

1 Elliptic Equation

The linearised elliptic equation is

$$G = uh3\frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

$$G = uh - h^2h_xu_x - \frac{h^3}{3}u_{xx}$$

now we replace

$$u = U(t)e^{ikx}$$

$$h = H(t)e^{ikx}$$

$$G = UHe^{2ikx} - H^2e^{2ikx}ikHe^{ikx}ikUe^{ikx} - \frac{1}{3}H^3e^{3ikx}(-k^2)Ue^{ikx}$$

$$G = UHe^{2ikx} + k^2H^3Ue^{4ikx} + \frac{k^2}{3}UH^3e^{4ikx}$$

$$G = \left(1 + \frac{4}{3}k^2H^2e^{2ikx}\right)UHe^{2ikx}$$

$$G_j = \left(1 + \frac{4}{3}k^2H^2e^{2ikx_j}\right)UHe^{2ikx_j}$$

2 Finite Difference

we have the derivatives

$$\left(\frac{\partial q}{\partial x}\right)_j = \frac{q_{j+1} - q_{j-1}}{2\Delta x} = \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}Q(t)e^{ikx_j} = \frac{i \sin(k\Delta x)}{\Delta x}Q(t)e^{ikx_j}$$

$$\left(\frac{\partial^2 q}{\partial x^2}\right)_j = \frac{q_{j+1} - 2q_j + q_{j-1}}{\Delta x^2} = \frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{\Delta x^2}Q(t)e^{ikx_j} = \frac{2 \cos(k\Delta x) - 2}{\Delta x^2}Q(t)e^{ikx_j}$$

$$\left(\frac{\partial^2 q}{\partial x^2}\right)_j = -4 \frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^2} Q(t) e^{ikx_j} = - \left(\frac{2 \sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2 Q(t) e^{ikx_j}$$

$$G = uh - h^2 h_x u_x - \frac{h^3}{3} u_{xx}$$

$$G_j = UH e^{2ikx_j} - H^2 e^{2ikx_j} \frac{i \sin(k\Delta x)}{\Delta x} H e^{ikx_j} \frac{i \sin(k\Delta x)}{\Delta x} U e^{ikx_j} + \frac{1}{3} H^3 e^{3ikx_j} \left(\frac{2 \sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2 U e^{ikx_j}$$

$$G_j = UH e^{2ikx_j} + UH^3 e^{4ikx_j} \left(\frac{\sin(k\Delta x)}{\Delta x}\right)^2 + \frac{1}{3} UH^3 e^{4ikx_j} \left(\frac{2 \sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2$$

$$G_j = \left(1 + H^2 e^{2ikx_j} \left(\frac{\sin(k\Delta x)}{\Delta x}\right)^2 + \frac{1}{3} H^2 e^{2ikx_j} \left(\frac{2 \sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2\right) UH e^{2ikx_j}$$

$$G_j = \left(1 + \left[\left(\frac{\sin(k\Delta x)}{\Delta x}\right)^2 + \frac{1}{3} \left(\frac{2 \sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2\right] H^2 e^{2ikx_j}\right) UH e^{2ikx_j}$$

So we want

$$\left[\left(\frac{\sin(k\Delta x)}{\Delta x}\right)^2 + \frac{1}{3} \left(\frac{2 \sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2\right] \approx \frac{4}{3} k^2$$

Wolfram Alpha has

$$\left[\left(\frac{\sin(k\Delta x)}{\Delta x}\right)^2 + \frac{1}{3} \left(\frac{2 \sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2\right] = \frac{4k^2}{3} - \frac{13k^4 \Delta x^2}{36} + \frac{49k^6 \Delta x^4}{1080} + O(\Delta x^6)$$

So we can see this scheme is second order.

3 Finite Element

$$G = uh - \frac{\partial}{\partial x} \left(\frac{h^3}{3} u_x \right)$$

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = uhv - \frac{\partial}{\partial x} \left(\frac{h^3}{3} u_x \right) v$$

and then we integrate over the entire domain to get

$$\int_{\Omega} Gv dx = \int_{\Omega} uhv dx - \int_{\Omega} \frac{\partial}{\partial x} \left(\frac{h^3}{3} u_x \right) v dx$$

for all v

We then make use of integration by parts, with Dirichlet boundaries to get

$$\int_{\Omega} Gv dx = \int_{\Omega} uhv dx + \int_{\Omega} \frac{h^3}{3} u_x v_x dx$$

For u we are going to use $x_{j+1/2}$ as the nodes, which generate the basis functions $\phi_{j+1/2}$, which for us will be the space of continuous linear elements. While for G and h we will choose basis functions w that are linear from $[x_{j-1/2}, x_{j+1/2}]$ but discontinuous at the edges.

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gv dx = \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} uhv dx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^3}{3} u_x v_x dx$$

for all v

4 P1 FEM

We are going to coordinate tranform from x space the interval $[x_{j-1/2}, x_{j+1/2}, x_{j+3/2}]$ to the ξ space interval $[-1, 0, 1]$. To accomplish this we have the following relation

$$x = \xi \Delta x + x_{j+1/2}$$

Taking the derivatives we see
 $dx = d\xi \Delta x$, $\frac{dx}{d\xi} = \Delta x$, $\frac{d\xi}{dx} = \frac{1}{\Delta x}$.

We can describe the basis functions in the ξ space

$$\phi_{j+1/2} = \begin{cases} 1 + \xi & \xi < 0 \\ 1 - \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

$$\phi_{j-1/2} = \begin{cases} -\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

$$\phi_{j+3/2} = \begin{cases} \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

While the descriptions for w 's is

$$w_{j+1/2}^+ = \begin{cases} 1 - \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

$$w_{j+1/2}^- = \begin{cases} 1 + \xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

$$w_{j-1/2}^+ = \begin{cases} -\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

$$w_{j+3/2}^- = \begin{cases} \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

We now replace our functions by our approximations to them

$$G \approx G' = \sum_j G_{j+1/2} w_{j+1/2}$$

$$u \approx u' = \sum_j u_{j+1/2} \phi_{j+1/2}$$

$$h \approx h' = \sum_j h_{j+1/2} w_{j+1/2}$$

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} G' \phi_{j+1/2} dx - \int_{x_{j-1/2}}^{x_{j+3/2}} u' h' \phi_{j+1/2} dx - \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{(h')^3}{3} u'_x (\phi_x)_{j+1/2} dx = 0$$

