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Chapter 3

Hybrid Finite Volume Methods

3.1 Structure Overview

In this section we will give the general structure for how the hybrid finite volume methods take the array of cell average values at time t^n ; $\bar{\mathbf{h}}^n$ and $\bar{\mathbf{G}}^n$ and evolve the system to the cell average values at time t^{n+1} ; $\bar{\mathbf{h}}^{n+1}$ and $\bar{\mathbf{G}}^{n+1}$.

- The cell average values are transformed into nodal values by \mathcal{M}

$$\bar{\mathbf{h}}^n, \bar{\mathbf{G}}^n \xrightarrow{\mathcal{M}} \mathbf{h}^n, \mathbf{G}^n$$

- \mathbf{u}^n is found by solving the elliptic equation in Def. 2.2 by \mathcal{A}

$$\mathbf{h}^n, \mathbf{G}^n, \mathbf{b} \xrightarrow{\mathcal{A}} \mathbf{u}^n$$

- The conservation equations (2.2) can now be solved by \mathcal{F}

$$\bar{\mathbf{h}}^n, \bar{\mathbf{G}}^n, \mathbf{b}, \mathbf{u}^n \xrightarrow{\mathcal{F}} \bar{\mathbf{h}}^{n+1}, \bar{\mathbf{G}}^{n+1}$$

Definition 3.1. \mathcal{E} is the single Euler step given by the procedure above that transforms updates the array of cell average values at time t^n ; $\bar{\mathbf{h}}^n$ and $\bar{\mathbf{G}}^n$ to the cell average values at time t^{n+1} ; $\bar{\mathbf{h}}^{n+1}$.

$$\bar{\mathbf{h}}^n, \bar{\mathbf{G}}^n, \mathbf{b} \xrightarrow{\mathcal{E}} \bar{\mathbf{h}}^{n+1}, \bar{\mathbf{G}}^{n+1}$$

3.2 Transformation Between Nodal Values and Cell Averages

For first and second order methods \mathcal{M} is just the identity map as the cell average values are equal to the nodal values.

For higher order methods this is not the case, hence why there is a need to incorporate the process \mathcal{M} into our methods, as assuming that \mathcal{M} is the identity map will lead to a loss of accuracy in the method.

From quadratic interpolation we have the formula relating the cell averages and nodal values of a quantity q with third order accuracy

$$q_j = \frac{-\bar{q}_{j+1} + 26\bar{q}_j - \bar{q}_{j-1}}{24}.$$

Therefore

$$\mathbf{q} = \frac{1}{24} \begin{bmatrix} 26 & -1 & & & \\ -1 & 26 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 26 & -1 \\ & & & -1 & 26 \end{bmatrix} \bar{\mathbf{q}}.$$

Dirichlet boundary conditions at edges.

So for the third-order method \mathcal{M} is a multiplication by the above matrix.

3.3 Elliptic Equation

The elliptic equation that relates the conserved variables h and G to the primitive variable u was given in Def 2.2 and is presented here to remind the reader

$$G = uh \left(1 + \frac{\partial h}{\partial x} \frac{\partial b}{\partial x} + \frac{1}{2} h \frac{\partial^2 b}{\partial x^2} + \frac{\partial b^2}{\partial x} \right) - \frac{\partial}{\partial x} \left(\frac{1}{3} h^3 \frac{\partial u}{\partial x} \right).$$

3.3.1 Finite Difference Methods

One way to approximate this ordinary differential equation is to replace all the derivatives with finite differences as has been done to second order accuracy in []. We have expanded this work by building a fourth order accurate finite difference method, where all derivatives were replaced with their centred fourth order approximation. This results in the following equation for each row of a the matrix \mathbf{A}

$$G_j = A_{j,j-2} u_{j-2} + A_{j,j-1} u_{j-1} + A_{j,j} u_j + A_{j,j+1} u_{j+1} + A_{j,j+2} u_{j+2} \quad (3.1)$$

