Thesis title

Your name

Month Year

A thesis submitted for the degree of Doctor of Philosophy
of the Australian National University

To my mother and father who have provided me with everything.

Declaration

The work in this thesis is my own except where otherwise stated.

Jordan Pitt

Acknowledgements

Abstract

In this thesis we will read paers then talk about them

Contents

Notation and terminology

Some preliminary description here? Eg, "In the following, G is a group, H is a subgroup of G, ..."

Notation

notation	definition text goes here definition text goes here definition text goes here
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Terminology

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Chapter 1

Introduction

1.1 Objectives of the Thesis

1.2 Original Contribution of the Thesis

1.3 Organisation of the Thesis

Chapter 2

The Serre Equations

The Serre equations are partial differential equations that describe the behaviour of waves of free surface flows of fluids for a wide array of wave properties.

In fact they are considered to be one of the best models for free surface flows up to wave breaking []. For this reason we are interested in using the Serre equations to model wave hazards such as tsunamis and storm surges.

The Serre equations can be derived asymptotically [] or via depth integration [] of the full Navier-Stokes equations. They are evolution type equations, however they are not strictly parabolic or hyperbolic and naively are not in conservation law form.

2.1 The Serre Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{2.1a}$$

and

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right) + \frac{\partial b}{\partial x}\left(gh + h\Psi + \frac{h^2}{2}\Phi\right) = 0 \tag{2.1b}$$

where Φ and Ψ are defined as

Definition 2.1.

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2 \frac{\partial b}{\partial x}$$

and

$$\Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

2.1.1 Alternative form of the Serre Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{2.2a}$$

and

$$\frac{\partial}{\partial t}(G) + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3\frac{\partial u}{\partial x}^2 + h^2 u\frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right) + \frac{1}{2}h^2 u\frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} - hu^2\frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} + gh\frac{\partial b}{\partial x} = 0 \tag{2.2b}$$

With G defined as

Definition 2.2.

$$G = hu\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \frac{\partial b}{\partial x}^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right)$$

2.2 Properties of the Serre Equations

2.2.1 Conservation Properties

The total amount of a quantity q in a system occurring on the interval $[a, b]$ at time t is measured by

Definition 2.3.

$$\mathcal{C}_q(t) = \int_a^b q(x,t)\,dx$$

Conservation of a quantity q implies that $\mathcal{C}_q(0) = \mathcal{C}_q(t)\ \forall t$ provided the interval is fixed and the system is closed

For any bed profile the Serre equations has two quantities that must be conserved the mass h, and the Hamiltonian $\mathcal{H}$. Where the Hamiltonian is

Definition 2.4.

$$\mathcal{H}(x,t) = \frac{1}{2}\left(hu^2 + \frac{h^3}{3}\left(\frac{\partial u}{\partial x}\right)^2 + gh^2 + 2ghb + u^2 h\frac{\partial b}{\partial x} - uh^2\frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right)$$

For horizontal bed profiles where $b(x) = 0\ \forall x$, the source terms in [] are 0 and the Serre equations also conserve momentum and G in the same way provided that boundary conditions for the water depths are equal at both ends.

2.2.2 Dispersion Relation

It was demonstrated in [] that the dispersion relation for the linearised Serre equations is

$$\omega = Uk \pm k\sqrt{gH}\sqrt{\frac{3}{(kH)^2 + 3}} \tag{2.3}$$

From this we get the phase velocity $v_p = \omega/k$ and the group velocity $v_g = d\omega/dk$.

$$v_p = U \pm \sqrt{gH}\sqrt{\frac{3}{(kH)^2 + 3}} \tag{2.4a}$$

$$v_g = U \pm \sqrt{gH}\left(\sqrt{\frac{3}{(kH)^2 + 3}} \mp (kH)^2\sqrt{\frac{3}{\left((kH)^2 + 3\right)^3}}\right) \tag{2.4b}$$

Chapter 3

Hybrid Finite Volume Methods

3.1 Structure Overview

In this section we will give the general structure for how the hybrid finite volume methods take the array of cell average values at time t^n; $\bar{\boldsymbol{h}}^n$ and $\bar{\boldsymbol{G}}^n$ and evolve the system to the cell average values at time t^{n+1}; $\bar{\boldsymbol{h}}^{n+1}$ and $\bar{\boldsymbol{G}}^{n+1}$.

- The cell average values are transformed into nodal values by $\mathcal{M}$
$$\bar{\boldsymbol{h}}^n, \bar{\boldsymbol{G}}^n \xrightarrow{\mathcal{M}} \boldsymbol{h}^n, \boldsymbol{G}^n$$

- $\boldsymbol{u}^n$ is found by solving the elliptic equation in Def. 2.2 by $\mathcal{A}$
$$\boldsymbol{h}^n, \boldsymbol{G}^n, \boldsymbol{b} \xrightarrow{\mathcal{G}} \boldsymbol{u}^n$$

- The conservation equations (2.2) can now be solved by $\mathcal{F}$
$$\bar{\boldsymbol{h}}^n, \bar{\boldsymbol{G}}^n, \boldsymbol{b}, \boldsymbol{u}^n \xrightarrow{\mathcal{F}} \bar{\boldsymbol{h}}^{n+1}, \bar{\boldsymbol{G}}^{n+1}$$

Definition 3.1. $\mathcal{E}$ is the single Euler step given by the procedure above that transforms updates the array of cell average values at time t^n; $\bar{\boldsymbol{h}}^n$ and $\bar{\boldsymbol{G}}^n$ to the cell average values at time t^{n+1}; $\bar{\boldsymbol{h}}^{n+1}$.

$$\bar{\boldsymbol{h}}^n, \bar{\boldsymbol{G}}^n, \boldsymbol{b} \xrightarrow{\mathcal{E}} \bar{\boldsymbol{h}}^{n+1}, \bar{\boldsymbol{G}}^{n+1}$$

3.2 Transformation Between Nodal Values and Cell Averages

For first and second order methods $\mathcal{M}$ is just the identity map as the cell average values are equal to the nodal values.

For higher order methods this is not the case, hence why there is a need to incorporate the process $\mathcal{M}$ into our methods, as assuming that $\mathcal{M}$ is the identity map will lead to a loss of accuracy in the method.

From quadratic interpolation we have the formula relating the cell averages and nodal values of a quantity q with third order accuracy

$$q_j = \frac{-\bar{q}_{j-1} + 26\bar{q}_j - \bar{q}_{j+1}}{24}.$$

Therefore

$$\boldsymbol{q} = \frac{1}{24}\begin{bmatrix} 26 & -1 & & & \\ -1 & 26 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 26 & -1 \\ & & & -1 & 26 \end{bmatrix}\bar{\boldsymbol{q}}.$$

Dirichlet boundary conditions at edges.

So for the third-order method $\mathcal{M}$ is a multiplication by the above matrix.

3.3 Elliptic Equation

The elliptic equation that relates the conserved variables h and G to the primitive variable u was given in Def 2.2 and is presented here to remind the reader

$$G = uh\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \frac{\partial b}{\partial x}^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right).$$

3.3.1 Finite Difference Methods

One way to approximate this ordinary differential equation is to replace all the derivatives with finite differences as has been done to second order accuracy in []. We have expanded this work by building a fourth order accurate finite difference method, where all derivatives were replaced with their centred fourth order approximation. This results in the following equation for each row of a the matrix $\boldsymbol{A}$

$$G_j = A_{j,j-2}\, u_{j-2} + A_{j,j-1}\, u_{j-1} + A_{j,j}\, u_j + A_{j,j+1}\, u_{j+1} + A_{j,j+2}\, u_{j+2} \tag{3.1}$$

where

$$
\begin{aligned}
A_{j,j-2} &= -h_j^2 \left(\frac{h_{j-2} - 8h_{j-1} + 8h_{j+1} - h_{j+2}}{144\Delta x^2} \right) + \frac{h_j^3}{36\Delta x^2}, \\
A_{j,j-1} &= h_j^2 \left(\frac{h_{j-2} - 8h_{j-1} + 8h_{j+1} - h_{j+2}}{18\Delta x^2} \right) - \frac{4h_j^3}{9\Delta x^2}, \\
A_{j,j} &= h_j + \frac{5h_j^3}{6\Delta x^2} + h_j \left(\frac{(h_{j-2} - 8h_{j-1} + 8h_{j+1} - h_{j+2})\,(b_{j-2} - 8b_{j-1} + 8b_{j+1} - b_{j+2})}{144\Delta x^2} \right) \\
&\quad + h_j \left(\frac{-b_{j-2} + 16b_{j-1} - 30b_{j-1} + 16b_{j+1} - b_{j+2}}{24\Delta x^2} \right) + h_j \left(\frac{b_{j-2} - 8b_{j-1} + 8b_{j+1} - b_{j+2}}{144\Delta x^2} \right), \\
A_{j,j+1} &= -h_j^2 \left(\frac{h_{j-2} - 8h_{j-1} + 8h_{j+1} - h_{j+2}}{18\Delta x^2} \right) - \frac{4h_j^3}{9\Delta x^2}, \\
A_{j,j+2} &= h_j^2 \left(\frac{h_{j-2} - 8h_{j-1} + 8h_{j+1} - h_{j+2}}{144\Delta x^2} \right) + \frac{h_j^3}{36\Delta x^2},
\end{aligned}
$$

This can be written for the whole domain

$$
\boldsymbol{G} = \boldsymbol{A}\boldsymbol{u}.
$$

Dirichlet boundary conditions

Therefore $\mathcal{G}$ is the solution of this matrix problem with $\boldsymbol{G}$ known and $\boldsymbol{u}$ unknown.

For completeness the second-order method finite difference approximation which is used for both the first and second order FDVM [] is given by

$$
G_j = A_{j,j-1}\, u_{j-1} + A_{j,j}\, u_j + A_{j,j+1}\, u_{j+1}
$$

where

$$
\begin{aligned}
A_{j,j-1} &= h_j^2 \frac{-h_{j-1} + h_{j+1}}{4\Delta x^2} - \frac{h_j^3}{3\Delta x^2}, \\
A_{j,j} &= h_j + h_j \frac{-h_{j-1} + h_{j+1}}{2\Delta x} \frac{-b_{j-1} + b_{j+1}}{2\Delta x} + h_j^2 \frac{b_{j-1} - 2b_j + b_{j+1}}{2\Delta x^2} + h_j \left(\frac{-b_{j-1} + b_{j+1}}{2\Delta x} \right)^2 \\
&\quad + \frac{2h_j^3}{3\Delta x^2},
\end{aligned}
$$

$$
A_{j,j+1} = -h_j^2 \frac{-h_{j-1} + h_{j+1}}{4\Delta x^2} - \frac{h_j^3}{3\Delta x^2}.
$$

3.3.2 Finite Element Methods

For a finite element method we take the weak form of the elliptic equation in Def 2.2 which is

$$\int_\Omega Gv\ dx = \int_\Omega uh\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \frac{\partial b}{\partial x}^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right) v\ dx.$$

Which after rearranging, using integration by parts and assuming Dirichlet boundary conditions becomes

$$\int_\Omega Gv\ dx = \int_\Omega uh\left(1 + \frac{\partial b}{\partial x}^2\right) v\ dx + \int_\Omega \frac{1}{3}h^3\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\ dx$$
$$- \int_\Omega \frac{1}{2}h^2\frac{\partial b}{\partial x}u\frac{\partial v}{\partial x}\ dx. - \int_\Omega \frac{1}{2}h^2\frac{\partial b}{\partial x}\frac{\partial u}{\partial x}v\ dx.$$

Should be able to handle non smooth h and G but requires first derivative of b. We can simplify this further by only performing integration over elements and then assemble them together to get the integral equation for the entire domain. As we use a finite volume method to solve the conservation equations [] our natural element choice are the cells $\left[x_{j-1/2}, x_{j+1/2}\right]$. Therefore we get that

$$\sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \left[\left(uh\left(1 + \frac{\partial b}{\partial x}^2\right) - \frac{1}{2}h^2\frac{\partial b}{\partial x}\frac{\partial u}{\partial x} - G\right) v \right.$$
$$\left. + \left(\frac{1}{3}h^3\frac{\partial u}{\partial x} - \frac{1}{2}h^2\frac{\partial b}{\partial x}u\right)\frac{\partial v}{\partial x}\right] dx = 0. \quad (3.2)$$

for each test function v. The next step is to replace the functions for the quantities h, G, b and u as well as the test function v with the appropriate order of accuracy piece-wise polynomial function. This is typically done using basis functions and this process turns our integral equations into matrix equations.

