1 Elliptic Equation

The linearised elliptic equation is

$$G = Hu - \frac{H^3}{3} u_{xx}$$

Want to find out the FEM approximation factor $\mathcal{G}_{FE_1}$ such that

$$G = \mathcal{G}_{FE_1} u$$

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = Huv - \frac{H^3}{3} u_{xx} v$$

and then we integrate over the entire domain to get

$$\int_{\Omega} Gv dx = \int_{\Omega} Huv dx - \int_{\Omega} \frac{H^3}{3} u_{xx} v dx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_{\Omega} Gv dx = \int_{\Omega} Huv dx + \int_{\Omega} \frac{H^3}{3} u_x v_x dx$$

For G and u we will choose basis functions w that are linear from $[x_{j-1/2}, x_{j+1/2}]$ but discontinuous at the edges.

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gv dx = \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Huv dx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{H^3}{3} u_x v_x dx$$

for all v

2 P1 FEM

We are going to cordinate tranform from x space the interval $[x_{j-1/2}, x_{j+1/2}, x_{j+3/2}]$ to the ξ space interval $[-1, 0, 1]$. To accomplish this we have the following relation

$$x = \xi \Delta x + x_{j+1/2}$$

Taking the derivatives we see
$dx = d\xi \Delta x$, $\frac{dx}{d\xi} = \Delta x$, $\frac{d\xi}{dx} = \frac{1}{\Delta x}$.

We can describe the basis functions in the ξ space

$$w^+_{j+1/2} = \begin{cases} 1-\xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$w^-_{j+1/2} = \begin{cases} 1+\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$w^+_{j-1/2} = \begin{cases} -\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$w^-_{j+3/2} = \begin{cases} \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Therefore taking the derivative of this

$$(w_x)^+_{j+1/2} = \begin{cases} -1 & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$(w_x)^-_{j+1/2} = \begin{cases} 1 & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$(w_x)^+_{j-1/2} = \begin{cases} -1 & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

$$(w_x)^-_{j+3/2} = \begin{cases} 1 & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

We now replace our functions by our approximations to them

$$G \approx G' = \sum_j G_{j+1/2} w_{j+1/2}$$

$$u \approx u' = \sum_j u_{j+1/2} w_{j+1/2}$$

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} G' w_{j+1/2} dx - H \int_{x_{j-1/2}}^{x_{j+3/2}} u' w_{j+1/2} dx - \frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+3/2}} u'_x (w_x)_{j+1/2} dx = 0$$

For all $w^{\pm}_{j+1/2}$. For this analysis we choose a particular basis function $w^{+}_{j+1/2}$ and we look at all the integrals. Begining from the right

$$\int_{x_{j-1/2}}^{x_{j+3/2}} G'(x) w_{j+1/2} dx = \int_{-1}^{1} G'(\xi) w_{j+1/2}(\xi) \frac{dx}{d\xi} d\xi$$

$$= \Delta x \int_{-1}^{1} \left(G^{+}_{j-1/2} w^{+}_{j-1/2} + G^{-}_{j+1/2} w^{-}_{j+1/2} + G^{+}_{j+1/2} w^{+}_{j+1/2} + G^{-}_{j-3/2} w^{-}_{j-3/2} \right) w^{+}_{j+1/2} d\xi$$

$$\begin{aligned} = \Delta x [G^{+}_{j-1/2} \int_{-1}^{1} w^{+}_{j-1/2} w^{+}_{j+1/2} d\xi + G^{-}_{j+1/2} \int_{-1}^{1} w^{-}_{j+1/2} w^{+}_{j+1/2} d\xi \\ + G^{+}_{j+1/2} \int_{-1}^{1} w^{+}_{j+1/2} w^{+}_{j+1/2} d\xi + G^{-}_{j+3/2} \int_{-1}^{1} w^{-}_{j+3/2} w^{+}_{j+1/2} d\xi] \end{aligned} \quad (9)$$

