1 Elliptic Equation

The elliptic equation is

$$G = uh - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

with bed terms it is

$$G = uh + u\left(\frac{\partial}{\partial x}\left(\frac{h^2}{2}b_x\right) + hb_x^2\right) - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

2 Finite Element

$$G = uh + u\left(\frac{\partial}{\partial x}\left(\frac{h^2}{2}b_x\right) + hb_x^2\right) - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = uhv + uv\left(\frac{\partial}{\partial x}\left(\frac{h^2}{2}b_x\right) + hb_x^2\right) - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)v$$

and then we integrate over the entire domain to get

$$\int_\Omega Gv\ dx = \int_\Omega uhv\ dx + \int_\Omega u\left(\frac{\partial}{\partial x}\left(\frac{h^2}{2}b_x\right) + hb_x^2\right)v\ dx - \int_\Omega \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)v\ dx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_\Omega Gvdx = \int_\Omega uhvdx + \int_\Omega \frac{h^3}{3}u_x v_x dx + \int_\Omega u\left(\frac{\partial}{\partial x}\left(\frac{h^2}{2}b_x\right) + hb_x^2\right)v\ dx$$

$$\int_\Omega Gvdx = \int_\Omega uhvdx + \int_\Omega \frac{h^3}{3}u_x v_x dx + \int_\Omega \frac{\partial}{\partial x}\left(\frac{h^2}{2}b_x\right)uv\ dx + \int_\Omega uhb_x^2 v\ dx$$

$$\int_{\Omega} Gv dx = \int_{\Omega} uhv dx + \int_{\Omega} \frac{h^3}{3} u_x v_x dx - \int_{\Omega} \frac{h^2}{2} b_x \frac{\partial}{\partial x} \left(uv\right)\ dx + \int_{\Omega} uhb_x^2 v\ dx$$

$$\int_{\Omega} Gv dx = \int_{\Omega} uhv dx + \int_{\Omega} \frac{h^3}{3} u_x v_x dx - \int_{\Omega} \frac{h^2}{2} b_x u_x v\ dx - \int_{\Omega} \frac{h^2}{2} b_x u v_x\ dx + \int_{\Omega} uhb_x^2 v\ dx$$

So importantly we just require that b_x is well behaved, and in particular it must be continuous

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gv dx = \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} uhv dx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^3}{3} u_x v_x dx$$
$$- \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^2}{2} b_x u_x v dx - \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^2}{2} b_x u v_x dx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} uhb_x^2 v dx$$

3 P1 FEM

We are going to cordinate tranform from x space the interval $[x_{j-1/2}, x_{j+1/2}, x_{j+3/2}]$ to the ξ space interval $[-1, 0, 1]$. To accomplish this we have the following relation

$$x = \xi \Delta x + x_{j+1/2}$$

Taking the derivatives we see
$dx = d\xi \Delta x$, $\frac{dx}{d\xi} = \Delta x$, $\frac{d\xi}{dx} = \frac{1}{\Delta x}$.

We can describe the basis functions in the ξ space

$$\phi_{j+1/2} = \begin{cases} 1+\xi & \xi < 0 \\ 1-\xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$\phi_{j-1/2} = \begin{cases} -\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$\phi_{j+3/2} = \begin{cases} \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

While the descriptions for w's is

$$w^{+}_{j+1/2} = \begin{cases} 1-\xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$w^{-}_{j+1/2} = \begin{cases} 1+\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$w^{+}_{j-1/2} = \begin{cases} -\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$w^{-}_{j+3/2} = \begin{cases} \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

We now replace our functions by our approximations to them

$$G \approx G' = \sum_j G_{j+1/2} w_{j+1/2}$$

$$u \approx u' = \sum_j u_{j+1/2} \phi_{j+1/2}$$

$$b \approx b' = \sum_j b_{j+1/2} \phi_{j+1/2}$$

$$h \approx h' = \sum_j h_{j+1/2} w_{j+1/2}$$

For all $\phi_{j+1/2}$. For this analysis we choose a particular basis function $\phi_{j+1/2}$ and we look at all the integrals. Now we have already done the integrals with no bed term, so we focus on the bed terms now

which are

$$-\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^2}{2} b_x u_x v dx - \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^2}{2} b_x u v_x dx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} u h b_x^2 v dx$$

Lets begin with the rightmost, first we change variable from x to ξ

$$\int_{x_{j-1/2}}^{x_{j+3/2}} \frac{(h'(x))^2}{2} b_x(x) u_x(x) \phi_{j+1/2}(x) dx = \int_{-1}^{1} \frac{(h'(\xi))^2}{2} b'(\xi)_\xi \frac{d\xi}{dx} u'(\xi)_\xi \frac{d\xi}{dx} \phi_{j+1/2}(\xi) \frac{dx}{d\xi} dx$$

$$= \frac{1}{\Delta x} \int_{-1}^{1} \frac{(h')^2}{2} b'_\xi u'_\xi \phi_{j+1/2} d\xi$$

Now we expand to the P1 approximations to include only where $\phi_{j+1/2}$ is non zero, also we use $'$ to denote derivatives

which is

$$h' = h^+_{j-1/2} w^+_{j-1/2} + h^-_{j+1/2} w^-_{j+1/2} + h^+_{j+1/2} w^+_{j+1/2} + h^-_{j+3/2} w^-_{j+3/2}$$

$$b'_\xi = b'_{j-1/2}\phi'_{j-1/2} + b'_{j+1/2}\phi'_{j+1/2} + b'_{j+3/2}\phi'_{j+3/2}$$

$$u'_\xi = u'_{j-1/2}\phi'_{j-1/2} + u'_{j+1/2}\phi'_{j+1/2} + u'_{j+3/2}\phi'_{j+3/2}$$

$$\begin{aligned} = \frac{1}{2\Delta x} \int_{-1}^{1} & \left[h^+_{j-1/2} w^+_{j-1/2} + h^-_{j+1/2} w^-_{j+1/2} + h^+_{j+1/2} w^+_{j+1/2} + h^-_{j+3/2} w^-_{j+3/2}\right]^2 \times \\ & \left[b'_{j-1/2}\phi'_{j-1/2} + b'_{j+1/2}\phi'_{j+1/2} + b'_{j+3/2}\phi'_{j+3/2}\right] \times \\ & \left[u'_{j-1/2}\phi'_{j-1/2} + u'_{j+1/2}\phi'_{j+1/2} + u'_{j+3/2}\phi'_{j+3/2}\right] \phi_{j+1/2} d\xi \end{aligned} \quad (8)$$

Note that we can square $a+b+c+d$ simply because $ac = ad = bc = bd = 0$ so we get that

$$[a+b+c+d]^2 = (a+b)^2 + (c+d)^2$$

therefore

$$\begin{aligned} = \frac{1}{2\Delta x} \int_{-1}^{1} & \Bigg[\left(h^+_{j-1/2} w^+_{j-1/2}\right)^2 + 2h^+_{j-1/2} w^+_{j-1/2} h^-_{j+1/2} w^-_{j+1/2} + \left(h^-_{j+1/2} w^-_{j+1/2}\right)^2 \\ & + \left(h^+_{j+1/2} w^+_{j+1/2}\right)^2 + 2h^+_{j+1/2} w^+_{j+1/2} h^-_{j+3/2} w^-_{j+3/2} + \left(h^-_{j+3/2} w^-_{j+3/2}\right)^2 \Bigg] \times \\ & \Big[b'_{j-1/2}\phi'_{j-1/2} u'_{j-1/2}\phi'_{j-1/2} + b'_{j-1/2}\phi'_{j-1/2} u'_{j+1/2}\phi'_{j+1/2} + b'_{j-1/2}\phi'_{j-1/2} u'_{j+3/2}\phi'_{j+3/2} \\ & + b'_{j+1/2}\phi'_{j+1/2} u'_{j-1/2}\phi'_{j-1/2} + b'_{j+1/2}\phi'_{j+1/2} u'_{j+1/2}\phi'_{j+1/2} + b'_{j+1/2}\phi'_{j+1/2} u'_{j+3/2}\phi'_{j+3/2} \\ & + b'_{j+3/2}\phi'_{j+3/2} u'_{j-1/2}\phi'_{j-1/2} + b'_{j+3/2}\phi'_{j+3/2} u'_{j+1/2}\phi'_{j+1/2} + b'_{j+3/2}\phi'_{j+3/2} u'_{j+3/2}\phi'_{j+3/2} \Big] \phi_{j+1/2} d\xi \end{aligned} \quad (9)$$