Second Order

We desire a method for the elliptic equation [] that is at least second-order and that can locally reconstruct the necessary physical quantities to update the evolution equations [] within each cell individually. This allows the scheme to be simply scaled into situations where nearby cell data may not be quickly accessible such as when using this method on parrallel cpus. From our equations and

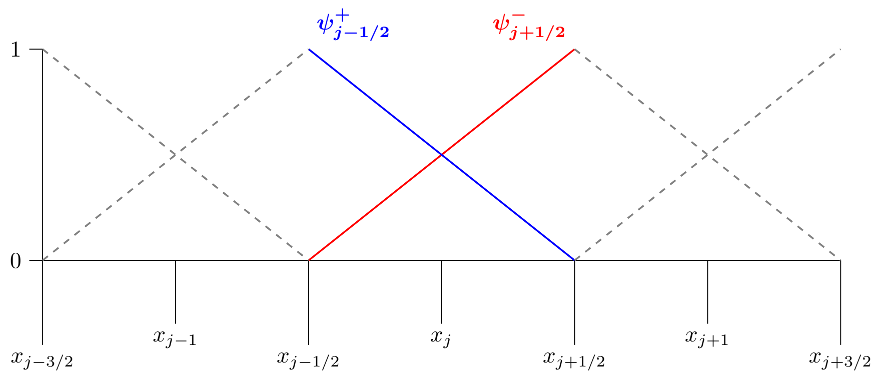

Figure 3.1: Basis functions for discontinuous linear elements.

smoothness conditions we know that if the bed is smooth enough, then we can allow discontinuous h and G and have u have a well defined derivative. Since in the evolution equations we do not require derivatives of these quantities for h and G we will use linear reconstruction that is discontinuous at the edges for h and G in our finite element method. The basis functions for this reconstruction of h and G will be represented by ψ. This reconstruction will use cell values at the edges $x_{j-1/2}$ and $x_{j+1/2}$ to represent the linear function over the cell.

So we have the following representation for h and G in our finite element method

$$h = \sum_j h^+_{j-1/2}\psi^+_{j-1/2} + h^-_{j+1/2}\psi^-_{j+1/2}$$
$$G = \sum_j G^+_{j-1/2}\psi^+_{j-1/2} + G^-_{j+1/2}\psi^-_{j+1/2}$$

Where we calculate $h^+_{j-1/2}$, $h^-_{j+1/2}$, $G^+_{j-1/2}$ and $G^-_{j+1/2}$ using our second-order reconstruction. Which for our method is the generalised minmod limiter of [**?**], which for a general quantity q is

$$q^-_{j+1/2} = q_j + a_j \frac{\Delta x}{2} \tag{3.3a}$$

and

$$q^+_{j+1/2} = q_{j+1} - a_{j+1} \frac{\Delta x}{2} \tag{3.3b}$$

where

$$a_j = \text{minmod}\left\{\theta\frac{-q_j + q_{j+1}}{\Delta x}, \frac{-q_{j-1} + q_{j+1}}{2\Delta x}, \theta\frac{-q_{j-1} + q_j}{\Delta x}\right\} \quad \text{for } \theta \in [1,2] \tag{3.3c}$$

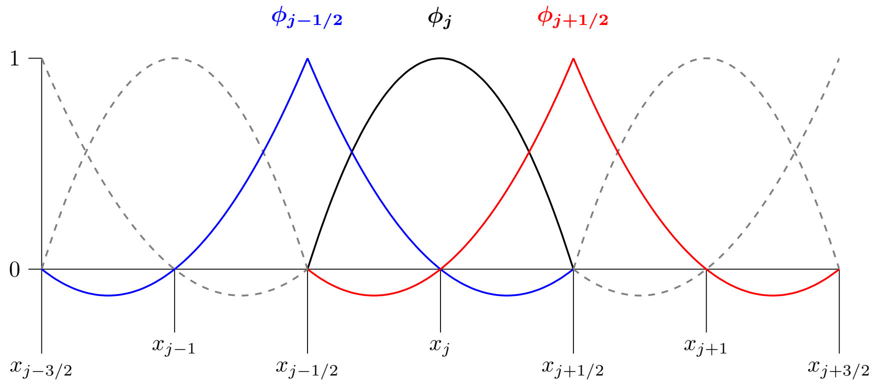

Figure 3.2: Basis functions for continuous piecewise quadratic elements.

For the velocity term our FVM requires a local second-order approximation of the first derivative. To do this will require a quadratic representation of u in each cell. Since u is guaranteed to have continuous representation across the cells we use a quadratic reconstruction that is continuous at the cell edges. The basis functions for this reconstruction of u will be represented by ϕ. We will also use these basis functions as our test functions v. This reconstruction will use cell values at $x_{j-1/2}$, x_j and $x_{j+1/2}$ to represent the quadratic function over the cell.

$$u = \sum_j u_{j-1/2}\phi_{j-1/2} + u_j\phi_j + u_{j+1/2}\phi_{j+1/2} \tag{3.4}$$

Since we calculate u from the other quantities we do not require a reconstruction scheme for $u_{j-1/2}$, u_j and $u_{j+1/2}$.

For the bed term our FVM requires a second-order approximation of the second derivative of the bed terms that must be done locally. To do this we will require a cubic representation of the bed in each cell. Since by our smoothness conditions we do not allow for discontinuous beds, we will assume the bed is continuous. Therefore we will use a cubic reconstruction for the bed which is continuous across the cell edges. The basis functions for this reconstruction of b will be represented by γ. This reconstruction will use cell values at $x_{j-1/2}$, $x_{j-1/6}$, $x_{j+1/6}$ and $x_{j+1/2}$ to represent the cubic function over the cell.

$$b = \sum_j b_{j-1/2}\gamma_{j-1/2} + b_{j-1/6}\gamma_{j-1/6} + b_{j+1/6}\gamma_{j+1/6} + b_{j+1/2}\gamma_{j+1/2} \tag{3.5}$$

The values $b_{j-1/2}$, $b_{j-1/6}$, $b_{j+1/6}$ and $b_{j+1/2}$ are reconstructed from the cubic that passes through the nodal values b_{j-2}, b_{j-1}, b_{j+1} and b_{j+2}. Which is given by

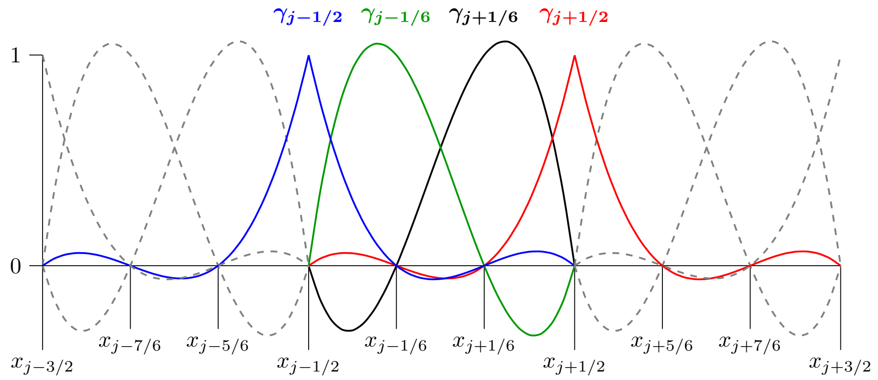

Figure 3.3: Basis functions for continuous piecewise cubic elements.

$$\begin{aligned} Q_j^b(x) = &\frac{-b_{j-2} + 2b_{j-1} - 2b_{j+1} + b_{j+2}}{12\Delta x^3}(x - x_j)^3 \\ &+ \frac{b_{j-2} - b_{j-1} - b_{j+1} + b_{j+2}}{6\Delta x^2}(x - x_j)^2 + \frac{b_{j-2} - 8b_{j-1} + 8b_{j+1} - b_{j+2}}{12\Delta x}(x - x_j) \\ &+ \frac{-b_{j-2} + 4b_{j-1} + 4b_{j+1} - b_{j+2}}{6} \end{aligned} \tag{3.6}$$

then we have the following

$$\begin{aligned} b_{j-1/2} &= Q_j^b(x_{j-1/2}) \\ b_{j-1/6} &= Q_j^b(x_{j-1/6}) \\ b_{j+1/6} &= Q_j^b(x_{j+1/6}) \\ b_{j+1/2} &= Q_j^b(x_{j+1/2}). \end{aligned}$$

Element wise

Since the integral equation (3.2) must be true for all v, in particular it must hold for $\phi_{j-1/2}$, ϕ_j and $\phi_{j+1/2}$. Since only these basis functions are non-zero over the element $[x_{j-1/2}, x_{j+1/2}]$, we can calculate the jth term in the sum (3.2) completely like so

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \left[\left(uh \left(1 + \frac{\partial b}{\partial x}^2 \right) - \frac{1}{2} h^2 \frac{\partial b}{\partial x} \frac{\partial u}{\partial x} - G \right) \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} \right.$$
$$\left. + \left(\frac{1}{3} h^3 \frac{\partial u}{\partial x} - \frac{1}{2} h^2 \frac{\partial b}{\partial x} u \right) \frac{\partial}{\partial x} \left(\begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} \right) \right] dx \quad (3.7)$$

where we use our finite element approximations for h, u, G and b. Because our basis functions over one element are just translations of the the other basis functions, this integral can be generalised by moving to what is commonly called the ξ space. For our current definitions the mapping from the x space to the ξ space is

$$x = x_j + \xi \frac{\Delta x}{2}$$

Making the change of variables the integral becomes

$$\frac{\Delta x}{2} \int_{-1}^{1} \left[\left(uh \left(1 + \frac{4}{\Delta x^2} \frac{\partial b}{\partial \xi}^2 \right) - \frac{1}{2} h^2 \frac{4}{\Delta x^2} \frac{\partial b}{\partial \xi} \frac{\partial u}{\partial \xi} - G \right) \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} \right.$$
$$\left. + \left(\frac{1}{3} h^3 \frac{2}{\Delta x} \frac{\partial u}{\partial \xi} - \frac{1}{2} h^2 \frac{2}{\Delta x} \frac{\partial b}{\partial \xi} u \right) \frac{2}{\Delta x} \frac{\partial}{\partial \xi} \left(\begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} \right) \right] d\xi \quad (3.8)$$

where all functions have ξ as there variable instead of x. We will now demonstrate the rest of this process for the term uh and then give the remaining matrices. Focusing on the uh term we have to compute

$$\frac{\Delta x}{2} \int_{-1}^{1} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi \quad (3.9)$$