For all $\phi_{j+1/2}$. For this analysis we choose a particular basis function $\phi_{j+1/2}$ and we look at all the integrals. Begining from the right

$$\begin{aligned}
& \int_{x_{j-1/2}}^{x_{j+3/2}} G'(x) \phi_{j+1/2} dx = \int_{-1}^1 G'(\xi) \phi_{j+1/2}(\xi) \frac{dx}{d\xi} d\xi \\
& = \Delta x \int_{-1}^1 \left(G_{j-1/2}^+ w_{j-1/2}^+ + G_{j+1/2}^- w_{j+1/2}^- + G_{j+1/2}^+ w_{j+1/2}^+ + G_{j-3/2}^- w_{j-3/2}^- \right) \phi_{j+1/2} d\xi \\
& = \Delta x [G_{j-1/2}^+ \int_{-1}^1 w_{j-1/2}^+ \phi_{j+1/2} d\xi + G_{j+1/2}^- \int_{-1}^1 w_{j+1/2}^- \phi_{j+1/2} d\xi \\
& \quad + G_{j+1/2}^+ \int_{-1}^1 w_{j+1/2}^+ \phi_{j+1/2} d\xi + G_{j+3/2}^- \int_{-1}^1 w_{j+3/2}^- \phi_{j+1/2} d\xi] \quad (8)
\end{aligned}$$

We have that

$$\int_{-1}^1 w_{j-1/2}^+ \phi_{j+1/2} d\xi = \int_{-1}^1 w_{j+3/2}^- \phi_{j+1/2} d\xi = \frac{1}{6}$$

and

$$\int_{-1}^1 w_{j+1/2}^- \phi_{j+1/2} d\xi = \int_{-1}^1 w_{j+1/2}^+ \phi_{j+1/2} d\xi = \frac{1}{3}$$

So

$$\begin{aligned}
& = \Delta x \left[\frac{1}{6} G_{j-1/2}^+ + \frac{1}{3} G_{j+1/2}^- + \frac{1}{3} G_{j+1/2}^+ + \frac{1}{6} G_{j+3/2}^- \right] \\
& = \frac{\Delta x}{6} \left[G_{j-1/2}^+ + 2G_{j+1/2}^- + 2G_{j+1/2}^+ + G_{j+3/2}^- \right]
\end{aligned}$$

Next we have

$$\begin{aligned}
& \int_{x_{j-1/2}}^{x_{j+3/2}} h' u' \phi_{j+1/2} dx = \Delta x \int_{-1}^1 h'(\xi) u'(\xi) \phi_{j+1/2}(\xi) d\xi \\
& = \Delta x \int_{-1}^1 \left(h_{j-1/2}^+ w_{j-1/2}^+ + h_{j+1/2}^- w_{j+1/2}^- + h_{j+1/2}^+ w_{j+1/2}^+ + h_{j-3/2}^- w_{j-3/2}^- \right) \\
& \quad (u_{j-1/2} \phi_{j-1/2} + u_{j+1/2} \phi_{j+1/2} + u_{j+3/2} \phi_{j+3/2}) \phi_{j+1/2} d\xi \quad (9)
\end{aligned}$$

$$\begin{aligned}
&= \Delta x \int_{-1}^1 \left(h_{j-1/2}^+ w_{j-1/2}^+ + h_{j+1/2}^- w_{j+1/2}^- + h_{j+1/2}^+ w_{j+1/2}^+ + h_{j-3/2}^- w_{j-3/2}^- \right) u_{j-1/2} \phi_{j-1/2} \phi_{j+1/2} \\
&+ \left(h_{j-1/2}^+ w_{j-1/2}^+ + h_{j+1/2}^- w_{j+1/2}^- + h_{j+1/2}^+ w_{j+1/2}^+ + h_{j-3/2}^- w_{j-3/2}^- \right) u_{j+1/2} \phi_{j+1/2} \phi_{j+1/2} \\
&+ \left(h_{j-1/2}^+ w_{j-1/2}^+ + h_{j+1/2}^- w_{j+1/2}^- + h_{j+1/2}^+ w_{j+1/2}^+ + h_{j-3/2}^- w_{j-3/2}^- \right) u_{j+3/2} \phi_{j+3/2} \phi_{j+1/2} d\xi
\end{aligned} \tag{10}$$

If one of the terms w_k , ϕ_l , ϕ_m is 0 then $w_k \phi_l \phi_m = 0$

$$\begin{aligned}
&= \Delta x \int_{-1}^1 \left(h_{j-1/2}^+ w_{j-1/2}^+ + h_{j+1/2}^- w_{j+1/2}^- \right) u_{j-1/2} \phi_{j-1/2} \phi_{j+1/2} \\
&+ \left(h_{j-1/2}^+ w_{j-1/2}^+ + h_{j+1/2}^- w_{j+1/2}^- + h_{j+1/2}^+ w_{j+1/2}^+ + h_{j-3/2}^- w_{j-3/2}^- \right) u_{j+1/2} \phi_{j+1/2} \phi_{j+1/2} \\
&\quad + \left(h_{j+1/2}^+ w_{j+1/2}^+ + h_{j-3/2}^- w_{j-3/2}^- \right) u_{j+3/2} \phi_{j+3/2} \phi_{j+1/2} d\xi \tag{11}
\end{aligned}$$

$$\begin{aligned}
&= \Delta x \int_{-1}^1 u_{j-1/2} h_{j-1/2}^+ w_{j-1/2}^+ \phi_{j-1/2} \phi_{j+1/2} + u_{j-1/2} h_{j+1/2}^- w_{j+1/2}^- \phi_{j-1/2} \phi_{j+1/2} \\
&\quad + u_{j+1/2} h_{j-1/2}^+ w_{j-1/2}^+ \phi_{j+1/2} \phi_{j+1/2} + u_{j+1/2} h_{j+1/2}^- w_{j+1/2}^- \phi_{j+1/2} \phi_{j+1/2} \\
&\quad + u_{j+1/2} h_{j+1/2}^+ w_{j+1/2}^+ \phi_{j+1/2} \phi_{j+1/2} + u_{j+1/2} h_{j-3/2}^- w_{j-3/2}^- \phi_{j+1/2} \phi_{j+1/2} \\
&\quad + u_{j+3/2} h_{j+1/2}^+ w_{j+1/2}^+ \phi_{j+3/2} \phi_{j+1/2} + u_{j+3/2} h_{j-3/2}^- w_{j-3/2}^- \phi_{j+3/2} \phi_{j+1/2} d\xi \tag{12}
\end{aligned}$$