$$
\begin{aligned}
&= \frac{1}{2\Delta x}\int_{-1}^{1}\Bigg[\left(h^+_{j-1/2}w^+_{j-1/2}\right)^2 + 2h^+_{j-1/2}h^-_{j+1/2}w^+_{j-1/2}w^-_{j+1/2} + \left(h^-_{j+1/2}w^-_{j+1/2}\right)^2 \\
&\quad + \left(h^+_{j+1/2}w^+_{j+1/2}\right)^2 + 2h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2} + \left(h^-_{j+3/2}w^-_{j+3/2}\right)^2\Bigg]\times \\
&\Big[b'_{j-1/2}u'_{j-1/2}\phi'_{j-1/2}\phi'_{j-1/2} + b'_{j-1/2}u'_{j+1/2}\phi'_{j-1/2}\phi'_{j+1/2} + b'_{j-1/2}u'_{j+3/2}\phi'_{j-1/2}\phi'_{j+3/2} \\
&+ b'_{j+1/2}u'_{j-1/2}\phi'_{j+1/2}\phi'_{j-1/2} + b'_{j+1/2}u'_{j+1/2}\phi'_{j+1/2}\phi'_{j+1/2} + b'_{j+1/2}u'_{j+3/2}\phi'_{j+1/2}\phi'_{j+3/2} \\
&+ b'_{j+3/2}u'_{j-1/2}\phi'_{j+3/2}\phi'_{j-1/2} + b'_{j+3/2}u'_{j+1/2}\phi'_{j+3/2}\phi'_{j+1/2} + b'_{j+3/2}\phi'_{j+3/2}\phi'_{j+3/2}u'_{j+3/2}\Big]\phi_{j+1/2}d\xi
\end{aligned} \tag{10}
$$

but $\phi'_{j+3/2}\phi'_{j-1/2}$ is zero everywhere

$$
\begin{aligned}
&= \frac{1}{2\Delta x}\int_{-1}^{1}\Bigg[\left(h^+_{j-1/2}w^+_{j-1/2}\right)^2 + 2h^+_{j-1/2}h^-_{j+1/2}w^+_{j-1/2}w^-_{j+1/2} + \left(h^-_{j+1/2}w^-_{j+1/2}\right)^2 \\
&\quad + \left(h^+_{j+1/2}w^+_{j+1/2}\right)^2 + 2h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2} + \left(h^-_{j+3/2}w^-_{j+3/2}\right)^2\Bigg]\times \\
&\Big[b'_{j-1/2}u'_{j-1/2}\phi'_{j-1/2}\phi'_{j-1/2} + b'_{j-1/2}u'_{j+1/2}\phi'_{j-1/2}\phi'_{j+1/2} + b'_{j+1/2}u'_{j-1/2}\phi'_{j+1/2}\phi'_{j-1/2} \\
&\qquad + b'_{j+1/2}u'_{j+1/2}\phi'_{j+1/2}\phi'_{j+1/2} \\
&+ b'_{j+1/2}u'_{j+3/2}\phi'_{j+1/2}\phi'_{j+3/2} + b'_{j+3/2}u'_{j+1/2}\phi'_{j+3/2}\phi'_{j+1/2} + b'_{j+3/2}\phi'_{j+3/2}\phi'_{j+3/2}u'_{j+3/2}\Big]\phi_{j+1/2}d\xi
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
&= \frac{1}{2\Delta x}\int_{-1}^{1} \\
&\Bigg(\left[h^+_{j-1/2}h^+_{j-1/2}w^+_{j-1/2}w^+_{j-1/2}+2h^+_{j-1/2}h^-_{j+1/2}w^+_{j-1/2}w^-_{j+1/2}+h^-_{j+1/2}h^-_{j+1/2}w^-_{j+1/2}w^-_{j+1/2}\right]\times \\
&[b'_{j-1/2}u'_{j-1/2}\phi'_{j-1/2}\phi'_{j-1/2}+b'_{j-1/2}u'_{j+1/2}\phi'_{j-1/2}\phi'_{j+1/2}+b'_{j+1/2}u'_{j-1/2}\phi'_{j+1/2}\phi'_{j-1/2} \\
&\quad + b'_{j+1/2}u'_{j+1/2}\phi'_{j+1/2}\phi'_{j+1/2}] \\
&+\left[h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}+2h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2}+h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\right]\times \\
&\quad [b'_{j+1/2}u'_{j+1/2}\phi'_{j+1/2}\phi'_{j+1/2} \\
&+b'_{j+1/2}u'_{j+3/2}\phi'_{j+1/2}\phi'_{j+3/2}+b'_{j+3/2}u'_{j+1/2}\phi'_{j+3/2}\phi'_{j+1/2}+b'_{j+3/2}\phi'_{j+3/2}\phi'_{j+3/2}u'_{j+3/2}]\Bigg)\phi_{j+1/2}d\xi
\end{aligned}
\tag{12}
$$

breaking this down we have two integrals over seperate domains in particular the first term is only non zero on $\xi \in [-1, 0]$ and the second is only non-zero on $\xi \in [0, 1]$

For the first term when $\xi \in [-1, 0]$ then

$$\phi'_{j-1/2}\phi'_{j-1/2} = \phi'_{j+1/2}\phi'_{j+1/2} = 1$$

and

$$\phi'_{j-1/2}\phi'_{j+1/2} = -1$$

So we can simplify it to be

$$
\begin{aligned}
&\left[h^+_{j-1/2}h^+_{j-1/2}w^+_{j-1/2}w^+_{j-1/2}+2h^+_{j-1/2}h^-_{j+1/2}w^+_{j-1/2}w^-_{j+1/2}+h^-_{j+1/2}h^-_{j+1/2}w^-_{j+1/2}w^-_{j+1/2}\right]\times \\
&\quad [b'_{j-1/2}u'_{j-1/2} - b'_{j-1/2}u'_{j+1/2} - b'_{j+1/2}u'_{j-1/2} + b'_{j+1/2}u'_{j+1/2}]
\end{aligned}
$$

Thus

$$= \Big[b'_{j-1/2} u'_{j-1/2} h^+_{j-1/2} h^+_{j-1/2} w^+_{j-1/2} w^+_{j-1/2} + 2 b'_{j-1/2} u'_{j-1/2} h^+_{j-1/2} h^-_{j+1/2} w^+_{j-1/2} w^-_{j+1/2} + b'_{j-1/2} u'_{j-1/2} h^-_{j+1/2} h^-_{j+1/2} w^-_{j+1/2} w^-_{j+1/2} \Big]$$

$$- \Big[b'_{j-1/2} u'_{j+1/2} h^+_{j-1/2} h^+_{j-1/2} w^+_{j-1/2} w^+_{j-1/2} + 2 b'_{j-1/2} u'_{j+1/2} h^+_{j-1/2} h^-_{j+1/2} w^+_{j-1/2} w^-_{j+1/2} + b'_{j-1/2} u'_{j+1/2} h^-_{j+1/2} h^-_{j+1/2} w^-_{j+1/2} w^-_{j+1/2} \Big]$$

$$- \Big[b'_{j+1/2} u'_{j-1/2} h^+_{j-1/2} h^+_{j-1/2} w^+_{j-1/2} w^+_{j-1/2} + 2 b'_{j+1/2} u'_{j-1/2} h^+_{j-1/2} h^-_{j+1/2} w^+_{j-1/2} w^-_{j+1/2} + b'_{j+1/2} u'_{j-1/2} h^-_{j+1/2} h^-_{j+1/2} w^-_{j+1/2} w^-_{j+1/2} \Big]$$

$$+ \Big[b'_{j+1/2} u'_{j+1/2} h^+_{j-1/2} h^+_{j-1/2} w^+_{j-1/2} w^+_{j-1/2} + 2 b'_{j+1/2} u'_{j+1/2} h^+_{j-1/2} h^-_{j+1/2} w^+_{j-1/2} w^-_{j+1/2} + b'_{j+1/2} u'_{j+1/2} h^-_{j+1/2} h^-_{j+1/2} w^-_{j+1/2} w^-_{j+1/2} \Big]$$

for the second term when $\xi \in [0, 0]$ then

$$\phi'_{j+1/2} \phi'_{j+1/2} = \phi'_{j+3/2} \phi'_{j+3/2} = 1$$

and

$$\phi'_{j+1/2} \phi'_{j+3/2} = -1$$

$$\Big[h^+_{j+1/2} h^+_{j+1/2} w^+_{j+1/2} w^+_{j+1/2} + 2 h^+_{j+1/2} h^-_{j+3/2} w^+_{j+1/2} w^-_{j+3/2} + h^-_{j+3/2} h^-_{j+3/2} w^-_{j+3/2} w^-_{j+3/2} \Big] \times [b'_{j+1/2} u'_{j+1/2} - b'_{j+1/2} u'_{j+3/2} - b'_{j+3/2} u'_{j+1/2} + b'_{j+3/2} u'_{j+3/2}]$$

