1 Serre Equations

The Serre Equations read (height/mass)

$$\frac{\partial h}{\partial t} + \frac{\partial uh}{\partial x} = 0$$

Phi

$$\Phi = \frac{\partial b}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2} + \frac{\partial b}{\partial x}\frac{\partial u}{\partial t}$$

Gamma

$$\Gamma = \left(\frac{\partial u}{\partial x}\right)^2 - u\left(\frac{\partial^2 u}{\partial x^2}\right) - \left(\frac{\partial^2 u}{\partial x \partial t}\right)$$

Pressure

$$p|_\xi = p_a + \rho g\xi + \frac{\rho}{2}\xi\left(2h - \xi\right)\Gamma + \rho\xi\Phi$$

Momentum(velocity) x

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^3}{3}\Gamma + \frac{h^2}{2}\Phi\right) + h\frac{\partial b}{\partial x}\left(g + \frac{h}{2}\Gamma + \Phi\right) = 0$$

z

$$w|_z = \frac{z-b}{h}\frac{\partial h}{\partial t} + u\frac{\partial b}{\partial x}$$

So

For the conservation of mass

$$\frac{\partial h}{\partial t} + \frac{\partial uh}{\partial x} = 0$$

$$h_t + uh_x + hu_x = 0$$

For the conservation ofg momentum

$$u_t h + uh_t + \frac{\partial}{\partial x}\left[u^2 h + \frac{gh^2}{2} + \frac{h^3}{3}\left(\left(\frac{\partial u}{\partial x}\right)^2 - u\left(\frac{\partial^2 u}{\partial x^2}\right) - \left(\frac{\partial^2 u}{\partial x \partial t}\right)\right) + \frac{h^2}{2}\left(\frac{\partial b}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2} + \frac{\partial b}{\partial x}\frac{\partial u}{\partial t}\right)\right]$$
$$+h\frac{\partial b}{\partial x}\left(g + \frac{h}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 - u\left(\frac{\partial^2 u}{\partial x^2}\right) - \left(\frac{\partial^2 u}{\partial x \partial t}\right)\right) + \left(\frac{\partial b}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2} + \frac{\partial b}{\partial x}\frac{\partial u}{\partial t}\right)\right) = 0$$

$$
\begin{aligned}
&u_t h + u h_t + 2 u u_x h + u^2 h_x + g h h_x \\
&\qquad + \frac{\partial}{\partial x}\left[\frac{h^3}{3}\left(u_x^2 - u u_{xx} - u_{xt}\right) + \frac{h^2}{2}\left(u u_x b_x + u^2 b_{xx} + b_x u_t\right) \right] \\
&\qquad + h \frac{\partial b}{\partial x}\left(g + \frac{h}{2}\left(u_x^2 - u u_{xx} - u_{xt}\right) + \left(u u_x b_x + u^2 b_{xx} + b_x u_t\right) \right) = 0
\end{aligned}
$$

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\begin{aligned}
&u_t h + u h_t + 2 u u_x h + u^2 h_x + g h h_x \\
&\qquad + \left[h^2 h_x \left(u_x^2 - u u_{xx} - u_{xt}\right) + h h_x \left(u u_x b_x + u^2 b_{xx} + b_x u_t\right) \right] \\
&\qquad + \left[\frac{h^3}{3}\left(u_x^2 - u u_{xx} - u_{xt}\right)_x + \frac{h^2}{2}\left(u u_x b_x + u^2 b_{xx} + b_x u_t\right)_x \right] \\
&\qquad + h \frac{\partial b}{\partial x}\left(g + \frac{h}{2}\left(u_x^2 - u u_{xx} - u_{xt}\right) + \left(u u_x b_x + u^2 b_{xx} + b_x u_t\right) \right) = 0
\end{aligned}
$$