where

$$\begin{aligned}
A_{j,j-2} &= -h_j^2 \left(\frac{-h_{j+2} + 8h_{j+1} - 8h_{j-1} + h_{j-2}}{144\Delta x^2} \right) + \frac{h_j^3}{36\Delta x^2}, \\
A_{j,j-1} &= h_j^2 \left(\frac{-h_{j+2} + 8h_{j+1} - 8h_{j-1} + h_{j-2}}{18\Delta x^2} \right) - \frac{4h_j^3}{9\Delta x^2}, \\
A_{j,j} &= h_j + \frac{5h_j^3}{6\Delta x^2} + h_j \left(\frac{(-h_{j+2} + 8h_{j+1} - 8h_{j-1} + h_{j-2})(-b_{j+2} + 8b_{j+1} - 8b_{j-1} + b_{j-2})}{144\Delta x^2} \right) \\
&\quad + h_j \left(\frac{-b_{j+2} + 16b_{j+1} - 30b_{j-1} + 16b_{j-1} - b_{j-2}}{24\Delta x^2} \right) + h_j \left(\frac{-b_{j+2} + 8b_{j+1} - 8b_{j-1} + b_{j-2}}{144\Delta x^2} \right), \\
A_{j,j+1} &= -h_j^2 \left(\frac{-h_{j+2} + 8h_{j+1} - 8h_{j-1} + h_{j-2}}{18\Delta x^2} \right) - \frac{4h_j^3}{9\Delta x^2}, \\
A_{j,j+2} &= h_j^2 \left(\frac{-h_{j+2} + 8h_{j+1} - 8h_{j-1} + h_{j-2}}{144\Delta x^2} \right) + \frac{h_j^3}{36\Delta x^2},
\end{aligned}$$

This can be written for the whole domain

$$\mathbf{G} = \mathbf{A}\mathbf{u}.$$

Dirichlet boundary conditions

Therefore \mathcal{G} is the solution of this matrix problem with \mathbf{G} known and \mathbf{u} unknown.

3.3.2 Finite Element Methods

For a finite element method we take the weak form of the elliptic equation in Def 2.2 which is

$$\int_{\Omega} Gv \, dx = \int_{\Omega} uh \left(1 + \frac{\partial h}{\partial x} \frac{\partial b}{\partial x} + \frac{1}{2} h \frac{\partial^2 b}{\partial x^2} + \frac{\partial b^2}{\partial x} \right) - \frac{\partial}{\partial x} \left(\frac{1}{3} h^3 \frac{\partial u}{\partial x} \right) v \, dx.$$

Which after rearranging, using integration by parts and assuming Dirichlet boundary conditions becomes

$$\begin{aligned}
\int_{\Omega} Gv \, dx &= \int_{\Omega} uh \left(1 + \frac{\partial b^2}{\partial x} \right) v \, dx + \int_{\Omega} \frac{1}{3} h^3 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx \\
&\quad - \int_{\Omega} \frac{1}{2} h^2 \frac{\partial b}{\partial x} u \frac{\partial v}{\partial x} \, dx - \int_{\Omega} \frac{1}{2} h^2 \frac{\partial b}{\partial x} \frac{\partial u}{\partial x} v \, dx. \quad (3.2)
\end{aligned}$$

Should be able to handle non smooth h and G but requires first derivative of b .

The next step is to replace the functions for the quantities h , G , b and u as well as the test function v with the appropriate order of accuracy piece-wise polynomial function. This is typically done using basis functions and this process turns our integral equations into matrix equations.

Second Order

We desire a method for the elliptic equation \square that is at least second-order and that can locally reconstruct the necessary physical quantities to update the evolution equations \square within each cell individually. This allows the scheme to be simply scaled into situations where nearby cell data may not be quickly accessible such as when using this method on parallel cpus. From our equations and smoothness conditions we know that if the bed is smooth enough, then we can allow discontinuous h and G and have u have a well defined derivative. Since in the evolution equations we do not require derivatives of these quantities for h and G we will use linear reconstruction that is discontinuous at the edges for h and G in our finite element method. The basis functions for this reconstruction of h and G will be represented by ψ . This reconstruction will use cell values at the edges $x_{j-1/2}$ and $x_{j+1/2}$ to represent the linear function over the cell.

