

Undular Bores of the Serre Equations

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Undular Bores

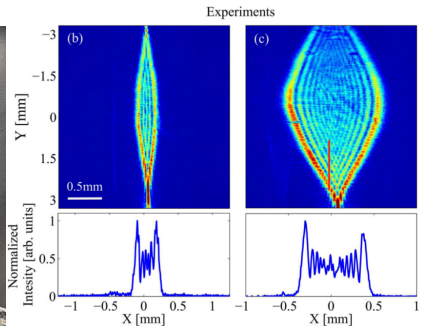


Figure: examples of undular bores from tidal flows to even optics.

Dam Break

Fluid depth (h) :

$$h(x, 0) = \begin{cases} h_1 & x \leq x_0 \\ h_0 & x > x_0 \end{cases}$$

Fluid velocity (u) :

$$u(x, 0) = 0.0.$$

Smoothing

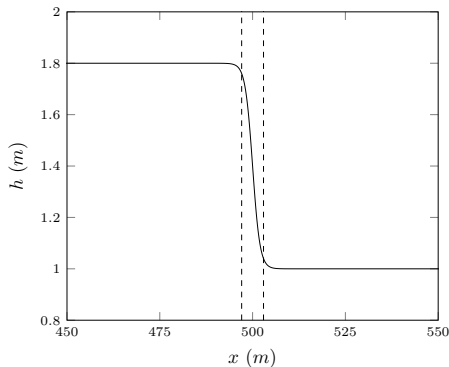


Figure: Example of water profile of a smoothed dam break with a transition width β of 5.8888.

Serre Equations

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0,$$

$$\underbrace{\frac{\partial(uh)}{\partial t} + \frac{\partial}{\partial x} \left(u^2 h + \frac{gh^2}{2} \right)}_{\text{Shallow Water Wave Equations}} + \underbrace{\frac{\partial}{\partial x} \left(\frac{h^3}{3} \left[\frac{\partial u}{\partial x} \frac{\partial u}{\partial x} - u \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t} \right] \right)}_{\text{Dispersion Terms}} = 0$$

Serre Equations

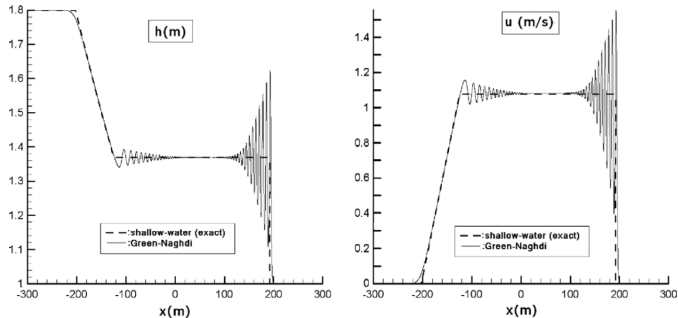


Figure: Fluid depth at 48s obtained from numerical method by Le Métayer (Le Métayer et al., 2010) (The Serre equations are also known as the Green Naghdi equations).

Numerical

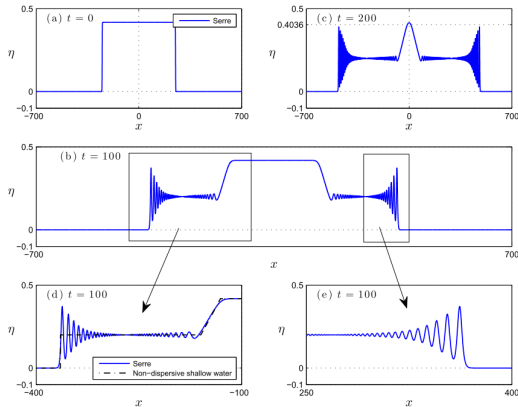


Figure: Wave height at various times for the smoothed dam break problem obtained from numerical method by Mitsotakis (Mitsotakis et al., 2014).

