1 Cell Averaged G

We have:

$$G = uh - \frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial u}{\partial x}\right)$$

For a soliton we have:

$$h(x,t) = a_0 + a_1 sech^2\left(\kappa\left(x - ct\right)\right)$$

$$u(x,t) = c\left(1 - \frac{a_0}{h(x,t)}\right)$$

$$\kappa = \frac{\sqrt{3a_1}}{2a_0\sqrt{a_0 + a_1}}$$

$$c = \sqrt{g(a_0 + a_1)}$$

Let's focus on $t = 0$

$$h(x) = a_0 + a_1 sech^2\left(\kappa x\right)$$

$$u(x) = c\left(1 - \frac{a_0}{h(x)}\right)$$

$$\kappa = \frac{\sqrt{3a_1}}{2\sqrt{a_0\left(a_0 + a_1\right)}}$$

$$c = \sqrt{g(a_0 + a_1)}$$

So we have

$$\frac{\partial}{\partial x}u(x,t) = -ca_0\frac{\partial}{\partial x}\left(\frac{1}{h(x,t)}\right)$$

$$= ca_0\left[\frac{\partial h}{\partial x}\frac{1}{h^2}\right]$$

Multiplying by $\frac{h^3}{3}$

$$\frac{h^3}{3}\frac{\partial}{\partial x}u(x,t) = \frac{ca_0}{3}h\left[\frac{\partial h}{\partial x}\right]$$

Taking the derivative:

$$\frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial}{\partial x}u\right) = \frac{\partial}{\partial x}\left(\frac{ca_0}{3}h\frac{\partial h}{\partial x}\right)$$

$$= \frac{ca_0}{3}\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)$$

$$= \frac{ca_0}{3}\left(h\frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^2\right)$$

So we have:

$$\frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial}{\partial x}u\right) = \frac{ca_0}{3}\left(h\frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^2\right)$$

Looking at momentum we have:

$$uh = c\left(h - a_0\right)$$

So G is:

$$G = c\left(h - a_0\right) - \frac{ca_0}{3}\left(h\frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^2\right)$$

We will now calculate the derivatives of h:

$$h = a_0 + a_1 sech^2\left(\kappa x\right)$$

$$\frac{\partial h}{\partial x} = a_1\frac{\partial}{\partial x}sech^2\left(\kappa x\right)$$

$$= a_1 \times 2 \times \kappa \times -tanh(\kappa x)sech(\kappa x) \times sech(\kappa x)$$

$$\frac{\partial h}{\partial x} = -2a_1\kappa tanh(\kappa x)sech^2(\kappa x)$$

$$\frac{\partial^2 h}{\partial x^2} = -2a_1\kappa\frac{\partial}{\partial x}\left(tanh(\kappa x)sech^2(\kappa(x))\right)$$

$$\frac{\partial^2 h}{\partial x^2} = -2a_1\kappa\left(-\kappa\left(cosh(2\kappa x) - 2\right)sech^4(\kappa x)\right)$$

$$\frac{\partial^2 h}{\partial x^2} = 2a_1\kappa^2\left(\left(cosh(2\kappa x) - 2\right)sech^4(\kappa x)\right)$$

From before we had:

$$\frac{\partial^2 h}{\partial x^2} = -2a_1\kappa\frac{\partial}{\partial x}\left(tanh(\kappa x)sech^2(\kappa(x)\right)$$

$$\frac{\partial^2 h}{\partial x^2} = -2a_1\kappa\left[\frac{\partial}{\partial x}\left(tanh(\kappa x)\right)sech^2(\kappa x) + \frac{\partial}{\partial x}\left(sech^2(\kappa x)\right)tanh(\kappa x)\right]$$

$$\frac{\partial^2 h}{\partial x^2} = -2a_1\kappa^2\left[\left(1 - tanh^2(\kappa x)\right)sech^2(\kappa x) - 2\left(sech^2(\kappa x)\right)tanh^2(\kappa x)\right]$$

$$\frac{\partial^2 h}{\partial x^2} = -2a_1\kappa^2 sech^2(\kappa x)\left[(1 - 3tanh^2(\kappa x)\right]$$