1 Finite Element

$$Guh - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = uhv - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)v$$

and then we integrate over the entire domain to get

$$\int_\Omega Gvdx = \int_\Omega uhvdx - \int_\Omega \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)vdx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_\Omega Gvdx = \int_\Omega uhvdx + \int_\Omega \frac{h^3}{3}u_x v_x dx$$

For u we are going to use $x_{j-1/2}$, x_j and $x_{j+1/2}$ as the nodes, which generate the basis functions $\phi_{j\pm 1/2}$ and ϕ_j , which for us will be the space of continuous quadratic elements.

While for G and h we will choose basis functions w that are linear from $[x_{j-1/2}, x_{j+1/2}]$ but discontinuous at the edges.

We are going to look at the entire area where the basis functions are non-zero for $\phi_{j-1/2}$, ϕ_j and $\phi_{j+1/2}$. Which is the interval from $x_{j-3/2}$ to $x_{j+3/2}$. So we focus on the integrals on $[x_{j-3/2}, x_{j+3/2}]$ as

$$\begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}$$

$$\int_\Omega G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \int_\Omega uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx + \int_\Omega \frac{h^3}{3}u_x \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_x dx$$

is

$$\sum_j \int_{x_{j-3/2}}^{x_{j+3/2}} G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \sum_j \int_{x_{j-3/2}}^{x_{j+3/2}} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx + \sum_j \int_{x_{j-3/2}}^{x_{j+3/2}} \frac{h^3}{3} u_x \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_x dx$$

$$x = \frac{1}{2}\xi \Delta x + x_j$$

Taking the derivatives we see
$dx = d\frac{1\xi}{2}\Delta x$, $\frac{dx}{d\xi} = \frac{1\Delta x}{2}$, $\frac{d\xi}{dx} = \frac{2}{\Delta x}$.

We can describe the basis functions in the ξ space, where they are non-zero

1.1 G

$$\int_{x_{j-3/2}}^{x_{j+3/2}} G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \frac{\Delta x}{2} \int_{-1}^{1} G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx =$$

$$\frac{\Delta x}{2} \begin{bmatrix} \frac{1}{3} G^+_{j-1/2} \\ \frac{2}{3} G^+_{j-1/2} + \frac{2}{3} G^-_{j+1/2} \\ \frac{1}{3} G^-_{j+1/2} \end{bmatrix}$$

$$= \frac{\Delta x}{6} \begin{bmatrix} G^+_{j-1/2} \\ 2G^+_{j-1/2} + 2G^-_{j+1/2} \\ G^-_{j+1/2} \end{bmatrix}$$

1.2 uh

$$\int_{x_{j-3/2}}^{x_{j+3/2}} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \frac{\Delta x}{2} \int_{-1}^{1} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx =$$

$$\frac{\Delta x}{2} \begin{bmatrix} \frac{7}{30} h^+_{j-1/2} + \frac{1}{30} h^-_{j+1/2} & \frac{4}{30} h^+_{j-1/2} & -\frac{1}{30} h^+_{j-1/2} - \frac{1}{30} h^-_{j+1/2} \\ \frac{4}{30} h^+_{j-1/2} & \frac{16}{30} h^+_{j-1/2} + \frac{16}{30} h^-_{j+1/2} & \frac{4}{30} h^-_{j+1/2} \\ -\frac{1}{30} h^+_{j-1/2} - \frac{1}{30} h^-_{j+1/2} & \frac{4}{30} h^-_{j+1/2} & \frac{1}{30} h^+_{j-1/2} \frac{7}{30} h^-_{j+1/2} \end{bmatrix} \begin{bmatrix} u_{j-1/2} \\ u_j \\ u_{j+1/2} \end{bmatrix} =$$

$$\frac{\Delta x}{60} \begin{bmatrix} 7h^+_{j-1/2} + h^-_{j+1/2} & 4h^+_{j-1/2} & -h^+_{j-1/2} - h^-_{j+1/2} \\ 4h^+_{j-1/2} & 16h^+_{j-1/2} + 16h^-_{j+1/2} & 4h^-_{j+1/2} \\ -h^+_{j-1/2} - h^-_{j+1/2} & 4h^-_{j+1/2} & h^+_{j-1/2} + 7h^-_{j+1/2} \end{bmatrix} \begin{bmatrix} u_{j-1/2} \\ u_j \\ u_{j+1/2} \end{bmatrix}$$