SWW equations

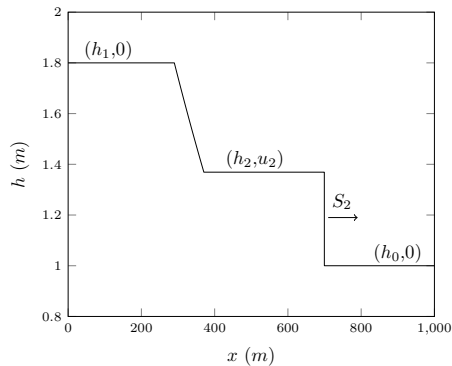


Figure: SWW analytic solution to dam break problem.

$$h_2 = \frac{h_0}{2} \left[\sqrt{1 + 8 \left(\frac{2h_2}{h_2 - h_0} \frac{\sqrt{gh_1} - \sqrt{gh_2}}{\sqrt{gh_0}} \right)^2} - 1 \right],$$

$$u_2 = 2 \left(\sqrt{gh_1} - \sqrt{gh_2} \right),$$

$$S_2 = \frac{h_2 u_2}{h_2 - h_0}.$$

El and Grimshaws Whitham Modulation

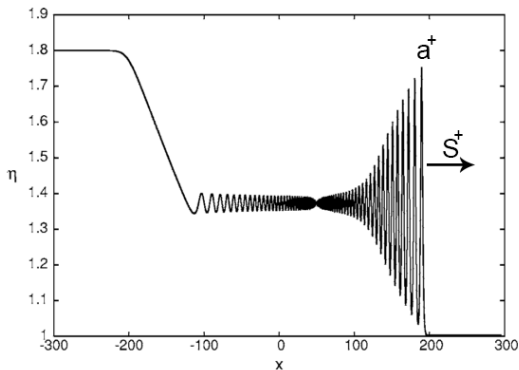


Figure: Whitham modulation values demonstrated on El and Grimshaws numerical results

$$\frac{\Delta}{(a^+ + 1)^{1/4}} - \left(\frac{3}{4 - \sqrt{a^+ + 1}} \right)^{21/10} \left(\frac{2}{1 + \sqrt{a^+ + 1}} \right)^{2/5} = 0$$

$$S^+ = \sqrt{g(a^+ + 1)}$$

where $\Delta = \frac{h_1 - h_0}{h_0}$. Appropriate when $\Delta \leq 1.43$.

Findings

Literature

- ▶ El and Grimshaws numerical and analytic results supported, but do not give the full picture.
- ▶ Le Métayers first order scheme is too diffusive.
- ▶ Mistotakis initial conditions were not sufficiently steep.
- ▶ SWW analytic solution is a useful guide for the mean behaviour of the fluid.

Methods

Le Métayer methods.

- ▶ First order
- ▶ Second order
- ▶ Third order

Finite Difference Method

- ▶ El and Grimshaws

Initial Conditions

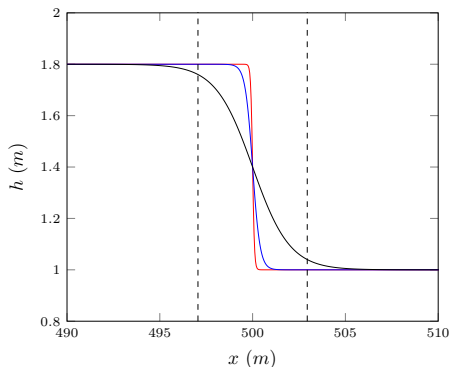


Figure: Initial Conditions where $\beta = 0.294$ (—), $\beta = 1.17778$ (—), $\beta = 5.8888$ (—) with reference β interval(— —).

$$\beta = 5.8888$$

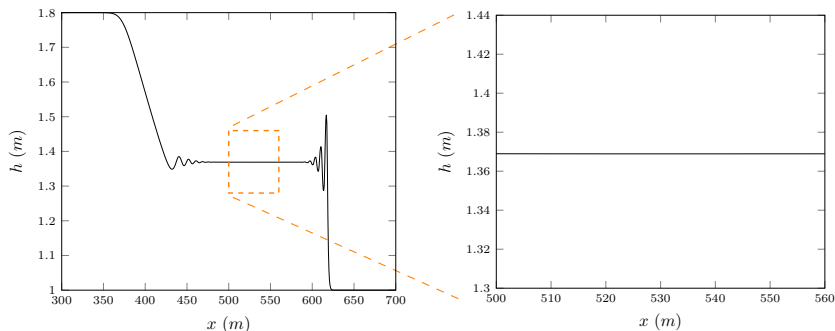


Figure: Numerical results of third order Le Métayer method at 30s with $\Delta x = \frac{10}{24}$ (-).

