1 Finite Element

$$G = uh - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = uhv - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)v$$

and then we integrate over the entire domain to get

$$\int_\Omega Gvdx = \int_\Omega uhvdx - \int_\Omega \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)vdx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_\Omega Gvdx = \int_\Omega uhvdx + \int_\Omega \frac{h^3}{3}u_x v_x dx$$

For u we are going to use $x_{j-1/2}$, x_j and $x_{j+1/2}$ as the nodes, which generate the basis functions $\phi_{j\pm1/2}$ and ϕ_j , which for us will be the space of continuous quadratic elements.

While for G and h we will choose basis functions w that are linear from $[x_{j-1/2}, x_{j+1/2}]$ but discontinuous at the edges.

There are two types of basis functions in this set up the ϕ_j which are non-zero on $[x_{j-1/2}, x_{j+1/2}]$ and the $\phi_{j\pm1/2}$, which we can reduce to just doing it once, but with a translation, so we focus on the $\phi_{j+1/2}$ which is non-zero on $[x_{j-1/2}, x_{j+3/2}]$

2 ϕ_j

In this section we focus on the test function $v = \phi_j$ and thus we focus on the integrals on $[x_{j-1/2}, x_{j+1/2}]$ as

$$\int_\Omega Gvdx = \int_\Omega uhvdx + \int_\Omega \frac{h^3}{3}u_x v_x dx$$

is

$$\sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} G\phi_j dx = \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} uh\phi_j dx + \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{h^3}{3} u_x (\phi_j)_x dx$$

$$x = \frac{1}{2}\xi \Delta x + x_j$$

Taking the derivatives we see
$dx = d\frac{\xi}{2}\Delta x$, $\frac{dx}{d\xi} = \frac{\Delta x}{2}$, $\frac{d\xi}{dx} = \frac{2}{\Delta x}$.

We can describe the basis functions in the ξ space, where they are non-zero

$$\phi_j = 1 - \xi^2 \tag{1}$$

$$\phi_{j-1/2} = \frac{1}{2}\left(\xi^2 - \xi\right) \tag{2}$$

$$\phi_{j+1/2} = \frac{1}{2}\left(\xi^2 + \xi\right) \tag{3}$$

$$\phi'_j = -2\xi \tag{4}$$

$$\phi'_{j-1/2} = \frac{1}{2}\left(2\xi - 1\right) \tag{5}$$

$$\phi'_{j+1/2} = \frac{1}{2}\left(2\xi + 1\right) \tag{6}$$

$$w^+_{j-1/2} = \frac{1}{2}\left(1 - \xi\right) \tag{7}$$

$$w^-_{j-1/2} = \frac{1}{2}\left(1 + \xi\right) \tag{8}$$

$$G \approx G' = \sum_j G_{j+1/2} w_{j+1/2}$$

$$u \approx u' = \sum_j \left[u_{j-1/2}\phi_{j-1/2} + u_j\phi_j + u_{j+1/2}\phi_{j+1/2}\right]$$

$$h \approx h' = \sum_j h_{j+1/2} w_{j+1/2}$$

2.1 First Integral

So now we do the substitution for all integrals firstly we do

$$\begin{aligned}
\int_{x_{j-1/2}}^{x_{j+1/2}} G\phi_j dx &= \int_{-1}^{1} G'(\xi)\phi_j(\xi)\frac{dx}{d\xi}d\xi \\
&= \frac{\Delta x}{2}\int_{-1}^{1} G'(\xi)\phi_j(\xi)d\xi
\end{aligned}$$

So we focus in on the integral

$$\begin{aligned}
\int_{-1}^{1} G'(\xi)\phi_j(\xi)d\xi &= \int_{-1}^{1}\left(G^{+}_{j-1/2}w^{+}_{j-1/2} + G^{-}_{j+1/2}w^{-}_{j+1/2}\right)\phi_j d\xi \\
&= G^{+}_{j-1/2}\int_{-1}^{1} w^{+}_{j-1/2}\phi_j d\xi + G^{-}_{j+1/2}\int_{-1}^{1} w^{-}_{j+1/2}\phi_j d\xi
\end{aligned}$$

we have

$$\int_{-1}^{1} w^{+}_{j-1/2}\phi_j d\xi = \int_{-1}^{1}\frac{1}{2}\left(1-\xi\right)\left(1-\xi^2\right)d\xi = \frac{1}{2}\frac{4}{3} = \frac{2}{3}$$