$$
\begin{aligned}
&\Big[b'_{j+1/2}u'_{j+1/2}h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}+2b'_{j+1/2}u'_{j+1/2}h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2}\\
&\qquad + b'_{j+1/2}u'_{j+1/2}h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\Big]\\
-&\Big[b'_{j+1/2}u'_{j+3/2}h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}+2b'_{j+1/2}u'_{j+3/2}h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2}\\
&\qquad + b'_{j+1/2}u'_{j+3/2}h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\Big]\\
-&\Big[b'_{j+3/2}u'_{j+1/2}h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}+2b'_{j+3/2}u'_{j+1/2}h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2}\\
&\qquad + b'_{j+3/2}u'_{j+1/2}h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\Big]\\
+&\Big[b'_{j+3/2}u'_{j+3/2}h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}+2b'_{j+3/2}u'_{j+3/2}h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2}\\
&\qquad + b'_{j+3/2}u'_{j+3/2}h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\Big]
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\frac{1}{2\Delta x}\int_{-1}^{1}(h')^2 b'_\xi u'_\xi \phi_{j+1/2}d\xi = \frac{1}{2\Delta x}\times \\
&\Bigg(\int_{-1}^{0}\Bigg\{\Bigg[b'_{j-1/2}u'_{j-1/2}h^+_{j-1/2}h^+_{j-1/2}w^+_{j-1/2}w^+_{j-1/2}+2b'_{j-1/2}u'_{j-1/2}h^+_{j-1/2}h^-_{j+1/2}w^+_{j-1/2}w^-_{j+1/2} \\
&\qquad + b'_{j-1/2}u'_{j-1/2}h^-_{j+1/2}h^-_{j+1/2}w^-_{j+1/2}w^-_{j+1/2}\Bigg] \\
&-\Bigg[b'_{j-1/2}u'_{j+1/2}h^+_{j-1/2}h^+_{j-1/2}w^+_{j-1/2}w^+_{j-1/2}+2b'_{j-1/2}u'_{j+1/2}h^+_{j-1/2}h^-_{j+1/2}w^+_{j-1/2}w^-_{j+1/2} \\
&\qquad + b'_{j-1/2}u'_{j+1/2}h^-_{j+1/2}h^-_{j+1/2}w^-_{j+1/2}w^-_{j+1/2}\Bigg] \\
&-\Bigg[b'_{j+1/2}u'_{j-1/2}h^+_{j-1/2}h^+_{j-1/2}w^+_{j-1/2}w^+_{j-1/2}+2b'_{j+1/2}u'_{j-1/2}h^+_{j-1/2}h^-_{j+1/2}w^+_{j-1/2}w^-_{j+1/2} \\
&\qquad + b'_{j+1/2}u'_{j-1/2}h^-_{j+1/2}h^-_{j+1/2}w^-_{j+1/2}w^-_{j+1/2}\Bigg] \\
&+\Bigg[b'_{j+1/2}u'_{j+1/2}h^+_{j-1/2}h^+_{j-1/2}w^+_{j-1/2}w^+_{j-1/2}+2b'_{j+1/2}u'_{j+1/2}h^+_{j-1/2}h^-_{j+1/2}w^+_{j-1/2}w^-_{j+1/2} \\
&\qquad + b'_{j+1/2}u'_{j+1/2}h^-_{j+1/2}h^-_{j+1/2}w^-_{j+1/2}w^-_{j+1/2}\Bigg]\Bigg\}\phi_{j+1/2}d\xi \\
&+\int_{0}^{1}\Bigg\{\Bigg[b'_{j+1/2}u'_{j+1/2}h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}+2b'_{j+1/2}u'_{j+1/2}h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2} \\
&\qquad + b'_{j+1/2}u'_{j+1/2}h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\Bigg] \\
&-\Bigg[b'_{j+1/2}u'_{j+3/2}h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}+2b'_{j+1/2}u'_{j+3/2}h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2} \\
&\qquad + b'_{j+1/2}u'_{j+3/2}h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\Bigg] \\
&-\Bigg[b'_{j+3/2}u'_{j+1/2}h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}+2b'_{j+3/2}u'_{j+1/2}h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2} \\
&\qquad + b'_{j+3/2}u'_{j+1/2}h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\Bigg] \\
&+\Bigg[b'_{j+3/2}u'_{j+3/2}h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}+2b'_{j+3/2}u'_{j+3/2}h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2} \\
&\qquad + b'_{j+3/2}u'_{j+3/2}h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\Bigg]\Bigg\}\phi_{j+1/2}d\xi\Bigg) \qquad (13)
\end{aligned}
$$

Only the basis functions are indeed functions so we just compute them

$$\int_{-1}^{0} w^+_{j-1/2} w^+_{j-1/2} \phi_{j+1/2} = \int_{-1}^{0} (\xi)(\xi)(\xi+1) = \frac{1}{12}$$
$$\int_{-1}^{0} w^+_{j-1/2} w^-_{j+1/2} \phi_{j+1/2} = \int_{-1}^{0} (\xi)(\xi+1)(\xi+1) = -\frac{1}{12}$$
$$\int_{-1}^{0} w^-_{j+1/2} w^-_{j+1/2} \phi_{j+1/2} = \int_{-1}^{0} (\xi+1)(\xi+1)(\xi+1) = \frac{1}{4}$$

$$\int_{0}^{1} w^+_{j+1/2} w^+_{j+1/2} \phi_{j+1/2} d\xi = \int_{0}^{1} (\xi-1)(\xi-1)(\xi-1) d\xi = -\frac{1}{4}$$
$$\int_{0}^{1} w^-_{j+3/2} w^+_{j+1/2} \phi_{j+1/2} d\xi = \int_{0}^{1} (\xi)(\xi-1)(\xi-1) d\xi = \frac{1}{12}$$
$$\int_{0}^{1} w^-_{j+3/2} w^-_{j+3/2} \phi_{j+1/2} d\xi = \int_{0}^{1} (\xi)(\xi)(\xi-1) d\xi = -\frac{1}{12}$$

