Thesis title

Your name

Month Year

A thesis submitted for the degree of Doctor of Philosophy
of the Australian National University

To my mother and father who have provided me with everything.

Declaration

The work in this thesis is my own except where otherwise stated.

Jordan Pitt

Acknowledgements

Abstract

In this thesis we will read paers then talk about them

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Notation and terminology

Some preliminary description here? Eg, "In the following, G is a group, H is a subgroup of G, ..."

Notation

notation	definition text goes here definition text goes here definition text goes here
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Terminology

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Chapter 1

Introduction

1.1 Objectives of the Thesis

1.2 Original Contribution of the Thesis

1.3 Organisation of the Thesis

Chapter 2

The Serre Equations

The Serre equations are partial differential equations that describe the behaviour of waves of free surface flows of fluids for a wide array of wave properties.

In fact they are considered to be one of the best models for free surface flows up to wave breaking []. For this reason we are interested in using the Serre equations to model wave hazards such as tsunamis and storm surges.

The Serre equations can be derived asymptotically [] or via depth integration [] of the full Navier-Stokes equations. They are evolution type equations, however they are not strictly parabolic or hyperbolic and naively are not in conservation law form.

2.1 The Serre Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{2.1a}$$

and

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right) + \frac{\partial b}{\partial x}\left(gh + h\Psi + \frac{h^2}{2}\Phi\right) = 0 \tag{2.1b}$$

where Φ and Ψ are defined as

Definition 2.1.

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2 \frac{\partial b}{\partial x}$$

and

$$\Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

2.1.1 Alternative form of the Serre Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{2.2a}$$

and

$$\frac{\partial}{\partial t}(G) + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3\frac{\partial u}{\partial x}^2 + h^2 u\frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right) + \frac{1}{2}h^2 u\frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} - hu^2\frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} + gh\frac{\partial b}{\partial x} = 0 \tag{2.2b}$$

With G defined as

Definition 2.2.

$$G = hu\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \frac{\partial b}{\partial x}^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right)$$

2.2 Properties of the Serre Equations

2.2.1 Conservation Properties

The total amount of a quantity q in a system occurring on the interval $[a, b]$ at time t is measured by

Definition 2.3.

$$\mathcal{C}_q(t) = \int_a^b q(x,t)\,dx$$

Conservation of a quantity q implies that $\mathcal{C}_q(0) = \mathcal{C}_q(t)\ \forall t$ provided the interval is fixed and the system is closed

For any bed profile the Serre equations has two quantities that must be conserved the mass h, and the Hamiltonian $\mathcal{H}$. Where the Hamiltonian is

Definition 2.4.

$$\mathcal{H}(x,t) = \frac{1}{2}\left(hu^2 + \frac{h^3}{3}\left(\frac{\partial u}{\partial x}\right)^2 + gh^2 + 2ghb + u^2 h\frac{\partial b}{\partial x} - uh^2\frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right)$$

For horizontal bed profiles where $b(x) = 0\ \forall x$, the source terms in [] are 0 and the Serre equations also conserve momentum and G in the same way provided that boundary conditions for the water depths are equal at both ends.

2.2.2 Dispersion Relation

It was demonstrated in [] that the dispersion relation for the linearised Serre equations is

$$\omega = Uk \pm k\sqrt{gH}\sqrt{\frac{3}{(kH)^2+3}} \tag{2.3}$$

From this we get the phase velocity $v_p = \omega/k$ and the group velocity $v_g = d\omega/dk$.

$$v_p = U \pm \sqrt{gH}\sqrt{\frac{3}{(kH)^2+3}} \tag{2.4a}$$

$$v_g = U \pm \sqrt{gH}\left(\sqrt{\frac{3}{(kH)^2+3}} \mp (kH)^2\sqrt{\frac{3}{\left((kH)^2+3\right)^3}}\right) \tag{2.4b}$$

Chapter 3

Hybrid Finite Volume Methods

3.1 Structure Overview

In this section we will give the general structure for how the hybrid finite volume methods take the array of cell average values at time t^n; $\bar{\boldsymbol{h}}^n$ and $\bar{\boldsymbol{G}}^n$ and evolve the system to the cell average values at time t^{n+1}; $\bar{\boldsymbol{h}}^{n+1}$ and $\bar{\boldsymbol{G}}^{n+1}$.

- The cell average values are transformed into nodal values by $\mathcal{M}$
$$\bar{\boldsymbol{h}}^n, \bar{\boldsymbol{G}}^n \xrightarrow{\mathcal{M}} \boldsymbol{h}^n, \boldsymbol{G}^n$$

- $\boldsymbol{u}^n$ is found by solving the elliptic equation in Def. 2.2 by $\mathcal{A}$
$$\boldsymbol{h}^n, \boldsymbol{G}^n, \boldsymbol{b} \xrightarrow{\mathcal{G}} \boldsymbol{u}^n$$

- The conservation equations (2.2) can now be solved by $\mathcal{F}$
$$\bar{\boldsymbol{h}}^n, \bar{\boldsymbol{G}}^n, \boldsymbol{b}, \boldsymbol{u}^n \xrightarrow{\mathcal{F}} \bar{\boldsymbol{h}}^{n+1}, \bar{\boldsymbol{G}}^{n+1}$$

Definition 3.1. $\mathcal{E}$ is the single Euler step given by the procedure above that transforms updates the array of cell average values at time t^n; $\bar{\boldsymbol{h}}^n$ and $\bar{\boldsymbol{G}}^n$ to the cell average values at time t^{n+1}; $\bar{\boldsymbol{h}}^{n+1}$.

$$\bar{\boldsymbol{h}}^n, \bar{\boldsymbol{G}}^n, \boldsymbol{b} \xrightarrow{\mathcal{E}} \bar{\boldsymbol{h}}^{n+1}, \bar{\boldsymbol{G}}^{n+1}$$

3.2 Transformation Between Nodal Values and Cell Averages

For first and second order methods $\mathcal{M}$ is just the identity map as the cell average values are equal to the nodal values.

For higher order methods this is not the case, hence why there is a need to incorporate the process $\mathcal{M}$ into our methods, as assuming that $\mathcal{M}$ is the identity map will lead to a loss of accuracy in the method.

From quadratic interpolation we have the formula relating the cell averages and nodal values of a quantity q with third order accuracy

$$q_j = \frac{-\bar{q}_{j+1} + 26\bar{q}_j - \bar{q}_{j-1}}{24}.$$

Therefore

$$\boldsymbol{q} = \frac{1}{24}\begin{bmatrix} 26 & -1 & & & \\ -1 & 26 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 26 & -1 \\ & & & -1 & 26 \end{bmatrix} \bar{\boldsymbol{q}}.$$

Dirichlet boundary conditions at edges.

So for the third-order method $\mathcal{M}$ is a multiplication by the above matrix.