$$\int_{-1}^{1} w^{-}_{j+1/2}\phi_j d\xi = \int_{-1}^{1}\frac{1}{2}\left(1+\xi\right)\left(1-\xi^2\right)d\xi = \frac{1}{2}\frac{4}{3} = \frac{2}{3}$$

so then

$$\int_{-1}^{1} G'(\xi)\phi_j(\xi)d\xi = \frac{2}{3}G^{+}_{j-1/2} + \frac{2}{3}G^{-}_{j+1/2}$$

so we have

$$\int_{x_{j-1/2}}^{x_{j+1/2}} G\phi_j dx = \frac{\Delta x}{2}\left[\frac{2}{3}G^{+}_{j-1/2} + \frac{2}{3}G^{-}_{j+1/2}\right] = \frac{\Delta x}{3}\left[G^{+}_{j-1/2} + G^{-}_{j+1/2}\right]$$

2.2 Second Integral

$$\int_{x_{j-1/2}}^{x_{j+1/2}} uh\phi_j dx = \int_{-1}^{1} u(\xi)h(\xi)\phi_j(\xi)\frac{dx}{d\xi}d\xi = \frac{\Delta x}{2}\int_{-1}^{1} u'h'\phi_j d\xi$$

focusing on the integral

$$\int_{-1}^{1} u'h'\phi_j d\xi = \int_{-1}^{1} \left(u_{j-1/2}\phi_{j-1/2} + u_j\phi_j + u_{j+1/2}\phi_{j+1/2}\right) \left(h^{+}_{j-1/2}w^{+}_{j-1/2} + h^{-}_{j+1/2}w^{-}_{j+1/2}\right) \phi_j d\xi$$

$$= \int_{-1}^{1} \Bigg(u_{j-1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j-1/2} + u_j h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_j + u_{j+1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j+1/2}$$
$$+ u_{j-1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j-1/2} + u_j h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_j + u_{j+1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j+1/2} \Bigg)\phi_j d\xi$$

$$= u_{j-1/2}h^{+}_{j-1/2}\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j-1/2}\phi_j d\xi + u_j h^{+}_{j-1/2}\int_{-1}^{1} w^{+}_{j-1/2}\phi_j\phi_j d\xi$$
$$+ u_{j+1/2}h^{+}_{j-1/2}\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j+1/2}\phi_j d\xi$$
$$+ u_{j-1/2}h^{-}_{j+1/2}\int_{-1}^{1} w^{-}_{j+1/2}\phi_{j-1/2}\phi_j d\xi + u_j h^{-}_{j+1/2}\int_{-1}^{1} w^{-}_{j+1/2}\phi_j\phi_j d\xi$$
$$+ u_{j+1/2}h^{-}_{j+1/2}\int_{-1}^{1} w^{-}_{j+1/2}\phi_{j+1/2}\phi_j d\xi$$

Now we calculate the integrals

$$\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j-1/2}\phi_j d\xi = \int_{-1}^{1} \frac{1}{2}(1-\xi)\frac{1}{2}\left(\xi^2-\xi\right)\left(1-\xi^2\right) d\xi = \frac{1}{4}\left[\frac{8}{15}\right] = \frac{2}{15}$$

$$\int_{-1}^{1} w^{+}_{j-1/2}\phi_j\phi_j d\xi = \int_{-1}^{1} \frac{1}{2}(1-\xi)\left(1-\xi^2\right)\left(1-\xi^2\right) d\xi = \frac{1}{2}\left[\frac{16}{15}\right] = \frac{8}{15}$$

$$\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j+1/2}\phi_j d\xi = \int_{-1}^{1} \frac{1}{2}(1-\xi)\frac{1}{2}\left(\xi^2+\xi\right)\left(1-\xi^2\right) = \frac{1}{4}\times 0 = 0$$

$$\int_{-1}^{1} w^{-}_{j+1/2}\phi_{j-1/2}\phi_j d\xi = \int_{-1}^{1} \frac{1}{2}(\xi+1)\frac{1}{2}\left(\xi^2-\xi\right)\left(1-\xi^2\right) d\xi = \frac{1}{4}\times 0 = 0$$

$$\int_{-1}^{1} w^{-}_{j+1/2}\phi_j\phi_j d\xi = \int_{-1}^{1} \frac{1}{2}\left(\xi+1\right)\left(1-\xi^2\right)\left(1-\xi^2\right)d\xi = \frac{1}{2}\left[\frac{16}{15}\right] = \frac{8}{15}$$

$$\int_{-1}^{1} w^{-}_{j+1/2}\phi_{j+1/2}\phi_j d\xi = \int_{-1}^{1} \frac{1}{2}\left(\xi+1\right)\frac{1}{2}\left(\xi^2+\xi\right)\left(1-\xi^2\right) = \frac{1}{4}\left[\frac{8}{15}\right] = \frac{2}{15}$$

