1 Cell Averaged G

We have a quartic:

$$Q_j(x) = a_j(x - x_j)^4 + b_j(x - x_j)^3 + c_j(x - x_j)^2 + d_j(x - x_j) + e_j$$

Satisfying the equations

$$Q_j(x_j) = q_j$$
$$Q_j(x_{j+1}) = q_{j+1}$$
$$Q_j(x_{j+2}) = q_{j+2}$$
$$Q_j(x_{j-1}) = q_{j-1}$$
$$Q_j(x_{j-2}) = q_{j-2}$$

In particular we have that

$$q_j = e_j \tag{1}$$

$$q_{j+1} = a_j(\Delta x)^4 + b_j(\Delta x)^3 + c_j(\Delta x)^2 + d_j(\Delta x) + e_j \tag{2}$$

$$q_{j+2} = 16a_j(\Delta x)^4 + 8b_j(\Delta x)^3 + 4c_j(\Delta x)^2 + 2d_j(\Delta x) + e_j \tag{3}$$

$$q_{j-1} = a_j(\Delta x)^4 - b_j(\Delta x)^3 + c_j(\Delta x)^2 - d_j(\Delta x) + e_j \tag{4}$$

$$q_{j-2} = 16a_j(\Delta x)^4 - 8b_j(\Delta x)^3 + 4c_j(\Delta x)^2 - 2d_j(\Delta x) + e_j \tag{5}$$

Adding (2) and (4) gives

$$q_{j+1} + q_{j-1} = 2a_j(\Delta x)^4 + 2c_j(\Delta x)^2 + 2q_j \tag{6}$$

Adding (3) and (5) gives

$$q_{j+2} + q_{j-2} = 32a_j(\Delta x)^4 + 8c_j(\Delta x)^2 + 2q_j \tag{7}$$

(7) - 4 × (6):

$$q_{j+2} + q_{j-2} - 4q_{j+1} - 4q_{j-1} = 24a_j(\Delta x)^4 - 6q_j$$

$$q_{j+2} - 4q_{j+1} + 6q_j - 4q_{j-1} + q_{j-2} = 24a_j(\Delta x)^4$$

$$a_j = \frac{q_{j+2} - 4q_{j+1} + 6q_j - 4q_{j-1} + q_{j-2}}{24\Delta x^4} \tag{8}$$

Subbing the known values into (6) gives

$$q_{j+1} + q_{j-1} = 2\frac{q_{j+2} - 4q_{j+1} + 6q_j - 4q_{j-1} + q_{j-2}}{24\Delta x^4}(\Delta x)^4 + 2c_j(\Delta x)^2 + 2q_j$$

$$q_{j+1} + q_{j-1} = \frac{q_{j+2} - 4q_{j+1} + 6q_j - 4q_{j-1} + q_{j-2}}{12} + 2c_j(\Delta x)^2 + 2q_j$$

$$q_{j+1} + q_{j-1} - 2q_j + \frac{-q_{j+2} + 4q_{j+1} - 6q_j + 4q_{j-1} - q_{j-2}}{12} = 2c_j(\Delta x)^2$$

$$\frac{12q_{j+1} + 12q_{j-1} - 24q_j - q_{j+2} + 4q_{j+1} - 6q_j + 4q_{j-1} - q_{j-2}}{12} = 2c_j(\Delta x)^2$$

$$\frac{-q_{j+2} + 16q_{j+1} - 30q_j + 16q_{j-1} - q_{j-2}}{12} = 2c_j(\Delta x)^2$$

$$c_j = \frac{-q_{j+2} + 16q_{j+1} - 30q_j + 16q_{j-1} - q_{j-2}}{24\Delta x^2} \tag{9}$$

