1 Finite Element

$$Guh - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = uhv - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)v$$

and then we integrate over the entire domain to get

$$\int_\Omega Gvdx = \int_\Omega uhvdx - \int_\Omega \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)vdx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_\Omega Gvdx = \int_\Omega uhvdx + \int_\Omega \frac{h^3}{3}u_x v_x dx$$

For u we are going to use $x_{j-1/2}$, x_j and $x_{j+1/2}$ as the nodes, which generate the basis functions $\phi_{j\pm1/2}$ and ϕ_j , which for us will be the space of continuous quadratic elements.

While for G and h we will choose basis functions w that are linear from $[x_{j-1/2}, x_{j+1/2}]$ but discontinuous at the edges.

We are going to look at the entire area where the basis functions are non-zero for $\phi_{j-1/2}$, ϕ_j and $\phi_{j+1/2}$. Which is the interval from $x_{j-3/2}$ to $x_{j+3/2}$. So we focus on the integrals on $[x_{j-3/2}, x_{j+3/2}]$ as

$$\begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}$$

$$\int_\Omega G\begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}dx = \int_\Omega uh\begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}dx + \int_\Omega \frac{h^3}{3}u_x\begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_x dx$$

is

$$\sum_j \int_{x_{j-3/2}}^{x_{j+3/2}} G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \sum_j \int_{x_{j-3/2}}^{x_{j+3/2}} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx + \sum_j \int_{x_{j-3/2}}^{x_{j+3/2}} \frac{h^3}{3} u_x \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_x dx$$

$$x = \frac{3}{2}\xi\Delta x + x_j$$

Taking the derivatives we see
$dx = d\frac{3\xi}{2}\Delta x$, $\frac{dx}{d\xi} = \frac{3\Delta x}{2}$, $\frac{d\xi}{dx} = \frac{2}{3\Delta x}$.

We can describe the basis functions in the ξ space, where they are non-zero

1.1 P2

$$\phi_{j-3/2} = \begin{cases} \frac{9}{2}\left(\xi + \frac{2}{3}\right)\left(\xi + \frac{1}{3}\right) & -1 \le \xi \le -\frac{1}{3} \\ 0 & else \end{cases}$$

$$\phi_{j-1} = \begin{cases} -9\left(\xi + 1\right)\left(\xi + \frac{1}{3}\right) & -1 \le \xi \le -\frac{1}{3} \\ 0 & else \end{cases}$$

$$\phi_{j-1/2} = \begin{cases} \frac{9}{2}\left(\xi + \frac{2}{3}\right)\left(\xi + 1\right) & -1 \le \xi \le -\frac{1}{3} \\ \frac{9}{2}\xi\left(\xi - \frac{1}{3}\right) & -\frac{1}{3} \le \xi \le \frac{1}{3} \\ 0 & else \end{cases}$$

$$\phi_j = \begin{cases} -9\left(\xi + \frac{1}{3}\right)\left(\xi - \frac{1}{3}\right) & -\frac{1}{3} \le \xi \le \frac{1}{3} \\ 0 & else \end{cases}$$

$$\phi_{j+1/2} = \begin{cases} \frac{9}{2}\xi\left(\xi + \frac{1}{3}\right) & -\frac{1}{3} \le \xi \le \frac{1}{3} \\ \frac{9}{2}\left(\xi - 1\right)\left(\xi - \frac{2}{3}\right) & \frac{1}{3} \le \xi \le 1 \\ 0 & else \end{cases}$$

$$\phi_{j+3/2} = \begin{cases} \frac{9}{2}\left(\xi - \frac{1}{3}\right)\left(\xi - \frac{2}{3}\right) & \frac{1}{3} \le \xi \le 1 \\ 0 & else \end{cases}$$

1.2 P1

$$w^+_{j-3/2} = \begin{cases} -\frac{3}{2}\left(\xi + \frac{1}{3}\right) & -1 \le \xi \le -\frac{1}{3} \\ 0 & else \end{cases}$$

$$w^-_{j-1/2} = \begin{cases} \frac{3}{2}\left(\xi + 1\right) & -1 \le \xi \le -\frac{1}{3} \\ 0 & else \end{cases}$$

$$w^+_{j-1/2} = \begin{cases} -\frac{3}{2}\left(\xi - \frac{1}{3}\right) & -\frac{1}{3} \le \xi \le \frac{1}{3} \\ 0 & else \end{cases}$$

$$w^-_{j+1/2} = \begin{cases} \frac{3}{2}\left(\xi + \frac{1}{3}\right) & -\frac{1}{3} \le \xi \le \frac{1}{3} \\ 0 & else \end{cases}$$

$$w^+_{j+1/2} = \begin{cases} -\frac{3}{2}\left(\xi - 1\right) & \frac{1}{3} \le \xi \le 1 \\ 0 & else \end{cases}$$

$$w^-_{j+3/2} = \begin{cases} \frac{3}{2}\left(\xi - \frac{1}{3}\right) & \frac{1}{3} \le \xi \le 1 \\ 0 & else \end{cases}$$

2 Integrals

So now we can use this to move the integral into the P2/P1 space

$$\sum_j \int_{x_{j-3/2}}^{x_{j+3/2}} G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \sum_j \int_{x_{j-3/2}}^{x_{j+3/2}} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx + \sum_j \int_{x_{j-3/2}}^{x_{j+3/2}} \frac{h^3}{3} u_x \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_x dx$$

2.1 First Integral

We have

$$\int_{x_{j-3/2}}^{x_{j+3/2}} G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \int_{-1}^{1} G'(\xi) \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} \frac{dx}{d\xi} d\xi = \frac{3\Delta x}{2} \int_{-1}^{1} G'(\xi) \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi$$

$$\int_{-1}^{1} G'(\xi) \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi =$$
$$\int_{-1}^{1} \Big[G^+_{j-3/2} w^+_{j-3/2} + G^-_{j-1/2} w^-_{j-1/2} + G^+_{j-1/2} w^+_{j-1/2}$$
$$+ G^-_{j+1/2} w^-_{j+1/2} + G^+_{j+1/2} w^+_{j+1/2} + G^-_{j+3/2} w^-_{j+3/2} \Big] \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi$$