$$
\begin{aligned}
&\frac{1}{2\Delta x}\int_{-1}^{1}(h')^2 b'_\xi u'_\xi \phi_{j+1/2} d\xi = \frac{1}{24\Delta x}\times\\
&\Bigg(\Big[b'_{j-1/2}u'_{j-1/2}h^+_{j-1/2}h^+_{j-1/2} - 2b'_{j-1/2}u'_{j-1/2}h^+_{j-1/2}h^-_{j+1/2} + 3b'_{j-1/2}u'_{j-1/2}h^-_{j+1/2}h^-_{j+1/2}\Big]\\
&-\Big[b'_{j-1/2}u'_{j+1/2}h^+_{j-1/2}h^+_{j-1/2} - 2b'_{j-1/2}u'_{j+1/2}h^+_{j-1/2}h^-_{j+1/2} + 3b'_{j-1/2}u'_{j+1/2}h^-_{j+1/2}h^-_{j+1/2}\Big]\\
&-\Big[b'_{j+1/2}u'_{j-1/2}h^+_{j-1/2}h^+_{j-1/2} - 2b'_{j+1/2}u'_{j-1/2}h^+_{j-1/2}h^-_{j+1/2} + 3b'_{j+1/2}u'_{j-1/2}h^-_{j+1/2}h^-_{j+1/2}\Big]\\
&+\Big[b'_{j+1/2}u'_{j+1/2}h^+_{j-1/2}h^+_{j-1/2} - 2b'_{j+1/2}u'_{j+1/2}h^+_{j-1/2}h^-_{j+1/2} + 3b'_{j+1/2}u'_{j+1/2}h^-_{j+1/2}h^-_{j+1/2}\Big]\\
&+\Big[-3b'_{j+1/2}u'_{j+1/2}h^+_{j+1/2}h^+_{j+1/2} + 2b'_{j+1/2}u'_{j+1/2}h^+_{j+1/2}h^-_{j+3/2} - b'_{j+1/2}u'_{j+1/2}h^-_{j+3/2}h^-_{j+3/2}\Big]\\
&-\Big[-3b'_{j+1/2}u'_{j+3/2}h^+_{j+1/2}h^+_{j+1/2} + 2b'_{j+1/2}u'_{j+3/2}h^+_{j+1/2}h^-_{j+3/2} - b'_{j+1/2}u'_{j+3/2}h^-_{j+3/2}h^-_{j+3/2}\Big]\\
&-\Big[-3b'_{j+3/2}u'_{j+1/2}h^+_{j+1/2}h^+_{j+1/2} + 2b'_{j+3/2}u'_{j+1/2}h^+_{j+1/2}h^-_{j+3/2} - b'_{j+3/2}u'_{j+1/2}h^-_{j+3/2}h^-_{j+3/2}\Big]\\
&+\Big[-3b'_{j+3/2}u'_{j+3/2}h^+_{j+1/2}h^+_{j+1/2} + 2b'_{j+3/2}u'_{j+3/2}h^+_{j+1/2}h^-_{j+3/2} - b'_{j+3/2}u'_{j+3/2}h^-_{j+3/2}h^-_{j+3/2}\Big]\Bigg)
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&\frac{1}{2\Delta x}\int_{-1}^{1}(h')^2 b'_\xi u'_\xi \phi_{j+1/2} d\xi = \frac{1}{24\Delta x}\times \\
&\Bigg(\Bigg[b'_{j-1/2}u'_{j-1/2}h^+_{j-1/2}h^+_{j-1/2} - 2b'_{j-1/2}u'_{j-1/2}h^+_{j-1/2}h^-_{j+1/2} + 3b'_{j-1/2}u'_{j-1/2}h^-_{j+1/2}h^-_{j+1/2}\Bigg] \\
&-\Bigg[b'_{j-1/2}u'_{j+1/2}h^+_{j-1/2}h^+_{j-1/2} - 2b'_{j-1/2}u'_{j+1/2}h^+_{j-1/2}h^-_{j+1/2} + 3b'_{j-1/2}u'_{j+1/2}h^-_{j+1/2}h^-_{j+1/2}\Bigg] \\
&-\Bigg[b'_{j+1/2}u'_{j-1/2}h^+_{j-1/2}h^+_{j-1/2} - 2b'_{j+1/2}u'_{j-1/2}h^+_{j-1/2}h^-_{j+1/2} + 3b'_{j+1/2}u'_{j-1/2}h^-_{j+1/2}h^-_{j+1/2}\Bigg] \\
&+\Bigg[b'_{j+1/2}u'_{j+1/2}h^+_{j-1/2}h^+_{j-1/2} - 2b'_{j+1/2}u'_{j+1/2}h^+_{j-1/2}h^-_{j+1/2} + 3b'_{j+1/2}u'_{j+1/2}h^-_{j+1/2}h^-_{j+1/2}\Bigg] \\
&+\Bigg[-3b'_{j+1/2}u'_{j+1/2}h^+_{j+1/2}h^+_{j+1/2} + 2b'_{j+1/2}u'_{j+1/2}h^+_{j+1/2}h^-_{j+3/2} - b'_{j+1/2}u'_{j+1/2}h^-_{j+3/2}h^-_{j+3/2}\Bigg] \\
&-\Bigg[-3b'_{j+1/2}u'_{j+3/2}h^+_{j+1/2}h^+_{j+1/2} + 2b'_{j+1/2}u'_{j+3/2}h^+_{j+1/2}h^-_{j+3/2} - b'_{j+1/2}u'_{j+3/2}h^-_{j+3/2}h^-_{j+3/2}\Bigg] \\
&-\Bigg[-3b'_{j+3/2}u'_{j+1/2}h^+_{j+1/2}h^+_{j+1/2} + 2b'_{j+3/2}u'_{j+1/2}h^+_{j+1/2}h^-_{j+3/2} - b'_{j+3/2}u'_{j+1/2}h^-_{j+3/2}h^-_{j+3/2}\Bigg] \\
&+\Bigg[-3b'_{j+3/2}u'_{j+3/2}h^+_{j+1/2}h^+_{j+1/2} + 2b'_{j+3/2}u'_{j+3/2}h^+_{j+1/2}h^-_{j+3/2} - b'_{j+3/2}u'_{j+3/2}h^-_{j+3/2}h^-_{j+3/2}\Bigg]\Bigg)
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&= \frac{1}{24\Delta x}\times \\
&\Bigg(\Big[b'_{j-1/2}u'_{j-1/2}h^{+}_{j-1/2}h^{+}_{j-1/2}-2b'_{j-1/2}u'_{j-1/2}h^{+}_{j-1/2}h^{-}_{j+1/2}+3b'_{j-1/2}u'_{j-1/2}h^{-}_{j+1/2}h^{-}_{j+1/2}\Big] \\
&+\Big[-b'_{j-1/2}u'_{j+1/2}h^{+}_{j-1/2}h^{+}_{j-1/2}+2b'_{j-1/2}u'_{j+1/2}h^{+}_{j-1/2}h^{-}_{j+1/2}-3b'_{j-1/2}u'_{j+1/2}h^{-}_{j+1/2}h^{-}_{j+1/2}\Big] \\
&+\Big[-b'_{j+1/2}u'_{j-1/2}h^{+}_{j-1/2}h^{+}_{j-1/2}+2b'_{j+1/2}u'_{j-1/2}h^{+}_{j-1/2}h^{-}_{j+1/2}-3b'_{j+1/2}u'_{j-1/2}h^{-}_{j+1/2}h^{-}_{j+1/2}\Big] \\
&+\Big[b'_{j+1/2}u'_{j+1/2}h^{+}_{j-1/2}h^{+}_{j-1/2}-2b'_{j+1/2}u'_{j+1/2}h^{+}_{j-1/2}h^{-}_{j+1/2}+3b'_{j+1/2}u'_{j+1/2}h^{-}_{j+1/2}h^{-}_{j+1/2}\Big] \\
&+\Big[-3b'_{j+1/2}u'_{j+1/2}h^{+}_{j+1/2}h^{+}_{j+1/2}+2b'_{j+1/2}u'_{j+1/2}h^{+}_{j+1/2}h^{-}_{j+3/2}-b'_{j+1/2}u'_{j+1/2}h^{-}_{j+3/2}h^{-}_{j+3/2}\Big] \\
&+\Big[3b'_{j+1/2}u'_{j+3/2}h^{+}_{j+1/2}h^{+}_{j+1/2}-2b'_{j+1/2}u'_{j+3/2}h^{+}_{j+1/2}h^{-}_{j+3/2}+b'_{j+1/2}u'_{j+3/2}h^{-}_{j+3/2}h^{-}_{j+3/2}\Big] \\
&+\Big[3b'_{j+3/2}u'_{j+1/2}h^{+}_{j+1/2}h^{+}_{j+1/2}-2b'_{j+3/2}u'_{j+1/2}h^{+}_{j+1/2}h^{-}_{j+3/2}+b'_{j+3/2}u'_{j+1/2}h^{-}_{j+3/2}h^{-}_{j+3/2}\Big] \\
&+\Big[-3b'_{j+3/2}u'_{j+3/2}h^{+}_{j+1/2}h^{+}_{j+1/2}+2b'_{j+3/2}u'_{j+3/2}h^{+}_{j+1/2}h^{-}_{j+3/2}-b'_{j+3/2}u'_{j+3/2}h^{-}_{j+3/2}h^{-}_{j+3/2}\Big]\Bigg)
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&= \frac{1}{24\Delta x} \times \Big(\\
&b'_{j-1/2}u'_{j-1/2}h^+_{j-1/2}h^+_{j-1/2} - 2b'_{j-1/2}u'_{j-1/2}h^+_{j-1/2}h^-_{j+1/2} + 3b'_{j-1/2}u'_{j-1/2}h^-_{j+1/2}h^-_{j+1/2} \\
&- b'_{j-1/2}u'_{j+1/2}h^+_{j-1/2}h^+_{j-1/2} + 2b'_{j-1/2}u'_{j+1/2}h^+_{j-1/2}h^-_{j+1/2} - 3b'_{j-1/2}u'_{j+1/2}h^-_{j+1/2}h^-_{j+1/2} \\
&- b'_{j+1/2}u'_{j-1/2}h^+_{j-1/2}h^+_{j-1/2} + 2b'_{j+1/2}u'_{j-1/2}h^+_{j-1/2}h^-_{j+1/2} - 3b'_{j+1/2}u'_{j-1/2}h^-_{j+1/2}h^-_{j+1/2} \\
&+ b'_{j+1/2}u'_{j+1/2}h^+_{j-1/2}h^+_{j-1/2} - 2b'_{j+1/2}u'_{j+1/2}h^+_{j-1/2}h^-_{j+1/2} + 3b'_{j+1/2}u'_{j+1/2}h^-_{j+1/2}h^-_{j+1/2} \\
&- 3b'_{j+1/2}u'_{j+1/2}h^+_{j+1/2}h^+_{j+1/2} + 2b'_{j+1/2}u'_{j+1/2}h^+_{j+1/2}h^-_{j+3/2} - b'_{j+1/2}u'_{j+1/2}h^-_{j+3/2}h^-_{j+3/2} \\
&+ 3b'_{j+1/2}u'_{j+3/2}h^+_{j+1/2}h^+_{j+1/2} - 2b'_{j+1/2}u'_{j+3/2}h^+_{j+1/2}h^-_{j+3/2} + b'_{j+1/2}u'_{j+3/2}h^-_{j+3/2}h^-_{j+3/2} \\
&+ 3b'_{j+3/2}u'_{j+1/2}h^+_{j+1/2}h^+_{j+1/2} - 2b'_{j+3/2}u'_{j+1/2}h^+_{j+1/2}h^-_{j+3/2} + b'_{j+3/2}u'_{j+1/2}h^-_{j+3/2}h^-_{j+3/2} \\
&- 3b'_{j+3/2}u'_{j+3/2}h^+_{j+1/2}h^+_{j+1/2} + 2b'_{j+3/2}u'_{j+3/2}h^+_{j+1/2}h^-_{j+3/2} - b'_{j+3/2}u'_{j+3/2}h^-_{j+3/2}h^-_{j+3/2} \Big)
\end{aligned}
\tag{17}
$$

Working our way across first we change variable from x to ξ

$$
\int_{x_{j-1/2}}^{x_{j+3/2}} \frac{(h'(x))^2}{2} b_x(x) u(x) \phi_{j+1/2_x}(x) dx = \int_{-1}^{1} \frac{(h'(\xi))^2}{2} b'(\xi)_\xi \frac{d\xi}{dx} u'(\xi) \phi_{j+1/2_\xi}(\xi) \frac{d\xi}{dx} \frac{dx}{d\xi} dx
$$

$$
= \frac{1}{2\Delta x} \int_{-1}^{1} (h'(\xi))^2 b'(\xi)_\xi u'(\xi) \phi_{j+1/2_\xi}(\xi) dx
$$

