Behaviour of the Dam-Break Problem for the Serre Equations

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ABSTRACT

Keywords: dispersive waves, conservation laws, Serre equation, finite volume method, finite difference method

INTRODUCTION

SERRE EQUATIONS

The Serre equations can derived as an approximation to the full Euler equations by depth integration similar to (Su and Gardner 1969). They can also be seen as an asymptotic expansion to the Euler equations as well (Lannes and Bonneton 2009). The former is more consistent with the perspective from which numerical methods will be developed while the latter indicates the appropriate regions in which to use these equations as a model for fluid flow. The set up of the scenario under which the Serre approximation is made consists of a two dimensional $\mathbf{x} = (x, z)$ fluid over a bottom topography as in Figure 1 acting under gravity. Consider a fluid particle at depth $\xi(\mathbf{x}, t) = z - h(x, t) - z_b(x)$ below the water surface, see Figure 1. Where the water depth is $h(x, t)$ and $z_b(x)$ is the bed elevation. The fluid particle is subject to the pressure, $p(\mathbf{x}, t)$ and gravitational acceleration, $\mathbf{g} = (0, g)^T$ and has a velocity $\mathbf{u} = (u(\mathbf{x}, t), w(\mathbf{x}, t))$, where $u(\mathbf{x}, t)$ is the velocity in the x-coordinate and $w(\mathbf{x}, t)$ is the velocity in the z-coordinate and t is time. Assuming that $z_b(x)$ is constant the Serre equations read (Li et al. 2014)

$$\frac{\partial h}{\partial t} + \frac{\partial (\bar{u}h)}{\partial x} = 0 \tag{1a}$$

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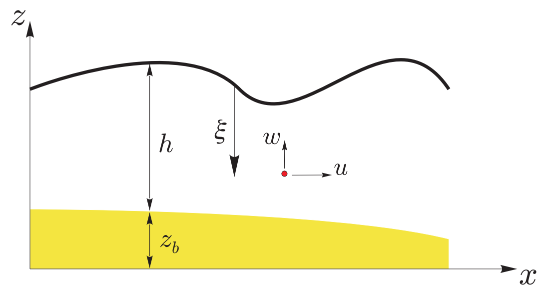

FIG. 1: The notation used for one-dimensional flow governed by the Serre equation.

$$\underbrace{\underbrace{\frac{\partial(\bar{u}h)}{\partial t} + \frac{\partial}{\partial x}\left(\bar{u}^2 h + \frac{gh^2}{2}\right)}_{\text{Shallow Water Wave Equations}} + \underbrace{\frac{\partial}{\partial x}\left(\frac{h^3}{3}\left[\frac{\partial \bar{u}}{\partial x}\frac{\partial \bar{u}}{\partial x} - \bar{u}\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{\partial^2 \bar{u}}{\partial x \partial t}\right]\right)}_{\text{Dispersion Terms}}}_{\text{Serre Equations}} = 0. \tag{1b}$$

Where $\bar{u}$ means the average of u over the depth of water.

FINITE DIFFERENCE AND LAX WENDROFF

This method was used in El et al. (2006) for the Serre equations. It consists of a lax-wendroff update for h and a spatio-temporal second order approximation to [] which results in a fully second-order method. To make this method precise it will be presented here in sufficient replicable detail.

Note that [] is in conservative law form for h where the Jacobian is u, where the bar has been dropped to simplify the notation. Thus assuming a fixed resolution discretisation for space and time which will be represented as follows $q_i^n = q(x_i, t^n)$ for some quantity q the lax-wendroff update for h obtained is

$$\begin{aligned} h_i^{n+1} &= h_i^n - \frac{\Delta t}{2\Delta x}\left((uh)_{i+1}^n - (uh)_{i-1}^n\right) \\ &+ \frac{\Delta t^2}{2\Delta x^2}\left(\frac{u_{i+1}^n - u_i^n}{2}\left((uh)_{i+1}^n - (uh)_i^n\right) - \frac{u_i^n - u_{i-1}^n}{2}\left((uh)_i^n - (uh)_{i-1}^n\right)\right) \end{aligned} \tag{2}$$

To get a second-order approximation to [] is built by first expanding all the derivatives out and making use of the continuity equation [], this results in:

$$h\frac{\partial u}{\partial t} + X - h^2\frac{\partial^2 u}{\partial x \partial t} - \frac{h^3}{3}\frac{\partial^3 u}{\partial x^2 \partial t} = 0 \tag{3a}$$

where X contains only spatial derivatives and is

$$X = uh\frac{\partial u}{\partial x} + gh\frac{\partial h}{\partial x} + h^2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{h^3}{3}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - h^2 u\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3}u\frac{\partial^3 u}{\partial x^3}. \tag{3b}$$

Then taking second-order approximations to the time derivatives for [] gives

$$h^n\frac{u^{n+1} - u^{n-1}}{2\Delta t} + X^n - (h^n)^2\,\frac{\left(\frac{\partial u}{\partial x}\right)^{n+1} - \left(\frac{\partial u}{\partial x}\right)^{n-1}}{2\Delta t} - \frac{(h^n)^3}{3}\frac{\left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1} - \left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1}}{2\Delta t} = 0 \tag{4}$$

$$h^n\left(u^{n+1} - u^{n-1}\right) + 2\Delta t X^n - (h^n)^2\left(\left(\frac{\partial u}{\partial x}\right)^{n+1} - \left(\frac{\partial u}{\partial x}\right)^{n-1}\right) - \frac{(h^n)^3}{3}\left(\left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1} - \left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1}\right) = 0 \tag{5}$$

$$h^n u^{n+1} - (h^n)^2\left(\frac{\partial u}{\partial x}\right)^{n+1} - \frac{(h^n)^3}{3}\left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1} + 2\Delta t X^n - h^n u^{n-1} + (h^n)^2\left(\frac{\partial u}{\partial x}\right)^{n-1} + \frac{(h^n)^3}{3}\left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1} = 0 \tag{6}$$

Let

$$Y^n = 2\Delta t X^n - h^n u^{n-1} + (h^n)^2\left(\frac{\partial u}{\partial x}\right)^{n-1} + \frac{(h^n)^3}{3}\left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1} \tag{7}$$