$\phi_{j-1/2}$ means that anything only non-zero on the last cell is neglected (equal to 0 on the domain) gives:

$$\int_{-1}^{1} \Big[G^+_{j-3/2} w^+_{j-3/2} + G^-_{j-1/2} w^-_{j-1/2} + G^+_{j-1/2} w^+_{j-1/2}$$
$$+ G^-_{j+1/2} w^-_{j+1/2} + G^+_{j+1/2} w^+_{j+1/2} + G^-_{j+3/2} w^-_{j+3/2} \Big] \phi_{j-1/2} d\xi =$$
$$\int_{-1}^{1} \Big[G^+_{j-3/2} w^+_{j-3/2} + G^-_{j-1/2} w^-_{j-1/2} + G^+_{j-1/2} w^+_{j-1/2}$$
$$+ G^-_{j+1/2} w^-_{j+1/2} \Big] \phi_{j-1/2} d\xi$$

$$= \int_{-1}^{1} G^+_{j-3/2} w^+_{j-3/2} \phi_{j-1/2} + G^-_{j-1/2} w^-_{j-1/2} \phi_{j-1/2} + G^+_{j-1/2} w^+_{j-1/2} \phi_{j-1/2}$$
$$+ G^-_{j+1/2} w^-_{j+1/2} \phi_{j-1/2} d\xi$$

$$= G^+_{j-3/2} \int_{-1}^{1} w^+_{j-3/2} \phi_{j-1/2} d\xi + G^-_{j-1/2} \int_{-1}^{1} w^-_{j-1/2} \phi_{j-1/2} d\xi + G^+_{j-1/2} \int_{-1}^{1} w^+_{j-1/2} \phi_{j-1/2} d\xi$$
$$+ G^-_{j+1/2} \int_{-1}^{1} w^-_{j+1/2} \phi_{j-1/2} d\xi$$

These integrals are

$$\int_{-1}^{1} w^{+}_{j-3/2}\phi_{j-1/2}d\xi = \int_{-1}^{-\frac{1}{3}} -\frac{3}{2}\left(\xi+\frac{1}{3}\right)\frac{9}{2}\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi$$
$$= \int_{-1}^{-\frac{1}{3}} -\frac{27}{4}\left(\xi+\frac{1}{3}\right)\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi = 0$$

$$\int_{-1}^{1} w^{-}_{j-1/2}\phi_{j-1/2}d\xi = \int_{-1}^{-\frac{1}{3}} \frac{3}{2}(\xi+1)\frac{9}{2}\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi$$
$$= \int_{-1}^{-\frac{1}{3}} \frac{27}{4}(\xi+1)\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi = \frac{1}{9}$$

$$\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j-1/2}d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{3}{2}\left(\xi-\frac{1}{3}\right)\frac{9}{2}\xi\left(\xi-\frac{1}{3}\right)d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{27}{4}\left(\xi-\frac{1}{3}\right)\xi\left(\xi-\frac{1}{3}\right)d\xi$$
$$= \frac{1}{9}$$

$$\int_{-1}^{1} w^{-}_{j+1/2}\phi_{j-1/2}d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{3}{2}\left(\xi+\frac{1}{3}\right)\frac{9}{2}\xi\left(\xi-\frac{1}{3}\right)d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{27}{4}\left(\xi+\frac{1}{3}\right)\xi\left(\xi-\frac{1}{3}\right)d\xi$$
$$= 0$$

So we have

$$\int_{-1}^{1}\Big[G^{+}_{j-3/2}w^{+}_{j-3/2} + G^{-}_{j-1/2}w^{-}_{j-1/2} + G^{+}_{j-1/2}w^{+}_{j-1/2}$$
$$+ G^{-}_{j+1/2}w^{-}_{j+1/2} + G^{+}_{j+1/2}w^{+}_{j+1/2} + G^{-}_{j+3/2}w^{-}_{j+3/2}\Big]\phi_{j-1/2}d\xi$$
$$= \frac{1}{9}G^{-}_{j-1/2} + \frac{1}{9}G^{+}_{j-1/2}$$

ϕ_j means that anything only non-zero on the first and last cell is neglected (equal to 0 on the domain) gives:

$$\int_{-1}^{1}\Big[G^{+}_{j-3/2}w^{+}_{j-3/2}+G^{-}_{j-1/2}w^{-}_{j-1/2}+G^{+}_{j-1/2}w^{+}_{j-1/2}$$
$$+G^{-}_{j+1/2}w^{-}_{j+1/2}+G^{+}_{j+1/2}w^{+}_{j+1/2}+G^{-}_{j+3/2}w^{-}_{j+3/2}\Big]\phi_j d\xi$$
$$=\int_{-1}^{1}\Big[G^{+}_{j-1/2}w^{+}_{j-1/2}+G^{-}_{j+1/2}w^{-}_{j+1/2}\Big]\phi_j d\xi$$

$$\int_{-1}^{1}\Big[G^{+}_{j-1/2}w^{+}_{j-1/2}+G^{-}_{j+1/2}w^{-}_{j+1/2}\Big]\phi_j d\xi =$$
$$G^{+}_{j-1/2}\int_{-1}^{1}w^{+}_{j-1/2}\phi_j d\xi+G^{-}_{j+1/2}\int_{-1}^{1}w^{-}_{j+1/2}\phi_j d\xi$$

These integrals are

$$\int_{-1}^{1}w^{+}_{j-1/2}\phi_j d\xi=\int_{-\frac{1}{3}}^{\frac{1}{3}}w^{+}_{j-1/2}\phi_j d\xi=\int_{-\frac{1}{3}}^{\frac{1}{3}}-\frac{3}{2}\left(\xi-\frac{1}{3}\right)(-9)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi$$
$$=\int_{-\frac{1}{3}}^{\frac{1}{3}}\frac{27}{2}\left(\xi-\frac{1}{3}\right)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi=\frac{2}{9}$$

$$\int_{-1}^{1}w^{-}_{j+1/2}\phi_j d\xi=\int_{-\frac{1}{3}}^{\frac{1}{3}}w^{-}_{j+1/2}\phi_j d\xi=\int_{-\frac{1}{3}}^{\frac{1}{3}}\frac{3}{2}\left(\xi+\frac{1}{3}\right)(-9)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi$$
$$=\int_{-\frac{1}{3}}^{\frac{1}{3}}-\frac{27}{2}\left(\xi+\frac{1}{3}\right)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi=\frac{2}{9}$$

So we have

$$\int_{-1}^{1}\Big[G^{+}_{j-3/2}w^{+}_{j-3/2}+G^{-}_{j-1/2}w^{-}_{j-1/2}+G^{+}_{j-1/2}w^{+}_{j-1/2}$$
$$+G^{-}_{j+1/2}w^{-}_{j+1/2}+G^{+}_{j+1/2}w^{+}_{j+1/2}+G^{-}_{j+3/2}w^{-}_{j+3/2}\Big]\phi_j d\xi$$
$$=\frac{2}{9}\Big[G^{+}_{j-1/2}+G^{-}_{j+1/2}\Big]$$