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\begin{aligned}
&u_t h + u h_t + 2 u u_x h + u^2 h_x + g h h_x \\
&\qquad + h^2 h_x \left(u_x^2 - u u_{xx} - u_{xt}\right) + h h_x \left(u u_x b_x + u^2 b_{xx} + b_x u_t\right) \\
&\qquad\qquad + \left[\frac{h^3}{3}\left(2 u_x u_{xx} - u_x u_{xx} - u u_{xxx} - u_{xtx}\right) \right. \\
&\qquad \left. + \frac{h^2}{2}\left(u_x^2 b_x + u u_{xx} b_x + u u_x b_{xx} + 2 u u_x b_{xx} + u^2 b_{xxx} + b_{xx} u_t + b_x u_{tx}\right) \right] \\
&\qquad + h b_x \left(g + \frac{h}{2}\left(u_x^2 - u u_{xx} - u_{xt}\right) + \left(u u_x b_x + u^2 b_{xx} + b_x u_t\right) \right) = 0
\end{aligned}
$$

$$
\begin{aligned}
&u_t h + u h_t + 2 u u_x h + u^2 h_x + g h h_x \\
&\qquad + h^2 h_x \left(u_x^2 - u u_{xx} - u_{xt} \right) + h h_x \left(u u_x b_x + u^2 b_{xx} + b_x u_t \right) \\
&\qquad\qquad + \frac{h^3}{3} \left(u_x u_{xx} - u u_{xxx} - u_{xtx} \right) \\
&\qquad + \frac{h^2}{2} \left(u_x^2 b_x + u u_{xx} b_x + 3 u u_x b_{xx} + u^2 b_{xxx} + b_{xx} u_t + b_x u_{tx} \right) \\
&\qquad + h b_x \left(g + \frac{h}{2} \left(u_x^2 - u u_{xx} - u_{xt} \right) + \left(u u_x b_x + u^2 b_{xx} + b_x u_t \right) \right) = 0
\end{aligned}
$$

Since $h_t = -u h_x - h u_x$

$$
\begin{aligned}
&u_t h - u^2 h_x - u h u_x + 2 u u_x h + u^2 h_x + g h h_x \\
&\qquad + h^2 h_x \left(u_x^2 - u u_{xx} - u_{xt} \right) + h h_x \left(u u_x b_x + u^2 b_{xx} + b_x u_t \right) \\
&\qquad\qquad + \frac{h^3}{3} \left(u_x u_{xx} - u u_{xxx} - u_{xtx} \right) \\
&\qquad + \frac{h^2}{2} \left(u_x^2 b_x + u u_{xx} b_x + 3 u u_x b_{xx} + u^2 b_{xxx} + b_{xx} u_t + b_x u_{tx} \right) \\
&\qquad + h b_x \left(g + \frac{h}{2} \left(u_x^2 - u u_{xx} - u_{xt} \right) + \left(u u_x b_x + u^2 b_{xx} + b_x u_t \right) \right) = 0
\end{aligned}
$$

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\begin{aligned}
&u_t h + u u_x h + g h h_x \\
&\quad + h^2 h_x u_x^2 - h^2 h_x u u_{xx} - h^2 h_x u_{xt} + h h_x u u_x b_x + h h_x u^2 b_{xx} + h h_x b_x u_t \\
&\qquad\qquad + \frac{h^3}{3} u_x u_{xx} - \frac{h^3}{3} u u_{xxx} - \frac{h^3}{3} u_{xtx} \\
&\quad + \frac{h^2}{2} u_x^2 b_x + \frac{h^2}{2} u u_{xx} b_x + 3 \frac{h^2}{2} u u_x b_{xx} + \frac{h^2}{2} u^2 b_{xxx} + \frac{h^2}{2} b_{xx} u_t + \frac{h^2}{2} b_x u_{tx} \\
&+ g h b_x + \frac{h}{2} h b_x u_x^2 - h \frac{h}{2} b_x u u_{xx} - \frac{h}{2} h b_x u_{xt} + h b_x u u_x b_x + u^2 h b_x b_{xx} + h b_x b_x u_t = 0
\end{aligned}
$$