Taking second-order approximations to the spatial derivatives gives

$$h_i^n u_i^{n+1} - (h_i^n)^2\left(\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x}\right) - \frac{\left(h_i^n\right)^3}{3}\left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}\right) = -Y_i^n \tag{8}$$

This can be rearranged into a tri-diagonal matrix that updates u given its current and previous values. So that

$$\begin{bmatrix} u_0^{n+1} \\ \vdots \\ u_m^{n+1} \end{bmatrix} = A^{-1}\begin{bmatrix} -Y_0^n \\ \vdots \\ -Y_m^n \end{bmatrix} =: \mathcal{G}_u\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n-1}, \Delta t\right).$$

Where

$$A = \begin{bmatrix} b_0 & c_0 & & & & & \\ a_0 & b_1 & c_1 & & & & \\ & a_1 & b_2 & c_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & a_{m-3} & b_{m-2} & c_{m-2} & \\ & & & & a_{m-2} & b_{m-1} & c_{m-1} \\ & & & & & a_{m-1} & b_m \end{bmatrix}$$

with

$$a_{i-1} = \frac{\left(h_i^n\right)^2}{2\Delta x}\frac{h_{i+1}^n - h_{i-1}^n}{2\Delta x} - \frac{\left(h_i^n\right)^3}{3\Delta x^2}, \tag{9a}$$

$$b_i = h_i^n + \frac{2h_i^n}{3\Delta x^2} \tag{9b}$$

and

$$c_i = -\frac{\left(h_i^n\right)^2}{2\Delta x}\frac{h_{i+1}^n - h_{i-1}^n}{2\Delta x} - \frac{\left(h_i^n\right)^3}{3\Delta x^2}. \tag{9c}$$

Lastly the final expression for Y_i^n is given by:

$$Y_i^n = 2\Delta t X_i^n - h_i^n u_i^{n-1} + (h_i^n)^2\,\frac{u_{i+1}^{n-1} - u_{i-1}^{n-1}}{2\Delta x} + \frac{\left(h_i^n\right)^3}{3}\frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{\Delta x^2} \tag{10}$$

$$\begin{aligned}
Y_i^n = 2\Delta t\Bigg[& u_i^n h_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + g h_i^n \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{2\Delta x} + (h_i^n)^2 \left(\frac{u_{i+1}^{n-1} - u_{i-1}^{n-1}}{2\Delta x}\right)^2 \\
& + \frac{\left(h_i^n\right)^3}{3}\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} - (h_i^n)^2\, u_i^n \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \\
& - \frac{\left(h_i^n\right)^3}{3} u_i^n \frac{u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n}{2\Delta x^3}\Bigg] \\
& - h_i^n u_i^{n-1} + (h_i^n)^2\,\frac{u_{i+1}^{n-1} - u_{i-1}^{n-1}}{2\Delta x} + \frac{\left(h_i^n\right)^3}{3}\frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{\Delta x^2}
\end{aligned} \tag{11}$$

SECOND ORDER FINITE DIFFERENCE METHOD

Above a second order finite difference method for updating u was given, thus replacing the numerical method for h by replacing derivatives with second order finite differences will give another full finite difference method. From (1a) we expand derivatives and then approximate them by second order finite differences to give

$$\frac{h_i^{n+1} - h_i^{n-1}}{2\Delta t} + u_i^n \frac{h_{i+1}^n - h_{i-1}^n}{2\Delta x} + h_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0 \tag{12}$$

After rearranging this to give an update formula one obtains

$$h_i^{n+1} = h_i^{n-1} - \Delta t \left(u_i^n \frac{h_{i+1}^n - h_{i-1}^n}{\Delta x} + h_i^n \frac{u_{i+1}^n - u_{i-1}^n}{\Delta x} \right) \tag{13}$$

Combining this with the update formula for u [] gives a full finite difference method for the Serre equations.

A HYBRID FINITE DIFFERENCE-VOLUME METHOD FOR SERRE EQUATIONS IN CONSERVATIVE FORM

[] also offer another family of numerical methods which can be constructed by first rearranging the equations into conservative form and then using both a finite difference and a finite volume method to solve these equations. This paper will make use of the first-, second- and third-order versions of this method as set out in []. These have been validated for both smooth and discontinuous problems and their orders of accuracy have been verified for smooth solutions so they are of particular interest for the comparisons that will be investigated in this paper.

NUMERICAL SIMULATIONS

In this section the methods introduced in this paper will be validated by using them to approximate an analytic solution of the Serre equations, this will also be used to verify their order of accuracy. Then an in depth comparison of using these methods for a smooth approximation to the discontinuous dam break problem will be provided to investigate the behaviour of these equations in the presence of discontinuities. This is a problem that so far has only received a proper treatment in (El et al. 2006), with other research giving only a cursory look into the topic.

SOLITON

Currently cnoidal waves are the only family of analytic solutions to the Serre equations (Carter and Cienfuegos 2011). Solitons are a particular instance of cnoidal waves that travel without deformation and have been used to verify the convergence rates of the described methods in this paper.

For the Serre equations the solitons have the following form

$$h(x,t) = a_0 + a_1 \text{sech}^2 \left(\kappa \left(x - ct \right) \right), \tag{14a}$$

$$u(x,t) = c \left(1 - \frac{a_0}{h(x,t)} \right), \tag{14b}$$

$$\kappa = \frac{\sqrt{3a_1}}{2a_0\sqrt{a_0 + a_1}} \tag{14c}$$

and

$$c = \sqrt{g\left(a_0 + a_1\right)} \tag{14d}$$

where a_0 and a_1 are input parameters that determine the depth of the quiescent water and the maximum height of the soliton above that respectively. In the simulation $a_0 = 10$m, $a_1 = 1$m for $x \in [-500\text{m}, 1500\text{m}]$ and $t \in [0\text{s}, 100\text{s}]$. With $\Delta t = 0.01\Delta x$ which satisfies [] and $\theta = 1.2$ for the second-order finite difference-volume method.