$\phi_{j+1/2}$ means that anything only non-zero on the first cell is neglected (equal to 0 on the domain) gives:

$$\int_{-1}^{1} \Big[G^+_{j-3/2} w^+_{j-3/2} + G^-_{j-1/2} w^-_{j-1/2} + G^+_{j-1/2} w^+_{j-1/2} + G^-_{j+1/2} w^-_{j+1/2} + G^+_{j+1/2} w^+_{j+1/2} + G^-_{j+3/2} w^-_{j+3/2} \Big] \phi_{j+1/2} d\xi$$
$$= \int_{-1}^{1} \Big[G^+_{j-1/2} w^+_{j-1/2} + G^-_{j+1/2} w^-_{j+1/2} + G^+_{j+1/2} w^+_{j+1/2} + G^-_{j+3/2} w^-_{j+3/2} \Big] \phi_{j+1/2} d\xi$$

$$\int_{-1}^{1} \Big[G^+_{j-1/2} w^+_{j-1/2} + G^-_{j+1/2} w^-_{j+1/2} + G^+_{j+1/2} w^+_{j+1/2} + G^-_{j+3/2} w^-_{j+3/2} \Big] \phi_{j+1/2} d\xi$$
$$= \int_{-1}^{1} G^+_{j-1/2} w^+_{j-1/2} \phi_{j+1/2} + G^-_{j+1/2} w^-_{j+1/2} \phi_{j+1/2} + G^+_{j+1/2} w^+_{j+1/2} \phi_{j+1/2} + G^-_{j+3/2} w^-_{j+3/2} \phi_{j+1/2} d\xi$$

$$\int_{-1}^{1} G^+_{j-1/2} w^+_{j-1/2} \phi_{j+1/2} + G^-_{j+1/2} w^-_{j+1/2} \phi_{j+1/2} + G^+_{j+1/2} w^+_{j+1/2} \phi_{j+1/2} + G^-_{j+3/2} w^-_{j+3/2} \phi_{j+1/2} d\xi$$
$$= G^+_{j-1/2} \int_{-1}^{1} w^+_{j-1/2} \phi_{j+1/2} d\xi + G^-_{j+1/2} \int_{-1}^{1} w^-_{j+1/2} \phi_{j+1/2} d\xi + G^+_{j+1/2} \int_{-1}^{1} w^+_{j+1/2} \phi_{j+1/2} d\xi + G^-_{j+3/2} \int_{-1}^{1} w^-_{j+3/2} \phi_{j+1/2} d\xi$$

These integrals are

$$\int_{-1}^{1} w^+_{j-1/2} \phi_{j+1/2} d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} w^+_{j-1/2} \phi_{j+1/2} d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{3}{2} \left(\xi - \frac{1}{3} \right) \frac{9}{2} \xi \left(\xi + \frac{1}{3} \right) d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{27}{4} \left(\xi - \frac{1}{3} \right) \xi \left(\xi + \frac{1}{3} \right) d\xi = 0$$

$$\int_{-1}^{1} w^-_{j+1/2} \phi_{j+1/2} d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} w^-_{j+1/2} \phi_{j+1/2} d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{3}{2} \left(\xi + \frac{1}{3} \right) \frac{9}{2} \xi \left(\xi + \frac{1}{3} \right) d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{27}{4} \left(\xi + \frac{1}{3} \right) \xi \left(\xi + \frac{1}{3} \right) d\xi = \frac{1}{9}$$

$$\int_{-1}^{1} w^+_{j+1/2}\phi_{j+1/2}d\xi = \int_{\frac{1}{3}}^{1} w^+_{j+1/2}\phi_{j+1/2}d\xi = \int_{\frac{1}{3}}^{1} -\frac{3}{2}(\xi-1)\frac{9}{2}(\xi-1)\left(\xi-\frac{2}{3}\right)d\xi$$
$$= \int_{\frac{1}{3}}^{1} -\frac{27}{4}(\xi-1)(\xi-1)\left(\xi-\frac{2}{3}\right)d\xi = \frac{1}{9}$$

$$\int_{-1}^{1} w^-_{j+3/2}\phi_{j+1/2}d\xi = \int_{\frac{1}{3}}^{1} w^-_{j+3/2}\phi_{j+1/2}d\xi = \int_{\frac{1}{3}}^{1} \frac{3}{2}\left(\xi-\frac{1}{3}\right)\frac{9}{2}(\xi-1)\left(\xi-\frac{2}{3}\right)d\xi$$
$$= \int_{\frac{1}{3}}^{1} \frac{27}{4}\left(\xi-\frac{1}{3}\right)(\xi-1)\left(\xi-\frac{2}{3}\right)d\xi = 0$$

Thus we have

$$\int_{-1}^{1} G^+_{j-1/2}w^+_{j-1/2}\phi_{j+1/2}+G^-_{j+1/2}w^-_{j+1/2}\phi_{j+1/2}+G^+_{j+1/2}w^+_{j+1/2}\phi_{j+1/2}+G^-_{j+3/2}w^-_{j+3/2}\phi_{j+1/2}d\xi$$
$$= \frac{1}{9}G^-_{j+1/2} + \frac{1}{9}G^+_{j+1/2}$$

Therefore

$$\int_{-1}^{1} G'(\xi)\begin{bmatrix}\phi_{j-1/2}\\ \phi_j\\ \phi_{j+1/2}\end{bmatrix}d\xi = \begin{bmatrix}\frac{1}{9}G^-_{j-1/2}+\frac{1}{9}G^+_{j-1/2}\\ \frac{2}{9}G^+_{j-1/2}+\frac{2}{9}G^-_{j+1/2}\\ \frac{1}{9}G^-_{j+1/2}+\frac{1}{9}G^+_{j+1/2}\end{bmatrix}^T\begin{bmatrix}\phi_{j-1/2}\\ \phi_j\\ \phi_{j+1/2}\end{bmatrix}$$

So

$$\int_{x_{j-3/2}}^{x_{j+3/2}} G\begin{bmatrix}\phi_{j-1/2}\\ \phi_j\\ \phi_{j+1/2}\end{bmatrix} = \frac{3\Delta x}{2}\begin{bmatrix}\frac{1}{9}G^-_{j-1/2}+\frac{1}{9}G^+_{j-1/2}\\ \frac{2}{9}G^+_{j-1/2}+\frac{2}{9}G^-_{j+1/2}\\ \frac{1}{9}G^-_{j+1/2}+\frac{1}{9}G^+_{j+1/2}\end{bmatrix}^T\begin{bmatrix}\phi_{j-1/2}\\ \phi_j\\ \phi_{j+1/2}\end{bmatrix}$$

$$\int_{x_{j-3/2}}^{x_{j+3/2}} G\begin{bmatrix}\phi_{j-1/2}\\ \phi_j\\ \phi_{j+1/2}\end{bmatrix} = \frac{\Delta x}{6}\begin{bmatrix}G^-_{j-1/2}+G^+_{j-1/2}\\ 2G^+_{j-1/2}+2G^-_{j+1/2}\\ G^-_{j+1/2}+G^+_{j+1/2}\end{bmatrix}^T\begin{bmatrix}\phi_{j-1/2}\\ \phi_j\\ \phi_{j+1/2}\end{bmatrix}$$

