1 Linearised Equations

We begin with the linearised equations from Chris's thesis/papers.

continuity:

$$\frac{\partial h_1}{\partial t} + h_0 \frac{\partial u_1}{\partial x} + u_0 \frac{\partial h_1}{\partial x} = 0$$

velocity:

$$\frac{\partial u_1}{\partial t} + g \frac{\partial h_1}{\partial x} + u_0 \frac{\partial u_1}{\partial x} - \frac{h_0^2}{3} \left(u_0 \frac{\partial^3 u_1}{\partial x^3} + \frac{\partial^3 u_1}{\partial x^3 \partial t} \right) = 0$$

Also G

$$G = u_0 h_0 + u_0 h_1 + h_0 u_1 - \frac{h_0^3}{3} \frac{\partial^2 u_1}{\partial x^2}$$

Now for simplicity, and because its all we need, we assume the water is still (except for the pertubations) so that $u_0 = 0$ thus we get:

$$\frac{\partial h_1}{\partial t} + h_0 \frac{\partial u_1}{\partial x} = 0$$

$$\frac{\partial u_1}{\partial t} + g \frac{\partial h_1}{\partial x} - \frac{h_0^2}{3} \frac{\partial^3 u_1}{\partial x^3 \partial t} = 0$$

$$G = h_0 u_1 - \frac{h_0^3}{3} \frac{\partial^2 u_1}{\partial x^2}$$

Importantly by multiplying the velocity by h_0 to get the momentum equation we have

$$\frac{\partial u_1}{\partial t} h_0 + g \frac{\partial h_1}{\partial x} h_0 - \frac{h_0^3}{3} \frac{\partial^3 u_1}{\partial x^3 \partial t} = 0$$

and thus

$$\frac{\partial G}{\partial t} + g \frac{\partial h_1}{\partial x} h_0 = 0$$

So we finally have

$$\frac{\partial h_1}{\partial t} + h_0 \frac{\partial u_1}{\partial x} = 0$$

$$\frac{\partial G}{\partial t} + g\frac{\partial h_1}{\partial x}h_0 = 0$$

$$G = h_0 u_1 - \frac{h_0^3}{3}\frac{\partial^2 u_1}{\partial x^2}$$

For convenience I will make the following notational changes $H = h_0$, $h = h_1$ and $u = u_1$. So that

$$\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial G}{\partial t} + gH\frac{\partial h}{\partial x} = 0$$

$$G = Hu - \frac{H^3}{3}\frac{\partial^2 u}{\partial x^2}$$

We know that the dispersion relation is given by

$$\omega = \pm k\sqrt{gH}\sqrt{\frac{3}{k^2H^2+3}}$$

2 Source Term in Mass Equation

We introduce a source term to the mass equation to get

$$h_t + Hu_x = f(x,t)$$

$$u_t H + gHh_x - \frac{H^3}{3}u_{xxt} = 0$$

$$G = Hu - \frac{H^3}{3}\frac{\partial^2 u}{\partial x^2}$$

We introduce a velocity potential $u = \Phi_x$ so

$$h_t + H\Phi_{xx} = f(x,t)$$

$$\Phi_{tx}H + gHh_x - \frac{H^3}{3}\Phi_{xxtx} = 0$$

So the equations become

$$h_t + H\Phi_{xx} = f(x,t)$$
$$\Phi_t H + gHh - \frac{H^3}{3}\Phi_{xxt} = 0$$

So

$$h = \frac{\frac{H^3}{3}\Phi_{xxt} - \Phi_t H}{gH}$$

Subbing this into the continuity equation this becomes

$$\left[\frac{\frac{H^2}{3}\Phi_{xxt} - \Phi_t}{g}\right]_t + H\Phi_{xx} = f(x,t)$$
$$\frac{1}{g}\left[\frac{H^2}{3}\Phi_{xxtt} - \Phi_{tt}\right] + H\Phi_{xx} = f(x,t)$$

So

$$\Phi_{tt} - \frac{H^2}{3}\Phi_{xxtt} - gH\Phi_{xx} = -gf(x,t)$$

We assume that things are nice enough to perform fourier transforms on Φ and f and we get that

$$\Phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\Phi}(x,\omega)\exp\left(-i\omega t\right)\ d\omega$$
$$f(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(x,\omega)\exp\left(-i\omega t\right)\ d\omega$$
$$\omega^2\hat{\Phi} - \frac{H^2}{3}\omega^2\hat{\Phi}'' - gH\hat{\Phi}'' = -g\hat{f}(x,t)$$
$$\omega^2\hat{\Phi} - \left[\frac{H^2}{3}\omega^2 + gH\right]\hat{\Phi}'' = -g\hat{f}(x,t)$$

So from here, its more clear that we have the same equations as them but with $\alpha = 1/3$ So that we have the solution.

$$\hat{\Phi} = \int_{-\infty}^{\infty} G(x,x') g \hat{f}(x) dx'$$
$$= \int_{-\infty}^{x} G_{+}(x,x') g \hat{f}(x) dx' + \int_{x}^{\infty} G_{-}(x,x') g \hat{f}(x) dx'$$

where

$$G(x,x') = \begin{cases} a e^{il(x-x')} & x > x' \\ a e^{il(x'-x)} & x < x' \end{cases}$$

2.1 Their Example

$$\hat{f} = D \exp\left(-\beta x^2\right)$$

which gives

$$\hat{\Phi} = g D a I_1 \exp\left(ilx\right)$$

$$D = \frac{2\eta_0 \omega}{I_1 k (1 - \alpha (kh)^2)}$$
$$I_1 = \sqrt{\frac{\pi}{\beta}} \exp\left(-\frac{l^2}{4\beta}\right)$$