3.3 Elliptic Equation

The elliptic equation that relates the conserved variables h and G to the primitive variable u was given in Def 2.2 and is presented here to remind the reader

$$G = uh\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \frac{\partial b}{\partial x}^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right).$$

3.3.1 Finite Difference Methods

One way to approximate this ordinary differential equation is to replace all the derivatives with finite differences as has been done to second order accuracy in []. We have expanded this work by building a fourth order accurate finite difference method, where all derivatives were replaced with their centred fourth order approximation. This results in the following equation for each row of a the matrix $\boldsymbol{A}$

$$G_j = A_{j,j-2}\, u_{j-2} + A_{j,j-1}\, u_{j-1} + A_{j,j}\, u_j + A_{j,j+1}\, u_{j+1} + A_{j,j+2}\, u_{j+2} \qquad (3.1)$$

where

$$A_{j,j-2} = -h_j^2\left(\frac{-h_{j+2}+8h_{j+1}-8h_{j-1}+h_{j-2}}{144\Delta x^2}\right)+\frac{h_j^3}{36\Delta x^2},$$
$$A_{j,j-1} = h_j^2\left(\frac{-h_{j+2}+8h_{j+1}-8h_{j-1}+h_{j-2}}{18\Delta x^2}\right)-\frac{4h_j^3}{9\Delta x^2},$$
$$A_{j,j} = h_j+\frac{5h_j^3}{6\Delta x^2}+h_j\left(\frac{(-h_{j+2}+8h_{j+1}-8h_{j-1}+h_{j-2})(-b_{j+2}+8b_{j+1}-8b_{j-1}+b_{j-2})}{144\Delta x^2}\right)$$
$$+h_j\left(\frac{-b_{j+2}+16b_{j+1}-30b_{j-1}+16b_{j-1}-b_{j-2}}{24\Delta x^2}\right)+h_j\left(\frac{-b_{j+2}+8b_{j+1}-8b_{j-1}+b_{j-2}}{144\Delta x^2}\right),$$

$$A_{j,j+1} = -h_j^2\left(\frac{-h_{j+2}+8h_{j+1}-8h_{j-1}+h_{j-2}}{18\Delta x^2}\right)-\frac{4h_j^3}{9\Delta x^2},$$
$$A_{j,j+2} = h_j^2\left(\frac{-h_{j+2}+8h_{j+1}-8h_{j-1}+h_{j-2}}{144\Delta x^2}\right)+\frac{h_j^3}{36\Delta x^2},$$

This can be written for the whole domain

$$\boldsymbol{G} = \boldsymbol{A}\boldsymbol{u}.$$

Dirichlet boundary conditions

Therefore $\mathcal{G}$ is the solution of this matrix problem with $\boldsymbol{G}$ known and $\boldsymbol{u}$ unknown.

3.3.2 Finite Element Methods

For a finite element method we take the weak form of the elliptic equation in Def 2.2 which is

$$\int_\Omega Gv\,dx = \int_\Omega uh\left(1+\frac{\partial h}{\partial x}\frac{\partial b}{\partial x}+\frac{1}{2}h\frac{\partial^2 b}{\partial x^2}+\frac{\partial b}{\partial x}^2\right)-\frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right)v\,dx.$$

Which after rearranging, using integration by parts and assuming Dirichlet boundary conditions becomes

$$\int_\Omega Gv\,dx = \int_\Omega uh\left(1+\frac{\partial b}{\partial x}^2\right)v\,dx+\int_\Omega\frac{1}{3}h^3\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx$$
$$-\int_\Omega\frac{1}{2}h^2\frac{\partial b}{\partial x}u\frac{\partial v}{\partial x}\,dx.-\int_\Omega\frac{1}{2}h^2\frac{\partial b}{\partial x}\frac{\partial u}{\partial x}v\,dx. \quad (3.2)$$

Should be able to handle non smooth h and G but requires first derivative of b.

Second Order

Third Order

3.4 Evolution Equations

The evolution equations in the alternative form of the Serre equations (2.2) are

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

and

$$\frac{\partial}{\partial t}(G) + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3\frac{\partial u}{\partial x}^2 + h^2 u\frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right) = -\frac{1}{2}h^2 u\frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} + hu^2\frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} - gh\frac{\partial b}{\partial x}$$

Because these equations are in conservation law form and we have an estimate for the maximum and minimum wave speeds [], Kurganovs method [] can be employed to estimate the fluxes across the boundary. This leads to the following update scheme for a quantity q

$$\bar{q}_j^{n+1} = \bar{q}_j^n - \frac{\Delta t}{\Delta x}\left[F_{j+1/2}^n - F_{j-1/2}^n\right] + \Delta t S_j^n. \tag{3.3}$$

Where $F_{j+1/2}^n$ and $F_{j-1/2}^n$ are approximations to the average fluxes across the boundary of the cell with midpoint x_i from time t^n to t^{n+1}. While S_j is an approximation to the average source term contribution in the cell from time t^n to t^{n+1}, which since this time-stepping is first-order we can just take to be constant over the time step.

3.4.1 Kurganovs Method

Kurganovs method is a finite volume method that can handle discontinuities across the boundary and only requires an estimate of the maximum and minimum wave speeds instead of the characteristics like other methods []. This makes it a good choice for the Serre equations as we do not have an expression for the characteristics but we do have estimates on the maximum and minimum wave speeds [].

The equation which approximates $F_{j+1/2}^n$ in (3.3) for a quantity q at a particular time t^n is

$$F_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}} f\left(q^-_{j+\frac{1}{2}}\right) - a^-_{j+\frac{1}{2}} f\left(q^+_{j+\frac{1}{2}}\right)}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}}\, a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\left[q^+_{j+\frac{1}{2}} - q^-_{j+\frac{1}{2}}\right] \tag{3.4}$$

where $a^+_{j+\frac{1}{2}}\ a^-_{j+\frac{1}{2}}$ are given by the wave speed bounds [], for the Serre equations we have

$$\begin{aligned} a^-_{j+\frac{1}{2}} &= \min\left\{0\ ,\ u^-_{j+1/2} - \sqrt{gh^-_{j+1/2}}\ ,\ u^+_{j+1/2} - \sqrt{gh^+_{j+1/2}}\right\}, \\ a^+_{j+\frac{1}{2}} &= \max\left\{0\ ,\ u^-_{j+1/2} + \sqrt{gh^-_{j+1/2}}\ ,\ u^+_{j+1/2} + \sqrt{gh^+_{j+1/2}}\right\}, \end{aligned}$$

