University of Waterloo
EARTH 437
Rock Mechanics

Computer Software Project
Poroelastic Finite Element Method Model

Prepared for
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Summary

Poroelasticity is the reaction of the porous structure within a rock as it is being subjected to loading. It is an important geomechanical property due to the fact that it must be taken into consideration in many industries for its effects on rocks and land formations.

To further understand the effects poroelasticity has on geotextiles, an open-source software was created. The software was designed to create a simulation of poroelastic changes using a mathematical model that was derived from the two-dimensional finite element method. The FEM model was then used to create solutions for coupled linear poroelastic problems, using the Galerkin method.

The software can be used by professionals from many different fields to create effective long-term models of poroelastic effects and can be customized based on the individual's' needs due to the open-sourced nature of the program. Although the provided program only computes steady conditions, the source code for transient conditions is available and can be implemented if needed. Problems that can utilize this software include any that involves a change in pore pressure which results in the displacement of solid and fluid within the system. Such include dewatering problems, depletion of gas reservoir, borehole drilling, and rock tunneling.

Introduction

Poroelasticity is a rock property that describes how the porous structure within a rock reacts to an external load. This leads to movement of fluids within the porous space as well as deformation of the solid components of the rock.

Poroelasticity is an essential property due to its industrial application, especially in oil and gas exploration. Since the reservoirs containing oil and gas are porous medium, the exploration of the natural resources from the reservoirs could cause changes in pore pressure and solids. The two components interact with each other, and could lead to drastic alterations to the formations.

During the depletion of the reservoirs, the pore pressure would be decreased due to the oil and gas being extracted. The lowered pore pressure would be compensated by an increase of the effective stress, closing the pore spaces. This could often result in subsidence in the area. There had been numerous incidents where the extraction of natural resources caused the surrounding areas to subside, including Lake Maracaibo in Venezuela, Goose Creek Oil Field in Texas, and the Po Delta in northern Italy (Lewis et al., 1998).

In order to gain more insight into how poroelasticity affects rocks, a software needs to be developed to create a comprehensive model on how the rocks react to changes in pore pressure and effective stress. Traditionally, the one-dimensional theory by Terzaghi is used for the analysis. As a result, most softwares available commercially employ the Terzaghi theory. However, in recent years, the Biot's theory, which includes two- and three-dimensional aspects, is being more frequently used, as evident in a model introduced by rocscience in 2010.

For the case study, an open-source software package and a mathematical model based on Biot's theory are created. The model utilizes a two-dimensional finite element method in order to accommodate the problem involving coupled linear poroelasticity. The resulting linear elastic model can be an accurate method to produce a long-term estimate for the amount of subsidence, as conclusively demonstrated through the case studies conducted by Lewis et al, (1998). The software can be used by engineers, hydrogeologists, researchers, and students alike to efficiently and accurately simulate mathematical models involving poroelasticity. The open source allows individual users to make modifications in order to suit their needs.

Method

The software utilizes Biot's theory of poroelasticity to solve saturated flow inside a porous rock. Biot's poroelastic theory uses the the following stress-strain equation and fluid mass balance equation to describe a fluid flow system.

$$T = c : \varepsilon - \alpha I p \tag{1}$$

$$\varsigma = \alpha \nabla \cdot u + \frac{p}{M} \tag{2}$$

where T is the total stress, c is the elasticity tensor, ε is the strain tensor, α is Biot's constant, I is the identity matrix and p is the pore pressure. In equation (2), ζ is the increment fluid content, u is the matrix displacement, and $\frac{1}{M}$ is the specific storage coefficient at a constant strain. This coefficient relates the fluid and solid compressibilities and assumes both phases are compressible. As a result, the specific storage coefficient is calculated using the following equation.

$$\frac{1}{M} = \frac{\alpha - n}{Ks} + \frac{n}{Kf} \tag{3}$$

where n is porosity, Ks is the solid bulk modulus and Kf is the fluid bulk modulus.