Water Profile

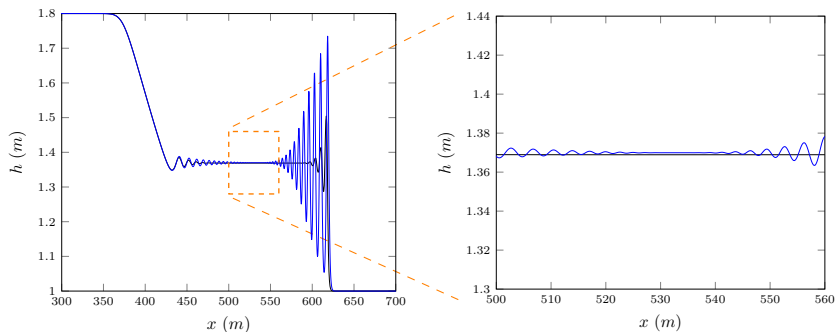


Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{2^4}$ (—) and $\frac{10}{2^7}$ (—).

Water Profile

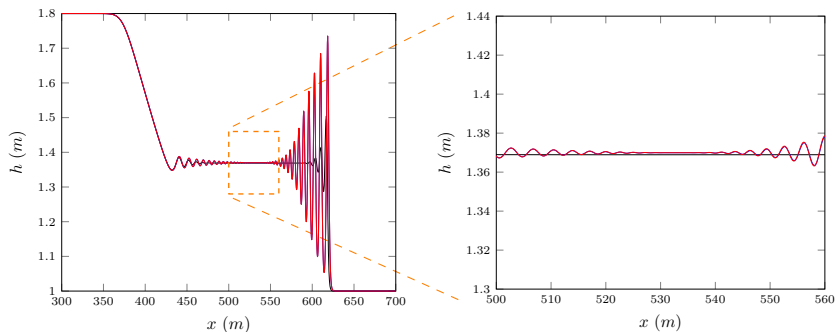


Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{2^4}$ (—), $\frac{10}{2^7}$ (—) and $\frac{10}{2^{10}}$ (—).

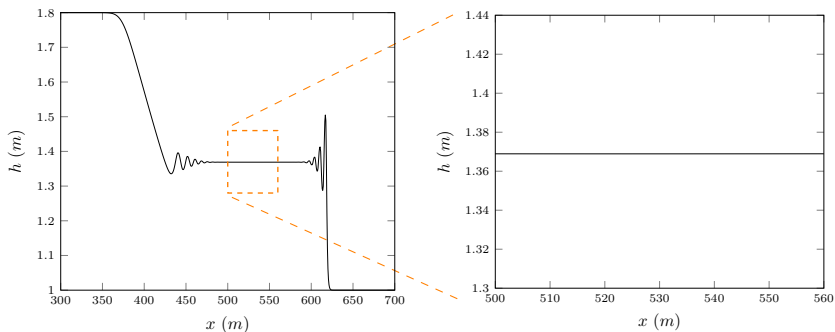


Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{24}$ (—).