1.3 h3ux

$$\int_{x_{j-3/2}}^{x_{j+3/2}} \frac{h^3}{3} u_x \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_x dx = \frac{2}{3\Delta x} \int_{-1}^{1} h^3 u_x \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_x dx$$
$$= \frac{2}{3\Delta x} \int_{-1}^{1} \left(h^+_{j-1/2} + h^-_{j+1/2} \right)^3 u_x \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_x dx$$

We have

$$\int_{-1}^{1} \left(\left(h^+_{j-1/2} \right)^3 + 3 \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) + 3 \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 + \left(h^-_{j+1/2} \right)^3 \right) u_x \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_x dx$$
$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} u_{j-1/2} \\ u_j \\ u_{j+1/2} \end{bmatrix}$$

$$a_{11} = -\frac{12}{35} \left(h^+_{j-1/2} \right)^3 - \frac{51}{420} \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) - \frac{3}{105} \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 - \frac{1}{140} \left(h^-_{j+1/2} \right)^3$$

$$a_{12} = \frac{44}{105} \left(h^+_{j-1/2} \right)^3 + \frac{3}{21} \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) + \frac{6}{105} \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 + \frac{1}{21} \left(h^-_{j+1/2} \right)^3$$

$$a_{13} = -\frac{8}{105} \left(h^+_{j-1/2} \right)^3 - \frac{3}{140} \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) - \frac{3}{105} \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 - \frac{17}{420} \left(h^-_{j+1/2} \right)^3$$

$$a_{21} = -\frac{26}{105} \left(h^+_{j-1/2} \right)^3 - \frac{9}{35} \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) - \frac{3}{21} \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 - \frac{2}{105} \left(h^-_{j+1/2} \right)^3$$

$$a_{22} = \frac{8}{35} \left(h^+_{j-1/2} \right)^3 + \frac{12}{105} \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) - \frac{12}{105} \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 - \frac{8}{35} \left(h^-_{j+1/2} \right)^3$$

$$a_{23} = \frac{2}{105}\left(h^+_{j-1/2}\right)^3 + \frac{3}{21}\left(h^+_{j-1/2}\right)^2\left(h^-_{j+1/2}\right) + \frac{9}{35}\left(h^+_{j-1/2}\right)\left(h^-_{j+1/2}\right)^2 + \frac{26}{105}\left(h^-_{j+1/2}\right)^3$$

$$a_{31} = \frac{17}{420}\left(h^+_{j-1/2}\right)^3 + \frac{3}{105}\left(h^+_{j-1/2}\right)^2\left(h^-_{j+1/2}\right) + \frac{3}{140}\left(h^+_{j-1/2}\right)\left(h^-_{j+1/2}\right)^2 + \frac{8}{105}\left(h^-_{j+1/2}\right)^3$$

$$a_{32} = -\frac{1}{21}\left(h^+_{j-1/2}\right)^3 - \frac{6}{105}\left(h^+_{j-1/2}\right)^2\left(h^-_{j+1/2}\right) - \frac{3}{21}\left(h^+_{j-1/2}\right)\left(h^-_{j+1/2}\right)^2 - \frac{44}{105}\left(h^-_{j+1/2}\right)^3$$

$$a_{33} = \frac{1}{140}\left(h^+_{j-1/2}\right)^3 + \frac{3}{105}\left(h^+_{j-1/2}\right)^2\left(h^-_{j+1/2}\right) + \frac{51}{420}\left(h^+_{j-1/2}\right)\left(h^-_{j+1/2}\right)^2 + \frac{12}{35}\left(h^-_{j+1/2}\right)^3$$