SMOOTHED DAM-BREAK

The discontinuous dam-break problem can be approximated by a smooth function using the hyperbolic tan function []. Such an approximation will be called a smoothed dam-break problem and will be defined as such

$$h(x,0) = h_0 + \frac{h_1 - h_0}{2}\left(1 + \tanh\left(a\left(x_0 - x\right)\right)\right),$$

$$u(x,0) = 0.0m/s.$$

Where a is given and controls the width of the transition between the two dam-break heights of h_0 and h_1. For large a the width is small and vice versa. For a fixed Δx there are large enough a values such that the transition width is zero. This experiment was run for both of the methods described in this paper and the 3 different order finite difference-volume methods described in []. In this particular simulation $h_0 = 1.0m$, $h_1 = 1.8m$ on $x \in [0m, 1000m]$ for $t \in [0s, 30s]$ with $x_0 = 500m$. The simulations were run changing both Δx and a and for stability $\Delta t = 0.01\Delta x$ while for the second order finite volume method $\theta = 1.2$.

Changing a

Changing dx

Comparison of Models

CONCLUSIONS

ACKNOWLEDGEMENTS

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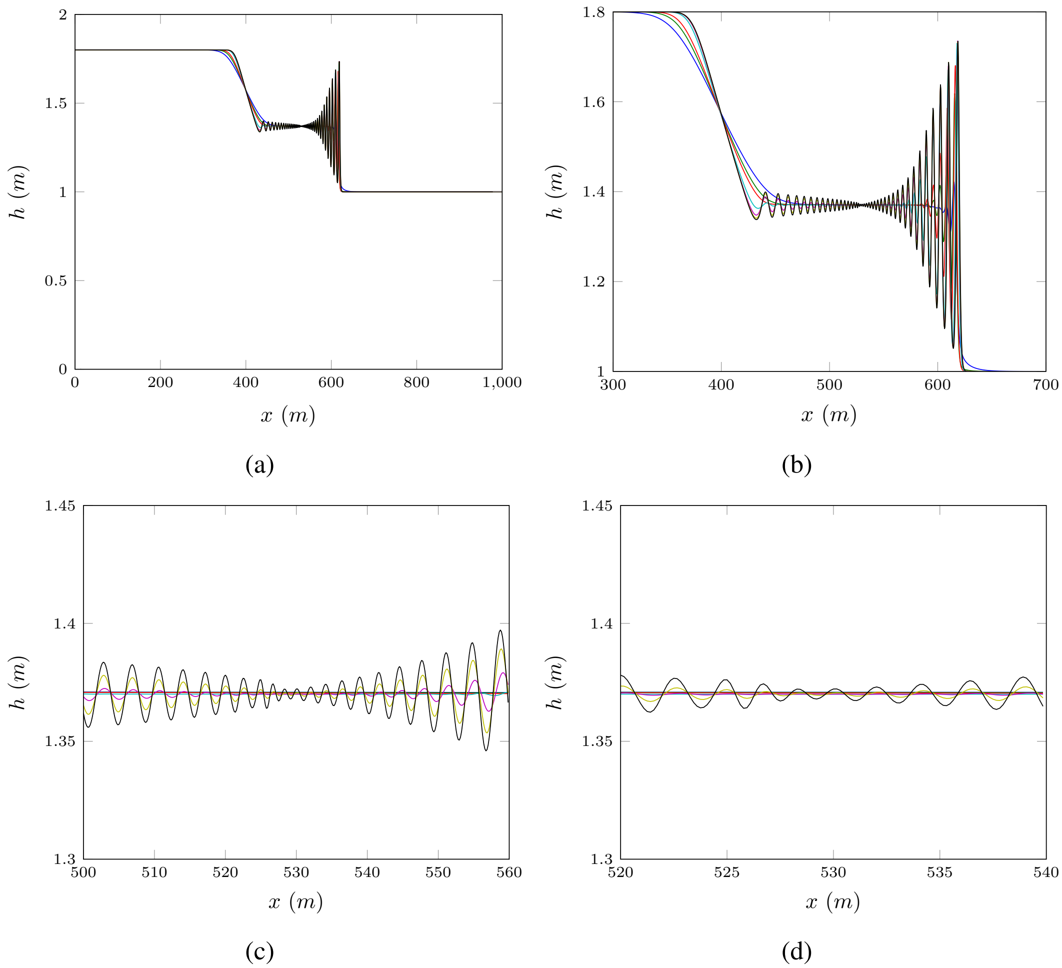

FIG. 2: Smooth dam break problem for FDcent [] with $dx = 10/2^6 m$ for $a = 0.05$ (− blue), $a = 0.075$ (− green), $a = 0.1$ (− red), $a = 0.25$ (− cyan), $a = 0.5$ (− magenta), $a = 0.75$ (− yellow), $a = 1.00$ (− black)

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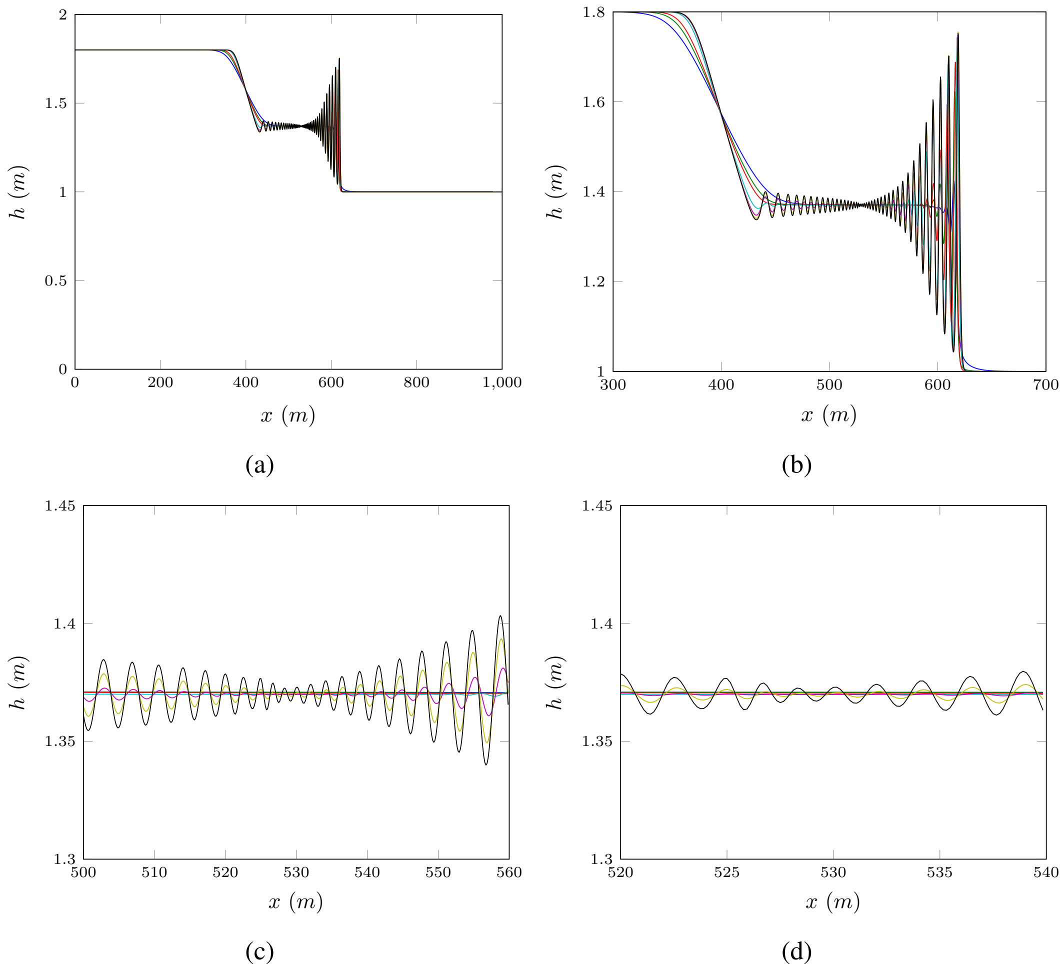