While $f(q^-_{j+\frac{1}{2}})$ and $f(q^+_{j+\frac{1}{2}})$ are the evaluations of the flux function on the left and right side of the cell interface respectively. For h in the Serre equations we have

$$\begin{aligned} f\left(h^-_{j+\frac{1}{2}}\right) &= u^-_{j+1/2} h^-_{j+1/2}, \\ f\left(h^+_{j+\frac{1}{2}}\right) &= u^+_{j+1/2} h^+_{j+1/2}, \end{aligned}$$

while for G we have

$$\begin{aligned} f\left(G^-_{j+\frac{1}{2}}\right) &= u^-_{j+1/2} G^-_{j+1/2} + \frac{g}{2}\left(h^-_{j+1/2}\right)^2 - \frac{2}{3}\left(h^-_{j+1/2}\right)^3 \left[\left(\frac{\partial u}{\partial x}\right)^-_{j+1/2}\right]^2 \\ &\quad + \left(h^-_{j+1/2}\right)^2 u^-_{j+1/2}\left(\frac{\partial u}{\partial x}\right)^-_{j+1/2}\left(\frac{\partial b}{\partial x}\right)^-_{j+1/2}, \\ f\left(G^+_{j+\frac{1}{2}}\right) &= u^+_{j+1/2} G^+_{j+1/2} + \frac{g}{2}\left(h^+_{j+1/2}\right)^2 - \frac{2}{3}\left(h^+_{j+1/2}\right)^3 \left[\left(\frac{\partial u}{\partial x}\right)^+_{j+1/2}\right]^2 \\ &\quad + \left(h^+_{j+1/2}\right)^2 u^+_{j+1/2}\left(\frac{\partial u}{\partial x}\right)^+_{j+1/2}\left(\frac{\partial b}{\partial x}\right)^+_{j+1/2}. \end{aligned}$$

We now only have to have some appropriate order of accuracy method to calculate the quantities we need at the boundaries from the cell averages of h, G and b and the nodal values of u.

Second Order

For the second order finite volume method we use the minmmod limiter to reconstruct h, G and b at the cell edges. For a general quantity q the generalised minmod limiter produces the following reconstruction

[?]

$$q^-_{j+1/2} = q_j + a_j \frac{\Delta x}{2} \tag{3.5a}$$

and

$$q^+_{j+1/2} = q_{j+1} - a_{j+1} \frac{\Delta x}{2} \tag{3.5b}$$

where

$$a_j = \text{minmod}\left\{\theta \frac{q_{j+1} - q_j}{\Delta x}, \frac{q_{j+1} - q_{j-1}}{2\Delta x}, \theta \frac{q_j - q_{j-1}}{\Delta x}\right\} \quad \text{for } \theta \in [1, 2] \tag{3.5c}$$

While for u the following reconstruction is used

$$u^+_{j+\frac{1}{2}} = u^-_{j+\frac{1}{2}} = \frac{u_{j+1} + u_j}{2}. \tag{3.6}$$

We approximate the derivatives in the following way

$$\left(\frac{\partial b}{\partial x}\right)^+_{j+1/2} = \frac{b^+_{j+3/2} - b^+_{j+1/2}}{\Delta x}, \tag{3.7}$$

$$\left(\frac{\partial b}{\partial x}\right)^-_{j+1/2} = \frac{b^-_{j+1/2} - b^-_{j-1/2}}{\Delta x}, \tag{3.8}$$

$$\left(\frac{\partial u}{\partial x}\right)^+_{j+1/2} = \left(\frac{\partial u}{\partial x}\right)^-_{j+1/2} = \frac{u_{j+1} - u_j}{\Delta x} \tag{3.9}$$

Third Order

For the third-order finite volume method we use the Koren limiter [?] to reconstruct the cell edges for h, G and b. For a general quantity q the reconstruction based on the Koren limiter is

$$q^-_{j+1/2} = \bar{q}_j + \phi^-(r_j)(\bar{q}_j - \bar{q}_{j-1})/2 \tag{3.10a}$$

and

$$q^+_{j+1/2} = \bar{q}_{j+1} - \phi^+(r_{j+1})(\bar{q}_{j+1} - \bar{q}_j)/2 \tag{3.10b}$$

where

$$\phi^{-}(r_j) = \max\left[0, \min\left[2r_j, \frac{1+2r_j}{3}, 2\right]\right], \tag{3.10c}$$

$$\phi^{+}(r_j) = \max\left[0, \min\left[2r_j, \frac{2+r_j}{3}, 2\right]\right] \tag{3.10d}$$

with

$$r_j = (\bar{q}_{j+1} - \bar{q}_j)/(\bar{q}_j - \bar{q}_{j-1}). \tag{3.10e}$$

While for u the following reconstruction is used

$$u_{j+\frac{1}{2}} = \frac{-3u_{j+2} + 27u_{j+1} + 27u_j - 3u_{j-1}}{48}. \tag{3.11}$$

We approximate the derivatives in the following way

$$\left(\frac{\partial b}{\partial x}\right)^{+}_{j+1/2} = \frac{-b^{+}_{j+5/2} + 4b^{+}_{j+3/2} + 3b^{+}_{j+1/2}}{\Delta x}, \tag{3.12}$$

$$\left(\frac{\partial b}{\partial x}\right)^{-}_{j+1/2} = \frac{3b^{-}_{j+1/2} - 4b^{-}_{j-1/2} + b^{-}_{j-3/2}}{\Delta x}, \tag{3.13}$$

$$\left(\frac{\partial u}{\partial x}\right)^{+}_{j+1/2} = \left(\frac{\partial u}{\partial x}\right)^{-}_{j+1/2} = \frac{-u_{j+2} + 27u_{j+1} - 27u_j + u_{j-1}}{24\Delta x}, \tag{3.14}$$

3.4.2 Source Terms and Well Balancing

3.5 Runge-Kutta Time-Stepping

The method $\mathcal{E}$ is only first order in time, one strategy for increasing our order of accuracy in time is to use SSP Runge Kutta time stepping [].

For the first order method our current method is sufficient and so

$$\bar{\boldsymbol{h}}^{n+1}, \bar{\boldsymbol{G}}^{n+1} = \mathcal{E}\left(\bar{\boldsymbol{h}}^{n}, \bar{\boldsymbol{G}}^{n}, \boldsymbol{b}\right) \tag{3.15}$$

For the second order method we have

$$\bar{\boldsymbol{h}}', \bar{\boldsymbol{G}}' = \mathcal{E}\left(\bar{\boldsymbol{h}}^{n}, \bar{\boldsymbol{G}}^{n}, \boldsymbol{b}\right) \tag{3.16a}$$

$$\bar{\boldsymbol{h}}'', \bar{\boldsymbol{G}}'' = \mathcal{E}\left(\bar{\boldsymbol{h}}', \bar{\boldsymbol{G}}', \boldsymbol{b}\right) \tag{3.16b}$$

$$\bar{\boldsymbol{h}}^{n+1}, \bar{\boldsymbol{G}}^{n+1} = \frac{1}{2}\left(\bar{\boldsymbol{h}}^{n} + \bar{\boldsymbol{h}}''\right), \frac{1}{2}\left(\bar{\boldsymbol{G}}^{n} + \bar{\boldsymbol{G}}''\right) \tag{3.16c}$$

