h_1 = 1.8m$  on  $x \in [0m, 1000m]$  for  $t \in [0s, 30s]$  with  $x_0 = 500m$ . The simulations were run changing both  $\Delta x$  and  $a$  and for stability  $\Delta t = 0.01\Delta x$  while for the second order finite volume method  $\theta = 1.2$ .

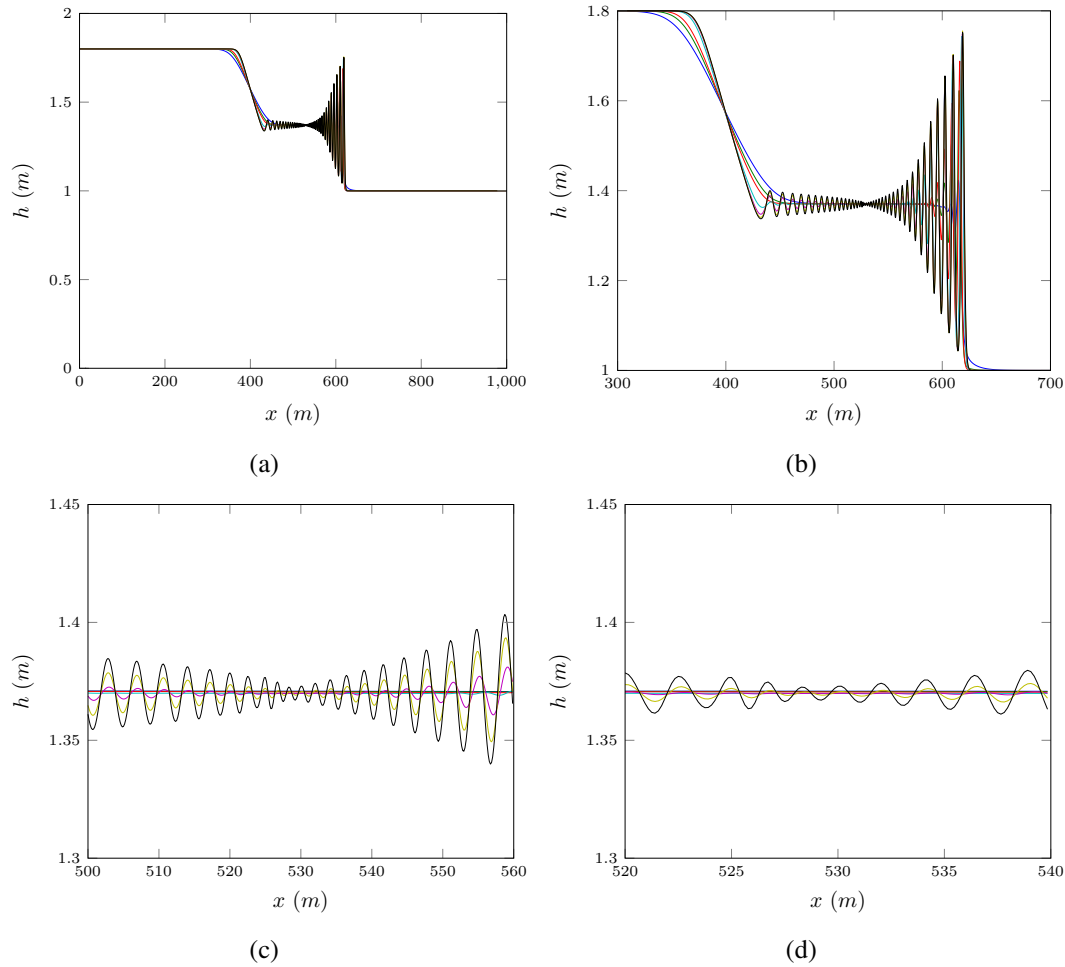


FIG. 3: Smooth dam break problem for grim [] with  $dx = 10/2^6 m$  for  $a = 0.05$  (– blue),  $a = 0.075$  (– green),  $a = 0.1$  (– red),  $a = 0.25$  (– cyan),  $a = 0.5$  (– magenta),  $a = 0.75$  (– yellow),  $a = 1.00$  (– black)

