1 Numerical Method for dispersion error Break Down

To do the first analysis we first construct the update matrix F, which plays a similar role to λ I showed you in the space continuos case. We then diagonalise this so that we get back to just factors (basically). This also means I would like to update what we present in the table, I think we should present all the elements of the matrices, for $\mathcal{F}^{h,u}$ and $\mathcal{F}^{h,h}$ there is no change but for $\mathcal{F}^{u,u}$ and $\mathcal{F}^{u,h}$ this means also dividing it by $\mathcal{G}$ as well as $\mathcal{M}$ and Δx. Anyway onto the method.

1.1 $\boldsymbol{F}$

$\boldsymbol{F}$ comes from the FVM update scheme which for us is

$$\bar{q}_j^{n+1} = \bar{q}_j^n - \frac{\Delta t}{\Delta x}\left[F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right]$$

This converts to (both analytical and numerical)

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j - \mathcal{F}^{q,v}v_{j-1} - \mathcal{F}^{q,q}q_{j-1}\right]$$

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j - \mathcal{F}^{q,v}e^{-ik\Delta x}v_j - \mathcal{F}^{q,q}e^{-ik\Delta x}q_j\right]$$

Defining $\mathcal{D}_x = 1 - e^{-ik\Delta x}$

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{D}_x\mathcal{F}^{q,v}v_j + \mathcal{D}_x\mathcal{F}^{q,q}q_j\right]$$

So we have

$$q_j^{n+1} = q_j^n - \frac{\mathcal{D}_x\Delta t}{\mathcal{M}\Delta x}\left[\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j\right]$$

Thus we have

$$\begin{bmatrix} h \\ \mathcal{G}u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ \mathcal{G}u \end{bmatrix}_j^n - \frac{\mathcal{D}_x\Delta t}{\mathcal{M}\Delta x}\begin{bmatrix} \mathcal{F}^{h,h} & \mathcal{F}^{h,u} \\ \mathcal{F}^{u,h} & \mathcal{F}^{u,u} \end{bmatrix}\begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ u \end{bmatrix}_j^{n} - \frac{\mathcal{D}_x \Delta t}{\mathcal{M}\Delta x} \begin{bmatrix} \mathcal{F}^{h,h} & \mathcal{F}^{h,u} \\ \frac{1}{\mathcal{G}}\mathcal{F}^{u,h} & \frac{1}{\mathcal{G}}\mathcal{F}^{u,u} \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

Lets define

$$\boldsymbol{F} = \frac{\mathcal{D}_x}{\mathcal{M}\Delta x} \begin{bmatrix} \mathcal{F}^{h,h} & \mathcal{F}^{h,u} \\ \frac{1}{\mathcal{G}}\mathcal{F}^{u,h} & \frac{1}{\mathcal{G}}\mathcal{F}^{u,u} \end{bmatrix}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ u \end{bmatrix}_j^{n} - \Delta t \boldsymbol{F} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = (\boldsymbol{I} - \Delta t \boldsymbol{F}) \begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

So we can see that ever time step produces a factor $(\boldsymbol{I} - \Delta t \boldsymbol{F})$, I will now demonstrate the method with the hardest example, third order.

1.2 Third Order

For third order the time stepping algorithm is

$$\begin{bmatrix} h \\ u \end{bmatrix}^{1} = (\boldsymbol{I} - \Delta t \boldsymbol{F}_3) \begin{bmatrix} h \\ u \end{bmatrix}^{n}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{2} = (\boldsymbol{I} - \Delta t \boldsymbol{F}_3) \begin{bmatrix} h \\ u \end{bmatrix}^{1}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{3} = \frac{3}{4} \begin{bmatrix} h \\ u \end{bmatrix}^{n} + \frac{1}{4} \begin{bmatrix} h \\ u \end{bmatrix}^{2}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{4} = (\boldsymbol{I} - \Delta t \boldsymbol{F}_3) \begin{bmatrix} h \\ u \end{bmatrix}^{3}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \frac{1}{3} \begin{bmatrix} h \\ u \end{bmatrix}^{n} + \frac{2}{3} \begin{bmatrix} h \\ u \end{bmatrix}^{4}$$