For the third order method we have

$$\bar{\boldsymbol{h}}', \bar{\boldsymbol{G}}' = \mathcal{E}\left(\bar{\boldsymbol{h}}^{n}, \bar{\boldsymbol{G}}^{n}, \boldsymbol{b}\right) \tag{3.17a}$$

$$\bar{\boldsymbol{h}}'', \bar{\boldsymbol{G}}'' = \mathcal{E}\left(\bar{\boldsymbol{h}}', \bar{\boldsymbol{G}}', \boldsymbol{b}\right) \tag{3.17b}$$

$$\bar{\boldsymbol{h}}''', \bar{\boldsymbol{G}}''' = \frac{3}{4}\bar{\boldsymbol{h}}^{n} + \frac{1}{4}\bar{\boldsymbol{h}}'', \frac{3}{4}\bar{\boldsymbol{G}}^{n} + \frac{1}{4}\bar{\boldsymbol{G}}'' \tag{3.17c}$$

$$\bar{\boldsymbol{h}}'''', \bar{\boldsymbol{G}}'''' = \mathcal{E}\left(\bar{\boldsymbol{h}}''', \bar{\boldsymbol{G}}''', \boldsymbol{b}\right) \tag{3.17d}$$

$$\bar{\boldsymbol{h}}^{n+1}, \bar{\boldsymbol{G}}^{n+1} = \frac{1}{3}\bar{\boldsymbol{h}}^{n} + \frac{2}{3}\bar{\boldsymbol{h}}'''', \frac{1}{3}\bar{\boldsymbol{G}}^{n} + \frac{2}{3}\bar{\boldsymbol{G}}'''' \tag{3.17e}$$

Chapter 4

Finite Difference Methods

The methods $\mathcal{W}$ and $\mathcal{D}$ use the centred second-order finite difference approximation to the momentum equation (2.1b), denoted as $\mathcal{D}_u$. For the mass equation (2.1a) $\mathcal{W}$ uses the two step Lax-Wendroff method, denoted as $\mathcal{W}_h$ while $\mathcal{D}$ uses a centred second-order finite difference approximation, denoted as $\mathcal{D}_h$.

4.1 Naive Second Order Finite Difference Approximation to the Momentum Equation

First (2.1b) is expanded to get

$$h\frac{\partial u}{\partial t} - h^2\frac{\partial^2 u}{\partial x \partial t} - \frac{h^3}{3}\frac{\partial^3 u}{\partial x^2 \partial t} = -X$$

where X contains only spatial derivatives and is

$$X = uh\frac{\partial u}{\partial x} + gh\frac{\partial h}{\partial x} + h^2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{h^3}{3}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - h^2 u\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3}u\frac{\partial^3 u}{\partial x^3}.$$

All derivatives are approximated by second-order centred finite difference approximations on a uniform grid in space and time, which after rearranging into an update formula becomes

$$h_j^n u_j^{n+1} - \left(h_j^n\right)^2\left(\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x}\right) - \frac{\left(h_j^n\right)^3}{3}\left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}\right) = -Y_j^n \quad (4.1)$$

where

$$Y_j^n = 2\Delta t X_j^n - h_j^n u_j^{n-1} + \left(h_j^n\right)^2\left(\frac{u_{j+1}^{n-1} - u_{j-1}^{n-1}}{2\Delta x}\right) + \frac{\left(h_j^n\right)^3}{3}\left(\frac{u_{j+1}^{n-1} - 2u_j^{n-1} + u_{j-1}^{n-1}}{\Delta x^2}\right)$$

and

$$X_j^n = u_j^n h_j^n \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + g h_j^n \frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} + \left(h_j^n\right)^2 \left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right)^2 + \frac{\left(h_j^n\right)^3}{3} \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} - \left(h_j^n\right)^2 u_j^n \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} - \frac{\left(h_j^n\right)^3}{3} u_j^n \frac{u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n}{2\Delta x^3}.$$

Equation (4.1) can be rearranged into an explicit update scheme $\mathcal{D}_u$ for u given its current and previous values, so that

$$\boldsymbol{u}^{n+1} = A^{-1} \begin{bmatrix} -Y_0^n \\ \vdots \\ -Y_m^n \end{bmatrix} =: \mathcal{D}_u\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n-1}, \Delta x, \Delta t\right) \tag{4.2}$$

[] where A is a tri-diagonal matrix.

with

$$A_{j,j-1} = \frac{\left(h_j^n\right)^2}{2\Delta x} \frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} - \frac{\left(h_j^n\right)^3}{3\Delta x^2}, \tag{4.3}$$

$$A_{j,j} = h_j^n + \frac{2h_j^n}{3\Delta x^2}, \tag{4.4}$$

$$A_{j,j+1} = -\frac{\left(h_j^n\right)^2}{2\Delta x} \frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} - \frac{\left(h_j^n\right)^3}{3\Delta x^2}. \tag{4.5}$$

4.2 Numerical Methods for the Mass Equation

4.2.1 Lax-Wendroff Method

The two step Lax-Wendroff update $\mathcal{W}_h$ for h is

$$h_{j+1/2}^{n+1/2} = \frac{1}{2}\left(h_{j+1}^n + h_j^n\right) - \frac{\Delta t}{2\Delta x}\left(u_{j+1}^n h_{j+1}^n - h_j^n u_j^n\right),$$

$$h_{j-1/2}^{n+1/2} = \frac{1}{2}\left(h_j^n + h_{j-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(u_j^n h_j^n - h_{j-1}^n u_{j-1}^n\right)$$

and

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x}\left(u_{j+1/2}^{n+1/2} h_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2} h_{j-1/2}^{n+1/2}\right).$$

The quantities $u_{j\pm1/2}^{n+1/2}$ are calculated using $\boldsymbol{u}^{n+1}$ obtained by applying $\mathcal{D}_u$ (4.2) to $\boldsymbol{u}^n$ then linearly interpolating in space and time to give

$$u_{j+1/2}^{n+1/2} = \frac{u_{j+1}^{n+1} + u_{j+1}^{n} + u_{j}^{n+1} + u_{j}^{n}}{4}$$

and

$$u_{j-1/2}^{n+1/2} = \frac{u_{j}^{n+1} + u_{j}^{n} + u_{j-1}^{n+1} + u_{j-1}^{n}}{4}.$$

Thus we have the following update scheme $\mathcal{W}_h$ for (2.1a)

$$\boldsymbol{h}^{n+1} = \mathcal{W}_h\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n+1}, \Delta x, \Delta t\right). \tag{4.6}$$

4.2.2 Second Order Finite Difference Approximation

The second order centered finite difference approximation to the conservation of mass equation (2.1a) is

$$h_j^{n+1} = h_j^{n-1} - \Delta t\left(u_j^n \frac{h_{j+1}^n - h_{j-1}^n}{\Delta x} + h_j^n \frac{u_{j+1}^n - u_{j-1}^n}{\Delta x}\right).$$

Thus we have an update scheme $\mathcal{D}_h$ for all i

$$\boldsymbol{h}^{n+1} = \mathcal{D}_h\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{h}^{n-1}, \Delta x, \Delta t\right). \tag{4.7}$$