Evaluating the integral

$$\begin{aligned}
&\int_{-1}^1 u_{j-1/2} h_{j-1/2}^+ w_{j-1/2}^+ \phi_{j-1/2} \phi_{j+1/2} + u_{j-1/2} h_{j+1/2}^- w_{j+1/2}^- \phi_{j-1/2} \phi_{j+1/2} \\
&\quad + u_{j+1/2} h_{j-1/2}^+ w_{j-1/2}^+ \phi_{j+1/2} \phi_{j+1/2} + u_{j+1/2} h_{j+1/2}^- w_{j+1/2}^- \phi_{j+1/2} \phi_{j+1/2} \\
&\quad + u_{j+1/2} h_{j+1/2}^+ w_{j+1/2}^+ \phi_{j+1/2} \phi_{j+1/2} + u_{j+1/2} h_{j-3/2}^- w_{j-3/2}^- \phi_{j+1/2} \phi_{j+1/2} \\
&\quad + u_{j+3/2} h_{j+1/2}^+ w_{j+1/2}^+ \phi_{j+3/2} \phi_{j+1/2} + u_{j+3/2} h_{j-3/2}^- w_{j-3/2}^- \phi_{j+3/2} \phi_{j+1/2} d\xi \tag{13}
\end{aligned}$$

$$\begin{aligned}
&= u_{j-1/2} h_{j-1/2}^+ \int_{-1}^1 w_{j-1/2}^+ \phi_{j-1/2} \phi_{j+1/2} d\xi + u_{j-1/2} h_{j+1/2}^- \int_{-1}^1 w_{j+1/2}^- \phi_{j-1/2} \phi_{j+1/2} d\xi \\
&+ u_{j+1/2} h_{j-1/2}^+ \int_{-1}^1 w_{j-1/2}^+ \phi_{j+1/2} \phi_{j+1/2} d\xi + u_{j+1/2} h_{j+1/2}^- \int_{-1}^1 w_{j+1/2}^- \phi_{j+1/2} \phi_{j+1/2} d\xi \\
&+ u_{j+1/2} h_{j+1/2}^+ \int_{-1}^1 w_{j+1/2}^+ \phi_{j+1/2} \phi_{j+1/2} d\xi + u_{j+1/2} h_{j-3/2}^- \int_{-1}^1 w_{j-3/2}^- \phi_{j+1/2} \phi_{j+1/2} d\xi \\
&+ u_{j+3/2} h_{j+1/2}^+ \int_{-1}^1 w_{j+1/2}^+ \phi_{j+3/2} \phi_{j+1/2} d\xi + u_{j+3/2} h_{j-3/2}^- \int_{-1}^1 w_{j-3/2}^- \phi_{j+3/2} \phi_{j+1/2} d\xi
\end{aligned} \tag{14}$$

Now we evaluate the integrals

$$\begin{aligned}
\int_{-1}^1 w_{j-1/2}^+ \phi_{j-1/2} \phi_{j+1/2} d\xi &= \int_{-1}^0 (-\xi) (-\xi) (1 + \xi) d\xi = \frac{1}{12} \\
\int_{-1}^1 w_{j+1/2}^- \phi_{j-1/2} \phi_{j+1/2} d\xi &= \int_{-1}^0 (1 + \xi) (-\xi) (1 + \xi) d\xi = \frac{1}{12} \\
\int_{-1}^1 w_{j-1/2}^+ \phi_{j+1/2} \phi_{j+1/2} d\xi &= \int_{-1}^0 (-\xi) (1 + \xi) (1 + \xi) d\xi = \frac{1}{12} \\
\int_{-1}^1 w_{j+1/2}^- \phi_{j+1/2} \phi_{j+1/2} d\xi &= \int_{-1}^0 (1 + \xi) (1 + \xi) (1 + \xi) d\xi = \frac{1}{4} \\
\int_{-1}^1 w_{j+1/2}^+ \phi_{j+1/2} \phi_{j+1/2} d\xi &= \int_0^1 (1 - \xi) (1 - \xi) (1 - \xi) d\xi = \frac{1}{4} \\
\int_{-1}^1 w_{j-3/2}^- \phi_{j+1/2} \phi_{j+1/2} d\xi &= \int_0^1 (\xi) (1 - \xi) (1 - \xi) d\xi = \frac{1}{12} \\
\int_{-1}^1 w_{j+1/2}^+ \phi_{j+3/2} \phi_{j+1/2} d\xi &= \int_0^1 (1 - \xi) (\xi) (1 - \xi) d\xi = \frac{1}{12} \\
\int_{-1}^1 w_{j-3/2}^- \phi_{j+3/2} \phi_{j+1/2} d\xi &= \int_0^1 (\xi) (\xi) (1 - \xi) d\xi = \frac{1}{12}
\end{aligned}$$

Note that these sum to the same fractions as in the linear case if the h is constant.

$$\begin{aligned}
&= u_{j-1/2} h_{j-1/2}^+ \frac{1}{12} + u_{j-1/2} h_{j+1/2}^- \frac{1}{12} \\
&\quad + u_{j+1/2} h_{j-1/2}^+ \frac{1}{12} + u_{j+1/2} h_{j+1/2}^- \frac{1}{4} \\
&\quad + u_{j+1/2} h_{j+1/2}^+ \frac{1}{4} + u_{j+1/2} h_{j-3/2}^- \frac{1}{12} \\
&\quad + u_{j+3/2} h_{j+1/2}^+ \frac{1}{12} + u_{j+3/2} h_{j-3/2}^- \frac{1}{12} \quad (15)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{12} \left[u_{j-1/2} h_{j-1/2}^+ + u_{j-1/2} h_{j+1/2}^- + u_{j+1/2} h_{j-1/2}^+ + 3u_{j+1/2} h_{j+1/2}^- \right. \\
&\quad \left. + 3u_{j+1/2} h_{j+1/2}^+ + u_{j+1/2} h_{j-3/2}^- + u_{j+3/2} h_{j+1/2}^+ + u_{j+3/2} h_{j-3/2}^- \right] \quad (16)
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{x_{j-1/2}}^{x_{j+3/2}} h' u' \phi_{j+1/2} dx &= \frac{\Delta x}{12} \left[u_{j-1/2} h_{j-1/2}^+ + u_{j-1/2} h_{j+1/2}^- + u_{j+1/2} h_{j-1/2}^+ + 3u_{j+1/2} h_{j+1/2}^- \right. \\
&\quad \left. + 3u_{j+1/2} h_{j+1/2}^+ + u_{j+1/2} h_{j-3/2}^- + u_{j+3/2} h_{j+1/2}^+ + u_{j+3/2} h_{j-3/2}^- \right] \quad (17)
\end{aligned}$$