Thus

$$\begin{bmatrix} h \\ u \end{bmatrix}^{3} = \frac{3}{4} \begin{bmatrix} h \\ u \end{bmatrix}^{n} + \frac{1}{4}(\boldsymbol{I} - \Delta t \boldsymbol{F}_3)^2 \begin{bmatrix} h \\ u \end{bmatrix}^{n}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^3 = \left(\frac{3}{4}\boldsymbol{I} + \frac{1}{4}(\boldsymbol{I} - \Delta t\boldsymbol{F}_3)^2\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \frac{1}{3}\left[\begin{array}{c} h \\ u \end{array}\right]^n + \frac{2}{3}(\boldsymbol{I} - \Delta t\boldsymbol{F}_3)\left(\frac{3}{4}\boldsymbol{I} + \frac{1}{4}\left(\boldsymbol{I} - \Delta t\boldsymbol{F}_3\right)^2\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\frac{1}{3}\boldsymbol{I} + \frac{2}{3}(\boldsymbol{I} - \Delta t\boldsymbol{F}_3)\left(\frac{3}{4}\boldsymbol{I} + \frac{1}{4}\left(\boldsymbol{I} - \Delta t\boldsymbol{F}_3\right)^2\right)\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\frac{1}{3}\boldsymbol{I} + (\boldsymbol{I} - \Delta t\boldsymbol{F}_3)\left(\frac{1}{2}\boldsymbol{I} + \frac{1}{6}\left(\boldsymbol{I} - 2\Delta t\boldsymbol{F}_3 + \Delta t^2\boldsymbol{F}_3^2\right)\right)\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\frac{1}{3}\boldsymbol{I} + (\boldsymbol{I} - \Delta t\boldsymbol{F}_3)\left(\frac{2}{3}\boldsymbol{I} - \frac{1}{3}\Delta t\boldsymbol{F}_3 + \frac{1}{6}\Delta t^2\boldsymbol{F}_3^2\right)\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\frac{1}{3}\boldsymbol{I} + \frac{2}{3}\boldsymbol{I} - \frac{1}{3}\Delta t\boldsymbol{F}_3 + \frac{1}{6}\Delta t^2\boldsymbol{F}_3^2 + (-\Delta t\boldsymbol{F}_3)\left(\frac{2}{3}\boldsymbol{I} - \frac{1}{3}\Delta t\boldsymbol{F}_3 + \frac{1}{6}\Delta t^2\boldsymbol{F}_3^2\right)\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\boldsymbol{I} - \frac{1}{3}\Delta t\boldsymbol{F}_3 + \frac{1}{6}\Delta t^2\boldsymbol{F}_3^2 - \frac{2}{3}\Delta t\boldsymbol{F}_3 + \frac{1}{3}\Delta t\boldsymbol{F}_3\Delta t\boldsymbol{F}_3 - \frac{1}{6}\Delta t^2\boldsymbol{F}_3^2\Delta t\boldsymbol{F}_3\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\boldsymbol{I} - \boldsymbol{F}_3 + \frac{1}{6}\Delta t^2\boldsymbol{F}_3^2 + \frac{1}{3}\Delta t^2\boldsymbol{F}_3^2 - \frac{1}{6}\Delta t^3\boldsymbol{F}_3^3\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

Finally

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\boldsymbol{I} - \Delta t\boldsymbol{F}_3 + \frac{1}{2}\Delta t^2\boldsymbol{F}_3^2 - \frac{1}{6}\Delta t^3\boldsymbol{F}_3^3\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

lets say we have an eigenvalue decomposition for $\boldsymbol{F}_3$

$$\boldsymbol{F}_3 = \boldsymbol{S}_3 \boldsymbol{D}_3 \boldsymbol{S}_3^{-1}$$

where

$$\boldsymbol{D}_3 = \begin{bmatrix} \lambda_{3,-} & 0 \\ 0 & \lambda_{3,+} \end{bmatrix}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \left(\boldsymbol{I} - \Delta t \boldsymbol{S}_3 \boldsymbol{D}_3 \boldsymbol{S}_3^{-1} + \frac{1}{2} \Delta t^2 \boldsymbol{S}_3 \boldsymbol{D}_3^2 \boldsymbol{S}_3^{-1} - \frac{1}{6} \Delta t^3 \boldsymbol{S}_3 \boldsymbol{D}_3^3 \boldsymbol{S}_3^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}^n$$