FIG. 3: Smooth dam break problem for grim [] with $dx = 10/2^6 m$ for $a = 0.05$ (− blue), $a = 0.075$ (− green), $a = 0.1$ (− red), $a = 0.25$ (− cyan), $a = 0.5$ (− magenta), $a = 0.75$ (− yellow), $a = 1.00$ (− black)

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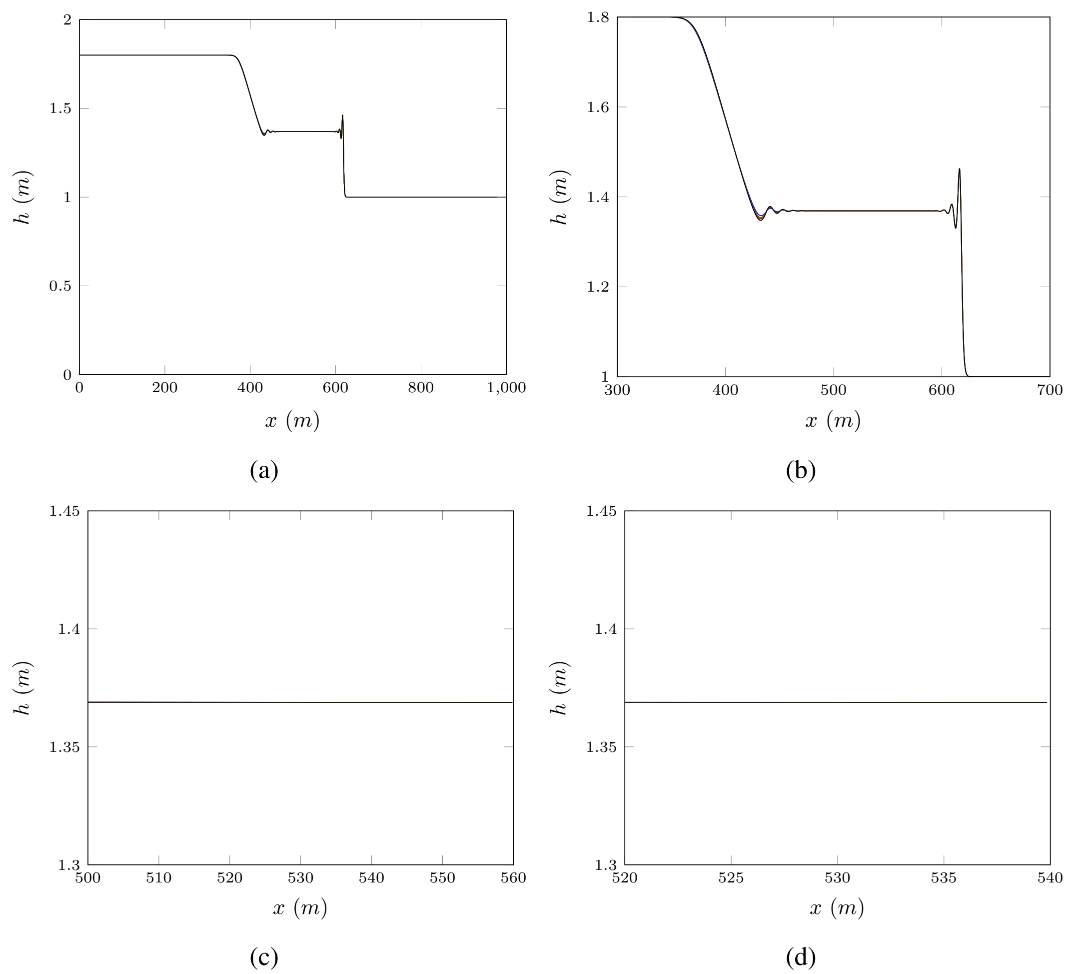


FIG. 4: Smooth dam break problem for o1 [] with $dx = 10/2^6 m$ for $a = 0.5$ (– blue), $a = 0.75$ (– green), $a = 1.0$ (– red), $a = 2.5$ (– cyan), $a = 5.0$ (– magenta), $a = 7.5$ (– yellow), $a = 10.0$ (– black)