So we have the following representation for h and G in our finite element method

$$\begin{aligned} h &= \sum_j h_{j-1/2}^+ \psi_{j-1/2}^+ + h_{j+1/2}^- \psi_{j+1/2}^- \\ G &= \sum_j G_{j-1/2}^+ \psi_{j-1/2}^+ + G_{j+1/2}^- \psi_{j+1/2}^- \end{aligned}$$

For the velocity term our FVM requires a local second-order approximation of the first derivative. To do this will require a quadratic representation of u in each cell. Since u is guaranteed to have continuous representation across the cells we use a quadratic reconstruction that is continuous at the cell edges. The basis functions for this reconstruction of u will be represented by ϕ . We will also use this reconstruction for our test function v . This reconstruction will use cell values at $x_{j-1/2}$, x_j and $x_{j+1/2}$ to represent the cubic function over the cell.

$$u = \sum_j u_{j-1/2} \phi_{j-1/2} + u_j \phi_j + u_{j+1/2} \phi_{j+1/2} \quad (3.3)$$

For the bed term our FVM requires a second-order approximation of the second derivative of the bed terms that must be done locally. To do this we will require a cubic representation of the bed in each cell. Since by our smoothness conditions we do not allow for discontinuous beds, we will assume the bed is continuous. Therefore we will use a cubic reconstruction for the bed which is continuous across the cell edges. The basis functions for this reconstruction of b will be represented by γ . This reconstruction will use cell values at $x_{j-1/2}$, $x_{j-1/6}$, $x_{j+1/6}$ and $x_{j+1/2}$ to represent the cubic function over the cell.

$$b = \sum_j b_{j-1/2} \gamma_{j-1/2} + b_{j-1/6} \gamma_{j-1/6} + b_{j+1/6} \gamma_{j+1/6} + b_{j+1/2} \gamma_{j+1/2} \quad (3.4)$$

3.4 Evolution Equations

The evolution equations in the alternative form of the Serre equations (2.2) are

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0$$

and

$$\begin{aligned} \frac{\partial}{\partial t}(G) + \frac{\partial}{\partial x} \left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3 \frac{\partial u}{\partial x} + h^2 u \frac{\partial u}{\partial x} \frac{\partial b}{\partial x} \right) \\ = -\frac{1}{2}h^2 u \frac{\partial u}{\partial x} \frac{\partial^2 b}{\partial x^2} + hu^2 \frac{\partial b}{\partial x} \frac{\partial^2 b}{\partial x^2} - gh \frac{\partial b}{\partial x} \end{aligned}$$

Because these equations are in conservation law form and we have an estimate for the maximum and minimum wave speeds [], Kurganovs method [] can be employed to estimate the fluxes across the boundary. This leads to the following update scheme for a quantity q

$$\bar{q}_j^{n+1} = \bar{q}_j^n - \frac{\Delta t}{\Delta x} [F_{j+1/2}^n - F_{j-1/2}^n] + \Delta t S_j^n. \quad (3.5)$$

Where $F_{j+1/2}^n$ and $F_{j-1/2}^n$ are approximations to the average fluxes across the boundary of the cell with midpoint x_i from time t^n to t^{n+1} . While S_j is an approximation to the average source term contribution in the cell from time t^n to t^{n+1} , which since this time-stepping is first-order we can just take to be constant over the time step.

3.4.1 Kurganovs Method

Kurganovs method is a finite volume method that can handle discontinuities across the boundary and only requires an estimate of the maximum and minimum wave speeds instead of the characteristics like other methods []. This makes it a good choice for the Serre equations as we do not have an expression for the characteristics but we do have estimates on the maximum and minimum wave speeds [].

The equation which approximates $F_{j+1/2}^n$ in (3.5) for a quantity q at a particular time t^n is

$$F_{j+\frac{1}{2}} = \frac{a_{j+\frac{1}{2}}^+ f(q_{j+\frac{1}{2}}^-) - a_{j+\frac{1}{2}}^- f(q_{j+\frac{1}{2}}^+)}{a_{j+\frac{1}{2}}^+ - a_{j+\frac{1}{2}}^-} + \frac{a_{j+\frac{1}{2}}^+ a_{j+\frac{1}{2}}^-}{a_{j+\frac{1}{2}}^+ - a_{j+\frac{1}{2}}^-} [q_{j+\frac{1}{2}}^+ - q_{j+\frac{1}{2}}^-] \quad (3.6)$$

where $a_{j+\frac{1}{2}}^+$ $a_{j+\frac{1}{2}}^-$ are given by the wave speed bounds [], for the Serre equations we have

$$\begin{aligned} a_{j+\frac{1}{2}}^- &= \min \left\{ 0, u_{j+1/2}^- - \sqrt{gh_{j+1/2}^-}, u_{j+1/2}^+ - \sqrt{gh_{j+1/2}^+} \right\}, \\ a_{j+\frac{1}{2}}^+ &= \max \left\{ 0, u_{j+1/2}^- + \sqrt{gh_{j+1/2}^-}, u_{j+1/2}^+ + \sqrt{gh_{j+1/2}^+} \right\}, \end{aligned}$$