Water Profile

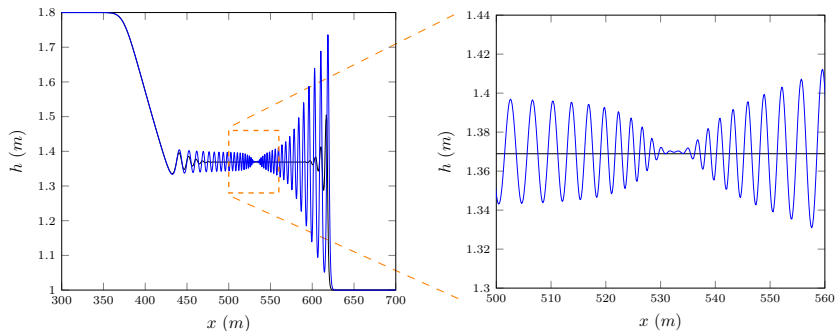


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Water Profile

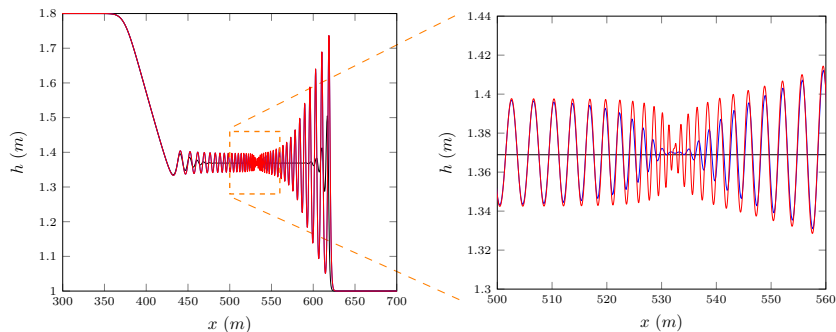


Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{2^4}$ (—), $\frac{10}{2^7}$ (—) and $\frac{10}{2^{10}}$ (—).

Dispersion Relation

The dispersion relation for the linearised Serre equations is

$$\omega = u_0 k \pm k \sqrt{gh_0} \sqrt{\frac{3}{h_0^2 k^2 + 3}}$$

Thus the phase speed is

$$v_p = u_0 \pm \sqrt{gh_0} \sqrt{\frac{3}{h_0^2 k^2 + 3}}$$

Taking $k \rightarrow 0$ we see $v_p \rightarrow u_0 \pm \sqrt{gh_0}$

Taking $k \rightarrow \infty$ we see $v_p \rightarrow u_0$

Contact Discontinuity

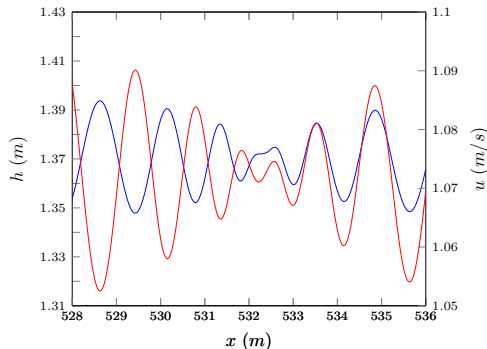


Figure: plot of h (—) and u (—) around contact discontinuity for third order Le Métayer method with $\Delta x = \frac{10}{2^{10}}$ at 30s.

Introduction

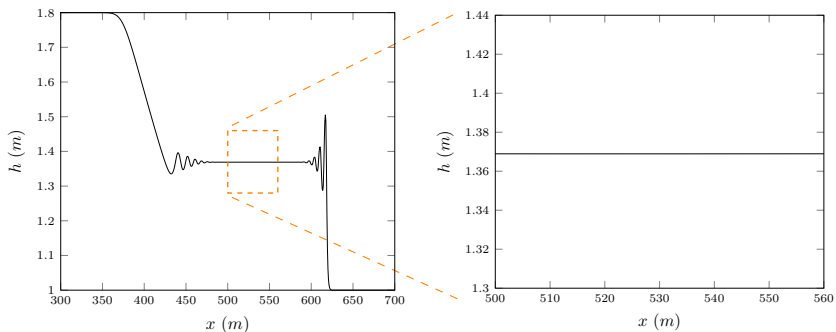


Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{24}$ (—).

Water Profile

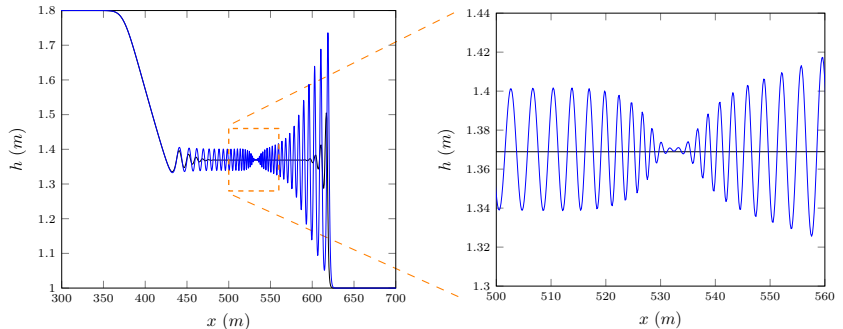


Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{2^4}$ (—) and $\frac{10}{2^7}$ (---).