Replacing with our approximations, switching $'$to mean derivatives then

$$
h' = h^+_{j-1/2} w^+_{j-1/2} + h^-_{j+1/2} w^-_{j+1/2} + h^+_{j+1/2} w^+_{j+1/2} + h^-_{j+3/2} w^-_{j+3/2}
$$

$$
b'_\xi = b'_{j-1/2}\phi'_{j-1/2} + b'_{j+1/2}\phi'_{j+1/2} + b'_{j+3/2}\phi'_{j+3/2}
$$

$$
u' = u_{j-1/2}\phi_{j-1/2} + u_{j+1/2}\phi_{j+1/2} + u_{j+3/2}\phi_{j+3/2}
$$

$$= \frac{1}{2\Delta x}\int_{-1}^{1}\left(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2} + h^+_{j+1/2}w^+_{j+1/2} + h^-_{j+3/2}w^-_{j+3/2}\right)^2 \times \left(b'_{j-1/2}\phi'_{j-1/2} + b'_{j+1/2}\phi'_{j+1/2} + b'_{j+3/2}\phi'_{j+3/2}\right) \times \left(u_{j-1/2}\phi_{j-1/2} + u_{j+1/2}\phi_{j+1/2} + u_{j+3/2}\phi_{j+3/2}\right)\phi'_{j+1/2}d\xi$$

This is very similar to before but derivatives on u and v switched So we have

$$\begin{aligned}
&= \frac{1}{2\Delta x}\int_{-1}^{1} \\
&\Bigg(\left[h^+_{j-1/2}h^+_{j-1/2}w^+_{j-1/2}w^+_{j-1/2} + 2h^+_{j-1/2}h^-_{j+1/2}w^+_{j-1/2}w^-_{j+1/2} + h^-_{j+1/2}h^-_{j+1/2}w^-_{j+1/2}w^-_{j+1/2}\right]\times \\
&[b'_{j-1/2}u_{j-1/2}\phi'_{j-1/2}\phi_{j-1/2} + b'_{j-1/2}u_{j+1/2}\phi'_{j-1/2}\phi_{j+1/2} + b'_{j+1/2}u_{j-1/2}\phi'_{j+1/2}\phi_{j-1/2} \\
&\quad + b'_{j+1/2}u_{j+1/2}\phi'_{j+1/2}\phi_{j+1/2}] \\
&+\left[h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2} + 2h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2} + h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\right]\times \\
&[b'_{j+1/2}u_{j+1/2}\phi'_{j+1/2}\phi_{j+1/2} \\
&+b'_{j+1/2}u_{j+3/2}\phi'_{j+1/2}\phi_{j+3/2} + b'_{j+3/2}u_{j+1/2}\phi'_{j+3/2}\phi_{j+1/2} + b'_{j+3/2}u_{j+3/2}\phi'_{j+3/2}\phi_{j+3/2}]\Bigg)\phi'_{j+1/2}d\xi
\end{aligned} \tag{18}$$

where again we have that the two terms here are only nonzero on each half of our domain, so we can use the gradient properties again and we have

$$
\begin{aligned}
&= \frac{1}{2\Delta x}\Bigg\{ \int_{-1}^{0} \\
&\Bigg(\Bigg[h^{+}_{j-1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}w^{+}_{j-1/2}+2h^{+}_{j-1/2}h^{-}_{j+1/2}w^{+}_{j-1/2}w^{-}_{j+1/2}+h^{-}_{j+1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}w^{-}_{j+1/2}\Bigg]\times \\
&[b'_{j-1/2}u_{j-1/2}\phi'_{j-1/2}\phi_{j-1/2}+b'_{j-1/2}u_{j+1/2}\phi'_{j-1/2}\phi_{j+1/2}+b'_{j+1/2}u_{j-1/2}\phi'_{j+1/2}\phi_{j-1/2} \\
&\qquad + b'_{j+1/2}u_{j+1/2}\phi'_{j+1/2}\phi_{j+1/2}]\phi'_{j+1/2}d\xi\Bigg) \\
&+\int_{0}^{1}\Bigg(\Bigg[h^{+}_{j+1/2}h^{+}_{j+1/2}w^{+}_{j+1/2}w^{+}_{j+1/2}+2h^{+}_{j+1/2}h^{-}_{j+3/2}w^{+}_{j+1/2}w^{-}_{j+3/2}+h^{-}_{j+3/2}h^{-}_{j+3/2}w^{-}_{j+3/2}w^{-}_{j+3/2}\Bigg]\times \\
&\qquad [b'_{j+1/2}u_{j+1/2}\phi'_{j+1/2}\phi_{j+1/2} \\
&+b'_{j+1/2}u_{j+3/2}\phi'_{j+1/2}\phi_{j+3/2}+b'_{j+3/2}u_{j+1/2}\phi'_{j+3/2}\phi_{j+1/2}+b'_{j+3/2}u_{j+3/2}\phi'_{j+3/2}\phi_{j+3/2}]\Bigg)\phi'_{j+1/2}d\xi\Bigg\}
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&= \frac{1}{2\Delta x}\Bigg\{ \int_{-1}^{0} \\
&\Bigg(\Bigg[h^{+}_{j-1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}w^{+}_{j-1/2}+2h^{+}_{j-1/2}h^{-}_{j+1/2}w^{+}_{j-1/2}w^{-}_{j+1/2}+h^{-}_{j+1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}w^{-}_{j+1/2}\Bigg]\times \\
&\qquad [-b'_{j-1/2}u_{j-1/2}\phi_{j-1/2}-b'_{j-1/2}u_{j+1/2}\phi_{j+1/2}+b'_{j+1/2}u_{j-1/2}\phi_{j-1/2} \\
&\qquad + b'_{j+1/2}u_{j+1/2}\phi_{j+1/2}]d\xi\Bigg) \\
&+\int_{0}^{1}\Bigg(\Bigg[h^{+}_{j+1/2}h^{+}_{j+1/2}w^{+}_{j+1/2}w^{+}_{j+1/2}+2h^{+}_{j+1/2}h^{-}_{j+3/2}w^{+}_{j+1/2}w^{-}_{j+3/2}+h^{-}_{j+3/2}h^{-}_{j+3/2}w^{-}_{j+3/2}w^{-}_{j+3/2}\Bigg]\times \\
&\qquad [b'_{j+1/2}u_{j+1/2}\phi_{j+1/2} \\
&+ b'_{j+1/2}u_{j+3/2}\phi_{j+3/2}-b'_{j+3/2}u_{j+1/2}\phi_{j+1/2}-b'_{j+3/2}u_{j+3/2}\phi_{j+3/2}]\Bigg)d\xi\Bigg\}
\end{aligned}
\tag{20}
$$

lets split this, so the first term is

$$\int_{-1}^{0} \Bigg(\Bigg[h^{+}_{j-1/2} h^{+}_{j-1/2} w^{+}_{j-1/2} w^{+}_{j-1/2} + 2 h^{+}_{j-1/2} h^{-}_{j+1/2} w^{+}_{j-1/2} w^{-}_{j+1/2} + h^{-}_{j+1/2} h^{-}_{j+1/2} w^{-}_{j+1/2} w^{-}_{j+1/2} \Bigg] \times$$

$$[- b'_{j-1/2} u_{j-1/2} \phi_{j-1/2} - b'_{j-1/2} u_{j+1/2} \phi_{j+1/2} + b'_{j+1/2} u_{j-1/2} \phi_{j-1/2}$$

$$+ b'_{j+1/2} u_{j+1/2} \phi_{j+1/2}] d\xi \Bigg) \quad (21)$$

$$= \int_{-1}^{0}$$

$$- b'_{j-1/2} u_{j-1/2} \phi_{j-1/2} \Bigg[h^{+}_{j-1/2} h^{+}_{j-1/2} w^{+}_{j-1/2} w^{+}_{j-1/2} + 2 h^{+}_{j-1/2} h^{-}_{j+1/2} w^{+}_{j-1/2} w^{-}_{j+1/2}$$

$$+ h^{-}_{j+1/2} h^{-}_{j+1/2} w^{-}_{j+1/2} w^{-}_{j+1/2} \Bigg]$$

$$- b'_{j-1/2} u_{j+1/2} \phi_{j+1/2} \Bigg[h^{+}_{j-1/2} h^{+}_{j-1/2} w^{+}_{j-1/2} w^{+}_{j-1/2} + 2 h^{+}_{j-1/2} h^{-}_{j+1/2} w^{+}_{j-1/2} w^{-}_{j+1/2}$$

