

1 Finite Element

$$Guh - \frac{\partial}{\partial x} \left(\frac{h^3}{3} u_x \right)$$

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = uhv - \frac{\partial}{\partial x} \left(\frac{h^3}{3} u_x \right) v$$

and then we integrate over the entire domain to get

$$\int_{\Omega} Gv dx = \int_{\Omega} uhv dx - \int_{\Omega} \frac{\partial}{\partial x} \left(\frac{h^3}{3} u_x \right) v dx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_{\Omega} Gv dx = \int_{\Omega} uhv dx + \int_{\Omega} \frac{h^3}{3} u_x v_x dx$$

For u we are going to use $x_{j-1/2}$, x_j and $x_{j+1/2}$ as the nodes, which generate the basis functions $\phi_{j\pm 1/2}$ and ϕ_j , which for us will be the space of continuous quadratic elements.

While for G and h we will choose basis functions w that are linear from $[x_{j-1/2}, x_{j+1/2}]$ but discontinuous at the edges.

We are going to look at the entire area where the basis functions are non-zero for $\phi_{j-1/2}$, ϕ_j and $\phi_{j+1/2}$. Which is the interval from $x_{j-3/2}$ to $x_{j+3/2}$. So we focus on the integrals on $[x_{j-3/2}, x_{j+3/2}]$ as

$$\begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}$$

$$\int_{\Omega} G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \int_{\Omega} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx + \int_{\Omega} \frac{h^3}{3} u_x \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_x dx$$

is

$$\sum_j \int_{x_{j-3/2}}^{x_{j+3/2}} G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \sum_j \int_{x_{j-3/2}}^{x_{j+3/2}} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx + \sum_j \int_{x_{j-3/2}}^{x_{j+3/2}} \frac{h^3}{3} u_x \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_x dx$$

$$x = \frac{3}{2}\xi\Delta x + x_j$$

Taking the derivatives we see

$$dx = d\frac{3\xi}{2}\Delta x, \quad \frac{dx}{d\xi} = \frac{3\Delta x}{2}, \quad \frac{d\xi}{dx} = \frac{2}{3\Delta x}.$$

We can describe the basis functions in the ξ space, where they are non-zero

2 Integrals

2.1 G

$$\int_{x_{j-3/2}}^{x_{j+3/2}} G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \frac{3\Delta x}{2} \int_{-1}^1 G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi \quad (1)$$

We have

$$\int_{-1}^1 G \phi_{j-1/2} d\xi = 0G_{j-3/2}^+ + \frac{1}{9}G_{j-1/2}^- + \frac{1}{9}G_{j-1/2}^+ + 0G_{j+1/2}^- = \frac{1}{9}G_{j-1/2}^- + \frac{1}{9}G_{j-1/2}^+$$

$$\int_{-1}^1 G \phi_j d\xi = \frac{2}{9}G_{j-1/2}^+ + \frac{2}{9}G_{j+1/2}^-$$

$$\int_{-1}^1 G \phi_{j+1/2} d\xi = 0G_{j-1/2}^+ + \frac{1}{9}G_{j+1/2}^- + \frac{1}{9}G_{j+1/2}^+ + 0G_{j+3/2}^- = \frac{1}{9}G_{j+1/2}^- + \frac{1}{9}G_{j+1/2}^+$$

So

$$\int_{-1}^1 G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi = \begin{bmatrix} \frac{1}{9}G_{j-1/2}^- + \frac{1}{9}G_{j-1/2}^+ \\ \frac{2}{9}G_{j-1/2}^+ + \frac{2}{9}G_{j+1/2}^- \\ \frac{1}{9}G_{j+1/2}^- + \frac{1}{9}G_{j+1/2}^+ \end{bmatrix}$$

So

$$\int_{x_{j-3/2}}^{x_{j+3/2}} G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \frac{3\Delta x}{2} \begin{bmatrix} \frac{1}{9}G_{j-1/2}^- + \frac{1}{9}G_{j-1/2}^+ \\ \frac{2}{9}G_{j-1/2}^+ + \frac{2}{9}G_{j+1/2}^- \\ \frac{1}{9}G_{j+1/2}^- + \frac{1}{9}G_{j+1/2}^+ \end{bmatrix} = \frac{\Delta x}{6} \begin{bmatrix} G_{j-1/2}^- + G_{j-1/2}^+ \\ 2G_{j-1/2}^+ + 2G_{j+1/2}^- \\ G_{j+1/2}^- + G_{j+1/2}^+ \end{bmatrix} \quad (2)$$