2.2 First Integral

$$\int_{x_{j-3/2}}^{x_{j+3/2}} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \frac{3\Delta x}{2} \int_{-1}^{1} u(\xi)h(\xi) \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi$$

$$= \frac{3\Delta x}{2} \int_{-1}^{1} \Big(u_{j-3/2}\phi_{j-3/2} + u_{j-1}\phi_{j-1} + u_{j-1/2}\phi_{j-1/2} + u_j\phi_j + u_{j+1/2}\phi_{j+1/2} + u_{j+1}\phi_{j+1} + u_{j+3/2}\phi_{j+3/2} \Big) \Big[h^+_{j-3/2}w^+_{j-3/2} + h^-_{j-1/2}w^-_{j-1/2} + h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2} + h^+_{j+1/2}w^+_{j+1/2} + h^-_{j+3/2}w^-_{j+3/2} \Big] \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi$$

$$\int_{-1}^{1} \Big(u_{j-3/2}\phi_{j-3/2} + u_{j-1}\phi_{j-1} + u_{j-1/2}\phi_{j-1/2} + u_j\phi_j + u_{j+1/2}\phi_{j+1/2} + u_{j+1}\phi_{j+1} + u_{j+3/2}\phi_{j+3/2} \Big) \Big[h^+_{j-3/2}w^+_{j-3/2} + h^-_{j-1/2}w^-_{j-1/2} + h^+_{j-1/2}w^+_{j-1/2} + h^-_{j+1/2}w^-_{j+1/2} + h^+_{j+1/2}w^+_{j+1/2} + h^-_{j+3/2}w^-_{j+3/2} \Big] \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi$$

$$
\begin{aligned}
= \int_{-1}^{1} \Bigg\{ \Bigg(& u_{j-3/2}\phi_{j-3/2} + u_{j-1}\phi_{j-1} + u_{j-1/2}\phi_{j-1/2} + u_j\phi_j \\
& + u_{j+1/2}\phi_{j+1/2} + u_{j+1}\phi_{j+1} + u_{j+3/2}\phi_{j+3/2} \Bigg) h^+_{j-3/2} w^+_{j-3/2} \\
& + \Bigg(u_{j-3/2}\phi_{j-3/2} + u_{j-1}\phi_{j-1} + u_{j-1/2}\phi_{j-1/2} + u_j\phi_j \\
& + u_{j+1/2}\phi_{j+1/2} + u_{j+1}\phi_{j+1} + u_{j+3/2}\phi_{j+3/2} \Bigg) h^-_{j-1/2} w^-_{j-1/2} \\
& + \Bigg(u_{j-3/2}\phi_{j-3/2} + u_{j-1}\phi_{j-1} + u_{j-1/2}\phi_{j-1/2} + u_j\phi_j \\
& + u_{j+1/2}\phi_{j+1/2} + u_{j+1}\phi_{j+1} + u_{j+3/2}\phi_{j+3/2} \Bigg) h^+_{j-1/2} w^+_{j-1/2} \\
& + \Bigg(u_{j-3/2}\phi_{j-3/2} + u_{j-1}\phi_{j-1} + u_{j-1/2}\phi_{j-1/2} + u_j\phi_j \\
& + u_{j+1/2}\phi_{j+1/2} + u_{j+1}\phi_{j+1} + u_{j+3/2}\phi_{j+3/2} \Bigg) h^-_{j+1/2} w^-_{j+1/2} \\
& + \Bigg(u_{j-3/2}\phi_{j-3/2} + u_{j-1}\phi_{j-1} + u_{j-1/2}\phi_{j-1/2} + u_j\phi_j \\
& + u_{j+1/2}\phi_{j+1/2} + u_{j+1}\phi_{j+1} + u_{j+3/2}\phi_{j+3/2} \Bigg) h^+_{j+1/2} w^+_{j+1/2} \\
& + \Bigg(u_{j-3/2}\phi_{j-3/2} + u_{j-1}\phi_{j-1} + u_{j-1/2}\phi_{j-1/2} + u_j\phi_j \\
& + u_{j+1/2}\phi_{j+1/2} + u_{j+1}\phi_{j+1} + u_{j+3/2}\phi_{j+3/2} \Bigg) h^-_{j+3/2} w^-_{j+3/2} \Bigg\} \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi
\end{aligned}
$$

We can reduce based on the non-zero region of the P1 basis functions

$$
\begin{aligned}
&= \int_{-1}^{1} \Bigg\{ \Big(u_{j-3/2}\phi_{j-3/2} + u_{j-1}\phi_{j-1} + u_{j-1/2}\phi_{j-1/2} \Big) h^{+}_{j-3/2} w^{+}_{j-3/2} \\
&\qquad + \Big(u_{j-3/2}\phi_{j-3/2} + u_{j-1}\phi_{j-1} + u_{j-1/2}\phi_{j-1/2} \Big) h^{-}_{j-1/2} w^{-}_{j-1/2} \\
&\qquad\quad + \Big(u_{j-1/2}\phi_{j-1/2} + u_{j}\phi_{j} + u_{j+1/2}\phi_{j+1/2} \Big) h^{+}_{j-1/2} w^{+}_{j-1/2} \\
&\qquad\quad + \Big(u_{j-1/2}\phi_{j-1/2} + u_{j}\phi_{j} + u_{j+1/2}\phi_{j+1/2} \Big) h^{-}_{j+1/2} w^{-}_{j+1/2} \\
&\qquad + \Big(u_{j+1/2}\phi_{j+1/2} + u_{j+1}\phi_{j+1} + u_{j+3/2}\phi_{j+3/2} \Big) h^{+}_{j+1/2} w^{+}_{j+1/2} \\
&+ \Big(u_{j+1/2}\phi_{j+1/2} + u_{j+1}\phi_{j+1} + u_{j+3/2}\phi_{j+3/2} \Big) h^{-}_{j+3/2} w^{-}_{j+3/2} \Bigg\} \begin{bmatrix} \phi_{j-1/2} \\ \phi_{j} \\ \phi_{j+1/2} \end{bmatrix} d\xi
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-1}^{1} \Big\{ u_{j-3/2} h^{+}_{j-3/2} w^{+}_{j-3/2} \phi_{j-3/2} + u_{j-1} h^{+}_{j-3/2} w^{+}_{j-3/2} \phi_{j-1} + u_{j-1/2} h^{+}_{j-3/2} w^{+}_{j-3/2} \phi_{j-1/2} \\
&+ u_{j-3/2} h^{-}_{j-1/2} w^{-}_{j-1/2} \phi_{j-3/2} + u_{j-1} h^{-}_{j-1/2} w^{-}_{j-1/2} \phi_{j-1} + u_{j-1/2} h^{-}_{j-1/2} w^{-}_{j-1/2} \phi_{j-1/2} \\
&\quad + u_{j-1/2} h^{+}_{j-1/2} w^{+}_{j-1/2} \phi_{j-1/2} + u_{j} h^{+}_{j-1/2} w^{+}_{j-1/2} \phi_{j} + u_{j+1/2} h^{+}_{j-1/2} w^{+}_{j-1/2} \phi_{j+1/2} \\
&\quad + u_{j-1/2} h^{-}_{j+1/2} w^{-}_{j+1/2} \phi_{j-1/2} + u_{j} h^{-}_{j+1/2} w^{-}_{j+1/2} \phi_{j} + u_{j+1/2} h^{-}_{j+1/2} w^{-}_{j+1/2} \phi_{j+1/2} \\
&+ u_{j+1/2} h^{+}_{j+1/2} w^{+}_{j+1/2} \phi_{j+1/2} + u_{j+1} h^{+}_{j+1/2} w^{+}_{j+1/2} \phi_{j+1} + u_{j+3/2} h^{+}_{j+1/2} w^{+}_{j+1/2} \phi_{j+3/2} \\
&+ u_{j+1/2} h^{-}_{j+3/2} w^{-}_{j+3/2} \phi_{j+1/2} + u_{j+1} h^{-}_{j+3/2} w^{-}_{j+3/2} \phi_{j+1} + u_{j+3/2} h^{-}_{j+3/2} w^{-}_{j+3/2} \phi_{j+3/2} \Big\} \\
&\qquad\qquad \begin{bmatrix} \phi_{j-1/2} \\ \phi_{j} \\ \phi_{j+1/2} \end{bmatrix} d\xi
\end{aligned}
$$

