

1 All Definitions for Numerical Versions

$$\mathcal{C}_2 = \frac{2 \cos(k\Delta x) - 2}{\Delta x^2}$$

$$\mathcal{C}_4 = \frac{-2 \cos(2k\Delta x) + 32 \cos(k\Delta x) - 30}{12\Delta x^2}$$

$$\mathcal{G} = \left[H - \frac{H^3}{3} \mathcal{C} \right]$$

$$\mathcal{M}_3 = \frac{24}{26 - 2 \cos(k\Delta x)}$$

$$\mathcal{M}_1 = \mathcal{M}_2 = 1$$

$$\mathcal{R}_1^+ = e^{ik\Delta x} \quad , \quad \mathcal{R}_1^- = 1$$

$$\mathcal{R}_2^- = 1 + \frac{i \sin(k\Delta x)}{2}$$

$$\mathcal{R}_2^+ = e^{ik\Delta x} \left(1 - \frac{i \sin(k\Delta x)}{2} \right)$$

$$\mathcal{R}_3^- = \frac{\mathcal{M}_3}{6} [5 + -e^{-ik\Delta x} + 2e^{ik\Delta x}]$$

$$\mathcal{R}_3^+ = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}]$$

$$\mathcal{R}_2^u = \frac{e^{ik\Delta x} + 1}{2}$$

$$\mathcal{R}_3^u = \frac{-3e^{2ik\Delta x} + 27e^{ik\Delta x} + 27 - 3e^{-ik\Delta x}}{48}$$

$$\mathcal{F}^{h,u} = H\mathcal{R}^u$$

$$\mathcal{F}^{h,h} = -\frac{\sqrt{gH}}{2} [\mathcal{R}^+ - \mathcal{R}^-]$$

$$\mathcal{F}^{u,u} = -\frac{\sqrt{gH}}{2} \mathcal{G} [\mathcal{R}^+ - \mathcal{R}^-]$$

$$\mathcal{F}^{u,h} = \frac{gH}{2} (\mathcal{R}^+ + \mathcal{R}^-)$$

$$\mathcal{D} = 1 - e^{-ik\Delta x}$$

2 Taylor Expansions Of Analytic Values

We denote exact/analytic version with a subscript a

$$\mathcal{G}_a = H + \frac{H^3}{3} k^2$$

$$\mathcal{M}_a = \frac{2}{k\Delta x} \sin\left(\frac{k\Delta x}{2}\right)$$

$$\mathcal{M}_a = 1 - \frac{k^2}{24}(\Delta x)^2 + \frac{k^4}{1920}(\Delta x)^4 - \frac{k^6}{322560}(\Delta x)^6 + O(x^8)$$

$$\mathcal{R}_a = e^{i\frac{k\Delta x}{2}}$$

$$\mathcal{R}_a^+ = \mathcal{R}_a^- = \mathcal{R}_a = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{8}\Delta x^2 - \frac{ik^3}{48}\Delta x^3 + \frac{k^4}{384}\Delta x^4 + \frac{ik^5}{3840}\Delta x^5 + O(x^6)$$

For the fluxes I think its best to report the elements of our matrix \mathbf{F} , which now encapsulates all space approximations and is the update matrix for our system

$$\frac{\mathcal{D}_a}{\Delta x \mathcal{M}_a} \mathcal{F}_a^{h,u} = ikH$$

$$\frac{\mathcal{D}_a}{\Delta x \mathcal{M}_a} \mathcal{F}_a^{h,h} = 0$$

$$\frac{\mathcal{D}_a}{\mathcal{G}_a \Delta x \mathcal{M}_a} \mathcal{F}_a^{u,h} = \frac{ikgH}{H + \frac{H^3}{3}k^2} = igk \frac{3}{3 + H^2k^2} = i \frac{\omega^2}{kH}$$

$$\frac{\mathcal{D}_a}{\mathcal{G}_a \Delta x \mathcal{M}_a} \mathcal{F}_a^{u,u} = 0$$

3 First Order Values

$$\mathcal{G}_1 = H - \frac{H^3}{3} \left(\frac{2 \cos(k\Delta x) - 2}{\Delta x^2} \right)$$

$$\mathcal{G}_1 = H + \frac{H^3 k^2}{3} - \frac{H^3 k^4}{36} (\Delta x)^2 + \frac{H^3 k^6}{1080} (\Delta x)^4 + O(x^6)$$

$$\mathcal{M}_1 = 1$$

$$\mathcal{R}_1^- = 1$$

$$\mathcal{R}_1^+ = e^{ik\Delta x}$$

$$\mathcal{R}_1^+ = 1 + ik\Delta x - \frac{k^2}{2} (\Delta x)^2 - \frac{ik^3}{6} (\Delta x)^3 + \frac{k^4}{24} (\Delta x)^4 + O(\Delta x^5)$$