**Changing  $a$**

**Changing  $dx$**

**Comparison of Models**

**CONCLUSIONS**

**ACKNOWLEDGEMENTS**

**REFERENCES**

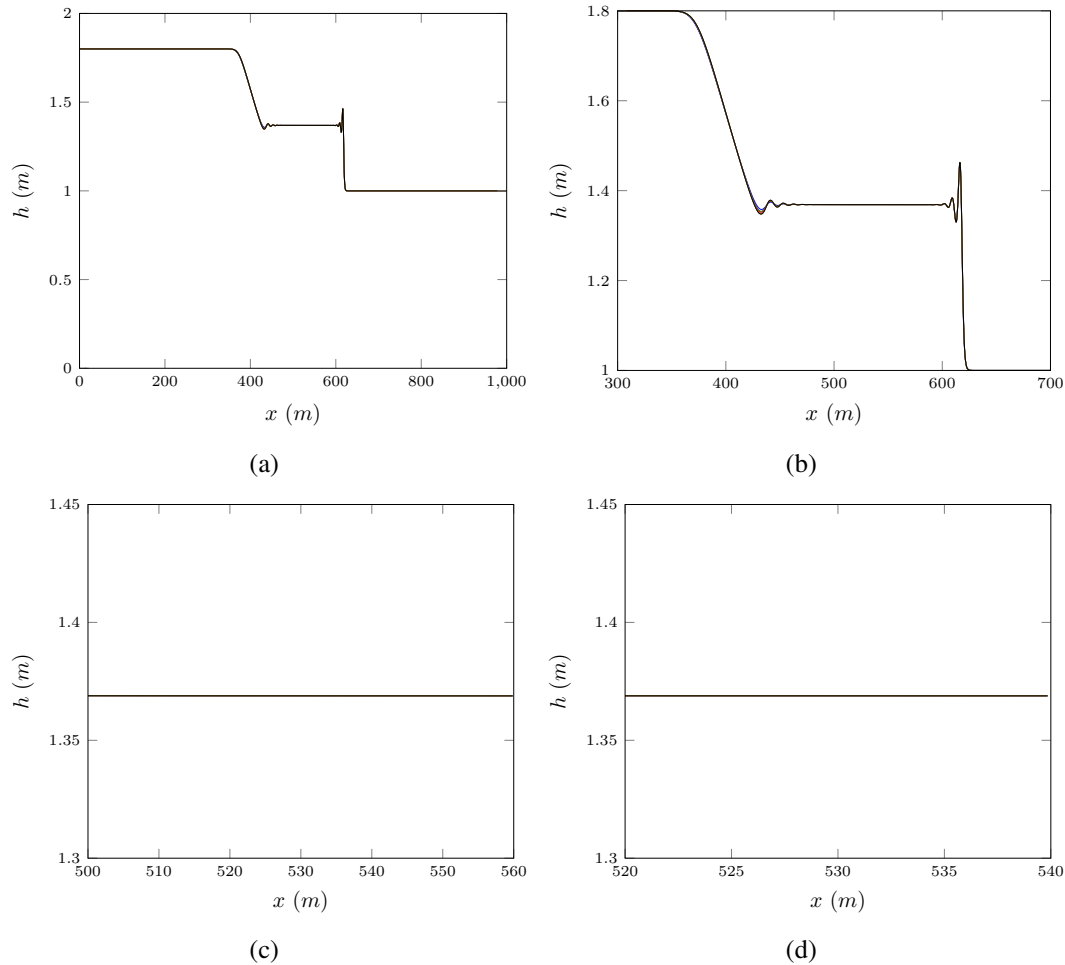


FIG. 4: Smooth dam break problem for  $\alpha_1$  [ ] with  $dx = 10/2^6 m$  for  $a = 0.5$  (– blue),  $a = 0.75$  (– green),  $a = 1.0$  (– red),  $a = 2.5$  (– cyan),  $a = 5.0$  (– magenta),  $a = 7.5$  (– yellow),  $a = 10.0$  (– black)

