1 Energy in DB

$$\mathcal{H}(t) = \frac{1}{2}\int_{-\infty}^{\infty} hu^2 + gh^2 + \frac{h^3}{3}\left(\frac{\partial u}{\partial x}\right)^2 \, dx$$

We split it so that

$$\mathcal{H}_1(t) = \int_{-\infty}^{\infty} hu^2 \, dx$$

$$\mathcal{H}_2(t) = \int_{-\infty}^{\infty} gh^2 \, dx$$

$$\mathcal{H}_3(t) = \int_{-\infty}^{\infty} \frac{h^3}{3}\left(\frac{\partial u}{\partial x}\right)^2 \, dx$$

For the Soliton at $t = 0$

$$h(x,0) = a_0 + a_1 sech^2\left(\kappa x\right)$$

$$u(x,0) = c\left(1 - \frac{a_0}{a_0 + a_1 sech^2\left(\kappa x\right)}\right)$$

$$u(x,0) = c\left(1 - \frac{a_0}{h}\right)$$

$$u_x(x,0) = c\left(1 - \frac{a_0}{h}\right) = ac\frac{h_x}{h^2}$$

$$\kappa = \frac{\sqrt{3a_1}}{2a_0\sqrt{a_0 + a_1}} \tag{1}$$

and

$$c = \sqrt{g\left(a_0 + a_1\right)} \tag{2}$$

Integrating from x_0 to x_1 Potential

$$\mathcal{H}_2(t) = g\int_{x_0}^{x_1} \left(a_0 + a_1 sech^2\left(\kappa x\right)\right)^2 \, dx$$

$$\mathcal{H}_2(t) = g\int_{x_0}^{x_1} a_0^2 + 2a_0a_1 sech^2\left(\kappa x\right) + a_1^2 sech^4\left(\kappa x\right) \, dx$$

$$\mathcal{H}_2(t) = g\left(a_0^2\left(x_1 - x_0\right) + 2a_0a_1\left[\frac{\tanh\left(\kappa x\right)}{\kappa}\right]_{x_0}^{x_1} + a_1^2\int_{x_0}^{x_1} sech^4\left(\kappa x\right)\, dx\right)$$

$$\mathcal{H}_2(t) = g\left(a_0^2\left(x_1 - x_0\right) + 2a_0a_1\left[\frac{\tanh\left(\kappa x\right)}{\kappa}\right]_{x_0}^{x_1} + a_1^2\left[\frac{\tanh\left(\kappa x\right)\left(sech^2\left(\kappa x\right) + 2\right)}{3\kappa}\right]_{x_0}^{x_1}\right)$$

Momentum

$$\mathcal{H}_1(t) = \int_{x_0}^{x_1} hc^2\left(1 - \frac{a_0}{h}\right)^2\, dx$$

$$\mathcal{H}_1(t) = \int_{x_0}^{x_1} hc^2\left(1 - \frac{2a_0}{h} + \frac{a_0^2}{h^2}\right)\, dx$$

$$\mathcal{H}_1(t) = \int_{x_0}^{x_1} c^2\left(h - 2a_0 + \frac{a_0^2}{h}\right)\, dx$$

$$\mathcal{H}_1(t) = \int_{x_0}^{x_1} c^2\left(a_0 + a_1 sech^2\left(\kappa x\right) - 2a_0 + \frac{a_0^2}{a_0 + a_1 sech^2\left(\kappa x\right)}\right)\, dx$$

$$\mathcal{H}_1(t) = \int_{x_0}^{x_1} c^2\left(a_1 sech^2\left(\kappa x\right) - a_0 + \frac{a_0^2}{a_0 + a_1 sech^2\left(\kappa x\right)}\right)\, dx$$

$$\mathcal{H}_1(t) = c^2\left(a_1\left[\frac{\tanh\left(\kappa x\right)}{\kappa}\right]_{x_0}^{x_1} - a_0(x_1 - x_0) + a_0^2\int_{x_0}^{x_1}\frac{1}{a_0 + a_1 sech^2\left(\kappa x\right)}\right)\, dx$$

Momentum Vertical

$$\mathcal{H}_3(t) = \int_{-\infty}^{\infty}\frac{h^3}{3}\left(\frac{\partial u}{\partial x}\right)^2\, dx$$