Multiply on the LHS by $\boldsymbol{S}_3^{-1}$

$$\boldsymbol{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \left(\boldsymbol{S}_3^{-1} - \Delta t \boldsymbol{D}_3 \boldsymbol{S}_3^{-1} + \frac{1}{2} \Delta t^2 \boldsymbol{D}_3^2 \boldsymbol{S}_3^{-1} - \frac{1}{6} \Delta t^3 \boldsymbol{D}_3^3 \boldsymbol{S}_3^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$\boldsymbol{S}_3^{-1} e^{i\omega \Delta t} \begin{bmatrix} h \\ u \end{bmatrix}^n = \left(\boldsymbol{I} - \Delta t \boldsymbol{D}_3 + \frac{1}{2} \Delta t^2 \boldsymbol{D}_3^2 - \frac{1}{6} \Delta t^3 \boldsymbol{D}_3^3 \right) \boldsymbol{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$e^{i\omega \Delta t} \boldsymbol{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^n = \left(\boldsymbol{I} - \Delta t \boldsymbol{D}_3 + \frac{1}{2} \Delta t^2 \boldsymbol{D}_3^2 - \frac{1}{6} \Delta t^3 \boldsymbol{D}_3^3 \right) \boldsymbol{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$e^{i\omega \Delta t} \boldsymbol{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^n = \begin{bmatrix} 1 - \Delta t \lambda_{3,-} + \frac{\Delta t^2}{2} \lambda_{3,-}^2 - \frac{\Delta t^3}{6} \lambda_{3,-}^3 & 0 \\ 0 & 1 - \Delta t \lambda_{3,+} + \frac{\Delta t^2}{2} \lambda_{3,+}^2 - \frac{\Delta t^3}{6} \lambda_{3,+}^3 \end{bmatrix} \boldsymbol{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^n \tag{1}$$

Since $\boldsymbol{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^n$ is just a vector then $e^{i\omega \Delta t}$ an eigenvalue of this diagonal matrix, so it must be that

$$e^{i\omega \Delta t} = 1 - \Delta t \lambda_{3,\pm} + \frac{\Delta t^2}{2} \lambda_{3,\pm}^2 - \frac{\Delta t^3}{6} \lambda_{3,\pm}^3$$

$$i\omega\Delta t = \ln\left(1 - \Delta t\lambda_{3,\pm} + \frac{\Delta t^2}{2}\lambda_{3,\pm}^2 - \frac{\Delta t^3}{6}\lambda_{3,\pm}^3\right)$$

$$\omega = \frac{1}{i\Delta t}\ln\left(1 - \Delta t\lambda_{3,\pm} + \frac{\Delta t^2}{2}\lambda_{3,\pm}^2 - \frac{\Delta t^3}{6}\lambda_{3,\pm}^3\right)$$

We note that $\lambda_{3,\pm}$ means we get both the positive and negative ω. So then our method is to form the matrix $\boldsymbol{F}_3$ and calculate the eigenvlaues numerically, then use the above operations to calculate ω for the numerical method.

1.3 First Order

We repeat this method for first order

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = (\boldsymbol{I} - \Delta t\boldsymbol{F_1})\begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

lets say we have

$$\boldsymbol{F}_1 = \boldsymbol{S}_1\begin{bmatrix} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{bmatrix}\boldsymbol{S}_1^{-1}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \left(\boldsymbol{I} - \Delta t\boldsymbol{S}_1\begin{bmatrix} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{bmatrix}\boldsymbol{S}_1^{-1}\right)\begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$\boldsymbol{S}_1^{-1}\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \left(\boldsymbol{S}_1^{-1} - \Delta t\begin{bmatrix} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{bmatrix}\boldsymbol{S}_1^{-1}\right)\begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$e^{i\omega\Delta t}\boldsymbol{S}_1^{-1}\begin{bmatrix} h \\ u \end{bmatrix}_j^n = \left(\boldsymbol{I} - \Delta t\begin{bmatrix} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{bmatrix}\right)\boldsymbol{S}_1^{-1}\begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$e^{i\omega\Delta t}\boldsymbol{S}_1^{-1}\begin{bmatrix} h \\ u \end{bmatrix}_j^n = \begin{bmatrix} 1 - \Delta t\lambda_{1,-} & 0 \\ 0 & 1 - \Delta t\lambda_{1,+} \end{bmatrix}\boldsymbol{S}_1^{-1}\begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