$$\mathcal{R}_1^u = \frac{e^{ik\Delta x} + 1}{2}$$

$$\mathcal{R}_1^u = 1 + \frac{ik}{2} \Delta x - \frac{k^2}{4} (\Delta x)^2 - \frac{ik^3}{12} (\Delta x)^3 + \frac{k^4}{48} (\Delta x)^4 + O(\Delta x^5)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1} \mathcal{F}_1^{h,u} = \frac{1 - e^{-ik\Delta x}}{\Delta x} H \frac{e^{ik\Delta x} + 1}{2}$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1} \mathcal{F}_1^{h,u} = iHk - \frac{iHk^3}{6} (\Delta x)^2 + \frac{iHk^5}{120} (\Delta x)^4 + O(\Delta x^6)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1} \mathcal{F}_1^{h,h} = -\frac{\sqrt{gH}}{2} \frac{1 - e^{-ik\Delta x}}{\Delta x} [e^{ik\Delta x} - 1]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1} \mathcal{F}_1^{h,h} = \frac{k^2 \sqrt{gH}}{2} \Delta x - \frac{k^4 \sqrt{gH}}{24} \Delta x^3 + \frac{k^6 \sqrt{gH}}{720} \Delta x^5 + O(\Delta x^7)$$

$$\frac{\mathcal{D}}{\mathcal{G}_1 \Delta x \mathcal{M}_1} \mathcal{F}_1^{u,h} = \frac{gH}{2} \frac{1 - e^{-ik\Delta x}}{\Delta x} (1 + e^{ik\Delta x}) \left[H - \frac{H^3}{3} \left(\frac{2 \cos(k\Delta x) - 2}{\Delta x^2} \right) \right]^{-1}$$

$$\frac{\mathcal{D}}{\mathcal{G}_1 \Delta x \mathcal{M}_1} \mathcal{F}_1^{u,h} = igk \frac{3}{3 + H^3 k^3} - \frac{igk^3 (H^2 k^2 + 6)}{4 (3 + H^2 k^2)^2} \Delta x^2 - \frac{igk^5 (H^4 k^4 - 54)}{240 (3 + H^2 k^2)^3} \Delta x^4 + O(\Delta x^6)$$

$$\frac{\mathcal{D}}{\mathcal{G}_1 \Delta x \mathcal{M}_1} \mathcal{F}_1^{u,u} = -\frac{\sqrt{gH}}{2} \frac{1 - e^{-ik\Delta x}}{\Delta x} [e^{ik\Delta x} - 1]$$

$$\frac{\mathcal{D}}{\mathcal{G}_1 \Delta x \mathcal{M}_1} \mathcal{F}_1^{u,u} = \frac{k^2 \sqrt{gH}}{2} \Delta x - \frac{k^4 \sqrt{gH}}{24} \Delta x^3 + \frac{k^6 \sqrt{gH}}{720} \Delta x^5 + O(\Delta x^7)$$

4 Second Order Values

$$\mathcal{G}_2 = H - \frac{H^3}{3} \left(\frac{2 \cos(k\Delta x) - 2}{\Delta x^2} \right)$$

$$\mathcal{G}_2 = H + \frac{H^3 k^2}{3} - \frac{H^3 k^4}{36} (\Delta x)^2 + \frac{H^3 k^6}{1080} (\Delta x)^4 + O(x^6)$$

$$\mathcal{M}_2 = 1$$

$$\mathcal{R}_2^- = 1 + \frac{i \sin(k\Delta x)}{2}$$

$$\mathcal{R}_2^- = 1 + \frac{ik}{2} (\Delta x) - \frac{ik^3}{12} (\Delta x)^3 + \frac{ik^5}{240} (\Delta x)^5 + O(x^7)$$

$$\mathcal{R}_2^+ = e^{ik\Delta x} \left(1 - \frac{i \sin(k\Delta x)}{2} \right)$$

$$\mathcal{R}_2^+ = 1 + \frac{ik}{2} \Delta x + \frac{ik^3}{6} \Delta x^3 - \frac{k^4}{8} \Delta x^4 - \frac{7ik^5}{120} \Delta x^5 + O(\Delta x^6)$$