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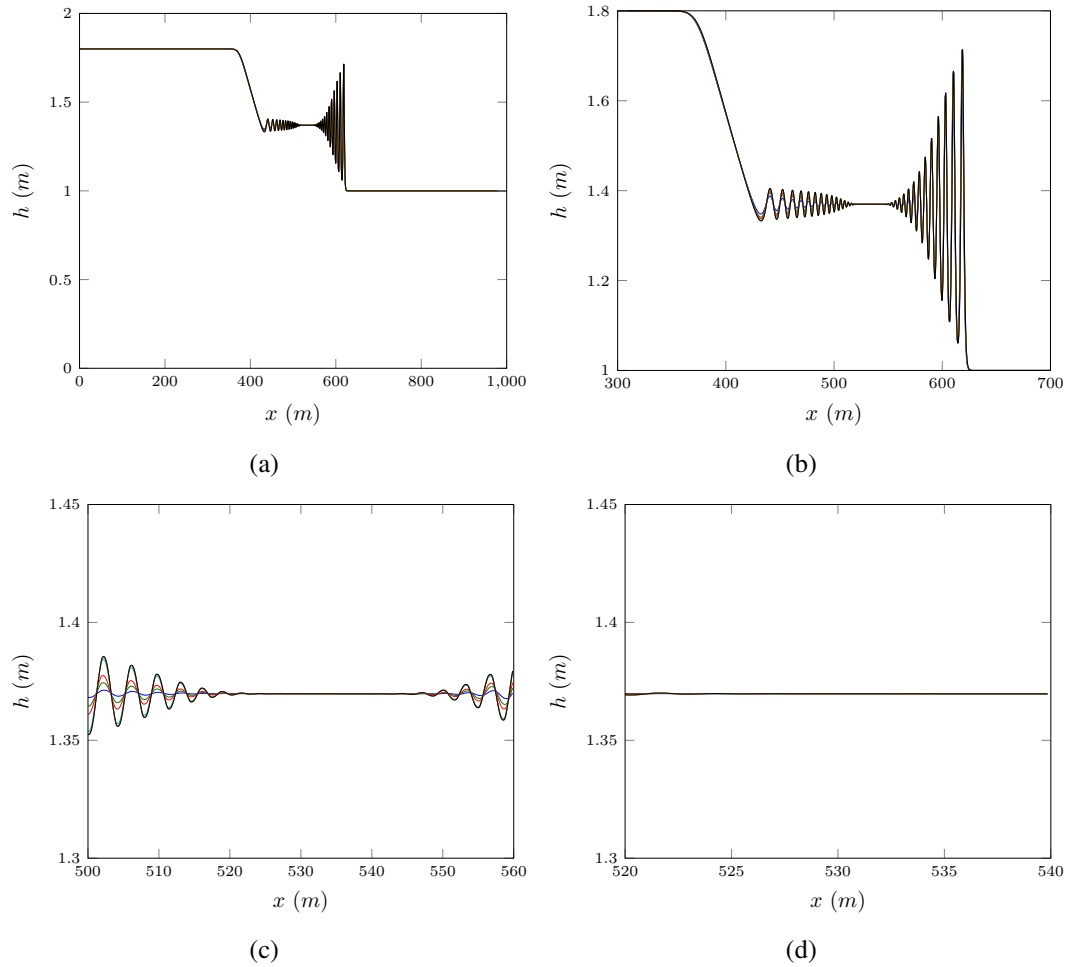


FIG. 5: Smooth dam break problem for  $\alpha_2$  [ ] with  $dx = 10/2^6 m$  for  $a = 0.5$  (– blue),  $a = 0.75$  (– green),  $a = 1.0$  (– red),  $a = 2.5$  (– cyan),  $a = 5.0$  (– magenta),  $a = 7.5$  (– yellow),  $a = 10.0$  (– black)

