1 Serre Equations

The Serre Equations read (height/mass)

$$\frac{\partial h}{\partial t} + \frac{\partial uh}{\partial x} = 0$$

Phi

$$\Phi = \frac{\partial b}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2} + \frac{\partial b}{\partial x}\frac{\partial u}{\partial t}$$

Gamma

$$\Gamma = \left(\frac{\partial u}{\partial x}\right)^2 - u\left(\frac{\partial^2 u}{\partial x^2}\right) - \left(\frac{\partial^2 u}{\partial x \partial t}\right)$$

Pressure

$$p|_{\xi} = p_a + \rho g \xi + \frac{\rho}{2}\xi\left(2h - \xi\right)\Gamma + \rho\xi\Phi$$

Momentum(velocity) x

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^3}{3}\Gamma + \frac{h^2}{2}\Phi\right) + h\frac{\partial b}{\partial x}\left(g + \frac{h}{2}\Gamma + \Phi\right) = 0$$

z

$$w|_z = \frac{z-b}{h}\frac{\partial h}{\partial t} + u\frac{\partial b}{\partial x}$$

The total energy of the Euler system is $E = \frac{1}{2}|\vec{u}|^2 + g(z-b)$:

$$\frac{\partial E}{\partial t} + \nabla \cdot \left(E\vec{u} + p\vec{u}\right) = 0$$

Integrating over depth $s = h + b$

$$\int_b^s \left(\frac{\partial E}{\partial t} + \nabla \cdot \left(E\vec{u} + p\vec{u}\right)\right) dz = 0$$

$$\int_b^s \frac{\partial E}{\partial t}\, dz + \int_b^s \frac{\partial}{\partial x}\left(Eu + pu\right)\, dz + \int_b^s \frac{\partial}{\partial z}\left(Ew + pw\right)\, dz = 0$$

$$\int_b^s \frac{\partial E}{\partial t}\, dz + \int_b^s \frac{\partial}{\partial x}\left(Eu + pu\right)\, dz + E(s)w(s) + p(s)w(s) - E(b)w(b) - p(b)w(b) = 0$$

at b $w = 0$ (no slip)

$$\int_b^s \frac{\partial E}{\partial t}\,dz + \int_b^s \frac{\partial}{\partial x}\left(Eu + pu\right)\,dz + E(s)w(s) + p(s)w(s) = 0$$

First term (Leibeniz):

$$\int_b^s \frac{\partial E}{\partial t}\,dz = \frac{\partial}{\partial t}\int_b^s E\,dz - E(s)\frac{\partial (s)}{\partial t} + E(b)\frac{\partial (b)}{\partial t}$$

$$\int_b^s \frac{\partial E}{\partial t}\,dz = \frac{\partial}{\partial t}\int_b^s E\,dz - E(s)\frac{\partial h}{\partial t}$$

So need to calculate the integral of the energy over depth:

$$\int_b^s E\,dz = \int_b^s \frac{1}{2}\left(u^2 + w^2\right) + g(z-b)\,dz$$

$$\int_b^s \frac{1}{2}\left(u^2 + w^2\right) + g(z-b)\,dz = \frac{1}{2}\left(\bar{u}^2 h + \int_b^s w^2\,dz\right) + \int_b^s g(z-b)\,dz$$

Calculating the P.E first (simplest)

$$\int_b^s g(z-b)\,dz = g\left(\int_b^s z\,dz - b\int_b^s dz\right) = g\left(\left[\frac{1}{2}z^2\right]_b^s - b\left[z\right]_b^s\right) = \frac{g}{2}\left(h^2 + 2hb\right) - gbh = \frac{gh^2}{2}$$

Calculating the vertical velocity

$$\int_b^s w^2\,dz = \int_b^s \left(\frac{z-b}{h}\frac{\partial h}{\partial t} + u\frac{\partial b}{\partial x}\right)^2 dz$$

$$= \int_b^s \left(\frac{z-b}{h}\frac{\partial h}{\partial t}\right)^2 + 2\left(\frac{z-b}{h}\frac{\partial h}{\partial t}\right)\left(u\frac{\partial b}{\partial x}\right) + \left(u\frac{\partial b}{\partial x}\right)^2 dz$$

$$= \int_b^s \left(\frac{z-b}{h}\frac{\partial h}{\partial t}\right)^2 dz + \int_b^s 2\left(\frac{z-b}{h}\frac{\partial h}{\partial t}\right)\left(u\frac{\partial b}{\partial x}\right) dz + h\left(\bar{u}\frac{\partial b}{\partial x}\right)^2$$

$$= \left(\frac{1}{h}\frac{\partial h}{\partial t}\right)^2 \int_b^s (z-b)^2 \; dz + 2\frac{1}{h}\frac{\partial h}{\partial t}\frac{\partial b}{\partial x}\int_b^s (z-b)\,(u) \; dz + h\left(\bar{u}\frac{\partial b}{\partial x}\right)^2$$