$$+ h^{-}_{j+1/2} h^{-}_{j+1/2} w^{-}_{j+1/2} w^{-}_{j+1/2} \Bigg]$$

$$+ b'_{j+1/2} u_{j-1/2} \phi_{j-1/2} \Bigg[h^{+}_{j-1/2} h^{+}_{j-1/2} w^{+}_{j-1/2} w^{+}_{j-1/2} + 2 h^{+}_{j-1/2} h^{-}_{j+1/2} w^{+}_{j-1/2} w^{-}_{j+1/2}$$

$$+ h^{-}_{j+1/2} h^{-}_{j+1/2} w^{-}_{j+1/2} w^{-}_{j+1/2} \Bigg]$$

$$+ b'_{j+1/2} u_{j+1/2} \phi_{j+1/2} \Bigg[h^{+}_{j-1/2} h^{+}_{j-1/2} w^{+}_{j-1/2} w^{+}_{j-1/2} + 2 h^{+}_{j-1/2} h^{-}_{j+1/2} w^{+}_{j-1/2} w^{-}_{j+1/2}$$

$$+ h^{-}_{j+1/2} h^{-}_{j+1/2} w^{-}_{j+1/2} w^{-}_{j+1/2} \Bigg] d\xi \quad (22)$$

$$= \int_{-1}^{0}$$

$$-b'_{j-1/2}u_{j-1/2}\Big[h^+_{j-1/2}h^+_{j-1/2}w^+_{j-1/2}w^+_{j-1/2}\phi_{j-1/2}+2h^+_{j-1/2}h^-_{j+1/2}w^+_{j-1/2}w^-_{j+1/2}\phi_{j-1/2}$$
$$+ h^-_{j+1/2}h^-_{j+1/2}w^-_{j+1/2}w^-_{j+1/2}\phi_{j-1/2}\Big]$$
$$-b'_{j-1/2}u_{j+1/2}\Big[h^+_{j-1/2}h^+_{j-1/2}w^+_{j-1/2}w^+_{j-1/2}\phi_{j+1/2}+2h^+_{j-1/2}h^-_{j+1/2}w^+_{j-1/2}w^-_{j+1/2}\phi_{j+1/2}$$
$$+ h^-_{j+1/2}h^-_{j+1/2}w^-_{j+1/2}w^-_{j+1/2}\phi_{j+1/2}\Big]$$
$$+b'_{j+1/2}u_{j-1/2}\Big[h^+_{j-1/2}h^+_{j-1/2}w^+_{j-1/2}w^+_{j-1/2}\phi_{j-1/2}+2h^+_{j-1/2}h^-_{j+1/2}w^+_{j-1/2}w^-_{j+1/2}\phi_{j-1/2}$$
$$+ h^-_{j+1/2}h^-_{j+1/2}w^-_{j+1/2}w^-_{j+1/2}\phi_{j-1/2}\Big]$$
$$+b'_{j+1/2}u_{j+1/2}\Big[h^+_{j-1/2}h^+_{j-1/2}w^+_{j-1/2}w^+_{j-1/2}\phi_{j+1/2}+2h^+_{j-1/2}h^-_{j+1/2}w^+_{j-1/2}w^-_{j+1/2}\phi_{j+1/2}$$
$$+ h^-_{j+1/2}h^-_{j+1/2}w^-_{j+1/2}w^-_{j+1/2}\phi_{j+1/2}\Big]d\xi \quad (23)$$

$$\int_{-1}^{0} w^+_{j-1/2}w^+_{j-1/2}\phi_{j-1/2} = \int_{-1}^{0}(\xi)(\xi)(-\xi) = \frac{1}{4}$$
$$\int_{-1}^{0} w^+_{j-1/2}w^-_{j+1/2}\phi_{j-1/2} = \int_{-1}^{0}(\xi)(\xi+1)(-\xi) = -\frac{1}{12}$$
$$\int_{-1}^{0} w^-_{j+1/2}w^-_{j+1/2}\phi_{j-1/2} = \int_{-1}^{0}(\xi+1)(\xi+1)(-\xi) = \frac{1}{12}$$
$$\int_{-1}^{0} w^+_{j-1/2}w^+_{j-1/2}\phi_{j+1/2} = \int_{-1}^{0}(\xi)(\xi)(\xi+1) = \frac{1}{12}$$
$$\int_{-1}^{0} w^+_{j-1/2}w^-_{j+1/2}\phi_{j+1/2} = \int_{-1}^{0}(\xi)(\xi+1)(\xi+1) = -\frac{1}{12}$$
$$\int_{-1}^{0} w^-_{j+1/2}w^-_{j+1/2}\phi_{j+1/2} = \int_{-1}^{0}(\xi+1)(\xi+1)(\xi+1) = \frac{1}{4}$$

So

$$
\begin{aligned}
= \frac{1}{12}\Bigg(&- b'_{j-1/2} u_{j-1/2} \Big[3h^+_{j-1/2} h^+_{j-1/2} - 2h^+_{j-1/2} h^-_{j+1/2} + h^-_{j+1/2} h^-_{j+1/2} \Big] \\
&- b'_{j-1/2} u_{j+1/2} \Big[h^+_{j-1/2} h^+_{j-1/2} - 2h^+_{j-1/2} h^-_{j+1/2} + 3h^-_{j+1/2} h^-_{j+1/2} \Big] \\
&+ b'_{j+1/2} u_{j-1/2} \Big[3h^+_{j-1/2} h^+_{j-1/2} - 2h^+_{j-1/2} h^-_{j+1/2} + h^-_{j+1/2} h^-_{j+1/2} \Big] \\
&+ b'_{j+1/2} u_{j+1/2} \Big[h^+_{j-1/2} h^+_{j-1/2} - 2h^+_{j-1/2} h^-_{j+1/2} + 3h^-_{j+1/2} h^-_{j+1/2} \Big] \Bigg) \quad (24)
\end{aligned}
$$

The second term is

$$
\begin{aligned}
\int_0^1 \Bigg(\Big[& h^+_{j+1/2} h^+_{j+1/2} w^+_{j+1/2} w^+_{j+1/2} + 2h^+_{j+1/2} h^-_{j+3/2} w^+_{j+1/2} w^-_{j+3/2} + h^-_{j+3/2} h^-_{j+3/2} w^-_{j+3/2} w^-_{j+3/2} \Big] \times \\
& [b'_{j+1/2} u_{j+1/2} \phi_{j+1/2} \\
& + b'_{j+1/2} u_{j+3/2} \phi_{j+3/2} - b'_{j+3/2} u_{j+1/2} \phi_{j+1/2} - b'_{j+3/2} u_{j+3/2} \phi_{j+3/2}] \Bigg) d\xi \quad (25)
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 b'_{j+1/2}u_{j+1/2}\phi_{j+1/2}\Big[&h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}+2h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2}\\
&+h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\Big]\\
+b'_{j+1/2}u_{j+3/2}\phi_{j+3/2}\Big[&h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}+2h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2}\\
&+h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\Big]\\
-b'_{j+3/2}u_{j+1/2}\phi_{j+1/2}\Big[&h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}+2h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2}\\
&+h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\Big]\\
-b'_{j+3/2}u_{j+3/2}\phi_{j+3/2}\Big[&h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}+2h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2}\\
&+h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\Big]d\xi \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 b'_{j+1/2}u_{j+1/2}\Big[&h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}\phi_{j+1/2}+2h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2}\phi_{j+1/2}\\
&+h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\phi_{j+1/2}\Big]\\
+b'_{j+1/2}u_{j+3/2}\Big[&h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}\phi_{j+3/2}+2h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2}\phi_{j+3/2}\\
&+h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\phi_{j+3/2}\Big]\\
-b'_{j+3/2}u_{j+1/2}\Big[&h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}\phi_{j+1/2}+2h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2}\phi_{j+1/2}\\
&+h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\phi_{j+1/2}\Big]\\
-b'_{j+3/2}u_{j+3/2}\Big[&h^+_{j+1/2}h^+_{j+1/2}w^+_{j+1/2}w^+_{j+1/2}\phi_{j+3/2}+2h^+_{j+1/2}h^-_{j+3/2}w^+_{j+1/2}w^-_{j+3/2}\phi_{j+3/2}\\
&+h^-_{j+3/2}h^-_{j+3/2}w^-_{j+3/2}w^-_{j+3/2}\phi_{j+3/2}\Big]d\xi \qquad (27)
\end{aligned}
$$

$$\int_0^1 w^+_{j+1/2}w^+_{j+1/2}\phi_{j+1/2}d\xi=\int_0^1(\xi-1)(\xi-1)(\xi-1)d\xi=-\frac{1}{4}$$

$$\int_0^1 w^-_{j+3/2}w^+_{j+1/2}\phi_{j+1/2}d\xi=\int_0^1(\xi)(\xi-1)(\xi-1)d\xi=\frac{1}{12}$$

$$\int_0^1 w^-_{j+3/2}w^-_{j+3/2}\phi_{j+1/2}d\xi=\int_0^1(\xi)(\xi)(\xi-1)d\xi=-\frac{1}{12}$$

