1 Finite Element

$$G = uh - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = uhv - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)v$$

and then we integrate over the entire domain to get

$$\int_\Omega Gvdx = \int_\Omega uhvdx - \int_\Omega \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)vdx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_\Omega Gvdx = \int_\Omega uhvdx + \int_\Omega \frac{h^3}{3}u_x v_x dx$$

For u we are going to use $x_{j-1/2}$, x_j and $x_{j+1/2}$ as the nodes, which generate the basis functions $\phi_{j\pm1/2}$ and ϕ_j , which for us will be the space of continuous quadratic elements.

While for G and h we will choose basis functions w that are linear from $[x_{j-1/2}, x_{j+1/2}]$ but discontinuous at the edges.

There are two types of basis functions in this set up the ϕ_j which are non-zero on $[x_{j-1/2}, x_{j+1/2}]$ and the $\phi_{j\pm1/2}$, which we can reduce to just doing it once, but with a translation, so we focus on the $\phi_{j+1/2}$ which is non-zero on $[x_{j-1/2}, x_{j+3/2}]$

2 ϕ_j

In this section we focus on the test function $v = \phi_j$ and thus we focus on the integrals on $[x_{j-1/2}, x_{j+1/2}]$ as

$$\int_\Omega Gvdx = \int_\Omega uhvdx + \int_\Omega \frac{h^3}{3}u_x v_x dx$$

is

$$\sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} G\phi_j dx = \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} uh\phi_j dx + \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{h^3}{3} u_x (\phi_j)_x dx$$

$$x = \frac{1}{2}\xi\Delta x + x_j$$

Taking the derivatives we see
$dx = d\frac{\xi}{2}\Delta x$, $\frac{dx}{d\xi} = \frac{\Delta x}{2}$, $\frac{d\xi}{dx} = \frac{2}{\Delta x}$.

We can describe the basis functions in the ξ space, where they are non-zero

$$\phi_j = 1 - \xi^2 \tag{1}$$

$$\phi_{j-1/2} = \frac{1}{2}\left(\xi^2 - \xi\right) \tag{2}$$

$$\phi_{j+1/2} = \frac{1}{2}\left(\xi^2 + \xi\right) \tag{3}$$

$$w^+_{j-1/2} = \frac{1}{2}\left(1 - \xi\right) \tag{4}$$

$$w^-_{j-1/2} = \frac{1}{2}\left(1 + \xi\right) \tag{5}$$

$$G \approx G' = \sum_j G_{j+1/2} w_{j+1/2}$$

$$u \approx u' = \sum_j \left[u_{j-1/2}\phi_{j-1/2} + u_j\phi_j + u_{j+1/2}\phi_{j+1/2}\right]$$

$$h \approx h' = \sum_j h_{j+1/2} w_{j+1/2}$$

2.1 First Integral

So now we do the substitution for all integrals firstly we do

$$\begin{aligned}\int_{x_{j-1/2}}^{x_{j+1/2}} G\phi_j dx &= \int_{-1}^{1} G'(\xi)\phi_j(\xi)\frac{dx}{d\xi}d\xi \\ &= \frac{\Delta x}{2}\int_{-1}^{1} G'(\xi)\phi_j(\xi)d\xi\end{aligned}$$

So we focus in on the integral

$$\begin{aligned}\int_{-1}^{1} G'(\xi)\phi_j(\xi)d\xi &= \int_{-1}^{1}\left(G^{+}_{j-1/2}w^{+}_{j-1/2} + G^{-}_{j+1/2}w^{-}_{j+1/2}\right)\phi_j d\xi \\ &= G^{+}_{j-1/2}\int_{-1}^{1} w^{+}_{j-1/2}\phi_j d\xi + G^{-}_{j+1/2}\int_{-1}^{1} w^{-}_{j+1/2}\phi_j d\xi\end{aligned}$$

we have

$$\int_{-1}^{1} w^{+}_{j-1/2}\phi_j d\xi = \int_{-1}^{1}\frac{1}{2}(1-\xi)\left(1-\xi^2\right)d\xi = \frac{1}{2}\frac{4}{3} = \frac{2}{3}$$