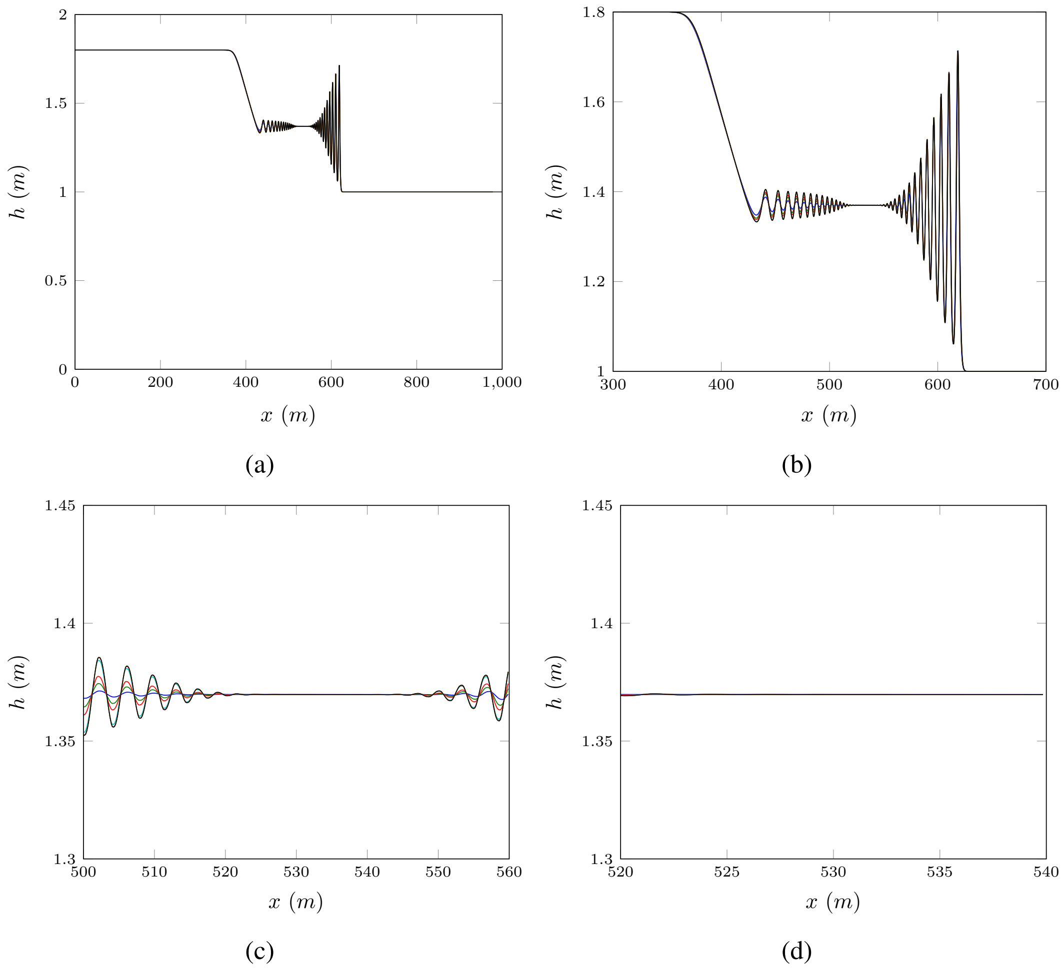


FIG. 5: Smooth dam break problem for o2 [] with $dx = 10/2^6 m$ for $a = 0.5$ (– blue), $a = 0.75$ (– green), $a = 1.0$ (– red), $a = 2.5$ (– cyan), $a = 5.0$ (– magenta), $a = 7.5$ (– yellow), $a = 10.0$ (– black)

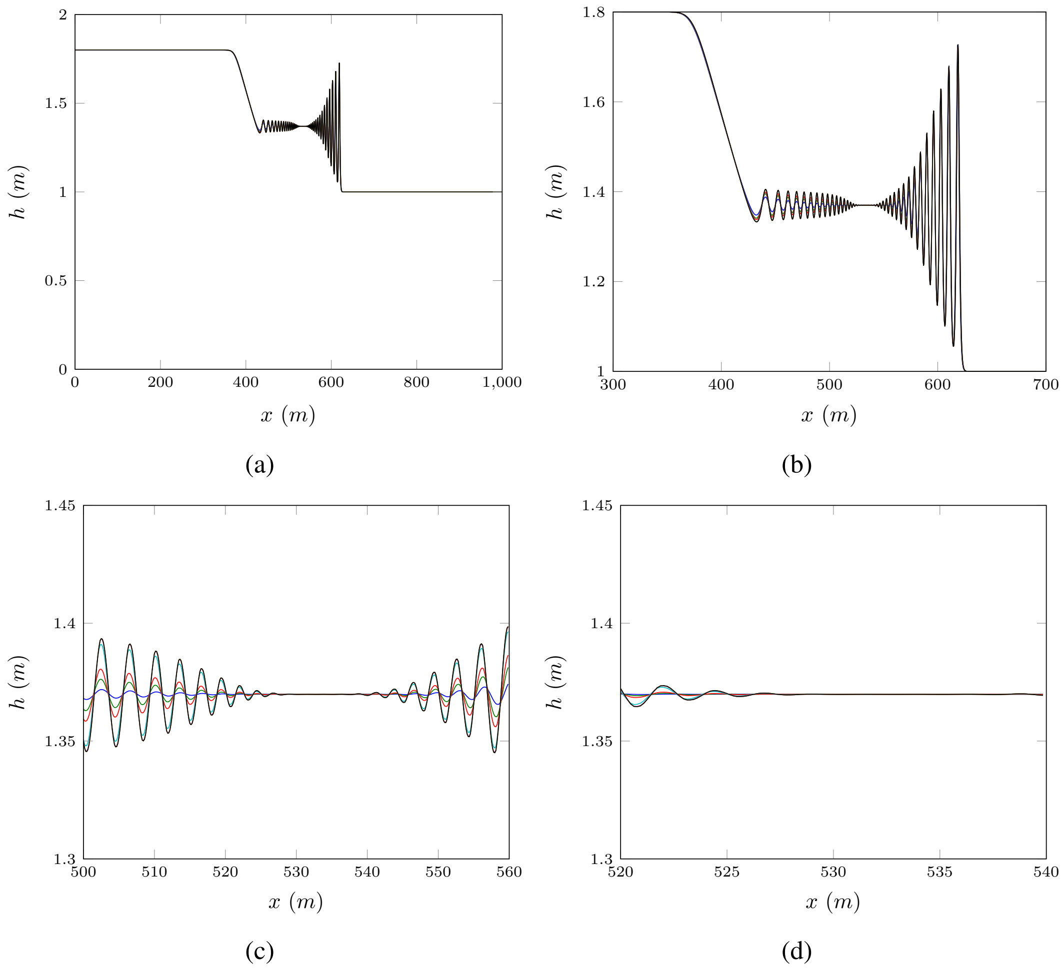


FIG. 6: Smooth dam break problem for o3 [] with $dx = 10/2^6 m$ for $a = 0.5$ (− blue), $a = 0.75$ (− green), $a = 1.0$ (− red), $a = 2.5$ (− cyan), $a = 5.0$ (− magenta), $a = 7.5$ (− yellow), $a = 10.0$ (− black)

