Derivation of Eulers Equations for Incompressible Fluid

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July 7, 2014

Abstract

borrows heavily from the books I need to reference, in fact its pretty much just that copy pasted

1 Derivation

I will begin by deriving Eulers equations in the case where fluid is incompressible from first principles in a Eulerian coordinate system. In this derivation we are following the path laid out as in [] and in fact the following is just a paraphrased vesion of that derivation.

The flow of a fluid is determined by the velocty field $\vec{v} = \vec{v}(x, y, z, t)$ and any other 2 thermodynamical quantities, canonically we usually choose the pressure quantity $p = p(x, y, z, t)$ and density $\rho = \rho(x, y, z, t)$.

[picture will go here]

Conservation of Mass:

Consider the mass of fluid inside V, it can be calculated as:

$$\int_V \rho \, dV \tag{1}$$

Also consider the flow of mass out of V which is:

$$-\oint_{\partial V} \rho\vec{v} \cdot \vec{n} \, dS \tag{2}$$

Figure 1: Simulation Results

Conservation of mass dictates that the only change in mass comes from the flow out so:

$$\frac{\partial}{\partial t}\int_V \rho \, dV = -\oint_{\partial V} \rho \vec{v} \cdot \vec{n} \, dS \tag{3}$$

appyling Greens Formula gives:

$$-\oint_{\partial V} \rho \vec{v} \cdot \vec{n} \, dS = -\int_V \nabla \cdot (\rho \vec{v}) \, dV \tag{4}$$

this gives:

$$\frac{\partial}{\partial t}\int_V \rho \, dV = -\int_V \nabla \cdot (\rho \vec{v}) \, dV \tag{5}$$

bringing the derivative inside, assuming the density function is smooth, since we assume constant pressure later on this won't make any difference for us. Then brining everything together gives:

$$\int_V \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \, dV = 0 \tag{6}$$

This is true for any V we choose, and so it must be that the integrand is identically zero for all points, so we get that:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{7}$$

This is the conservation of mass in a general fluid, now applying a constant density we get that, the change in density is constant, and we can divide the density out of the second term as well giving:

$$\nabla \cdot \vec{v} = 0 \tag{8}$$

Conservation of Momentum:

To get the conservation of momentum we consider the force acting on V, this is given by:

$$-\oint_{\partial V} p \cdot \vec{n} \, dS \tag{9}$$

applying Greens formula gives:

$$-\oint_{\partial V} p \cdot \vec{n} \, dS = \int_V \nabla p \, dV \tag{10}$$

This says that for a volume element dV a force is applied of magnitude $-\nabla p \, dV$, this means that for a unit volume of fluid there is a force of $-\nabla p$m acting on it. Applying the derivative form of Newtons second law gives:

$$\frac{D\ (\rho \vec{v})}{Dt} = -\nabla p \tag{11}$$

Where $\frac{D\vec{v}}{Dt}$ is the substantial time derivative. Which instead of taking the derivative for a constant point in space, it does so over a particles path. To put this in terms of our regular time derivative notice that a change of velocity $d\vec{v}$ during the time dt is composed of parts, namely the change during dt in the velocity at a fixed point in space and the difference between the velocities (at the same instant) at two points dr apart, where dr is the distance moved by the particle during dt. The first part gives us the standard partial derivative $\frac{\partial\vec{v}}{\partial t}$ the second is:

$$dx\frac{\partial\vec{v}}{\partial x} + dy\frac{\partial\vec{v}}{\partial y} + dz\frac{\partial\vec{v}}{\partial z} = (dr\cdot\nabla)\vec{v} \tag{12}$$

Therefore:

$$\frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} \tag{13}$$

applying the incompressibility of the fluid. expanding the dubstantial time derivative and adding an acceleration fue to gravity called $\vec{g} = (0, 0, g)$ where g is the acceleration due to gravity. gives us:

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{\nabla p}{\rho} + \vec{g} \tag{14}$$

Thus giving the Euler Equations for incompressible flow, representing the conservation of mass and momentum respectivly:

$$\nabla\cdot\vec{v} = 0 \tag{15}$$

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{\nabla p}{\rho} + \vec{g} \tag{16}$$