So we have

$$\begin{aligned} u_{j-1/2}h^{+}_{j-1/2}\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j-1/2}\phi_j d\xi + u_j h^{+}_{j-1/2}\int_{-1}^{1} w^{+}_{j-1/2}\phi_j\phi_j d\xi \\ + u_{j+1/2}h^{+}_{j-1/2}\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j+1/2}\phi_j d\xi \\ + u_{j-1/2}h^{-}_{j+1/2}\int_{-1}^{1} w^{-}_{j+1/2}\phi_{j-1/2}\phi_j d\xi + u_j h^{-}_{j+1/2}\int_{-1}^{1} w^{-}_{j+1/2}\phi_j\phi_j d\xi \\ + u_{j+1/2}h^{-}_{j+1/2}\int_{-1}^{1} w^{-}_{j+1/2}\phi_{j+1/2}\phi_j d\xi \end{aligned}$$

$$= \quad \frac{2}{15}h^{+}_{j-1/2}u_{j-1/2} \ + \ \frac{8}{15}h^{+}_{j-1/2}u_j \ + \ \frac{8}{15}h^{-}_{j+1/2}u_j \ + \ \frac{2}{15}h^{-}_{j+1/2}u_{j+1/2}$$

So

$$\begin{aligned} &\int_{x_{j-1/2}}^{x_{j+1/2}} uh\phi_j dx = \\ &\frac{\Delta x}{2}\left[\frac{2}{15}h^{+}_{j-1/2}u_{j-1/2} + \frac{8}{15}h^{+}_{j-1/2}u_j + \frac{8}{15}h^{-}_{j+1/2}u_j + \frac{2}{15}h^{-}_{j+1/2}u_{j+1/2}\right] \\ &\qquad = \frac{\Delta x}{15}\left[h^{+}_{j-1/2}u_{j-1/2} + 4h^{+}_{j-1/2}u_j + 4h^{-}_{j+1/2}u_j + h^{-}_{j+1/2}u_{j+1/2}\right] \end{aligned}$$

2.3 Third Integral

Lastly we have

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{h^3}{3} u_x (\phi_j)_x dx = \frac{1}{3\Delta x} \int_{-1}^{1} h^3 u_\xi (\phi_j)_\xi d\xi = \frac{1}{3\Delta x} \int_{-1}^{1} h^3 u_\xi (\phi_j)_\xi d\xi \quad (9)$$

$$\int_{-1}^{1} h^3 u_\xi (\phi_j)_\xi d\xi = \int_{-1}^{1} \left(h^+_{j-1/2} w^+_{j-1/2} + h^-_{j+1/2} w^-_{j+1/2} \right)^3 \times \left(u_{j-1/2} \phi'_{j-1/2} + u_j \phi'_j + u_{j+1/2} \phi'_{j+1/2} \right) (\phi'_j) d\xi \quad (10)$$

$$\int_{-1}^{1} \Bigg[\left(h^+_{j-1/2} w^+_{j-1/2} \right)^3 + 3 \left(h^+_{j-1/2} w^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} w^-_{j+1/2} \right) + 3 \left(h^+_{j-1/2} w^+_{j-1/2} \right) \left(h^-_{j+1/2} w^-_{j+1/2} \right)^2 + \left(h^-_{j+1/2} w^-_{j+1/2} \right)^3 \Bigg] \times \left(u_{j-1/2} \phi'_{j-1/2} \phi'_j + u_j \phi'_j \phi'_j + u_{j+1/2} \phi'_{j+1/2} \phi'_j \right) d\xi \quad (11)$$

