

1 Serre Equations

The elliptic equation we want to solve in strong form is:

$$\vec{G} = h\vec{u} - \nabla \left[\frac{h^3}{3} (\nabla \cdot \vec{u}) \right]$$

First step is to take this equations dot product with a vector of smooth test functions \vec{v} .

$$\vec{G} \cdot \vec{v} = h\vec{u} \cdot \vec{v} - \nabla \left[\frac{h^3}{3} (\nabla \cdot \vec{u}) \right] \cdot \vec{v}$$

Integrate over the domain Ω

$$\int_{\Omega} \vec{G} \cdot \vec{v} dx = \int_{\Omega} h\vec{u} \cdot \vec{v} dx - \int_{\Omega} \nabla \left[\frac{h^3}{3} (\nabla \cdot \vec{u}) \right] \cdot \vec{v} dx$$

Now we can apply integration by parts to the bottom equation, we will use Dirichlet boundaries (with zero velocities), so we lose the surface integral.

$$\int_{\Omega} \vec{G} \cdot \vec{v} dx = \int_{\Omega} h\vec{u} \cdot \vec{v} dx + \int_{\Omega} \frac{h^3}{3} (\nabla \cdot \vec{u}) (\nabla \cdot \vec{v}) dx$$

So our problem becomes.

Find $\vec{u} \in V$ such that

$$\begin{aligned} \vec{u} &= 0 \quad \text{on } \partial\Omega \\ \int_{\Omega} \vec{G} \cdot \vec{v} dx &= \int_{\Omega} h\vec{u} \cdot \vec{v} dx + \int_{\Omega} \frac{h^3}{3} (\nabla \cdot \vec{u}) (\nabla \cdot \vec{v}) dx \end{aligned}$$

$\forall \vec{v}$ who are smooth \mathbb{R}^2 valued functions.

In particular this analysis gives that \vec{u} is in $H(\text{div})$ as long as \vec{G} and h are square integrable, (proof?).

2 Is this sufficient?

Our equations for conservation are:

$$h_t + (uh)_x + (vh)_y = 0 \tag{1}$$

(1) we can expand this to get

$$h_t + h(u_x + v_y) + h_x u + h_y v = 0$$

$$h_t + \nabla \cdot (h\vec{u}) = 0$$

$$F^x = \begin{bmatrix} G^x u + \frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 - v \frac{\partial}{\partial y} \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \\ uvh \end{bmatrix} \quad (2)$$

$$F^y = \begin{bmatrix} uvh \\ G^y v + \frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 - u \frac{\partial}{\partial x} \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \end{bmatrix} \quad (3)$$

Such that

$$\frac{\partial \vec{G}}{\partial t} + \nabla \cdot \vec{F} = 0$$

Expanding

$$\frac{\partial G^x}{\partial t} + \frac{\partial}{\partial x} \left(G^x u + \frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 - v \frac{\partial}{\partial y} \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \right) + \frac{\partial}{\partial y} (uvh) = 0$$

$$\frac{\partial G^y}{\partial t} + \frac{\partial}{\partial x} (uvh) + \frac{\partial}{\partial y} \left(G^y v + \frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 - u \frac{\partial}{\partial x} \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \right) = 0$$

We can group things together a little differently. For instance

$$\begin{aligned} & \left[\frac{\partial}{\partial x} \left(G^x u + \frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 - v \frac{\partial}{\partial y} \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \right) \right. \\ & \left. \frac{\partial}{\partial y} \left(G^y v + \frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 - u \frac{\partial}{\partial x} \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \right) \right] = \\ & \nabla \left(\vec{G} \cdot \vec{u} + \left(\frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\vec{u} \cdot \nabla \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \right) \right) \end{aligned} \quad (4)$$

And

$$\begin{bmatrix} \frac{\partial}{\partial y} (uvh) \\ \frac{\partial}{\partial x} (uvh) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \nabla (uvh) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \nabla \left(h\vec{u} \cdot \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix} \vec{u} \right) \quad (5)$$

So we have

$$\begin{aligned} \frac{\partial \vec{G}}{\partial t} + \nabla \left(\vec{G} \cdot \vec{u} + \left(\frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\vec{u} \cdot \nabla \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \right) \right) \\ + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \nabla (uvh) = 0 \end{aligned} \quad (6)$$

Taking the dot product of this with a test function \vec{w}