The next integral is

$$\begin{aligned}
\int_{x_{j-1/2}}^{x_{j+3/2}} \frac{(h')^3}{3} u'_x(\phi_x)_{j+1/2} dx &= \frac{1}{\Delta x} \int_{-1}^1 \frac{(h'(\xi))^3}{3} u'_\xi(\phi_\xi)_{j+1/2} d\xi \\
&= \frac{1}{3\Delta x} \int_{-1}^1 (h'(\xi))^3 u'_\xi(\phi_\xi)_{j+1/2} d\xi
\end{aligned}$$

were now going to expand and use the superscript $'$ to denote derivatives

$$\begin{aligned}
&= \frac{1}{3\Delta x} \int_{-1}^1 \left(h_{j-1/2}^+ w_{j-1/2}^+ + h_{j+1/2}^- w_{j+1/2}^- + h_{j+1/2}^+ w_{j+1/2}^+ + h_{j-3/2}^- w_{j-3/2}^- \right)^3 \\
&\quad (u_{j-1/2} \phi'_{j-1/2} + u_{j+1/2} \phi'_{j+1/2} + u_{j+3/2} \phi'_{j+3/2}) \phi'_{j+1/2} d\xi \quad (18)
\end{aligned}$$

$$\begin{aligned}
& \begin{pmatrix} h_{j-1/2}^+ w_{j-1/2}^+ \\ h_{j+1/2}^- w_{j+1/2}^- \\ h_{j+1/2}^+ w_{j+1/2}^+ \\ h_{j-3/2}^- w_{j-3/2}^- \end{pmatrix} \\
&= \frac{1}{3\Delta x} \int_{-1}^1 \left[\left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- w_{j+1/2}^- \right) \right. \\
&\quad + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^+ w_{j+1/2}^+ \right) + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^2 \left(h_{j-3/2}^- w_{j-3/2}^- \right) \\
&\quad + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right) \left(h_{j+1/2}^- w_{j+1/2}^- \right)^2 + 6 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right) \left(h_{j+1/2}^- w_{j+1/2}^- \right) \left(h_{j+1/2}^+ w_{j+1/2}^+ \right) \\
&\quad + 6 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right) \left(h_{j+1/2}^- w_{j+1/2}^- \right) \left(h_{j-3/2}^- w_{j-3/2}^- \right) + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right) \left(h_{j+1/2}^+ w_{j+1/2}^+ \right)^2 \\
&\quad + 6 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right) \left(h_{j+1/2}^+ w_{j+1/2}^+ \right) \left(h_{j-3/2}^- w_{j-3/2}^- \right) + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right) \left(h_{j-3/2}^- w_{j-3/2}^- \right)^2 \\
&\quad + \left(h_{j+1/2}^- w_{j+1/2}^- \right)^3 + 3 \left(h_{j+1/2}^- w_{j+1/2}^- \right)^2 \left(h_{j+1/2}^+ w_{j+1/2}^+ \right) + 3 \left(h_{j+1/2}^- w_{j+1/2}^- \right)^2 \left(h_{j-3/2}^- w_{j-3/2}^- \right) \\
&\quad + 3 \left(h_{j+1/2}^- w_{j+1/2}^- \right) \left(h_{j+1/2}^+ w_{j+1/2}^+ \right)^2 + 6 \left(h_{j+1/2}^- w_{j+1/2}^- \right) \left(h_{j+1/2}^+ w_{j+1/2}^+ \right) \left(h_{j-3/2}^- w_{j-3/2}^- \right) \\
&\quad + 3 \left(h_{j+1/2}^- w_{j+1/2}^- \right) \left(h_{j-3/2}^- w_{j-3/2}^- \right)^2 + \left(h_{j+1/2}^+ w_{j+1/2}^+ \right)^3 + 3 \left(h_{j+1/2}^+ w_{j+1/2}^+ \right)^2 \left(h_{j-3/2}^- w_{j-3/2}^- \right) \\
&\quad \left. + 3 \left(h_{j+1/2}^+ w_{j+1/2}^+ \right) \left(h_{j-3/2}^- w_{j-3/2}^- \right)^2 + \left(h_{j-3/2}^- w_{j-3/2}^- \right)^3 \right] \\
&\quad (u_{j-1/2} \phi'_{j-1/2} + u_{j+1/2} \phi'_{j+1/2} + u_{j+3/2} \phi'_{j+3/2}) \phi'_{j+1/2} d\xi \quad (19)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3\Delta x} \int_{-1}^1 \left[\left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- w_{j+1/2}^- \right) \right. \\
&\quad + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right) \left(h_{j+1/2}^- w_{j+1/2}^- \right)^2 \\
&\quad \quad \quad + \left(h_{j+1/2}^- w_{j+1/2}^- \right)^3 \\
&\quad + \left(h_{j+1/2}^+ w_{j+1/2}^+ \right)^3 + 3 \left(h_{j+1/2}^+ w_{j+1/2}^+ \right)^2 \left(h_{j-3/2}^- w_{j-3/2}^- \right) \\
&\quad \left. + 3 \left(h_{j+1/2}^+ w_{j+1/2}^+ \right) \left(h_{j-3/2}^- w_{j-3/2}^- \right)^2 + \left(h_{j-3/2}^- w_{j-3/2}^- \right)^3 \right] \\
&\quad \quad \quad (u_{j-1/2} \phi'_{j-1/2} + u_{j+1/2} \phi'_{j+1/2} + u_{j+3/2} \phi'_{j+3/2}) \phi'_{j+1/2} d\xi \quad (20)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3\Delta x} \int_{-1}^1 \left[\left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- w_{j+1/2}^- \right) \right. \\
&\quad + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right) \left(h_{j+1/2}^- w_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- w_{j+1/2}^- \right)^3 \\
&\quad + \left(h_{j+1/2}^+ w_{j+1/2}^+ \right)^3 + 3 \left(h_{j+1/2}^+ w_{j+1/2}^+ \right)^2 \left(h_{j-3/2}^- w_{j-3/2}^- \right) \\
&\quad \left. + 3 \left(h_{j+1/2}^+ w_{j+1/2}^+ \right) \left(h_{j-3/2}^- w_{j-3/2}^- \right)^2 + \left(h_{j-3/2}^- w_{j-3/2}^- \right)^3 \right] \\
&\quad \quad \quad (u_{j-1/2} \phi'_{j-1/2} + u_{j+1/2} \phi'_{j+1/2} + u_{j+3/2} \phi'_{j+3/2}) \phi'_{j+1/2} d\xi \quad (21)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3\Delta x} \int_{-1}^1 \left[\left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- w_{j+1/2}^- \right) \right. \\
&\quad \left. + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right) \left(h_{j+1/2}^- w_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- w_{j+1/2}^- \right)^3 \right] \\
&\quad (u_{j-1/2} \phi'_{j-1/2}) \phi'_{j+1/2} \\
&+ \left[\left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- w_{j+1/2}^- \right) \right. \\
&\quad + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right) \left(h_{j+1/2}^- w_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- w_{j+1/2}^- \right)^3 \\
&\quad + \left(h_{j+1/2}^+ w_{j+1/2}^+ \right)^3 + 3 \left(h_{j+1/2}^+ w_{j+1/2}^+ \right)^2 \left(h_{j-3/2}^- w_{j-3/2}^- \right) \\
&\quad \left. + 3 \left(h_{j+1/2}^+ w_{j+1/2}^+ \right) \left(h_{j-3/2}^- w_{j-3/2}^- \right)^2 + \left(h_{j-3/2}^- w_{j-3/2}^- \right)^3 \right] \\
&\quad (u_{j+1/2} \phi'_{j+1/2}) \phi'_{j+1/2} \\
&+ \left[\left(h_{j+1/2}^+ w_{j+1/2}^+ \right)^3 + 3 \left(h_{j+1/2}^+ w_{j+1/2}^+ \right)^2 \left(h_{j-3/2}^- w_{j-3/2}^- \right) \right. \\
&\quad \left. + 3 \left(h_{j+1/2}^+ w_{j+1/2}^+ \right) \left(h_{j-3/2}^- w_{j-3/2}^- \right)^2 + \left(h_{j-3/2}^- w_{j-3/2}^- \right)^3 \right] \\
&\quad (u_{j+3/2} \phi'_{j+3/2}) \phi'_{j+1/2} d\xi \quad (22)
\end{aligned}$$