4.3 Complete Method

The method $\mathcal{W}$ is the combination of (4.6) for (2.1a) and (4.2) for (2.1b) in the following way

$$\left.\begin{array}{ll} \boldsymbol{u}^{n+1} & = \mathcal{D}_u\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n-1}, \Delta x, \Delta t\right) \\ \boldsymbol{h}^{n+1} & = \mathcal{W}_h\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n+1}, \Delta x, \Delta t\right) \end{array}\right\} \mathcal{W}\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n-1}, \boldsymbol{h}^{n-1}, \Delta x, \Delta t\right). \tag{4.8}$$

The method $\mathcal{D}$ is the combination of (4.7) for (2.1a) and (4.2) for (2.1b) in the following way

$$\left.\begin{array}{ll} \boldsymbol{h}^{n+1} & = \mathcal{D}_h\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{h}^{n-1}\Delta x, \Delta t\right) \\ \boldsymbol{u}^{n+1} & = \mathcal{D}_u\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n-1}, \Delta x, \Delta t\right) \end{array}\right\} \mathcal{D}\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n-1}, \boldsymbol{h}^{n-1}, \Delta x, \Delta t\right). \tag{4.9}$$

Chapter 5

Analysis of Numerical Methods

There are a variety of ways to analyse the numerical methods presented in this paper []. There are also a variety of properties that numerical methods can possess. Chief among these is convergence, which for linear partial differential equations can be broken up into consistency and stability using the Lax equivalence theorem [].

For dispersive equations such as the Serre equations other more specific properties may be of interest such as the error in the dispersion relation introduced by the numerical methods as investigated in [].

The main obstacle for analysing the numerical methods for the Serre equations is that the Serre equations are non-linear, which means the techniques employed to investigate the convergence and dispersion error of our numerical methods are no longer valid. However, some insights can still be gained by instead examining the linearised version of the Serre equations as has been done []. This is the approach we take in this section of the thesis.

Horizontal bed

We will use the following notation

H base, η perturbation

U base, υ perturbation

$$\frac{\partial \eta}{\partial t} + H\frac{\partial \upsilon}{\partial x} + U\frac{\partial \eta}{\partial x} = 0 \tag{5.1a}$$

$$H\frac{\partial \upsilon}{\partial t} + gH\frac{\partial \eta}{\partial x} + UH\frac{\partial \upsilon}{\partial x} - \frac{H^3}{3}\left(U\frac{\partial^3 \upsilon}{\partial x^3} + \frac{\partial^3 \upsilon}{\partial x^3 \partial t}\right) = 0 \tag{5.1b}$$

Also G

$$G = U\left(H + \eta\right) + H\upsilon - \frac{H^3}{3}\frac{\partial^2 \upsilon}{\partial x^2} \tag{5.2}$$

5.1 Dispersion Error

To study the error in the dispersion relation caused by the numerical methods we will follow the example of [] who used different methods on a different reformulation of the Serre equations. This allows us to compare our methods to those of Phillipne [].

In keeping with [], we will assume that $U = 0$ so that (5.1) and (5.2) reduce to

$$\frac{\partial \eta}{\partial t} + H\frac{\partial \upsilon}{\partial x} = 0 \tag{5.3a}$$

$$h_0\frac{\partial \upsilon}{\partial t} + gH\frac{\partial \eta}{\partial x} - \frac{H^3}{3}\left(\frac{\partial^3 \upsilon}{\partial x^3 \partial t}\right) = 0 \tag{5.3b}$$

Also G

$$G = H\upsilon - \frac{H^3}{3}\frac{\partial^2 \upsilon}{\partial x^2} \tag{5.4}$$

Combining (5.3) and (5.4) the linearised equations with $u_0 = 0$ can be written as

$$\frac{\partial \eta}{\partial t} + H\frac{\partial \upsilon}{\partial x} = 0 \tag{5.5a}$$

$$\frac{\partial G}{\partial t} + gH\frac{\partial \eta}{\partial x} = 0. \tag{5.5b}$$

For brevity we will only demonstrate our analysis of the dispersion error for our hybrid finite volume methods, through singular examples for the steps required to finally attain the dispersion error.

To perform the dispersion error we replace both η and υ by fourier modes, which for some quantity q is given by so that

$$q(x,t) = q(0,0)e^{i(\omega t + kx)} \tag{5.6}$$

Therefore because we use uniform spatial grids so that $q_j^n = q(x_j, t^n)$ we have that

$$q_{j\pm l}^n = q_j^n e^{\pm ikl\Delta x} \qquad \text{and} \qquad q_j^{n\pm l} = q_j^n e^{\pm i\omega l\Delta t} \tag{5.7}$$

For the hybrid finite volume methods we break this process up into the three parts of these methods, the elliptic equation which we use to solve for u, the evolution equation we use to update h and G and the runge-kutta steps we use to increase the order of accuracy of the method in time. [Choose second-order method as our example, simplest method which highlights the RK steps]

[need notation]

5.1.1 Elliptic Equation

The elliptic equation (5.4) at a particular grid point x_j is

$$G_j = H\upsilon_j - \frac{H^3}{3}\left(\frac{\partial^2 \upsilon}{\partial x^2}\right)_j$$

For the finite difference method the derivative of u_1 is approximated by a finite difference, in particular the second order method uses the approximation

$$\left(\frac{\partial^2 \upsilon}{\partial x^2}\right)_j = \frac{\upsilon_{j+1} - 2\upsilon_j + \upsilon_{j-1}}{\Delta x^2}$$

Which making use of (**??**) becomes

$$\left(\frac{\partial^2 \upsilon}{\partial x^2}\right)_j = \frac{\upsilon_j e^{ik\Delta x} - 2\upsilon_j + \upsilon_j e^{-ik\Delta x}}{\Delta x^2}$$

Which reduces to

$$\left(\frac{\partial^2 \upsilon}{\partial x^2}\right)_j = \frac{2\cos(k\Delta x) - 2}{\Delta x^2}\upsilon_j$$

Substituting this approximation into our ellitpic equation one obtains

$$G_j = \left(H - \frac{H^3}{3}\frac{2\cos(k\Delta x) - 2}{\Delta x^2}\right)\upsilon_j$$

We then define

$$\mathcal{G}_{FD2} = \left(H - \frac{H^3}{3}\frac{2\cos(k\Delta x) - 2}{\Delta x^2}\right),$$

where the $\mathcal{G}$ denotes that this is the error introduced by transforming from υ to G and the subscript denotes that it is the factor for the second-order finite different approximation to this transformation.

5.1.2 Conservation Equation

Finite volume methods have the following update scheme to approximate equations in conservation law form [] for some quantity q

$$\bar{q}_j^{n+1} = \bar{q}_j^n - \frac{\Delta t}{\Delta x}\left[F_{j+1/2}^n - F_{j-1/2}^n\right].$$

Where the bar denotes that it is the cell average of the quantity q and $F_{j+1/2}^n$ and $F_{j-1/2}^n$ are the approximations to the average fluxes across the cell boundary between the times t^n and t^{n+1}.

In our methods there is some transformation between the nodal value q_j and the cell average $\bar{q}_j$, which will introduce some error factor $\mathcal{M}$. For first and second order methods $\mathcal{M}_1 = \mathcal{M}_2 = 1$, however for higher-order methods $\mathcal{M} \neq 1$.

