

1 Elliptic Equation

The elliptic equation is

$$G = uh - \frac{\partial}{\partial x} \left(\frac{h^3}{3} u_x \right)$$

with bed terms it is

$$G = uh + u \left(\frac{\partial}{\partial x} \left(\frac{h^2}{2} b_x \right) + hb_x^2 \right) - \frac{\partial}{\partial x} \left(\frac{h^3}{3} u_x \right)$$

2 Finite Element

$$G = uh + u \left(\frac{\partial}{\partial x} \left(\frac{h^2}{2} b_x \right) + hb_x^2 \right) - \frac{\partial}{\partial x} \left(\frac{h^3}{3} u_x \right)$$

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = uhv + uv \left(\frac{\partial}{\partial x} \left(\frac{h^2}{2} b_x \right) + hb_x^2 \right) - \frac{\partial}{\partial x} \left(\frac{h^3}{3} u_x \right) v$$

and then we integrate over the entire domain to get

$$\int_{\Omega} Gv \, dx = \int_{\Omega} uhv \, dx + \int_{\Omega} u \left(\frac{\partial}{\partial x} \left(\frac{h^2}{2} b_x \right) + hb_x^2 \right) v \, dx - \int_{\Omega} \frac{\partial}{\partial x} \left(\frac{h^3}{3} u_x \right) v \, dx$$

for all v

We then make use of integration by parts, with Dirichlet boundaries to get

$$\int_{\Omega} Gv \, dx = \int_{\Omega} uhv \, dx + \int_{\Omega} \frac{h^3}{3} u_x v_x \, dx + \int_{\Omega} u \left(\frac{\partial}{\partial x} \left(\frac{h^2}{2} b_x \right) + hb_x^2 \right) v \, dx$$

$$\int_{\Omega} Gv \, dx = \int_{\Omega} uhv \, dx + \int_{\Omega} \frac{h^3}{3} u_x v_x \, dx + \int_{\Omega} \frac{\partial}{\partial x} \left(\frac{h^2}{2} b_x \right) uv \, dx + \int_{\Omega} uhb_x^2 v \, dx$$

$$\int_{\Omega} G v dx = \int_{\Omega} u h v dx + \int_{\Omega} \frac{h^3}{3} u_x v_x dx - \int_{\Omega} \frac{h^2}{2} b_x \frac{\partial}{\partial x} (u v) dx + \int_{\Omega} u h b_x^2 v dx$$

$$\int_{\Omega} G v dx = \int_{\Omega} u h v dx + \int_{\Omega} \frac{h^3}{3} u_x v_x dx - \int_{\Omega} \frac{h^2}{2} b_x u_x v dx - \int_{\Omega} \frac{h^2}{2} b_x u v_x dx + \int_{\Omega} u h b_x^2 v dx$$

So importantly we just require that b_x is well behaved, and in particular it must be continuous

$$\begin{aligned} \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} G v dx &= \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} u h v dx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^3}{3} u_x v_x dx \\ &- \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^2}{2} b_x u_x v dx - \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^2}{2} b_x u v_x dx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} u h b_x^2 v dx \end{aligned}$$

3 P1 FEM

Here are the terms for j for $v = \phi_{j+1/2}$

3.1 LHS

The LHS for each j is

$$\frac{\Delta x}{6} \left[G_{j-1/2}^+ + 2G_{j+1/2}^- + 2G_{j+1/2}^+ + G_{j+3/2}^- \right]$$

3.2 RHS

3.2.1 First Term

$$\begin{aligned} &= \frac{\Delta x}{12} \left[u_{j-1/2} h_{j-1/2}^+ + u_{j-1/2} h_{j+1/2}^- + u_{j+1/2} h_{j-1/2}^+ + 3u_{j+1/2} h_{j+1/2}^- \right. \\ &\quad \left. + 3u_{j+1/2} h_{j+1/2}^+ + u_{j+1/2} h_{j-3/2}^- + u_{j+3/2} h_{j+1/2}^+ + u_{j+3/2} h_{j-3/2}^- \right] \quad (1) \end{aligned}$$

Which breaking up into u terms gives

$$\begin{aligned}
&= \frac{\Delta x}{12} \left[u_{j-1/2} \left(h_{j-1/2}^+ + h_{j+1/2}^- \right) \right. \\
&\quad + u_{j+1/2} \left(h_{j-1/2}^+ + 3h_{j+1/2}^- + 3h_{j+1/2}^+ + h_{j-3/2}^- \right) \\
&\quad \left. + u_{j+3/2} \left(h_{j+1/2}^+ + h_{j-3/2}^- \right) \right] \quad (2)
\end{aligned}$$