We now focus on $\phi_{j-1/2}$, so any terms that have there basis on the last cell are 0, and dont effect the integral, so

$$\int_{-1}^{1}\Big\{u_{j-3/2}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-3/2}+u_{j-1}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-1}+u_{j-1/2}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-1/2}$$
$$+u_{j-3/2}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-3/2}+u_{j-1}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-1}+u_{j-1/2}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-1/2}$$
$$+u_{j-1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j-1/2}+u_{j}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j}+u_{j+1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j+1/2}$$
$$+u_{j-1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j-1/2}+u_{j}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j}+u_{j+1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j+1/2}$$
$$+u_{j+1/2}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+1/2}+u_{j+1}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+1}+u_{j+3/2}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+3/2}$$
$$+u_{j+1/2}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+1/2}+u_{j+1}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+1}+u_{j+3/2}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+3/2}\Big\}\phi_{j-1/2}$$

$$=\int_{-1}^{1}\Big\{u_{j-3/2}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-3/2}+u_{j-1}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-1}+u_{j-1/2}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-1/2}$$
$$+u_{j-3/2}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-3/2}+u_{j-1}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-1}+u_{j-1/2}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-1/2}$$
$$+u_{j-1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j-1/2}+u_{j}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j}+u_{j+1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j+1/2}$$
$$+u_{j-1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j-1/2}+u_{j}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j}+u_{j+1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j+1/2}\Big\}\phi_{j-1/2}$$

$$\begin{aligned}=\int_{-1}^{1}&u_{j-3/2}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-3/2}\phi_{j-1/2}+u_{j-1}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-1}\phi_{j-1/2}\\
&+u_{j-1/2}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-1/2}\phi_{j-1/2}\\
&+u_{j-3/2}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-3/2}\phi_{j-1/2}+u_{j-1}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-1}\phi_{j-1/2}\\
&+u_{j-1/2}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-1/2}\phi_{j-1/2}\\
&+u_{j-1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j-1/2}\phi_{j-1/2}+u_{j}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j}\phi_{j-1/2}\\
&+u_{j+1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j+1/2}\phi_{j-1/2}\\
&+u_{j-1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j-1/2}\phi_{j-1/2}+u_{j}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j}\phi_{j-1/2}\\
&+u_{j+1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j+1/2}\phi_{j-1/2}d\xi\end{aligned}$$

$$
\begin{aligned}
&= u_{j-3/2}h^+_{j-3/2}\int_{-1}^{1} w^+_{j-3/2}\phi_{j-3/2}\phi_{j-1/2}d\xi + u_{j-1}h^+_{j-3/2}\int_{-1}^{1} w^+_{j-3/2}\phi_{j-1}\phi_{j-1/2}d\xi \\
&\qquad + u_{j-1/2}h^+_{j-3/2}\int_{-1}^{1} w^+_{j-3/2}\phi_{j-1/2}\phi_{j-1/2}d\xi \\
&+ u_{j-3/2}h^-_{j-1/2}\int_{-1}^{1} w^-_{j-1/2}\phi_{j-3/2}\phi_{j-1/2}d\xi + u_{j-1}h^-_{j-1/2}\int_{-1}^{1} w^-_{j-1/2}\phi_{j-1}\phi_{j-1/2}d\xi \\
&\qquad + u_{j-1/2}h^-_{j-1/2}\int_{-1}^{1} w^-_{j-1/2}\phi_{j-1/2}\phi_{j-1/2}d\xi \\
&+ u_{j-1/2}h^+_{j-1/2}\int_{-1}^{1} w^+_{j-1/2}\phi_{j-1/2}\phi_{j-1/2}d\xi + u_{j}h^+_{j-1/2}\int_{-1}^{1} w^+_{j-1/2}\phi_{j}\phi_{j-1/2}d\xi \\
&\qquad + u_{j+1/2}h^+_{j-1/2}\int_{-1}^{1} w^+_{j-1/2}\phi_{j+1/2}\phi_{j-1/2}d\xi \\
&+ u_{j-1/2}h^-_{j+1/2}\int_{-1}^{1} w^-_{j+1/2}\phi_{j-1/2}\phi_{j-1/2}d\xi + u_{j}h^-_{j+1/2}\int_{-1}^{1} w^-_{j+1/2}\phi_{j}\phi_{j-1/2}d\xi \\
&\qquad + u_{j+1/2}h^-_{j+1/2}\int_{-1}^{1} w^-_{j+1/2}\phi_{j+1/2}\phi_{j-1/2}d\xi
\end{aligned}
$$

These integrals are

$$
\begin{aligned}
\int_{-1}^{1} w^+_{j-3/2}\phi_{j-3/2}\phi_{j-1/2}d\xi &= \int_{-1}^{-\frac{1}{3}} w^+_{j-3/2}\phi_{j-3/2}\phi_{j-1/2}d\xi \\
&= \int_{-1}^{-\frac{1}{3}} -\frac{3}{2}\left(\xi+\frac{1}{3}\right)\frac{9}{2}\left(\xi+\frac{2}{3}\right)\left(\xi+\frac{1}{3}\right)\frac{9}{2}\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi \\
&= \int_{-1}^{-\frac{1}{3}} -\frac{243}{8}\left(\xi+\frac{1}{3}\right)\left(\xi+\frac{2}{3}\right)\left(\xi+\frac{1}{3}\right)\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi = -\frac{1}{90}
\end{aligned}
$$

$$\int_{-1}^{1} w^{+}_{j-3/2}\phi_{j-1}\phi_{j-1/2}d\xi = \int_{-1}^{-\frac{1}{3}} w^{+}_{j-3/2}\phi_{j-1}\phi_{j-1/2}d\xi$$
$$= \int_{-1}^{-\frac{1}{3}} -\frac{3}{2}\left(\xi+\frac{1}{3}\right)(-9)(\xi+1)\left(\xi+\frac{1}{3}\right)\frac{9}{2}\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi$$
$$= \int_{-1}^{-\frac{1}{3}} \frac{243}{4}\left(\xi+\frac{1}{3}\right)(\xi+1)\left(\xi+\frac{1}{3}\right)\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi = 0$$