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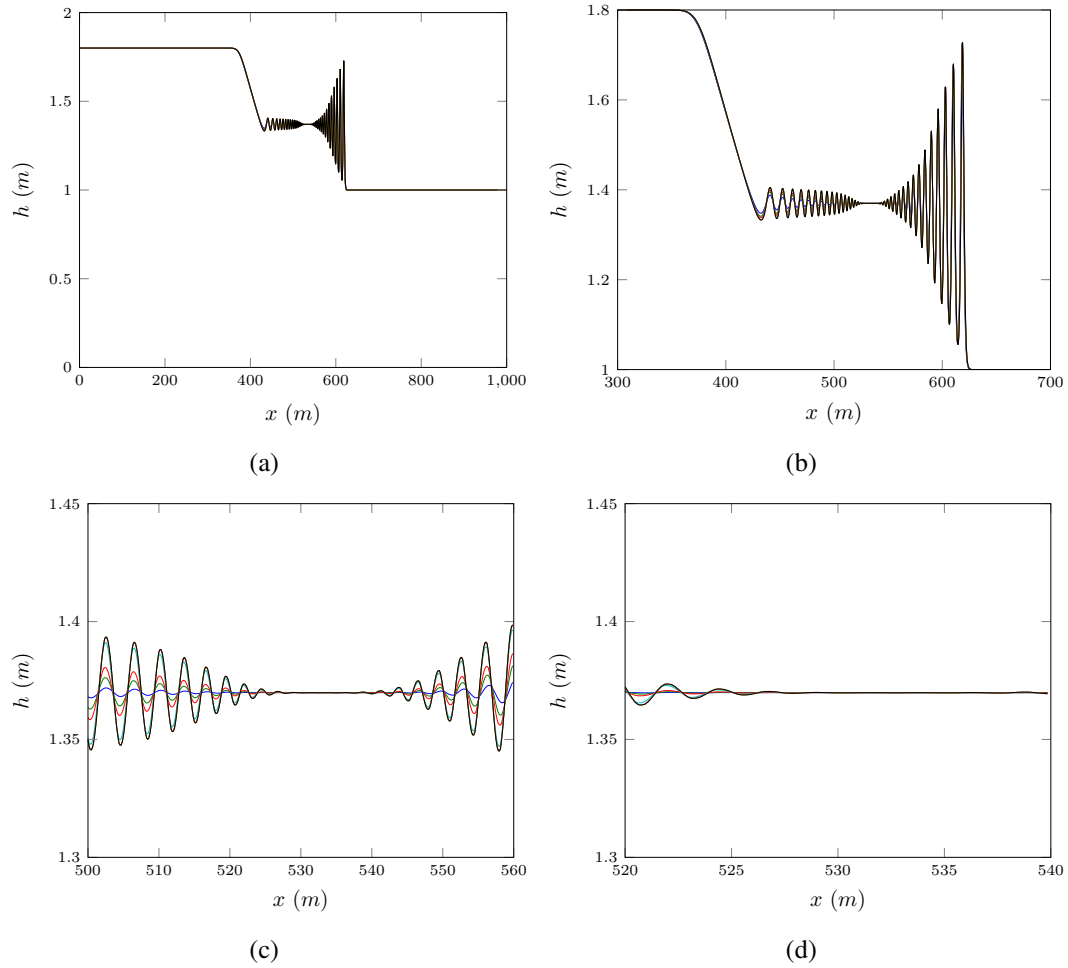


FIG. 6: Smooth dam break problem for o3 [] with  $dx = 10/2^6 m$  for  $a = 0.5$  (– blue),  $a = 0.75$  (– green),  $a = 1.0$  (– red),  $a = 2.5$  (– cyan),  $a = 5.0$  (– magenta),  $a = 7.5$  (– yellow),  $a = 10.0$  (– black)