Since we are computing the integral over $\left[x_{j-1/2}, x_{j+1/2}\right]$ the only non-zero contributions from the finite element approximation to h and u are

$$\frac{\Delta x}{2}\int_{-1}^{1}\left(u_{j-1/2}\phi_{j-1/2}+u_j\phi_j+u_{j+1/2}\phi_{j+1/2}\right)\left(h^+_{j-1/2}\psi^+_{j-1/2}+h^-_{j+1/2}\psi^-_{j+1/2}\right)\begin{bmatrix}\phi_{j-1/2}\\ \phi_j\\ \phi_{j+1/2}\end{bmatrix}d\xi \tag{3.10}$$

$$\frac{\Delta x}{2}\int_{-1}^{1}\begin{bmatrix}\phi_{j-1/2} & \phi_j & \phi_{j+1/2}\end{bmatrix}\begin{bmatrix}u_{j-1/2}\\ u_j\\ u_{j+1/2}\end{bmatrix}\left(h^+_{j-1/2}\psi^+_{j-1/2}+h^-_{j+1/2}\psi^-_{j+1/2}\right)\begin{bmatrix}\phi_{j-1/2}\\ \phi_j\\ \phi_{j+1/2}\end{bmatrix}d\xi \tag{3.11}$$

$$\left(\frac{\Delta x}{2}\int_{-1}^{1}\left(h^+_{j-1/2}\psi^+_{j-1/2}+h^-_{j+1/2}\psi^-_{j+1/2}\right)\begin{bmatrix}\phi_{j-1/2}\phi_{j-1/2} & \phi_j\phi_{j-1/2} & \phi_{j+1/2}\phi_{j-1/2}\\ \phi_{j-1/2}\phi_j & \phi_j\phi_j & \phi_{j+1/2}\phi_j\\ \phi_{j+1/2}\phi_{j-1/2} & \phi_{j+1/2}\phi_j & \phi_{j+1/2}\phi_{j+1/2}\end{bmatrix}d\xi\right)\begin{bmatrix}u_{j-1/2}\\ u_j\\ u_{j+1/2}\end{bmatrix} \tag{3.12}$$

$$\begin{aligned}\frac{\Delta x}{2}\Bigg(&h^+_{j-1/2}\int_{-1}^{1}\psi^+_{j-1/2}\begin{bmatrix}\phi_{j-1/2}\phi_{j-1/2} & \phi_j\phi_{j-1/2} & \phi_{j+1/2}\phi_{j-1/2}\\ \phi_{j-1/2}\phi_j & \phi_j\phi_j & \phi_{j+1/2}\phi_j\\ \phi_{j+1/2}\phi_{j-1/2} & \phi_{j+1/2}\phi_j & \phi_{j+1/2}\phi_{j+1/2}\end{bmatrix}d\xi\\ &h^-_{j+1/2}\int_{-1}^{1}\psi^-_{j+1/2}\begin{bmatrix}\phi_{j-1/2}\phi_{j-1/2} & \phi_j\phi_{j-1/2} & \phi_{j+1/2}\phi_{j-1/2}\\ \phi_{j-1/2}\phi_j & \phi_j\phi_j & \phi_{j+1/2}\phi_j\\ \phi_{j+1/2}\phi_{j-1/2} & \phi_{j+1/2}\phi_j & \phi_{j+1/2}\phi_{j+1/2}\end{bmatrix}d\xi\Bigg)\\ &\begin{bmatrix}u_{j-1/2}\\ u_j\\ u_{j+1/2}\end{bmatrix}\end{aligned} \tag{3.13}$$

Calculating these integrals we get that

$$\begin{aligned}&\frac{\Delta x}{2}\int_{-1}^{1}uh\begin{bmatrix}\phi_{j-1/2}\\ \phi_j\\ \phi_{j+1/2}\end{bmatrix}d\xi=\\ &\frac{\Delta x}{2}\begin{bmatrix}\frac{7}{30}h^+_{j-1/2}+\frac{1}{30}h^-_{j+1/2} & \frac{4}{30}h^+_{j-1/2} & -\frac{1}{30}h^+_{j-1/2}-\frac{1}{30}h^-_{j+1/2}\\ \frac{4}{30}h^+_{j-1/2} & \frac{16}{30}h^+_{j-1/2}+\frac{16}{30}h^-_{j+1/2} & \frac{4}{30}h^-_{j+1/2}\\ -\frac{1}{30}h^+_{j-1/2}-\frac{1}{30}h^-_{j+1/2} & \frac{4}{30}h^-_{j+1/2} & \frac{1}{30}h^+_{j-1/2}+\frac{7}{30}h^-_{j+1/2}\end{bmatrix}\begin{bmatrix}u_{j-1/2}\\ u_j\\ u_{j+1/2}\end{bmatrix}\end{aligned} \tag{3.14}$$

We construct all these element matrices for u values at the edges, which are given in the appendix and then combine them to form the matrix equation

$$\boldsymbol{G}_j - \boldsymbol{A}_j \boldsymbol{u}_j.$$

for each element. Then from (3.2) we have that

$$\sum_j \boldsymbol{G}_j - \boldsymbol{A}_j \boldsymbol{u}_j = 0.$$

and so our equation becomes

$$\sum_j \boldsymbol{G}_j = \sum_j \boldsymbol{A}_j \boldsymbol{u}_j.$$

For the second-order method this is a tri-diagonal matrix equation, and we can use standard banded matrix solution techniques.

3.4 Evolution Equations

The evolution equations in the alternative form of the Serre equations (2.2) are

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

and

$$\frac{\partial}{\partial t}(G) + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3 \frac{\partial u}{\partial x}^2 + h^2 u \frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right) = -\frac{1}{2}h^2 u \frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} + hu^2 \frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} - gh\frac{\partial b}{\partial x}$$

Because these equations are in conservation law form and we have an estimate for the maximum and minimum wave speeds [], Kurganovs method [] can be employed to estimate the fluxes across the boundary. This leads to the following update scheme for a quantity q

$$\bar{q}_j^{n+1} = \bar{q}_j^n - \frac{\Delta t}{\Delta x}\left[F^n_{j+1/2} - F^n_{j-1/2}\right] + \Delta t S^n_j. \tag{3.15}$$

Where $F^n_{j+1/2}$ and $F^n_{j-1/2}$ are approximations to the average fluxes across the boundary of the cell with midpoint x_i from time t^n to t^{n+1}. While S_j is an approximation to the average source term contribution in the cell from time t^n to t^{n+1}.

3.4.1 Kurganovs Method

Kurganovs method is a finite volume method that can handle discontinuities across the boundary and only requires an estimate of the maximum and minimum wave speeds instead of the characteristics like other methods []. This makes it a good choice for the Serre equations as we do not have an expression for the characteristics but we do have estimates on the maximum and minimum wave speeds [].

The equation which approximates $F^n_{j+1/2}$ in (3.15) for a quantity q at a particular time t^n is

$$F_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}} f\left(q^-_{j+\frac{1}{2}}\right) - a^-_{j+\frac{1}{2}} f\left(q^+_{j+\frac{1}{2}}\right)}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}}\, a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} \left[q^+_{j+\frac{1}{2}} - q^-_{j+\frac{1}{2}}\right] \tag{3.16}$$

where $a^+_{j+\frac{1}{2}}$ $a^-_{j+\frac{1}{2}}$ are given by the wave speed bounds [], for the Serre equations we have

$$\begin{aligned} a^-_{j+\frac{1}{2}} &= \min\left\{0\ ,\ u^-_{j+1/2} - \sqrt{gh^-_{j+1/2}}\ ,\ u^+_{j+1/2} - \sqrt{gh^+_{j+1/2}}\right\}, \\ a^+_{j+\frac{1}{2}} &= \max\left\{0\ ,\ u^-_{j+1/2} + \sqrt{gh^-_{j+1/2}}\ ,\ u^+_{j+1/2} + \sqrt{gh^+_{j+1/2}}\right\}, \end{aligned}$$

While $f(q^-_{j+\frac{1}{2}})$ and $f(q^+_{j+\frac{1}{2}})$ are the evaluations of the flux function on the left and right side of the cell interface respectively. For h in the Serre equations we have

$$\begin{aligned} f\left(h^-_{j+\frac{1}{2}}\right) &= u^-_{j+1/2} h^-_{j+1/2}, \\ f\left(h^+_{j+\frac{1}{2}}\right) &= u^+_{j+1/2} h^+_{j+1/2}, \end{aligned}$$

while for G we have

$$\begin{aligned} f\left(G^-_{j+\frac{1}{2}}\right) &= u^-_{j+1/2} G^-_{j+1/2} + \frac{g}{2}\left(h^-_{j+1/2}\right)^2 - \frac{2}{3}\left(h^-_{j+1/2}\right)^3 \left[\left(\frac{\partial u}{\partial x}\right)^-_{j+1/2}\right]^2 \\ &\quad + \left(h^-_{j+1/2}\right)^2 u^-_{j+1/2} \left(\frac{\partial u}{\partial x}\right)^-_{j+1/2} \left(\frac{\partial b}{\partial x}\right)^-_{j+1/2}, \\ f\left(G^+_{j+\frac{1}{2}}\right) &= u^+_{j+1/2} G^+_{j+1/2} + \frac{g}{2}\left(h^+_{j+1/2}\right)^2 - \frac{2}{3}\left(h^+_{j+1/2}\right)^3 \left[\left(\frac{\partial u}{\partial x}\right)^+_{j+1/2}\right]^2 \\ &\quad + \left(h^+_{j+1/2}\right)^2 u^+_{j+1/2} \left(\frac{\partial u}{\partial x}\right)^+_{j+1/2} \left(\frac{\partial b}{\partial x}\right)^+_{j+1/2}. \end{aligned}$$

We now only have to have some appropriate order of accuracy method to calculate the quantities we need at the boundaries from the cell averages of h, G and b and the nodal values of u.

First Order

For h and G we use that constant approximation inside a cell which For a general quantity q is

$$q^-_{j+1/2} = q^+_{j+1/2} = \bar{q}_j \tag{3.17}$$

For the other quantities we use the values outline below for the second order method.

Second Order

For the second order finite volume method we use the minmmod limiter to reconstruct h, G and b at the cell edges. We show how this is done for h and G above [] in the FEM section.