$$\int_0^1 w^+_{j+1/2}w^+_{j+1/2}\phi_{j+3/2}d\xi=\int_0^1(\xi-1)(\xi-1)(\xi)d\xi=\frac{1}{12}$$

$$\int_0^1 w^-_{j+3/2}w^+_{j+1/2}\phi_{j+3/2}d\xi=\int_0^1(\xi)(\xi-1)(\xi)d\xi=-\frac{1}{12}$$

$$\int_0^1 w^-_{j+3/2}w^-_{j+3/2}\phi_{j+3/2}d\xi=\int_0^1(\xi)(\xi)(\xi)d\xi=\frac{1}{4}$$

$$= \frac{1}{12}\Bigg(b'_{j+1/2}u_{j+1/2}\Big[-3h^+_{j+1/2}h^+_{j+1/2} + 2h^+_{j+1/2}h^-_{j+3/2} - h^-_{j+3/2}h^-_{j+3/2}\Big]$$
$$+ b'_{j+1/2}u_{j+3/2}\Big[h^+_{j+1/2}h^+_{j+1/2} - 2h^+_{j+1/2}h^-_{j+3/2} + 3h^-_{j+3/2}h^-_{j+3/2}\Big]$$
$$- b'_{j+3/2}u_{j+1/2}\Big[-3h^+_{j+1/2}h^+_{j+1/2} + 2h^+_{j+1/2}h^-_{j+3/2} - h^-_{j+3/2}h^-_{j+3/2}\Big]$$
$$- b'_{j+3/2}u_{j+3/2}\Big[h^+_{j+1/2}h^+_{j+1/2} - 2h^+_{j+1/2}h^-_{j+3/2} + 3h^-_{j+3/2}h^-_{j+3/2}\Big]\Bigg) \quad (28)$$

So we have

$$\frac{1}{2\Delta x}\int_{-1}^{1}(h'(\xi))^2 b'(\xi)_\xi u'(\xi)\phi_{j+1/2_\xi}(\xi)dx =$$
$$\frac{1}{24\Delta x}\Bigg(-b'_{j-1/2}u_{j-1/2}\Big[3h^+_{j-1/2}h^+_{j-1/2} - 2h^+_{j-1/2}h^-_{j+1/2} + h^-_{j+1/2}h^-_{j+1/2}\Big]$$
$$- b'_{j-1/2}u_{j+1/2}\Big[h^+_{j-1/2}h^+_{j-1/2} - 2h^+_{j-1/2}h^-_{j+1/2} + 3h^-_{j+1/2}h^-_{j+1/2}\Big]$$
$$+ b'_{j+1/2}u_{j-1/2}\Big[3h^+_{j-1/2}h^+_{j-1/2} - 2h^+_{j-1/2}h^-_{j+1/2} + h^-_{j+1/2}h^-_{j+1/2}\Big]$$
$$+ b'_{j+1/2}u_{j+1/2}\Big[h^+_{j-1/2}h^+_{j-1/2} - 2h^+_{j-1/2}h^-_{j+1/2} + 3h^-_{j+1/2}h^-_{j+1/2}\Big]$$
$$+ b'_{j+1/2}u_{j+1/2}\Big[-3h^+_{j+1/2}h^+_{j+1/2} + 2h^+_{j+1/2}h^-_{j+3/2} - h^-_{j+3/2}h^-_{j+3/2}\Big]$$
$$+ b'_{j+1/2}u_{j+3/2}\Big[h^+_{j+1/2}h^+_{j+1/2} - 2h^+_{j+1/2}h^-_{j+3/2} + 3h^-_{j+3/2}h^-_{j+3/2}\Big]$$
$$- b'_{j+3/2}u_{j+1/2}\Big[-3h^+_{j+1/2}h^+_{j+1/2} + 2h^+_{j+1/2}h^-_{j+3/2} - h^-_{j+3/2}h^-_{j+3/2}\Big]$$
$$- b'_{j+3/2}u_{j+3/2}\Big[h^+_{j+1/2}h^+_{j+1/2} - 2h^+_{j+1/2}h^-_{j+3/2} + 3h^-_{j+3/2}h^-_{j+3/2}\Big]\Bigg) \quad (29)$$

Now for the last term

$$\int_{x_{j-1/2}}^{x_{j+3/2}} uhb_x^2\phi_{j+1/2}dx = \int_{-1}^{1} u'h'(b'_\xi)^2\left(\frac{d\xi}{dx}\right)^2\phi_{j+1/2}\frac{dx}{d\xi}d\xi = \frac{1}{\Delta x}\int_{-1}^{1} u'h'(b'_\xi)^2\phi_{j+1/2}d\xi$$

$$h' = h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2} + h^+_{j+1/2}w^+_{j+1/2} + h^-_{j+3/2}w^-_{j+3/2}$$

$$b'_\xi = b'_{j-1/2}\phi'_{j-1/2} + b'_{j+1/2}\phi'_{j+1/2} + b'_{j+3/2}\phi'_{j+3/2}$$

$$u = u_{j-1/2}\phi_{j-1/2} + u_{j+1/2}\phi_{j+1/2} + u_{j+3/2}\phi_{j+3/2}$$

$$\begin{aligned}\frac{1}{\Delta x}\int_{-1}^{1} &\left(u_{j-1/2}\phi_{j-1/2} + u_{j+1/2}\phi_{j+1/2} + u_{j+3/2}\phi_{j+3/2}\right)\\ &\times \left(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2} + h^+_{j+1/2}w^+_{j+1/2} + h^-_{j+3/2}w^-_{j+3/2}\right)\\ &\qquad \times \left(b'_{j-1/2}\phi'_{j-1/2} + b'_{j+1/2}\phi'_{j+1/2} + b'_{j+3/2}\phi'_{j+3/2}\right)^2 \phi_{j+1/2}d\xi\end{aligned}$$

$$\begin{aligned}\frac{1}{\Delta x}\int_{-1}^{1} &\left(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2} + h^+_{j+1/2}w^+_{j+1/2} + h^-_{j+3/2}w^-_{j+3/2}\right)\\ &\times \left(u_{j-1/2}\phi_{j-1/2} + u_{j+1/2}\phi_{j+1/2} + u_{j+3/2}\phi_{j+3/2}\right)\\ \times(&b'_{j-1/2}b'_{j-1/2}\phi'_{j-1/2}\phi'_{j-1/2} + 2b'_{j+1/2}b'_{j-1/2}\phi'_{j+1/2}\phi'_{j-1/2} + b'_{j+1/2}b'_{j+1/2}\phi'_{j+1/2}\phi'_{j+1/2}\\ &+ 2b'_{j+3/2}b'_{j+1/2}\phi'_{j+3/2}\phi'_{j+1/2} + b'_{j+3/2}b'_{j+3/2}\phi'_{j+3/2}\phi'_{j+3/2})\phi_{j+1/2}d\xi\end{aligned}$$

using when w's are zero

$$\begin{aligned}\frac{1}{\Delta x}\int_{-1}^{1} &\Bigg(\left(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2}\right)\left(u_{j-1/2}\phi_{j-1/2} + u_{j+1/2}\phi_{j+1/2}\right)\\ &+ \left(h^+_{j+1/2}w^+_{j+1/2} + h^-_{j+3/2}w^-_{j+3/2}\right)\left(u_{j+1/2}\phi_{j+1/2} + u_{j+3/2}\phi_{j+3/2}\right)\Bigg)\\ \times(&b'_{j-1/2}b'_{j-1/2}\phi'_{j-1/2}\phi'_{j-1/2} + 2b'_{j+1/2}b'_{j-1/2}\phi'_{j+1/2}\phi'_{j-1/2} + b'_{j+1/2}b'_{j+1/2}\phi'_{j+1/2}\phi'_{j+1/2}\\ &+ 2b'_{j+3/2}b'_{j+1/2}\phi'_{j+3/2}\phi'_{j+1/2} + b'_{j+3/2}b'_{j+3/2}\phi'_{j+3/2}\phi'_{j+3/2})\phi_{j+1/2}d\xi\end{aligned}$$