3.2.2 Second Term

$$\begin{aligned}
&\frac{1}{12\Delta x} \left[-u_{j-1/2} \left[\left(h_{j-1/2}^+ \right)^3 + \left(h_{j-1/2}^+ \right)^2 h_{j+1/2}^- \right. \right. \\
&\quad \left. \left. + h_{j-1/2}^+ \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \right] \right. \\
&\quad + u_{j+1/2} \left[\left(h_{j-1/2}^+ \right)^3 + \left(h_{j-1/2}^+ \right)^2 h_{j+1/2}^- \right. \\
&\quad \left. + h_{j-1/2}^+ \left(h_{j+1/2}^- \right)^2 + \left(h_{j+1/2}^- \right)^3 \right. \\
&\quad \left. + \left(h_{j+1/2}^+ \right)^3 + 3 \left(h_{j+1/2}^+ \right)^2 \left(h_{j-3/2}^- \right) \right. \\
&\quad \left. + 3 \left(h_{j+1/2}^+ \right) \left(h_{j-3/2}^- \right)^2 + \left(h_{j-3/2}^- \right)^3 \right] \\
&\quad - u_{j+3/2} \left[\left(h_{j+1/2}^+ \right)^3 + 3 \left(h_{j+1/2}^+ \right)^2 \left(h_{j-3/2}^- \right) \right. \\
&\quad \left. \left. + 3 \left(h_{j+1/2}^+ \right) \left(h_{j-3/2}^- \right)^2 + \left(h_{j-3/2}^- \right)^3 \right] \right] \quad (3)
\end{aligned}$$

3.2.3 Third Term

$$\begin{aligned}
& \frac{1}{24\Delta x} \times \left(\right. \\
& b_{j-1/2}u_{j-1/2}h_{j-1/2}^+h_{j-1/2}^+ - 2b_{j-1/2}u_{j-1/2}h_{j-1/2}^+h_{j+1/2}^- + 3b_{j-1/2}u_{j-1/2}h_{j+1/2}^-h_{j+1/2}^- \\
& - b_{j-1/2}u_{j+1/2}h_{j-1/2}^+h_{j-1/2}^+ + 2b_{j-1/2}u_{j+1/2}h_{j-1/2}^+h_{j+1/2}^- - 3b_{j-1/2}u_{j+1/2}h_{j+1/2}^-h_{j+1/2}^- \\
& - b_{j+1/2}u_{j-1/2}h_{j-1/2}^+h_{j-1/2}^+ + 2b_{j+1/2}u_{j-1/2}h_{j-1/2}^+h_{j+1/2}^- - 3b_{j+1/2}u_{j-1/2}h_{j+1/2}^-h_{j+1/2}^- \\
& + b_{j+1/2}u_{j+1/2}h_{j-1/2}^+h_{j-1/2}^+ - 2b_{j+1/2}u_{j+1/2}h_{j-1/2}^+h_{j+1/2}^- + 3b_{j+1/2}u_{j+1/2}h_{j+1/2}^-h_{j+1/2}^- \\
& - 3b_{j+1/2}u_{j+1/2}h_{j+1/2}^+h_{j+1/2}^+ + 2b_{j+1/2}u_{j+1/2}h_{j+1/2}^+h_{j+3/2}^- - b_{j+1/2}u_{j+1/2}h_{j+3/2}^-h_{j+3/2}^- \\
& + 3b_{j+1/2}u_{j+3/2}h_{j+1/2}^+h_{j+1/2}^+ - 2b_{j+1/2}u_{j+3/2}h_{j+1/2}^+h_{j+3/2}^- + b_{j+1/2}u_{j+3/2}h_{j+3/2}^-h_{j+3/2}^- \\
& + 3b_{j+3/2}u_{j+1/2}h_{j+1/2}^+h_{j+1/2}^+ - 2b_{j+3/2}u_{j+1/2}h_{j+1/2}^+h_{j+3/2}^- + b_{j+3/2}u_{j+1/2}h_{j+3/2}^-h_{j+3/2}^- \\
& \left. - 3b_{j+3/2}u_{j+3/2}h_{j+1/2}^+h_{j+1/2}^+ + 2b_{j+3/2}u_{j+3/2}h_{j+1/2}^+h_{j+3/2}^- - b_{j+3/2}u_{j+3/2}h_{j+3/2}^-h_{j+3/2}^- \right) \quad (4)
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{24\Delta x} \times \left(\right. \\
& u_{j-1/2} \left(b_{j-1/2}h_{j-1/2}^+h_{j-1/2}^+ - 2b_{j-1/2}h_{j-1/2}^+h_{j+1/2}^- + 3b_{j-1/2}h_{j+1/2}^-h_{j+1/2}^- \right) \\
& + u_{j+1/2} \left(-b_{j-1/2}h_{j-1/2}^+h_{j-1/2}^+ + 2b_{j-1/2}h_{j-1/2}^+h_{j+1/2}^- - 3b_{j-1/2}h_{j+1/2}^-h_{j+1/2}^- \right) \\
& + u_{j-1/2} \left(-b_{j+1/2}h_{j-1/2}^+h_{j-1/2}^+ + 2b_{j+1/2}h_{j-1/2}^+h_{j+1/2}^- - 3b_{j+1/2}h_{j+1/2}^-h_{j+1/2}^- \right) \\
& + u_{j+1/2} \left(b_{j+1/2}h_{j-1/2}^+h_{j-1/2}^+ - 2b_{j+1/2}h_{j-1/2}^+h_{j+1/2}^- + 3b_{j+1/2}h_{j+1/2}^-h_{j+1/2}^- \right) \\
& + u_{j+1/2} \left(-3b_{j+1/2}h_{j+1/2}^+h_{j+1/2}^+ + 2b_{j+1/2}h_{j+1/2}^+h_{j+3/2}^- - b_{j+1/2}h_{j+3/2}^-h_{j+3/2}^- \right) \\
& + u_{j+3/2} \left(3b_{j+1/2}h_{j+1/2}^+h_{j+1/2}^+ - 2b_{j+1/2}h_{j+1/2}^+h_{j+3/2}^- + b_{j+1/2}h_{j+3/2}^-h_{j+3/2}^- \right) \\
& + u_{j+1/2} \left(3b_{j+3/2}h_{j+1/2}^+h_{j+1/2}^+ - 2b_{j+3/2}h_{j+1/2}^+h_{j+3/2}^- + b_{j+3/2}h_{j+3/2}^-h_{j+3/2}^- \right) \\
& \left. + u_{j+3/2} \left(-3b_{j+3/2}h_{j+1/2}^+h_{j+1/2}^+ + 2b_{j+3/2}h_{j+1/2}^+h_{j+3/2}^- - b_{j+3/2}h_{j+3/2}^-h_{j+3/2}^- \right) \right) \quad (5)
\end{aligned}$$