While for u the following reconstruction is used

$$u^-_{j+\frac{1}{2}} = u^+_{j+\frac{1}{2}} = \frac{u_{j+1} + u_j}{2}. \tag{3.18}$$

We approximate the derivatives in the following way

$$\left(\frac{\partial b}{\partial x}\right)^-_{j+1/2} = \frac{-b^-_{j-1/2} + b^-_{j+1/2}}{\Delta x}, \tag{3.19}$$

$$\left(\frac{\partial b}{\partial x}\right)^+_{j+1/2} = \frac{-b^+_{j+1/2} + b^+_{j+3/2}}{\Delta x}, \tag{3.20}$$

$$\left(\frac{\partial u}{\partial x}\right)^-_{j+1/2} = \left(\frac{\partial u}{\partial x}\right)^+_{j+1/2} = \frac{-u_j + u_{j+1}}{\Delta x} \tag{3.21}$$

Third Order

For the third-order finite volume method we use the Koren limiter [?] to reconstruct the cell edges for h, G and b. For a general quantity q the reconstruction based on the Koren limiter is

$$q^-_{j+1/2} = \bar{q}_j + \phi^-\left(r_j\right)\left(\bar{q}_j - \bar{q}_{j-1}\right)/2 \tag{3.22a}$$

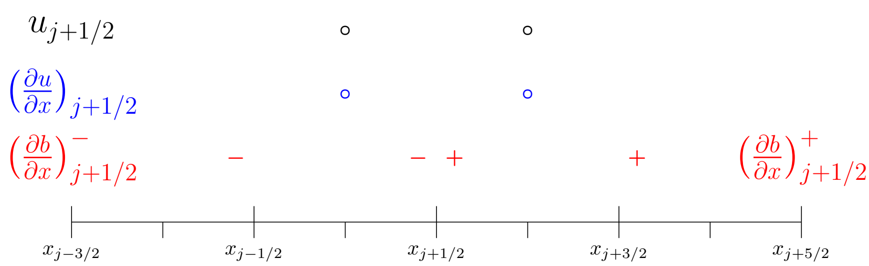

Figure 3.4: Summary of values necessary to calculate values of interest at cell boundary for the second-order finite difference volume method.

and

$$q^{+}_{j+1/2} = \bar{q}_{j+1} - \phi^{+}\left(r_{j+1}\right)\left(\bar{q}_{j+1} - \bar{q}_{j}\right)/2 \tag{3.22b}$$

where

$$\phi^{-}\left(r_{j}\right) = \max\left[0, \min\left[2r_{j}, \frac{1+2r_{j}}{3}, 2\right]\right], \tag{3.22c}$$

$$\phi^{+}\left(r_{j}\right) = \max\left[0, \min\left[2r_{j}, \frac{2+r_{j}}{3}, 2\right]\right] \tag{3.22d}$$

with

$$r_{j} = (\bar{q}_{j+1} - \bar{q}_{j})/(\bar{q}_{j} - \bar{q}_{j-1}). \tag{3.22e}$$

While for u the following reconstruction is used

$$u_{j+\frac{1}{2}} = \frac{-3u_{j-1} + 27u_{j} + 27u_{j+1} - 3u_{j+2}}{48}. \tag{3.23}$$

We approximate the derivatives in the following way

$$\left(\frac{\partial b}{\partial x}\right)^{-}_{j+1/2} = \frac{b^{-}_{j-3/2} - 4b^{-}_{j-1/2} + 3b^{-}_{j+1/2}}{\Delta x}, \tag{3.24}$$

$$\left(\frac{\partial b}{\partial x}\right)^{+}_{j+1/2} = \frac{3b^{+}_{j+1/2} + 4b^{+}_{j+3/2} - b^{+}_{j+5/2}}{\Delta x}, \tag{3.25}$$

$$\left(\frac{\partial u}{\partial x}\right)^{-}_{j+1/2} = \left(\frac{\partial u}{\partial x}\right)^{+}_{j+1/2} = \frac{u_{j-1} - 27u_{j} + 27u_{j+1} - u_{j+2}}{24\Delta x}, \tag{3.26}$$

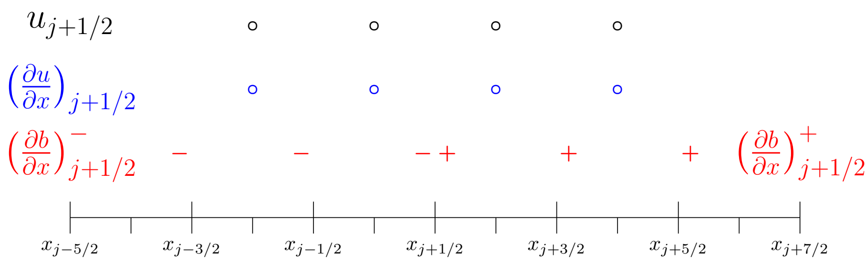

Figure 3.5: Summary of values necessary to calculate values of interest at cell boundary for the third-order finite difference volume method.

3.4.2 Source Terms and Well Balancing

In [] we demonstrated that second-order accuracy is sufficient and so we have only included source terms and well balancing for the second-order method. So this section will only demonstrate the second-order method.

From (3.15) the only term that we have not computed is S_j^n according to the development of our finite volume method we should have

$$S_j^n \approx \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} S(x,t) dx.$$

Because our time-stepping method for the flux evaluation is only first-order and higher-order temporal schemes are developed using Runge-Kutta steps we can just take the approximation that $S(x,t)$ is constant in time so that

$$S_j^n \approx \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} S(x,t^n) dx.$$

Where we now only have to ensure that the approximation to the integral of the source term has the appropriate spatial order of accuracy. With the midpoint approximation we have

$$\frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} S(x,t^n) dx = S(x_j,t^n) + \mathcal{O}\left(\Delta x^2\right).$$

So in our second-order method we take

$$S_j^n = S(x_j,t^n).$$

This means that

$$S_j^n = -\frac{1}{2}\left(h_j^n\right)^2 u_j^n \left(\frac{\partial u}{\partial x}\right)_j^n \left(\frac{\partial^2 b}{\partial x^2}\right)_j^n + h_j^n \left(u_j^n\right)^2 \left(\frac{\partial b}{\partial x}\right)_j^n \left(\frac{\partial^2 b}{\partial x^2}\right)_j^n - g h_j^n \left(\frac{\partial b}{\partial x}\right)_j^n \tag{3.27}$$

where the derivatives and the nodal values must have the appropriate order of accuracy. For the second-order method we already have that the $\bar{h}_j^n = h_j^n$ and $\bar{u}_j^n = u_j^n$ so we turn our attention to the derivatives.

Finite Element Method

In the finite element method we use the polynomial over the cell to approximate the derivative.

For u the quadratic over the cell $\left[x_{j-1/2}, x_{j+1/2}\right]$ is

$$P_j^u(x) = 2\frac{u_{j-1/2} - 2u_j + u_{j+1/2}}{\Delta x^2}\left(x - x_j\right)^2 + \frac{-u_{j-1/2} + u_{j+1/2}}{\Delta x}\left(x - x_j\right) + u_j \tag{3.28}$$

while for b the cubic over the cell is

$$\begin{aligned} P_j^b(x) = {}& 9\frac{-b_{j-1/2} + 3b_{j-1/6} - 3b_{j+1/6} + b_{j+1/2}}{2\Delta x^3}\left(x - x_j\right)^3 \\ & + 9\frac{b_{j-1/2} - b_{j-1/6} - b_{j+1/6} + b_{j+1/2}}{4\Delta x^2}\left(x - x_j\right)^2 \\ & + \frac{b_{j-1/2} - 27b_{j-1/6} + 27b_{j+1/6} - b_{j+1/2}}{8\Delta x}\left(x - x_j\right) \\ & + \frac{1}{16}\left(-b_{j-1/2} + 9b_{j-1/6} + 9b_{j+1/6} - b_{j+1/2}\right) \end{aligned} \tag{3.29}$$

so we have , to make the scheme well balanced we will rely on a slightly different approximation to the bed term that is still second order

$$\begin{aligned} \left(\frac{\partial u}{\partial x}\right)_j^n &= \left.\frac{\partial P_j^u}{\partial x}\right|_{x=x_j} = \frac{-u_{j-1/2} + u_{j+1/2}}{\Delta x} \\ \left(\frac{\partial b}{\partial x}\right)_j^n &= \left.\frac{\partial P_j^b}{\partial x}\right|_{x=x_j} = \frac{b_{j-1/2} - 27b_{j-1/6} + 27b_{j+1/6} - b_{j+1/2}}{8\Delta x} \\ \left(\frac{\partial^2 b}{\partial x^2}\right)_j^n &= \left.\frac{\partial^2 P_j^b}{\partial x^2}\right|_{x=x_j} = 9\frac{b_{j-1/2} - b_{j-1/6} - b_{j+1/6} + b_{j+1/2}}{2\Delta x^2} \end{aligned}$$

Finite Difference Method

For the finite difference method we use the following approximations

$$\left(\frac{\partial u}{\partial x}\right)_j^n = \frac{-u_{j-1/2} + u_{j+1/2}}{\Delta x}$$
$$\left(\frac{\partial b}{\partial x}\right)_j^n = \frac{-b_{j-1/2} + b_{j+1/2}}{\Delta x}$$
$$\left(\frac{\partial^2 b}{\partial x^2}\right)_j^n = \frac{b_{j-1} - 2b_j + b_{j+1}}{\Delta x^2}$$

Well balancing

In [] we demonstrated that the well balancing method of [] for the SWWE can be applied to the Serre equations. To achieve a well balanced method which recovers the lake at rest steady state we use the same method. For completeness we present it here

[]

Algorithm 3.2.

1. Use the appropriate reconstruction to get the interface values for both u and G giving $u_{i+\frac{1}{2}}$, $G^-_{i+\frac{1}{2}}$ and $G^+_{i+\frac{1}{2}}$.
2. Use the second order reconstruction to get the interface values for $h+b$ and h giving $(h+b)^-_{i+\frac{1}{2}}$, $(h+b)^+_{i+\frac{1}{2}}$, $h^-_{i+\frac{1}{2}}$ and $h^+_{i+\frac{1}{2}}$.
3. Calculate the interface values for b by using the reconstructed values of $h+b$ and h above by simply taking the difference giving $b^-_{i+\frac{1}{2}}$ and $b^+_{i+\frac{1}{2}}$.
4. Define $\acute{b}_{i+\frac{1}{2}} = \max\left(b^-_{i+\frac{1}{2}}, b^+_{i+\frac{1}{2}}\right)$
5. Define $\acute{h}^-_{i+\frac{1}{2}} = \max\left(0, (h+b)^-_{i+\frac{1}{2}} - \acute{b}_{i+\frac{1}{2}}\right)$ and $\acute{h}^+_{i+\frac{1}{2}} = \max\left(0, (h+b)^+_{i+\frac{1}{2}} - \acute{b}_{i+\frac{1}{2}}\right)$
6. Calculate the flux as in chapter **??** at $x_{i+\frac{1}{2}}$ using $\acute{U}^+_{i+\frac{1}{2}} = \begin{pmatrix} \acute{h}^+_{i+\frac{1}{2}} \\ H^+_{i+\frac{1}{2}} \end{pmatrix}$ and $\acute{U}^-_{i+\frac{1}{2}} = \begin{pmatrix} \acute{h}^-_{i+\frac{1}{2}} \\ H^-_{i+\frac{1}{2}} \end{pmatrix}$ as the conserved variables on the right and left respectively. With the $u^+_{i+\frac{1}{2}}$ and $b^+_{i+\frac{1}{2}}$ giving the right velocity and bed. While $u^-_{i+\frac{1}{2}}$ and $b^-_{i+\frac{1}{2}}$ gives the left velocity and bed.