To calculate the fluxes $F_{j+1/2}^n$ and $F_{j-1/2}^n$ we use Kurganovs method [superscript dropped][]

$$F_{j+\frac{1}{2}} = \frac{a_{j+\frac{1}{2}}^+ f\left(q_{j+\frac{1}{2}}^-\right) - a_{j+\frac{1}{2}}^- f\left(q_{j+\frac{1}{2}}^+\right)}{a_{j+\frac{1}{2}}^+ - a_{j+\frac{1}{2}}^-} + \frac{a_{j+\frac{1}{2}}^+ a_{j+\frac{1}{2}}^-}{a_{j+\frac{1}{2}}^+ - a_{j+\frac{1}{2}}^-}\left[q_{j+\frac{1}{2}}^+ - q_{j+\frac{1}{2}}^-\right]$$

where $a_{j+\frac{1}{2}}^+$ and $a_{j+\frac{1}{2}}^-$ are given by the wave speed bounds [], so that

$$a_{j+1/2}^- = -\sqrt{gH}$$

$$a_{j+1/2}^+ = \sqrt{gH}.$$

Substituting these values into Kurganovs flux approximation we obtain

$$F_{j+\frac{1}{2}} = \frac{f\left(q_{j+\frac{1}{2}}^-\right) + f\left(q_{j+\frac{1}{2}}^+\right)}{2} - \frac{\sqrt{gH}}{2}\left[q_{j+\frac{1}{2}}^+ - q_{j+\frac{1}{2}}^-\right] \tag{5.8}$$

For η our Kurganov approximation to the flux of (5.5a) is then

$$F_{j+\frac{1}{2}}^\eta = \frac{H\upsilon_{j+\frac{1}{2}}^- + H\upsilon_{j+\frac{1}{2}}^+}{2} - \frac{\sqrt{gH}}{2}\left[\eta_{j+\frac{1}{2}}^+ - \eta_{j+\frac{1}{2}}^-\right] \tag{5.9}$$

The missing piece here is the error introduced by reconstruction of the edge values $\upsilon^-_{j+\frac{1}{2}}$, $\upsilon^+_{j+\frac{1}{2}}$, $\eta^-_{j+\frac{1}{2}}$ and $\eta^+_{j+\frac{1}{2}}$ from the cell averages $\bar{\upsilon}_j$ and $\bar{\eta}_j$. Because our quantities are smooth the nonlinear limiters can be neglected so we have for the second-order reconstruction of η

$$\eta^-_{j+\frac{1}{2}} = \bar{\eta}_j + \frac{\bar{\eta}_{j+1} - \bar{\eta}_{j-1}}{4}$$
$$\eta^+_{j+\frac{1}{2}} = \bar{\eta}_{j+1} + \frac{\bar{\eta}_{j+2} - \bar{\eta}_j}{4}.$$

Using (**??**) these equations become

$$\eta^-_{j+\frac{1}{2}} = \mathcal{M}_2\eta_j + \frac{\mathcal{M}_2\eta_j e^{ik\Delta x} - \mathcal{M}_2\eta_j e^{-ik\Delta x}}{4}$$
$$\eta^+_{j+\frac{1}{2}} = \mathcal{M}_2\eta_j e^{ik\Delta x} + \frac{\mathcal{M}_2\eta_j e^{2ik\Delta x} - \mathcal{M}_2\eta_j}{4}.$$

For the second order case $\mathcal{M}_2 = 1$ and these equations can be reduced to

$$\eta^-_{j+\frac{1}{2}} = \left(1 + \frac{i\sin(k\Delta x)}{2}\right)\eta_j \tag{5.10a}$$
$$\eta^+_{j+\frac{1}{2}} = e^{ik\Delta x}\left(1 - \frac{i\sin(k\Delta x)}{2}\right)\eta_j. \tag{5.10b}$$

From these we introduce the second order reconstruction factors $\mathcal{R}^+_2 = e^{ik\Delta x}\left(1 - \frac{i\sin(k\Delta x)}{2}\right)$ and $\mathcal{R}^-_2 = 1 + \frac{i\sin(k\Delta x)}{2}$ for both η and G. So that we have

$$\eta^-_{j+\frac{1}{2}} = \mathcal{R}^-_2\eta_j$$
$$\eta^+_{j+\frac{1}{2}} = \mathcal{R}^+_2\eta_j.$$

In our numerical methods our reconstruction of υ is slightly different as $\upsilon^-_{j+\frac{1}{2}}$ and $\upsilon^+_{j+\frac{1}{2}}$ are equal as we assume υ is continuous. For the second order method we have

$$u^-_{j+1/2} = u^+_{j+1/2} = \frac{u_{j+1} + u_j}{2}$$

Using (**??**) and rearranging gives

$$u^-_{j+1/2} = u^+_{j+1/2} = \frac{e^{ik\Delta x} + 1}{2}u_j. \tag{5.11}$$

We also introduce the second order reconstruction error factor $\mathcal{R}^u_2 = \frac{e^{ik\Delta x}+1}{2}$

We now have all the pieces to substitute into (5.9) which for the second order method results in

$$F^{\eta}_{j+\frac{1}{2}} = \frac{H\mathcal{R}^u_2\upsilon_j + H\mathcal{R}^u_2\upsilon_j}{2} - \frac{\sqrt{gH}}{2}\left[\mathcal{R}^+_2\eta_j - \mathcal{R}^-_2\eta_j\right]$$

Which becomes

$$F^{\eta}_{j+\frac{1}{2}} = H\mathcal{R}^u_2\upsilon_j - \frac{\sqrt{gH}}{2}\left[\mathcal{R}^+_2 - \mathcal{R}^-_2\right]\eta_j$$

We then introduce the factors $\mathcal{F}^{\eta,\upsilon}_2$ and $\mathcal{F}^{\eta,\eta}_2$ so that

$$F^{\eta}_{j+\frac{1}{2}} = \mathcal{F}^{\eta,\upsilon}_2\upsilon_j + \mathcal{F}^{\eta,\eta}_2\eta_j. \tag{5.12}$$

Repeating this process for G using [] and [] we get that

$$F^{G}_{j+\frac{1}{2}} = \frac{gHh^-_{j+\frac{1}{2}} + gHh^+_{j+\frac{1}{2}}}{2} - \frac{\sqrt{gH}}{2}\left[G^+_{j+\frac{1}{2}} - G^-_{j+\frac{1}{2}}\right] \tag{5.13}$$

Using our reconstruction factors this becomes:

$$F^{G}_{j+\frac{1}{2}} = \frac{gH\mathcal{R}^-_2h_j + gH\mathcal{R}^+_2h_j}{2} - \frac{\sqrt{gH}}{2}\left[\mathcal{R}^+_2G_j - \mathcal{R}^-_2G_j\right]$$

which by factoring and using the factor $\mathcal{G}_{FD2}$ becomes

$$F^{G}_{j+\frac{1}{2}} = gH\frac{\mathcal{R}^-_2 + \mathcal{R}^+_2}{2}h_j - \frac{\sqrt{gH}}{2}\left[\mathcal{R}^+_2 - \mathcal{R}^-_2\right]\mathcal{G}_{FD2}\upsilon_j$$