3.2.4 Fourth Term

$$\begin{aligned}
& \frac{1}{24\Delta x} \left(-b_{j-1/2}u_{j-1/2} \left[3h_{j-1/2}^+h_{j-1/2}^+ - 2h_{j-1/2}^+h_{j+1/2}^- + h_{j+1/2}^-h_{j+1/2}^- \right] \right. \\
& - b_{j-1/2}u_{j+1/2} \left[h_{j-1/2}^+h_{j-1/2}^+ - 2h_{j-1/2}^+h_{j+1/2}^- + 3h_{j+1/2}^-h_{j+1/2}^- \right] \\
& + b_{j+1/2}u_{j-1/2} \left[3h_{j-1/2}^+h_{j-1/2}^+ - 2h_{j-1/2}^+h_{j+1/2}^- + h_{j+1/2}^-h_{j+1/2}^- \right] \\
& + b_{j+1/2}u_{j+1/2} \left[h_{j-1/2}^+h_{j-1/2}^+ - 2h_{j-1/2}^+h_{j+1/2}^- + 3h_{j+1/2}^-h_{j+1/2}^- \right] \\
& + b_{j+1/2}u_{j+1/2} \left[-3h_{j+1/2}^+h_{j+1/2}^+ + 2h_{j+1/2}^+h_{j+3/2}^- - h_{j+3/2}^-h_{j+3/2}^- \right] \\
& + b_{j+1/2}u_{j+3/2} \left[h_{j+1/2}^+h_{j+1/2}^+ - 2h_{j+1/2}^+h_{j+3/2}^- + 3h_{j+3/2}^-h_{j+3/2}^- \right] \\
& - b_{j+3/2}u_{j+1/2} \left[-3h_{j+1/2}^+h_{j+1/2}^+ + 2h_{j+1/2}^+h_{j+3/2}^- - h_{j+3/2}^-h_{j+3/2}^- \right] \\
& \left. - b_{j+3/2}u_{j+3/2} \left[h_{j+1/2}^+h_{j+1/2}^+ - 2h_{j+1/2}^+h_{j+3/2}^- + 3h_{j+3/2}^-h_{j+3/2}^- \right] \right) \quad (6)
\end{aligned}$$

3.2.5 Fifth Term

$$\begin{aligned}
& \frac{1}{12\Delta x} \left\{ \left(b_{j-1/2}b_{j-1/2} - 2b_{j+1/2}b_{j-1/2} + b_{j+1/2}b_{j+1/2} - 2b_{j+3/2}b_{j+1/2} + b_{j+3/2}b_{j+3/2} \right) \right. \\
& \times \left(u_{j-1/2}h_{j-1/2}^+ + u_{j-1/2}h_{j+1/2}^- + u_{j+1/2}h_{j-1/2}^+ + 3u_{j+1/2}h_{j+1/2}^- \right. \\
& \left. \left. + 3u_{j+1/2}h_{j+1/2}^+ + u_{j+1/2}h_{j+3/2}^- + u_{j+3/2}h_{j+1/2}^+ + u_{j+3/2}h_{j+3/2}^- \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{12\Delta x} \left\{ \left(b_{j-1/2}b_{j-1/2} - 2b_{j+1/2}b_{j-1/2} + b_{j+1/2}b_{j+1/2} - 2b_{j+3/2}b_{j+1/2} + b_{j+3/2}b_{j+3/2} \right) \right. \\
& \quad \times \left(u_{j-1/2} \left(h_{j-1/2}^+ + h_{j+1/2}^- \right) \right. \\
& \quad + u_{j+1/2} \left(h_{j-1/2}^+ + 3h_{j+1/2}^- + 3h_{j+1/2}^+ + h_{j+3/2}^- \right) \\
& \quad \left. \left. \left. + u_{j+3/2} \left(h_{j+1/2}^+ + h_{j+3/2}^- \right) \right) \right) \right\}
\end{aligned}$$