Now to we break it up into the separate ϕ terms so that

4.0.1 p1

$$\begin{aligned}
&\int_{-1}^1 \left[\left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- w_{j+1/2}^- \right) \right. \\
&\quad \left. + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right) \left(h_{j+1/2}^- w_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- w_{j+1/2}^- \right)^3 \right] \\
&\quad (u_{j-1/2} \phi'_{j-1/2}) \phi'_{j+1/2} \quad (23)
\end{aligned}$$

$$\begin{aligned}
&= \int_{-1}^1 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^3 u_{j-1/2} \phi'_{j-1/2} \phi'_{j+1/2} \\
&\quad + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- w_{j+1/2}^- \right) u_{j-1/2} \phi'_{j-1/2} \phi'_{j+1/2} \\
&\quad + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right) \left(h_{j+1/2}^- w_{j+1/2}^- \right)^2 u_{j-1/2} \phi'_{j-1/2} \phi'_{j+1/2} \\
&\quad + \left(h_{j+1/2}^- w_{j+1/2}^- \right)^3 u_{j-1/2} \phi'_{j-1/2} \phi'_{j+1/2} d\xi \quad (24)
\end{aligned}$$

$$\begin{aligned}
&= \left(h_{j-1/2}^+ \right)^3 u_{j-1/2} \int_{-1}^1 \left(w_{j-1/2}^+ \right)^3 \phi'_{j-1/2} \phi'_{j+1/2} d\xi \\
&\quad + 3 \left(h_{j-1/2}^+ \right)^2 h_{j+1/2}^- u_{j-1/2} \int_{-1}^1 \left(w_{j-1/2}^+ \right)^2 w_{j+1/2}^- \phi'_{j-1/2} \phi'_{j+1/2} d\xi \\
&\quad + 3 h_{j-1/2}^+ \left(h_{j+1/2}^- \right)^2 u_{j-1/2} \int_{-1}^1 w_{j-1/2}^+ \left(w_{j+1/2}^- \right)^2 \phi'_{j-1/2} \phi'_{j+1/2} d\xi \\
&\quad + \left(h_{j+1/2}^- \right)^3 u_{j-1/2} \int_{-1}^1 \left(w_{j+1/2}^- \right)^3 \phi'_{j-1/2} \phi'_{j+1/2} d\xi \quad (25)
\end{aligned}$$

$$\int_{-1}^1 \left(w_{j-1/2}^+ \right)^3 \phi'_{j-1/2} \phi'_{j+1/2} d\xi = \int_{-1}^0 (-\xi)^3 (-1)(1) d\xi = -\frac{1}{4}$$

$$\int_{-1}^1 \left(w_{j-1/2}^+ \right)^2 w_{j+1/2}^- \phi'_{j-1/2} \phi'_{j+1/2} d\xi = \int_{-1}^0 (-\xi)^2 (1+\xi) (-1)(1) d\xi = -\frac{1}{12}$$

$$\int_{-1}^1 w_{j-1/2}^+ \left(w_{j+1/2}^- \right)^2 \phi'_{j-1/2} \phi'_{j+1/2} d\xi = \int_{-1}^1 (-\xi) (1+\xi)^2 (-1)(1) d\xi = -\frac{1}{12}$$

$$\int_{-1}^1 \left(w_{j+1/2}^- \right)^3 \phi'_{j-1/2} \phi'_{j+1/2} d\xi = \int_{-1}^1 (1+\xi)^3 (-1)(1) d\xi = -\frac{1}{4}$$