$$\int_{-1}^{1} w^{+}_{j-3/2}\phi_{j-1/2}\phi_{j-1/2}d\xi = \int_{-1}^{-\frac{1}{3}} w^{+}_{j-3/2}\phi_{j-1/2}\phi_{j-1/2}d\xi$$
$$= \int_{-1}^{-\frac{1}{3}} -\frac{3}{2}\left(\xi+\frac{1}{3}\right)\frac{9}{2}\left(\xi+\frac{2}{3}\right)(\xi+1)\frac{9}{2}\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi$$
$$= \int_{-1}^{-\frac{1}{3}} -\frac{243}{8}\left(\xi+\frac{1}{3}\right)\left(\xi+\frac{2}{3}\right)(\xi+1)\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi = \frac{1}{90}$$

$$\int_{-1}^{1} w^{-}_{j-1/2}\phi_{j-3/2}\phi_{j-1/2}d\xi = \int_{-1}^{-\frac{1}{3}} w^{+}_{j-1/2}\phi_{j-3/2}\phi_{j-1/2}d\xi$$
$$= \int_{-1}^{-\frac{1}{3}} \frac{3}{2}(\xi+1)\frac{9}{2}\left(\xi+\frac{2}{3}\right)\left(\xi+\frac{1}{3}\right)\frac{9}{2}\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi$$
$$= \int_{-1}^{-\frac{1}{3}} \frac{243}{8}(\xi+1)\left(\xi+\frac{2}{3}\right)\left(\xi+\frac{1}{3}\right)\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi = -\frac{1}{90}$$

$$\int_{-1}^{1} w^{-}_{j-1/2}\phi_{j-1}\phi_{j-1/2}d\xi = \int_{-1}^{-\frac{1}{3}} w^{-}_{j-1/2}\phi_{j-1}\phi_{j-1/2}d\xi$$
$$= \int_{-1}^{-\frac{1}{3}} \frac{3}{2}(\xi+1)(-9)(\xi+1)\left(\xi+\frac{1}{3}\right)\frac{9}{2}\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi$$
$$= \int_{-1}^{-\frac{1}{3}} -\frac{243}{4}(\xi+1)(\xi+1)\left(\xi+\frac{1}{3}\right)\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi = \frac{2}{45}$$

$$\int_{-1}^{1} w^{-}_{j-1/2}\phi_{j-1/2}\phi_{j-1/2}d\xi = \int_{-1}^{-\frac{1}{3}} w^{-}_{j-1/2}\phi_{j-1/2}\phi_{j-1/2}d\xi$$
$$= \int_{-1}^{-\frac{1}{3}} \frac{3}{2}(\xi+1)\frac{9}{2}\left(\xi+\frac{2}{3}\right)(\xi+1)\frac{9}{2}\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi$$
$$= \int_{-1}^{-\frac{1}{3}} \frac{243}{8}(\xi+1)\left(\xi+\frac{2}{3}\right)(\xi+1)\left(\xi+\frac{2}{3}\right)(\xi+1)\,d\xi = \frac{7}{90}$$

$$\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j-1/2}\phi_{j-1/2}d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} w^{+}_{j-1/2}\phi_{j-1/2}\phi_{j-1/2}d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{3}{2}\left(\xi-\frac{1}{3}\right)\frac{9}{2}(\xi)\left(\xi-\frac{1}{3}\right)\frac{9}{2}(\xi)\left(\xi-\frac{1}{3}\right)d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{243}{8}\left(\xi-\frac{1}{3}\right)(\xi)\left(\xi-\frac{1}{3}\right)(\xi)\left(\xi-\frac{1}{3}\right)d\xi = \frac{7}{90}$$

$$\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j}\phi_{j-1/2}d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} w^{+}_{j-1/2}\phi_{j}\phi_{j-1/2}d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{3}{2}\left(\xi-\frac{1}{3}\right)(-9)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)\frac{9}{2}(\xi)\left(\xi-\frac{1}{3}\right)d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{243}{4}\left(\xi-\frac{1}{3}\right)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)(\xi)\left(\xi-\frac{1}{3}\right)d\xi = \frac{2}{45}$$

$$\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j+1/2}\phi_{j-1/2}d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} w^{+}_{j-1/2}\phi_{j+1/2}\phi_{j-1/2}d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{3}{2}\left(\xi-\frac{1}{3}\right)\frac{9}{2}(\xi)\left(\xi+\frac{1}{3}\right)\frac{9}{2}(\xi)\left(\xi-\frac{1}{3}\right)d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{243}{8}\left(\xi-\frac{1}{3}\right)(\xi)\left(\xi+\frac{1}{3}\right)(\xi)\left(\xi-\frac{1}{3}\right)d\xi = -\frac{1}{90}$$

$$\int_{-1}^{1} w_{j+1/2}^{-}\phi_{j-1/2}\phi_{j-1/2}d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} w_{j+1/2}^{-}\phi_{j-1/2}\phi_{j-1/2}d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{3}{2}\left(\xi+\frac{1}{3}\right)\frac{9}{2}(\xi)\left(\xi-\frac{1}{3}\right)\frac{9}{2}(\xi)\left(\xi-\frac{1}{3}\right)d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{243}{8}\left(\xi+\frac{1}{3}\right)(\xi)\left(\xi-\frac{1}{3}\right)(\xi)\left(\xi-\frac{1}{3}\right)d\xi = \frac{1}{90}$$

$$\int_{-1}^{1} w_{j+1/2}^{-}\phi_{j}\phi_{j-1/2}d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} w_{j+1/2}^{-}\phi_{j}\phi_{j-1/2}d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{3}{2}\left(\xi+\frac{1}{3}\right)(-9)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)\frac{9}{2}(\xi)\left(\xi-\frac{1}{3}\right)d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{243}{4}\left(\xi+\frac{1}{3}\right)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)(\xi)\left(\xi-\frac{1}{3}\right)d\xi = 0$$

$$\int_{-1}^{1} w_{j+1/2}^{-}\phi_{j+1/2}\phi_{j-1/2}d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} w_{j+1/2}^{-}\phi_{j+1/2}\phi_{j-1/2}d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{3}{2}\left(\xi+\frac{1}{3}\right)\frac{9}{2}(\xi)\left(\xi+\frac{1}{3}\right)\frac{9}{2}(\xi)\left(\xi-\frac{1}{3}\right)d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{243}{8}\left(\xi+\frac{1}{3}\right)(\xi)\left(\xi+\frac{1}{3}\right)(\xi)\left(\xi-\frac{1}{3}\right)d\xi = -\frac{1}{90}$$