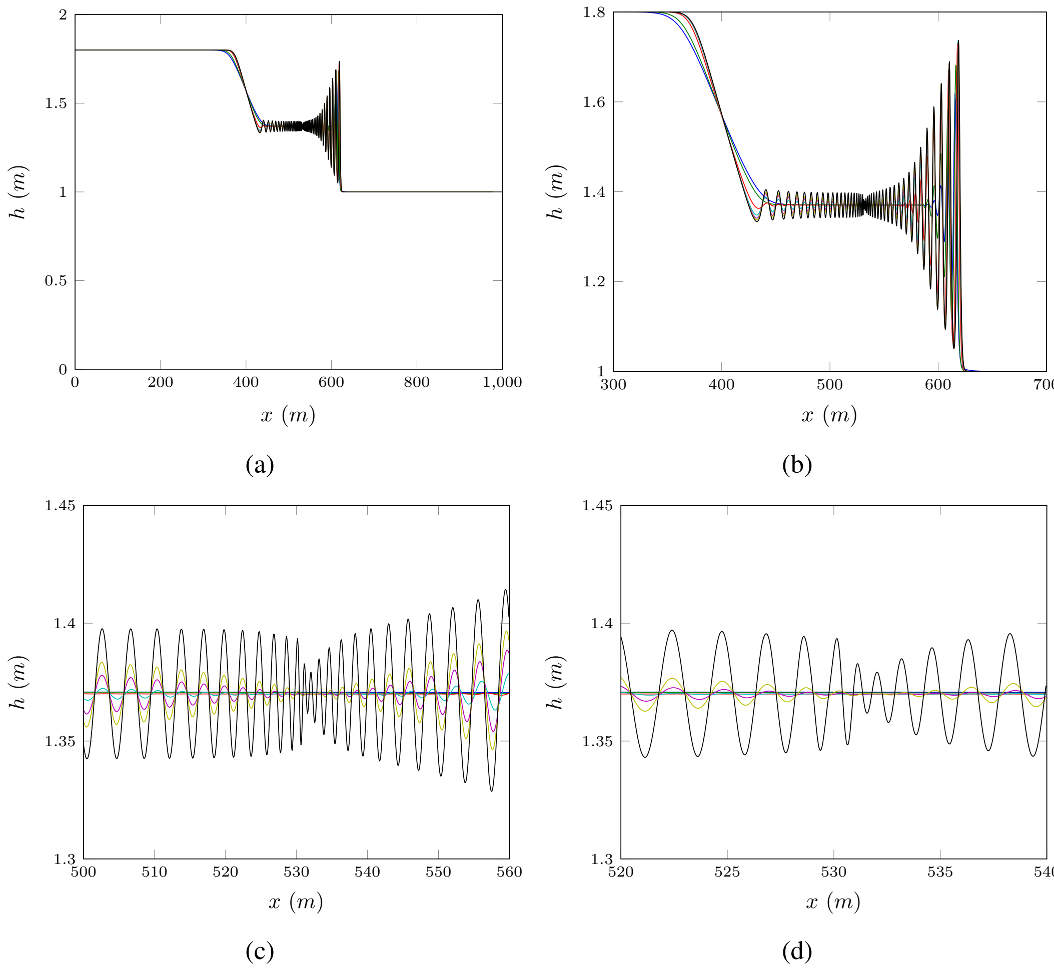


FIG. 7: Smooth dam break problem for FDcent [] with $dx = 10/2^8 m$ for $a = 0.075$ (– blue), $a = 0.1$ (– green), $a = 0.25$ (– red), $a = 0.5$ (– cyan), $a = 0.75$ (– magenta), $a = 1.0$ (– yellow), $a = 2.5$ (– black)