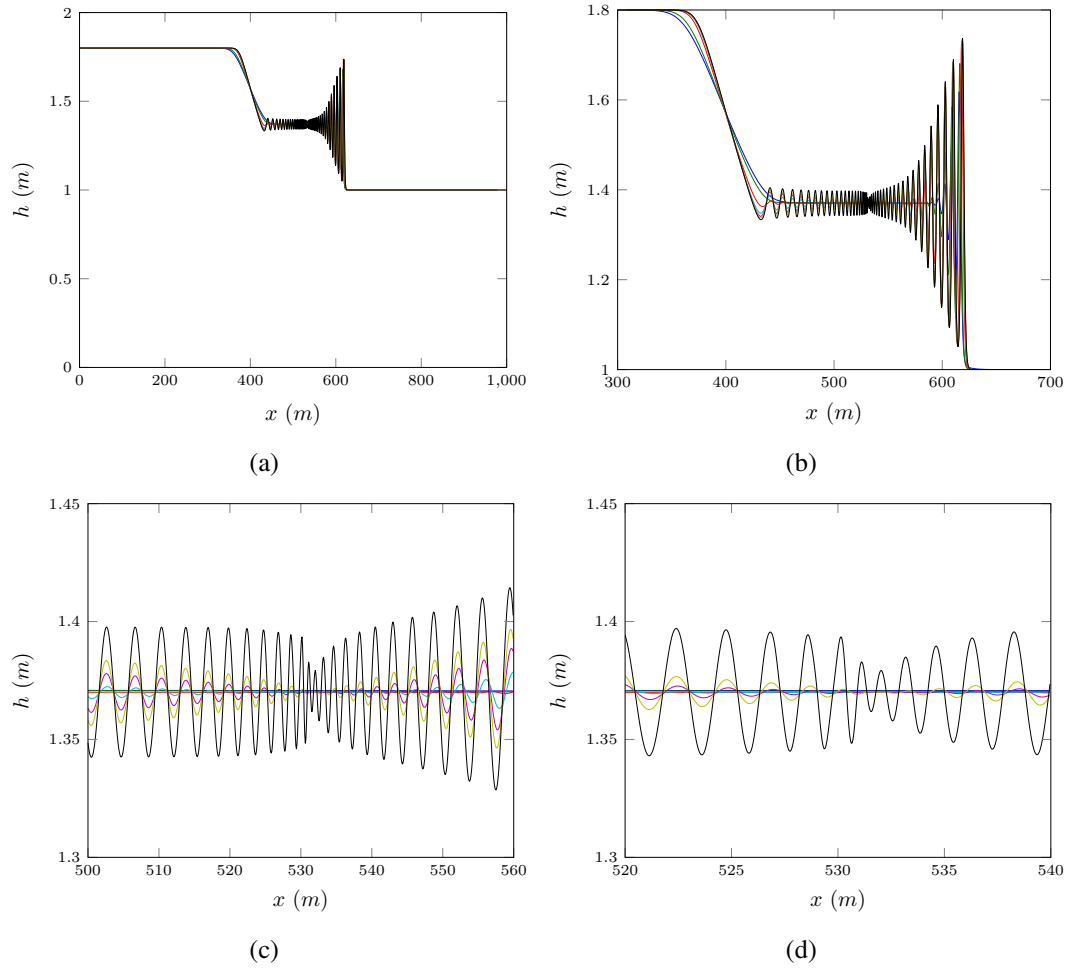


FIG. 7: Smooth dam break problem for FDcent [] with  $dx = 10/2^8 m$  for  $a = 0.075$  (– blue),  $a = 0.1$  (– green),  $a = 0.25$  (– red),  $a = 0.5$  (– cyan),  $a = 0.75$  (– magenta),  $a = 1.0$  (– yellow),  $a = 2.5$  (– black)