So we have

$$
\begin{aligned}
&u_{j-3/2}h^+_{j-3/2}\int_{-1}^{1} w^+_{j-3/2}\phi_{j-3/2}\phi_{j-1/2}d\xi + u_{j-1}h^+_{j-3/2}\int_{-1}^{1} w^+_{j-3/2}\phi_{j-1}\phi_{j-1/2}d\xi \\
&\qquad + u_{j-1/2}h^+_{j-3/2}\int_{-1}^{1} w^+_{j-3/2}\phi_{j-1/2}\phi_{j-1/2}d\xi \\
&+ u_{j-3/2}h^-_{j-1/2}\int_{-1}^{1} w^-_{j-1/2}\phi_{j-3/2}\phi_{j-1/2}d\xi + u_{j-1}h^-_{j-1/2}\int_{-1}^{1} w^-_{j-1/2}\phi_{j-1}\phi_{j-1/2}d\xi \\
&\qquad + u_{j-1/2}h^-_{j-1/2}\int_{-1}^{1} w^-_{j-1/2}\phi_{j-1/2}\phi_{j-1/2}d\xi \\
&+ u_{j-1/2}h^+_{j-1/2}\int_{-1}^{1} w^+_{j-1/2}\phi_{j-1/2}\phi_{j-1/2}d\xi + u_{j}h^+_{j-1/2}\int_{-1}^{1} w^+_{j-1/2}\phi_{j}\phi_{j-1/2}d\xi \\
&\qquad + u_{j+1/2}h^+_{j-1/2}\int_{-1}^{1} w^+_{j-1/2}\phi_{j+1/2}\phi_{j-1/2}d\xi \\
&+ u_{j-1/2}h^-_{j+1/2}\int_{-1}^{1} w^-_{j+1/2}\phi_{j-1/2}\phi_{j-1/2}d\xi + u_{j}h^-_{j+1/2}\int_{-1}^{1} w^-_{j+1/2}\phi_{j}\phi_{j-1/2}d\xi \\
&\qquad + u_{j+1/2}h^-_{j+1/2}\int_{-1}^{1} w^-_{j+1/2}\phi_{j+1/2}\phi_{j-1/2}d\xi \\
&\qquad = -\frac{1}{90}u_{j-3/2}h^+_{j-3/2} + \frac{1}{90}u_{j-1/2}h^+_{j-3/2} \\
&\quad - \frac{1}{90}u_{j-3/2}h^-_{j-1/2} + \frac{4}{90}u_{j-1}h^-_{j-1/2} + \frac{7}{90}u_{j-1/2}h^-_{j-1/2} \\
&\quad + \frac{7}{90}u_{j-1/2}h^+_{j-1/2} + \frac{4}{90}u_{j}h^+_{j-1/2} - \frac{1}{90}u_{j+1/2}h^+_{j-1/2} \\
&\qquad + \frac{1}{90}u_{j-1/2}h^-_{j+1/2} - \frac{1}{90}u_{j+1/2}h^-_{j+1/2}
\end{aligned}
$$

We now focus on ϕ_j, so any terms that have there basis on the first and last cell are 0, and dont effect the integral, so

$$\int_{-1}^{1}\Big\{u_{j-3/2}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-3/2}+u_{j-1}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-1}+u_{j-1/2}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-1/2}$$
$$+u_{j-3/2}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-3/2}+u_{j-1}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-1}+u_{j-1/2}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-1/2}$$
$$+u_{j-1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j-1/2}+u_{j}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j}+u_{j+1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j+1/2}$$
$$+u_{j-1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j-1/2}+u_{j}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j}+u_{j+1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j+1/2}$$
$$+u_{j+1/2}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+1/2}+u_{j+1}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+1}+u_{j+3/2}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+3/2}$$
$$+u_{j+1/2}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+1/2}+u_{j+1}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+1}+u_{j+3/2}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+3/2}\Big\}\phi_j$$

$$=\int_{-1}^{1}\Big\{u_{j-1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j-1/2}+u_{j}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j}+u_{j+1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j+1/2}$$
$$+u_{j-1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j-1/2}+u_{j}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j}+u_{j+1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j+1/2}\Big\}\phi_j$$

$$=\int_{-1}^{1}u_{j-1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j-1/2}\phi_j+u_{j}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_j\phi_j+u_{j+1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j+1/2}\phi_j$$
$$+u_{j-1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j-1/2}\phi_j+u_{j}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_j\phi_j+u_{j+1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j+1/2}\phi_j d\xi$$

$$\begin{aligned}=u_{j-1/2}h^{+}_{j-1/2}\int_{-1}^{1}w^{+}_{j-1/2}\phi_{j-1/2}\phi_j d\xi+u_{j}h^{+}_{j-1/2}\int_{-1}^{1}w^{+}_{j-1/2}\phi_j\phi_j d\xi\\ +u_{j+1/2}h^{+}_{j-1/2}\int_{-1}^{1}w^{+}_{j-1/2}\phi_{j+1/2}\phi_j d\xi\\ +u_{j-1/2}h^{-}_{j+1/2}\int_{-1}^{1}w^{-}_{j+1/2}\phi_{j-1/2}\phi_j d\xi+u_{j}h^{-}_{j+1/2}\int_{-1}^{1}w^{-}_{j+1/2}\phi_j\phi_j d\xi\\ +u_{j+1/2}h^{-}_{j+1/2}\int_{-1}^{1}w^{-}_{j+1/2}\phi_{j+1/2}\phi_j d\xi\end{aligned}$$

These integrals are

$$\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j-1/2}\phi_j d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} w^{+}_{j-1/2}\phi_{j-1/2}\phi_j d\xi$$

$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{3}{2}\left(\xi-\frac{1}{3}\right)\frac{9}{2}(\xi)\left(\xi-\frac{1}{3}\right)(-9)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi$$

$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{243}{4}\left(\xi-\frac{1}{3}\right)(\xi)\left(\xi-\frac{1}{3}\right)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi = \frac{2}{45}$$

$$\int_{-1}^{1} w^{+}_{j-1/2}\phi_j\phi_j d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} w^{+}_{j-1/2}\phi_j\phi_j d\xi$$

$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{3}{2}\left(\xi-\frac{1}{3}\right)(-9)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)(-9)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi$$

$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{243}{2}\left(\xi-\frac{1}{3}\right)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi = \frac{8}{45}$$

$$\int_{-1}^{1} w^{+}_{j-1/2}\phi_{j+1/2}\phi_j d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} w^{+}_{j-1/2}\phi_{j+1/2}\phi_j d\xi$$

$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{3}{2}\left(\xi-\frac{1}{3}\right)\frac{9}{2}(\xi)\left(\xi+\frac{1}{3}\right)(-9)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi$$

$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{243}{4}\left(\xi-\frac{1}{3}\right)(\xi)\left(\xi+\frac{1}{3}\right)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi = 0$$

$$\int_{-1}^{1} w^{-}_{j+1/2}\phi_{j-1/2}\phi_j d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} w^{-}_{j+1/2}\phi_{j-1/2}\phi_j d\xi$$

$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{3}{2}\left(\xi+\frac{1}{3}\right)\frac{9}{2}(\xi)\left(\xi-\frac{1}{3}\right)(-9)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi$$