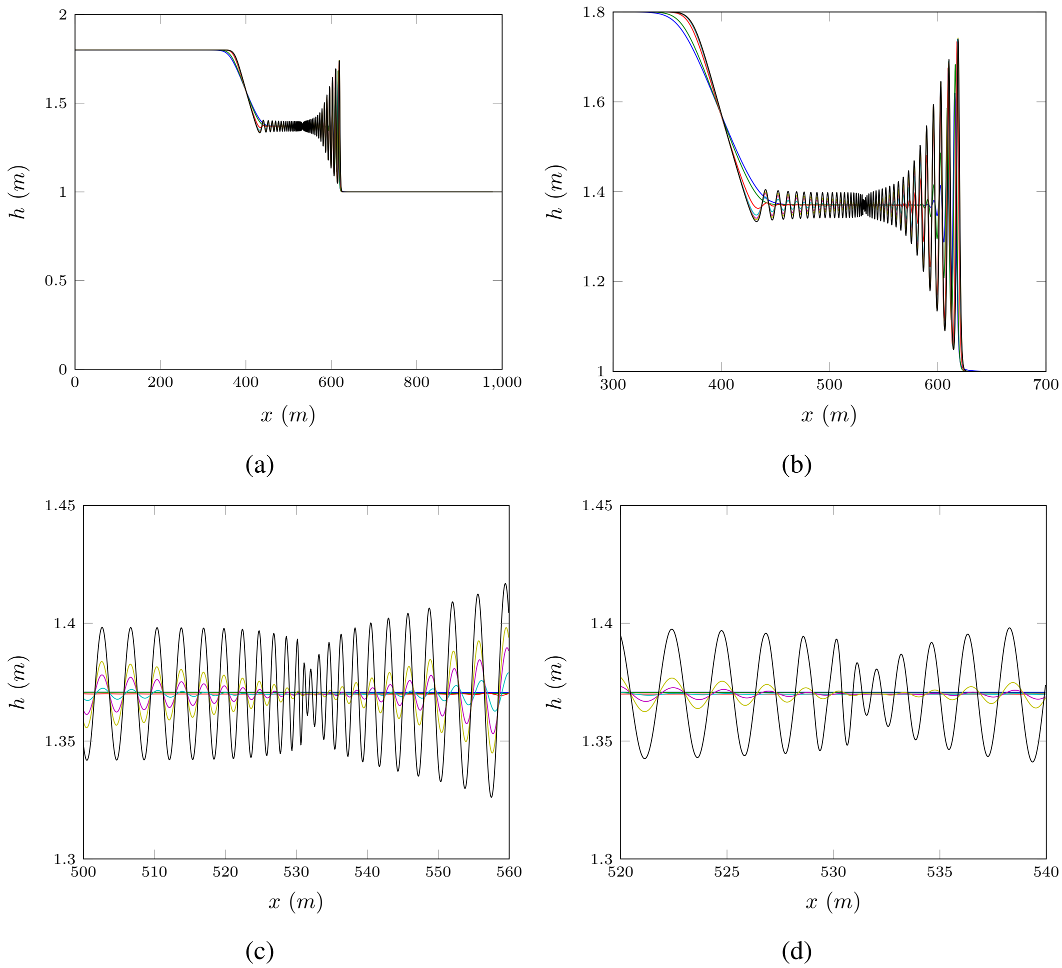
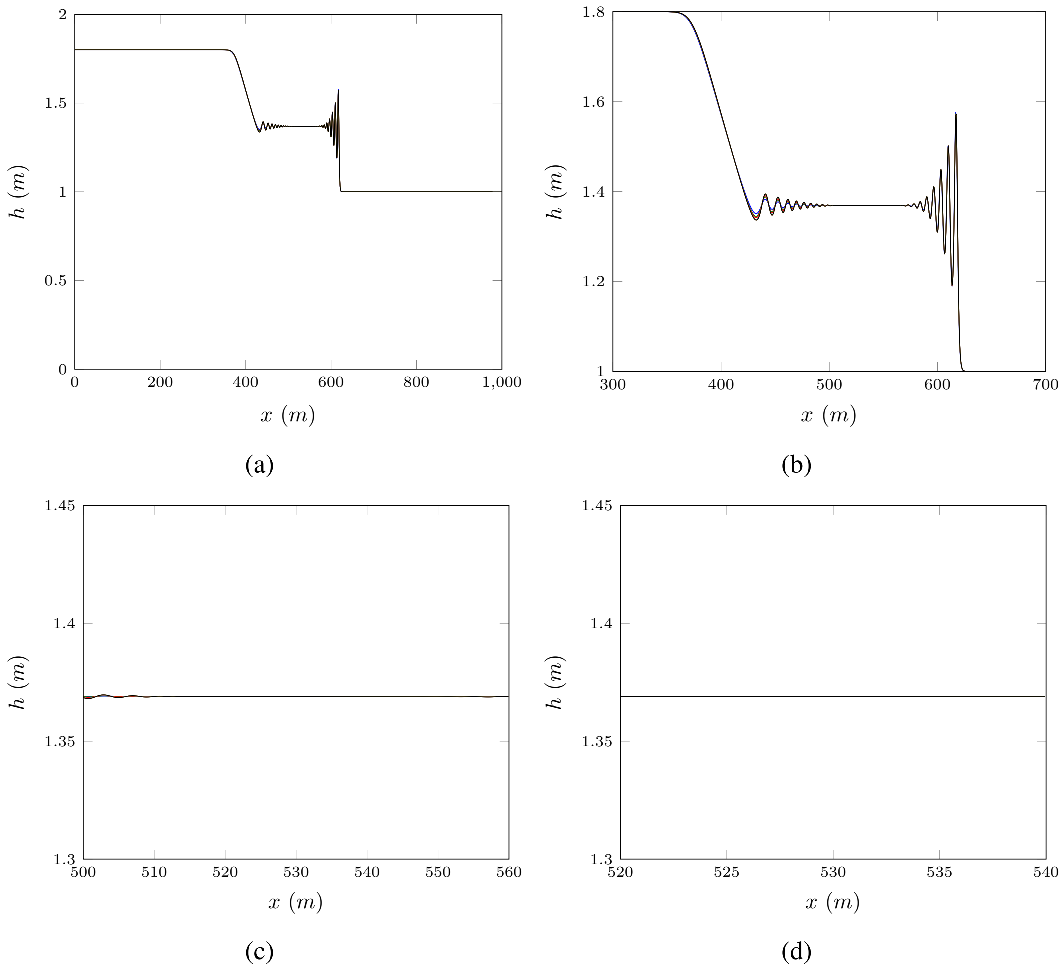


FIG. 8: Smooth dam break problem for grim [] with $dx = 10/2^8 m$ for $a = 0.075$ (– blue), $a = 0.1$ (– green), $a = 0.25$ (– red), $a = 0.5$ (– cyan), $a = 0.75$ (– magenta), $a = 1.0$ (– yellow), $a = 2.5$ (– black)



(a) (b) (c) (d)

FIG. 9: Smooth dam break problem for o1 [] with $dx = 10/2^8 m$ for $a = 0.5$ (– blue), $a = 0.75$ (– green), $a = 1.0$ (– red), $a = 2.5$ (– cyan), $a = 5.0$ (– magenta), $a = 7.5$ (– yellow), $a = 10.0$ (– black)