So then

$$\begin{aligned}
&= \left(h_{j-1/2}^+\right)^3 u_{j-1/2} \left(-\frac{1}{4}\right) \\
&\quad + 3 \left(h_{j-1/2}^+\right)^2 h_{j+1/2}^- u_{j-1/2} \left(-\frac{1}{12}\right) \\
&\quad + 3 h_{j-1/2}^+ \left(h_{j+1/2}^-\right)^2 u_{j-1/2} \left(-\frac{1}{12}\right) \\
&\quad + \left(h_{j+1/2}^-\right)^3 u_{j-1/2} \left(-\frac{1}{4}\right) \quad (26)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{4} \left(\left(h_{j-1/2}^+\right)^3 u_{j-1/2} + \left(h_{j-1/2}^+\right)^2 h_{j+1/2}^- u_{j-1/2} \right. \\
&\quad \left. + h_{j-1/2}^+ \left(h_{j+1/2}^-\right)^2 u_{j-1/2} + \left(h_{j+1/2}^-\right)^3 u_{j-1/2} \right) \quad (27)
\end{aligned}$$

4.0.2 p2

$$\begin{aligned}
&\int_{-1}^1 \left[\left(h_{j-1/2}^+ w_{j-1/2}^+\right)^3 + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+\right)^2 \left(h_{j+1/2}^- w_{j+1/2}^-\right) \right. \\
&\quad + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+\right) \left(h_{j+1/2}^- w_{j+1/2}^-\right)^2 + \left(h_{j+1/2}^- w_{j+1/2}^-\right)^3 \\
&\quad + \left(h_{j+1/2}^+ w_{j+1/2}^+\right)^3 + 3 \left(h_{j+1/2}^+ w_{j+1/2}^+\right)^2 \left(h_{j-3/2}^- w_{j-3/2}^-\right) \\
&\quad \left. + 3 \left(h_{j+1/2}^+ w_{j+1/2}^+\right) \left(h_{j-3/2}^- w_{j-3/2}^-\right)^2 + \left(h_{j-3/2}^- w_{j-3/2}^-\right)^3 \right] \\
&\quad u_{j+1/2} \left(\phi'_{j+1/2}\right)^2 \quad (28)
\end{aligned}$$

$$\begin{aligned}
&= \int_{-1}^1 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^3 u_{j+1/2} (\phi'_{j+1/2})^2 \\
&\quad + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- w_{j+1/2}^- \right) u_{j+1/2} (\phi'_{j+1/2})^2 \\
&\quad + 3 \left(h_{j-1/2}^+ w_{j-1/2}^+ \right) \left(h_{j+1/2}^- w_{j+1/2}^- \right)^2 u_{j+1/2} (\phi'_{j+1/2})^2 \\
&\quad \quad + \left(h_{j+1/2}^- w_{j+1/2}^- \right)^3 u_{j+1/2} (\phi'_{j+1/2})^2 \\
&\quad \quad + \left(h_{j+1/2}^+ w_{j+1/2}^+ \right)^3 u_{j+1/2} (\phi'_{j+1/2})^2 \\
&\quad + 3 \left(h_{j+1/2}^+ w_{j+1/2}^+ \right)^2 \left(h_{j-3/2}^- w_{j-3/2}^- \right) u_{j+1/2} (\phi'_{j+1/2})^2 \\
&\quad + 3 \left(h_{j+1/2}^+ w_{j+1/2}^+ \right) \left(h_{j-3/2}^- w_{j-3/2}^- \right)^2 u_{j+1/2} (\phi'_{j+1/2})^2 \\
&\quad \quad + \left(h_{j-3/2}^- w_{j-3/2}^- \right)^3 u_{j+1/2} (\phi'_{j+1/2})^2 d\xi \quad (29)
\end{aligned}$$

$$\begin{aligned}
&= \left(h_{j-1/2}^+ \right)^3 u_{j+1/2} \int_{-1}^1 \left(w_{j-1/2}^+ \right)^3 (\phi'_{j+1/2})^2 d\xi \\
&\quad + 3 \left(h_{j-1/2}^+ \right)^2 h_{j+1/2}^- u_{j+1/2} \int_{-1}^1 \left(w_{j-1/2}^+ \right)^2 \left(w_{j+1/2}^- \right) (\phi'_{j+1/2})^2 d\xi \\
&\quad + 3 h_{j-1/2}^+ \left(h_{j+1/2}^- \right)^2 u_{j+1/2} \int_{-1}^1 w_{j-1/2}^+ \left(w_{j+1/2}^- \right)^2 (\phi'_{j+1/2})^2 d\xi \\
&\quad \quad + \left(h_{j+1/2}^- \right)^3 u_{j+1/2} \int_{-1}^1 \left(w_{j+1/2}^- \right)^3 (\phi'_{j+1/2})^2 d\xi \\
&\quad \quad + \left(h_{j+1/2}^+ \right)^3 u_{j+1/2} \int_{-1}^1 \left(w_{j+1/2}^+ \right)^3 (\phi'_{j+1/2})^2 d\xi \\
&\quad + 3 \left(h_{j+1/2}^+ \right)^2 \left(h_{j-3/2}^- \right) u_{j+1/2} \int_{-1}^1 \left(w_{j+1/2}^+ \right)^2 \left(w_{j-3/2}^- \right) (\phi'_{j+1/2})^2 d\xi \\
&\quad + 3 \left(h_{j+1/2}^+ \right) \left(h_{j-3/2}^- \right)^2 u_{j+1/2} \int_{-1}^1 \left(w_{j+1/2}^+ \right) \left(w_{j-3/2}^- \right)^2 (\phi'_{j+1/2})^2 d\xi \\
&\quad \quad + \left(h_{j-3/2}^- \right)^3 u_{j+1/2} \int_{-1}^1 \left(w_{j-3/2}^- \right)^3 (\phi'_{j+1/2})^2 d\xi \quad (30)
\end{aligned}$$

$$\int_{-1}^1 \left(w_{j-1/2}^+\right)^3 \left(\phi'_{j+1/2}\right)^2 d\xi = \int_{-1}^0 (-\xi)^3 (1)^2 d\xi = \frac{1}{4}$$