To solve the differential poroelastic system, continuous galerkin finite element method is used. It is assumed that an equilibrium state is maintained at all times within the system to result is a force balance equation as the following

$$\nabla \cdot T + \rho g = 0 \tag{4}$$

where ρ is the density and g is the gravitational acceleration. Equation (1) and (4) combined will then form

$$\nabla \cdot (c : \varepsilon) - \alpha \nabla p + \rho g = 0 \tag{5}$$

Darcy's law is used to define the slow movement of fluid and is defined as

$$q = -\rho_f \frac{1}{\mu} k \cdot (\nabla p - \rho_f g) \tag{6}$$

where q is the fluid flux, k is permeability, ρ_f is the fluid density, and μ is the dynamic viscosity. A relationship between the fluid flux and increment fluid content can be made since the increment fluid content is simply the volume of fluid added into a control volume. This relationship is defined as

$$\frac{\partial \zeta}{\partial t} = -\frac{1}{\rho_f} \nabla \cdot q = \nabla \cdot [\frac{1}{\mu} k \cdot (\nabla p - \rho_f g)] \tag{7}$$

The fluid mass balance equation can be achieved by combining equation (7) and (2) to get

$$(\frac{\alpha - n}{Ks} + \frac{n}{Kf}) \frac{\partial p}{\partial t} + \alpha \nabla \cdot \frac{\partial u}{\partial t} + \nabla \cdot [\frac{1}{\mu} k \cdot (-\nabla p - \rho_f g)] = 0 \tag{8}$$

Utilizing equations (5) and (8), the finite element method can be used. The solid phase displacement (u) and pore pressure are solved simultaneously in this method. The stress and strain are written in vector form as the following

$$T = \{\tau_{xx}, \tau_{yy}, \tau_{xy}\}^T \tag{9}$$

$$\varepsilon = \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}\}^T \tag{10}$$

As a result, the stress divergence is defined with a differential operator L

$$\nabla \cdot T = L^T T \tag{11}$$

and the strain is defined as

$$\varepsilon = Lu \tag{12}$$

Using equation (12), equation (1) can be expressed as the following

$$T = c(Lu) - \alpha mp \tag{13}$$

where m is a vector as

$$m = \{1,1,0\}^T \tag{14}$$

In the finite element method, the displacement and pore pressure are defined as

$$u = N_u \bar{u} \tag{15}$$

and

$$p = N_p \bar{p} \tag{16}$$

where $\bar{u}$ and $\bar{p}$ are the unknown vectors of the displacement and pore pressure nodes in space. N_u and N_p are the displacement and pore pressure interpolation function matrices.
The Galerkin finite element method is produced from the manipulation of equations (5) and (8) to result in the following linear system for each element

$$\begin{bmatrix} 0 & 0 \\ Q^T & S \end{bmatrix} \frac{d}{dt} \begin{Bmatrix} \bar{u} \\ \bar{p} \end{Bmatrix} + \begin{bmatrix} K & -Q \\ 0 & H \end{bmatrix} \begin{Bmatrix} \bar{u} \\ \bar{p} \end{Bmatrix} = \begin{Bmatrix} F_u \\ F_p \end{Bmatrix} \tag{17}$$

Where the discontinuous piecewise linear interpolation functions are

$$K = \int_\Omega B^T c B \, d\Omega \tag{18}$$

$$B = \int_\Omega Lu \, d\Omega \tag{19}$$

$$Q = \int_\Omega B^T \alpha m N_p \, d\Omega \tag{20}$$

$$S = \int_\Omega d\Omega \tag{21}$$

$$H = \int_\Omega k \, \mu^{-1} \Delta N_p \, d\Omega \tag{22}$$

$$F_u = \int_\Omega d\Omega \tag{23}$$

$$F_p = \int_\Omega d\Omega \tag{24}$$

For this software, it is assumed that the derivative with respect to time is negligible therfore, equation (17) is simplified into the following

$$\begin{bmatrix} K & -Q \\ 0 & H \end{bmatrix} \begin{Bmatrix} \bar{u} \\ \bar{p} \end{Bmatrix} = \begin{Bmatrix} F_u \\ F_p \end{Bmatrix} \tag{25}$$

FEM Element Formulation for Q9/Q4 Elements (Taylor-R Elements)

It is important to note the assumptions made in the formulation:

-The solid and fluid are assumed compressible
-The system is in isothermal equilibrium and inertial forces are negligible
-Darcy's Law is valid
-Use of small strain theory
-Partially saturated zone
-Flow of air is without resistance
-Atmospheric pressure is always the reference pressure

$$\begin{bmatrix} K & -Q \\ 0 & H \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{u}} \\ \bar{\mathbf{p}} \end{Bmatrix} = \begin{Bmatrix} F_u \\ F_p \end{Bmatrix}$$