We then introduce the factors $\mathcal{F}^{G,\upsilon}_2$ and $\mathcal{F}^{G,\eta}_2$ so that

$$F^{G}_{j+\frac{1}{2}} = \mathcal{F}^{G,\eta}_2\eta_j + \mathcal{F}^{G,\upsilon}_2\upsilon_j \tag{5.14}$$

By substituting (5.12), (5.14) and $\mathcal{M}_2$ into [] our finite volume method can be written as

$$\mathcal{M}_2\eta^{n+1}_j = \mathcal{M}_2\eta^n_j - \frac{\Delta t}{\Delta x}\left[\left(1 - e^{ik\Delta x}\right)\left(\mathcal{F}^{\eta,\eta}_2h_j + \mathcal{F}^{\eta,\upsilon}_2\upsilon_j\right)\right]$$

$$\mathcal{M}_2G^{n+1}_j = \mathcal{M}_2G^n_j - \frac{\Delta t}{\Delta x}\left[\left(1 - e^{ik\Delta x}\right)\left(\mathcal{F}^{G,\eta}_2\eta_j + \mathcal{F}^{G,\upsilon}_2\upsilon_j\right)\right]$$

Furthermore by transforming the G's into υ's using our second order finite volume factor $\mathcal{G}_{FD2}$ and using $\mathcal{M}_2 = 1$ we obtain

$$\eta^{n+1}_j = \eta^n_j - \frac{\Delta t}{\Delta x}\left[\left(1 - e^{ik\Delta x}\right)\left(\mathcal{F}^{\eta,\eta}_2\eta_j + \mathcal{F}^{\eta,\upsilon}_2\upsilon_j\right)\right]$$

$$\upsilon_j^{n+1} = \upsilon_j^n - \frac{1}{\mathcal{G}_{FD2}} \frac{\Delta t}{\Delta x} \left[\left(1 - e^{ik\Delta x}\right) \left(\mathcal{F}_2^{G,\eta} \eta_j + \mathcal{F}_2^{G,\upsilon} \upsilon_j \right) \right]$$

This can be written in matrix form as

$$\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n+1} = \begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n - \frac{\left(1 - e^{ik\Delta x}\right) \Delta t}{\Delta x} \begin{bmatrix} \mathcal{F}_2^{\eta,\eta} & \mathcal{F}_2^{\eta,\upsilon} \\ \frac{1}{\mathcal{G}} \mathcal{F}_2^{\upsilon,\eta} & \frac{1}{\mathcal{G}} \mathcal{F}_2^{\upsilon,\upsilon} \end{bmatrix} \begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n$$

Introducing

$$\boldsymbol{F}_2 = \frac{\left(1 - e^{ik\Delta x}\right)}{\Delta x} \begin{bmatrix} \mathcal{F}_2^{\eta,\eta} & \mathcal{F}_2^{\eta,\upsilon} \\ \frac{1}{\mathcal{G}} \mathcal{F}_2^{\upsilon,\eta} & \frac{1}{\mathcal{G}} \mathcal{F}_2^{\upsilon,\upsilon} \end{bmatrix}$$

this becomes

$$\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n+1} = (\boldsymbol{I} - \Delta t \boldsymbol{F}_2) \begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n$$

5.1.3 Runge-Kutta Time Stepping

The above analysis does not include the Runge-Kutta steps that make allow our schemes to be higher order in time. However, extending this analysis to Runge-Kutta steps is not difficult now that our method is in the form (5.1.2). For second order time stepping the Runge Kutta steps are then

$$\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{1} = (\boldsymbol{I} - \Delta t \boldsymbol{F}_2) \begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n \tag{5.15a}$$

$$\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{2} = (\boldsymbol{I} - \Delta t \boldsymbol{F}_2) \begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^1 \tag{5.15b}$$

$$\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n+1} = \frac{1}{2} \left(\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n + \begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^2 \right) \tag{5.15c}$$

Substituting (5.15a) and (5.15b) into (5.15c) gives

$$\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n+1} = \frac{1}{2} \left(\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n + (\boldsymbol{I} - \Delta t \boldsymbol{F}_2)^2 \begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n \right)$$

Expanding this we get

$$\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n+1} = \frac{1}{2} \left(2\boldsymbol{I} - 2\Delta t \boldsymbol{F}_2 + \Delta t^2 \boldsymbol{F}_2^2 \right) \begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n$$

lets say we have an eigenvalue decomposition $\boldsymbol{F}_2 = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$ then this can be rewritten as

$$\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n+1} = \frac{1}{2}\left(2\boldsymbol{I} - 2\Delta t\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1} + \Delta t^2\boldsymbol{P}\boldsymbol{\Lambda}^2\boldsymbol{P}^{-1}\right)\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n}$$

Multiplying by $\boldsymbol{P}^{-1}$ on the left and rearranging this get that

$$\boldsymbol{P}^{-1}\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n+1} = \frac{1}{2}\left(2 - 2\Delta t\boldsymbol{\Lambda} + \Delta t^2\boldsymbol{\Lambda}^2\right)\boldsymbol{P}^{-1}\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n}$$

Since η and υ are Fourier modes we have

$$e^{i\omega\Delta t}\left(\boldsymbol{P}^{-1}\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n}\right) = \left(1 - 1\Delta t\boldsymbol{\Lambda} + \frac{1}{2}\Delta t^2\boldsymbol{\Lambda}^2\right)\left(\boldsymbol{P}^{-1}\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n}\right)$$

Since $\boldsymbol{\Lambda}$ is a diagonal matrix of the eigenvalues λ_1 and λ_2 we have that

$$\begin{aligned} e^{i\omega\Delta t} &= 1 + \frac{1}{2}\Delta t^2\lambda_1^2 - \Delta t\lambda_1, \\ e^{i\omega\Delta t} &= 1 + \frac{1}{2}\Delta t^2\lambda_2^2 - \Delta t\lambda_2 \end{aligned}$$

So that the dispersion relation for the second order finite difference finite volume method is

$$\omega = \frac{1}{i\Delta t}\ln\left(1 + \frac{1}{2}\Delta t^2\lambda_1^2 - \Delta t\lambda_1\right), \tag{5.16}$$

$$\omega = \frac{1}{i\Delta t}\ln\left(1 + \frac{1}{2}\Delta t^2\lambda_2^2 - \Delta t\lambda_2\right), \tag{5.17}$$

Comparing this with the dispersion relation (2.3) of the Serre equations we can then determine the error in dispersion caused by the particular method. We perform this computationally by finding the eigenvalues of $\boldsymbol{F}$, substituting them into (5.16) and comparing that to the dispersion relation of the Serre equations for a particular H and k value.