While $f(q_{j+\frac{1}{2}}^-)$ and $f(q_{j+\frac{1}{2}}^+)$ are the evaluations of the flux function on the left and right side of the cell interface respectively. For h in the Serre equations we have

$$\begin{aligned} f(h_{j+\frac{1}{2}}^-) &= u_{j+1/2}^- h_{j+1/2}^-, \\ f(h_{j+\frac{1}{2}}^+) &= u_{j+1/2}^+ h_{j+1/2}^+, \end{aligned}$$

while for G we have

$$\begin{aligned} f(G_{j+\frac{1}{2}}^-) &= u_{j+1/2}^- G_{j+1/2}^- + \frac{g}{2} (h_{j+1/2}^-)^2 - \frac{2}{3} (h_{j+1/2}^-)^3 \left[\left(\frac{\partial u}{\partial x} \right)_{j+1/2}^- \right]^2 \\ &\quad + (h_{j+1/2}^-)^2 u_{j+1/2}^- \left(\frac{\partial u}{\partial x} \right)_{j+1/2}^- \left(\frac{\partial b}{\partial x} \right)_{j+1/2}^-, \\ f(G_{j+\frac{1}{2}}^+) &= u_{j+1/2}^+ G_{j+1/2}^+ + \frac{g}{2} (h_{j+1/2}^+)^2 - \frac{2}{3} (h_{j+1/2}^+)^3 \left[\left(\frac{\partial u}{\partial x} \right)_{j+1/2}^+ \right]^2 \\ &\quad + (h_{j+1/2}^+)^2 u_{j+1/2}^+ \left(\frac{\partial u}{\partial x} \right)_{j+1/2}^+ \left(\frac{\partial b}{\partial x} \right)_{j+1/2}^+. \end{aligned}$$

We now only have to have some appropriate order of accuracy method to calculate the quantities we need at the boundaries from the cell averages of h , G and b and the nodal values of u .

Second Order

For the second order finite volume method we use the minmod limiter to reconstruct h , G and b at the cell edges. For a general quantity q the generalised minmod limiter produces the following reconstruction

[?]

$$q_{j+1/2}^- = q_j + a_j \frac{\Delta x}{2} \quad (3.7a)$$

and

$$q_{j+1/2}^+ = q_{j+1} - a_{j+1} \frac{\Delta x}{2} \quad (3.7b)$$

where

$$a_j = \text{minmod} \left\{ \theta \frac{q_{j+1} - q_j}{\Delta x}, \frac{q_{j+1} - q_{j-1}}{2\Delta x}, \theta \frac{q_j - q_{j-1}}{\Delta x} \right\} \quad \text{for } \theta \in [1, 2] \quad (3.7c)$$

While for u the following reconstruction is used

$$u_{j+1/2}^+ = u_{j+1/2}^- = \frac{u_{j+1} + u_j}{2}. \quad (3.8)$$

We approximate the derivatives in the following way

$$\left(\frac{\partial b}{\partial x} \right)_{j+1/2}^+ = \frac{b_{j+3/2}^+ - b_{j+1/2}^+}{\Delta x}, \quad (3.9)$$

$$\left(\frac{\partial b}{\partial x} \right)_{j+1/2}^- = \frac{b_{j+1/2}^- - b_{j-1/2}^-}{\Delta x}, \quad (3.10)$$

$$\left(\frac{\partial u}{\partial x} \right)_{j+1/2}^+ = \left(\frac{\partial u}{\partial x} \right)_{j+1/2}^- = \frac{u_{j+1} - u_j}{\Delta x} \quad (3.11)$$