Water Profile

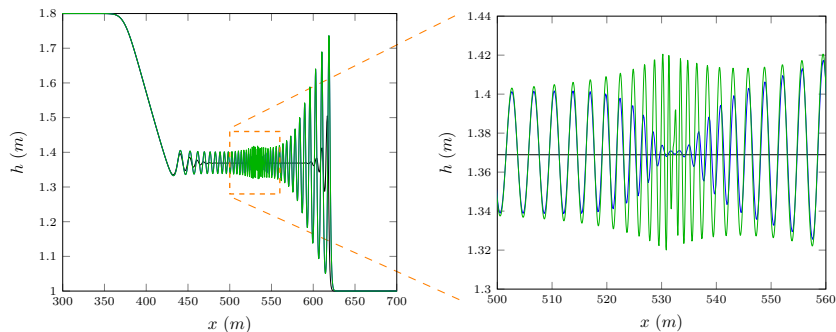


Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{2^4}$ (—), $\frac{10}{2^7}$ (—) and $\frac{10}{2^9}$ (—).

Water Profile

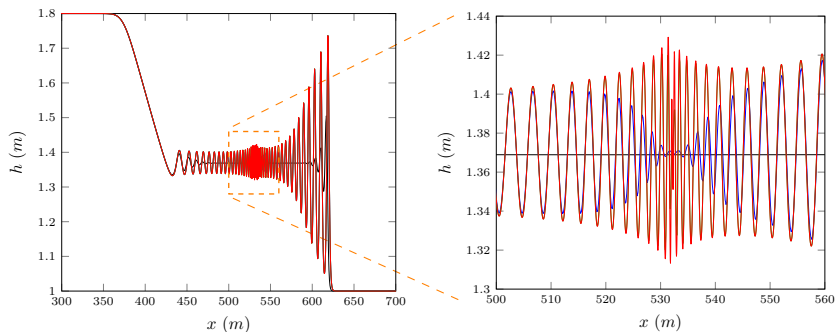


Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{2^4}$ (—), $\frac{10}{2^7}$ (—), $\frac{10}{2^9}$ (—) and $\frac{10}{2^{10}}$ (—).

$\beta = 0.294$ Various Models

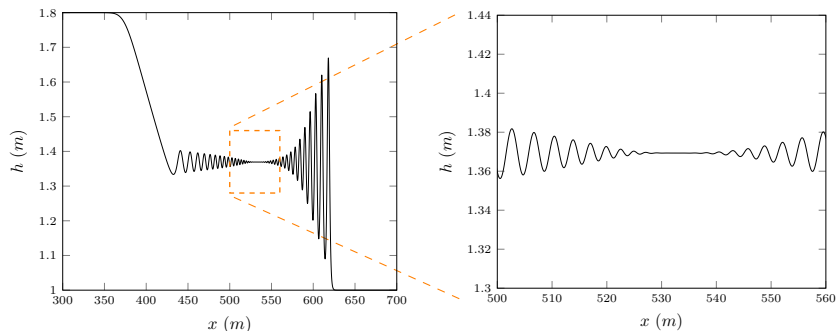


Figure: Numerical results at 30s with a $\Delta x = \frac{10}{2^{10}}$ for the first order(—) Le Métayer method.

Water Profile

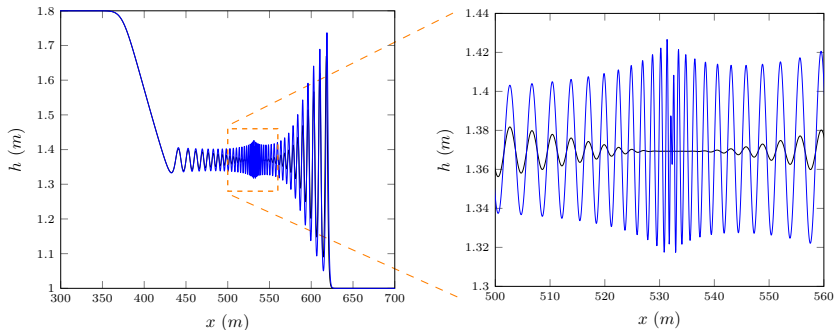


Figure: Numerical results at 30s with a $\Delta x = \frac{10}{2^{10}}$ for the first order (—) and second order (—) Le Métayer method.