$$\int_{-1}^{1} w^{-}_{j+1/2}\phi_j d\xi = \int_{-1}^{1}\frac{1}{2}(1+\xi)\left(1-\xi^2\right)d\xi = \frac{1}{2}\frac{4}{3} = \frac{2}{3}$$

so then

$$\int_{-1}^{1} G'(\xi)\phi_j(\xi)d\xi = \frac{2}{3}G^{+}_{j-1/2} + \frac{2}{3}G^{-}_{j+1/2}$$

so we have

$$\int_{x_{j-1/2}}^{x_{j+1/2}} G\phi_j dx = \frac{\Delta x}{2}\left[\frac{2}{3}G^{+}_{j-1/2} + \frac{2}{3}G^{-}_{j+1/2}\right] = \frac{\Delta x}{3}\left[G^{+}_{j-1/2} + G^{-}_{j+1/2}\right]$$

2.2 Second Integral

$$\int_{x_{j-1/2}}^{x_{j+1/2}} uh\phi_j dx = \int_{-1}^{1} u(\xi)h(\xi)\phi_j(\xi)\frac{dx}{d\xi}d\xi = \frac{\Delta x}{2}\int_{-1}^{1} u'h'\phi_j d\xi$$

focusing on the integral

$$\int_{-1}^{1} u'h'\phi_j d\xi = \int_{-1}^{1} \left(u_{j-1/2}\phi_{j-1/2} + u_j\phi_j + u_{j+1/2}\phi_{j+1/2}\right)\left(h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2}\right)\phi_j d\xi$$

$$= \int_{-1}^{1} \Bigg(u_{j-1/2}h^+_{j-1/2}w^+_{j-1/2}\phi_{j-1/2} + u_j h^+_{j-1/2}w^+_{j-1/2}\phi_j + u_{j+1/2}h^+_{j-1/2}w^+_{j-1/2}\phi_{j+1/2}$$
$$+ u_{j-1/2}h^-_{j+1/2}w^-_{j+1/2}\phi_{j-1/2} + u_j h^-_{j+1/2}w^-_{j+1/2}\phi_j + u_{j+1/2}h^-_{j+1/2}w^-_{j+1/2}\phi_{j+1/2} \Bigg)\phi_j d\xi$$

$$= u_{j-1/2}h^+_{j-1/2}\int_{-1}^{1} w^+_{j-1/2}\phi_{j-1/2}\phi_j d\xi + u_j h^+_{j-1/2}\int_{-1}^{1} w^+_{j-1/2}\phi_j\phi_j d\xi$$
$$+ u_{j+1/2}h^+_{j-1/2}\int_{-1}^{1} w^+_{j-1/2}\phi_{j+1/2}\phi_j d\xi$$
$$+ u_{j-1/2}h^-_{j+1/2}\int_{-1}^{1} w^-_{j+1/2}\phi_{j-1/2}\phi_j d\xi + u_j h^-_{j+1/2}\int_{-1}^{1} w^-_{j+1/2}\phi_j\phi_j d\xi$$
$$+ u_{j+1/2}h^-_{j+1/2}\int_{-1}^{1} w^-_{j+1/2}\phi_{j+1/2}\phi_j d\xi$$

Now we calculate the integrals

$$\int_{-1}^{1} w^+_{j-1/2}\phi_{j-1/2}\phi_j d\xi = \int_{-1}^{1} \frac{1}{2}(1-\xi)\frac{1}{2}\left(\xi^2-\xi\right)\left(1-\xi^2\right)d\xi = \frac{1}{4}\left[\frac{8}{15}\right] = \frac{2}{15}$$

$$\int_{-1}^{1} w^+_{j-1/2}\phi_j\phi_j d\xi = \int_{-1}^{1} \frac{1}{2}(1-\xi)\left(1-\xi^2\right)\left(1-\xi^2\right)d\xi = \frac{1}{2}\left[\frac{16}{15}\right] = \frac{8}{15}$$