So we have

$$e^{i\omega\Delta t} = 1 - \Delta t\lambda_{1,-}$$

$$\omega = \frac{1}{i\Delta t}\ln\left(1 - \Delta t\lambda_{1,-}\right)$$

1.4 Second Order

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^1 = (\boldsymbol{I} - \Delta t \boldsymbol{F}_2) \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^2 = (\boldsymbol{I} - \Delta t \boldsymbol{F}_2) \begin{bmatrix} h \\ u \end{bmatrix}_j^1$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \frac{1}{2} \left(\begin{bmatrix} h \\ u \end{bmatrix}_j^n + \begin{bmatrix} h \\ u \end{bmatrix}_j^2 \right)$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \frac{1}{2} \left(\begin{bmatrix} h \\ u \end{bmatrix}_j^n + (\boldsymbol{I} - \Delta t \boldsymbol{F}_2)^2 \begin{bmatrix} h \\ u \end{bmatrix}_j^n \right)$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \frac{1}{2} \left(\boldsymbol{I} + (\boldsymbol{I} - \Delta t \boldsymbol{F}_2)^2 \right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \frac{1}{2} \left(\boldsymbol{I} + \boldsymbol{I} - 2\Delta t \boldsymbol{F}_2 + \Delta t^2 \boldsymbol{F}_2^2 \right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \frac{1}{2} \left(2\boldsymbol{I} - 2\Delta t \boldsymbol{F}_2 + \Delta t^2 \boldsymbol{F}_2^2 \right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

lets say we have

$$\boldsymbol{F}_2 = \boldsymbol{S}_2 \begin{bmatrix} \lambda_{2,-} & 0 \\ 0 & \lambda_{2,+} \end{bmatrix} \boldsymbol{S}_2^{-1}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \frac{1}{2} \left(2\boldsymbol{I} - 2\Delta t \boldsymbol{S}_2 \begin{bmatrix} \lambda_{2,-} & 0 \\ 0 & \lambda_{2,+} \end{bmatrix} \boldsymbol{S}_2^{-1} + \Delta t^2 \boldsymbol{S}_2 \begin{bmatrix} \lambda_{2,-}^2 & 0 \\ 0 & \lambda_{2,+}^2 \end{bmatrix} \boldsymbol{S}_2^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$\boldsymbol{S}_2^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \frac{1}{2} \left(2\boldsymbol{S}_2^{-1} - 2\Delta t \begin{bmatrix} \lambda_{2,-} & 0 \\ 0 & \lambda_{2,+} \end{bmatrix} \boldsymbol{S}_2^{-1} + \Delta t^2 \begin{bmatrix} \lambda_{2,-}^2 & 0 \\ 0 & \lambda_{2,+}^2 \end{bmatrix} \boldsymbol{S}_2^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$\boldsymbol{S}_2^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \frac{1}{2} \left(2\boldsymbol{S}_2^{-1} + \begin{bmatrix} \Delta t^2 \lambda_{2,-}^2 - 2\Delta t \lambda_{2,-} & 0 \\ 0 & \Delta t^2 \lambda_{2,+}^2 - 2\Delta t \lambda_{2,-} \end{bmatrix} \boldsymbol{S}_2^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$e^{i\omega\Delta t}\left(\boldsymbol{S}_2^{-1}\left[\begin{array}{c} h \\ u \end{array}\right]_j^n\right) = \frac{1}{2}\left(2+\left[\begin{array}{cc} \Delta t^2\lambda_{2,-}^2 - 2\Delta t\lambda_{2,-} & 0 \\ 0 & \Delta t^2\lambda_{2,+}^2 - 2\Delta t\lambda_{2,-} \end{array}\right]\right)\left(\boldsymbol{S}_2^{-1}\left[\begin{array}{c} h \\ u \end{array}\right]_j^n\right)$$

$$e^{i\omega\Delta t}\left(\boldsymbol{S}_2^{-1}\left[\begin{array}{c} h \\ u \end{array}\right]_j^n\right) = \frac{1}{2}\left[\begin{array}{cc} 2+\Delta t^2\lambda_{2,-}^2 - 2\Delta t\lambda_{2,-} & 0 \\ 0 & 2+\Delta t^2\lambda_{2,+}^2 - 2\Delta t\lambda_{2,-} \end{array}\right]\left(\boldsymbol{S}_2^{-1}\left[\begin{array}{c} h \\ u \end{array}\right]_j^n\right)$$

So we have

$$e^{i\omega\Delta t} = 1 + \frac{1}{2}\Delta t^2\lambda_{2,\pm}^2 - \Delta t\lambda_{2,\pm}$$

$$\omega = \frac{1}{i\Delta t}\ln\left(1 + \frac{1}{2}\Delta t^2\lambda_{2,\pm}^2 - \Delta t\lambda_{2,\pm}\right)$$