$$
\begin{aligned}
&u_t h + u u_x h + g h h_x \\
&\quad + h^2 h_x u_x^2 - h^2 h_x u u_{xx} - h^2 h_x u_{xt} + u h h_x u_x b_x + u^2 h h_x b_{xx} + h h_x b_x u_t \\
&\qquad + \frac{h^3}{3} u_x u_{xx} - \frac{h^3}{3} u u_{xxx} - \frac{h^3}{3} u_{xtx} \\
&\quad + \frac{h^2}{2} u_x^2 b_x + \frac{h^2}{2} u u_{xx} b_x + \frac{3h^2}{2} u u_x b_{xx} + \frac{h^2}{2} u^2 b_{xxx} + \frac{h^2}{2} b_{xx} u_t + \frac{h^2}{2} b_x u_{tx} \\
&+ g h b_x + \frac{h^2}{2} b_x u_x^2 - u \frac{h^2}{2} b_x u_{xx} - \frac{h^2}{2} b_x u_{xt} + u h u_x b_x^2 + u^2 h b_x b_{xx} + h b_x^2 u_t = 0
\end{aligned}
$$

$$
\begin{aligned}
&u_t h + u u_x h + g h h_x \\
&\quad + h^2 h_x u_x^2 - h^2 h_x u u_{xx} - h^2 h_x u_{xt} + u h h_x u_x b_x + u^2 h h_x b_{xx} + h h_x b_x u_t \\
&\qquad + \frac{h^3}{3} u_x u_{xx} - \frac{h^3}{3} u u_{xxx} - \frac{h^3}{3} u_{xtx} \\
&\quad + h^2 u_x^2 b_x + \frac{3h^2}{2} u u_x b_{xx} + \frac{h^2}{2} u^2 b_{xxx} + \frac{h^2}{2} b_{xx} u_t \\
&\qquad + g h b_x + u h u_x b_x^2 + u^2 h b_x b_{xx} + h b_x^2 u_t = 0
\end{aligned}
$$

$$
\begin{aligned}
&u_t h + u u_x h + g h h_x + h^2 h_x u_x^2 - h^2 h_x u u_{xx} - h^2 h_x u_{xt} + u h h_x u_x b_x \\
&\quad + u^2 h h_x b_{xx} + h h_x b_x u_t + \frac{h^3}{3} u_x u_{xx} - \frac{h^3}{3} u u_{xxx} - \frac{h^3}{3} u_{xtx} \\
&\quad + h^2 u_x^2 b_x + \frac{3h^2}{2} u u_x b_{xx} + \frac{h^2}{2} u^2 b_{xxx} + \frac{h^2}{2} b_{xx} u_t + g h b_x \\
&\qquad + u h u_x b_x^2 + u^2 h b_x b_{xx} + h b_x^2 u_t = 0
\end{aligned}
$$

Divide through by h

$$
\begin{aligned}
&u_t + u u_x + g h_x + h h_x u_x^2 - h h_x u u_{xx} - h h_x u_{xt} + u h_x u_x b_x \\
&\quad + u^2 h_x b_{xx} + h_x b_x u_t + \frac{h^2}{3} u_x u_{xx} - \frac{h^2}{3} u u_{xxx} - \frac{h^2}{3} u_{xtx} \\
&\quad + h u_x^2 b_x + \frac{3h}{2} u u_x b_{xx} + \frac{h}{2} u^2 b_{xxx} + \frac{h}{2} b_{xx} u_t + g b_x \\
&\qquad + u u_x b_x^2 + u^2 b_x b_{xx} + b_x^2 u_t = 0
\end{aligned}
$$

$$u_t\left(1+h_xb_x+\frac{h}{2}b_{xx}+b_x^2u_t\right)-hh_xu_{xt}-\frac{h^2}{3}u_{xtx}+uu_x\left(1+h_xb_x+\frac{3h}{2}b_{xx}+b_x^2\right)+g\left(h_x+b_x\right)$$
$$+hh_xu_x^2-hh_xuu_{xx}+u^2h_xb_{xx}+\frac{h^2}{3}u_xu_{xx}-\frac{h^2}{3}uu_{xxx}+hu_x^2b_x+\frac{h}{2}u^2b_{xxx}+u^2b_xb_{xx}=0$$