5.1.4 Results

5.2 Neumann Stability

We again begin from the linearised Serre equations (5.1) and as for the hybrid finite volume methods we will only show the working for one example as the process is the same in both cases. For the Neumann Stability our example will be the naive second-order finite difference method $\mathcal{D}$ (4.9). Again we being by replacing both η and υ by Fourier nodes (5.6).

Because our approximations to derivatives is consistent for $\mathcal{D}$ we will provide all the factors for the second order centred finite difference approximations to derivatives of some quantity q generated by making use of (5.7).

$$\left(\frac{\partial q}{\partial x}\right)_j^n = \frac{q_{j+1}^n - q_{j-1}^n}{2\Delta x} = \frac{i\sin(k\Delta x)}{\Delta x} q_j^n \tag{5.18}$$

$$\left(\frac{\partial^2 q}{\partial x^2}\right)_j^n = \frac{q_{j+1}^n - 2q_j^n + q_{j-1}^n}{\Delta x^2} = \frac{2\cos(k\Delta x) - 2}{\Delta x^2} q_j^n \tag{5.19}$$

$$\left(\frac{\partial^3 q}{\partial x^3}\right)_j^n = \frac{q_{j+2}^n - 2q_{j+1}^n + 2q_{j-1}^n - q_{j-2}^n}{2\Delta x^3} = -4i\sin(k\Delta x)\frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^3} q_j^n \tag{5.20}$$

$$\frac{\partial \eta}{\partial t} + H\frac{\partial \upsilon}{\partial x} + U\frac{\partial \eta}{\partial x} = 0 \tag{5.21a}$$

$$H\frac{\partial \upsilon}{\partial t} + gH\frac{\partial \eta}{\partial x} + UH\frac{\partial \upsilon}{\partial x} - \frac{H^3}{3}\left(U\frac{\partial^3 \upsilon}{\partial x^3} + \frac{\partial^3 \upsilon}{\partial x^3 \partial t}\right) = 0 \tag{5.21b}$$

The factors we get for the temporal derivatives are very similar to this. The numerical method $\mathcal{D}$ is just attained from replacing all the derivatives in (5.1) with the approximations in (5.20). For the linearised equations the update formulas of $\mathcal{D}$ become

$$\eta_j^{n+1} = \eta_j^{n-1} - \Delta t\left(U\frac{\eta_{j+1}^n - \eta_{j-1}^n}{\Delta x} + H\frac{\upsilon_{j+1}^n - \upsilon_{j-1}^n}{\Delta x}\right). \tag{5.22a}$$

$$\upsilon_j^{n+1} - \frac{H^2}{3}\frac{\upsilon_{j+1}^{n+1} - 2\upsilon_j^{n+1} + \upsilon_{j-1}^{n+1}}{\Delta x^2} = \upsilon_j^{n-1} - \frac{H^2}{3}\frac{\upsilon_{j+1}^{n-1} - 2\upsilon_j^{n-1} + \upsilon_{j-1}^{n-1}}{\Delta x^2} + \Delta t\left(-g\frac{\eta_{j+1}^n - \eta_{j-1}^n}{\Delta x} - U\frac{\upsilon_{j+1}^n - \upsilon_{j-1}^n}{\Delta x} + \frac{H^2}{3}\left(U\frac{-\upsilon_{j-2}^n + 2\upsilon_{j-1}^n - 2\upsilon_{j+1}^n + \upsilon_{j+2}^n}{\Delta x^3}\right)\right) \tag{5.22b}$$

Since we have assumed that η and υ are fourier nodes, we can just replace the finite difference approximations with the appropriate factors from (5.20). After some rearranging we get that

$$\eta_j^{n+1} = \eta_j^{n-1} - \Delta t\left(U\frac{i\sin(k\Delta x)}{\Delta x}\eta_j^n + H\frac{i\sin(k\Delta x)}{\Delta x}\upsilon_j^n\right), \tag{5.23a}$$

$$\upsilon_j^{n+1} = \upsilon_j^{n-1} - \frac{3\Delta x^2\Delta t}{3\Delta x^2 - 2H^2(\cos(k\Delta x) - 1)}\left(g\frac{i\sin(k\Delta x)}{\Delta x}\right)\eta_j^n + U\frac{i\Delta t\sin(k\Delta x)}{\Delta x}\upsilon_j^n \tag{5.23b}$$

By setting

$$A_{0,0} = -\frac{2i\Delta t}{\Delta x}U\sin(k\Delta x) \tag{5.24}$$

$$A_{0,1} = -\frac{2i\Delta t}{\Delta x}H\sin(k\Delta x) \tag{5.25}$$

$$A_{1,0} = -\frac{6gi\Delta x\Delta t}{3\Delta x^2 - 2H^2(\cos(k\Delta x) - 1)}\sin(k\Delta x) \tag{5.26}$$

$$A_{1,1} = \frac{2i\Delta t}{\Delta x}U\sin(k\Delta x) \tag{5.27}$$

$$\begin{bmatrix}\eta_j^{n+1}\\ \upsilon_j^{n+1}\\ \eta_j^n\\ \upsilon_j^n\end{bmatrix} = \begin{bmatrix}A_{0,0} & A_{0,1} & 1 & 0\\ A_{1,0} & A_{1,1} & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\eta_j^n\\ \upsilon_j^n\\ \eta_j^{n-1}\\ \upsilon_j^{n-1}\end{bmatrix} \tag{5.28}$$

This matrix is the growth matrix and if its spectral radius is less than 1 then $\mathcal{D}$ is stable.

For $\mathcal{W}$ after following through with same process we get that

$$
\begin{aligned}
B_{0,0} &= 1 - \frac{\Delta t}{\Delta x} A_{1,0} H \frac{i \sin(k\Delta x)}{2} \\
&- \frac{\Delta t}{\Delta x} U \left((i \sin(k\Delta x)) - \frac{\Delta t}{\Delta x} U (\cos(ik\Delta x) - 1) \right) \\
B_{0,1} &= -\frac{\Delta t}{\Delta x} \left[H \frac{i \sin(k\Delta x)}{2} A_{1,1} - U \left(\frac{\Delta t}{\Delta x} H (\cos(ik\Delta x) - 1) \right) \right] \\
B_{0,4} &= -\frac{\Delta t}{\Delta x} H \frac{i \sin(k\Delta x)}{2} \\
B_{1,0} &= A_{1,0} \\
B_{1,1} &= A_{1,1}
\end{aligned}
$$

$$
\begin{bmatrix} h_j^{n+1} \\ u_j^{n+1} \\ h_j^n \\ u_j^n \end{bmatrix} = \begin{bmatrix} B_{0,0} & B_{0,1} & 0 & B_{0,4} \\ B_{1,0} & B_{1,1} & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_j^n \\ u_j^n \\ h_j^{n-1} \\ u_j^{n-1} \end{bmatrix}
$$

Again if this matrix has a spectral radius less than 1 then $\mathcal{W}$ is stable.

Chapter 6

Validation and Comparison

6.1 Analytic Validation

6.2 Experimental Validation

Chapter 7

The Dam-Break problem

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