2.2 uh

$$\int_{x_{j-3/2}}^{x_{j+3/2}} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \frac{3\Delta x}{2} \int_{-1}^1 uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi \quad (3)$$

We have

$$\begin{aligned} \int_{-1}^1 uh \phi_{j-1/2} d\xi = & \left(-\frac{1}{90}h_{j-3/2}^+ - \frac{1}{90}h_{j-1/2}^- \right) u_{j-3/2} \\ & + \left(0h_{j-3/2}^+ + \frac{4}{90}h_{j-1/2}^- \right) u_{j-1} \\ & + \left(\frac{1}{90}h_{j-3/2}^+ + \frac{7}{90}h_{j-1/2}^- + \frac{7}{90}h_{j-1/2}^+ + \frac{1}{90}h_{j+1/2}^- \right) u_{j-1/2} \\ & + \left(\frac{4}{90}h_{j-1/2}^+ + 0h_{j+1/2}^- \right) u_j \\ & + \left(-\frac{1}{90}h_{j-1/2}^+ - \frac{1}{90}h_{j+1/2}^- \right) u_{j+1/2} \end{aligned}$$

$$\begin{aligned}
\int_{-1}^1 uh\phi_j d\xi &= \\
&\left(\frac{4}{90}h_{j-1/2}^+ + 0h_{j+1/2}^-\right)u_{j-1/2} \\
&+ \left(\frac{16}{90}h_{j-1/2}^+ + \frac{16}{90}h_{j+1/2}^-\right)u_j \\
&+ \left(0h_{j-1/2}^+ + \frac{4}{90}h_{j+1/2}^-\right)u_{j+1/2} \\
&= \\
&\left(\frac{4}{90}h_{j-1/2}^+\right)u_{j-1/2} \\
&+ \left(\frac{16}{90}h_{j-1/2}^+ + \frac{16}{90}h_{j+1/2}^-\right)u_j \\
&\qquad\qquad\qquad + \left(\frac{4}{90}h_{j+1/2}^-\right)u_{j+1/2}
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^1 uh\phi_{j+1/2}d\xi = & \left(-\frac{1}{90}h_{j-1/2}^+ - \frac{1}{90}h_{j+1/2}^-\right)u_{j-1/2} \\
& + \left(0h_{j-1/2}^+ + \frac{4}{90}h_{j+1/2}^-\right)u_j \\
& + \left(\frac{1}{90}h_{j-1/2}^+ + \frac{7}{90}h_{j+1/2}^- + \frac{7}{90}h_{j+1/2}^+ + \frac{1}{90}h_{j+3/2}^-\right)u_{j+1/2} \\
& + \left(\frac{4}{90}h_{j+1/2}^+ + 0h_{j+3/2}^-\right)u_{j+1} \\
& + \left(-\frac{1}{90}h_{j+1/2}^+ - \frac{1}{90}h_{j+3/2}^-\right)u_{j+3/2} = \\
& \left(-\frac{1}{90}h_{j-1/2}^+ - \frac{1}{90}h_{j+1/2}^-\right)u_{j-1/2} \\
& + \left(\frac{4}{90}h_{j+1/2}^-\right)u_j \\
& + \left(\frac{1}{90}h_{j-1/2}^+ + \frac{7}{90}h_{j+1/2}^- + \frac{7}{90}h_{j+1/2}^+ + \frac{1}{90}h_{j+3/2}^-\right)u_{j+1/2} \\
& + \left(\frac{4}{90}h_{j+1/2}^+\right)u_{j+1} \\
& + \left(-\frac{1}{90}h_{j+1/2}^+ - \frac{1}{90}h_{j+3/2}^-\right)u_{j+3/2}
\end{aligned}$$