Water Profile

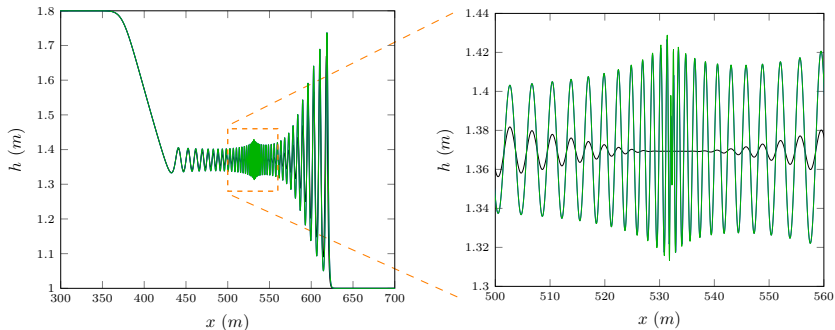


Figure: Numerical results at 30s with a $\Delta x = \frac{10}{2^{10}}$ for the first order (—), second order (—) and third order (—) Le Métayer method.

Water Profile

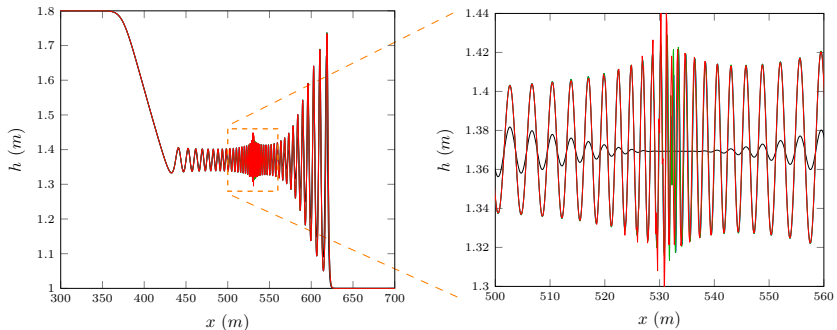


Figure: Numerical results at 30s with a $\Delta x = \frac{10}{2^{10}}$ for the first order (—), second order (—) and third order (—) Le Métayer method and El and Grimshaws method (—).

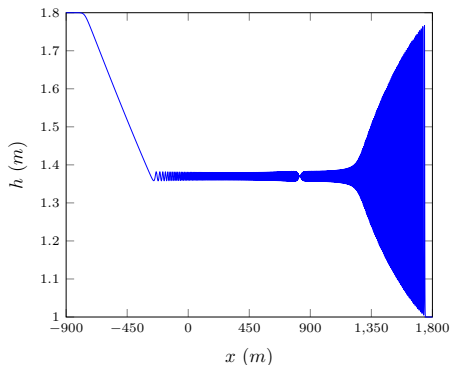
$\beta = 0.294$ Long Time

Figure: Numerical results at 300s with $\Delta x = 10/2^9$ for third-order Le Métayer Method.

Water Profile

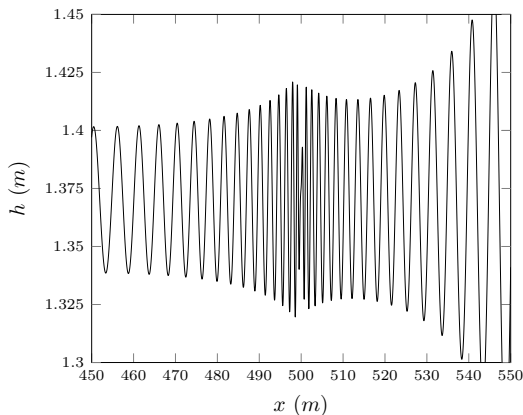


Figure: Translated numerical results with $\Delta x = 10/2^9$ at 30s (—) using third order Le Métayer method.

Water Profile

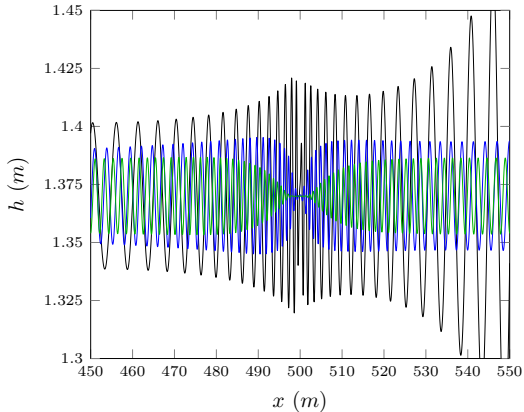


Figure: Translated numerical results with $\Delta x = 10/2^9$ at 30s (—) , 100s (—) , 200s (—) using third order Le Métayer method.