Can use that the velocity profile is constant over depth ($u|_x, z = \bar{u}(x)$)

$$= \left(\frac{1}{h}\frac{\partial h}{\partial t}\right)^2 \int_b^s (z-b)^2 \; dz + 2\frac{\bar{u}}{h}\frac{\partial h}{\partial t}\frac{\partial b}{\partial x}\int_b^s (z-b) \; dz + h\left(\bar{u}\frac{\partial b}{\partial x}\right)^2$$

Integrals:

$$\begin{aligned}
\int_b^s (z-b) \; dz &= \int_b^s z\,dx - \int_b^s b\,dz \\
&= \left[\frac{z^2}{2}\right]_b^s - b\int_b^s 1\,dz \\
&= \left[\frac{z^2}{2}\right]_b^s - b\,[s-b] \\
&= \frac{s^2}{2} - \frac{b^2}{2} - b\,[h+b-b] \\
&= \frac{h^2+2hb+b^2}{2} - \frac{b^2}{2} - bh \\
&= \frac{h^2+2hb}{2} - bh \\
&= \frac{h^2}{2}
\end{aligned}$$

Second

$$\begin{aligned}
\int_b^s (z-b)^2 \; dz &= \int_b^s z^2 - 2zb + b^2\,dz \\
&= \int_b^s z^2\,dz - \int_b^s 2zb\,dz + \int_b^s b^2\,dz \\
&= \int_b^s z^2\,dz - 2b\int_b^s z\,dz + b^2\int_b^s 1\,dz
\end{aligned}$$

$$= \left[\frac{z^3}{3}\right]_b^s - 2b\left[\frac{z^2}{2}\right]_b^s + b^2h$$

$$= \left(\frac{s^3}{3} - \frac{b^3}{3}\right) - 2b\left(\frac{h^2+2hb}{2}\right) + b^2h$$

$$= \left(\frac{h^3+3h^2b+3hb^2+b^3}{3} - \frac{b^3}{3}\right) - bh^2 - 2hb^2 + b^2h$$

$$= \left(\frac{h^3+3h^2b+3hb^2}{3}\right) - bh^2 - hb^2$$

$$= \left(\frac{h^3}{3}\right)$$

(identity that applies)
Back to it

$$= \left(\frac{1}{h}\frac{\partial h}{\partial t}\right)^2 \int_b^s (z-b)^2\, dz + 2\frac{\bar{u}}{h}\frac{\partial h}{\partial t}\frac{\partial b}{\partial x}\int_b^s (z-b)\, dz + h\left(\bar{u}\frac{\partial b}{\partial x}\right)^2$$

$$= \left(\frac{1}{h}\frac{\partial h}{\partial t}\right)^2 \left(\frac{h^3}{3}\right) + 2\frac{\bar{u}}{h}\frac{\partial h}{\partial t}\frac{\partial b}{\partial x}\left(\frac{h^2}{2}\right) + h\left(\bar{u}\frac{\partial b}{\partial x}\right)^2$$

$$= \frac{h}{3}\left(\frac{\partial h}{\partial t}\right)^2 + \bar{u}h\frac{\partial h}{\partial t}\frac{\partial b}{\partial x} + h\left(\bar{u}\frac{\partial b}{\partial x}\right)^2$$

$$= \frac{h}{3}\left(-\frac{\partial h\bar{u}}{\partial x}\right)^2 + \bar{u}h\left(-\frac{\partial h\bar{u}}{\partial x}\right)\frac{\partial b}{\partial x} + h\bar{u}^2\left(\frac{\partial b}{\partial x}\right)^2$$

$$= \frac{h}{3}\left(\frac{\partial\left(h\bar{u}\right)}{\partial x}\right)^2 - \bar{u}h\frac{\partial\left(h\bar{u}\right)}{\partial x}\frac{\partial b}{\partial x} + h\bar{u}^2\left(\frac{\partial b}{\partial x}\right)^2$$

So

$$\int_b^s E\, dz = \frac{1}{2}\left(\bar{u}^2h + \frac{h}{3}\left(\frac{\partial\left(h\bar{u}\right)}{\partial x}\right)^2 - \bar{u}h\frac{\partial\left(h\bar{u}\right)}{\partial x}\frac{\partial b}{\partial x} + h\bar{u}^2\left(\frac{\partial b}{\partial x}\right)^2\right) + \frac{gh^2}{2}$$

then

$$\int_b^s \frac{\partial E}{\partial t}\,dz = \frac{1}{2}\frac{\partial}{\partial t}\left[\bar{u}^2 h + \frac{h}{3}\left(\frac{\partial\left(h\bar{u}\right)}{\partial x}\right)^2 - \bar{u}h\frac{\partial\left(h\bar{u}\right)}{\partial x}\frac{\partial b}{\partial x} + h\bar{u}^2\left(\frac{\partial b}{\partial x}\right)^2 + \frac{gh^2}{2}\right] - E(s)\frac{\partial h}{\partial t}$$