$$\int_{-1}^1 uh\phi_{j-1/2}d\xi =$$

$$\begin{aligned} & \left(-\frac{1}{90}h_{j-3/2}^+ - \frac{1}{90}h_{j-1/2}^- \right) u_{j-3/2} \\ & + \left(\frac{4}{90}h_{j-1/2}^- \right) u_{j-1} \\ & + \left(\frac{1}{90}h_{j-3/2}^+ + \frac{7}{90}h_{j-1/2}^- + \frac{7}{90}h_{j-1/2}^+ + \frac{1}{90}h_{j+1/2}^- \right) u_{j-1/2} \\ & + \left(\frac{4}{90}h_{j-1/2}^+ \right) u_j \\ & + \left(-\frac{1}{90}h_{j-1/2}^+ - \frac{1}{90}h_{j+1/2}^- \right) u_{j+1/2} \end{aligned}$$

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$$= \frac{1}{90} \begin{bmatrix} -h_{j-3/2}^+ - h_{j-1/2}^- & 0 & 0 & 0 & 0 \\ 0 & 4h_{j-1/2}^- & 0 & 0 & 0 \\ 0 & 0 & h_{j-3/2}^+ + 7h_{j-1/2}^- + 7h_{j-1/2}^+ + h_{j+1/2}^- & 0 & 0 \\ 0 & 0 & 0 & 4h_{j-1/2}^+ & 0 \\ 0 & 0 & 0 & 0 & -h_{j-1/2}^+ - h_{j+1/2}^- \end{bmatrix} \begin{bmatrix} u_{j-3/2} \\ u_{j-1} \\ u_{j-1/2} \\ u_j \\ u_{j+1/2} \end{bmatrix}$$

$$\int_{-1}^1 uh\phi_j d\xi =$$

$$\begin{aligned} & \left(\frac{4}{90} h_{j-1/2}^+ \right) u_{j-1/2} \\ & + \left(\frac{16}{90} h_{j-1/2}^+ + \frac{16}{90} h_{j+1/2}^- \right) u_j \\ & + \left(\frac{4}{90} h_{j+1/2}^- \right) u_{j+1/2} \\ & = \frac{1}{90} \begin{bmatrix} 4h_{j-1/2}^+ & 0 & 0 \\ 0 & 16h_{j-1/2}^+ + 16h_{j+1/2}^- & 0 \\ 0 & 0 & 4h_{j+1/2}^- \end{bmatrix} \begin{bmatrix} u_{j-1/2} \\ u_j \\ u_{j+1/2} \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
& \int_{-1}^1 u h \phi_{j+1/2} d\xi = \\
& \left(-\frac{1}{90} h_{j-1/2}^+ - \frac{1}{90} h_{j+1/2}^- \right) u_{j-1/2} \\
& \quad + \left(\frac{4}{90} h_{j+1/2}^- \right) u_j \\
& + \left(\frac{1}{90} h_{j-1/2}^+ + \frac{7}{90} h_{j+1/2}^- + \frac{7}{90} h_{j+1/2}^+ + \frac{1}{90} h_{j+3/2}^- \right) u_{j+1/2} \\
& \quad + \left(\frac{4}{90} h_{j+1/2}^+ \right) u_{j+1} \\
& \quad + \left(-\frac{1}{90} h_{j+1/2}^+ - \frac{1}{90} h_{j+3/2}^- \right) u_{j+3/2} \\
& \infty \\
& = \frac{1}{90} \begin{bmatrix} -h_{j-1/2}^+ - h_{j+1/2}^- & 0 & 0 & 0 & 0 \\ 0 & 4h_{j+1/2}^- & 0 & 0 & 0 \\ 0 & 0 & h_{j-1/2}^+ + 7h_{j+1/2}^- + 7h_{j+1/2}^+ + h_{j+3/2}^- & 0 & 0 \\ 0 & 0 & 0 & 4h_{j+1/2}^+ & 0 \\ 0 & 0 & 0 & 0 & -h_{j+1/2}^+ - h_{j+3/2}^- \end{bmatrix} \begin{bmatrix} u_{j-1/2} \\ u_j \\ u_{j+1/2} \\ u_{j+1} \\ u_{j+3/2} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
& \int_{-1}^1 uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi = \\
& \frac{1}{90} \begin{bmatrix} -h_{j-3/2}^+ - h_{j-1/2}^- & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4h_{j-1/2}^- & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{j-3/2}^+ + 7h_{j-1/2}^- + 10h_{j-1/2}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20h_{j-1/2}^+ + 20h_{j+1/2}^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10h_{j+1/2}^- + 7h_{j+1/2}^+ + h_{j+3/2}^- & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4h_{j+1/2}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -h_{j+1/2}^+ - \end{bmatrix} \\
& \begin{bmatrix} u_{j-3/2} \\ u_{j-1} \\ u_{j-1/2} \\ u_j \\ u_{j+1/2} \\ u_{j+1} \\ u_{j+3/2} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
& \int_{-1}^1 uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi = \\
& \frac{1}{90} \begin{bmatrix} -h_{j-3/2}^+ - h_{j-1/2}^- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4h_{j-1/2}^- & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{j-3/2}^+ + 7h_{j-1/2}^- + 10h_{j-1/2}^+ & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20h_{j-1/2}^+ + 20h_{j+1/2}^- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10h_{j+1/2}^- + 7h_{j+1/2}^+ + h_{j+3/2}^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4h_{j+1/2}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_{j+1/2}^+ - h_{j+3/2}^- & 0 \end{bmatrix} \begin{bmatrix} u_{j-3/2} \\ u_{j-1} \\ u_{j-1/2} \\ u_j \\ u_{j+1/2} \\ u_{j+1} \\ u_{j+3/2} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
& \int_{x_{j-3/2}}^{x_{j+3/2}} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \\
& \frac{\Delta x}{60} \begin{bmatrix} -h_{j-3/2}^+ - h_{j-1/2}^- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4h_{j-1/2}^- & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{j-3/2}^+ + 7h_{j-1/2}^- + 10h_{j-1/2}^+ & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20h_{j-1/2}^+ + 20h_{j+1/2}^- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10h_{j+1/2}^- + 7h_{j+1/2}^+ + h_{j+3/2}^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4h_{j+1/2}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_{j+1/2}^+ - h_{j+3/2}^- & 0 \end{bmatrix} \begin{bmatrix} u_{j-3/2} \\ u_{j-1} \\ u_{j-1/2} \\ u_j \\ u_{j+1/2} \\ u_{j+1} \\ u_{j+3/2} \end{bmatrix}
\end{aligned}$$

2.3 h3 u

First

$$(a + b + c + d)^3$$

where $ad = ac = bd = bc = 0$

is

$$(a + b + c + d)^3 = a^3 + 3a^2b + 3ab^2 + b^3 + c^3 + 3c^2d + 3cd^2 + d^3$$

So that

$$\begin{aligned} & \left(h_{j-3/2}^+ + h_{j-1/2}^- + h_{j-1/2}^+ + h_{j+1/2}^- \right)^3 = \\ & \left(h_{j-3/2}^+ \right)^3 + 3 \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \\ & + \left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \end{aligned}$$

$$\int_{x_{j-3/2}}^{x_{j+3/2}} \frac{h^3}{3} u_x \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_x dx = \frac{2}{9\Delta x} \int_{-1}^1 h^3 u_\xi \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_\xi d\xi$$

$$\begin{aligned}
& \int_{-1}^1 h^3 u_{\xi} \phi_{j-1/2} d\xi = \\
& \left[\left(h_{j-3/2}^+ \right)^3 + 3 \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right. \\
& + \left. \left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right] u_{j-3/2} \\
& + \left[\left(h_{j-3/2}^+ \right)^3 + 3 \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right. \\
& + \left. \left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right] u_{j-1} \\
& + \left[\left(h_{j-3/2}^+ \right)^3 + 3 \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right. \\
& + \left. \left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right] u_{j-1/2} \\
& + \left[\left(h_{j-3/2}^+ \right)^3 + 3 \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right. \\
& + \left. \left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right] u_j \\
& + \left[\left(h_{j-3/2}^+ \right)^3 + 3 \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right. \\
& + \left. \left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right] u_{j+1/2}
\end{aligned}$$

Using where these values are not zero (messed up the subscripts for $h_{j+1/2}^-$)

$$\begin{aligned}
&= \left[\left(h_{j-3/2}^+ \right)^3 + 3 \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right] u_{j-3/2} \\
&+ \left[\left(h_{j-3/2}^+ \right)^3 + 3 \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right] u_{j-1} \\
&+ \left[\left(h_{j-3/2}^+ \right)^3 + 3 \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right. \\
&+ \left. \left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \right] u_{j-1/2} \\
&\left[\left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \right] u_j \\
&\left[\left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \right] u_{j+1/2}
\end{aligned}$$

$$\begin{aligned}
&= \left[\frac{17}{420} \left(h_{j-3/2}^+ \right)^3 + \frac{3}{105} \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + \frac{3}{140} \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \frac{8}{105} \left(h_{j-1/2}^- \right)^3 \right] u_{j-3/2} \\
&+ \left[-\frac{1}{21} \left(h_{j-3/2}^+ \right)^3 - \frac{6}{105} \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) - \frac{3}{21} \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 - \frac{44}{105} \left(h_{j-1/2}^- \right)^3 \right] u_{j-1} \\
&+ \left[\frac{1}{140} \left(h_{j-3/2}^+ \right)^3 + 3 \frac{1}{105} \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \frac{17}{420} \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \frac{12}{35} \left(h_{j-1/2}^- \right)^3 \right. \\
&- \frac{12}{35} \left(h_{j-1/2}^+ \right)^3 - 3 \frac{17}{420} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) - 3 \frac{1}{105} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 - \frac{1}{140} \left(h_{j+1/2}^- \right)^3 \left. \right] u_{j-1/2} \\
&\left[\frac{44}{105} \left(h_{j-1/2}^+ \right)^3 + 3 \frac{1}{21} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + 3 \frac{2}{105} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \frac{1}{21} \left(h_{j+1/2}^- \right)^3 \right] u_j \\
&\left[-\frac{8}{105} \left(h_{j-1/2}^+ \right)^3 - \frac{3}{140} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) - \frac{3}{105} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 - \frac{17}{420} \left(h_{j+1/2}^- \right)^3 \right] u_{j+1/2}
\end{aligned}$$

$$\begin{aligned}
&= \left[\frac{17}{420} \left(h_{j-3/2}^+ \right)^3 + \frac{3}{105} \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + \frac{3}{140} \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \frac{8}{105} \left(h_{j-1/2}^- \right)^3 \right] u_{j-3/2} \\
&+ \left[-\frac{1}{21} \left(h_{j-3/2}^+ \right)^3 - \frac{6}{105} \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) - \frac{3}{21} \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 - \frac{44}{105} \left(h_{j-1/2}^- \right)^3 \right] u_{j-1} \\
&+ \left[\frac{1}{140} \left(h_{j-3/2}^+ \right)^3 + \frac{3}{105} \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + \frac{51}{420} \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \frac{12}{35} \left(h_{j-1/2}^- \right)^3 \right. \\
&\quad \left. - \frac{12}{35} \left(h_{j-1/2}^+ \right)^3 - \frac{51}{420} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) - \frac{3}{105} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 - \frac{1}{140} \left(h_{j+1/2}^- \right)^3 \right] u_{j-1/2} \\
&\left[\frac{44}{105} \left(h_{j-1/2}^+ \right)^3 + \frac{3}{21} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + \frac{6}{105} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \frac{1}{21} \left(h_{j+1/2}^- \right)^3 \right] u_j \\
&\left[-\frac{8}{105} \left(h_{j-1/2}^+ \right)^3 - \frac{3}{140} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) - \frac{3}{105} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 - \frac{17}{420} \left(h_{j+1/2}^- \right)^3 \right] u_{j+1/2}
\end{aligned}$$

$$\begin{aligned}
&\int_{-1}^1 h^3 u_\xi \phi_{j\xi} d\xi = \\
&\left(\left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \right) u_{j-1/2} \\
&+ \left(\left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \right) u_j \\
&\left(\left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \right) u_{j+1/2}
\end{aligned}$$

$$\begin{aligned}
&\int_{-1}^1 h^3 u_\xi \phi_{j\xi} d\xi = \\
&\left(-\frac{26}{105} \left(h_{j-1/2}^+ \right)^3 - \frac{9}{35} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) - \frac{3}{21} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 - \frac{2}{105} \left(h_{j+1/2}^- \right)^3 \right) u_{j-1/2} \\
&+ \left(\frac{8}{35} \left(h_{j-1/2}^+ \right)^3 + \frac{12}{105} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) - \frac{12}{105} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 - \frac{8}{35} \left(h_{j+1/2}^- \right)^3 \right) u_j \\
&\left(\frac{2}{105} \left(h_{j-1/2}^+ \right)^3 + \frac{3}{21} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + \frac{9}{35} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \frac{26}{105} \left(h_{j+1/2}^- \right)^3 \right) u_{j+1/2}
\end{aligned}$$

$$\begin{aligned}
& \int_{-1}^1 h^3 u_\xi \phi_{j+1/2} d\xi = \\
& \left[\left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \right. \\
& + \left. \left(h_{j+1/2}^+ \right)^3 + 3 \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) + 3 \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 + \left(h_{j+3/2}^- \right)^3 \right] u_{j-1/2} \\
& + \left[\left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \right. \\
& + \left. \left(h_{j+1/2}^+ \right)^3 + 3 \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) + 3 \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 + \left(h_{j+3/2}^- \right)^3 \right] u_j \\
& + \left[\left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \right. \\
& + \left. \left(h_{j+1/2}^+ \right)^3 + 3 \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) + 3 \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 + \left(h_{j+3/2}^- \right)^3 \right] u_{j+1/2} \\
& + \left[\left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \right. \\
& + \left. \left(h_{j+1/2}^+ \right)^3 + 3 \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) + 3 \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 + \left(h_{j+3/2}^- \right)^3 \right] u_{j+1} \\
& + \left[\left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \right. \\
& + \left. \left(h_{j+1/2}^+ \right)^3 + 3 \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) + 3 \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 + \left(h_{j+3/2}^- \right)^3 \right] u_{j+3/2}
\end{aligned}$$

$$\begin{aligned}
& \int_{-1}^1 h^3 u_\xi \phi_{j+1/2} d\xi = \\
& \left[\frac{17}{420} \left(h_{j-1/2}^+ \right)^3 + \frac{3}{105} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + \frac{3}{140} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \frac{8}{105} \left(h_{j+1/2}^- \right)^3 \right] u_{j-1/2} \\
& + \left[-\frac{1}{21} \left(h_{j-1/2}^+ \right)^3 - \frac{6}{105} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) - \frac{3}{21} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 - \frac{44}{105} \left(h_{j+1/2}^- \right)^3 \right] u_j \\
& + \left[\frac{1}{140} \left(h_{j-1/2}^+ \right)^3 + \frac{3}{105} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + \frac{51}{420} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \frac{12}{35} \left(h_{j+1/2}^- \right)^3 \right. \\
& \left. - \frac{12}{35} \left(h_{j+1/2}^+ \right)^3 - \frac{51}{420} \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) - \frac{3}{105} \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 - \frac{1}{140} \left(h_{j+3/2}^- \right)^3 \right] u_{j+1/2} \\
& + \left[\frac{44}{105} \left(h_{j+1/2}^+ \right)^3 + \frac{3}{21} \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) + \frac{6}{105} \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 + \frac{1}{21} \left(h_{j+3/2}^- \right)^3 \right] u_{j+1} \\
& + \left[-\frac{8}{105} \left(h_{j+1/2}^+ \right)^3 - \frac{3}{140} \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) - \frac{3}{105} \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 - \frac{17}{420} \left(h_{j+3/2}^- \right)^3 \right] u_{j+3/2}
\end{aligned}$$