Subtracting (4) from (2) gives

$$q_{j+1} - q_{j-1} = 2b_j(\Delta x)^3 + 2d_j(\Delta x) \tag{10}$$

Subtracting (5) from (3) gives

$$q_{j+2} - q_{j-2} = 16b_j(\Delta x)^3 + 4d_j(\Delta x) \tag{11}$$

(11) - 2 × (10):

$$q_{j+2} - q_{j-2} - 2q_{j+1} + 2q_{j-1} = 12b_j(\Delta x)^3$$

$$b_j = \frac{q_{j+2} - 2q_{j+1} + 2q_{j-1} - q_{j-2}}{12\Delta x^3} \tag{12}$$

Subbing the known values in to (10) gives

$$q_{j+1} - q_{j-1} = \frac{q_{j+2} - 2q_{j+1} - q_{j-2} + 2q_{j-1}}{6} + 2d_j(\Delta x)$$

$$q_{j+1} - q_{j-1} + \frac{-q_{j+2} + 2q_{j+1} + q_{j-2} - 2q_{j-1}}{6} = +2d_j(\Delta x)$$

$$\frac{6q_{j+1} - 6q_{j-1} - q_{j+2} + 2q_{j+1} + q_{j-2} - 2q_{j-1}}{6} = 2d_j(\Delta x)$$

$$\frac{-q_{j+2} + 8q_{j+1} - 8q_{j-1} + q_{j-2}}{6} = 2d_j(\Delta x)$$

$$d_j = \frac{-q_{j+2} + 8q_{j+1} - 8q_{j-1} + q_{j-2}}{12\Delta x} \tag{13}$$

So to recall we have

$$a_j = \frac{q_{j+2} - 4q_{j+1} + 6q_j - 4q_{j-1} + q_{j-2}}{24\Delta x^4}$$

$$b_j = \frac{q_{j+2} - 2q_{j+1} + 2q_{j-1} - q_{j-2}}{12\Delta x^3}$$

$$c_j = \frac{-q_{j+2} + 16q_{j+1} - 30q_j + 16q_{j-1} - q_{j-2}}{24\Delta x^2}$$

$$d_j = \frac{-q_{j+2} + 8q_{j+1} - 8q_{j-1} + q_{j-2}}{12\Delta x}$$

$$e_j = q_j$$

test this for $Q_j(x_{j+1}) = q_{j+1}$

$$\frac{q_{j+2} - 4q_{j+1} + 6q_j - 4q_{j-1} + q_{j-2}}{24\Delta x^4}(\Delta x)^4 + \frac{q_{j+2} - 2q_{j+1} + 2q_{j-1} - q_{j-2}}{12\Delta x^3}(\Delta x)^3$$
$$+\frac{-q_{j+2} + 16q_{j+1} - 30q_j + 16q_{j-1} - q_{j-2}}{24\Delta x^2}(\Delta x)^2 + \frac{-q_{j+2} + 8q_{j+1} - 8q_{j-1} + q_{j-2}}{12\Delta x}(\Delta x) + q_j$$

$$= \frac{q_{j+2} - 4q_{j+1} + 6q_j - 4q_{j-1} + q_{j-2}}{24} + \frac{q_{j+2} - 2q_{j+1} + 2q_{j-1} - q_{j-2}}{12}$$
$$+\frac{-q_{j+2} + 16q_{j+1} - 30q_j + 16q_{j-1} - q_{j-2}}{24} + \frac{-q_{j+2} + 8q_{j+1} - 8q_{j-1} + q_{j-2}}{12} + q_j$$

$$= \frac{q_{j+2} - 4q_{j+1} + 6q_j - 4q_{j-1} + q_{j-2}}{24} + \frac{2q_{j+2} - 4q_{j+1} + 4q_{j-1} - 2q_{j-2}}{24}$$

$$+ \frac{-q_{j+2} + 16q_{j+1} - 30q_j + 16q_{j-1} - q_{j-2}}{24} + \frac{-2q_{j+2} + 16q_{j+1} - 16q_{j-1} + 2q_{j-2}}{24} + q_j$$

$$= \frac{24q_{j+1}}{24} = q_{j+1} \square$$