Again this lends itself to partitioning the interval

$$\frac{1}{\Delta x}\Bigg\{ \int_{-1}^{0} \bigg(\Big(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2}\Big) \left(u_{j-1/2}\phi_{j-1/2} + u_{j+1/2}\phi_{j+1/2}\right) \bigg)$$

$$\times(b'_{j-1/2}b'_{j-1/2}\phi'_{j-1/2}\phi'_{j-1/2}+2b'_{j+1/2}b'_{j-1/2}\phi'_{j+1/2}\phi'_{j-1/2}+b'_{j+1/2}b'_{j+1/2}\phi'_{j+1/2}\phi'_{j+1/2}$$

$$+ 2b'_{j+3/2}b'_{j+1/2}\phi'_{j+3/2}\phi'_{j+1/2} + b'_{j+3/2}b'_{j+3/2}\phi'_{j+3/2}\phi'_{j+3/2})\phi_{j+1/2}d\xi$$

$$+ \int_{0}^{1} \bigg(\Big(h^+_{j+1/2}w^+_{j+1/2} + h^-_{j+3/2}w^-_{j+3/2}\Big) \left(u_{j+1/2}\phi_{j+1/2} + u_{j+3/2}\phi_{j+3/2}\right) \bigg)$$

$$\times(b'_{j-1/2}b'_{j-1/2}\phi'_{j-1/2}\phi'_{j-1/2}+2b'_{j+1/2}b'_{j-1/2}\phi'_{j+1/2}\phi'_{j-1/2}+b'_{j+1/2}b'_{j+1/2}\phi'_{j+1/2}\phi'_{j+1/2}$$

$$+ 2b'_{j+3/2}b'_{j+1/2}\phi'_{j+3/2}\phi'_{j+1/2} + b'_{j+3/2}b'_{j+3/2}\phi'_{j+3/2}\phi'_{j+3/2})\phi_{j+1/2}d\xi\Bigg\}$$

using our gradient properties

$$\frac{1}{\Delta x}\Bigg\{ \int_{-1}^{0} \bigg(\Big(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2}\Big) \left(u_{j-1/2}\phi_{j-1/2} + u_{j+1/2}\phi_{j+1/2}\right) \bigg)$$

$$\times(b'_{j-1/2}b'_{j-1/2}-2b'_{j+1/2}b'_{j-1/2}+b'_{j+1/2}b'_{j+1/2}-2b'_{j+3/2}b'_{j+1/2}+b'_{j+3/2}b'_{j+3/2})\phi_{j+1/2}d\xi$$

$$+ \int_{0}^{1} \bigg(\Big(h^+_{j+1/2}w^+_{j+1/2} + h^-_{j+3/2}w^-_{j+3/2}\Big) \left(u_{j+1/2}\phi_{j+1/2} + u_{j+3/2}\phi_{j+3/2}\right) \bigg)$$

$$\times(b'_{j-1/2}b'_{j-1/2}-2b'_{j+1/2}b'_{j-1/2}+b'_{j+1/2}b'_{j+1/2}-2b'_{j+3/2}b'_{j+1/2}+b'_{j+3/2}b'_{j+3/2})\phi_{j+1/2}d\xi\Bigg\}$$

The first term is

$$\int_{-1}^{0} \bigg(\Big(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2}\Big) \left(u_{j-1/2}\phi_{j-1/2} + u_{j+1/2}\phi_{j+1/2}\right) \bigg)$$

$$\times(b'_{j-1/2}b'_{j-1/2}-2b'_{j+1/2}b'_{j-1/2}+b'_{j+1/2}b'_{j+1/2}-2b'_{j+3/2}b'_{j+1/2}+b'_{j+3/2}b'_{j+3/2})\phi_{j+1/2}d\xi$$

$$= \int_{-1}^{0} \bigg(\Big(u_{j-1/2}h^+_{j-1/2}\phi_{j-1/2}w^+_{j-1/2} + u_{j-1/2}h^-_{j+1/2}\phi_{j-1/2}w^-_{j+1/2}\Big)$$

$$+ \Big(u_{j+1/2}h^+_{j-1/2}\phi_{j+1/2}w^+_{j-1/2} + u_{j+1/2}h^-_{j+1/2}\phi_{j+1/2}w^-_{j+1/2}\Big) \bigg)$$

$$\times(b'_{j-1/2}b'_{j-1/2}-2b'_{j+1/2}b'_{j-1/2}+b'_{j+1/2}b'_{j+1/2}-2b'_{j+3/2}b'_{j+1/2}+b'_{j+3/2}b'_{j+3/2})\phi_{j+1/2}d\xi$$

$$\int_{-1}^{0} w^+_{j-1/2}\phi_{j-1/2}\phi_{j+1/2} = \int_{-1}^{0}(-\xi)(-\xi)(\xi+1) = \frac{1}{12}$$
$$\int_{-1}^{0} w^-_{j+1/2}\phi_{j-1/2}\phi_{j+1/2} = \int_{-1}^{0}(\xi+1)(-\xi)(\xi+1) = \frac{1}{12}$$
$$\int_{-1}^{0} w^+_{j-1/2}\phi_{j+1/2}\phi_{j+1/2} = \int_{-1}^{0}(-\xi)(\xi+1)(\xi+1) = \frac{1}{12}$$
$$\int_{-1}^{0} w^-_{j+1/2}\phi_{j+1/2}\phi_{j+1/2} = \int_{-1}^{0}(\xi+1)(\xi+1)(\xi+1) = \frac{1}{4}$$

So we have

$$= \frac{1}{12}\Bigg(\Big(u_{j-1/2}h^+_{j-1/2} + u_{j-1/2}h^-_{j+1/2}\Big) + \Big(u_{j+1/2}h^+_{j-1/2} + 3u_{j+1/2}h^-_{j+1/2}\Big) \Bigg)$$
$$\times (b'_{j-1/2}b'_{j-1/2} - 2b'_{j+1/2}b'_{j-1/2} + b'_{j+1/2}b'_{j+1/2} - 2b'_{j+3/2}b'_{j+1/2} + b'_{j+3/2}b'_{j+3/2})$$

The second term is

$$\int_0^1 \Bigg(\Big(h^+_{j+1/2}w^+_{j+1/2} + h^-_{j+3/2}w^-_{j+3/2}\Big)\left(u_{j+1/2}\phi_{j+1/2} + u_{j+3/2}\phi_{j+3/2}\right) \Bigg)$$
$$\times(b'_{j-1/2}b'_{j-1/2}-2b'_{j+1/2}b'_{j-1/2}+b'_{j+1/2}b'_{j+1/2}-2b'_{j+3/2}b'_{j+1/2}+b'_{j+3/2}b'_{j+3/2})\phi_{j+1/2}d\xi$$

$$\int_0^1 \Bigg(\Big(u_{j+1/2}h^+_{j+1/2}\phi_{j+1/2}w^+_{j+1/2} + u_{j+1/2}h^-_{j+3/2}\phi_{j+1/2}w^-_{j+3/2}\Big)$$
$$+ \Big(u_{j+3/2}h^+_{j+1/2}\phi_{j+3/2}w^+_{j+1/2} + u_{j+3/2}h^-_{j+3/2}\phi_{j+3/2}w^-_{j+3/2}\Big) \Bigg)$$
$$\times(b'_{j-1/2}b'_{j-1/2}-2b'_{j+1/2}b'_{j-1/2}+b'_{j+1/2}b'_{j+1/2}-2b'_{j+3/2}b'_{j+1/2}+b'_{j+3/2}b'_{j+3/2})\phi_{j+1/2}d\xi$$

$$\int_{-1}^{0} w^+_{j+1/2}\phi_{j+1/2}\phi_{j+1/2} = \int_{-1}^{0}(1-\xi)(1-\xi)(1-\xi) = \frac{1}{4}$$
$$\int_{-1}^{0} w^-_{j+3/2}\phi_{j+1/2}\phi_{j+1/2} = \int_{-1}^{0}(\xi)(1-\xi)(1-\xi) = \frac{1}{12}$$

$$\int_{-1}^{0} w^{+}_{j+1/2}\phi_{j+3/2}\phi_{j+1/2} = \int_{-1}^{0}(1-\xi)(\xi)(1-\xi) = \frac{1}{12}$$

$$\int_{-1}^{0} w^{-}_{j+3/2}\phi_{j+3/2}\phi_{j+1/2} = \int_{-1}^{0}(\xi)(\xi)(1-\xi) = \frac{1}{12}$$

$$\frac{1}{12}\left(\left(3u_{j+1/2}h^{+}_{j+1/2} + u_{j+1/2}h^{-}_{j+3/2} + u_{j+3/2}h^{+}_{j+1/2} + u_{j+3/2}h^{-}_{j+3/2}\right) \right)$$
$$\times \left(b'_{j-1/2}b'_{j-1/2} - 2b'_{j+1/2}b'_{j-1/2} + b'_{j+1/2}b'_{j+1/2} - 2b'_{j+3/2}b'_{j+1/2} + b'_{j+3/2}b'_{j+3/2}\right)$$

So

$$\frac{1}{\Delta x}\int_{-1}^{1} u'h'(b'_{\xi})^2\phi_{j+1/2}d\xi =$$
$$\frac{1}{12\Delta x}\Bigg\{ \left(b'_{j-1/2}b'_{j-1/2} - 2b'_{j+1/2}b'_{j-1/2} + b'_{j+1/2}b'_{j+1/2} - 2b'_{j+3/2}b'_{j+1/2} + b'_{j+3/2}b'_{j+3/2}\right)$$
$$\times \Big(u_{j-1/2}h^{+}_{j-1/2} + u_{j-1/2}h^{-}_{j+1/2} + u_{j+1/2}h^{+}_{j-1/2} + 3u_{j+1/2}h^{-}_{j+1/2}$$
$$+ 3u_{j+1/2}h^{+}_{j+1/2} + u_{j+1/2}h^{-}_{j+3/2} + u_{j+3/2}h^{+}_{j+1/2} + u_{j+3/2}h^{-}_{j+3/2} \Big) \Bigg\}$$