$$\begin{aligned} \int_{-1}^{1} \Bigg[& \left(h^+_{j-1/2} w^+_{j-1/2} \right)^3 + 3 \left(h^+_{j-1/2} w^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} w^-_{j+1/2} \right) + 3 \left(h^+_{j-1/2} w^+_{j-1/2} \right) \left(h^-_{j+1/2} w^-_{j+1/2} \right)^2 \\ & + \left(h^-_{j+1/2} w^-_{j+1/2} \right)^3 \Bigg] u_{j-1/2} \phi'_{j-1/2} \phi'_j \\ + \Bigg[& \left(h^+_{j-1/2} w^+_{j-1/2} \right)^3 + 3 \left(h^+_{j-1/2} w^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} w^-_{j+1/2} \right) + 3 \left(h^+_{j-1/2} w^+_{j-1/2} \right) \left(h^-_{j+1/2} w^-_{j+1/2} \right)^2 \\ & + \left(h^-_{j+1/2} w^-_{j+1/2} \right)^3 \Bigg] u_j \phi'_j \phi'_j \\ + \Bigg[& \left(h^+_{j-1/2} w^+_{j-1/2} \right)^3 + 3 \left(h^+_{j-1/2} w^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} w^-_{j+1/2} \right) + 3 \left(h^+_{j-1/2} w^+_{j-1/2} \right) \left(h^-_{j+1/2} w^-_{j+1/2} \right)^2 \\ & + \left(h^-_{j+1/2} w^-_{j+1/2} \right)^3 \Bigg] u_{j+1/2} \phi'_{j+1/2} \phi'_j d\xi \quad (12) \end{aligned}$$

$$
\begin{aligned}
\int_{-1}^{1} & u_{j-1/2} \left(h^+_{j-1/2}\right)^3 \left(w^+_{j-1/2}\right)^3 \phi'_{j-1/2}\phi'_j \\
&+ 3u_{j-1/2} \left(h^+_{j-1/2}\right)^2 h^-_{j+1/2} \left(w^+_{j-1/2}\right)^2 \left(w^-_{j+1/2}\right) \phi'_{j-1/2}\phi'_j \\
&+ 3u_{j-1/2} h^+_{j-1/2} \left(h^-_{j+1/2}\right)^2 w^+_{j-1/2} \left(w^-_{j+1/2}\right)^2 \phi'_{j-1/2}\phi'_j \\
&+ u_{j-1/2} \left(h^-_{j+1/2}\right)^3 \left(w^-_{j+1/2}\right)^3 \phi'_{j-1/2}\phi'_j \\
&+ u_j \left(h^+_{j-1/2}\right)^3 \left(w^+_{j-1/2}\right)^3 \phi'_j\phi'_j \\
&+ 3u_j \left(h^+_{j-1/2}\right)^2 h^-_{j+1/2} \left(w^+_{j-1/2}\right)^2 \left(w^-_{j+1/2}\right) \phi'_j\phi'_j \\
&+ 3u_j h^+_{j-1/2} \left(h^-_{j+1/2}\right)^2 w^+_{j-1/2} \left(w^-_{j+1/2}\right)^2 \phi'_j\phi'_j \\
&+ u_j \left(h^-_{j+1/2}\right)^3 \left(w^-_{j+1/2}\right)^3 \phi'_j\phi'_j \\
&+ u_{j+1/2} \left(h^+_{j-1/2}\right)^3 \left(w^+_{j-1/2}\right)^3 \phi'_{j+1/2}\phi'_j \\
&+ 3u_{j+1/2} \left(h^+_{j-1/2}\right)^2 h^-_{j+1/2} \left(w^+_{j-1/2}\right)^2 \left(w^-_{j+1/2}\right) \phi'_{j+1/2}\phi'_j \\
&+ 3u_{j+1/2} h^+_{j-1/2} \left(h^-_{j+1/2}\right)^2 w^+_{j-1/2} \left(w^-_{j+1/2}\right)^2 \phi'_{j+1/2}\phi'_j \\
&+ u_{j+1/2} \left(h^-_{j+1/2}\right)^3 \left(w^-_{j+1/2}\right)^3 \phi'_{j+1/2}\phi'_j d\xi \qquad (13)
\end{aligned}
$$

$$
\int_{-1}^{1} \left(w^+_{j-1/2}\right)^3 \phi'_{j-1/2}\phi'_j d\xi
$$

etc (print out reference for this)

3 $\phi_{j+1/2}$

In this section we focus on the test function $v = \phi_{j+1/2}$ and thus we focus on the integrals on $[x_{j-1/2}, x_{j+3/2}]$ as

$$
\int_\Omega G v dx = \int_\Omega u h v dx + \int_\Omega \frac{h^3}{3} u_x v_x dx
$$

is

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} G\phi_{j+1/2} dx = \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} uh\phi_{j+1/2} dx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^3}{3} u_x \left(\phi_{j+1/2}\right)_x dx$$

4 $\phi_{j-1/2}$

5 Combination