Water Profile

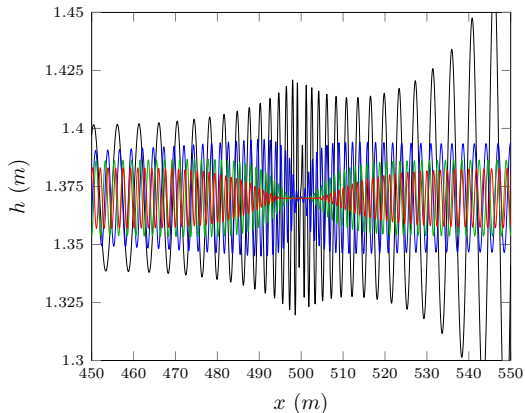


Figure: Translated numerical results with $\Delta x = 10/2^9$ at 30s (—) , 100s (—) , 200s (—) and 300s (—) using third order Le Métayer method.

SWWE Solution Comparison

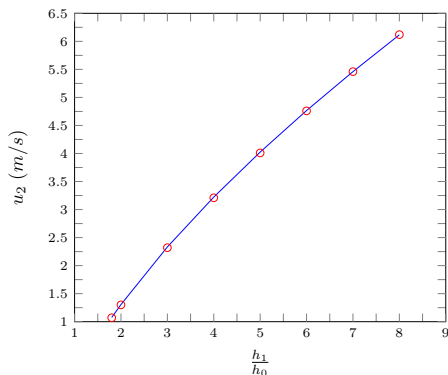


Figure: compares u_2 (—) to the average speed of the contact discontinuity (○) for third order Le Métayer method with $\Delta x = \frac{10}{2^9}$ at 300s.

SWWE Solution Comparison

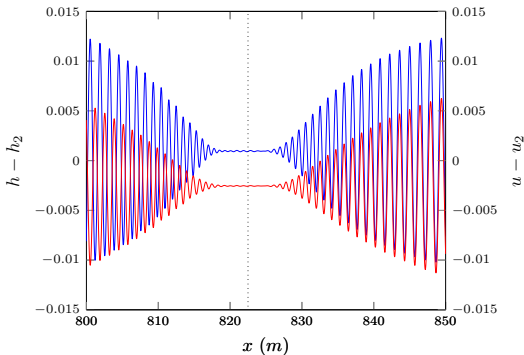


Figure: plot of $h - h_2$ (—) and $u - u_2$ (—) with x_2 (···) for third order Le Métayer method with $\Delta x = \frac{10}{2^9}$ at 300s.

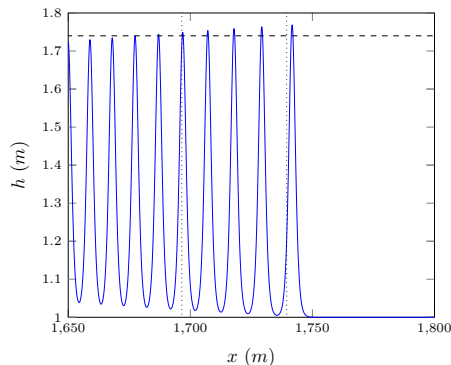


Figure: Plot comparing numerical results for shock front of the Serre equations to $S^+(\dots)$, $S_2(\dots)$ and $a^+(- -)$ for third order Le Métayer method with $\Delta x = \frac{10}{29}$ at 300s.

Els Analytic Comparison

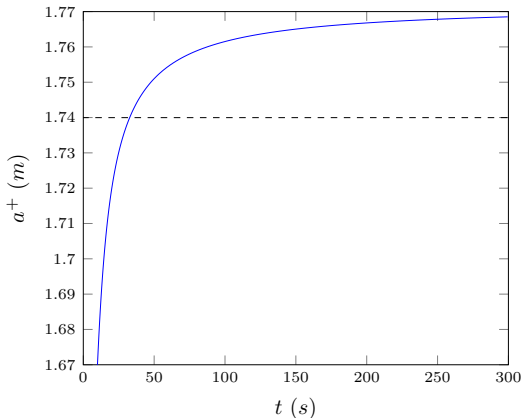


Figure: Plot of lead oscillation amplitude over time for third order Le Métayer method with $\Delta x = \frac{10}{29}$ at 300s with analytic comparison (— —)