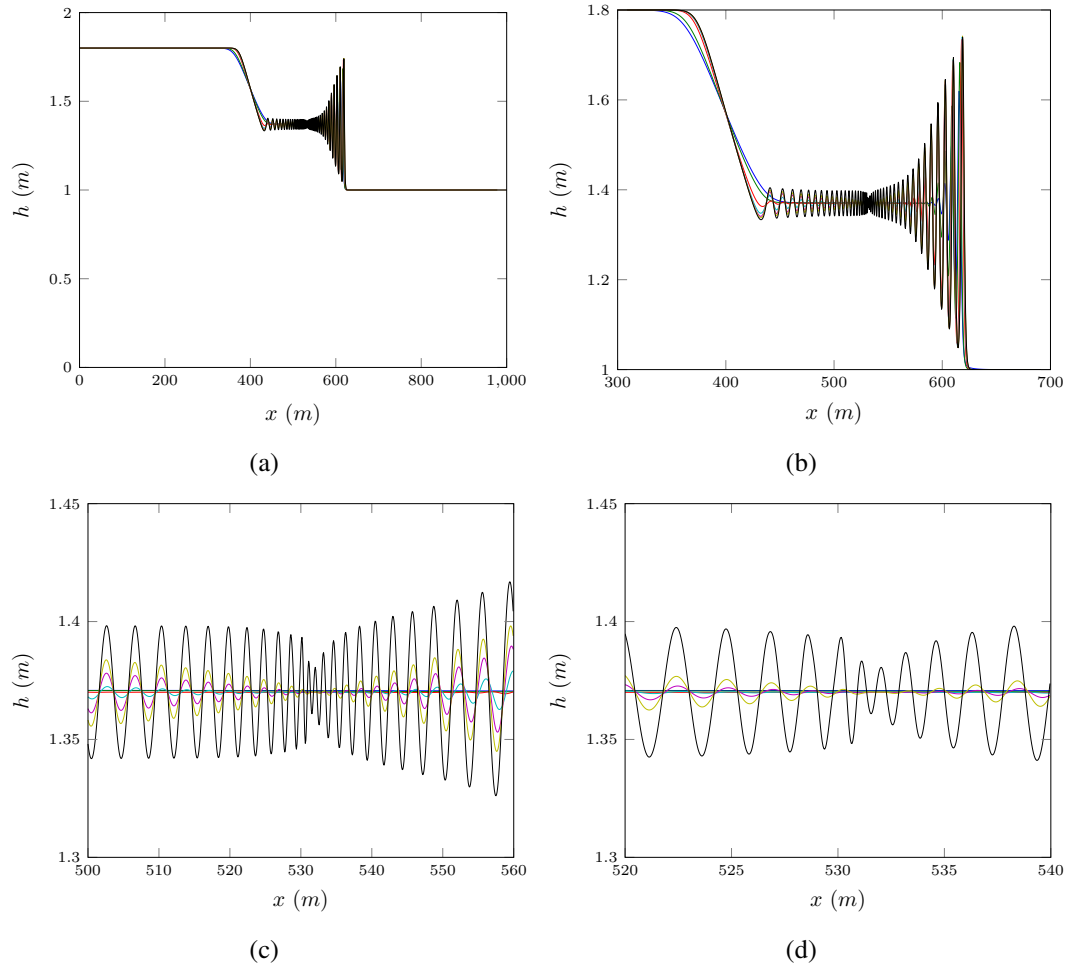


FIG. 8: Smooth dam break problem for grim [] with  $dx = 10/2^8 m$  for  $a = 0.075$  (– blue),  $a = 0.1$  (– green),  $a = 0.25$  (– red),  $a = 0.5$  (– cyan),  $a = 0.75$  (– magenta),  $a = 1.0$  (– yellow),  $a = 2.5$  (– black)