Third Order

For the third-order finite volume method we use the Koren limiter [?] to reconstruct the cell edges for h , G and b . For a general quantity q the reconstruction based on the Koren limiter is

$$q_{j+1/2}^- = \bar{q}_j + \phi^-(r_j)(\bar{q}_j - \bar{q}_{j-1})/2 \quad (3.12a)$$

and

$$q_{j+1/2}^+ = \bar{q}_{j+1} - \phi^+(r_{j+1})(\bar{q}_{j+1} - \bar{q}_j)/2 \quad (3.12b)$$

where

$$\phi^-(r_j) = \max \left[0, \min \left[2r_j, \frac{1+2r_j}{3}, 2 \right] \right], \quad (3.12c)$$

$$\phi^+(r_j) = \max \left[0, \min \left[2r_j, \frac{2+r_j}{3}, 2 \right] \right] \quad (3.12d)$$

with

$$r_j = (\bar{q}_{j+1} - \bar{q}_j)/(\bar{q}_j - \bar{q}_{j-1}). \quad (3.12e)$$

While for u the following reconstruction is used

$$u_{j+\frac{1}{2}} = \frac{-3u_{j+2} + 27u_{j+1} + 27u_j - 3u_{j-1}}{48}. \quad (3.13)$$

We approximate the derivatives in the following way

$$\left(\frac{\partial b}{\partial x} \right)_{j+1/2}^+ = \frac{-b_{j+5/2}^+ + 4b_{j+3/2}^+ + 3b_{j+1/2}^+}{\Delta x}, \quad (3.14)$$

$$\left(\frac{\partial b}{\partial x} \right)_{j+1/2}^- = \frac{3b_{j+1/2}^- - 4b_{j-1/2}^- + b_{j-3/2}^-}{\Delta x}, \quad (3.15)$$

$$\left(\frac{\partial u}{\partial x} \right)_{j+1/2}^+ = \left(\frac{\partial u}{\partial x} \right)_{j+1/2}^- = \frac{-u_{j+2} + 27u_{j+1} - 27u_j + u_{j-1}}{24\Delta x}, \quad (3.16)$$

3.4.2 Source Terms and Well Balancing

3.5 Runge-Kutta Time-Stepping

The method \mathcal{E} is only first order in time, one strategy for increasing our order of accuracy in time is to use SSP Runge Kutta time stepping [].

For the first order method our current method is sufficient and so

$$\bar{\mathbf{h}}^{n+1}, \bar{\mathbf{G}}^{n+1} = \mathcal{E}(\bar{\mathbf{h}}^n, \bar{\mathbf{G}}^n, \mathbf{b}) \quad (3.17)$$

For the second order method we have

$$\bar{\mathbf{h}}', \bar{\mathbf{G}}' = \mathcal{E}(\bar{\mathbf{h}}^n, \bar{\mathbf{G}}^n, \mathbf{b}) \quad (3.18a)$$

$$\bar{\mathbf{h}}'', \bar{\mathbf{G}}'' = \mathcal{E}(\bar{\mathbf{h}}', \bar{\mathbf{G}}', \mathbf{b}) \quad (3.18b)$$

$$\bar{\mathbf{h}}^{n+1}, \bar{\mathbf{G}}^{n+1} = \frac{1}{2}(\bar{\mathbf{h}}^n + \bar{\mathbf{h}}''), \frac{1}{2}(\bar{\mathbf{G}}^n + \bar{\mathbf{G}}'') \quad (3.18c)$$

For the third order method we have

$$\bar{\mathbf{h}}', \bar{\mathbf{G}}' = \mathcal{E}(\bar{\mathbf{h}}^n, \bar{\mathbf{G}}^n, \mathbf{b}) \quad (3.19a)$$

$$\bar{\mathbf{h}}'', \bar{\mathbf{G}}'' = \mathcal{E}(\bar{\mathbf{h}}', \bar{\mathbf{G}}', \mathbf{b}) \quad (3.19b)$$

$$\bar{\mathbf{h}}''', \bar{\mathbf{G}}''' = \frac{3}{4}\bar{\mathbf{h}}^n + \frac{1}{4}\bar{\mathbf{h}}'', \frac{3}{4}\bar{\mathbf{G}}^n + \frac{1}{4}\bar{\mathbf{G}}'' \quad (3.19c)$$

$$\bar{\mathbf{h}}''', \bar{\mathbf{G}}''' = \mathcal{E}(\bar{\mathbf{h}}''', \bar{\mathbf{G}}''', \mathbf{b}) \quad (3.19d)$$

$$\bar{\mathbf{h}}^{n+1}, \bar{\mathbf{G}}^{n+1} = \frac{1}{3}\bar{\mathbf{h}}^n + \frac{2}{3}\bar{\mathbf{h}}''', \frac{1}{3}\bar{\mathbf{G}}^n + \frac{2}{3}\bar{\mathbf{G}}''' \quad (3.19e)$$