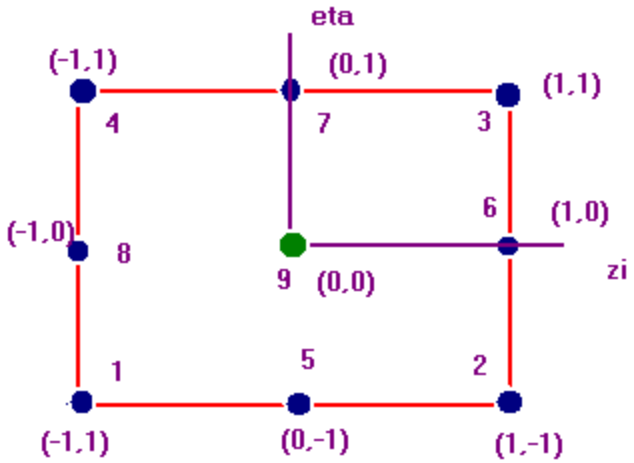

Figure 1. Q9 Reference Element (Biquadratic)

For this reference element, system of biparametric formulation is

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)\xi\eta$$

$$N_2 = -\frac{1}{4}(1+\xi)(1-\eta)\xi\eta$$

$$N_3 = \frac{1}{4}(\xi^2+\xi)(\eta^2+\eta)$$

$$N_4 = \frac{1}{4}(\xi^2-\xi)(\eta^2+\eta)$$

$$N_5 = -\frac{1}{2}(1-\xi^2)(1-\eta)\eta$$

$$N_6 = \frac{1}{2}(1+\xi)(1-\eta^2)\xi$$

$$N_7 = \frac{1}{2}(1-\xi^2)(\eta^2+\eta)$$

$$N_8 = \frac{1}{2}(\xi^2 - \xi)(1 - \eta^2)$$
$$N_9 = (1 - \xi^2)(1 - \eta^2)$$

$$N^{mat} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_9 & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_9 \end{bmatrix} \begin{bmatrix} u_{x-dir} \\ u_{x-dir} \end{bmatrix}$$

$$\vec{u} = N^{mat} u_\eta$$
$$\vec{u} = approximated\ u$$
$$u_\eta = basis\ function$$

To use reference element

* $\vec{x} = x_1 N_1 + x_2 N_2 + x_3 N_3 + \cdots$
  $\vec{y} = y_1 N_1 + y_2 N_2 + y_3 N_3 + \cdots$

for function f

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi} + \cdots$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta} + \cdots$$

$$\begin{pmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

In terms of real x, y co-ordinate

in terms of reference co-ordinate

Transform Matrix ≈ J

We know we can convert co-ordinate transformation

ex. $\int_\Omega \delta x \delta y \approx \int_\Omega \delta\xi \delta N |\det J|$

as long as the determinant of Jacobian > 0

* which is the case for isoparametric equations of any element.

then

$K = \int_\Omega B^T c B d\Omega$

$$B = LN_u$$

$$B = \begin{Bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} \dots \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_1}{\partial x} \end{Bmatrix}$$

B at

$$\begin{Bmatrix} \frac{\partial N_1}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{Bmatrix} \begin{Bmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \end{Bmatrix}$$

$$|J| = \frac{\partial \vec{x}}{\partial \xi}\frac{\partial \vec{y}}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}$$

$$\begin{Bmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial N_1}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} \end{pmatrix}$$

$$\begin{matrix} \delta x \\ \delta y \end{matrix}(N_1) = \frac{1}{|J|} \begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{pmatrix} \begin{pmatrix} \frac{\partial N_1}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} \end{pmatrix}$$

But we know all the terms on the right

$$\frac{\partial N_1}{\partial \xi} = \frac{\delta}{\delta \xi}\left(\frac{1}{4}(1-\xi)(1-\eta)\xi\eta\right)$$

and so on...

$$\frac{\partial \vec{x}}{\partial \xi} = x_1 \frac{\delta}{\delta \xi}(N_1) + x_2 \frac{\delta}{\delta \xi}(N_2) + \cdots$$

$$\frac{\partial \vec{y}}{\partial \eta} = y_1 \frac{\delta}{\delta \eta}(N_1) + y_2 \frac{\delta}{\delta \eta}(N_2) + \cdots$$

and so on to $\frac{\partial \vec{x}}{\partial \eta}, \frac{\partial \vec{y}}{\partial \xi}$

in this $x_1, x_2, \dots$

$y_1, y_2, \dots$

are co-ordinate of real <x, y> Q9 element

So now we can compute matrix B. Notice B is f(ξ, η) as $\frac{\delta N}{\delta x}$ are evaluated from (ξ, η).