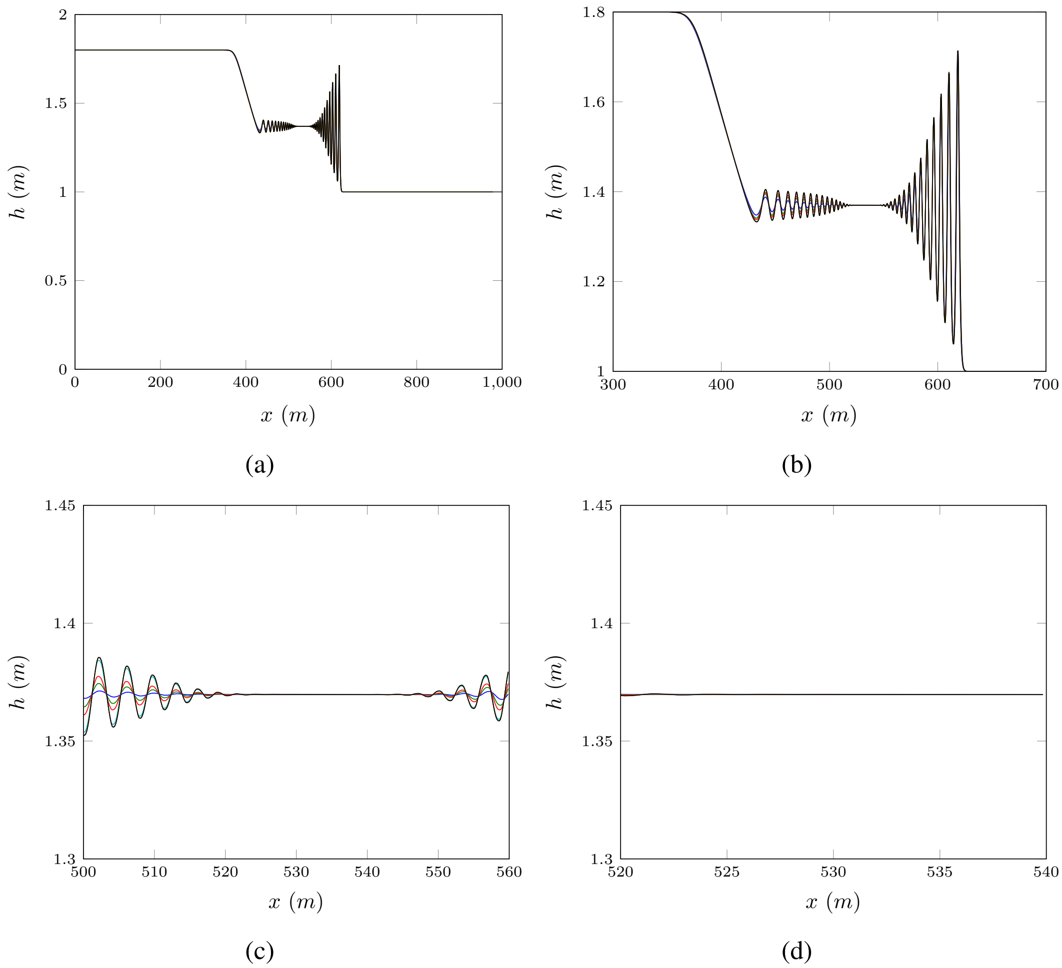


FIG. 10: Smooth dam break problem for o2 [] with $dx = 10/2^8 m$ for $a = 0.5$ (– blue), $a = 0.75$ (– green), $a = 1.0$ (– red), $a = 2.5$ (– cyan), $a = 5.0$ (– magenta), $a = 7.5$ (– yellow), $a = 10.0$ (– black)

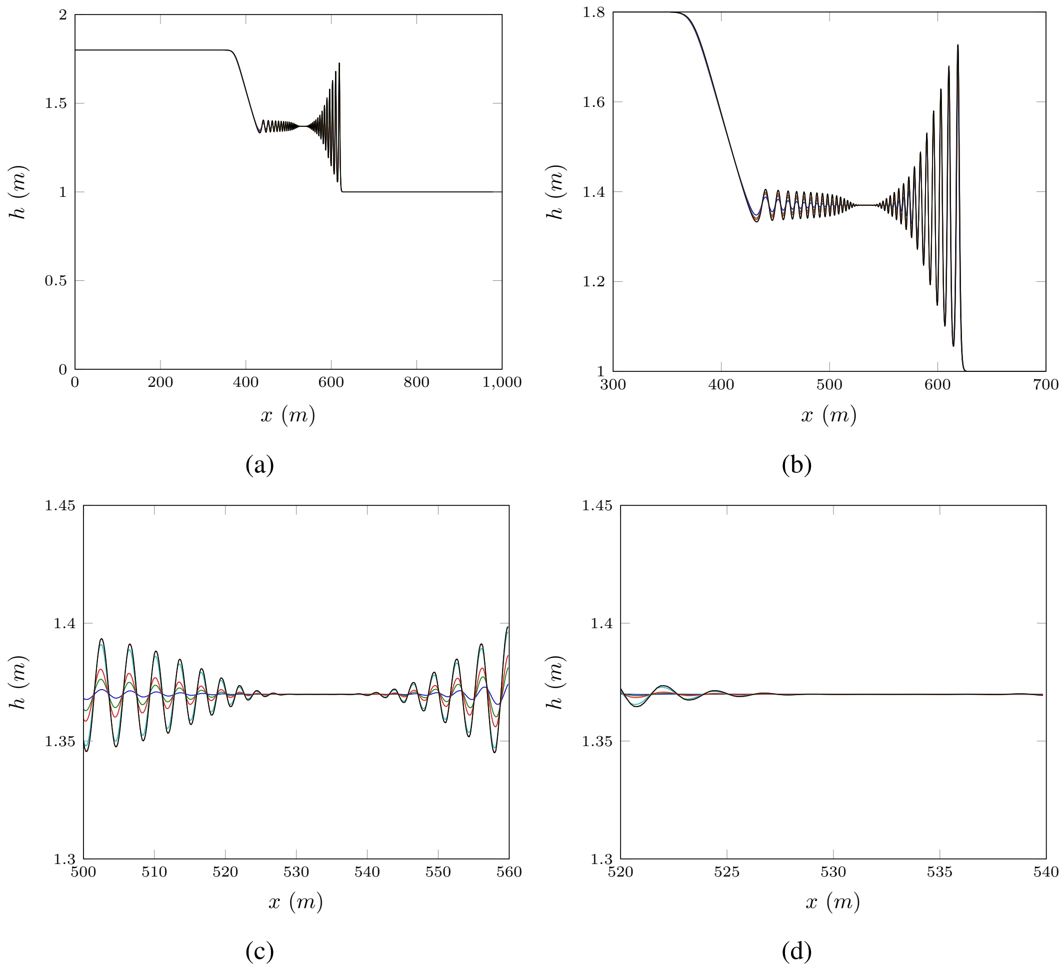


FIG. 11: Smooth dam break problem for o3 [] with $dx = 10/2^8 m$ for $a = 0.5$ (– blue), $a = 0.75$ (– green), $a = 1.0$ (– red), $a = 2.5$ (– cyan), $a = 5.0$ (– magenta), $a = 7.5$ (– yellow), $a = 10.0$ (– black)