$$\int_{-1}^1 \left(w_{j-1/2}^+\right)^2 \left(w_{j+1/2}^-\right) \left(\phi'_{j+1/2}\right)^2 d\xi = \int_{-1}^0 (-\xi)^2 (1+\xi) (1)^2 d\xi = \frac{1}{12}$$

$$\int_{-1}^1 w_{j-1/2}^+ \left(w_{j+1/2}^-\right)^2 \left(\phi'_{j+1/2}\right)^2 d\xi = \int_{-1}^0 (-\xi) (1+\xi)^2 (1)^2 d\xi = \frac{1}{12}$$

$$\int_{-1}^1 \left(w_{j+1/2}^-\right)^3 \left(\phi'_{j+1/2}\right)^2 d\xi = \int_{-1}^0 (1+\xi)^3 (1)^2 d\xi = \frac{1}{4}$$

$$\int_{-1}^1 \left(w_{j+1/2}^+\right)^3 \left(\phi'_{j+1/2}\right)^2 d\xi = \int_0^1 (1-\xi)^3 (-1)^2 d\xi = \frac{1}{4}$$

$$\int_{-1}^1 \left(w_{j+1/2}^+\right)^2 \left(w_{j-3/2}^-\right) \left(\phi'_{j+1/2}\right)^2 d\xi = \int_0^1 (1-\xi)^2 (\xi) (-1)^2 d\xi = \frac{1}{12}$$

$$\int_{-1}^1 \left(w_{j+1/2}^+\right) \left(w_{j-3/2}^-\right)^2 \left(\phi'_{j+1/2}\right)^2 d\xi = \int_0^1 (1-\xi) (\xi)^2 (-1)^2 d\xi = \frac{1}{12}$$

$$\int_{-1}^1 \left(w_{j-3/2}^-\right)^3 \left(\phi'_{j+1/2}\right)^2 d\xi = \int_0^1 (\xi)^3 (-1)^2 d\xi = \frac{1}{4}$$

So

$$\begin{aligned} &= \frac{1}{4} \left[\left(h_{j-1/2}^+\right)^3 u_{j+1/2} + \left(h_{j-1/2}^+\right)^2 h_{j+1/2}^- u_{j+1/2} \right. \\ &\quad + h_{j-1/2}^+ \left(h_{j+1/2}^-\right)^2 u_{j+1/2} + \left(h_{j+1/2}^-\right)^3 u_{j+1/2} \\ &\quad + \left(h_{j+1/2}^+\right)^3 u_{j+1/2} + 3 \left(h_{j+1/2}^+\right)^2 \left(h_{j-3/2}^-\right) u_{j+1/2} \\ &\quad \left. + 3 \left(h_{j+1/2}^+\right) \left(h_{j-3/2}^-\right)^2 u_{j+1/2} + \left(h_{j-3/2}^-\right)^3 u_{j+1/2} \right] \quad (31) \end{aligned}$$

4.0.3 p3

Similarly the integral separates into

$$\begin{aligned}
& \left(h_{j+1/2}^+\right)^3 u_{j+3/2} \int_{-1}^1 \left(w_{j+1/2}^+\right)^3 \phi'_{j+1/2} \phi'_{j+3/2} d\xi \\
& + 3 \left(h_{j+1/2}^+\right)^2 \left(h_{j-3/2}^-\right) u_{j+3/2} \int_{-1}^1 \left(w_{j+1/2}^+\right)^2 \left(w_{j-3/2}^-\right) \phi'_{j+1/2} \phi'_{j+3/2} d\xi \\
& + 3 \left(h_{j+1/2}^+\right) \left(h_{j-3/2}^-\right)^2 u_{j+3/2} \int_{-1}^1 \left(w_{j+1/2}^+\right) \left(w_{j-3/2}^-\right)^2 \phi'_{j+1/2} \phi'_{j+3/2} d\xi \\
& + \left(h_{j-3/2}^-\right)^3 u_{j+3/2} \int_{-1}^1 \left(w_{j-3/2}^-\right)^3 \phi'_{j+1/2} \phi'_{j+3/2} d\xi \quad (32)
\end{aligned}$$

We have that

$$\int_{-1}^1 \left(w_{j+1/2}^+\right)^3 \phi'_{j+1/2} \phi'_{j+3/2} d\xi = \int_0^1 (1-\xi)^3 (-1)(1) d\xi = -\frac{1}{4}$$

$$\int_{-1}^1 \left(w_{j+1/2}^+\right)^2 \left(w_{j-3/2}^-\right) \phi'_{j+1/2} \phi'_{j+3/2} d\xi = \int_0^1 (1-\xi)^2 (\xi) (-1)(1) d\xi = -\frac{1}{12}$$

$$\int_{-1}^1 \left(w_{j+1/2}^+\right) \left(w_{j-3/2}^-\right)^2 \phi'_{j+1/2} \phi'_{j+3/2} d\xi = \int_0^1 (1-\xi) (\xi)^2 (-1)(1) d\xi = -\frac{1}{12}$$

$$\int_{-1}^1 \left(w_{j-3/2}^-\right)^3 \phi'_{j+1/2} \phi'_{j+3/2} d\xi = \int_0^1 (\xi)^3 (-1)(1) d\xi = -\frac{1}{4}$$

So we have

$$\begin{aligned}
& = -\frac{1}{4} \left[\left(h_{j+1/2}^+\right)^3 u_{j+3/2} + 3 \left(h_{j+1/2}^+\right)^2 \left(h_{j-3/2}^-\right) u_{j+3/2} \right. \\
& \quad \left. + 3 \left(h_{j+1/2}^+\right) \left(h_{j-3/2}^-\right)^2 u_{j+3/2} + \left(h_{j-3/2}^-\right)^3 u_{j+3/2} \right] \quad (33)
\end{aligned}$$