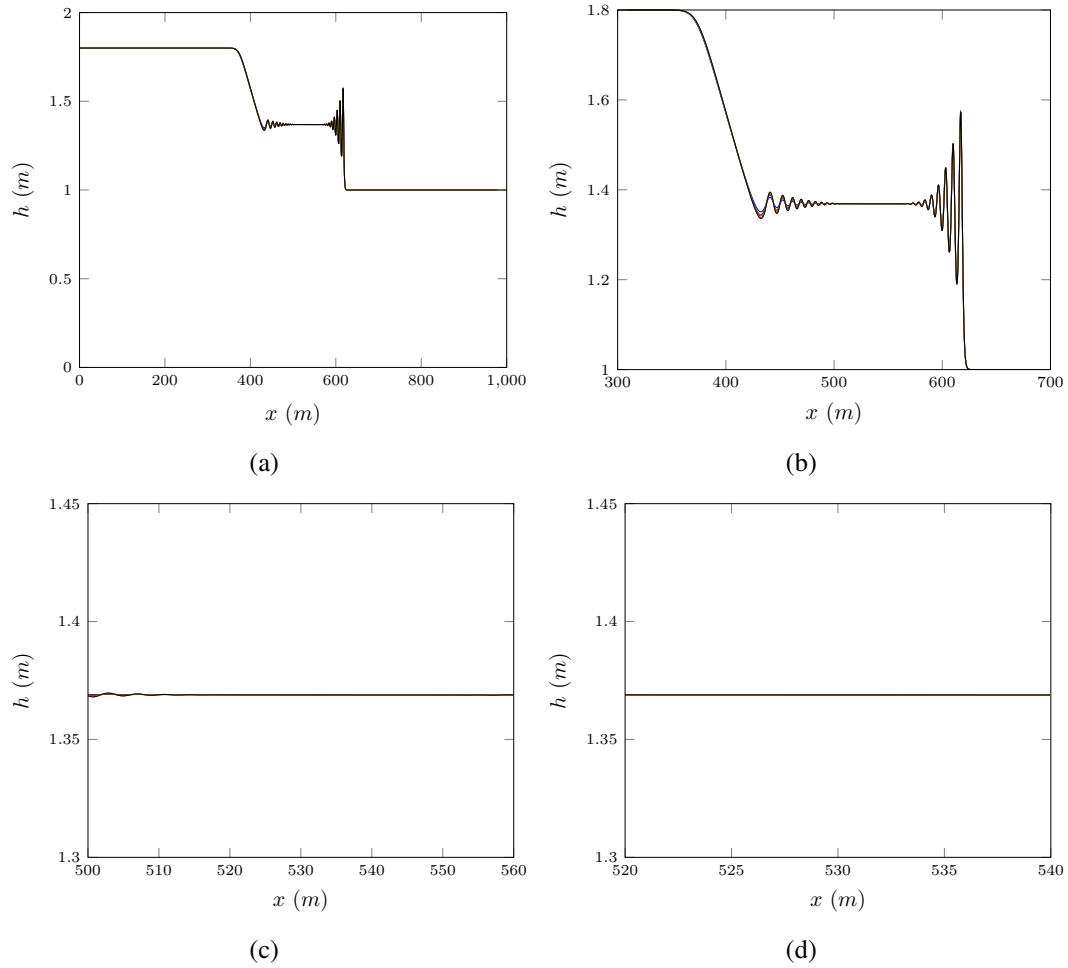


FIG. 9: Smooth dam break problem for  $\alpha_1$  [] with  $dx = 10/2^8 m$  for  $a = 0.5$  (– blue),  $a = 0.75$  (– green),  $a = 1.0$  (– red),  $a = 2.5$  (– cyan),  $a = 5.0$  (– magenta),  $a = 7.5$  (– yellow),  $a = 10.0$  (– black)

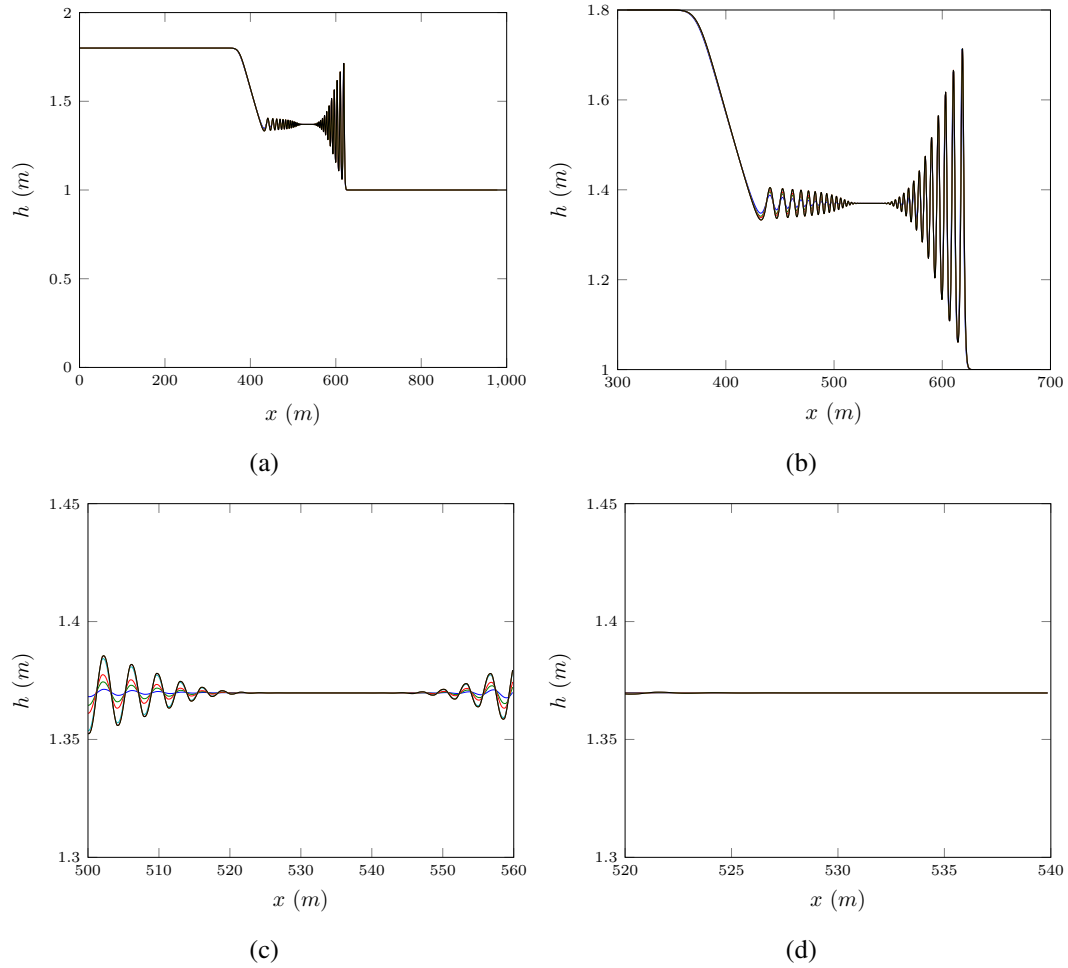


FIG. 10: Smooth dam break problem for  $\alpha_2$  with  $dx = 10/2^8 m$  for  $a = 0.5$  (– blue),  $a = 0.75$  (– green),  $a = 1.0$  (– red),  $a = 2.5$  (– cyan),  $a = 5.0$  (– magenta),  $a = 7.5$  (– yellow),  $a = 10.0$  (– black)