But we know $K = \int_\Omega B^T C B \delta\Omega$ in which C is elastic tensor, which is provided by the user, as he/she is required to enter Poisson, Young's parameters.

Now we can compute K, for an element

$K = \int_{\Omega}^{element} B^T C B \delta\Omega$ $K = \int_{\Omega} B^T C B \delta\xi\delta\eta |\det J|$

These are integrated numerically.

$$K = \sum_{i}^{1}\sum_{j}^{1} w_c w_j B^T(\xi,\eta) C B(\xi,\eta) \det|J|$$

$$\sum w_{ij} B^T(\xi,\eta) C B(\xi,\eta) \det|J|$$

use Gauss-quadratic weighted scheme

Gauss-Legrande Quadrature Rules is 2D defined on (-1,1) square,

3*3 rule			2*2 rule		
ξ	η	ω	ξ	η	Ω
$-\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{3}{5}}$	$\frac{25}{81}$	$-\frac{1}{\sqrt{3}}$	$-\frac{11}{\sqrt{3}}$	1
0	$-\sqrt{\frac{3}{5}}$	$\frac{40}{81}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	1
$\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{3}{5}}$	$\frac{25}{81}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	1
$-\sqrt{\frac{3}{5}}$	0	$\frac{40}{81}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	1
0	0	$\frac{64}{81}$			
$\sqrt{\frac{3}{5}}$	0	$\frac{40}{81}$			
$-\sqrt{\frac{3}{5}}$	$\sqrt{\frac{3}{5}}$	$\frac{25}{81}$			
0	$\sqrt{\frac{3}{5}}$	$\frac{40}{81}$			
$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{3}{5}}$	$\frac{25}{81}$			

Now we can completely compute

$$K = \sum_{\xi,\eta} B^T(\xi,\eta)\, CB(\xi,\eta)\, w(\xi,\eta)\, \det|J(\xi,\eta)|$$

$$Q = \int_{\Omega} B^T \alpha m N_p$$

N_p is for pressure which has a Q4 element shape.

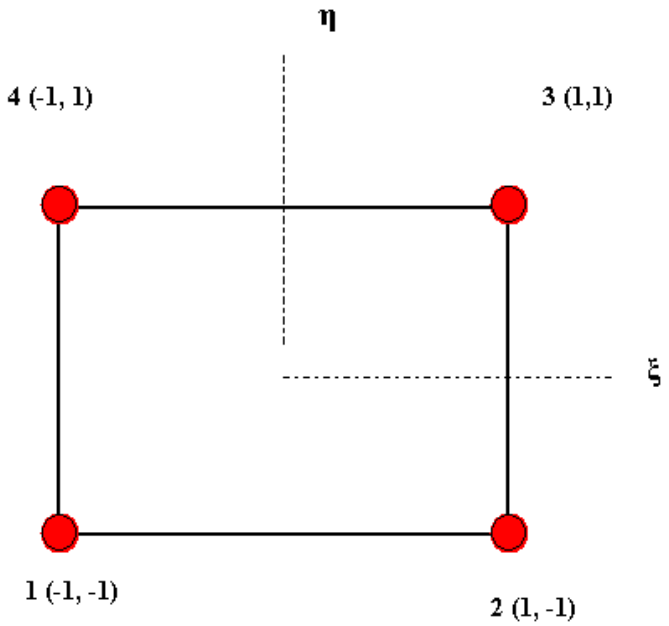

Figure 2. Q4 Reference Element

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

likewise for Q4 element, we can compute

$$N^{mat} = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

Pressure is at P(x,y,z) without direction.

then likewise with

$\vec{x}$ = " " " "