$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{243}{4}\left(\xi+\frac{1}{3}\right)(\xi)\left(\xi-\frac{1}{3}\right)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi = 0$$

$$\int_{-1}^{1} w^{-}_{j+1/2}\phi_j\phi_j d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} w^{-}_{j+1/2}\phi_j\phi_j d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{3}{2}\left(\xi+\frac{1}{3}\right)(-9)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)(-9)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{243}{2}\left(\xi+\frac{1}{3}\right)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi = \frac{8}{45}$$

$$\int_{-1}^{1} w^{-}_{j+1/2}\phi_{j+1/2}\phi_j d\xi = \int_{-\frac{1}{3}}^{\frac{1}{3}} w^{-}_{j+1/2}\phi_{j+1/2}\phi_j d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{3}{2}\left(\xi+\frac{1}{3}\right)\frac{9}{2}(\xi)\left(\xi+\frac{1}{3}\right)(-9)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi$$
$$= \int_{-\frac{1}{3}}^{\frac{1}{3}} -\frac{243}{4}\left(\xi+\frac{1}{3}\right)(\xi)\left(\xi+\frac{1}{3}\right)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)d\xi = \frac{2}{45}$$

So we have

$$\int_{-1}^{1}\Big\{u_{j-3/2}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-3/2}+u_{j-1}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-1}+u_{j-1/2}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-1/2}$$
$$+u_{j-3/2}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-3/2}+u_{j-1}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-1}+u_{j-1/2}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-1/2}$$
$$+u_{j-1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j-1/2}+u_jh^{+}_{j-1/2}w^{+}_{j-1/2}\phi_j+u_{j+1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j+1/2}$$
$$+u_{j-1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j-1/2}+u_jh^{-}_{j+1/2}w^{-}_{j+1/2}\phi_j+u_{j+1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j+1/2}$$
$$+u_{j+1/2}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+1/2}+u_{j+1}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+1}+u_{j+3/2}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+3/2}$$
$$+u_{j+1/2}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+1/2}+u_{j+1}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+1}+u_{j+3/2}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+3/2}\Big\}\phi_j$$
$$= \frac{2}{45}u_{j-1/2}h^{+}_{j-1/2}+\frac{8}{45}u_jh^{+}_{j-1/2}$$
$$+\frac{8}{45}u_jh^{-}_{j+1/2}+\frac{2}{45}u_{j+1/2}h^{-}_{j+1/2}$$

We now focus on $\phi_{j+1/2}$, so any terms that have there basis on the first cell are 0, and dont effect the integral, so

$$\int_{-1}^{1}\Big\{u_{j-3/2}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-3/2}+u_{j-1}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-1}+u_{j-1/2}h^{+}_{j-3/2}w^{+}_{j-3/2}\phi_{j-1/2}$$
$$+u_{j-3/2}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-3/2}+u_{j-1}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-1}+u_{j-1/2}h^{-}_{j-1/2}w^{-}_{j-1/2}\phi_{j-1/2}$$
$$+u_{j-1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j-1/2}+u_{j}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j}+u_{j+1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j+1/2}$$
$$+u_{j-1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j-1/2}+u_{j}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j}+u_{j+1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j+1/2}$$
$$+u_{j+1/2}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+1/2}+u_{j+1}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+1}+u_{j+3/2}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+3/2}$$
$$+u_{j+1/2}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+1/2}+u_{j+1}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+1}+u_{j+3/2}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+3/2}\Big\}\phi_{j+1/2}$$

$$=\int_{-1}^{1}\Big\{u_{j-1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j-1/2}+u_{j}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j}+u_{j+1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j+1/2}$$
$$+u_{j-1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j-1/2}+u_{j}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j}+u_{j+1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j+1/2}$$
$$+u_{j+1/2}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+1/2}+u_{j+1}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+1}+u_{j+3/2}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+3/2}$$
$$+u_{j+1/2}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+1/2}+u_{j+1}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+1}+u_{j+3/2}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+3/2}\Big\}\phi_{j+1/2}$$

$$=\int_{-1}^{1}$$
$$u_{j-1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j-1/2}\phi_{j+1/2}+u_{j}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j}\phi_{j+1/2}+u_{j+1/2}h^{+}_{j-1/2}w^{+}_{j-1/2}\phi_{j+1/2}\phi_{j+1/2}$$
$$+u_{j-1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j-1/2}\phi_{j+1/2}+u_{j}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j}\phi_{j+1/2}+u_{j+1/2}h^{-}_{j+1/2}w^{-}_{j+1/2}\phi_{j+1/2}\phi_{j+1/2}$$
$$+u_{j+1/2}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+1/2}\phi_{j+1/2}+u_{j+1}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+1}\phi_{j+1/2}+u_{j+3/2}h^{+}_{j+1/2}w^{+}_{j+1/2}\phi_{j+3/2}\phi_{j+1/2}$$
$$+u_{j+1/2}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+1/2}\phi_{j+1/2}+u_{j+1}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+1}\phi_{j+1/2}+u_{j+3/2}h^{-}_{j+3/2}w^{-}_{j+3/2}\phi_{j+3/2}\phi_{j+1/2}d\xi$$

$$
\begin{aligned}
&= u_{j-1/2}h^+_{j-1/2}\int_{-1}^{1} w^+_{j-1/2}\phi_{j-1/2}\phi_{j+1/2}d\xi + u_j h^+_{j-1/2}\int_{-1}^{1} w^+_{j-1/2}\phi_j\phi_{j+1/2}d\xi \\
&\qquad + u_{j+1/2}h^+_{j-1/2}\int_{-1}^{1} w^+_{j-1/2}\phi_{j+1/2}\phi_{j+1/2}d\xi \\
&+ u_{j-1/2}h^-_{j+1/2}\int_{-1}^{1} w^-_{j+1/2}\phi_{j-1/2}\phi_{j+1/2}d\xi + u_j h^-_{j+1/2}\int_{-1}^{1} w^-_{j+1/2}\phi_j\phi_{j+1/2}d\xi \\
&\qquad + u_{j+1/2}h^-_{j+1/2}\int_{-1}^{1} w^-_{j+1/2}\phi_{j+1/2}\phi_{j+1/2}d\xi \\
&+ u_{j+1/2}h^+_{j+1/2}\int_{-1}^{1} w^+_{j+1/2}\phi_{j+1/2}\phi_{j+1/2}d\xi + u_{j+1}h^+_{j+1/2}\int_{-1}^{1} w^+_{j+1/2}\phi_{j+1}\phi_{j+1/2}d\xi \\
&\qquad + u_{j+3/2}h^+_{j+1/2}\int_{-1}^{1} w^+_{j+1/2}\phi_{j+3/2}\phi_{j+1/2}d\xi \\
&+ u_{j+1/2}h^-_{j+3/2}\int_{-1}^{1} w^-_{j+3/2}\phi_{j+1/2}\phi_{j+1/2}d\xi + u_{j+1}h^-_{j+3/2}\int_{-1}^{1} w^-_{j+3/2}\phi_{j+1}\phi_{j+1/2}d\xi \\
&\qquad + u_{j+3/2}h^-_{j+3/2}\int_{-1}^{1} w^-_{j+3/2}\phi_{j+3/2}\phi_{j+1/2}d\xi
\end{aligned}
$$