1 Serre Equations

The Serre Equations read (height/mass)

$$\frac{\partial h}{\partial t} + \frac{\partial uh}{\partial x} = 0$$

Phi

$$\Phi = \frac{\partial b}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2} + \frac{\partial b}{\partial x}\frac{\partial u}{\partial t}$$

Gamma

$$\Gamma = \left(\frac{\partial u}{\partial x}\right)^2 - u\left(\frac{\partial^2 u}{\partial x^2}\right) - \left(\frac{\partial^2 u}{\partial x \partial t}\right)$$

Pressure

$$p|_{\xi}= p_a + \rho g \xi + \frac{\rho}{2}\xi\left(2h - \xi\right)\Gamma + \rho\xi\Phi$$

Momentum(velocity) x

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^3}{3}\Gamma + \frac{h^2}{2}\Phi\right) + h\frac{\partial b}{\partial x}\left(g + \frac{h}{2}\Gamma + \Phi\right) = 0$$

z

$$w|_z= \frac{z-b}{h}\frac{\partial h}{\partial t} + u\frac{\partial b}{\partial x}$$

The total energy of the Euler system is $E = \frac{1}{2}|\vec{u}|^2 + g(z-b)$:

$$\frac{\partial E}{\partial t} + \nabla\cdot\left(E\vec{u} + p\vec{u}\right) = 0$$

Integrating over depth $s = h + b$

$$\int_b^s E\,dz = \int_b^s \frac{1}{2}\left(u^2 + w^2\right) + g(z-b)\,dz$$

$$\int_b^s \frac{1}{2}\left(u^2 + w^2\right) + gz\,dz = \frac{1}{2}\left(\bar{u}^2 h + \int_b^s w^2\,dz\right) + \int_b^s g(z-b)\,dz$$

Calculating the P.E first (simplest)

$$\int_b^s g(z-b)\,dz = g\left[\frac{1}{2}z^2\right]_b^s - gb\left[z\right]_b^s = g\frac{h^2 + 2hb + b^2 - b^2}{2} - gb(h) = \frac{g}{2}\left(h^2\right)$$

Calculating the vertical velocity

$$\int_b^s w^2\, dz = \int_b^s \left(\frac{z-b}{h}\frac{\partial h}{\partial t} + u\frac{\partial b}{\partial x} \right)^2 \, dz$$

$$= \int_b^s \left(\frac{z-b}{h}\frac{\partial h}{\partial t} \right)^2 + 2\left(\frac{z-b}{h}\frac{\partial h}{\partial t} \right)\left(u\frac{\partial b}{\partial x} \right) + \left(u\frac{\partial b}{\partial x} \right)^2 \, dz$$

$$= \int_b^s \left(\frac{z-b}{h}\frac{\partial h}{\partial t} \right)^2 \, dz + \int_b^s 2\left(\frac{z-b}{h}\frac{\partial h}{\partial t} \right)\left(u\frac{\partial b}{\partial x} \right) \, dz + h\left(\bar{u}\frac{\partial b}{\partial x} \right)^2$$

$$= \left(\frac{1}{h}\frac{\partial h}{\partial t} \right)^2 \int_b^s (z-b)^2 \, dz + 2\frac{1}{h}\frac{\partial h}{\partial t}\frac{\partial b}{\partial x}\int_b^s (z-b)\,(u) \, dz + h\left(\bar{u}\frac{\partial b}{\partial x} \right)^2$$

Can use that the velocity profile is constant over depth $(u|_x, z = \bar{u}(x))$

$$= \left(\frac{1}{h}\frac{\partial h}{\partial t} \right)^2 \int_b^s (z-b)^2 \, dz + 2\frac{\bar{u}}{h}\frac{\partial h}{\partial t}\frac{\partial b}{\partial x}\int_b^s (z-b) \, dz + h\left(\bar{u}\frac{\partial b}{\partial x} \right)^2$$

Integrals:

$$\int_b^s (z-b) \, dz = \int_b^s z\, dx - \int_b^s b\, dz$$

$$= \left[\frac{z^2}{2} \right]_b^s - b\int_b^s 1\, dz$$

$$= \left[\frac{z^2}{2} \right]_b^s - b\,[s-b]$$

$$= \frac{s^2}{2} - \frac{b^2}{2} - b\,[h+b-b]$$

$$= \frac{h^2+2hb+b^2}{2} - \frac{b^2}{2} - bh$$

$$= \frac{h^2+2hb}{2} - bh$$

$$= \frac{h^2}{2}$$

Second

$$\int_b^s (z-b)^2 \ dz = \int_b^s z^2 - 2zb + b^2 \, dz$$

$$= \int_b^s z^2 \, dz - \int_b^s 2zb \, dz + \int_b^s b^2 \, dz$$

$$= \int_b^s z^2 \, dz - 2b \int_b^s z \, dz + b^2 \int_b^s 1 \, dz$$

$$= \left[\frac{z^3}{3}\right]_b^s - 2b \left[\frac{z^2}{2}\right]_b^s + b^2 h$$

$$= \left(\frac{s^3}{3} - \frac{b^3}{3}\right) - 2b \left(\frac{h^2 + 2hb}{2}\right) + b^2 h$$

$$= \left(\frac{h^3 + 3h^2 b + 3hb^2 + b^3}{3} - \frac{b^3}{3}\right) - bh^2 - 2hb^2 + b^2 h$$