We have that

$$\int_{-1}^{1} w^{+}_{j-1/2} w^{+}_{j+1/2} d\xi = 0$$

$$\int_{-1}^{1} w^{-}_{j+1/2} w^{+}_{j+1/2} d\xi = 0$$

$$\int_{-1}^{1} w^{-}_{j+3/2} \phi_{j+1/2} d\xi = \frac{1}{6}$$

and

$$\int_{-1}^{1} w^{+}_{j+1/2} w^{+}_{j+1/2} d\xi = \frac{1}{3}$$

So

$$= \Delta x \left[\frac{1}{3} G^{+}_{j+1/2} + \frac{1}{6} G^{-}_{j+3/2} \right]$$

$$= \frac{\Delta x}{6}\left[2G^{+}_{j+1/2} + G^{-}_{j+3/2}\right]$$

Taking the next integral is similar so

$$= H\frac{\Delta x}{6}\left[2u^{+}_{j+1/2} + u^{-}_{j+3/2}\right]$$

For the third integral we have

$$\frac{H^3}{3}\int_{x_{j-1/2}}^{x_{j+3/2}} u'_x (w^{+}_{j+1/2})_x dx = -\frac{H^3}{3}\int_{-1}^{1} u'_\xi \frac{d\xi}{dx} (w_\xi)^{+}_{j+1/2} \frac{d\xi}{dx}\frac{dx}{d\xi} d\xi$$

$$= \frac{H^3}{3\Delta x}\int_{-1}^{1} u'_\xi (w_\xi)^{+}_{j+1/2} d\xi$$

We will now use $'$ to denote derivative

$$= \frac{H^3}{3\Delta x}\int_{-1}^{1} \left(u^{+}_{j+1/2}{}' w^{+}_{j+1/2}{}' + u^{-}_{j+3/2}{}' w^{-}_{j+3/2}{}'\right) w^{+}_{j+1/2}{}' d\xi$$

$$= \frac{H^3}{3\Delta x}\left[u_{j-1/2}\int_{-1}^{1} \phi'_{j-1/2}\phi'_{j+1/2} d\xi + u_{j+1/2}\int_{-1}^{1} \phi'_{j+1/2}\phi'_{j+1/2} d\xi + u_{j+3/2}\int_{-1}^{1} \phi'_{j+3/2}\phi'_{j+1/2} d\xi\right]$$

We have that

$$\int_{-1}^{1} \phi'_{j-1/2}\phi'_{j+1/2} d\xi = -1 = \int_{-1}^{1} \phi'_{j+3/2}\phi'_{j+1/2} d\xi$$

$$\int_{-1}^{1} \phi'_{j+1/2}\phi'_{j+1/2} d\xi = 2$$

Therefore

$$= \frac{H^3}{3\Delta x}\left[-u_{j-1/2} + 2u_{j+1/2} - u_{j+3/2}\right]$$

Then our equation becomes

$$\frac{\Delta x}{6}\left[G^{+}_{j-1/2} + 2G^{-}_{j+1/2} + 2G^{+}_{j+1/2} + G^{-}_{j+3/2}\right] =$$
$$\frac{H\Delta x}{6}\left[u_{j-1/2} + 4u_{j+1/2} + u_{j+3/2}\right] + \frac{H^3}{3\Delta x}\left[-u_{j-1/2} + 2u_{j+1/2} - u_{j+3/2}\right] \quad (10)$$

$$\left[G^{+}_{j-1/2} + 2G^{-}_{j+1/2} + 2G^{+}_{j+1/2} + G^{-}_{j+3/2}\right] =$$
$$H\left[u_{j-1/2} + 4u_{j+1/2} + u_{j+3/2}\right] + \frac{2H^3}{\Delta x^2}\left[-u_{j-1/2} + 2u_{j+1/2} - u_{j+3/2}\right] \quad (11)$$

We now assume the following form for u and G.

Let quantity q is given by so that $q(x,t) = q(0,0)e^{i(\omega t + kx)}$. The use of this comes when we use our uniform grid in space, so that $q(x_j,t) = q_j$ then $q_{j\pm l} = q_j e^{\pm ikl\Delta x}$. With reconstructions from the previous dispersion analysis