7. Repeat for the other boundary at $x_{i-\frac{1}{2}}$

8. Calculate the source term S_{ci}

$$S_{ci} = \Delta x \begin{pmatrix} 0 \\ -gh\frac{\partial b}{\partial x} - \frac{1}{2}h^2 u \frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} + hu^2\frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} \end{pmatrix}$$

   by the appropriate method given above, with the following modification

$$\frac{\partial b}{\partial x} = \frac{-b^+_{i-\frac{1}{2}} + b^-_{i+\frac{1}{2}}}{\Delta x}$$

9. Calculate the corrective source term S_{bi} where $S_{bi} = S^-_{i+\frac{1}{2}} + S^+_{i-\frac{1}{2}}$

$$S^-_{i+\frac{1}{2}} = \begin{pmatrix} 0 \\ \frac{g}{2}\left(\acute{h}^-_{i+\frac{1}{2}}\right)^2 - \frac{g}{2}\left(h^-_{i+\frac{1}{2}}\right)^2 \end{pmatrix}$$

$$S^+_{i-\frac{1}{2}} = \begin{pmatrix} 0 \\ \frac{g}{2}\left(h^+_{i-\frac{1}{2}}\right)^2 - \frac{g}{2}\left(\acute{h}^+_{i-\frac{1}{2}}\right)^2 \end{pmatrix}$$

10. Update according to (**??**) with the original average values for U_i and the calculated values for the fluxes and $S_i = S_{bi} + S_{ci}$

3.5 Runge-Kutta Time-Stepping

The method $\mathcal{E}$ is only first order in time, one strategy for increasing our order of accuracy in time is to use SSP Runge Kutta time stepping [].

For the first order method our current method is sufficient and so

$$\bar{\boldsymbol{h}}^{n+1}, \bar{\boldsymbol{G}}^{n+1} = \mathcal{E}\left(\bar{\boldsymbol{h}}^n, \bar{\boldsymbol{G}}^n, \boldsymbol{b}\right) \tag{3.30}$$

For the second order method we have

$$\bar{\boldsymbol{h}}', \bar{\boldsymbol{G}}' = \mathcal{E}\left(\bar{\boldsymbol{h}}^n, \bar{\boldsymbol{G}}^n, \boldsymbol{b}\right) \tag{3.31a}$$

$$\bar{\boldsymbol{h}}'', \bar{\boldsymbol{G}}'' = \mathcal{E}\left(\bar{\boldsymbol{h}}', \bar{\boldsymbol{G}}', \boldsymbol{b}\right) \tag{3.31b}$$

$$\bar{\boldsymbol{h}}^{n+1}, \bar{\boldsymbol{G}}^{n+1} = \frac{1}{2}\left(\bar{\boldsymbol{h}}^n + \bar{\boldsymbol{h}}''\right), \frac{1}{2}\left(\bar{\boldsymbol{G}}^n + \bar{\boldsymbol{G}}''\right) \tag{3.31c}$$

For the third order method we have

$$\bar{\boldsymbol{h}}', \bar{\boldsymbol{G}}' = \mathcal{E}\left(\bar{\boldsymbol{h}}^n, \bar{\boldsymbol{G}}^n, \boldsymbol{b}\right) \tag{3.32a}$$

$$\bar{\boldsymbol{h}}'', \bar{\boldsymbol{G}}'' = \mathcal{E}\left(\bar{\boldsymbol{h}}', \bar{\boldsymbol{G}}', \boldsymbol{b}\right) \tag{3.32b}$$

$$\bar{\boldsymbol{h}}''', \bar{\boldsymbol{G}}''' = \frac{3}{4}\bar{\boldsymbol{h}}^n + \frac{1}{4}\bar{\boldsymbol{h}}'', \frac{3}{4}\bar{\boldsymbol{G}}^n + \frac{1}{4}\bar{\boldsymbol{G}}'' \tag{3.32c}$$

$$\bar{\boldsymbol{h}}'''', \bar{\boldsymbol{G}}'''' = \mathcal{E}\left(\bar{\boldsymbol{h}}''', \bar{\boldsymbol{G}}''', \boldsymbol{b}\right) \tag{3.32d}$$

$$\bar{\boldsymbol{h}}^{n+1}, \bar{\boldsymbol{G}}^{n+1} = \frac{1}{3}\bar{\boldsymbol{h}}^n + \frac{2}{3}\bar{\boldsymbol{h}}'''', \frac{1}{3}\bar{\boldsymbol{G}}^n + \frac{2}{3}\bar{\boldsymbol{G}}'''' \tag{3.32e}$$

Chapter 4

Finite Difference Methods

The methods $\mathcal{W}$ and $\mathcal{D}$ use the centred second-order finite difference approximation to the momentum equation (2.1b), denoted as $\mathcal{D}_u$. For the mass equation (2.1a) $\mathcal{W}$ uses the two step Lax-Wendroff method, denoted as $\mathcal{W}_h$ while $\mathcal{D}$ uses a centred second-order finite difference approximation, denoted as $\mathcal{D}_h$.

4.1 Naive Second Order Finite Difference Approximation to the Momentum Equation

First (2.1b) is expanded to get

$$h\frac{\partial u}{\partial t} - h^2\frac{\partial^2 u}{\partial x \partial t} - \frac{h^3}{3}\frac{\partial^3 u}{\partial x^2 \partial t} = -X$$

where X contains only spatial derivatives and is

$$X = uh\frac{\partial u}{\partial x} + gh\frac{\partial h}{\partial x} + h^2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{h^3}{3}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - h^2 u\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3}u\frac{\partial^3 u}{\partial x^3}.$$

All derivatives are approximated by second-order centred finite difference approximations on a uniform grid in space and time, which after rearranging into an update formula becomes

$$h_j^n u_j^{n+1} - \left(h_j^n\right)^2\left(\frac{-u_{j-1}^{n+1} + u_{j+1}^{n+1}}{2\Delta x}\right) - \frac{\left(h_j^n\right)^3}{3}\left(\frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2}\right) = -Y_j^n \quad (4.1)$$

where

$$Y_j^n = 2\Delta t X_j^n - h_j^n u_j^{n-1} + \left(h_j^n\right)^2\left(\frac{-u_{j-1}^{n-1} + u_{j+1}^{n-1}}{2\Delta x}\right) + \frac{\left(h_j^n\right)^3}{3}\left(\frac{u_{j-1}^{n-1} - 2u_j^{n-1} + u_{j+1}^{n-1}}{\Delta x^2}\right)$$

and

$$X_j^n = u_j^n h_j^n \frac{-u_{j-1}^n + u_{j+1}^n}{2\Delta x} + g h_j^n \frac{-h_{j-1}^n + h_{j+1}^n}{2\Delta x} + \left(h_j^n\right)^2 \left(\frac{-u_{j-1}^n + u_{j+1}^n}{2\Delta x}\right)^2$$
$$+ \frac{\left(h_j^n\right)^3}{3} \frac{-u_{j-1}^n + u_{j+1}^n}{2\Delta x} \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} - \left(h_j^n\right)^2 u_j^n \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2}$$
$$- \frac{\left(h_j^n\right)^3}{3} u_j^n \frac{-u_{j-2}^n + 2u_{j-1}^n - 2u_{j+1}^n + u_{j+2}^n}{2\Delta x^3}.$$

Equation (4.1) can be rearranged into an explicit update scheme $\mathcal{D}_u$ for u given its current and previous values, so that

$$\boldsymbol{u}^{n+1} = A^{-1} \begin{bmatrix} -Y_0^n \\ \vdots \\ -Y_m^n \end{bmatrix} =: \mathcal{D}_u \left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n-1}, \Delta x, \Delta t\right) \tag{4.2}$$

[] where A is a tri-diagonal matrix.
with

$$A_{j,j-1} = \frac{\left(h_j^n\right)^2}{2\Delta x} \frac{-h_{j-1}^n + h_{j+1}^n}{2\Delta x} - \frac{\left(h_j^n\right)^3}{3\Delta x^2}, \tag{4.3}$$
$$A_{j,j} = h_j^n + \frac{2h_j^n}{3\Delta x^2}, \tag{4.4}$$
$$A_{j,j+1} = -\frac{\left(h_j^n\right)^2}{2\Delta x} \frac{-h_{j-1}^n + h_{j+1}^n}{2\Delta x} - \frac{\left(h_j^n\right)^3}{3\Delta x^2}. \tag{4.5}$$

4.2 Numerical Methods for the Mass Equation

4.2.1 Lax-Wendroff Method

The two step Lax-Wendroff update $\mathcal{W}_h$ for h is

$$h_{j+1/2}^{n+1/2} = \frac{1}{2}\left(h_{j+1}^n + h_j^n\right) - \frac{\Delta t}{2\Delta x}\left(u_{j+1}^n h_{j+1}^n - h_j^n u_j^n\right),$$
$$h_{j-1/2}^{n+1/2} = \frac{1}{2}\left(h_j^n + h_{j-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(u_j^n h_j^n - h_{j-1}^n u_{j-1}^n\right)$$

and

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x}\left(u_{j+1/2}^{n+1/2} h_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2} h_{j-1/2}^{n+1/2}\right).$$

The quantities $u_{j\pm1/2}^{n+1/2}$ are calculated using $\boldsymbol{u}^{n+1}$ obtained by applying $\mathcal{D}_u$ (4.2) to $\boldsymbol{u}^n$ then linearly interpolating in space and time to give

$$u_{j+1/2}^{n+1/2} = \frac{u_j^n + u_{j+1}^n + u_j^{n+1} + u_{j+1}^{n+1}}{4}$$

and

$$u_{j-1/2}^{n+1/2} = \frac{u_{j-1}^n + u_j^n + u_{j-1}^{n+1} + u_j^{n+1}}{4}.$$

Thus we have the following update scheme $\mathcal{W}_h$ for (2.1a)

$$\boldsymbol{h}^{n+1} = \mathcal{W}_h\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n+1}, \Delta x, \Delta t\right). \tag{4.6}$$

4.2.2 Second Order Finite Difference Approximation

The second order centered finite difference approximation to the conservation of mass equation (2.1a) is

$$h_j^{n+1} = h_j^{n-1} - \Delta t\left(u_j^n \frac{-h_{j-1}^n + h_{j+1}^n}{\Delta x} + h_j^n \frac{-u_{j-1}^n + u_{j+1}^n}{\Delta x}\right).$$

Thus we have an update scheme $\mathcal{D}_h$ for all i

$$\boldsymbol{h}^{n+1} = \mathcal{D}_h\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{h}^{n-1}, \Delta x, \Delta t\right). \tag{4.7}$$

4.3 Complete Method

The method $\mathcal{W}$ is the combination of (4.6) for (2.1a) and (4.2) for (2.1b) in the following way

$$\left.\begin{aligned} \boldsymbol{u}^{n+1} &= \mathcal{D}_u\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n-1}, \Delta x, \Delta t\right) \\ \boldsymbol{h}^{n+1} &= \mathcal{W}_h\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n+1}, \Delta x, \Delta t\right) \end{aligned}\right\} \mathcal{W}\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n-1}, \boldsymbol{h}^{n-1}, \Delta x, \Delta t\right). \tag{4.8}$$

The method $\mathcal{D}$ is the combination of (4.7) for (2.1a) and (4.2) for (2.1b) in the following way

$$\left.\begin{aligned} \boldsymbol{h}^{n+1} &= \mathcal{D}_h\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{h}^{n-1} \Delta x, \Delta t\right) \\ \boldsymbol{u}^{n+1} &= \mathcal{D}_u\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n-1}, \Delta x, \Delta t\right) \end{aligned}\right\} \mathcal{D}\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n-1}, \boldsymbol{h}^{n-1}, \Delta x, \Delta t\right). \tag{4.9}$$

Chapter 5

Analysis of Numerical Methods

There are a variety of ways to analyse the numerical methods presented in this paper []. There are also a variety of properties that numerical methods can possess. Chief among these is convergence, which for linear partial differential equations can be broken up into consistency and stability using the Lax equivalence theorem [].