$$= \left(\frac{h^3 + 3h^2 b + 3hb^2}{3}\right) - bh^2 - hb^2$$

$$= \left(\frac{h^3}{3}\right)$$

(identity that applies)
Back to it

$$= \left(\frac{1}{h}\frac{\partial h}{\partial t}\right)^2 \int_b^s (z-b)^2 \ dz + 2\frac{\bar{u}}{h}\frac{\partial h}{\partial t}\frac{\partial b}{\partial x} \int_b^s (z-b) \ dz + h \left(\bar{u}\frac{\partial b}{\partial x}\right)^2$$

$$= \left(\frac{1}{h}\frac{\partial h}{\partial t}\right)^2 \left(\frac{h^3}{3}\right) + 2\frac{\bar{u}}{h}\frac{\partial h}{\partial t}\frac{\partial b}{\partial x} \left(\frac{h^2}{2}\right) + h \left(\bar{u}\frac{\partial b}{\partial x}\right)^2$$

$$= \frac{h}{3}\left(\frac{\partial h}{\partial t}\right)^2 + \bar{u}h\frac{\partial h}{\partial t}\frac{\partial b}{\partial x} + h \left(\bar{u}\frac{\partial b}{\partial x}\right)^2$$

$$= \frac{h}{3}\left(-\frac{\partial h\bar{u}}{\partial x}\right)^2 + \bar{u}h\left(-\frac{\partial h\bar{u}}{\partial x}\right)\frac{\partial b}{\partial x} + h\bar{u}^2\left(\frac{\partial b}{\partial x}\right)^2$$

$$= \frac{h}{3}\left(\frac{\partial\left(h\bar{u}\right)}{\partial x}\right)^2 - \bar{u}h\frac{\partial\left(h\bar{u}\right)}{\partial x}\frac{\partial b}{\partial x} + h\bar{u}^2\left(\frac{\partial b}{\partial x}\right)^2$$

So

$$\int_b^s E\,dz = \frac{1}{2}\left(\bar{u}^2 h + \frac{h}{3}\left(\frac{\partial\left(h\bar{u}\right)}{\partial x}\right)^2 - \bar{u}h\frac{\partial\left(h\bar{u}\right)}{\partial x}\frac{\partial b}{\partial x} + h\bar{u}^2\left(\frac{\partial b}{\partial x}\right)^2\right) + \frac{gh^2}{2}$$

$$\int_b^s E\,dz = \frac{1}{2}\left(\bar{u}^2 h + \frac{h}{3}\left(\frac{\partial\left(h\bar{u}\right)}{\partial x}\right)^2 - \bar{u}h\frac{\partial\left(h\bar{u}\right)}{\partial x}\frac{\partial b}{\partial x} + h\bar{u}^2\left(\frac{\partial b}{\partial x}\right)^2 + gh^2\right)$$

$$\int_b^s E\,dz = \frac{1}{2}\left(\bar{u}^2 h + \frac{h}{3}\left(\frac{\partial h}{\partial x}\bar{u} + \frac{\partial \bar{u}}{\partial x}h\right)^2 - \bar{u}h\left(\frac{\partial h}{\partial x}\bar{u} + \frac{\partial \bar{u}}{\partial x}h\right)\frac{\partial b}{\partial x} + h\bar{u}^2\left(\frac{\partial b}{\partial x}\right)^2 + gh^2\right)$$

$$= \frac{1}{2}\left(\bar{u}^2 h + \frac{h}{3}\left(\left(\frac{\partial h}{\partial x}\bar{u}\right)^2 + 2\left(\frac{\partial h}{\partial x}\bar{u}\right)\left(\frac{\partial \bar{u}}{\partial x}h\right) + \left(\frac{\partial \bar{u}}{\partial x}h\right)^2\right) - \bar{u}h\left(\frac{\partial h}{\partial x}\bar{u} + \frac{\partial \bar{u}}{\partial x}h\right)\frac{\partial b}{\partial x} + h\bar{u}^2\left(\frac{\partial b}{\partial x}\right.\right.$$

$$= \frac{1}{2}\left(\bar{u}^2 h + \frac{h}{3}\left(h_x^2\bar{u}^2 + 2h_x\bar{u}h\bar{u}_x + \bar{u}_x^2h^2\right) - \bar{u}h\left(h_x\bar{u} + \bar{u}_xh\right)b_x + h\bar{u}^2b_x^2 + gh^2\right)$$

$$= \frac{1}{2}\left(\bar{u}^2 h + \frac{h}{3}\left(h_x^2\bar{u}^2 + 2\bar{u}h\bar{u}_xh_x + \bar{u}_x^2h^2\right) - \bar{u}h\left(h_x\bar{u} + \bar{u}_xh\right)b_x + h\bar{u}^2b_x^2 + gh^2\right)$$

$$= \frac{1}{2}\left(\bar{u}^2 h + \frac{h}{3}\left(h_x^2\bar{u}^2 + 2\bar{u}h\bar{u}_xh_x + \bar{u}_x^2h^2\right) - \bar{u}hb_x\left(h_x\bar{u} + \bar{u}_xh\right) + h\bar{u}^2b_x^2 + gh^2\right)$$

If $b = 0$ then

$$= \frac{1}{2}\left(\bar{u}^2 h + \frac{h}{3}\left(h_x^2\bar{u}^2 + 2\bar{u}h\bar{u}_xh_x + \bar{u}_x^2h^2\right) + gh^2\right)$$