Then we have

$$\left[G_j e^{-ik\Delta x}\mathcal{R}^{+} + 2G_j\mathcal{R}^{-} + 2G_j\mathcal{R}^{+} + G_j e^{ik\Delta x}\mathcal{R}^{-}\right] =$$
$$H\left[u_j e^{-ik\Delta x/2} + 4u_j e^{ik\Delta x/2} + u_j e^{3ik\Delta x/2}\right] + \frac{2H^3}{\Delta x^2}\left[-u_j e^{-ik\Delta x/2} + 2u_j e^{ik\Delta x/2} - u_j e^{3ik\Delta x/2}\right] \quad (12)$$

$$\left[e^{-ik\Delta x}\mathcal{R}^{+} + 2\mathcal{R}^{-} + 2\mathcal{R}^{+} + e^{ik\Delta x}\mathcal{R}^{-}\right]G_j =$$
$$H\left[e^{-ik\Delta x} + 4 + e^{ik\Delta x}\right]u_j e^{ik\Delta x/2} + \frac{2H^3}{\Delta x^2}\left[-e^{ik\Delta x} + 2 - e^{ik\Delta x}\right]u_j e^{ik\Delta x/2} \quad (13)$$

$$\left[e^{-ik\Delta x}\mathcal{R}^{+} + 2\mathcal{R}^{-} + 2\mathcal{R}^{+} + e^{ik\Delta x}\mathcal{R}^{-}\right]G_j =$$
$$H\left[2\cos(k\Delta x) + 4\right]u_j e^{ik\Delta x/2} + \frac{2H^3}{\Delta x^2}\left[2 - 2\cos(k\Delta x)\right]u_j e^{ik\Delta x/2} \quad (14)$$

From the previous dispersion analysis we know that

$$\mathcal{R}^{-} = 1 + \frac{i\sin(k\Delta x)}{2}$$

$$\mathcal{R}^{+} = e^{ik\Delta x}\left(1 - \frac{i\sin(k\Delta x)}{2}\right)$$

So we have

$$\begin{aligned} &[\left(1 - \frac{i\sin(k\Delta x)}{2}\right) + 2\left[1 + \frac{i\sin(k\Delta x)}{2}\right] \\ &\quad + 2\left[e^{ik\Delta x}\left(1 - \frac{i\sin(k\Delta x)}{2}\right)\right] + e^{ik\Delta x}\left[1 + \frac{i\sin(k\Delta x)}{2}\right]]G_j = \\ &\quad H\left[2\cos(k\Delta x) + 4\right]u_j e^{ik\Delta x/2} + \frac{2H^3}{\Delta x^2}\left[2 - 2\cos(k\Delta x)\right]u_j e^{ik\Delta x/2} \end{aligned} \tag{15}$$

$$\begin{aligned} &\left[3 + \frac{i\sin(k\Delta x)}{2} + e^{ik\Delta x}\left(3 - \frac{i\sin(k\Delta x)}{2}\right)\right]G_j = \\ &\quad H\left[2\cos(k\Delta x) + 4\right]u_j e^{ik\Delta x/2} + \frac{2H^3}{\Delta x^2}\left[2 - 2\cos(k\Delta x)\right]u_j e^{ik\Delta x/2} \end{aligned} \tag{16}$$

$$\begin{aligned} G_j &= \\ &2He^{ik\Delta x/2}\frac{\cos(k\Delta x) + 2}{3 + \frac{i\sin(k\Delta x)}{2} + e^{ik\Delta x}\left(3 - \frac{i\sin(k\Delta x)}{2}\right)}u_j \\ &\quad + \frac{4H^3}{\Delta x^2}e^{ik\Delta x/2}\frac{1 - \cos(k\Delta x)}{3 + \frac{i\sin(k\Delta x)}{2} + e^{ik\Delta x}\left(3 - \frac{i\sin(k\Delta x)}{2}\right)}u_j \end{aligned} \tag{17}$$

$$\begin{aligned} G_j &= \\ &H2e^{ik\Delta x/2}\frac{\cos(k\Delta x) + 2}{3 + \frac{i\sin(k\Delta x)}{2} + e^{ik\Delta x}\left(3 - \frac{i\sin(k\Delta x)}{2}\right)}u_j \\ &\quad + \frac{H^3}{3}\frac{12}{\Delta x^2}e^{ik\Delta x/2}\frac{1 - \cos(k\Delta x)}{3 + \frac{i\sin(k\Delta x)}{2} + e^{ik\Delta x}\left(3 - \frac{i\sin(k\Delta x)}{2}\right)}u_j \end{aligned} \tag{18}$$