For dispersive equations such as the Serre equations other more specific properties may be of interest such as the error in the dispersion relation introduced by the numerical methods as investigated in [].

The main obstacle for analysing the numerical methods for the Serre equations is that the Serre equations are non-linear, which means the techniques employed to investigate the convergence and dispersion error of our numerical methods are no longer valid. However, some insights can still be gained by instead examining the linearised version of the Serre equations as has been done []. This is the approach we take in this section of the thesis.

In this thesis we will demonstrate the stability of the finite difference methods and calculate the dispersion error of the hybrid finite volume methods. In the finite difference methods we have a horizontal bed profile, while for the dispersion relation the bed terms have no contribution, so for the rest of this chapter we will assume $b(x) = 0$ for all x.

To linearise the equations we assume that

$$h(x,t) = H + \delta\eta(x,t) + \mathcal{O}\left(\delta^2\right), \tag{5.1}$$

$$u(x,t) = U + \delta\upsilon(x,t) + \mathcal{O}\left(\delta^2\right). \tag{5.2}$$

Where $\delta \ll 1$, so that we are modelling small waves on top of water with a mean height of H and a background mean flow of U and terms of order δ^2 are negligible. We substitute this into (2.2) and (2.2) and neglect terms of order δ^2 to obtain

$$\frac{\partial \eta}{\partial t} + H\frac{\partial \upsilon}{\partial x} + U\frac{\partial \eta}{\partial x} + = 0 \tag{5.3a}$$

and

$$H\frac{\partial \upsilon}{\partial t} + gH\frac{\partial \eta}{\partial x} + UH\frac{\partial \upsilon}{\partial x} - \frac{H^3}{3}\left(U\frac{\partial^3 \upsilon}{\partial x^3} + \frac{\partial^3 \upsilon}{\partial x^3 \partial t}\right) = 0 \tag{5.3b}$$

with

$$G = U\left(H + \eta\right) + H\upsilon - \frac{H^3}{3}\frac{\partial^2 \upsilon}{\partial x^2}. \tag{5.3c}$$

These are the linearised Serre equations.

5.1 Dispersion Error

To study the error in the dispersion relation caused by the numerical methods we will follow the work of [1] who used a range of numerical methods on a different reformulation of the Serre equations $U = 0$, this is a reasonable simplification because firstly we are interested in modelling waves on quiescent water and secondly the more complicated part of the dispersion error to correctly approximate is the term which dictates the behaviour of gravity waves on still water.

By assuming that $U = 0$ the equations (5.3) and (5.3c) reduce to

$$\frac{\partial \eta}{\partial t} + H\frac{\partial \upsilon}{\partial x} = 0 \tag{5.4a}$$

and

$$h_0\frac{\partial \upsilon}{\partial t} + gH\frac{\partial \eta}{\partial x} - \frac{H^3}{3}\left(\frac{\partial^3 \upsilon}{\partial x^3 \partial t}\right) = 0 \tag{5.4b}$$

with

$$G = H\upsilon - \frac{H^3}{3}\frac{\partial^2 \upsilon}{\partial x^2}. \tag{5.4c}$$

The linearised equations (5.4) can be reformulated into equations with η and G as conserved variables as in (2.2) to obtain

$$\frac{\partial \eta}{\partial t} + H\frac{\partial \upsilon}{\partial x} = 0, \tag{5.5a}$$

$$\frac{\partial G}{\partial t} + gH\frac{\partial \eta}{\partial x} = 0. \tag{5.5b}$$

For brevity we will only demonstrate our analysis of the dispersion error for our hybrid finite volume methods, through singular examples for the steps required to finally attain the dispersion error. In particular we will demonstrate how the dispersion error was obtained for both the second-order finite difference volume method and the second-order finite element volume method.

To perform the dispersion error we assume that η and υ are periodic functions in both space and time. In particular we assume that these both these quantities are fourier modes, which for some quantity q is given by

$$q(x,t) = q(0,0)e^{i(\omega t + kx)} \tag{5.6}$$

Therefore because we use uniform spatial grids so that $q_j^n = q(x_j, t^n)$ we have that

$$q_{j\pm l}^n = q_j^n e^{\pm ikl\Delta x} \quad \text{and} \quad q_j^{n\pm l} = q_j^n e^{\pm i\omega l\Delta t} \tag{5.7}$$

For the hybrid finite volume methods we break this process up into the three parts of these methods, the elliptic equation which relates the h and G to u, the evolution equation we use to update h and G and the Runge-Kutta steps we use to increase the order of accuracy of the method in time.

5.1.1 Elliptic Equation

Finite Difference

The elliptic equation (5.4c) at a particular grid point x_j is

$$G_j = H\upsilon_j - \frac{H^3}{3}\left(\frac{\partial^2 \upsilon}{\partial x^2}\right)_j$$

For the second-order finite difference method the derivative $\frac{\partial^2 \upsilon}{\partial x^2}$ is approximated by

$$\left(\frac{\partial^2 \upsilon}{\partial x^2}\right)_j = \frac{\upsilon_{j-1} - 2\upsilon_j + \upsilon_{j+1}}{\Delta x^2}.$$

Making use of (5.7) this becomes

$$\left(\frac{\partial^2 \upsilon}{\partial x^2}\right)_j = \frac{\upsilon_j e^{-ik\Delta x} - 2\upsilon_j + \upsilon_j e^{ik\Delta x}}{\Delta x^2}.$$

Which reduces to

$$\left(\frac{\partial^2 \upsilon}{\partial x^2}\right)_j = \frac{2\cos(k\Delta x) - 2}{\Delta x^2}\upsilon_j.$$

Substituting this approximation into our elliptic equation (5.4c) one obtains

$$G_j = \left(H - \frac{H^3}{3}\frac{2\cos(k\Delta x) - 2}{\Delta x^2}\right)\upsilon_j.$$

Therefore we an equation which is independent of j which gives the error introduced transforming between G_j and u_j of this form

$$G_j = \mathcal{G}\upsilon_j.$$

In particular for the centred second-order finite difference method we have

$$\mathcal{G}_{FD2} = \left(H - \frac{H^3}{3}\frac{2\cos(k\Delta x) - 2}{\Delta x^2}\right).$$

Where the subscript denotes that it is the factor for the second-order finite different approximation to this transformation.

Finite Element Method

Since finite difference methods are all very similar in how the error coefficient is found, it is sufficient to just show one example of the process used. Finite element methods however, are slightly different and so we now include the working done to obtain the error coefficient of the FEM as well for our mixed P1 P2 FEM.

Instead of a relation between G_j and υ_j as in the finite difference method above, we get $\upsilon_{j+1/2}$ by solving the finite element method. So we desire the error coefficient produced by the FEM solution of the elliptic equation given G_j and solving for $\upsilon_{j+1/2}$ which will be used in the conservation equation part below as our reconstruction error coefficient for υ at the cell interface. The matrix equation of the FEM for the linearised equations (5.4c) can be attained by just using $h^+_{j-/2} = H = h^-_{j+1/2}$ and so we obtain from []

$$\sum_j \frac{\Delta x}{6} \begin{bmatrix} G^+_{j-1/2} \\ 2G^+_{j-1/2} + 2G^-_{j+1/2} \\ G^-_{j+1/2} \end{bmatrix} =$$

$$\sum_j \left(H\frac{\Delta x}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} + \frac{H^3}{9\Delta x} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \right) \begin{bmatrix} \upsilon_{j-1/2} \\ \upsilon_j \\ \upsilon_{j+1/2} \end{bmatrix}$$

Using our relations from the periodic nature of u and G, and the minmod reconstruction used on G we get that

$$\sum_j \frac{\Delta x}{6} \begin{bmatrix} e^{-ik\Delta x}\mathcal{R}^+_2 \\ 2e^{-ik\Delta x}\mathcal{R}^+_2 + 2\mathcal{R}^-_2 \\ \mathcal{R}^-_2 \end{bmatrix} G_j =$$

$$\sum_j \left(H\frac{\Delta x}{30} \begin{bmatrix} 5i\sin\left(k\frac{\Delta x}{2}\right) + 3\cos\left(k\frac{\Delta x}{2}\right) + 2 \\ 16 + 4\cos\left(k\frac{\Delta x}{2}\right) \\ -5i\sin\left(k\frac{\Delta x}{2}\right) + 3\cos\left(k\frac{\Delta x}{2}\right) + 2 \end{bmatrix} \right.$$
$$\left. + \frac{H^3}{9\Delta x} \begin{bmatrix} 6i\sin\left(k\frac{\Delta x}{2}\right) + 8\cos\left(k\frac{\Delta x}{2}\right) - 8 \\ -16\cos\left(k\frac{\Delta x}{2}\right) + 16 \\ -6i\sin\left(k\frac{\Delta x}{2}\right) + 8\cos\left(k\frac{\Delta x}{2}\right) - 8 \end{bmatrix} \right) \upsilon_j \quad (5.8)$$

We can now add all the terms that overlap i.e the extra contributions from the functions $\phi_{j+1/2}$ and $\phi_{j-1/2}$ from outside the cell $\left[x_{j-1/2}, x_{j+1/2}\right]$, this then gives us a relation between the sub-vectors of the total vectors of the FEM. Doing this we can rewrite the matrix equation as []

$$\sum_j \frac{\Delta x}{6} \begin{bmatrix} 2 \\ \mathcal{R}^-_2 + \mathcal{R}^+_2 \end{bmatrix}^T \begin{bmatrix} G_j \\ G_j \end{bmatrix} =$$

$$\sum_j \left(H\frac{\Delta x}{30} \begin{bmatrix} 16 + 4\cos\left(k\frac{\Delta x}{2}\right) \\ 4\cos\left(\frac{k\Delta x}{2}\right) + 8\cos\left(k\Delta x\right) - 2 \end{bmatrix}^T \right.$$
$$\left. + \frac{H^3}{9\Delta x} \begin{bmatrix} -16\cos\left(k\frac{\Delta x}{2}\right) + 16 \\ -16\cos\left(\frac{k\Delta x}{2}\right) + 14\cos\left(k\Delta x\right) + 2 \end{bmatrix}^T \right) \begin{bmatrix} u_j \\ u_{j+1/2} \end{bmatrix} \quad (5.9)$$

So the equation for $u_{j+1/2}$ is

$$\frac{\Delta x}{6}\left(\mathcal{R}_2^+ + \mathcal{R}_2^-\right) G_j = \\ \left(H\frac{\Delta x}{30}\left(4\cos\left(\frac{k\Delta x}{2}\right) + 8\cos\left(k\Delta x\right) - 2\right) \right. \\ \left. + \frac{H^3}{9\Delta x}\left(-16\cos\left(\frac{k\Delta x}{2}\right) + 14\cos\left(k\Delta x\right) + 2\right)\right) u_{j+1/2} \quad (5.10)$$

so we have

$$G_j = \\ \frac{6}{\Delta x\left(\mathcal{R}_2^+ + \mathcal{R}_2^-\right)}\left(H\frac{\Delta x}{30}\left(4\cos\left(\frac{k\Delta x}{2}\right) + 8\cos\left(k\Delta x\right) - 2\right) \right. \\ \left. + \frac{H^3}{9\Delta x}\left(-16\cos\left(\frac{k\Delta x}{2}\right) + 14\cos\left(k\Delta x\right) + 2\right)\right) e^{ik\frac{\Delta x}{2}} u_j \quad (5.11)$$

So we have

$$\mathcal{G}_{FEMP1P2} = \\ \frac{6}{\Delta x\left(\mathcal{R}_2^+ + \mathcal{R}_2^-\right)}\left(H\frac{\Delta x}{30}\left(4\cos\left(\frac{k\Delta x}{2}\right) + 8\cos\left(k\Delta x\right) - 2\right) \right. \\ \left. + \frac{H^3}{9\Delta x}\left(-16\cos\left(\frac{k\Delta x}{2}\right) + 14\cos\left(k\Delta x\right) + 2\right)\right) e^{ik\frac{\Delta x}{2}} \quad (5.12)$$

This is the error introduced by calculating $u_{j+1/2}$ in our method, the only u value we need to numerically solve the linearised Serre equations.