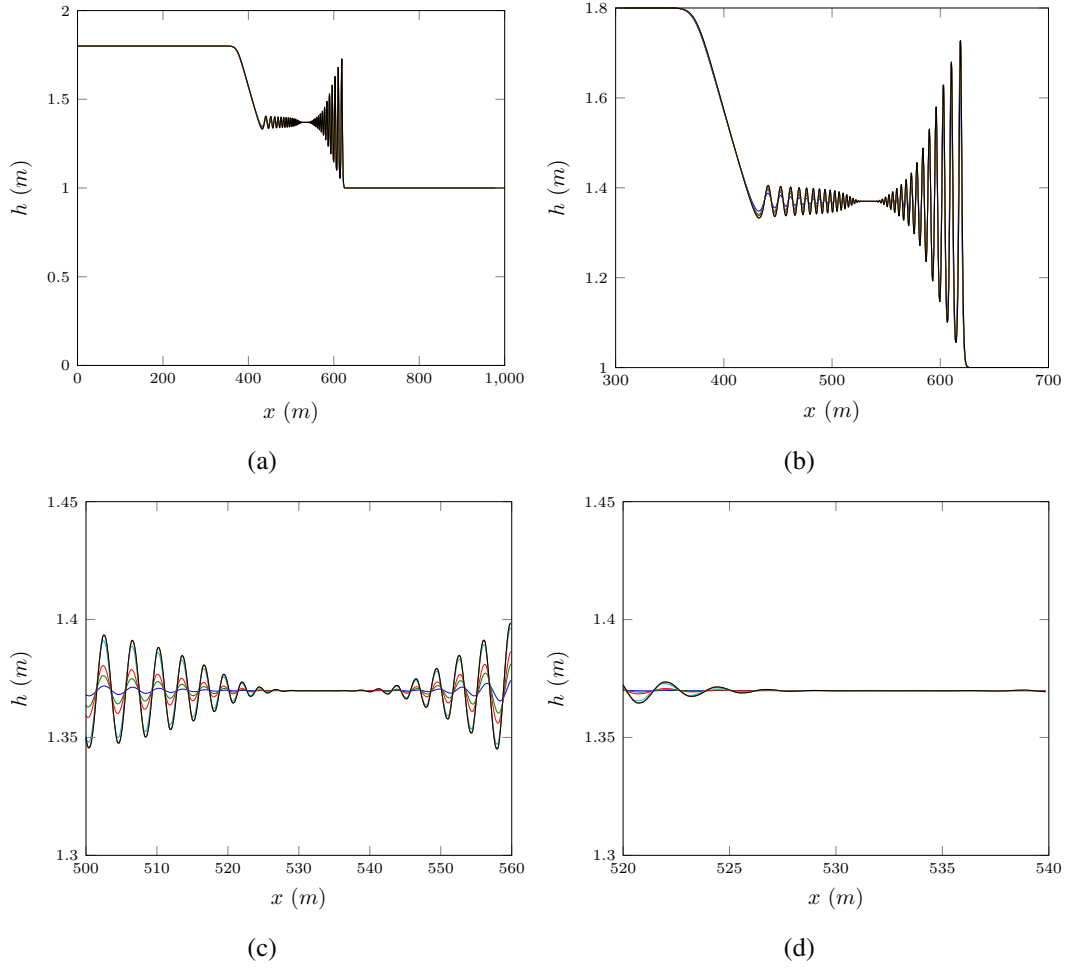


FIG. 11: Smooth dam break problem for o3 [] with  $dx = 10/2^8 m$  for  $a = 0.5$  (– blue),  $a = 0.75$  (– green),  $a = 1.0$  (– red),  $a = 2.5$  (– cyan),  $a = 5.0$  (– magenta),  $a = 7.5$  (– yellow),  $a = 10.0$  (– black)