Conclusions

Literature

- ▶ Supports Els results
 - ▶ Best numerical results for the dam break problem
 - ▶ a^+ seems to underestimate lead oscillation amplitude
 - ▶ S^+ underestimates speed
- ▶ Le Métayers first order scheme is too diffusive
- ▶ Mistotakis initial conditions were not sufficiently steep
- ▶ SWW analytic solution is a useful guide for the mean bore height h_2 (underestimate), mean bore velocity u_2 (overestimate) and speed of the bore S_2 (underestimate)

References I

El, G., Grimshaw, R. H. J., and Smyth, N. F. (2006).
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Physics of Fluids, 18(027104).

Le Métayer, O., Gavriluk, S., and Hank, S. (2010).
A numerical scheme for the GreenNaghdi model.
Journal of Computational Physics, 229(6):2034–2045.

Mitsotakis, D., Dutykh, D., and Carter, J. (2014).
On the nonlinear dynamics of the traveling-wave solutions of
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Supplementary Plots

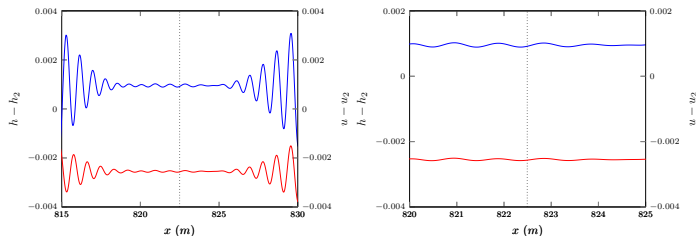
Zoom in on u and h 

Figure: plot of $h - h_2$ (—) and $u - u_2$ (—) with x_2 (···)

Supplementary Plots

Zoom in on all Models

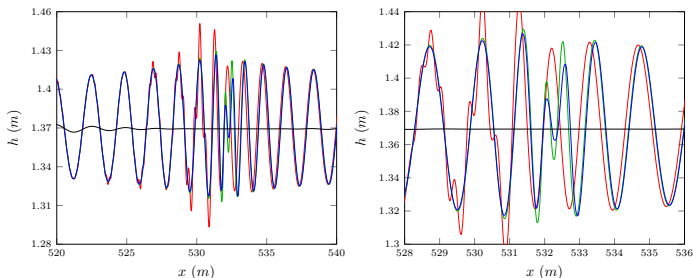


Figure: Numerical results at 30s with a $\Delta x = \frac{10}{2^{10}}$ for the first order (—), second order (—) and third order (—) Le Métayer method and El and Grimshaws method (—).

Serre Equations

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0,$$

$$\underbrace{\frac{\partial(uh)}{\partial t} + \frac{\partial}{\partial x} \left(u^2 h + \frac{gh^2}{2} \right)}_{\text{Shallow Water Wave Equations}} + \underbrace{\frac{\partial}{\partial x} \left(\frac{h^3}{3} \left[\frac{\partial u}{\partial x} \frac{\partial u}{\partial x} - u \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t} \right] \right)}_{\text{Dispersion Terms}} = 0$$

Serre Equations

Conservation Law Form

New conserved quantity

$$G = uh - h^2 \frac{\partial h}{\partial x} \frac{\partial u}{\partial x} - \frac{h^3}{3} \frac{\partial^2 u}{\partial x^2}. \quad (6)$$

Reformulated equations

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0 \quad (7a)$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x} \left(Gu + \frac{gh^2}{2} - \frac{2h^3}{3} \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} \right) = 0 \quad (7b)$$

Basic Overview

Vector of conserved quantities:

$$\mathbf{U} = \begin{bmatrix} h \\ G \end{bmatrix}$$

Algorithm:

$$\mathcal{H}(\bar{\mathbf{U}}^n, \Delta x, \Delta t) = \begin{cases} \mathbf{U}^n & = \mathcal{M}(\bar{\mathbf{U}}^n) \\ \mathbf{u}^n & = \mathcal{A}(\mathbf{U}^n, \Delta x) \\ \bar{\mathbf{U}}^{n+1} & = \mathcal{L}(\bar{\mathbf{U}}^n, \mathbf{u}^n, \Delta x, \Delta t) \end{cases} .$$