So we have

$$\begin{aligned}
& \int_{-1}^1 h^3 u_\xi \left[\begin{array}{c} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{array} \right]_\xi d\xi = \\
& \left[\frac{17}{420} (h_{j-3/2}^+)^3 + \frac{3}{105} (h_{j-3/2}^+)^2 (h_{j-1/2}^-) + \frac{3}{140} (h_{j-3/2}^+) (h_{j-1/2}^-)^2 + \frac{8}{105} (h_{j-1/2}^-)^3 \right] u_{j-3/2} \\
& + \left[-\frac{1}{21} (h_{j-3/2}^+)^3 - \frac{6}{105} (h_{j-3/2}^+)^2 (h_{j-1/2}^-) - \frac{3}{21} (h_{j-3/2}^+) (h_{j-1/2}^-)^2 - \frac{44}{105} (h_{j-1/2}^-)^3 \right] u_{j-1} \\
& + \left[\frac{1}{140} (h_{j-3/2}^+)^3 + \frac{3}{105} (h_{j-3/2}^+)^2 (h_{j-1/2}^-) + \frac{51}{420} (h_{j-3/2}^+) (h_{j-1/2}^-)^2 + \frac{12}{35} (h_{j-1/2}^-)^3 \right. \\
& \left. - \frac{12}{35} (h_{j-1/2}^+)^3 - \frac{51}{420} (h_{j-1/2}^+)^2 (h_{j+1/2}^-) - \frac{3}{105} (h_{j-1/2}^+) (h_{j+1/2}^-)^2 - \frac{1}{140} (h_{j+1/2}^-)^3 \right] u_{j-1/2} \\
& \left[\frac{44}{105} (h_{j-1/2}^+)^3 + \frac{3}{21} (h_{j-1/2}^+)^2 (h_{j+1/2}^-) + \frac{6}{105} (h_{j-1/2}^+) (h_{j+1/2}^-)^2 + \frac{1}{21} (h_{j+1/2}^-)^3 \right] u_j \\
& \left[-\frac{8}{105} (h_{j-1/2}^+)^3 - \frac{3}{140} (h_{j-1/2}^+)^2 (h_{j+1/2}^-) - \frac{3}{105} (h_{j-1/2}^+) (h_{j+1/2}^-)^2 - \frac{17}{420} (h_{j+1/2}^-)^3 \right] u_{j+1/2} \\
& + \left(-\frac{26}{105} (h_{j-1/2}^+)^3 - \frac{9}{35} (h_{j-1/2}^+)^2 (h_{j+1/2}^-) - \frac{3}{21} (h_{j-1/2}^+) (h_{j+1/2}^-)^2 - \frac{2}{105} (h_{j+1/2}^-)^3 \right) u_{j-1/2} \\
& + \left(\frac{8}{35} (h_{j-1/2}^+)^3 + \frac{12}{105} (h_{j-1/2}^+)^2 (h_{j+1/2}^-) - \frac{12}{105} (h_{j-1/2}^+) (h_{j+1/2}^-)^2 - \frac{8}{35} (h_{j+1/2}^-)^3 \right) u_j \\
& \left(\frac{2}{105} (h_{j-1/2}^+)^3 + \frac{3}{21} (h_{j-1/2}^+)^2 (h_{j+1/2}^-) + \frac{9}{35} (h_{j-1/2}^+) (h_{j+1/2}^-)^2 + \frac{26}{105} (h_{j+1/2}^-)^3 \right) u_{j+1/2} \\
& + \left[\frac{17}{420} (h_{j-1/2}^+)^3 + \frac{3}{105} (h_{j-1/2}^+)^2 (h_{j+1/2}^-) + \frac{3}{140} (h_{j-1/2}^+) (h_{j+1/2}^-)^2 + \frac{8}{105} (h_{j+1/2}^-)^3 \right] u_{j-1/2} \\
& + \left[-\frac{1}{21} (h_{j-1/2}^+)^3 - \frac{6}{105} (h_{j-1/2}^+)^2 (h_{j+1/2}^-) - \frac{3}{21} (h_{j-1/2}^+) (h_{j+1/2}^-)^2 - \frac{44}{105} (h_{j+1/2}^-)^3 \right] u_j \\
& + \left[\frac{1}{140} (h_{j-1/2}^+)^3 + \frac{3}{105} (h_{j-1/2}^+)^2 (h_{j+1/2}^-) + \frac{51}{420} (h_{j-1/2}^+) (h_{j+1/2}^-)^2 + \frac{12}{35} (h_{j+1/2}^-)^3 \right. \\
& \left. - \frac{12}{35} (h_{j+1/2}^+)^3 - \frac{51}{420} (h_{j+1/2}^+)^2 (h_{j+3/2}^-) - \frac{3}{105} (h_{j+1/2}^+) (h_{j+3/2}^-)^2 - \frac{1}{140} (h_{j+3/2}^-)^3 \right] u_{j+1/2} \\
& + \left[\frac{44}{105} (h_{j+1/2}^+)^3 + \frac{3}{21} (h_{j+1/2}^+)^2 (h_{j+3/2}^-) + \frac{6}{105} (h_{j+1/2}^+) (h_{j+3/2}^-)^2 + \frac{1}{21} (h_{j+3/2}^-)^3 \right] u_{j+1} \\
& + \left[-\frac{8}{105} (h_{j+1/2}^+)^3 - \frac{3}{140} (h_{j+1/2}^+)^2 (h_{j+3/2}^-) - \frac{3}{105} (h_{j+1/2}^+) (h_{j+3/2}^-)^2 - \frac{17}{420} (h_{j+3/2}^-)^3 \right] u_{j+3/2}
\end{aligned}$$