5.1.2 Conservation Equation

Finite volume methods have the following update scheme to approximate equations in conservation law form [] for some quantity q

$$\bar{q}_j^{n+1} = \bar{q}_j^n - \frac{\Delta t}{\Delta x}\left[F_{j+1/2}^n - F_{j-1/2}^n\right]. \quad (5.13)$$

Where the bar denotes that it is the cell average of the quantity q and $F_{j+1/2}^n$ and $F_{j-1/2}^n$ are the approximations to the average fluxes across the cell boundary between the times t^n and t^{n+1}.

In our methods there is some transformation between the nodal value q_j and the cell average $\bar{q}_j$, which will introduce some error factor $\mathcal{M}$. For first and second order methods $\mathcal{M}_1 = \mathcal{M}_2 = 1$, however for higher-order methods $\mathcal{M} \neq 1$.

To calculate the fluxes $F^n_{j+1/2}$ and $F^n_{j-1/2}$ we use Kurganov's method [2] which is

$$F_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}} f\left(q^-_{j+\frac{1}{2}}\right) - a^-_{j+\frac{1}{2}} f\left(q^+_{j+\frac{1}{2}}\right)}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}}\, a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} \left[q^+_{j+\frac{1}{2}} - q^-_{j+\frac{1}{2}}\right]$$

where $a^+_{j+\frac{1}{2}}$ and $a^-_{j+\frac{1}{2}}$ are given by the wave speed bounds [], so that

$$a^-_{j+1/2} = -\sqrt{gH}$$

$$a^+_{j+1/2} = \sqrt{gH}.$$

We have suppressed the superscripts denoting the time to simplify the notation as all times are now t^n in the flux calculation. Substituting these values into the flux approximation we obtain

$$F_{j+\frac{1}{2}} = \frac{f\left(q^-_{j+\frac{1}{2}}\right) + f\left(q^+_{j+\frac{1}{2}}\right)}{2} - \frac{\sqrt{gH}}{2}\left[q^+_{j+\frac{1}{2}} - q^-_{j+\frac{1}{2}}\right] \tag{5.14}$$

For η our Kurganov approximation to the flux of (5.5a) is then

$$F^\eta_{j+\frac{1}{2}} = \frac{H\upsilon^-_{j+\frac{1}{2}} + H\upsilon^+_{j+\frac{1}{2}}}{2} - \frac{\sqrt{gH}}{2}\left[\eta^+_{j+\frac{1}{2}} - \eta^-_{j+\frac{1}{2}}\right] \tag{5.15}$$

The missing pieces here are the errors introduced by reconstruction of the edge values $\upsilon^-_{j+\frac{1}{2}}$, $\upsilon^+_{j+\frac{1}{2}}$, $\eta^-_{j+\frac{1}{2}}$ and $\eta^+_{j+\frac{1}{2}}$ from the cell averages $\bar{\upsilon}_j$ and $\bar{\eta}_j$. Because our quantities are smooth the nonlinear limiters can be neglected so we have for the second-order reconstruction of η

$$\begin{aligned}
\eta^-_{j+\frac{1}{2}} &= \bar{\eta}_j + \frac{-\bar{\eta}_{j-1} + \bar{\eta}_{j+1}}{4} \\
\eta^+_{j+\frac{1}{2}} &= \bar{\eta}_{j+1} + \frac{-\bar{\eta}_j + \bar{\eta}_{j+2}}{4}.
\end{aligned}$$

Using (5.7) these equations become

$$\begin{aligned}
\eta^-_{j+\frac{1}{2}} &= \mathcal{M}_2\eta_j + \frac{-\mathcal{M}_2\eta_j e^{-ik\Delta x} + \mathcal{M}_2\eta_j e^{ik\Delta x}}{4} \\
\eta^+_{j+\frac{1}{2}} &= \mathcal{M}_2\eta_j e^{ik\Delta x} + \frac{-\mathcal{M}_2\eta_j + \mathcal{M}_2\eta_j e^{2ik\Delta x}}{4}.
\end{aligned}$$

For the second order case $\mathcal{M}_2 = 1$ and these equations can be reduced to

$$\eta^-_{j+\frac{1}{2}} = \left(1 + \frac{i\sin(k\Delta x)}{2}\right)\eta_j \tag{5.16a}$$

$$\eta^+_{j+\frac{1}{2}} = e^{ik\Delta x}\left(1 - \frac{i\sin(k\Delta x)}{2}\right)\eta_j. \tag{5.16b}$$

From these we introduce the second order reconstruction factors $\mathcal{R}^+_2 = e^{ik\Delta x}\left(1 - \frac{i\sin(k\Delta x)}{2}\right)$ and $\mathcal{R}^-_2 = 1 + \frac{i\sin(k\Delta x)}{2}$ for both η and G. So that we have

$$\eta^-_{j+\frac{1}{2}} = \mathcal{R}^-_2\eta_j$$
$$\eta^+_{j+\frac{1}{2}} = \mathcal{R}^+_2\eta_j.$$

In our numerical methods our reconstruction of υ is slightly different as $\upsilon^-_{j+\frac{1}{2}}$ and $\upsilon^+_{j+\frac{1}{2}}$ are equal as we assume υ is continuous. For the second order finite difference volume method we have

$$u^-_{j+1/2} = u^+_{j+1/2} = \frac{u_{j+1} + u_j}{2}$$

Using (5.7) and rearranging gives

$$u^-_{j+1/2} = u^+_{j+1/2} = \frac{e^{ik\Delta x} + 1}{2}u_j. \tag{5.17}$$

We therefore introduce the second order reconstruction error factor $\mathcal{R}^u_{FD2} = \frac{e^{ik\Delta x}+1}{2}$. Because our FEM for the elliptic equation [] calculates $u_{j+1/2}$ from G_j we do not need to reconstruct $u_{j+1/2}$ and so we have the $\mathcal{R}^u_{FEM2} = e^{ik\frac{\Delta x}{2}}$, which corresponds to calculating $u_{j+1/2}$ analytically given u_j.

We will now suppress the superscripts and subscripts in the flux approximation, as the rest of the calculations are independent of the particular reconstructions used. Substituting these error coefficients into into (5.15) results in

$$F^\eta_{j+\frac{1}{2}} = \frac{H\mathcal{R}^u\upsilon_j + H\mathcal{R}^u\upsilon_j}{2} - \frac{\sqrt{gH}}{2}\left[\mathcal{R}^+\eta_j - \mathcal{R}^-\eta_j\right]$$

Which becomes

$$F^\eta_{j+\frac{1}{2}} = H\mathcal{R}^u\upsilon_j - \frac{\sqrt{gH}}{2}\left[\mathcal{R}^+ - \mathcal{R}^-\right]\eta_j$$

We then introduce the factors $\mathcal{F}^{\eta,\upsilon}$ and $\mathcal{F}^{\eta,\eta}$

$$\mathcal{F}^{\eta,\eta} = -\frac{\sqrt{gH}}{2}\left[\mathcal{R}^+ - \mathcal{R}^-\right]$$
$$\mathcal{F}^{\eta,\upsilon} = H\mathcal{R}^u$$

so that

$$F^{\eta}_{j+\frac{1}{2}} = \mathcal{F}^{\eta,\upsilon}\upsilon_j + \mathcal{F}^{\eta,\eta}\eta_j. \tag{5.18}$$

Repeating this process for G using [] and [] we get that

$$F^{G}_{j+\frac{1}{2}} = \mathcal{F}^{G,\eta}\eta_j + \mathcal{F}^{G,\upsilon}\upsilon_j \tag{5.19}$$

$$\mathcal{F}^{G,\eta} = gH\frac{\mathcal{R}^-_2 + \mathcal{R}^+_2}{2}$$
$$\mathcal{F}^{G,\upsilon} = -\frac{\sqrt{gH}}{2}\left[\mathcal{R}^+ - \mathcal{R}^-\right]\mathcal{G}$$

By substituting (5.18), (5.19) into (5.13) our finite volume method can be written as

$$\mathcal{M}_2\eta^{n+1}_j = \mathcal{M}_2\eta^n_j - \frac{\Delta t}{\Delta x}\left[\left(1 - e^{ik\Delta x}\right)\left(\mathcal{F}^{\eta,\eta}_2 h_j + \mathcal{F}^{\eta,\upsilon}_2\upsilon_j\right)\right]$$
$$\mathcal{M}_2 G^{n+1}_j = \mathcal{M}_2 G^n_j - \frac{\Delta t}{\Delta x}\left[\left(1 - e^{ik\Delta x}\right)\left(\mathcal{F}^{G,\eta}_2\eta_j + \mathcal{F}^{G,\upsilon}_2\upsilon_j\right)\right]$$

Furthermore by transforming the G's into υ's using our second order finite volume factor $\mathcal{G}_{FD2}$ (or the FEM one) [] we obtain

$$\eta^{n+1}_j = \eta^n_j - \frac{1}{\mathcal{M}_2}\frac{\Delta t}{\Delta x}\left[\left(1 - e^{ik\Delta x}\right)\left(\mathcal{F}^{\eta,\eta}_2\eta_j + \mathcal{F}^{\eta,\upsilon}_2\upsilon_j\right)\right]$$
$$\upsilon^{n+1}_j = \upsilon^n_j - \frac{1}{\mathcal{G}_{FD2}\mathcal{M}_2}\frac{\Delta t}{\Delta x}\left[\left(1 - e^{ik\Delta x}\right)\left(\mathcal{F}^{G,\eta}_2\eta_j + \mathcal{F}^{G,\upsilon}_2\upsilon_j\right)\right]$$

This can be written in matrix form as

$$\begin{bmatrix}\eta\\ \upsilon\end{bmatrix}^{n+1}_j = \begin{bmatrix}\eta\\ \upsilon\end{bmatrix}^{n}_j - \frac{\left(1 - e^{ik\Delta x}\right)\Delta t}{\Delta x}\begin{bmatrix}\frac{1}{\mathcal{M}_2}\mathcal{F}^{\eta,\eta}_2 & \frac{1}{\mathcal{M}_2}\mathcal{F}^{\eta,\upsilon}_2\\ \frac{1}{\mathcal{G}_{FD2}\mathcal{M}_2}\mathcal{F}^{\upsilon,\eta}_2 & \frac{1}{\mathcal{G}_{FD2}\mathcal{M}_2}\mathcal{F}^{\upsilon,\upsilon}_2\end{bmatrix}\begin{bmatrix}\eta\\ \upsilon\end{bmatrix}^{n}_j$$