We want something like

$$1 \approx 2e^{ik\Delta x/2} \frac{\cos(k\Delta x) + 2}{3 + \frac{i\sin(k\Delta x)}{2} + e^{ik\Delta x}\left(3 - \frac{i\sin(k\Delta x)}{2}\right)}$$

and

$$k^2 \approx \frac{12}{\Delta x^2} e^{ik\Delta x/2} \frac{1 - \cos(k\Delta x)}{3 + \frac{i\sin(k\Delta x)}{2} + e^{ik\Delta x}\left(3 - \frac{i\sin(k\Delta x)}{2}\right)}$$

and we want to compare it to the FD approximation

$$k^2 \approx \frac{2}{\Delta x^2}\left(1 - \cos(k\Delta x)\right)$$

and

$$1 \approx 1$$

For the FEM we have the taylor series

$$\frac{12}{\Delta x^2} e^{ik\Delta x/2} \frac{1 - \cos(k\Delta x)}{3 + \frac{i\sin(k\Delta x)}{2} + e^{ik\Delta x}\left(3 - \frac{i\sin(k\Delta x)}{2}\right)} = \\ k^2 - \frac{k^4\Delta x^2}{24} + \frac{91k^6\Delta x^4}{5760} - \frac{1259k^8\Delta x^6}{967680} + \frac{44327k^{10}\Delta x^8}{1454828800} + O(\Delta x^{10}) \quad (19)$$

For the FD

$$\frac{2}{\Delta x^2}\left(1 - \cos(k\Delta x)\right) = \\ k^2 - \frac{k^4\Delta x^2}{12} + \frac{k^6\Delta x^4}{360} - \frac{k^8\Delta x^6}{20160} - \frac{k^{10}\Delta x^8}{1814400} + O(\Delta x^{10}) \quad (20)$$

We also have for the FEM

$$2e^{ik\Delta x/2} \frac{\cos(k\Delta x) + 2}{3 + \frac{i\sin(k\Delta x)}{2} + e^{ik\Delta x}\left(3 - \frac{i\sin(k\Delta x)}{2}\right)} = \\ 1 - \frac{k^2\Delta x^2}{8} + \frac{3k^4\Delta x^4}{128} - \frac{121k^6\Delta x^6}{46080} + \frac{14227k^8\Delta x^8}{30965760} + O(\Delta x^{10}) \quad (21)$$

In our experiments we set $H = 1$ and so our expression for the $\mathcal{G}$ factor is analytically

$$1 + \frac{k^3}{3}$$

For our FEM this is approximated by (taylor expansion)

$$1 - \frac{k^2\Delta x^2}{8} + \frac{3k^4\Delta x^4}{128} - \frac{121k^6\Delta x^6}{46080} + \frac{14227k^8\Delta x^8}{30965760} + O(\Delta x^{10}) + \frac{1}{3}\left(k^2 - \frac{k^4\Delta x^2}{24} + \frac{91k^6\Delta x^4}{5760} - \frac{1259k^8\Delta x^6}{967680} + \frac{44327k^{10}\Delta x^8}{1454828800} + O(\Delta x^{10})\right) \tag{22}$$

$$1 - \frac{k^2\Delta x^2}{8} + \frac{3k^4\Delta x^4}{128} - \frac{121k^6\Delta x^6}{46080} + \frac{14227k^8\Delta x^8}{30965760} + O(\Delta x^{10}) + \frac{k^2}{3} - \frac{k^4\Delta x^2}{72} + \frac{91k^6\Delta x^4}{17280} - \frac{1259k^8\Delta x^6}{2903040} + \frac{44327k^{10}\Delta x^8}{4364486400} + O(\Delta x^{10}) \tag{23}$$

$$1 + \frac{k^2}{3} - \frac{(k^4 + 9k^2)\Delta x^2}{72} + \frac{(91k^6 + 405k^4)\Delta x^4}{17280} - \frac{(1259k^8 + 7623k^6)\Delta x^6}{2903040} + O(\Delta x^8) \tag{24}$$

For our FD this is approximated by (taylor expansion)

$$1 + \frac{k^2}{3} - \frac{k^4\Delta x^2}{36} + \frac{k^6\Delta x^4}{1080} - \frac{k^8\Delta x^6}{60480} + O(\Delta x^{80}) \tag{25}$$

Taking $k = 0.5$
then we have

$$\frac{(k^4 + 9k^2)\Delta x^2}{72} = 0.03211805555 > 0.00173611111 = \frac{k^4\Delta x^2}{36}$$

And for $k = 2.5$ we have

$$\frac{(k^4 + 9k^2)\Delta x^2}{72} = 1.323784722 > 0.17361111111 = \frac{k^4\Delta x^2}{36}$$

So our FEM is worse