$\vec{y}$ = " " " "

as we did for Q9 element we can arrive at
$N_p(\xi,\eta) \xleftrightarrow[mapping]{} N_p(x,y)$

For N_p matrix we only need grad operator as

$$H = \int_\Omega (\nabla N_p)^T \frac{K}{\mu} (\nabla N_p) \delta\Omega$$

$$\nabla N_p = \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} < N_p^1 \quad N_p^2 \quad N_p^3 \quad N_p^4 >$$

so $$= \begin{Bmatrix} \delta x N_p & \delta x N_p^2 & \delta x N_p^3 & \delta x N_p^4 \\ \delta y N_p & \delta y N_p^2 & \delta y N_p^3 & \delta y N_p^4 \end{Bmatrix}$$

But we know that

$$\frac{\delta N_i}{\delta x} = J^{-1} \frac{\delta N_i}{\delta \xi}; \qquad \frac{\delta N_i}{\delta \eta} = J^{-1} \frac{\delta N_i}{\delta \eta}$$

$$= J^{-1} \begin{pmatrix} \frac{\delta N_1}{\delta \xi} & , & \frac{\delta N_2}{\delta \xi} & , & \dots \\ \frac{\delta N_1}{\delta \eta} & & \frac{\delta N_2}{\delta \eta} & & \end{pmatrix}$$

$$= J^{-1} < \frac{\delta \xi}{\delta \eta} >< N_1^p \quad N_1^p \quad N_1^p \quad N_1^p >$$

We know J^{-1} & $\nabla N_p(\xi,\eta)$

J^{-1} depends on x, y co-ordinate of real element for Q4.

$$H = \int_\Omega (\nabla N_p)^T \frac{K}{\mu} (\nabla N_p) \delta\Omega$$

$$H = \int_\Omega (\mathrm{J}^{-1} \nabla_{\xi,\eta} N_p)^T \frac{K}{\mu} (\mathrm{J}^{-1} \nabla_{\xi,\eta} N_p) \delta\xi\delta\eta |\det \mathrm{J}|$$

$$H = \sum_{\xi,\eta} w(\xi,\eta) (\mathrm{J}^{-1} \nabla_{\xi,\eta} N_p)^T \frac{K}{\mu} (\mathrm{J}^{-1} \nabla_{\xi,\eta} N_p) \det|\mathrm{J}|$$

$$S = \int_\Omega N_p^T \left(\frac{\alpha - n}{K_s} + \frac{n}{K_f}\right)$$

$$Q = \int_\Omega B^T \alpha m N_p \delta\Omega$$

$$F_u = \int_\Omega N_u^T \rho g d\Omega + \int_T N_u^T t dT$$

$$F_p = \int_\Omega (\nabla N_p)^T \frac{K}{\mu} \rho_f g d\Omega - \int_T N_p^T \frac{q}{\rho_f} dT$$

These can be computed using described method. We know B^T, N_p, ∇N_p in term of ξ, η we can then numerically integrate all the terms.

Acceptable Ranges

The parameters must be within a certain range, as out-of-range values would render the scenario unrealistic. The Biot's Constant as well as the specific storage coefficient must be in between 0 and 1. The shear modulus value must range from 0 to 100 GPa. The rock mass density must range from 0 to 5. The Poisson's ratio must be between -0.5 and 0.5.

Conclusion

The FEM poroelastic program shows that the code is capable of performing various simulations for 2-dimensional problems. As noted by Lewis et al. (1998), linearized elastic models have shown to be accurate to provide long term estimates of subsidence.

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Glossary

α	Biot's constant
e	Elasticity tensor
ε	Strain tensor
η	Vertical component of the isoparametric element
g	Gravitational acceleration
I	Identity matrix
J	Transform matrix
k	Permeability
Kf	Fluid bulk modulus
Ks	Solid bulk modulus
L	Stress divergence
$1/M$	Specific storage coefficient
μ	Dynamic viscosity
n	Porosity
N_p	Pore pressure interpolation function matrix
N_u	Displacement interpolation function matrix
p	Pore pressure
$\overline{p}$	Vector of pore pressure node
q	Fluid flux
ρ	Density
ρ_f	Fluid density
T	Total stress
u	Matrix displacement
$\overline{u}$:	Vector of displacement node
ξ:	Horizontal component of the isoparametric element
ζ	Increment fluid content