Introducing

$$\boldsymbol{F}_2 = \frac{\left(1 - e^{ik\Delta x}\right)}{\Delta x}\begin{bmatrix}\mathcal{F}^{\eta,\eta}_2 & \mathcal{F}^{\eta,\upsilon}_2\\ \frac{1}{\mathcal{G}}\mathcal{F}^{\upsilon,\eta}_2 & \frac{1}{\mathcal{G}}\mathcal{F}^{\upsilon,\upsilon}_2\end{bmatrix}$$

this becomes

$$\begin{bmatrix}\eta\\ \upsilon\end{bmatrix}^{n+1}_j = \left(\boldsymbol{I} - \Delta t\boldsymbol{F}_2\right)\begin{bmatrix}\eta\\ \upsilon\end{bmatrix}^{n}_j$$

5.1.3 Runge-Kutta Time Stepping

The above analysis does not include the Runge-Kutta steps that make allow our schemes to be higher order in time. However, extending this analysis to Runge-Kutta steps is not difficult now that our method is in the form (5.1.2). For second order time stepping the Runge Kutta steps are then

$$\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^1 = (\boldsymbol{I} - \Delta t \boldsymbol{F}_2) \begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n \tag{5.20a}$$

$$\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^2 = (\boldsymbol{I} - \Delta t \boldsymbol{F}_2) \begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^1 \tag{5.20b}$$

$$\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n+1} = \frac{1}{2}\left(\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n + \begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^2 \right) \tag{5.20c}$$

Substituting (5.20a) and (5.20b) into (5.20c) gives

$$\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n+1} = \frac{1}{2}\left(\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n + (\boldsymbol{I} - \Delta t \boldsymbol{F}_2)^2 \begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n \right)$$

Expanding this we get

$$\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n+1} = \frac{1}{2}\left(2\boldsymbol{I} - 2\Delta t \boldsymbol{F}_2 + \Delta t^2 \boldsymbol{F}_2^2\right) \begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n$$

lets say we have an eigenvalue decomposition $\boldsymbol{F}_2 = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$ then this can be rewritten as

$$\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n+1} = \frac{1}{2}\left(2\boldsymbol{I} - 2\Delta t \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1} + \Delta t^2 \boldsymbol{P}\boldsymbol{\Lambda}^2\boldsymbol{P}^{-1}\right) \begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n$$

Multiplying by $\boldsymbol{P}^{-1}$ on the left and rearranging this get that

$$\boldsymbol{P}^{-1}\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^{n+1} = \frac{1}{2}\left(2 - 2\Delta t \boldsymbol{\Lambda} + \Delta t^2 \boldsymbol{\Lambda}^2\right) \boldsymbol{P}^{-1}\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n$$

Since η and υ are Fourier modes we have

$$e^{i\omega\Delta t}\left(\boldsymbol{P}^{-1}\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n\right) = \left(1 - 1\Delta t \boldsymbol{\Lambda} + \frac{1}{2}\Delta t^2 \boldsymbol{\Lambda}^2\right)\left(\boldsymbol{P}^{-1}\begin{bmatrix} \eta \\ \upsilon \end{bmatrix}_j^n\right)$$

Since $\mathbf{\Lambda}$ is a diagonal matrix of the eigenvalues λ_1 and λ_2 we have that

$$
\begin{aligned}
e^{i\omega \Delta t} &= 1 + \frac{1}{2}\Delta t^2 \lambda_1^2 - \Delta t \lambda_1, \\
e^{i\omega \Delta t} &= 1 + \frac{1}{2}\Delta t^2 \lambda_2^2 - \Delta t \lambda_2
\end{aligned}
$$

So that the dispersion relation for the second order finite difference finite volume method is

$$
\omega = \frac{1}{i\Delta t} \ln\left(1 + \frac{1}{2}\Delta t^2 \lambda_1^2 - \Delta t \lambda_1\right), \tag{5.21}
$$

$$
\omega = \frac{1}{i\Delta t} \ln\left(1 + \frac{1}{2}\Delta t^2 \lambda_2^2 - \Delta t \lambda_2\right), \tag{5.22}
$$

Comparing this with the dispersion relation (2.3) of the Serre equations we can then determine the error in dispersion caused by the particular method. We perform this computationally by finding the eigenvalues of $\boldsymbol{F}$, substituting them into (5.21) and comparing that to the dispersion relation of the Serre equations for a particular H and k value.

5.1.4 Results

5.2 Neumann Stability

We again begin from the linearised Serre equations (5.3) and as for the hybrid finite volume methods we will only show the working for one example as the process is the same in both cases. For the Neumann Stability our example will be the naive second-order finite difference method $\mathcal{D}$ (4.9). Again we being by replacing both η and υ by Fourier nodes (5.6).

Because our approximations to derivatives is consistent for $\mathcal{D}$ we will provide all the factors for the second order centred finite difference approximations to derivatives of some quantity q generated by making use of (5.7).

$$\left(\frac{\partial q}{\partial x}\right)_j^n = \frac{-q_{j-1}^n + q_{j+1}^n}{2\Delta x} = \frac{i \sin(k\Delta x)}{\Delta x} q_j^n \tag{5.23}$$

$$\left(\frac{\partial^2 q}{\partial x^2}\right)_j^n = \frac{q_{j-1}^n - 2q_j^n + q_{j+1}^n}{\Delta x^2} = \frac{2\cos(k\Delta x) - 2}{\Delta x^2} q_j^n \tag{5.24}$$

$$\left(\frac{\partial^3 q}{\partial x^3}\right)_j^n = \frac{-q_{j-2}^n + 2q_{j-1}^n - 2q_{j+1}^n + q_{j+2}^n}{2\Delta x^3} = -4i \sin(k\Delta x) \frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^3} q_j^n \tag{5.25}$$

$$\frac{\partial \eta}{\partial t} + H\frac{\partial \upsilon}{\partial x} + U\frac{\partial \eta}{\partial x} = 0 \tag{5.26a}$$

$$H\frac{\partial \upsilon}{\partial t} + gH\frac{\partial \eta}{\partial x} + UH\frac{\partial \upsilon}{\partial x} - \frac{H^3}{3}\left(U\frac{\partial^3 \upsilon}{\partial x^3} + \frac{\partial^3 \upsilon}{\partial x^3 \partial t}\right) = 0 \tag{5.26b}$$

The factors we get for the temporal derivatives are very similar to this. The numerical method $\mathcal{D}$ is just attained from replacing all the derivatives in (5.3) with the approximations in (5.25). For the linearised equations the update formulas of $\mathcal{D}$ become

$$\eta_j^{n+1} = \eta_j^{n-1} - \Delta t \left(U\frac{-\eta_{j-1}^n + \eta_{j+1}^n}{\Delta x} + H\frac{-\upsilon_{j-1}^n + \upsilon_{j+1}^n}{\Delta x}\right). \tag{5.27a}$$

$$\begin{aligned}\upsilon_j^{n+1} - \frac{H^2}{3}\frac{\upsilon_{j-1}^{n+1} - 2\upsilon_j^{n+1} + \upsilon_{j+1}^{n+1}}{\Delta x^2} \\ = \upsilon_j^{n-1} - \frac{H^2}{3}\frac{\upsilon_{j-1}^{n-1} - 2\upsilon_j^{n-1} + \upsilon_{j+1}^{n-1}}{\Delta x^2} \\ +\Delta t\left(-g\frac{-\eta_{j-1}^n + \eta_{j+1}^n}{\Delta x} - U\frac{-\upsilon_{j-1}^n + \upsilon_{j+1}^n}{\Delta x} + \frac{H^2}{3}\left(U\frac{-\upsilon_{j-2}^n + 2\upsilon_{j-1}^n - 2\upsilon_{j+1}^n + \upsilon_{j+2}^n}{\Delta x^3}\right)\right)\end{aligned} \tag{5.27b}$$

Since we have assumed that η and υ are fourier nodes, we can just replace the finite difference approximations with the appropriate factors from (5.25). After some rearranging we get that

$$\eta_j^{n+1} = \eta_j^{n-1} - \Delta t \left(U\frac{i\sin(k\Delta x)}{\Delta x}\eta_j^n + H\frac{i\sin(k\Delta x)}{\Delta x}\upsilon_j^n\right), \tag{5.28a}$$

$$\begin{aligned}\upsilon_j^{n+1} = \upsilon_j^{n-1} - \frac{3\Delta x^2 \Delta t}{3\Delta x^2 - 2H^2\left(\cos\left(k\Delta x\right)-1\right)}\left(g\frac{i\sin\left(k\Delta x\right)}{\Delta x}\right)\eta_j^n \\ + U\frac{i\Delta t\sin\left(k\Delta x\right)}{\Delta x}\upsilon_j^n\end{aligned} \tag{5.28b}$$

By setting

$$A_{0,0} = -\frac{2i\Delta t}{\Delta x}U\sin\left(k\Delta x\right) \tag{5.29}$$

$$A_{0,1} = -\frac{2i\Delta t}{\Delta x}H\sin\left(k\Delta x\right) \tag{5.30}$$

$$A_{1,0} = -\frac{6gi\Delta x\Delta t}{3\Delta x^2 - 2H^2\left(\cos\left(k\Delta x\right)-1\right)}\sin\left(k\Delta x\right) \tag{5.31}$$

$$A_{1,1} = \frac{2i\Delta t}{\Delta x}U\sin\left(k\Delta x\right) \tag{5.32}$$

$$\begin{bmatrix}\eta_j^{n+1}\\ \upsilon_j^{n+1}\\ \eta_j^n\\ \upsilon_j^n\end{bmatrix} = \begin{bmatrix}A_{0,0} & A_{0,1} & 1 & 0\\ A_{1,0} & A_{1,1} & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\eta_j^n\\ \upsilon_j^n\\ \eta_j^{n-1}\\ \upsilon_j^{n-1}\end{bmatrix} \tag{5.33}$$

This matrix is the growth matrix and if its spectral radius is less than 1 then $\mathcal{D}$ is stable.

For $\mathcal{W}$ after following through with same process we get that

$$\begin{aligned}B_{0,0} &= 1 - \frac{\Delta t}{\Delta x}A_{1,0}H\frac{i\sin\left(k\Delta x\right)}{2}\\ &- \frac{\Delta t}{\Delta x}U\left(\left(i\sin\left(k\Delta x\right)\right) - \frac{\Delta t}{\Delta x}U\left(\cos\left(ik\Delta x\right)-1\right)\right)\\ B_{0,1} &= -\frac{\Delta t}{\Delta x}\left[H\frac{i\sin\left(k\Delta x\right)}{2}A_{1,1} - U\left(\frac{\Delta t}{\Delta x}H\left(\cos\left(ik\Delta x\right)-1\right)\right)\right]\\ B_{0,4} &= -\frac{\Delta t}{\Delta x}H\frac{i\sin\left(k\Delta x\right)}{2}\\ B_{1,0} &= A_{1,0}\\ B_{1,1} &= A_{1,1}\end{aligned}$$

$$\begin{bmatrix}h_j^{n+1}\\ u_j^{n+1}\\ h_j^n\\ u_j^n\end{bmatrix} = \begin{bmatrix}B_{0,0} & B_{0,1} & 0 & B_{0,4}\\ B_{1,0} & B_{1,1} & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}h_j^n\\ u_j^n\\ h_j^{n-1}\\ u_j^{n-1}\end{bmatrix}$$

Again if this matrix has a spectral radius less than 1 then $\mathcal{W}$ is stable.

Chapter 6

Validation and Comparison

6.1 Analytic Validation

6.1.1 Soliton

6.1.2 Forced Solutions

6.2 Experimental Validation

6.2.1 Beji

6.2.2 Roeber

Chapter 7

The Dam-Break problem

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