I just noticed that $j - 3/2$ is $j + 3/2$
 So our equation becomes

$$\begin{aligned}
 & \frac{\Delta x}{6} \left[G_{j-1/2}^+ + 2G_{j+1/2}^- + 2G_{j+1/2}^+ + G_{j+3/2}^- \right] \\
 &= \frac{\Delta x}{12} \left[u_{j-1/2} h_{j-1/2}^+ + u_{j-1/2} h_{j+1/2}^- + u_{j+1/2} h_{j-1/2}^+ + 3u_{j+1/2} h_{j+1/2}^- \right. \\
 & \quad \left. + 3u_{j+1/2} h_{j+1/2}^+ + u_{j+1/2} h_{j-3/2}^- + u_{j+3/2} h_{j+1/2}^+ + u_{j+3/2} h_{j-3/2}^- \right] \\
 & \quad + \frac{1}{12\Delta x} \left[- \left[\left(h_{j-1/2}^+ \right)^3 u_{j-1/2} + \left(h_{j-1/2}^+ \right)^2 h_{j+1/2}^- u_{j-1/2} \right. \right. \\
 & \quad \left. \left. + h_{j-1/2}^+ \left(h_{j+1/2}^- \right)^2 u_{j-1/2} + \left(h_{j+1/2}^- \right)^3 u_{j-1/2} \right] \right. \\
 & \quad \left. + \left[\left(h_{j-1/2}^+ \right)^3 u_{j+1/2} + \left(h_{j-1/2}^+ \right)^2 h_{j+1/2}^- u_{j+1/2} \right. \right. \\
 & \quad \left. \left. + h_{j-1/2}^+ \left(h_{j+1/2}^- \right)^2 u_{j+1/2} + \left(h_{j+1/2}^- \right)^3 u_{j+1/2} \right. \right. \\
 & \quad \left. \left. + \left(h_{j+1/2}^+ \right)^3 u_{j+1/2} + 3 \left(h_{j+1/2}^+ \right)^2 \left(h_{j-3/2}^- \right) u_{j+1/2} \right. \right. \\
 & \quad \left. \left. + 3 \left(h_{j+1/2}^+ \right) \left(h_{j-3/2}^- \right)^2 u_{j+1/2} + \left(h_{j-3/2}^- \right)^3 u_{j+1/2} \right] \right. \\
 & \quad \left. - \left[\left(h_{j+1/2}^+ \right)^3 u_{j+3/2} + 3 \left(h_{j+1/2}^+ \right)^2 \left(h_{j-3/2}^- \right) u_{j+3/2} \right. \right. \\
 & \quad \left. \left. + 3 \left(h_{j+1/2}^+ \right) \left(h_{j-3/2}^- \right)^2 u_{j+3/2} + \left(h_{j-3/2}^- \right)^3 u_{j+3/2} \right] \right] \quad (34)
 \end{aligned}$$

$$\begin{aligned}
& 2 \left[G_{j-1/2}^+ + 2G_{j+1/2}^- + 2G_{j+1/2}^+ + G_{j+3/2}^- \right] \\
&= \left[u_{j-1/2} h_{j-1/2}^+ + u_{j-1/2} h_{j+1/2}^- + u_{j+1/2} h_{j-1/2}^+ + 3u_{j+1/2} h_{j+1/2}^- \right. \\
&\quad \left. + 3u_{j+1/2} h_{j+1/2}^+ + u_{j+1/2} h_{j+3/2}^- + u_{j+3/2} h_{j+1/2}^+ + u_{j+3/2} h_{j+3/2}^- \right] \\
&\quad + \frac{1}{\Delta x^2} \left[- \left[\left(h_{j-1/2}^+ \right)^3 u_{j-1/2} + \left(h_{j-1/2}^+ \right)^2 h_{j+1/2}^- u_{j-1/2} \right. \right. \\
&\quad \left. \left. + h_{j-1/2}^+ \left(h_{j+1/2}^- \right)^2 u_{j-1/2} + \left(h_{j+1/2}^- \right)^3 u_{j-1/2} \right] \right. \\
&\quad \left. + \left[\left(h_{j-1/2}^+ \right)^3 u_{j+1/2} + \left(h_{j-1/2}^+ \right)^2 h_{j+1/2}^- u_{j+1/2} \right. \right. \\
&\quad \left. \left. + h_{j-1/2}^+ \left(h_{j+1/2}^- \right)^2 u_{j+1/2} + \left(h_{j+1/2}^- \right)^3 u_{j+1/2} \right. \right. \\
&\quad \left. \left. + \left(h_{j+1/2}^+ \right)^3 u_{j+1/2} + \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) u_{j+1/2} \right. \right. \\
&\quad \left. \left. + \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 u_{j+1/2} + \left(h_{j+3/2}^- \right)^3 u_{j+1/2} \right] \right. \\
&\quad \left. - \left[\left(h_{j+1/2}^+ \right)^3 u_{j+3/2} + \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) u_{j+3/2} \right. \right. \\
&\quad \left. \left. + \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 u_{j+3/2} + \left(h_{j+3/2}^- \right)^3 u_{j+3/2} \right] \right] \quad (35)
\end{aligned}$$

$$\begin{aligned}
& 2 \left[G_{j-1/2}^+ + 2G_{j+1/2}^- + 2G_{j+1/2}^+ + G_{j+3/2}^- \right] \\
&= \left[u_{j-1/2} h_{j-1/2}^+ + u_{j-1/2} h_{j+1/2}^- + u_{j+1/2} h_{j-1/2}^+ + 3u_{j+1/2} h_{j+1/2}^- \right. \\
&\quad \left. + 3u_{j+1/2} h_{j+1/2}^+ + u_{j+1/2} h_{j+3/2}^- + u_{j+3/2} h_{j+1/2}^+ + u_{j+3/2} h_{j+3/2}^- \right] \\
&+ \frac{1}{\Delta x^2} \left[- \left[\left(h_{j-1/2}^+ \right)^3 + \left(h_{j-1/2}^+ \right)^2 h_{j+1/2}^- + h_{j-1/2}^+ \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \right] u_{j-1/2} \right. \\
&\quad + \left[\left(h_{j-1/2}^+ \right)^3 + \left(h_{j-1/2}^+ \right)^2 h_{j+1/2}^- + h_{j-1/2}^+ \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \right. \\
&\quad \left. + \left(h_{j+1/2}^+ \right)^3 + \left(h_{j+1/2}^+ \right)^2 h_{j+3/2}^- + \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 + \left(h_{j+3/2}^- \right)^3 \right] u_{j+1/2} \\
&\quad \left. - \left[\left(h_{j+1/2}^+ \right)^3 + \left(h_{j+1/2}^+ \right)^2 h_{j+3/2}^- + \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 + \left(h_{j+3/2}^- \right)^3 \right] u_{j+3/2} \right] \\
&\hspace{15em} (36)
\end{aligned}$$

if h is constant then the coefficients are the same as before.