$$\mathcal{R}_2^u = \frac{e^{ik\Delta x} + 1}{2}$$

$$\mathcal{R}_2^u = 1 + \frac{ik}{2} \Delta x - \frac{k^2}{4} (\Delta x)^2 - \frac{ik^3}{12} (\Delta x)^3 + \frac{k^4}{48} (\Delta x)^4 + O(\Delta x^5)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_2} \mathcal{F}_2^{h,u} = \frac{1 - e^{-ik\Delta x}}{\Delta x} H \frac{e^{ik\Delta x} + 1}{2}$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_2} \mathcal{F}_2^{h,u} = iHk - \frac{iHk^3}{6} (\Delta x)^2 + \frac{iHk^5}{120} (\Delta x)^4 + O(\Delta x^6)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_2} \mathcal{F}_2^{h,h} = -\frac{\sqrt{gH}}{2} \frac{1 - e^{-ik\Delta x}}{\Delta x} \left[e^{ik\Delta x} \left(1 - \frac{i \sin(k\Delta x)}{2} \right) - \left(1 + \frac{i \sin(k\Delta x)}{2} \right) \right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_2} \mathcal{F}_2^{h,h} = \frac{k^4 \sqrt{gH}}{8} (\Delta x)^3 - \frac{k^6 \sqrt{gH}}{48} (\Delta x)^5 + O(\Delta x^7)$$

$$\frac{\mathcal{D}}{\mathcal{G}_2 \Delta x \mathcal{M}_2} \mathcal{F}_2^{u,u} = -\frac{\sqrt{gH}}{2} \frac{1 - e^{-ik\Delta x}}{\Delta x} \left[e^{ik\Delta x} \left(1 - \frac{i \sin(k\Delta x)}{2} \right) - \left(1 + \frac{i \sin(k\Delta x)}{2} \right) \right]$$

$$\frac{\mathcal{D}}{\mathcal{G}_2 \Delta x \mathcal{M}_2} \mathcal{F}_2^{u,u} = \frac{k^4 \sqrt{gH}}{8} (\Delta x)^3 - \frac{k^6 \sqrt{gH}}{48} (\Delta x)^5 + O(\Delta x^7)$$

$$\begin{aligned} \frac{\mathcal{D}}{\mathcal{G}_2 \Delta x \mathcal{M}_2} \mathcal{F}_2^{u,h} &= \frac{1 - e^{-ik\Delta x}}{\Delta x} \frac{gH}{2} \left[e^{ik\Delta x} \left(1 - \frac{i \sin(k\Delta x)}{2} \right) + \left(1 + \frac{i \sin(k\Delta x)}{2} \right) \right] \\ &\quad \times \left[H - \frac{H^3}{3} \left(\frac{2 \cos(k\Delta x) - 2}{\Delta x^2} \right) \right]^{-1} \end{aligned}$$

$$\begin{aligned} \frac{\mathcal{D}}{\mathcal{G}_2 \Delta x \mathcal{M}_2} \mathcal{F}_2^{u,h} &= \\ &= igk \frac{3}{3 + H^3 k^3} + \frac{igk^3 (2H^2 k^2 + 3)}{4(3 + H^2 k^2)^2} \Delta x^2 - \frac{igk^5 (31H^4 k^4 + 225H^2 k^2 + 351)}{240(3 + H^2 k^2)^3} \Delta x^4 + O(\Delta x^6) \end{aligned}$$

5 Taylor Expansions Of Third Order Values

$$\mathcal{G}_3 = H - \frac{H^3}{3} \frac{2 \cos(2k\Delta x) + 32 \cos(k\Delta x) - 30}{12\Delta x^2}$$

$$\mathcal{G}_3 = H + \frac{k^2 H^3}{3} - \frac{k^6 H^3}{270} (\Delta x)^4 + O(\Delta x^6)$$

$$\mathcal{M}_3 = \frac{24}{26 - 2 \cos(k\Delta x)}$$

$$\mathcal{M}_3 = 1 - \frac{k^2}{24} (\Delta x)^2 + \frac{k^4}{192} (\Delta x)^4 + O(\Delta x^6)$$

$$\mathcal{R}_3^- = \frac{\mathcal{M}_3}{6} [5 - e^{-ik\Delta x} + 2e^{ik\Delta x}]$$

$$R_3^- = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{8}(\Delta x)^2 - \frac{5ik^3}{48}(\Delta x)^3 + \frac{k^4}{64}(\Delta x)^4 + O(\Delta x^5)$$

$$\mathcal{R}_3^+ = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}]$$

$$R_3^+ = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{8}(\Delta x)^2 + \frac{ik^3}{16}(\Delta x)^3 - \frac{13k^4}{192}(\Delta x)^4 + O(\Delta x^5)$$

$$\mathcal{R}_3^u = \frac{-3e^{2ik\Delta x} + 27e^{ik\Delta x} + 27 - 3e^{-ik\Delta x}}{48}$$

$$R_3^u = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{8}(\Delta x)^2 - \frac{ik^3}{48}(\Delta x)^3 - \frac{k^4}{48}(\Delta x)^4 + O(\Delta x^5)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}^{h,u} = \frac{1 - e^{-ik\Delta x}}{\Delta x} \frac{26 - 2\cos(k\Delta x)}{24} H \mathcal{R}^u$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}_3^{h,u} = ikH - \frac{9ik^5 H}{320} \Delta x^4 - \frac{ik^7 H}{448} \Delta x^6 + O(\Delta x^9)$$