$$\begin{aligned}
& \int_{-1}^1 h^3 u_\xi \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_\xi d\xi = \\
& \left[\frac{17}{420} (h_{j-3/2}^+)^3 + \frac{3}{105} (h_{j-3/2}^+)^2 (h_{j-1/2}^-) + \frac{3}{140} (h_{j-3/2}^+) (h_{j-1/2}^-)^2 + \frac{8}{105} (h_{j-1/2}^-)^3 \right] u_{j-3/2} \\
& + \left[-\frac{1}{21} (h_{j-3/2}^+)^3 - \frac{6}{105} (h_{j-3/2}^+)^2 (h_{j-1/2}^-) - \frac{3}{21} (h_{j-3/2}^+) (h_{j-1/2}^-)^2 - \frac{44}{105} (h_{j-1/2}^-)^3 \right] u_{j-1} \\
& + \left[\frac{1}{140} (h_{j-3/2}^+)^3 + \frac{3}{105} (h_{j-3/2}^+)^2 (h_{j-1/2}^-) + \frac{51}{420} (h_{j-3/2}^+) (h_{j-1/2}^-)^2 + \frac{12}{35} (h_{j-1/2}^-)^3 \right. \\
& \left. - \frac{11}{20} (h_{j-1/2}^+)^3 - \frac{7}{20} (h_{j-1/2}^+)^2 (h_{j+1/2}^-) - \frac{3}{20} (h_{j-1/2}^+) (h_{j+1/2}^-)^2 + \frac{1}{20} (h_{j+1/2}^-)^3 \right] u_{j-1/2} \\
& \left[\frac{3}{5} (h_{j-1/2}^+)^3 + \frac{1}{5} (h_{j-1/2}^+)^2 (h_{j+1/2}^-) - \frac{1}{5} (h_{j-1/2}^+) (h_{j+1/2}^-)^2 - \frac{3}{5} (h_{j+1/2}^-)^3 \right] u_j \\
& + \left[-\frac{1}{20} (h_{j-1/2}^+)^3 + \frac{3}{20} (h_{j-1/2}^+)^2 (h_{j+1/2}^-) + \frac{7}{20} (h_{j-1/2}^+) (h_{j+1/2}^-)^2 + \frac{11}{20} (h_{j+1/2}^-)^3 \right. \\
& \left. - \frac{12}{35} (h_{j+1/2}^+)^3 - \frac{51}{420} (h_{j+1/2}^+)^2 (h_{j+3/2}^-) - \frac{3}{105} (h_{j+1/2}^+) (h_{j+3/2}^-)^2 - \frac{1}{140} (h_{j+3/2}^-)^3 \right] u_{j+1/2} \\
& + \left[\frac{44}{105} (h_{j+1/2}^+)^3 + \frac{3}{21} (h_{j+1/2}^+)^2 (h_{j+3/2}^-) + \frac{6}{105} (h_{j+1/2}^+) (h_{j+3/2}^-)^2 + \frac{1}{21} (h_{j+3/2}^-)^3 \right] u_{j+1} \\
& + \left[-\frac{8}{105} (h_{j+1/2}^+)^3 - \frac{3}{140} (h_{j+1/2}^+)^2 (h_{j+3/2}^-) - \frac{3}{105} (h_{j+1/2}^+) (h_{j+3/2}^-)^2 - \frac{17}{420} (h_{j+3/2}^-)^3 \right] u_{j+3/2}
\end{aligned}$$