$$\int_{-1}^{1} w^+_{j-1/2}\phi_{j+1/2}\phi_j d\xi = \int_{-1}^{1} \frac{1}{2}(1-\xi)\frac{1}{2}\left(\xi^2+\xi\right)\left(1-\xi^2\right) = \frac{1}{4}\times 0 = 0$$

$$\int_{-1}^{1} w^-_{j+1/2}\phi_{j-1/2}\phi_j d\xi = \int_{-1}^{1} \frac{1}{2}(\xi+1)\frac{1}{2}\left(\xi^2-\xi\right)\left(1-\xi^2\right)d\xi = \frac{1}{4}\times 0 = 0$$

$$\int_{-1}^{1} w^{-}_{j+1/2}\phi_j\phi_j d\xi = \int_{-1}^{1} \frac{1}{2}\left(\xi+1\right)\left(1-\xi^2\right)\left(1-\xi^2\right) d\xi = \frac{1}{2}\left[\frac{16}{15}\right] = \frac{8}{15}$$

$$\int_{-1}^{1} w^{-}_{j+1/2}\phi_{j+1/2}\phi_j d\xi = \int_{-1}^{1} \frac{1}{2}\left(\xi+1\right)\frac{1}{2}\left(\xi^2+\xi\right)\left(1-\xi^2\right) = \frac{1}{4}\left[\frac{8}{15}\right] = \frac{2}{15}$$

So we have

$$\begin{aligned} u_{j-1/2}h^{+}_{j-1/2}\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j-1/2}\phi_j d\xi + u_j h^{+}_{j-1/2}\int_{-1}^{1} w^{+}_{j-1/2}\phi_j\phi_j d\xi \\ + u_{j+1/2}h^{+}_{j-1/2}\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j+1/2}\phi_j d\xi \\ + u_{j-1/2}h^{-}_{j+1/2}\int_{-1}^{1} w^{-}_{j+1/2}\phi_{j-1/2}\phi_j d\xi + u_j h^{-}_{j+1/2}\int_{-1}^{1} w^{-}_{j+1/2}\phi_j\phi_j d\xi \\ + u_{j+1/2}h^{-}_{j+1/2}\int_{-1}^{1} w^{-}_{j+1/2}\phi_{j+1/2}\phi_j d\xi \end{aligned}$$

$$= \quad \frac{2}{15}h^{+}_{j-1/2}u_{j-1/2} \ + \ \frac{8}{15}h^{+}_{j-1/2}u_j \ + \ \frac{8}{15}h^{-}_{j+1/2}u_j \ + \ \frac{2}{15}h^{-}_{j+1/2}u_{j+1/2}$$

So

$$\begin{aligned} &\int_{x_{j-1/2}}^{x_{j+1/2}} uh\phi_j dx = \\ &\frac{\Delta x}{2}\left[\frac{2}{15}h^{+}_{j-1/2}u_{j-1/2} + \frac{8}{15}h^{+}_{j-1/2}u_j + \frac{8}{15}h^{-}_{j+1/2}u_j + \frac{2}{15}h^{-}_{j+1/2}u_{j+1/2}\right] \\ &= \frac{\Delta x}{15}\left[h^{+}_{j-1/2}u_{j-1/2} + 4h^{+}_{j-1/2}u_j + 4h^{-}_{j+1/2}u_j + h^{-}_{j+1/2}u_{j+1/2}\right] \end{aligned}$$

Lastly we have

3 $\phi_{j+1/2}$

In this section we focus on the test function $v = \phi_{j+1/2}$ and thus we focus on the integrals on $[x_{j-1/2}, x_{j+3/2}]$ as

$$\int_{\Omega} Gv dx = \int_{\Omega} uhv dx + \int_{\Omega} \frac{h^3}{3} u_x v_x dx$$

is

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} G\phi_{j+1/2} dx = \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} uh\phi_{j+1/2} dx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^3}{3} u_x \left(\phi_{j+1/2}\right)_x dx$$

4 Combination