$$\begin{aligned} \frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}^{h,h} &= -\frac{1 - e^{-ik\Delta x}}{\Delta x \mathcal{M}_3} \frac{\sqrt{gH}}{2} \\ &\times \left[\left(\frac{\mathcal{M}_3 e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}] \right) - \left(\frac{\mathcal{M}_3}{6} [5 - e^{-ik\Delta x} + 2e^{ik\Delta x}] \right) \right] \end{aligned}$$

$$\begin{aligned} \frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}^{h,h} &= -\frac{1 - e^{-ik\Delta x}}{\Delta x} \frac{\sqrt{gH}}{2} \\ &\times \left[\left(\frac{e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}] \right) - \left(\frac{1}{6} [5 - e^{-ik\Delta x} + 2e^{ik\Delta x}] \right) \right] \end{aligned}$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}_3^{h,h} = \frac{k^4 \sqrt{gH}}{12} \Delta x^3 - \frac{k^6 \sqrt{gH}}{72} \Delta x^5 + \frac{k^8 \sqrt{gH}}{960} \Delta x^7 + O(\Delta x^9)$$

$$\begin{aligned} \frac{\mathcal{D}}{\mathcal{G}_3 \Delta x \mathcal{M}_3} \mathcal{F}^{u,u} &= -\frac{1 - e^{-ik\Delta x}}{\Delta x \mathcal{M}_3} \frac{\sqrt{gH}}{2} \\ &\times \left[\left(\frac{\mathcal{M}_3 e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}] \right) - \left(\frac{\mathcal{M}_3}{6} [5 - e^{-ik\Delta x} + 2e^{ik\Delta x}] \right) \right] \end{aligned}$$

$$\begin{aligned} \frac{\mathcal{D}}{\mathcal{G}_3 \Delta x \mathcal{M}_3} \mathcal{F}^{u,u} &= -\frac{1 - e^{-ik\Delta x}}{\Delta x} \frac{\sqrt{gH}}{2} \\ &\times \left[\left(\frac{e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}] \right) - \left(\frac{1}{6} [5 - e^{-ik\Delta x} + 2e^{ik\Delta x}] \right) \right] \end{aligned}$$

$$\frac{\mathcal{D}}{\mathcal{G}_3 \Delta x \mathcal{M}_3} \mathcal{F}^{u,u} = \frac{k^4 \sqrt{gH}}{12} \Delta x^3 - \frac{k^6 \sqrt{gH}}{72} \Delta x^5 + \frac{k^8 \sqrt{gH}}{960} \Delta x^7 + O(\Delta x^9)$$

$$\begin{aligned} \frac{\mathcal{D}}{\mathcal{G}_3 \Delta x \mathcal{M}_3} \mathcal{F}^{u,h} &= \frac{gH}{2} \frac{1 - e^{-ik\Delta x}}{\Delta x \mathcal{M}_3} \left[H - \frac{H^3}{3} \frac{-2 \cos(2k\Delta x) + 32 \cos(k\Delta x) - 30}{12\Delta x^2} \right]^{-1} \\ &\times \left[\left(\frac{e^{ik\Delta x} \mathcal{M}_3}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}] \right) + \left(\frac{\mathcal{M}_3}{6} [5 - e^{-ik\Delta x} + 2e^{ik\Delta x}] \right) \right] \end{aligned}$$

$$\begin{aligned} \frac{\mathcal{D}}{\mathcal{G}_3 \Delta x \mathcal{M}_3} \mathcal{F}^{u,h} &= \frac{gH}{2} \frac{1 - e^{-ik\Delta x}}{\Delta x} \left[H - \frac{H^3}{3} \frac{-2 \cos(2k\Delta x) + 32 \cos(k\Delta x) - 30}{12\Delta x^2} \right]^{-1} \\ &\times \left[\left(\frac{e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}] \right) + \left(\frac{1}{6} [5 - e^{-ik\Delta x} + 2e^{ik\Delta x}] \right) \right] \end{aligned}$$

$$\frac{\mathcal{D}}{\mathcal{G}_3 \Delta x \mathcal{M}_3} \mathcal{F}^{u,h} = igk \frac{3}{3 + H^2 k^2} - \frac{igk^5 (2H^2 k^2 + 9)}{30 (H^2 k^2 + 3)^2} \Delta x^4 + \frac{igk^7 (H^2 k^2 + 4)}{112 (H^2 k^2 + 3)^2} \Delta x^6 + O(\Delta x^8)$$