$$= \frac{1}{2}\Bigg[2\bar{u}\bar{u}_t h + \bar{u}^2 h_t + \frac{h_t}{3}\left(\bar{u}h\right)_x + \frac{2h}{3}\left(\bar{u}h\right)_x\left(\bar{u}h\right)_{xt} - \left(\bar{u}h\right)\left(\bar{u}h\right)_{xt} b_x - \left(\bar{u}h\right)_t\left(\bar{u}h\right)_x b_x$$
$$+ 2\bar{u}\bar{u}_t h\left(b_x\right)^2 + \bar{u}^2 h_t\left(b_x\right)^2 + ghh_t\Bigg] - E(s)\frac{\partial h}{\partial t} \quad (1)$$

$$= \frac{1}{2}\Bigg[h_t\left(\bar{u}^2 + \frac{1}{3}\left(\bar{u}h\right)_x + gh + \bar{u}^2\left(b_x\right)^2\right) + u_t\left(2\bar{u}h + 2\bar{u}h\left(b_x\right)^2\right)$$
$$+ \frac{2h}{3}\left(\bar{u}h\right)_x\left(\bar{u}h\right)_{xt} - \left(\bar{u}h\right)\left(\bar{u}h\right)_{xt} b_x - \left(\bar{u}h\right)_t\left(\bar{u}h\right)_x b_x\Bigg] - E(s)\frac{\partial h}{\partial t} \quad (2)$$

$$= \frac{1}{2}\Bigg[h_t\left(\bar{u}^2 + \frac{1}{3}\left(\bar{u}h\right)_x + gh + \bar{u}^2\left(b_x\right)^2\right) + u_t\left(2\bar{u}h + 2\bar{u}h\left(b_x\right)^2\right)$$
$$+ \frac{2h}{3}\left(\bar{u}h\right)_x\left(\bar{u}h\right)_{xt} - \left(\bar{u}h\right)\left(\bar{u}h\right)_{xt} b_x - h_t\bar{u}\left(\bar{u}h\right)_x b_x - \bar{u}_t h\left(\bar{u}h\right)_x b_x\Bigg] - E(s)\frac{\partial h}{\partial t} \quad (3)$$

$$= \frac{1}{2}\Bigg[h_t\left(\bar{u}^2 + \frac{1}{3}\left(\bar{u}h\right)_x + gh + \bar{u}^2\left(b_x\right)^2 - \bar{u}\left(\bar{u}h\right)_x b_x\right) + u_t\left(2\bar{u}h + 2\bar{u}h\left(b_x\right)^2 - h\left(\bar{u}h\right)_x b_x\right)$$
$$+ \left(\bar{u}h\right)_{xt}\left(\frac{2h}{3}\left(\bar{u}h\right)_x - \left(\bar{u}h\right)b_x\right)\Bigg] - E(s)\frac{\partial h}{\partial t} \quad (4)$$

$$= \frac{1}{2}\Bigg[h_t\left(\bar{u}^2 + \frac{1}{3}\left(\bar{u}h\right)_x + gh + \bar{u}^2\left(b_x\right)^2 - \bar{u}\left(\bar{u}h\right)_x b_x\right) + u_t\left(2\bar{u}h + 2\bar{u}h\left(b_x\right)^2 - h\left(\bar{u}h\right)_x b_x\right)$$
$$+ \left(\bar{u}_{tx}h + \bar{u}_t h_x + \bar{u}_x h_t + \bar{u}h_{tx}\right)\left(\frac{2h}{3}\left(\bar{u}h\right)_x - \left(\bar{u}h\right)b_x\right)\Bigg] - E(s)\frac{\partial h}{\partial t} \quad (5)$$

$$= \frac{1}{2}\Bigg[h_t\left(\bar{u}^2 + \frac{1}{3}\left(\bar{u}h\right)_x + gh + \bar{u}^2\left(b_x\right)^2 - \bar{u}\left(\bar{u}h\right)_x b_x + \bar{u}_x\left(\frac{2h}{3}\left(\bar{u}h\right)_x - \left(\bar{u}h\right)b_x\right)\right)$$
$$+ u_t\left(2\bar{u}h + 2\bar{u}h\left(b_x\right)^2 - h\left(\bar{u}h\right)_x b_x + h_x\left(\frac{2h}{3}\left(\bar{u}h\right)_x - \left(\bar{u}h\right)b_x\right)\right)$$
$$+ \left(\bar{u}_{tx}h + \bar{u}h_{tx}\right)\left(\frac{2h}{3}\left(\bar{u}h\right)_x - \left(\bar{u}h\right)b_x\right)\Bigg] - E(s)\frac{\partial h}{\partial t} \quad (6)$$

From Serre equations $h_t = -(uh)_x$ and

$$u_t + uu_x + gh_x + hh_xu_x^2 - hh_xuu_{xx} - hh_xu_{xt} + uh_xu_xb_x$$
$$+ u^2h_xb_{xx} + h_xb_xu_t + \frac{h^2}{3}u_xu_{xx} - \frac{h^2}{3}uu_{xxx} - \frac{h^2}{3}u_{xtx}$$
$$+ hu_x^2b_x + \frac{3h}{2}uu_xb_{xx} + \frac{h}{2}u^2b_{xxx} + \frac{h}{2}b_{xx}u_t + gb_x$$
$$+ uu_xb_x^2 + u^2b_xb_{xx} + b_x^2u_t = 0$$

Thus: